chapter 2 Z - Transform

$$X(z) = \mathcal{Z}[x(n)] \triangleq \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{(two sided)}$$

z : complex variable

$$X_I(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \quad \text{(one sided)}$$

which we don't use

In polar form $\boxed{z = r e^{j\omega}}$ then

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) (re^{j\omega})^{-n} = \sum_n x(n) r^{-n} e^{-j\omega n}$$

For $r = 1$ (i.e., $|z| = 1$) $\mathcal{Z}$-trans. = F.T. of a seq.

$\therefore$ Z-transform is more general

Ex. $x(n) = a^n u(n) \qquad |a| > 1$

F. transform doesn't exist

$$X(z) = \sum_n a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

converges if $|z| > |a|$

Region of Convergence (R.C.)

$\mathcal{Z}[x(n)]$ is uniformly converged at $z_0 = r_0 e^{j\omega_0}$ iff $x(n) z_0^{-n}$ is absolutely summable

i.e., $\sum_n |x(n) z_0^{-n}| < \infty$

$$\Rightarrow \boxed{\sum_{n=-\infty}^{\infty} |x(n) r_0^{-n}| < \infty} \qquad (I)$$

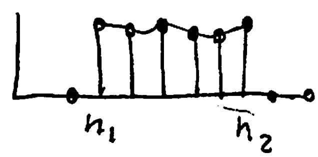
Def. : R.C. is the set of values of z for which (I) holds.

R.C. = Ring , annular region

$$= \{ z \,/\, R_- < |z| < R_+ \} \qquad \begin{matrix} R_- \geq 0 \\ R_+ \leq \infty \end{matrix}$$

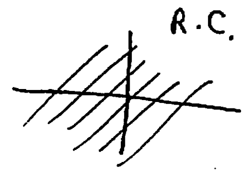
Consider the following cases

① $x(n)$ is a finite length seq.

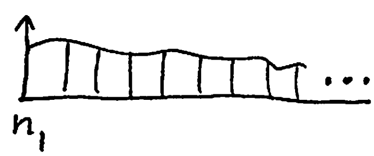


R.C. : all values of z

except $\begin{cases} z = \infty & \text{if } n_1 < 0 \\ z = 0 & \text{if } n_2 > 0 \end{cases}$

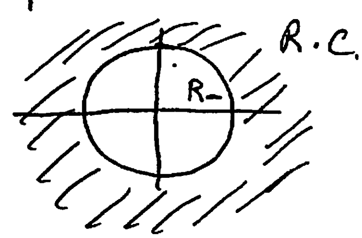


② $x(n)$ is a Right-Sided seq.

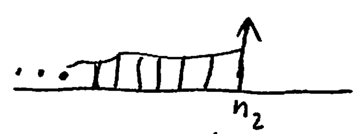


n_1 may be < 0

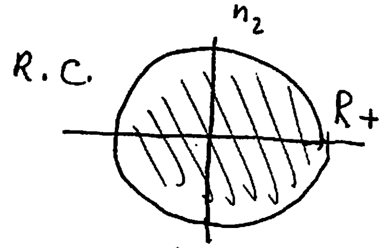
If $n_1 \geq 0$ R.C. includes $z = \infty$



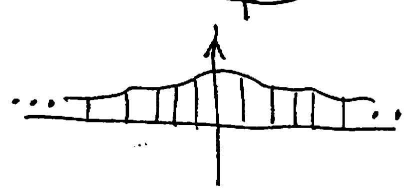
③ $x(n)$ is a Left-sided seq



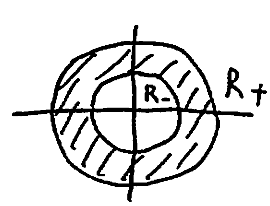
If $n_2 > 0$ R.C. does not include $z = 0$



④ Two sided sequence



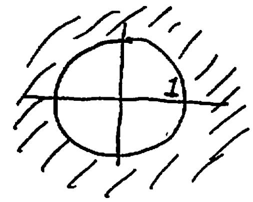
R.C.



