Handout H 4/19/2010

③ Bilinear Transformation

A better approximation for $\frac{d}{dt}$: Use Taylor's expansion

Ⓘ $$y(t-T) = y(t) - T y'(t) + \frac{T^2}{2} y''(t) - \frac{T^3}{6} y'''(t) + \dots$$

$$\Rightarrow \frac{y(t) - y(t-T)}{T} = y'(t) - \frac{T}{2} y''(t) + \frac{T^2}{6} y'''(t) \dots \quad ①$$

do same for $y'(t)$:

Ⓘⓘ $$y'(t-T) = y'(t) - T y''(t) + \frac{T^2}{2} y'''(t) - \frac{T^3}{6} y''''(t) + \dots$$

$$\Rightarrow y'(t) + y'(t-T) = 2y'(t) - T y''(t) + \frac{T^2}{2} y'''(t) - \frac{T^3}{6} y''''(t) + \dots$$

$$\Rightarrow \frac{y'(t) + y'(t-T)}{2} = y'(t) - \frac{T}{2} y''(t) + \frac{T^2}{4} y'''(t) - \dots \quad ②$$

$$① \,\&\, ② \Rightarrow \frac{y(t) - y(t-T)}{T} \simeq \frac{y'(t) + y'(t-T)}{2}$$

$$\Rightarrow \frac{1-D}{T} y(t) \cong \frac{1+D}{2} \frac{d}{dt}(y(t))$$

better approximation to derivative

$$\Rightarrow \frac{d}{dt} y(t) \cong \frac{2}{T} \left(\frac{1-D}{1+D} \right) y(t)$$

So bilinear Transform is better approximation to derivative

$$\Rightarrow \boxed{s \sim \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}$$

$$\Rightarrow \boxed{z = \frac{1 + s\frac{T}{2}}{1 - s\frac{T}{2}}}$$

Bilinear Transformation

A different way of getting this (trapezoidal approx.) is shown in the text

$$H(z) = H_a(s)\Big|_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Digital frequency response

$$z = \frac{1 + s\frac{T}{2}}{1 - s\frac{T}{2}}$$

Let $s = j\Omega \quad \Rightarrow \quad |z| = \left|\frac{1 + j\Omega T/2}{1 - j\Omega T/2}\right| = 1$

Let $z = e^{j\omega}$

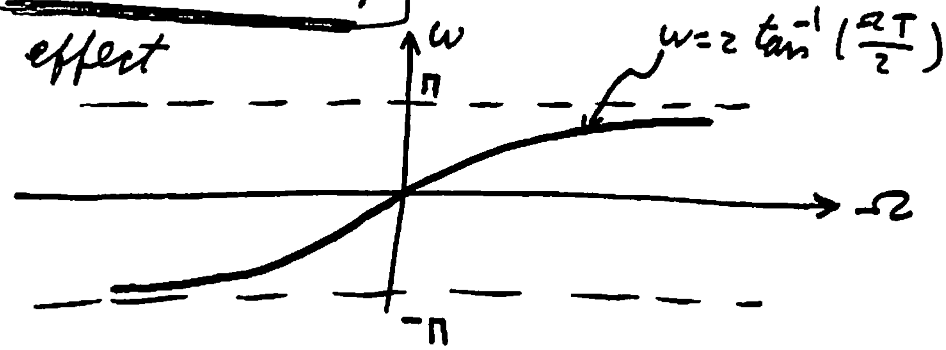
$$s = \frac{2}{T}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right) = \frac{2}{T}\,\frac{j\sin\omega/2}{\cos\omega/2} = \frac{2}{T}\, j\tan\frac{\omega}{2}$$

$s = \sigma + j\Omega$.

For z on unit circle $\sigma = 0$ and

$$\boxed{\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)} \Rightarrow \boxed{\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)}$$

