Flow Graph & Matrix Representation of Digital Filters

LSI digital system $\Leftrightarrow$ Lin. const.- coef. difference eqn.

Digital Network : N-th order diff. eqn.

$$y(n) = \sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

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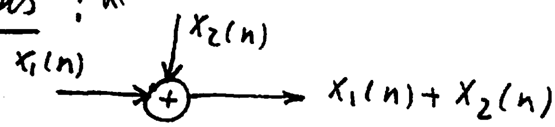
Transfer (system) function :

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

always

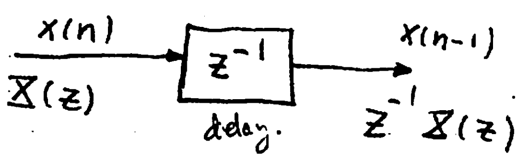
Basic Operations :

① Addition

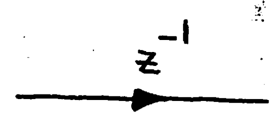


② Multiplication

③ Delay

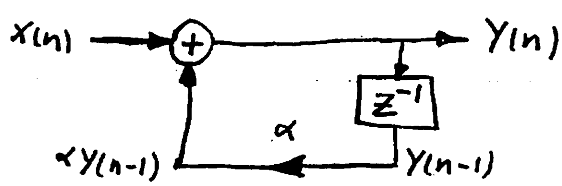


or

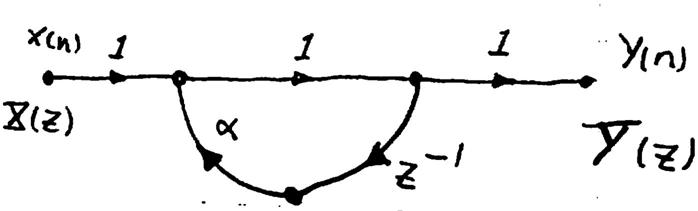


Ex. 1 $\quad y(n) = \alpha y(n-1) + x(n)$

Block diagram :

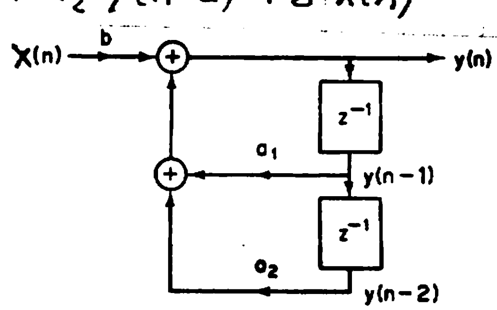


Signal flow graph :
nodes and Branches



Ex. 2 $\quad y(n) = a_1 y(n-1) + a_2 y(n-2) + b\, x(n)$ $\quad$ Second order system

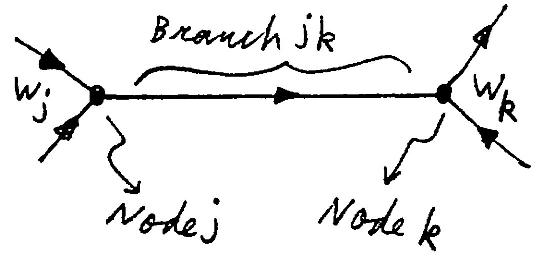
Block diagram :



<u>Flow graph</u> $\begin{cases} \text{a) directed branches} \\ \text{b) nodes} \end{cases}$

Node value : W_k : is the value associated with node k.

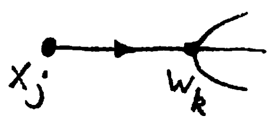
$\begin{cases} \text{Input to branch } jk = \boxed{W_j} \\ \text{Output of branch } jk \text{ to node } k = \boxed{V_{jk}} \end{cases}$



$V_{jk} = f_{jk}[W_j]$

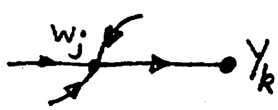
$f_{jk}[\cdot]$: transformation of a branch input into a branch output.

