The Discrete Fourier Transform (DFT)

Let $x(n)$ be a finite-duration seq. of length N

i.e., $x(n) = 0$ except for $0 \leqslant n \leqslant N-1$

~~A~~ periodic seq. for which $x(n)$ is one period is:

# of coeff. is always odd since 1 at zero

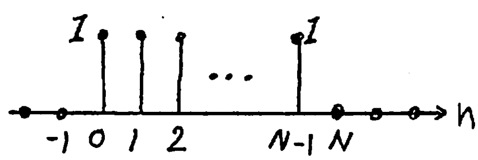
$$\begin{cases} \tilde{x}(n) = \sum_{r=-\infty}^{\infty} x(n+rN) \\ \text{or } \tilde{x}(n) = x(n \text{ modulo } N) \\ \text{or } \tilde{x}(n) = x((n))_N \qquad ((n))_N \Rightarrow n \text{ mod. } N \end{cases}$$

If $n = n_1 + n_2 N$ and $0 \leqslant n_1 \leqslant N-1$, then $n \text{ mod. } N = n_1$.

Now

$$x(n) = \begin{cases} \tilde{x}(n) & 0 \leqslant n \leqslant N-1 \\ 0 & \text{otherwise} \end{cases}$$

Let $$R_N(n) \triangleq \begin{cases} 1 & 0 \leqslant n \leqslant N-1 \\ 0 & \text{otherwise} \end{cases}$$



Then

$$\boxed{\begin{aligned} x(n) &= \tilde{x}(n) R_N(n) \\ \tilde{x}(n) &= x(n \text{ mod. } N) \end{aligned}}$$

In freq. domain

$$\boxed{\begin{aligned} X(k) &= \tilde{X}(k) R_N(k) \\ \tilde{X}(k) &= X(k \text{ mod. } N) \end{aligned}}$$

finite-duration

periodic

we had

D.F.S. pair
$$\begin{cases} \tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{kn} \\ \tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) W_N^{-kn} \end{cases}$$

D.F.T. pair
$$X(k) = \begin{cases} \sum_{n=0}^{N-1} x(n) W_N^{kn} & 0 \leq k \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$
$$x(n) = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

DFT $\quad x(n) \longleftrightarrow X(k)$

$$X(k) = \left[\sum_{n=0}^{N-1} x(n) W_N^{kn}\right] R_N(k) \quad \text{analysis transform}$$
$$x(n) = \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}\right] R_N(n) \quad \text{synthesis "}$$

Properties of DFT

1) Linearity

has to have same length

$$\text{DFT} \Rightarrow \begin{cases} x_3(n) = a\, x_1(n) + b\, x_2(n) \\ X_3(k) = a\, X_1(k) + b\, X_2(k) \end{cases}$$

$$\left.\begin{array}{l} x_1(n) : \text{duration } N_1 \\ x_2(n) : \text{duration } N_2 \end{array}\right\} \Rightarrow x_3(n) : \text{duration } N_3 = \max.[N_1, N_2]$$

In general DFT is computed with $N = N_3$

If $N_1 < N_2$ augment $x_1(n)$ by $(N_2 - N_1)$ zeros. i.e.,

$$X_1(k) = \sum_{n=0}^{N_1-1} x_1(n) W_{N_2}^{kn} \qquad 0 \leq k \leq N_2 - 1$$

$$X_2(k) = \sum_{n=0}^{N_2-1} x_2(n) W_{N_2}^{kn} \qquad 0 \leq k \leq N_2 - 1$$

2) Circular shift of a seq. (Time shift) $\boxed{D-2}$

shift $x(n)$ by m
$x_1(n)$ is obtained by extracting one period of $\tilde{x}(n+m)$