Warping effect



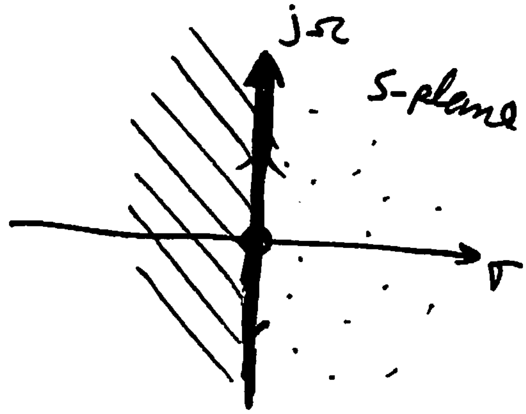
stability

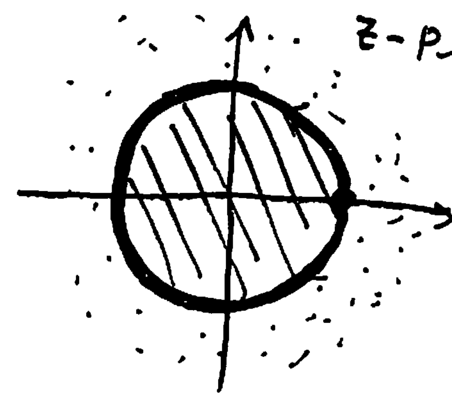
$$s = \sigma + j\Omega$$

$$z = \frac{1 + \frac{T}{2}\sigma + \frac{T}{2} j\Omega}{1 - \frac{T}{2}\sigma - \frac{T}{2} j\Omega}$$

If $\sigma < 0 \Rightarrow \left(1 + \frac{T}{2}\sigma\right) < \left(1 - \frac{T}{2}\sigma\right) \Rightarrow |z_k| < 1$

$\therefore$ stable





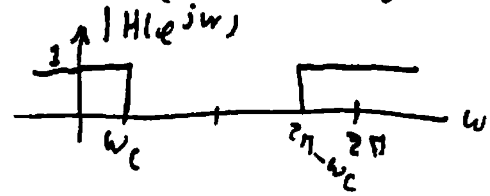
Remarks

① Warping in frequency domain, and the shape is changed in general

② Specifications $\Omega_p, \Omega_c, \cdots \rightarrow \omega_p, \omega_c, \cdots$ by the relationship

$$\boxed{\omega = 2\tan^{-1}\frac{\Omega T}{2}}$$

③ For piecewise constant frequency response the transformation is good (warping not too important)



④ Avoid aliasing problem

Design examples

1. Digital Frequency specifications
2. Convert to analog freq. specification
3. Design analog filter — butterworth
4. Transform back to digital filter < impulse inv. / bilinear.

Analog Butterworth filter

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{j\Omega}{j\Omega_c}\right)^{2N}} \rightarrow N: \text{order of the sys.}$$

Ω_c → cutoff freq

$\Omega = 0 \Rightarrow |H_a(j\Omega)|^2 = 1$

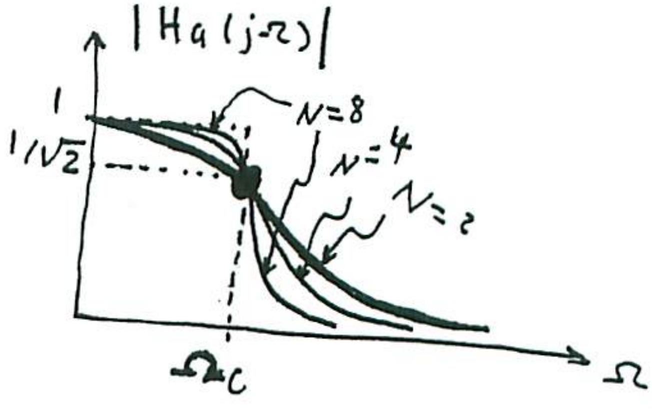
$\Omega = \Omega_c \Rightarrow |H_a(j\Omega)|^2 = 1/2$

$\Omega_c \equiv 3$ db drop.

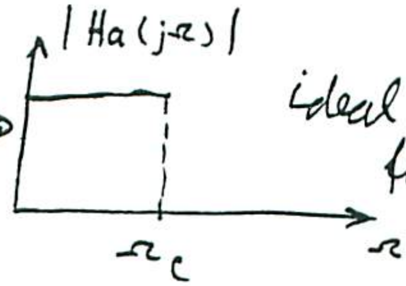
constant:

when $\Omega = \Omega_c \Rightarrow |H_a(j\Omega)|^2 = \frac{1}{2}$

so $|H_a(j\Omega)| = \frac{1}{\sqrt{2}}$



For $N \rightarrow \infty \Rightarrow$



ideal low pass filter

Poles of Butterworth filter

$$|H_a(j\Omega)|^2 = \frac{T^2}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}} = H_a(j\Omega)\,H_a(-j\Omega)$$

$$\Rightarrow H_a(s)\,H_a(-s) = \frac{T^2}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}}$$

$$1+\left(\frac{s}{j\Omega_c}\right)^{2N} = 0 \Rightarrow \text{Poles at} \quad s_p = (-1)^{1/2N}\,(j\Omega_c)$$

