Fourier Transform Properties

* For continuous-time Fourier transform ($x_a(t) \Leftrightarrow X_a(j\Omega)$)

a-1 Freq. transform

$$\mathcal{F}_a\left[e^{j\Omega_0 t} x_a(t)\right] = X_a(\Omega - \Omega_0)$$

a-2 Time transform

$$\mathcal{F}_a\left[x_a(t - t_0)\right] = X_a(j\Omega)\, e^{-j\Omega t_0}$$

* {For Fourier transform of a sequence $x(n) \Leftrightarrow X(e^{j\omega})$)
DTFT

S-1 $$\mathcal{F}_s\left[e^{j\omega_0 n} x(n)\right] = X\left(e^{j(\omega - \omega_0)}\right)$$

S-2 $$\mathcal{F}_s\left[x(n - n_0)\right] = X(e^{j\omega})\, e^{-j\omega n_0}$$

S-2 Pf.

$$\mathcal{F}_s\left[x(n - n_0)\right] = \sum_{n=-\infty}^{\infty} x(n - n_0)\, e^{-j\omega n}$$ Let $n - n_0 = n'$

$$= \sum_{n'=-\infty}^{\infty} x(n')\, e^{-j\omega(n' + n_0)}$$

$$= \sum_{n'=-\infty}^{\infty} x(n')\, e^{-j\omega n'}\, e^{-j\omega n_0}$$

$$= X(e^{j\omega})\, e^{-j\omega n_0}$$

* C.F.T.

a-3 Convolution

$$\mathcal{F}_a[f(t) * g(t)] = F_a(j\Omega)\,G_a(j\Omega)$$

a-4 Product

$$\mathcal{F}_a[f(t)\cdot g(t)] = \frac{1}{2\pi}\,F_a(j\Omega) * G_a(j\Omega)$$

* S.F.T. (DTFT)

discrete

S-3 $\mathcal{F}[f(n) * g(n)] = F(e^{j\omega})\,G(e^{j\omega})$

S-4 $\mathcal{F}[f(n)\cdot g(n)] = \frac{1}{2\pi}\,F(e^{j\omega}) \underbrace{*_p}_{\text{Periodic convolution}} G(e^{j\omega})$

~~or Circular Conv.~~

a windowed version of this sinus circular convolution

$$\triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\beta})\,G(e^{j(\omega-\beta)})\,d\beta$$

S-4 Pf.

$$\text{R.H.S.} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} f(n)\,e^{-j\beta n}\right)\left(\sum_{k=-\infty}^{\infty} g(k)\,e^{-j(\omega-\beta)k}\right)d\beta$$

$$= \sum_{n=-\infty}^{\infty}\sum_{k} f(n)\,g(k)\,e^{-j\omega k}\int_{-\pi}^{\pi}\frac{e^{-j\beta(n-k)}}{2\pi}\,d\beta$$

$$= \sum_{n}\sum_{k} f(n)\,g(k)\,e^{-j\omega k}\underbrace{\left(\frac{1}{-j(n-k)2\pi}\right)e^{-j\beta(n-k)}\Bigg|_{-\pi}^{\pi}}$$

so second sum collapses to just one term.

$$= \dots = \frac{\sin((n-k)\pi)}{(n-k)\pi} = \delta(n-k)$$

$$\therefore \text{R.H.S} = \sum_{n} f(n)\,g(n)\,e^{-j\omega n}$$

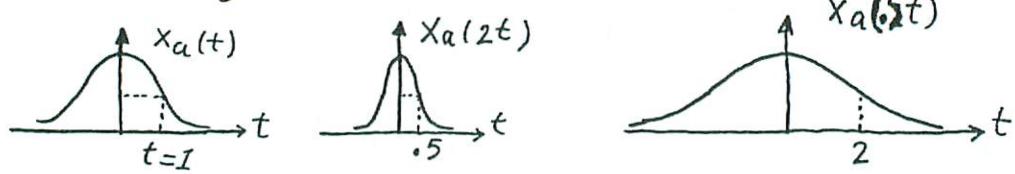
$$= \mathcal{F}[f(n)\,g(n)] = \text{L.H.S.}$$