$x(n) \rightarrow \tilde{x}(n) \rightarrow \tilde{x}(n+m) \rightarrow x_1(n)$

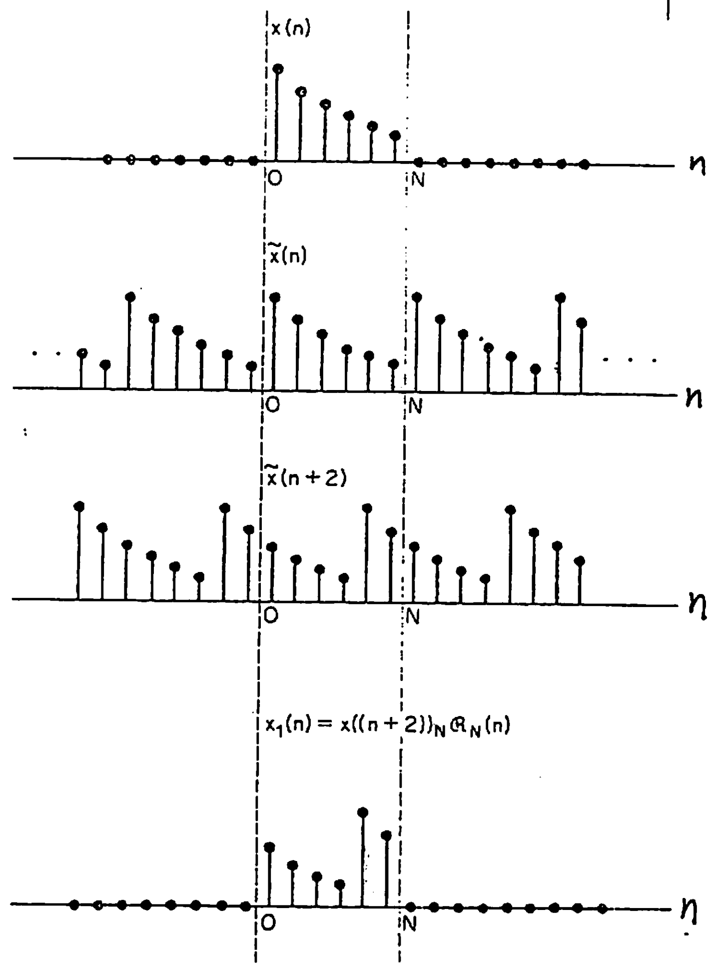


Fig. 3.9 Circular shift of a sequence.

Comparing $x_1(n)$ to $x(n)$

as a sample leaves the interval, $0 \leq n \leq N-1$, an identical sample enters the interval at the other end.

$$\tilde{x}_1(n) = \tilde{x}(n+m) = x((n+m))_N = x(n+m \text{ mod. } N)$$

$$\Rightarrow \boxed{x_1(n) = x((n+m))_N R_N(n)}$$

Relate DFT of $x(n)$ to DFT of $x_1(n)$

Recall : $\tilde{x}(n) \xrightarrow{D.F.S.} \tilde{X}(k)$

$$\tilde{x}_1(n) = \tilde{x}(n+m) \xrightarrow{D.F.S.} W_N^{-km} \tilde{X}(k) = \tilde{X}_1(k)$$

$$X(k) = \tilde{X}(k) R_N(k)$$

i.e. $\tilde{X}_1(k) = \tilde{X}(k) W_N^{-km}$

$$\Rightarrow \boxed{X_1(k) = W_N^{-km} X(k)}$$

i.e.,

$$\boxed{\mathcal{F}_D\left[x((n+m)\,\text{mod.}\,N)\right]R_N(n) = W_N^{-km}\,X(k)} \quad \text{(D-2)}$$

Because of duality between time & freq. domains, If

$$X_1(k) = X((k+\ell))_N\,R_N(k)$$

then $x_1(n) = W_N^{\ell n}\,x(n)$ i.e.,

$$\boxed{\mathcal{F}_D\left[W_N^{\ell n}\,x(n)\right] = X((k+\ell)\,\text{mod.}\,N)\,R_N(k)} \quad \text{(D-1)}$$