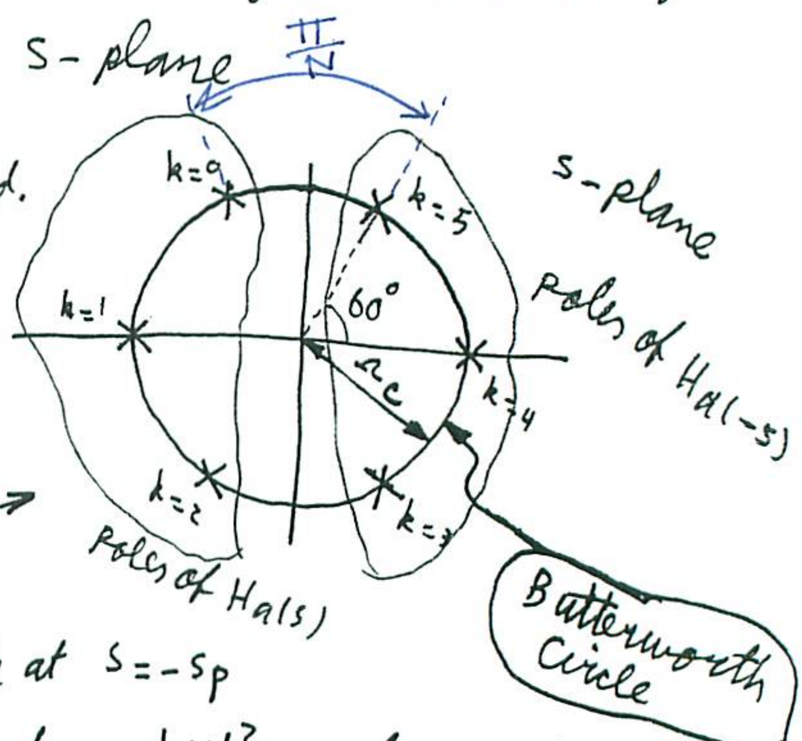
$$s_p = e^{j\left(\frac{\pi+2k\pi}{2N}\right)}(j\Omega_c) = e^{j\left(\frac{\pi+2k\pi}{2N}\right)}\,e^{j\frac{\pi}{2}}\,\Omega_c$$

$$k = 0, 1, \ldots, 2N-1$$

2N poles equally spaced in angle on a circle of radius Ω_c in the s-plane

{ Angular spacing $= \frac{\pi}{N}$ rad.
No poles on imaginary axis.
Pole on the real axis for <u>N odd</u> but not for <u>N even</u> }

Ex. $N = 3$



Pole at $s = s_p \Rightarrow$ Pole at $s = -s_p$

$\therefore$ to construct $H_a(s)$ from $|H|^2$ we <u>choose</u> one <u>pole from each pair</u>.

For <u>stable</u> & <u>causal</u> filter, choose <u>left-half-plane poles</u> for $H_a(s)$

In Bilinear transformation, we will have $2N$ zeros at $z=-1$ corresponding to $|H_a(j\Omega)|^2$

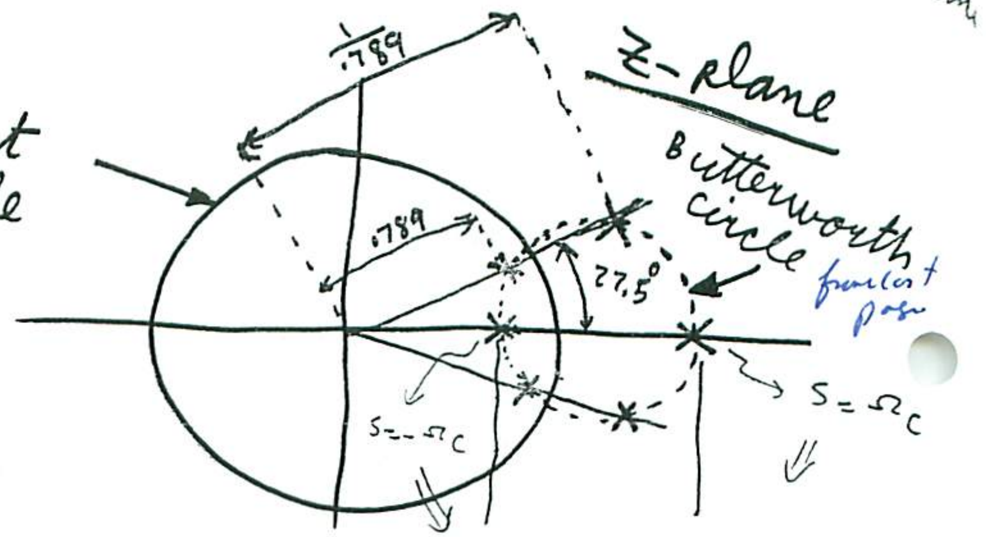
$$z = \frac{1+(\frac{T}{2})s}{1-(\frac{T}{2})s} \qquad \left\{ \begin{array}{l} s_p \longrightarrow z_p \\ -s_p \longrightarrow 1/z_p \end{array} \right.$$

