<u>System Function</u> (Transfer fnc.)

$$x(n) \longrightarrow \boxed{\text{L.S.I. Sys.}} \longrightarrow y(n) = x(n) * h(n)$$

$$(\underline{Z\text{-}3}) \Rightarrow \qquad Y(z) = X(z)\, H(z)$$

Z-transform of the unit-sample response, $(h(n))$, is $H(z)$ and is called <u>system function</u>.

$H(z)$ <u>evaluated on the unit circle</u> (i.e., $|z| = 1$) is the <u>freq. response</u> $H(e^{j\omega})$ $\blacksquare$

① A system is stable iff $h(n)$ is absolutely summable

② R.C. of Z-trans. contains those values of z for which $h(n)\, z^{-n}$ is abs. summable

① & ② $\Rightarrow$ If R.C. includes the unit circle, the system is <u>stable</u> and vice versa $\blacksquare$

Stable & Causal system : R.C. includes unit circle, Z-plane outside the unit circle, and $z = \infty$ $\blacksquare$

System described by an N-th order difference eqn.

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{r=0}^{M} b_r\, x(n-r)$$

Take Z-transform

$$\Rightarrow \sum_{k=0}^{N} a_k \underbrace{\mathcal{Z}[y(n-k)]}_{z^{-k} Y(z)} = \sum_{r=0}^{M} b_r \underbrace{\mathcal{Z}[x(n-r)]}_{z^{-r} X(z)}$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{r=0}^{M} b_r z^{-r}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$\boxed{H(z) = \frac{A \prod_{r=1}^{M} (1 - c_r z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}}$$

a_0 & $b_0 \neq 0$

each term produces a pole and a zero

$$(1 - c_r z^{-1}) \Rightarrow \begin{cases} \text{zero at } z = c_r \\ \text{Pole at } z = 0 \end{cases}$$

$$(1 - d_k z^{-1}) \Rightarrow \begin{cases} \text{Pole at } z = d_k \\ \text{zero at } z = 0 \end{cases}$$

① Causal system $\Rightarrow$ R.C. $= \{ |z| > \max \{d_k\} \}$

② stable system

d_i, for which $|d_i| > 1$, is a L.H. Pole

d_i, " " $|d_i| < 1$, " " R.H. "

i.e., annular R.C. includes the unit circle

③ Causal & stable system

$$\begin{cases} |d_i| < 1 \\ d_i \text{ is a R.H. pole} \end{cases} \quad \forall i$$

i.e., R.C. is exterior of a circle passing through the pole of $H(z)$ that is farther from the origin and includes the unit circle.

In terms of z

If $N > M$

$$H(z) = \frac{A\, z^{N-M} \prod_{r=1}^{M} (z - c_r)}{\prod_{k=1}^{N} (z - d_k)}$$

If $N < M$

$$H(z) = \frac{A \prod_{r=1}^{M} (z - c_r)}{z^{M-N} \prod_{k=1}^{N} (z - d_k)}$$

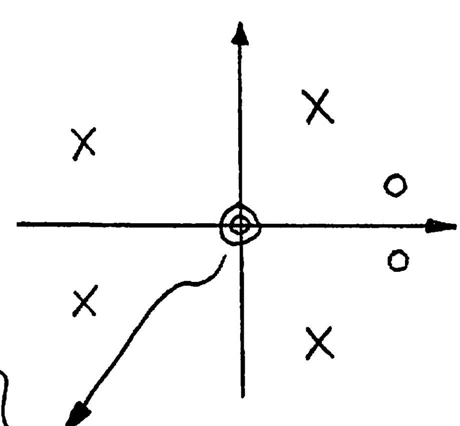
if we shift $X(z)$ then # pole ≠ # zeros

① If $A \neq 0$, # of zeros = # of poles

② $H(z) = \frac{b(z)}{a(z)}$ If both $a(z)$ and $b(z)$ have real coefficients; then, all complex poles (zeros) appear in pairs (i.e., pole + its conjugate and same for zeros)