Ex. 1 R. sided seq.

a) $x(n) = u(n)$

$$T = \sum_{n=-\infty}^{\infty} |x(n) r_0^{-n}| = \sum_{n=0}^{\infty} r_0^{-n} < \infty$$



$\Rightarrow$ R.C. is $r_0 = |z| > 1$

b) $x(n) = a^n u(n)$

$$T = \sum_{n=0}^{\infty} |a|^n r_0^{-n} \quad \Rightarrow \quad R.C. = r_0 = |z| > |a|$$

Ex. 2 L. sided seq.

a) $x(n) = u(-n)$

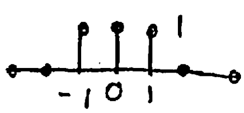
$$T = \sum_{n=-\infty}^{\infty} |x(n) r_0^{-n}| = \sum_{-\infty}^{0} r_0^{-n} < \infty \quad \Rightarrow \quad r_0 < 1$$

$\therefore$ R.C. $= 0 \leqslant |z| < 1$

b) $x(n) = a^n u(-n)$

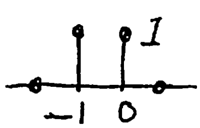
$$T = \sum_{-\infty}^{0} |a|^n r_0^{-n} \quad \Rightarrow |z| < |a|$$

Ex. 3 Finite length seq.

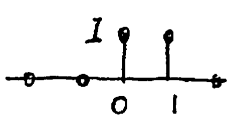
a) $x(n) =$ 

$T = r_0 + 1 + r_0^{-1}$

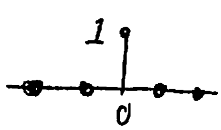
R.C. $= 0 < |z| < \infty$

b) 

$T = r_0 + 1 \quad \Rightarrow$ R.C. $= |z| < \infty$

c) 

$T = 1 + r_0^{-1}$, $r_0 \neq 0 \Rightarrow$ R.C. $= |z| > 0$

d) 

$T = 1$ R.C. : z-plane

$\delta(\cdot)$ Funtion converges everywhere

Ex. 4 Two sided

$$x(n) = \begin{cases} a^n & n \geq 0 \\ b^n & n < 0 \end{cases}$$

$$T = \sum_{n=-\infty}^{\infty} |x(n)\, r_0^{-n}| = \underbrace{\sum_{-\infty}^{-1} |b|^n r_0^{-n}}_{r_0 < |b|} + \underbrace{\sum_{0}^{\infty} |a|^n r_0^{-n}}_{r_0 > |a|}$$

R.C. $\Rightarrow |a| < |z| < |b|$ if $|a| < |b|$

otherwise no convergence

If $X(z) = \dfrac{a(z)}{b(z)}$

Roots of $a(z)$ are called zeros of $X(z)$

" " $b(z)$ " " Poles " "

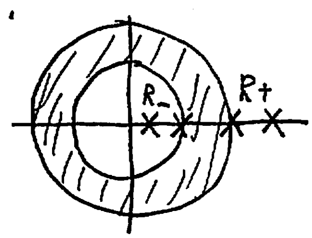
Properties of R.C.

1) For a causal seq. $R_+ = \infty$?

2) - Poles located exterior to the R.C. are L.H.S. poles

- " " interior " " " " R.H.S. "

- no poles located inside the R.C.



3) The Z-trans. must be analytic in the R.C.

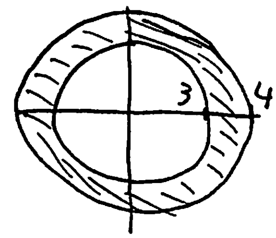
4) A system is stable iff the unit circle is included in its R.C.

stable sys. $\Rightarrow \begin{cases} \text{LHS poles} > 1 \\ \text{RHS poles} < 1 \end{cases}$

Ex.

$$x(n) = \begin{cases} 2^n + 3^n & n \geq 0 \\ 4^n + 5^n & n < 0 \end{cases}$$

not stable



5) If a sys. is stable then its F.T. exists.

Inverse Z-transform

Cauchy integral theorem

$$\frac{1}{2\pi j} \oint_C z^{k-1}\, dz = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} = \delta(k)$$

$$C \begin{cases} \text{1- counterclockwise contour} \\ \text{2- encircles the origin} \\ \text{3- entirely in the R.C. of } X(z) \end{cases}$$

$$\boxed{\mathcal{Z}^{-1}[X(z)] = x(n) = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz}$$

proof not needed

Calculation of the contour integral

Residue Theorem (For rational $X(z)$)

$$\boxed{x(n) = \sum [\text{residues of } X(z)\, z^{n-1} \text{ at the poles inside } C]}$$

Let $X(z)\, z^{n-1} = \dfrac{\Psi(z)}{(z - z_0)^s}$

where $X(z)\, z^{n-1}$ has s poles at $z = z_0$ & $\Psi(z)$ has no poles at $z = z_0$, then

$$\text{Res}\left[X(z)\, z^{n-1} \text{ at } z = z_0\right] = \frac{1}{(s-1)!} \left[\frac{d^{s-1}\, \Psi(z)}{d z^{s-1}}\right]_{z = z_0}$$