$|H_a(j\Omega)|^2 \Rightarrow 2N$ zeros at $s=\infty \Rightarrow 2N$ zeros at $z = \left.\frac{1+Ts/2}{1-Ts/2}\right|_{s=\infty} = -1$

Butterworth circle $\xrightarrow[\text{Bilinear}]{\text{maps to}}$ a circle in z-plane

$\downarrow$

conformal mapping $\Rightarrow$ a circle in z-plane



$$\underbrace{\frac{1-\Omega_c T/2}{1+\Omega_c T/2}}_{3/5} \qquad \underbrace{\frac{1+\Omega_c T/2}{1-\Omega_c T/2}}_{5/3}$$

Ex.

For $N=3 \longrightarrow$

Let $\Omega_c T = \frac{1}{2}$

Using $z = \frac{1+(\frac{T}{2})s}{1-(\frac{T}{2})s}$ $\nearrow$

called bilinear because it transform circle to circle

Transform of specifications

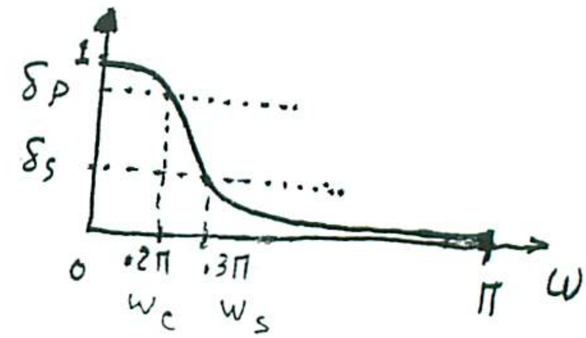
→ specs are given in db

Digital filter specifications:

example $T = \frac{1}{20{,}000}$ ← $f_s = 20{,}000\ Hz \sim \omega = 2\pi$

$f_p = 2000 \sim \omega_p = 0.2\pi$

Criterion $\underbrace{f_s = 3000}_{\text{given}}$, $\sim \omega_s = 0.3\pi$ → convert to digital.



$(1-\delta_p) \geq -1$ db

$\delta_s \leq -15$ db

or equivalently

Passband: $20 \log_{10} |H(e^{j0.2\pi})| \geq -1 \Rightarrow |H(e^{j0.2\pi})| \geq 10^{-0.05}$

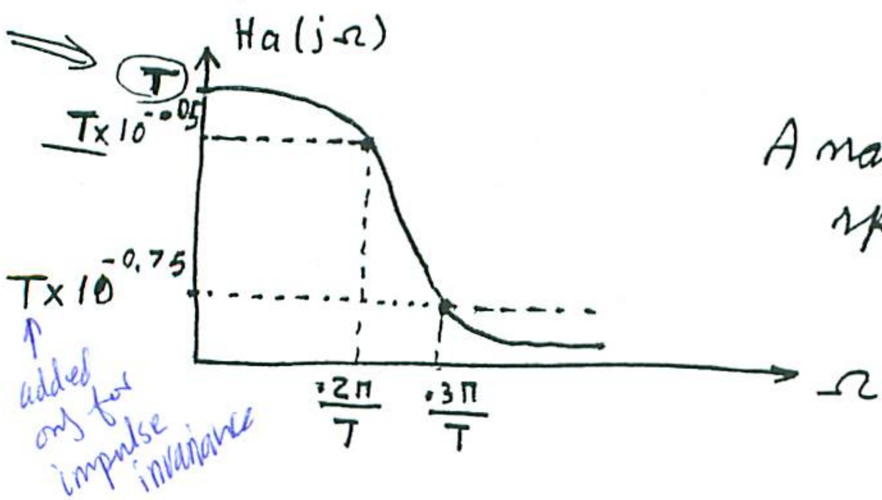
stopband $20 \log_{10} |H(e^{j0.3\pi})| \leq -15 \Rightarrow |H(e^{j0.3\pi})| \leq 10^{-0.75}$