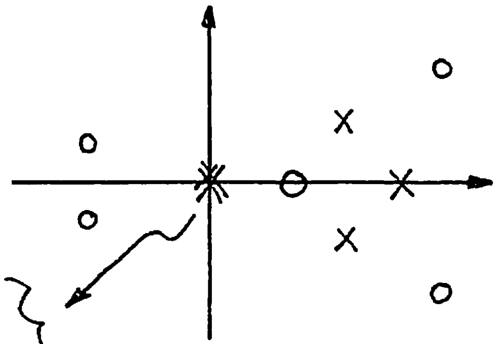
Ex. $N = 4$ $M = 2$

Pole-zero plot:



added to equalize # of poles & # of zeros

Ex. $N = 3$, $M = 5$



add 2 poles to make # of poles = # of zeros

Ex. A causal system

$$y(n) = a\,y(n-1) + x(n)$$

$$Y(z) = a z^{-1} Y(z) + X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a z^{-1}}$$: system function

Causality $\Rightarrow$ $|z| > |a|$ = R.C.

$\therefore$ Impulse response is $\boxed{h(n) = a^n u(n)}$

In equation

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{r=0}^{M} b_r x(n-r)$$

If $N=0$, then $a_0 y(n) = \sum_{r=0}^{M} b_r x(n-r)$

or $H(z) = A \prod_{r=1}^{M} (1 - C_r z^{-1})$

i.e., the system has no poles except at $z=0$, the unit sample response $(h(n))$ has a finite duration.

If $N>0$, the system has poles, each of which contributes an exponential sequence to the unit-sample response; hence, $h(n)$ is of infinite duration.

Geometric approach

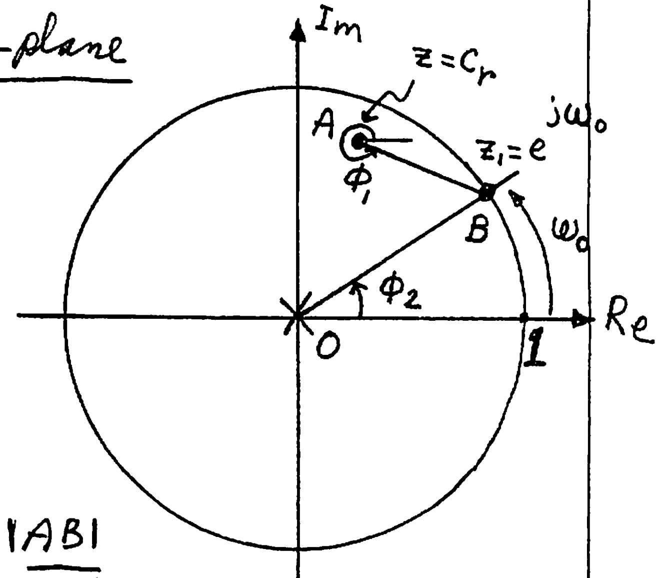
To determine the system function on the unit circle, corresponding to an excitation freq. ω_0, substitute $z = e^{j\omega_0}$ into

$$H(z) = \frac{A \prod_{r=1}^{M} (1 - c_r z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

i.e., find the freq. response, $H(e^{j\omega})$, for sinusoidal inputs. Recall that $H(e^{j\omega})$ is equal to behavior of $H(z)$ on the unit circle (i.e., $z = e^{j\omega}$)

Ex.

Consider a factor $(1 - c_r z^{-1}) \Rightarrow \begin{cases} \text{pole at } z = 0 \\ \text{zero at } z = c_r \end{cases}$