3) Symmetry properties

$x(n)$ with duration N

For an arbitrary seq. $y(n)$, we had

$$\begin{cases} \text{conjugate symmetric part} \quad y_e(n) = \frac{1}{2}\left[y(n) + y^*(-n)\right] \\ \quad \text{"} \quad \text{antisymmetric} \quad \text{"} \quad y_o(n) = \frac{1}{2}\left[y(n) - y^*(-n)\right] \\ \quad \text{such that} \begin{cases} y(n) = y_e(n) + y_o(n) \\ y_e(n) = y_e^*(-n) \\ y_o(n) = -y_o^*(-n) \end{cases} \end{cases}$$

Using the above equations for $x(n)$, with finite length N, both $x_e(n)$ and $x_o(n)$ will be of duration $(2N-1)$; hence, we look at $\tilde{x}(n)$:

$$\tilde{x}(n) = x((n))_N$$

$$\begin{cases} \tilde{x}_e(n) = \frac{1}{2}\left[\tilde{x}(n) + \tilde{x}^*(-n)\right] \\ \tilde{x}_o(n) = \frac{1}{2}\left[\tilde{x}(n) - \tilde{x}^*(-n)\right] \end{cases}$$

Then define

$$\begin{cases} x_{ep}(n) \triangleq \tilde{x}_e(n)\,R_N(n) & \text{Periodic conj. sym.} \\ x_{op}(n) \triangleq \tilde{x}_o(n)\,R_N(n) & \text{Periodic conj. antisym.} \end{cases}$$

// periodic even

periodic odd

i.e.,

$$\begin{cases} x_{ep}(n) = \frac{1}{2}\left[x((n))_N + x^*((-n))_N \right] R_N(n) \\ x_{op}(n) = \frac{1}{2}\left[x((n))_N - x^*((-n))_N \right] R_N(n) \end{cases}$$

or (see problem 3-17)

$$\boxed{\begin{aligned} x_{ep}(n) &= [x_e(n) + x_e(n-N)]\, R_N(n) \\ x_{op}(n) &= [x_o(n) + x_o(n-N)]\, R_N(n) \end{aligned}}$$

If $x_{ep}(n)$ and $x_{op}(n)$ were <u>real</u> we call them <u>periodic even</u> and <u>periodic odd</u> components. (<u>Note</u>: actually x_{ep} and x_{op} are <u>not</u> periodic, they just show one period of $\tilde{x}_e(n)$ and $\tilde{x}_o(n)$)

$$x(n) = \tilde{x}(n)\, R_N(n) = [\tilde{x}_e(n) + \tilde{x}_o(n)]\, R_N(n)$$

$$= \tilde{x}_e(n)\, R_N(n) + \tilde{x}_o(n)\, R_N(n)$$

$$\Rightarrow \boxed{x(n) = x_{ep}(n) + x_{op}(n)}$$

<u>Symmetry properties</u>: $x(n)$ of length N

$$x(n) \xrightarrow{DFT} X(k)$$

$$x^*(n) \xrightarrow{DFT} X^*((-k))_N\, R_N(k)$$

$$x^*((-n))_N\, R_N(n) \longrightarrow X^*(k)$$

$$Re[x(n)] \longrightarrow X_{ep}(k)$$

$$j\, Im[x(n)] \longrightarrow X_{op}(k)$$

$$x_{ep}(n) \longrightarrow Re[X(k)]$$

$$x_{op}(n) \longrightarrow j\, Im[X(k)]$$

whenever we do $x^*((-n))$ we need to window it! with $R_N(k)$

For $x(n)$ a <u>real</u> seq.

$Re[X(k)]$ and $|X(k)|$ are periodic ~~odd~~ even

i.e. $Re[X(k)] = Re[X((-k))_N]\, R_N(k)$

$Im[X(k)]$ and $Arg[X(k)]$ are periodic ~~~~ odd

$x_{ep}(n) \longrightarrow Re[X(k)]$

$x_{op}(n) \longrightarrow j\, Im[X(k)]$

4) Circular Convolution

$x_1(n)$ & $x_2(n)$ of duration N

$$\begin{cases} x_1(n) \xrightarrow{DFT} X_1(k) \\ x_2(n) \xrightarrow{DFT} X_2(k) \end{cases} \qquad \begin{cases} x_3(n) \xrightarrow{DFT} X_1(k)\, X_2(k) \\ x_3(n) = ? \end{cases}$$