Impulse Invariant Design

$$H(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(j\frac{\omega}{T} + j\frac{2\pi k}{T}\right)$$

(1) Find the analog Butterworth (Find N & Ω_c)

$\Omega = \frac{\omega}{T}$ (neglect aliasing)



Analog filter specifications

$$|H_a(j\Omega)|^2 = \frac{1 \times T^2}{1+\left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

$$\begin{cases} \dfrac{T^2}{1+\left(\frac{.2\pi/T}{\Omega_c}\right)^{2N}} \geq T^2 \times 10^{-0.1} & \text{Passband} \\ \dfrac{T^2}{1+\left(\frac{.3\pi/T}{\Omega_c}\right)^{2N}} \leq T^2 \times 10^{-1.5} & \text{Stopband} \end{cases}$$

unknowns are N and Ω_c

using equalities $\Longrightarrow$
$$\begin{cases} 1+\left(\frac{.2\pi/T}{\Omega_c}\right)^{2N} \overset{(\leq)}{=} 10^{0.1} & \text{(I)} \\ 1+\left(\frac{.3\pi/T}{\Omega_c}\right)^{2N} \overset{(\geq)}{=} 10^{1.5} & \text{(II)} \end{cases}$$

$$\left(\frac{0.2\pi/\Omega_c T}{0.3\pi/\Omega_c T}\right)^{2N} = \left(\frac{0.2}{0.3}\right)^{2N} = \frac{10^{0.1}-1}{10^{1.5}-1}$$

$$N = \frac{1}{2}\,\frac{\log[(10^{0.1}-1)/(10^{1.5}-1)]}{\log[.2/.3]} = \boxed{5.886} \quad \boxed{\text{Assuming } T=1}$$

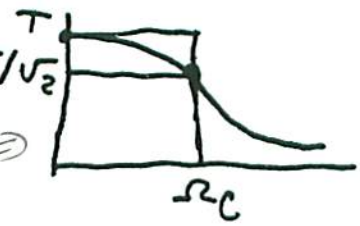
what is for $N=5.886$

$$\Rightarrow \left(\frac{.2\pi}{\Omega_c T}\right)^{2N} = 10^{0.1}-1 \quad \Rightarrow \Omega_c = \frac{0.2\pi}{T}\left(\frac{1}{(10^{0.1}-1)^{1/2N}}\right) = \boxed{0.704}$$

$\therefore$ Let $\boxed{N=6}$ to meet or exceed spec.'s

always round up

$N = 6$ in eq. (I) $\Rightarrow$ $\boxed{\Omega_c = 0.7032}$ 0.7032



$\begin{cases} \text{meet passband spec.'s} \\ \text{exceed stopband spec.'s} \end{cases}$

for impulse invariance pick 1st point

$$H_a(s)H_a(-s) = \frac{T^2}{1+\left(\frac{s}{j\,0.7032}\right)^{12}}$$

$T^2 \longrightarrow$ to count for $\frac{1}{T}\cdot\frac{1}{T}$ in $A \to D$

$\boxed{\text{Note: } T=1}$

(2) Transform of filter (I.I.) $A \to D$

$N=6$

80

$\longrightarrow$

do this symbolically

Poles of $H_a(s)\,H_a(-s)$:

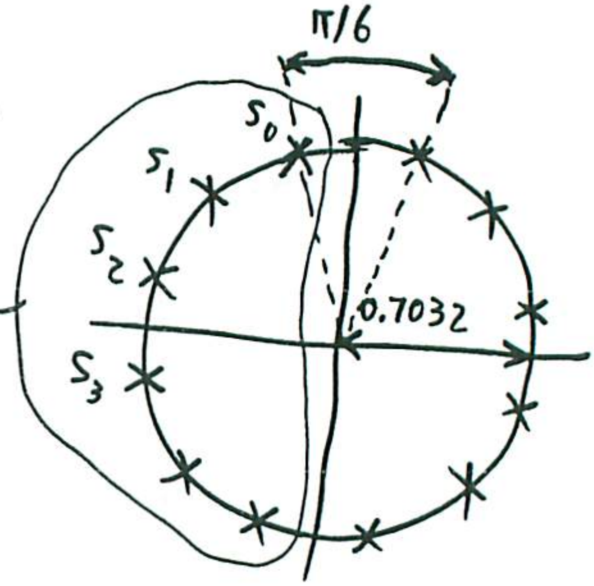
$$1 + \left(\frac{s}{j\,0.7032}\right)^{12} = 0$$