$\blacksquare$

* C.F.T.

a-5 Scaling

$x_a(\alpha t)$



$$\mathcal{F}_a[x_a(\alpha t)] = \frac{1}{|\alpha|} X_a\left(j\frac{\Omega}{\alpha}\right)$$

a-6 Parseval's theorem

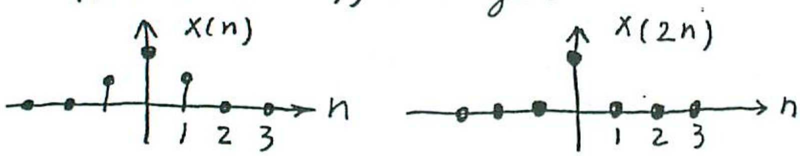
$$\int_{-\infty}^{\infty} |x_a(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_a(j\Omega)|^2 d\Omega$$

$$\int_{-\infty}^{\infty} x_a(t)\, y_a^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)\, Y^*(j\Omega)\, d\Omega$$

* S.F.T. (DTFT)

S-5 $$\mathcal{F}[x(\beta n)] = \frac{1}{\beta}\sum_{k=0}^{\beta-1} X\left(e^{j\frac{\omega-2\pi k}{\beta}}\right)$$

$\beta > 0$ $\quad$ β : integer



$\Leftarrow$ { Effect of scaling

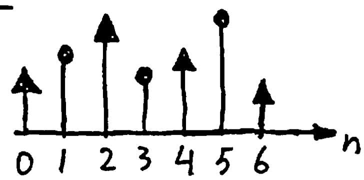
S-6 $$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

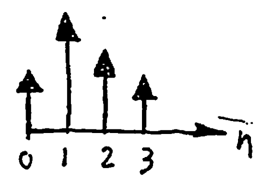
$$\sum_n x(n)\, y^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, Y^*(e^{j\omega})\, d\omega$$

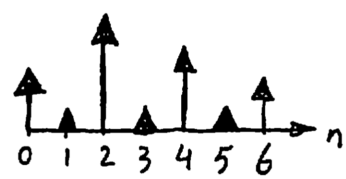
$$\begin{cases} g(n) = x(\beta n) \\ G(e^{j\omega}) = \dfrac{1}{\beta} \displaystyle\sum_{k=0}^{\beta-1} X\left(e^{j\frac{\omega - 2\pi k}{\beta}}\right) \end{cases}$$

Let $f(n) = \begin{cases} x(n) & n = m\beta \\ 0 & n \neq m\beta \end{cases}$

Ex. $\beta = 2$

For $x(n) \Rightarrow$ 

$g(n) \Rightarrow$ 

then $f(n)$ will be $\Rightarrow$ 

For $f(n)$ we will have

$$\begin{cases} ① \quad F(\omega) = G(\beta\omega) \\ ② \quad F(\omega) = \dfrac{1}{\beta} \displaystyle\sum_{k=0}^{\beta-1} X\left(e^{j\left(\omega - \frac{2\pi k}{\beta}\right)}\right) \end{cases}$$

Pf. ①

$$F(\omega) = \sum_{n=-\infty}^{\infty} f(n) e^{-j\omega n} = \sum_{n=m\beta} f(n) e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} f(m\beta) e^{-j\omega m\beta}$$

$$= \sum_{m} g(m) e^{-j(\omega\beta)m} = G(\beta\omega)$$

Pf. (2) $\qquad \frac{1}{B}\sum_{k=0}^{B-1} e^{j\frac{2\pi}{B}nk} = \begin{cases} 1 & n = mB \\ 0 & n \neq mB \end{cases}$

check this

$$\therefore\ f(n) = x(n)\cdot\frac{1}{B}\sum_{k=0}^{B-1} e^{j\frac{2\pi}{B}nk}$$

$$= \frac{1}{B}\sum_{k=0}^{B-1}\left[x(n)\cdot e^{j\frac{2\pi}{B}nk}\right]$$

$$F(\omega) \triangleq \sum_{n=-\infty}^{\infty}\frac{1}{B}\sum_{k=0}^{B-1} x(n)\, e^{j\frac{2\pi}{B}nk}\, e^{-j\omega n}$$