$$(1 - c_r z^{-1}) = \frac{z - c_r}{z}$$

$$\Rightarrow (1 - c_r e^{-j\omega_0}) = \frac{z_1 - c_r}{z_1}$$

where $z_1 = e^{j\omega_0}$

Then

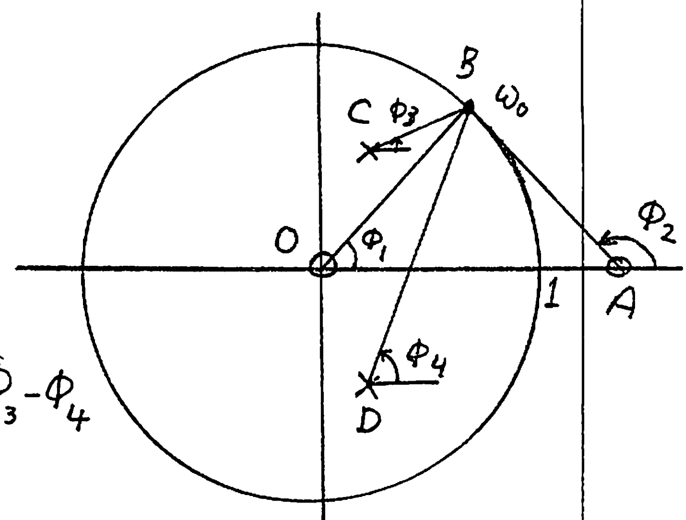
$$\begin{cases} \left| 1 - c_r z_1^{-1} \right| = \dfrac{|z_1 - c_r|}{|z_1|} = \dfrac{|AB|}{|OB|} \\ \text{Angle}\,(1 - c_r z_1^{-1}) = \measuredangle\,(\overrightarrow{z_1 - c_r}) - \measuredangle\,(\overrightarrow{z_1 - 0}) \end{cases}$$

$$= \phi_1 - \phi_2$$

In general

$$\begin{cases} |H(z_1)| = \dfrac{|A| \prod_{i=1}^{M} \text{length}\,(\overrightarrow{z_1 - \text{Zero}_i})}{\prod_{i=1}^{N} \text{length}\,(\overrightarrow{z_1 - \text{Pole}_i})} \\ \measuredangle H(z_1) = \measuredangle A + \sum_{i=1}^{M} \measuredangle (z_1 - \text{zero}_i) - \sum_{i=1}^{N} \measuredangle (z_1 - \text{Pole}_i) \end{cases}$$

Ex. 2^{nd} order system



$$\begin{cases} |H(e^{j\omega_0})| = |A| \dfrac{\overline{OB} \cdot \overline{AB}}{\overline{CB} \cdot \overline{DB}} \\ \text{Angle}\,(H(e^{j\omega})) = \measuredangle A + \Phi_1 + \Phi_2 - \Phi_3 - \Phi_4 \end{cases}$$

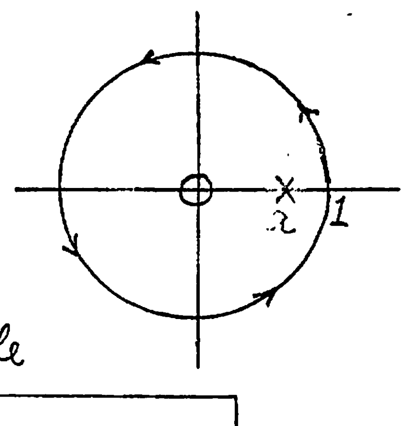
Note :

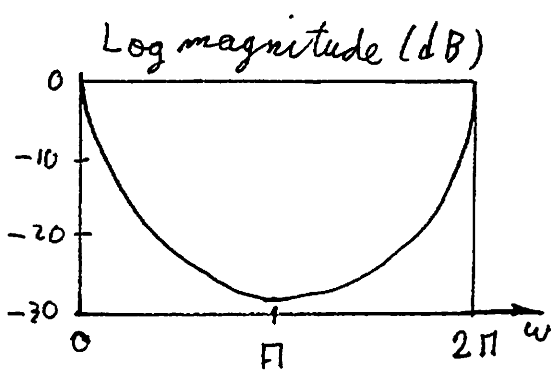
1) Poles ~~on~~ zeros at the origin offer no contribution to the $|H(e^{j\omega})|$ and introduce only a linear component in the phase

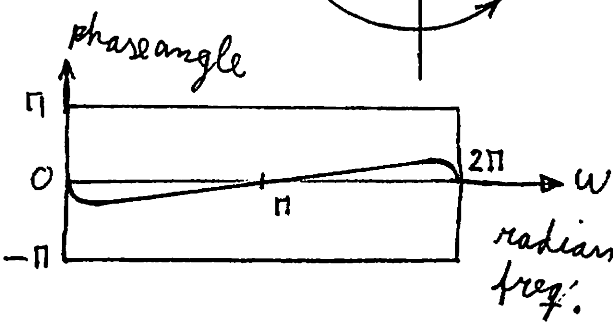
2) $|H(e^{j\omega_0})|$ peaks in the vicinity of the poles.