$$\left(\frac{s}{j\,0.7032}\right)^{12} = -1 = e^{+j(\pi + 2k\pi)}$$

$$s = j\,0.7032\; e^{+j\frac{\pi + 2k\pi}{12}} = 0.7032\; e^{j\left(\frac{\pi}{2} + \frac{\pi}{12} + \frac{2k\pi}{12}\right)}$$

$\boxed{\text{We have 12 poles}}$

s_k , $k = 0, 1, \ldots, 11$



Poles of $H_a(s)$

A → D

$$A \rightarrow H_a(s) = \sum \frac{A_k}{s - s_k} \quad \text{with left-half-plane poles}$$

$$D \rightarrow H(z) = \sum \frac{A_k}{1 - e^{s_k} z^{-1}}$$

See PP. 215–216 Text.

$$H_a(s)\,H_a(-s) = \frac{T^2}{1 + \left(\frac{s}{j\,0.7032}\right)^{12}}$$

$$H_a(s) = \frac{T\,K}{(s-s_0)(s-s_1)(s-s_2)\cdots(s-s_5)}$$

K → chose to make $H_a(0) = T$.

Bilinear Transformation design

Digital freq. specifications must be prewarped to the corresponding analog freq.

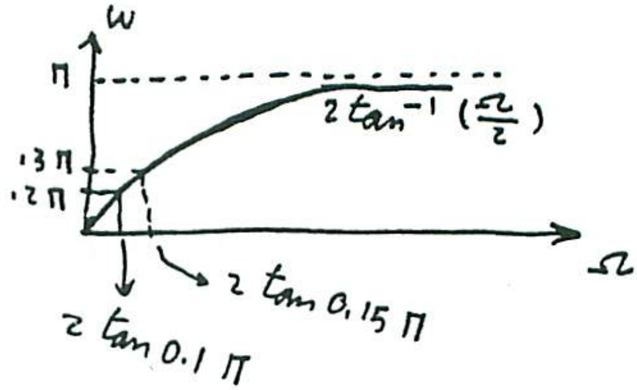
① Specification of Transformation

use this to go from ω to Ω

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

Assume $T = 1$



② Analog design

Passband : $20\log_{10}|H_a(j2\tan 0.1\pi)| \geq -1 \Rightarrow |H_a| \geqslant 10^{-0.05}$

Stopband : $20\log_{10}|H_a(j2\tan 0.15\pi)| \leqslant -15 \Rightarrow |H_a| \leqslant 10^{-.75}$

Solving with equality :

Recall $\quad |H_a(j\Omega)|^2 = \dfrac{1}{1+\left(\dfrac{j\Omega}{j\Omega_c}\right)^{2N}}$

Assume T = 1

$$\Rightarrow \begin{cases} 1+\left(\dfrac{\frac{1}{T}2\tan 0.1\pi}{\Omega_c}\right)^{2N} \overset{\leqslant}{=} 10^{0.1} & \text{I} \\ 1+\left(\dfrac{\frac{1}{T}2\tan 0.15\pi}{\Omega_c}\right)^{2N} \overset{\geqslant}{=} 10^{1.5} & \text{II} \end{cases}$$

want to find N & Ω_c

$$\left(\frac{\tan 0.1\pi}{\tan 0.15\pi}\right)^{2N} = \frac{10^{0.1}-1}{10^{1.5}-1}$$

$$N = \frac{1}{2}\,\frac{\log\left[(10^{0.1}-1)/(10^{1.5}-1)\right]}{\log\left[\tan(0.1\pi)/\tan 0.15\pi\right]} = 5.30466$$