<u>Source node</u>



$\begin{cases} \text{No input branch} \\ \text{shown by } \boxed{X_j} \\ \text{Output to node } k \text{ is } \boxed{S_{jk}} \end{cases}$

<u>Sink node</u>



$\begin{cases} \text{Only entering branches} \\ \text{shown by } \boxed{Y_k} \\ \text{Output of branch } jk \text{ to node } k \text{ is } \boxed{r_{jk}} \end{cases}$

$\begin{cases} \text{"N" network nodes numbered} & 1 \to N \\ \text{"M" source nodes} & 1 \to M \\ \text{"P" sink nodes} & 1 \to P \end{cases}$

Assuming there is a branch in each direction between each pair of network nodes and each source node is connected to each network node :

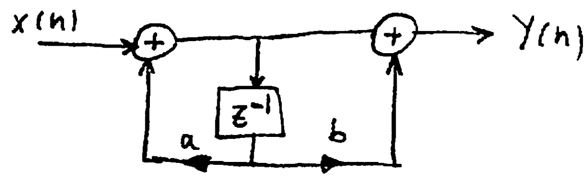
$$W_k = \underbrace{\sum_{j=1}^{N} V_{jk}}_{\text{Net. nodes}} + \underbrace{\sum_{j=1}^{M} S_{jk}}_{\text{Source nodes}} \qquad k = 1, 2, \ldots, N$$

Some of V_{jk} and S_{jk}'s are zero and

$$Y_k = \underbrace{\sum_{j'=1}^{N} r_{j'k}}_{\text{Network nodes}} \qquad k = 1, 2, \cdots, P$$

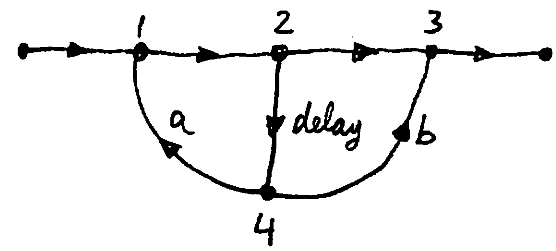
$v_{jk} = f_{jk}[w_j]$, In $\mathcal{Z}$-transform domain : $V_{jk}(z) = F_{jk}(z) W_j(\cdot$

Ex. Block diagram :



$\Downarrow$

signal flow graph



some of the equations :

$$\begin{cases} w_1(n) = S_{11}(n) + v_{41}(n) \\ w_2(n) = v_{12}(n) \\ y(n) = w_3(n) \end{cases}$$

some of the branch outputs :
$$\begin{cases} S_{11}(n) = x(n) \\ v_{43}(n) = f_{43}(w_4) = b\,w_4(n) \\ v_{24}(n) = f_{24}(w_2) = w_2(n-1) \quad \text{delay} \end{cases}$$

Matrix representation of Digital Networks

In $\mathcal{Z}$-domain

$$\begin{cases} W_k(z) = \sum_{j=1}^{N} V_{jk}(z) + \sum_{j=1}^{M} S_{jk}(z) & k = 1, \ldots, N \\ Y_k(z) = \sum_{j=1}^{N} R_{jk}(z) & k = 1, 2, \ldots, P \end{cases}$$

where
$$\begin{cases} V_{jk}(z) = F_{jk}(z)\, W_j(z) \\ S_{jk}(z) = b_{jk}\, X_j(z) \\ R_{jk}(z) = c_{jk}\, W_j(z) \end{cases}$$

Hence :

$$\begin{cases} W_k(z) = \sum_{j=1}^{N} F_{jk}(z)\, W_j(z) + \sum_{j=1}^{M} b_{jk}\, X_j(z) & k = 1, \dots, N \\ Y_k(z) = \sum_{j=1}^{N} C_{jk}\, W_j(z) & k = 1, \dots, P \end{cases}$$