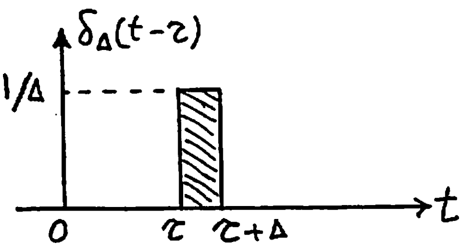
$$= \frac{1}{B}\sum_{k=0}^{B-1}\sum_{n} x(n)\, e^{-j\left(\omega-\frac{2\pi}{B}k\right)n}$$

$$= \frac{1}{B}\sum_{k=0}^{B-1} X\left(e^{j\left(\omega-\frac{2\pi}{B}k\right)}\right)$$

Relation between continuous time signals

$\delta(t)$ $\quad \begin{cases} (1) & \delta(t-\tau) = \begin{cases} 0 & t \neq \tau \\ \infty & t = \tau \end{cases} \\ (2) & \int_{-\infty}^{\infty}\delta(t-\tau)\,dt = 1 \end{cases}$

$$\delta(t-\tau) = \lim_{\Delta\to 0}\delta_\Delta(t-\tau)$$



Properties

(1) $\quad \delta(t-\tau) = \delta(\tau-t)$

(2) $\quad \int_{-\infty}^{\infty}\delta(t-\tau)\,f(t)\,dt = f(\tau)$

$$\Rightarrow \int_{-\infty}^{\infty}\delta(t-\tau)\,f(\tau)\,d\tau = f(t) \qquad (\mathrm{I})$$

Pf. 2

$$\delta(t-\tau) f(t) = \begin{cases} 0 & t \neq \tau \\ f(\tau)\delta(t-\tau) & t = \tau \end{cases}$$

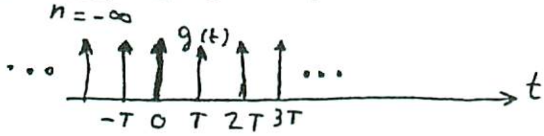
$$\therefore \text{L.H.S.} = \int_{-\infty}^{\infty} \delta(t-\tau) f(\tau)\, dt = f(\tau) \int_{-\infty}^{\infty} \delta(t-\tau)\, dt = f(\tau)$$

$$\text{Eqn. (I)} \Rightarrow \int_{-\infty}^{\infty} \delta(t-\tau) f(\tau)\, d\tau = \boxed{\delta(t) * f(t) = f(t)}$$

and $\boxed{\delta(t-\alpha) * f(t) = f(t-\alpha)}$

sampling function

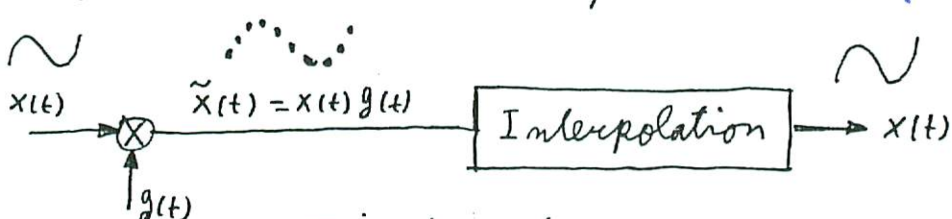
$$g(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t-nT) \qquad T \text{; sampling interval}$$



$$\mathcal{F}_c[\delta_c(t-\tau)] = \int_{-\infty}^{\infty} \delta_c(t-\tau) e^{-j\Omega t}\, dt = e^{-j\Omega \tau}$$