If $\tilde{x}_3(n) \xrightarrow{D.F.S.} \tilde{X}_1(k)\, \tilde{X}_2(k)$, then $x_3(n) = \tilde{x}_3(n)\, R_N(n)$

$$\therefore \quad x_3(n) = \underbrace{\left[\sum_{m=0}^{N-1} \tilde{x}_1(m)\, \tilde{x}_2(n-m) \right]}_{\tilde{x}_3(n)} R_N(n)$$

this is periodic conv.

$$\Rightarrow \quad x_3(n) = \left[\sum_{m=0}^{N-1} x_1((m))_N\, x_2((n-m))_N \right] R_N(n)$$

this is periodic convolution!

N-point circular convolution of $x_1(n)$ & $x_2(n)$ represented notationally by $x_1(n) \; Ⓝ \; x_2(n)$

Hence

$$\boxed{\mathcal{F}_D[x_1(n)\, Ⓝ\, x_2(n)] = X_1(k)\, X_2(k)} \qquad \text{(D-3)}$$

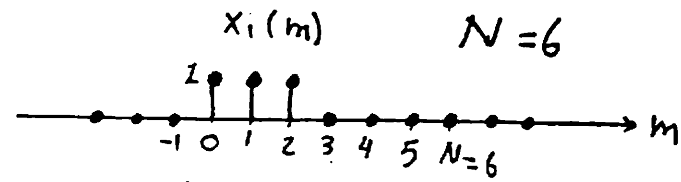
$$x_1(n)\; Ⓝ \; x_2(n) = \left[\sum_{m=0}^{N-1} x_1((m))_N\, x_2((n-m))_N \right] R_N(n)$$

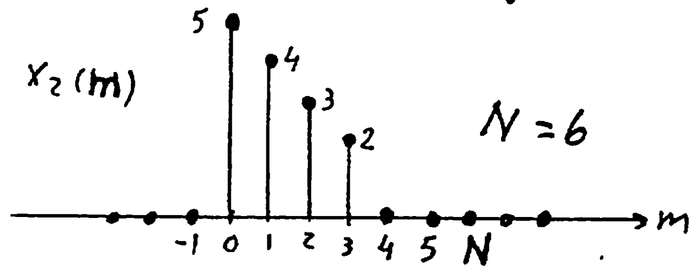
$\Rightarrow$

$$\boxed{\begin{aligned} x_1(n)\; Ⓝ \; x_2(n) &= \left[\sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N \right] R_N(n) \\ &= \left[x_1(n) * x_2((n))_N \right] R_N(n) \end{aligned}}$$

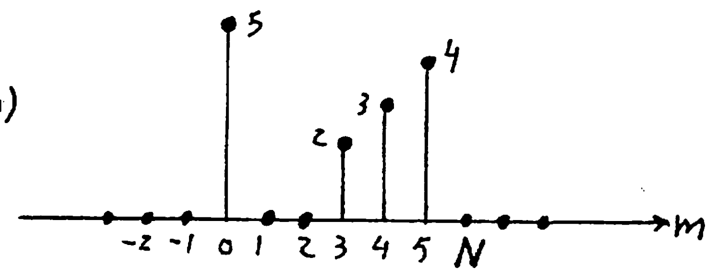
Ex. $N = 6$

Circular Conv.