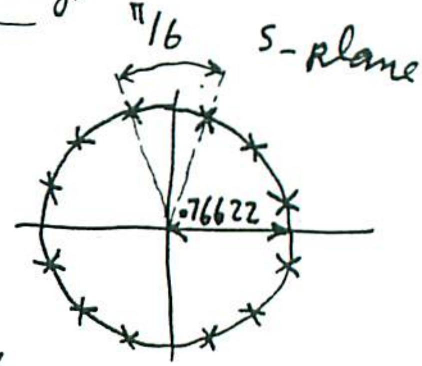
To meet the specifications let $N=6$

$N=6 \overset{\text{using II}}{\Rightarrow} \Omega_c = 0.76622$ with this Ω_c

Passband specifications are exceeded
stopband specifications are met exactly

rest same as with Impulse Inv.

$$H_a(s)\,H_a(-s) = \frac{1}{1+\left(\dfrac{s}{j\,0.76622}\right)^{2\times 6}}$$



$H_a(s)$ is formed from left-half-plane poles

$$H_a(s) = \frac{K}{[(s-s_0)(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)]}$$

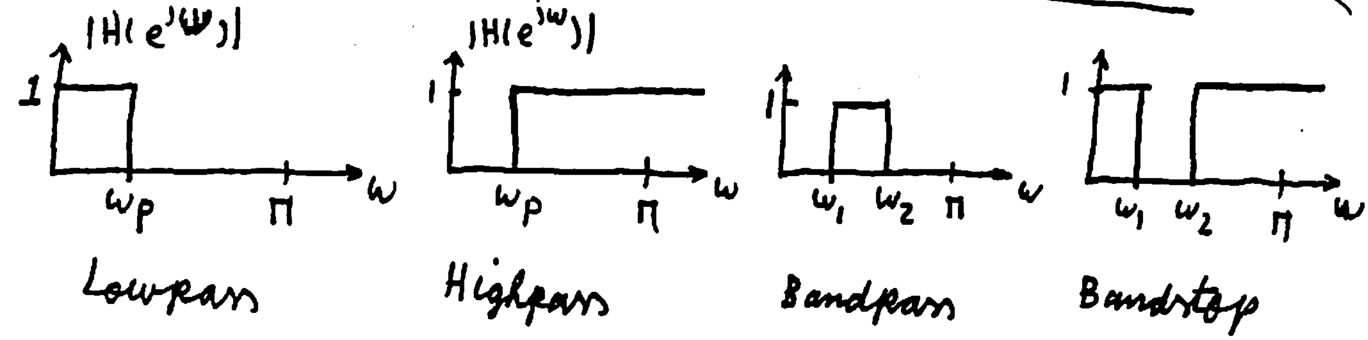
simplify Then use

③ $H_a(s)$

$$\Rightarrow H(z) = H_a(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} \quad \text{since } T=1$$

See PP. 217 – 219 Text.

Frequency transformation of Lowpass IIR filters



Freq. responses of Ideal

Design : For analog filters :

1 - Design a freq. normalized prototype lowpass

2 - Derive the desired lowpass, highpass, bandpass, or bandstop filter from the prototype lowpass filter

For digital filters :

1 - Design the digital prototype lowpass filter

2 - Transform the lowpass to the desired freq. selective digital filter using algebraic transformations

(If we want to transform analog filters to digital filters we can't transform highpass or bandstop using impulse - invariant trans. because of aliasing)

Let

$H(z)$: Transfer func. of the prototype low pass filter

$H_d(z_d)$: T.F. of the desired filter

desired

Then define a mapping from z-plane to z_d-plane of the form

$$z^{-1} = G(z_d^{-1})$$

such that $H_d(z_d) = H(G^{-1}(z_d^{-1}))$

where $z_d^{-1} = G^{-1}(z^{-1})$

We want

$$H(z) \begin{matrix}\text{rational}\\ \text{stable}\\ \text{causal}\end{matrix} \longrightarrow H_d(z_d) \begin{matrix}\text{rational}\\ \text{stable}\\ \text{causal}\end{matrix}$$

Hence $\begin{cases} \text{1- } G(z_d^{-1}) \text{ must be a rational fnc. of } z_d^{-1} \text{ (or } z_d) \\ \text{2- Inside of the unit circle in } z\text{-plane must map into} \\ \quad \text{" " " " " } z_d\text{-plane} \end{cases}$