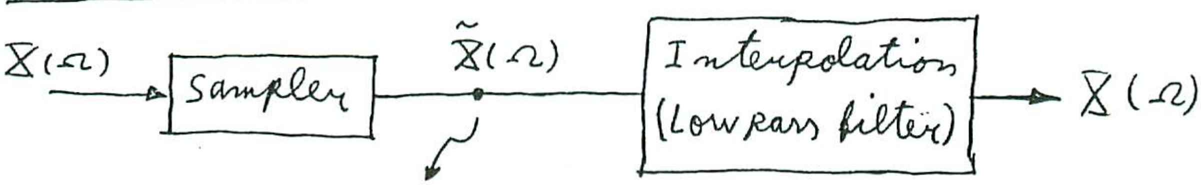
$$\Rightarrow \mathcal{F}_c[g(t)] = \sum_{n=-\infty}^{\infty} e^{-j\Omega nT} = \boxed{\frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi n}{T}\right)}$$

$\frac{2\pi}{T} = w_s$ = sampling frequency   Sampling frequency (radian)

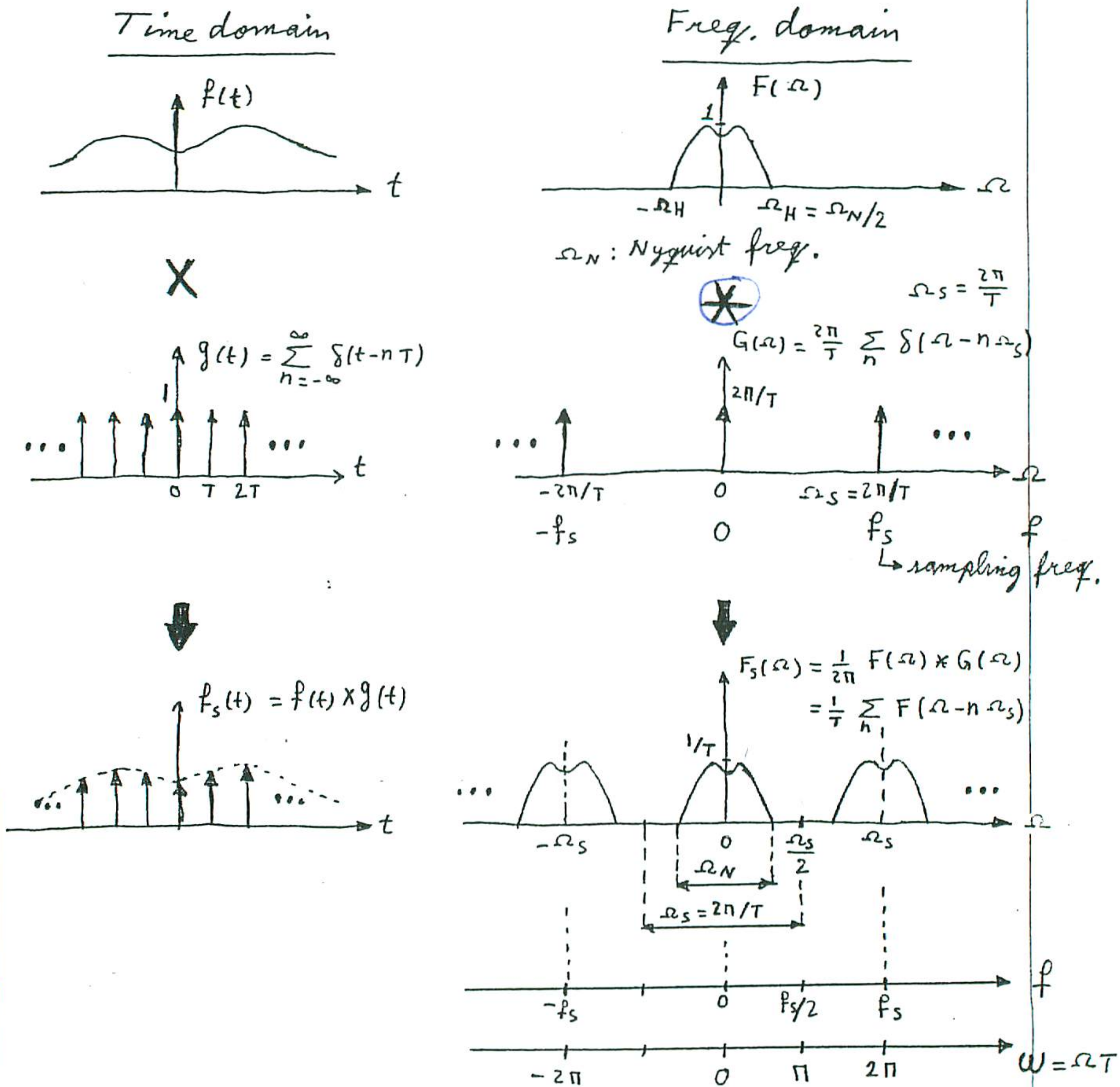


Time domain

Freq. domain



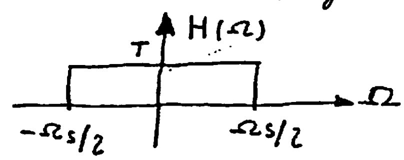
$$\tilde{X}(\Omega) = \frac{1}{2\pi} X(\Omega) * G(\Omega)$$



$$
\begin{cases}
\Omega_N \leqslant \Omega_s & \Rightarrow \text{No overlapping (No Aliasing)} \\
\Omega_N > \Omega_s & \Rightarrow \text{Aliasing}
\end{cases}
$$

Ω_s and Ω_N are sometimes shown by ω_s and ω_N but what they represent is the continuous radian freq. (not the discrete freq.)

To recover $F(\Omega)$ from $F_s(\Omega)$ we multiply $F_s(\Omega)$ by $H(\Omega)$ in freq. domain.