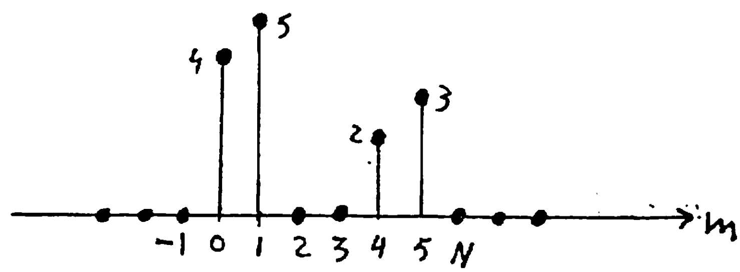




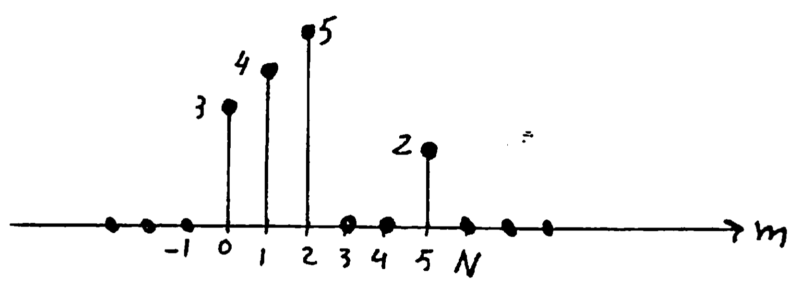
$x_2((0-m))_N \, R_N(m)$



$x_2((1-m))_N \, R_N(m)$



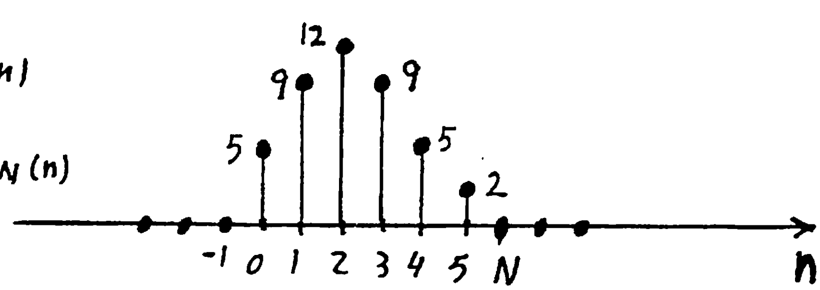
$x_2((2-m))_N \, R_N(m)$



$\bullet \quad \bullet \quad \bullet$

$$x_3(n) = x_1(n) \; \text{Ⓝ} \; x_2(n)$$

$$= \left[\sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N\right] R_N(n)$$



Ex. $x_1(n) = x_2(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$

$$X(k) \triangleq \begin{cases} \sum_{n=0}^{N-1} x(n)\, W_N^{kn} & 0 \le k \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow X_1(k) = X_2(k) = \sum_{n=0}^{N-1} W_N^{kn} \qquad 0 \le k \le N-1$$

but $\sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} N & k=0 \\ 0 & \text{otherwise} \end{cases} \Rightarrow X_1(k) = X_2(k) = \begin{cases} N, & k=0 \\ 0, & \text{otherwise} \end{cases}$

$$\therefore X_3(k) = X_1(k)\, X_2(k) = \begin{cases} N^2 & k=0 \\ 0 & \text{otherwise} \end{cases}$$

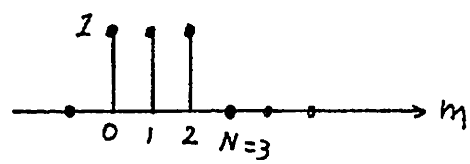
$$x(n) \triangleq \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn} & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \boxed{x_3(n) = N} \qquad 0 \le n \le N-1$$