Let $z = e^{jw}$ & $z_d = e^{jw_d}$

$$\Rightarrow \quad e^{-jw} = |G(e^{-jw_d})| \, e^{j \arg[G(e^{-jw_d})]}$$

$$\Rightarrow \begin{cases} |G(e^{-jw_d})| = 1 \\ w = -\arg G(e^{-jw_d}) \end{cases}$$

$$\therefore \boxed{G(z_d^{-1}) = \pm \prod_{k=1}^{N} \frac{z_d^{-1} - \alpha_k}{1 - \alpha_k z_d^{-1}}} \quad , \; |\alpha_k| < 1 \text{ for stability}$$

Ex. Transform a lowpass filter into another L.P. filter

$$z^{-1} = G(z_d^{-1}) = \frac{z_d^{-1} - \alpha}{1 - \alpha z_d^{-1}} \qquad \text{substitute } \begin{cases} z = e^{jw} \\ z_d = e^{jw_d} \end{cases}$$

$$\Rightarrow e^{-jw} = \frac{e^{-jw_d} - \alpha}{1 - \alpha e^{-jw_d}}$$

can show

$$\Rightarrow w_d = \arctan\left[\frac{(1-\alpha^2)\sin w}{2\alpha + (1+\alpha^2)\cos w}\right]$$

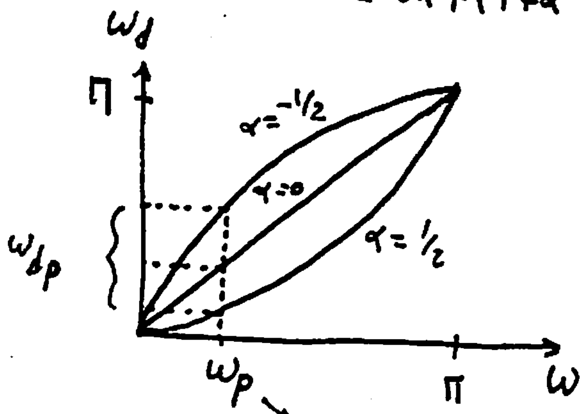
solve for α

$$\Rightarrow \alpha = \frac{\sin\left(\frac{w_p - w_{dp}}{2}\right)}{\sin\left(\frac{w_p + w_{dp}}{2}\right)}$$

ω_p is given
one we design for

w_d the desired one.

$$\Rightarrow H_d(z_d) = H(z)\Big|_{z^{-1} = \frac{z_d^{-1} - \alpha}{1 - \alpha z_d^{-1}}}$$



$w_p \rightarrow$ cutoff freq.

$\omega \rightarrow \theta_p$
$w_d \rightarrow w$

$Z = z_d.$

<u>Rdg</u>

PP. 214 - 218
226 - 230

Table 5.1

TRANSFORMATIONS FROM A LOWPASS-DIGITAL-FILTER PROTOTYPE OF CUTOFF FREQUENCY θ_p

Filter Type	*Transformation*	*Associated Design Formulas*
Lowpass	$z^{-1} = \frac{Z^{-1} - \alpha}{1 - \alpha Z^{-1}}$	$\alpha = \frac{\sin\left(\frac{\theta_p - \omega_p}{2}\right)}{\sin\left(\frac{\theta_p + \omega_p}{2}\right)}$ ω_p = desired cutoff frequency
Highpass	$-\frac{Z^{-1} + \alpha}{1 + \alpha Z^{-1}}$	$\alpha = -\frac{\cos\left(\frac{\omega_p + \theta_p}{2}\right)}{\cos\left(\frac{\omega_p - \theta_p}{2}\right)}$ ω_p = desired cutoff frequency
Bandpass	$-\frac{Z^{-2} - \frac{2\alpha k}{k+1}Z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1}Z^{-2} - \frac{2\alpha k}{k+1}Z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_2 + \omega_1}{2}\right)}{\cos\left(\frac{\omega_2 - \omega_1}{2}\right)}$ $k = \cot\left(\frac{\omega_2 - \omega_1}{2}\right)\tan\frac{\theta_p}{2}$ ω_2, ω_1 = desired upper and lower cutoff frequencies
Bandstop	$\frac{Z^{-2} - \frac{2\alpha}{1+k}Z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k}Z^{-2} - \frac{2\alpha}{1+k}Z^{-1} + 1}$	$\alpha = \frac{\cos\left(\frac{\omega_2 + \omega_1}{2}\right)}{\cos\left(\frac{\omega_2 - \omega_1}{2}\right)}$ $k = \tan\left(\frac{\omega_2 - \omega_1}{2}\right)\tan\frac{\theta_p}{2}$ ω_2, ω_1 = desired upper and lower cutoff frequencies