Let

$$\underline{W} = \begin{bmatrix} W_1 \\ \vdots \\ W_N \end{bmatrix}, \quad \underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_M \end{bmatrix}, \quad \underline{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_P \end{bmatrix}$$

node col. vector, input col. vector, output col. vector

$$F = \left[F_{kj}(z) \right]_{N \times N}, \quad B = [b_{kj}]_{M \times N}, \text{ and } C = [c_{kj}]_{N \times P}$$

transmittance matrix

Then in matrix form:

$$\begin{cases} \underline{W}(z) = F^t(z)\, \underline{W}(z) + B^t\, \underline{X}(z) \\ \underline{Y}(z) = C^t\, \underline{W}(z) \end{cases}$$

$$[I - F^t(z)]\, \underline{W}(z) = B^t\, \underline{X}(z)$$

$$\Rightarrow \quad \underline{W}(z) = \underbrace{[I - F^t(z)]^{-1} B^t}_{T^t(z)}\, \underline{X}(z)$$

$$T^t(z) = [T_{jk}(z)]$$: Transfer fnc. matrix

$$Y(z) = C^t\, \underline{W}(z) = \underbrace{C^t\, [I - F^t(z)]^{-1} B^t}_{H(z) = C^t T^t}\, X(z)$$

system fnc.

$$\boxed{\underline{Y}(z) = H(z)\, \underline{X}(z)}$$

Elements of $F^t(z)$ are $\begin{cases} \text{constant} \\ \text{constant times } z^{-1} \end{cases}$

$$\Rightarrow \quad F^t(z) = \underset{\text{const.}}{F_c^t} + z^{-1} \underset{\text{delay}}{F_d^t}$$

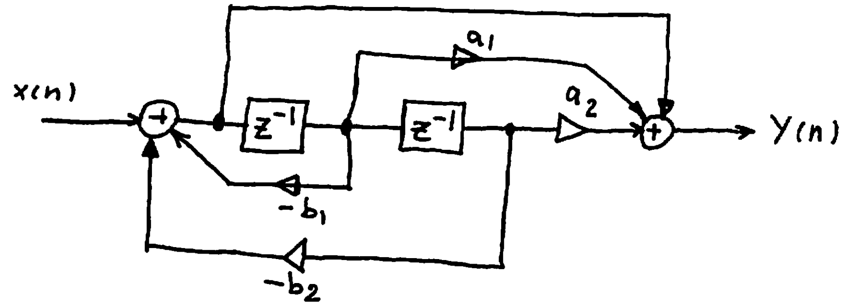
$N \times N$ matrices

$$\therefore \quad \underline{W}(z) = F_c^t\,\underline{W}(z) + z^{-1} F_d^t\,\underline{W}(z) + B^t\,\underline{X}(z)$$

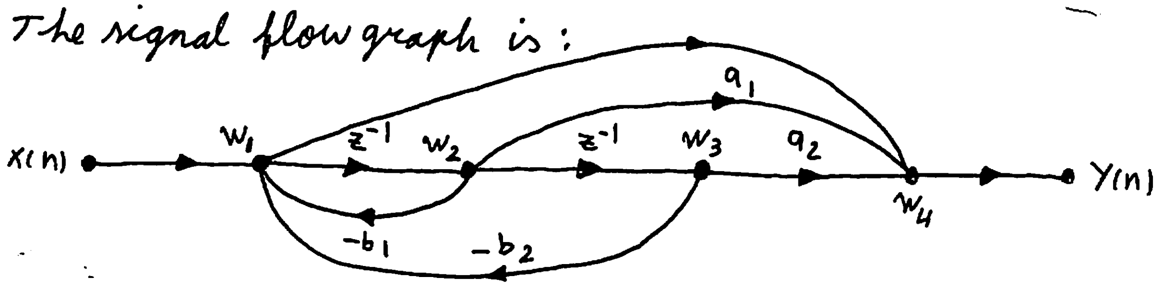
$$= \left[I - F_c^t - z^{-1} F_d^t\right]^{-1} B^t\,\underline{X}(z)$$

$$\Rightarrow \begin{cases} \underline{w}(n) = F_c^t\,\underline{w}(n) + F_d^t\,\underline{w}(n-1) + B^t\,\underline{x}(n) \\ \underline{y}(n) = c^t\,\underline{W}(n) \end{cases}$$

Ex.