Ex. 1 1^{st} order system

$$H(z) = \frac{1}{1 - a z^{-1}} \Leftrightarrow h(n) = a^n u(n)$$

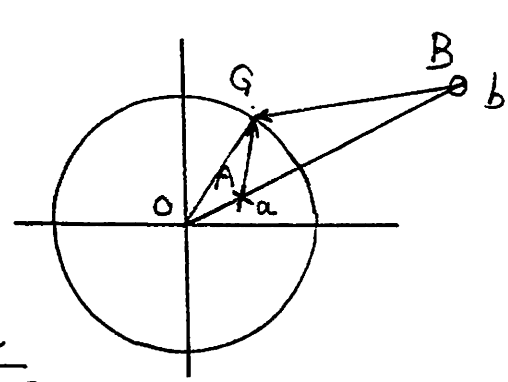






Ex. 2

$$H(z) = \frac{1 - b z^{-1}}{1 - a z^{-1}}$$



$$b = (a^*)^{-1}$$

If $a = \alpha + j\beta$

$$(a^*)^{-1} = \frac{1}{\alpha - j\beta} = \frac{\alpha + j\beta}{\alpha^2 + \beta^2} = \frac{a}{|a|^2}$$

$$|H(e^{j\omega})| = \frac{GB}{GA} = \text{constant} \rightarrow \underline{\text{Pf.}}$$

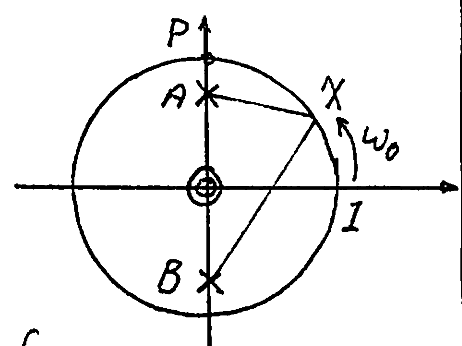
$$\left\{ \begin{aligned} &\overline{OA} \cdot \overline{OB} = |a||b| \\ &\qquad = |a| \cdot \frac{|a|}{|a|^2} = 1 \\ &\therefore \frac{OA}{OG} = \frac{OG}{OB} \\ &\Rightarrow \Delta\, OAG \sim \Delta\, OGB \\ &\Rightarrow \frac{GB}{GA} = \frac{OG}{OA} = \frac{1}{|a|} = \text{const.} \end{aligned} \right.$$

$\therefore$ Magnitude is constant for all values of ω_0

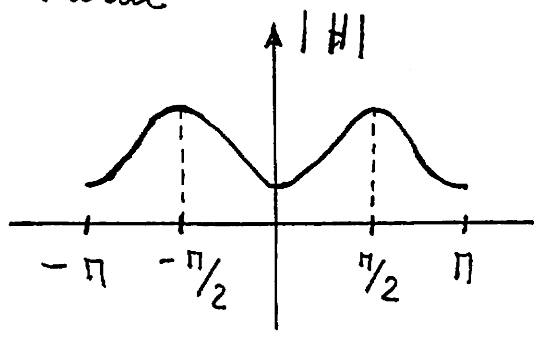
$\Rightarrow$ All-pass filter

Ex. 3

$$|H(e^{j\omega})| = \frac{1}{\overline{XA} \cdot \overline{XB}}$$



as x goes to point P, the denominator has its min. value (i.e., at $\omega = \pi/2$



$\therefore$ Band pass filter