For $S=1$

$$\text{Res}\,[\,X(z)\,z^{n-1} \text{ at } z=z_0\,] = \varphi(z_0)$$

Read PP. 54-55

Other methods

① **Power series** $\quad X(z) = \sum_n x(n)\,z^{-n}$

$x(n)$ is the coefficient of the terms with z^{-n} in the power series

Ex. $\quad X(z) = e^{z} \qquad R.C. = \{|z| \neq \infty\}$

We want $\mathcal{Z}^{-1}\{e^{z}\} =$ L. sided seq.

$$e^{z} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{k} = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-(-k)}$$

Let $k=-k' \Rightarrow e^{z} = \sum_{k'=0}^{-\infty} \frac{1}{(-k')!} z^{-k'} = \sum_{k'=-\infty}^{0} \frac{1}{(-k')!} e^{-k'}$

$$\therefore \; x(n) = \frac{1}{(-n)!}\,u(-n)$$

Long division

Ex. $\quad X(z) = \frac{1}{1-az^{-1}} \qquad |z| > |a|$

$\Downarrow$

Right-sided seq.

$z \to \infty \Rightarrow X(z) \to$ const. $\Rightarrow$ Causal seq.

$$\therefore \; \frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \cdots$$

$$\Rightarrow x(n) = a^n u(n)$$

② Partial-Fraction Expansion

Ex. a right-sided seq. with

$$X(z) = \frac{1}{(1-2z^{-1})(1-.5z^{-1})} = \cdots = \frac{4/3}{1-2z^{-1}} + \frac{-1/3}{1-.5z^{-1}},$$

$$\Rightarrow x(n) = (4/3)(2^n)u(n) - \frac{1}{3}(.5)^n u(n)$$

Properties of $\mathcal{Z}$-transform

Let $\mathcal{Z}[x(n)] = X(z)$ R.C.: $R_{x-} < |z| < R_{x+}$

Z-1 Multiplication by an exponential sequence

$\mathcal{Z}[a^n x(n)] = X(a^{-1}z)$ "a" may be complex

R.C. : $|a| R_{x-} < |z| < |a| R_{x+}$

Pole of $X(z)$ at $z = z_1$ $\Longrightarrow$ Pole of $X(a^{-1}z)$ at $z = a z_1$

Z-2 Shift of a sequence

$\mathcal{Z}[x(n+n_0)] = z^{n_0} X(z)$, $R_{x-} < |z| < R_{x+}$

R.C. identical to that of $X(z)$, with the possible exception of $z=0$ or $z=\infty$

Ex. $\mathcal{Z}[\delta(n)]$ converges $\forall z$

$\mathcal{Z}[\delta(n-1)]$ does not converge at $z=0$

Z-3 convolution of sequences

$\mathcal{Z}[f(n) * g(n)] = F(z)\, G(z)$

R.C. $\max(R_{f-}, R_{g-}) < |z| < \min(R_{f+}, R_{g+})$

Z-4 Complex convolution

$$\mathcal{Z}[f(n) \cdot g(n)] = \frac{1}{2\pi j} \oint_C F(v)\, G\left(\frac{z}{v}\right) v^{-1}\, dv$$

R.C. $R_{f-} R_{g-} < |z| < R_{f+} R_{g+}$ Common ROC

Z-5 $\mathcal{Z}[x(\beta n)] = \frac{1}{\beta}\, X\left(z^{1/\beta} e^{-j2\pi k/\beta}\right)$

R.C. : $(R_{x-})^{\beta} < |z| < (R_{x+})^{\beta}$

Z-6 Parseval's theorem

$$\sum_{n=-\infty}^{\infty} x(n)\, y^{*}(n) = \frac{1}{2\pi j} \oint_{C} X(v)\, Y(1/v)\, v^{-1}\, dv$$

Z-7 $\mathcal{Z}[x(-n)] = X(1/z)$ R.C. $\frac{1}{R_{x+}} < |z| < \frac{1}{R_{x-}}$

Z-8 conjugate of a complex sequence

$\mathcal{Z}[x^{*}(n)] = X^{*}(z^{*})$ R.C. $R_{x-} < |z| < R_{x+}$

Z-9 Linearity

$\mathcal{Z}[a\,x(n) + b\,y(n)] = a\,X(z) + b\,Y(z)$

R.C. $\max(R_{x-}, R_{y-}) < |z| < \min(R_{x+}, R_{y+})$ or larger (pole zero cancelation)

Z-10 Differentiation of $X(z)$

$\mathcal{Z}[n\,x(n)] = -z \dfrac{d X(z)}{d(z)}$ $R_{x-} < |z| < R_{x+}$

Z-11 Initial value theorem

If $x(n) = 0$, $n < 0$

Then $x(0) = \lim_{z \to \infty} X(z)$

For other properties see Table 2.1 , Pg. 67