The signal flow graph is:



$$\underbrace{\begin{bmatrix} w_1(n) \\ w_2(n) \\ w_3(n) \\ w_4(n) \end{bmatrix}}_{\underline{W}(n)} = \underbrace{\begin{bmatrix} 0 & -b_1 & -b_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & a_1 & a_2 & 0 \end{bmatrix}}_{F_c^t} \begin{bmatrix} w_1(n) \\ w_2(n) \\ w_3(n) \\ w_4(n) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{F_d^t} \underbrace{\begin{bmatrix} w_1(n-1) \\ w_2(n-1) \\ w_3(n-1) \\ w_4(n-1) \end{bmatrix}}_{\underline{W}(n-1)} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{B^t} x(n)$$

$$Y(n) = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}}_{C^t} \begin{bmatrix} w_1(n) \\ w_2(n) \\ w_3(n) \\ w_4(n) \end{bmatrix}$$

Can change ordering of nodes and hence get new F & B matrices for the same system

Network structure for Infinite Impulse Response (IIR) systems

Direct form : consider

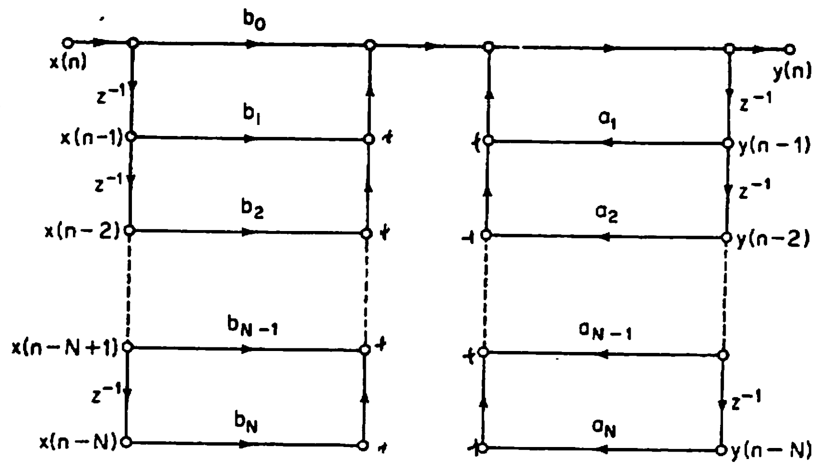
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} \qquad (I)$$

then

$$y(n) = \sum_{k=1}^{N} a_k \, y(n-k) + \sum_{k=0}^{M} b_k \, x(n-k) \qquad (II)$$

Since (II) can be written directly from system fn., (I), the network corresponding to eqn. (II) is called direct form I realization of the system in eqn. (I).

realizing zeros



realizing poles

Fig. 4.12 Direct form I realization of an Nth-order difference equation.

We have assumed $M=N$. If not true some of the branch transmittances are zero

In direct form I we realize zeros first and poles next.

Direct form II : Realize poles first and zeros next (The result is the same from an input output point of view)

Realizing Poles 1st:

We can combine the two strings of delays (to minimize the number of branches with transmission z^{-1}).

$\Rightarrow$ Direct form II

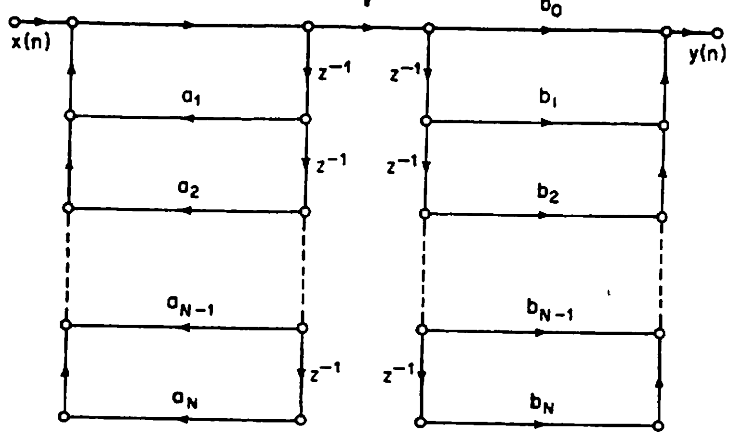


Fig. 4.13 Network of Fig. 4.12 with the order in which the poles and zeros are reverse cascaded.