Time domain :

$$h(t) = \mathcal{F}_c^{-1}(\underbrace{H(\Omega)}_{\text{L.P. filter}})$$

$$\boxed{h(t) = \frac{\sin\frac{\pi}{T}t}{\frac{\pi}{T}t}} = \operatorname{sinc}\frac{\pi}{T}t \quad : \text{interpolation function}$$

Using the notations shown on page 6

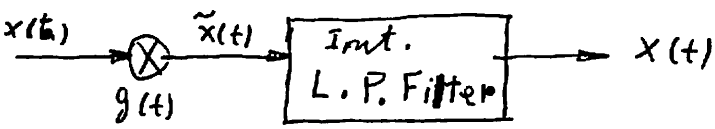
$$x(t) = \tilde{x}(t) * h(t) = [x(t)\underbrace{\sum_{n=-\infty}^{\infty}\delta(t-nT)}_{g(t)}] * h(t)$$

$$= [\sum_n x(t)\,\delta(t-nT)] * h(t) = [\sum_n \underbrace{x(nT)}_{\text{Const.}}\,\delta(t-nT)] * h(t)$$

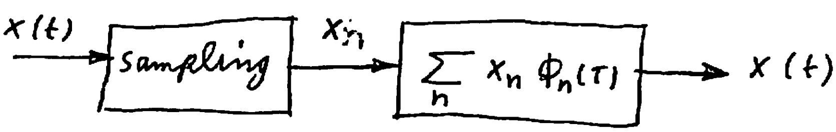
$$= \sum_n x(nT)\,h(t-nT) \qquad \text{(same as before)}$$

$$= \sum_{n=-\infty}^{\infty} x_n \cdot \phi_n(T) \qquad \begin{cases} x_n = x(nT) \\ \phi_n(T) = h(t-nT) \end{cases}$$

Math.



Practice



Sampling ⟹ repeated spectrum

Interpolation ⟹ truncated spectrum