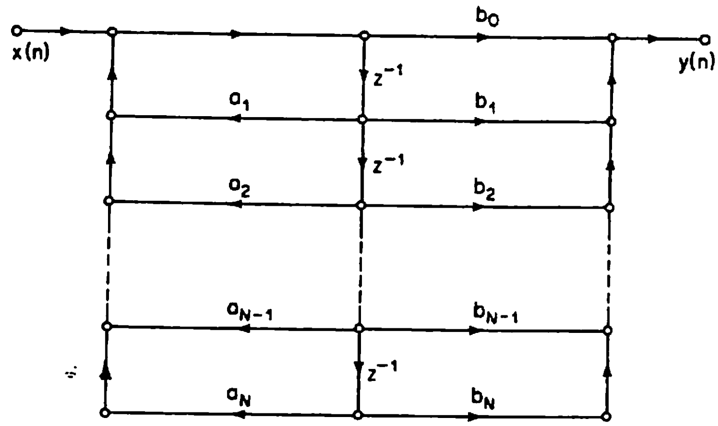
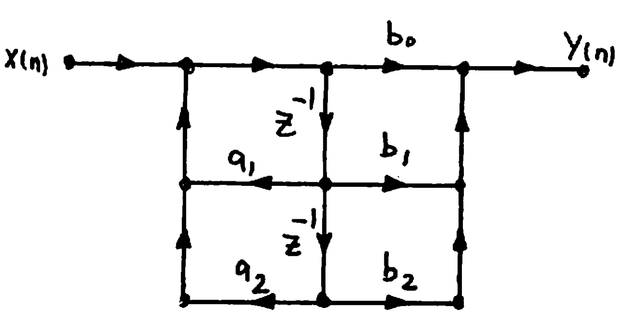
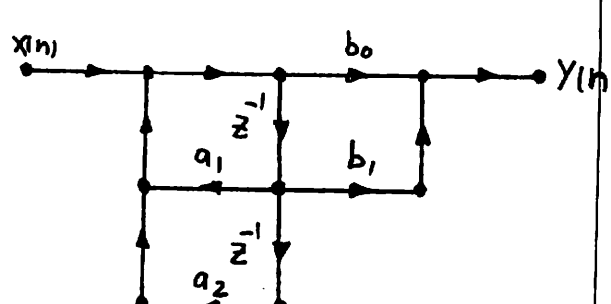


Fig. 4.14 Network of Fig. 4.13 with the two strings of delays combined into one. The resulting network form is referred to as the direct form II and has the minimum possible number of delays.

Ex. 2nd order form

$$\frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} , \qquad \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$





Cascade form   If $H(z) = A \prod_{k=1}^{N'} \frac{1+\beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1-\alpha_{1k} z^{-1} - \alpha_{2k} z^{-2}}$

(can have complex conjugate poles or zeros)

(N' is the largest integer part of $\frac{N+1}{2}$)   Then, we use direct form II to realize each 2^{nd} order term.

N' sections

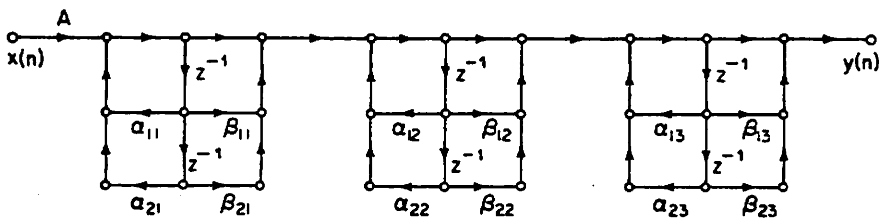


Fig. 4.16 Cascade structure with a direct form II realization of each second-order subsystem.

Parallel form   If $H(z) = \underbrace{\sum_{k=0}^{M-N} C_k z^{-k}}_{C} + \sum_{k=1}^{N'} \frac{\gamma_{0k} + \gamma_{1k} z^{-1}}{1-\alpha_{1k} z^{-1} - \alpha_{2k} z^{-2}}$

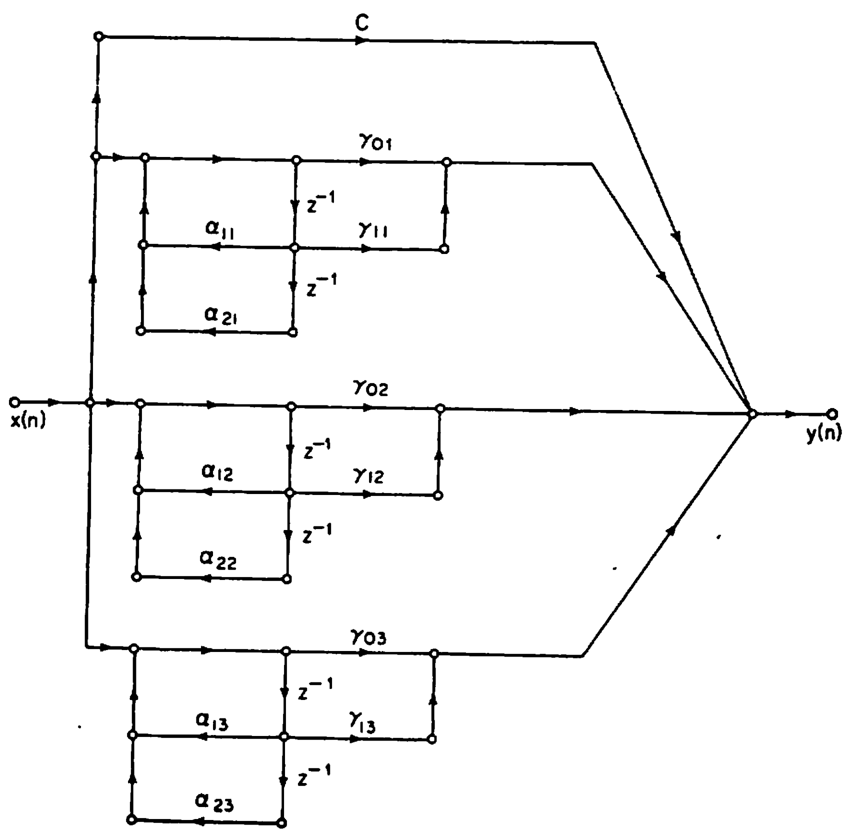


Fig. 4.17 Parallel-form realization with the real and complex poles grouped in pairs.

<u>Transpose form</u> (<u>transposition</u> or <u>flow graph reversal</u>) does not change the input output relation.

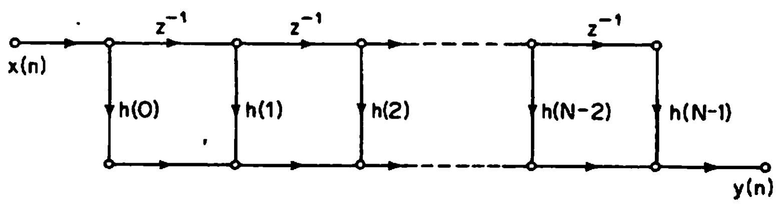
1. Reverse the direction of all branches
2. Interchange input and output
3. Summing nodes $\longrightarrow$ branching nodes.
4. Branching nodes $\longrightarrow$ summing nodes

<u>Transpose Theorem</u> : The transfer function remains the same after the transposition.

FIR systems

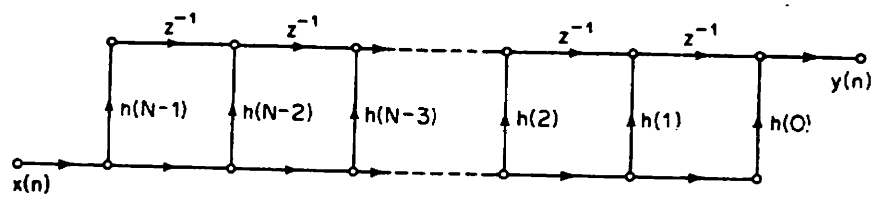
$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

<u>Direct form</u> $\quad y(n) = \sum_{k=0}^{N-1} h(k)\, x(n-k)$



Same as direct form II for IIR systems with a_k's $= 0$

<u>Transposition</u> of direct form $\Rightarrow$



<u>Cascade form</u>

$$H(z) = \prod_{k=1}^{[N/2]} [\beta_{0k} + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}]$$

$[N/2]$: integer part