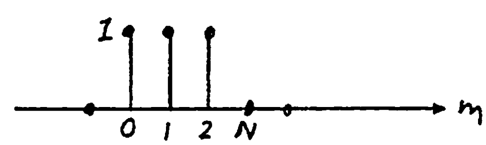
Ex. $N = 3$

using graphical method to show

$x_1(m)$

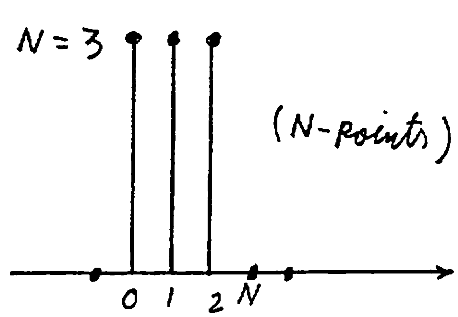


$x_2(m)$



$x_2((n-m))_N\, R_N(n)$

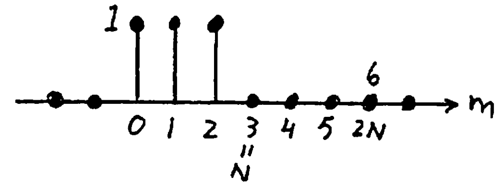
$x_3(n) = x_1(n)\; N\; x_2(n)$



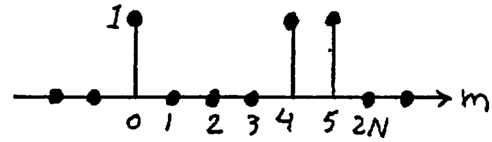
Linear convolution, using circular conv.

Augment $x_1(n)$ and $x_2(m)$ by adding N zeros making them $2N$-point sequences.

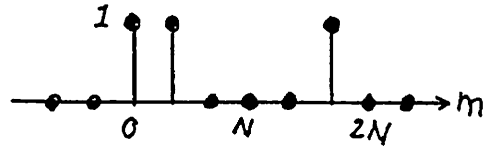
$x_1(m) = x_2(m)$



$x_2((0-m))_{2N} R_{2N}(m)$



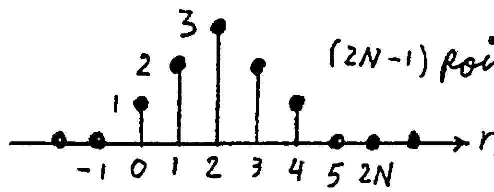
$x_2((1-m))_{2N} \ R_{2N}(m)$



. . .

$$x_3(n) = x_1(n) \, \text{(2N)} \, x_2(n)$$

$$= \sum_{m=0}^{2N-1} x_1(m) \, x_2((n-m))_{2N} \cdot R_{2N}(n)$$



$(2N-1)$ points

$\therefore$ Circular conv. of the augmented seq.'s is identical to linear conv. of $x_1(n)$ & $x_2(n)$

Consider $x_1(n)$ & $x_2(n)$ of duration N

$$\text{F.T.} \Rightarrow \begin{cases} X_1(e^{j\omega}) = \sum_{n=0}^{N-1} x_1(n) e^{-j\omega n} \\ X_2(e^{j\omega}) = \sum_{n=0}^{N-1} x_2(n) e^{-j\omega n} \end{cases}$$

Let $X_3(e^{j\omega}) = X_1(e^{j\omega}) \, X_2(e^{j\omega})$

then
$$\boxed{x_3(n) = \sum_{m=0}^{N-1} x_1(m) \, x_2(n-m)}$$
convolution

$2N-1$ samples

$$\text{DFT} \Rightarrow \begin{cases} X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, W_N^{nk} \\ X_2(k) = \sum_{n=0}^{N-1} x_2(n)\, W_N^{nk} \end{cases} \qquad W_N = e^{-j\frac{2\pi}{N}}$$

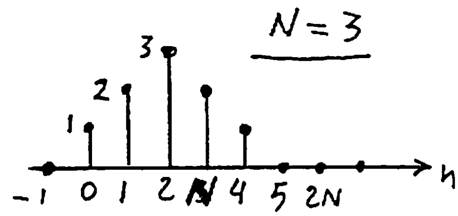
These are samples of $X_1(e^{j\omega})$ & $X_2(e^{j\omega})$ at $\omega_k = \frac{2\pi k}{N}$

Let $X_4(k) = X_1(k)\, X_2(k)$

then $\boxed{x_4(n) = \left[\sum_{r=-\infty}^{\infty} x_3(n+rN)\right] R_N(n)}$

$x_3(n)$ has a duration of $(2N-1)$ samples $\Rightarrow$ $x_4(n)$ will be an aliased version of $x_3(n)$

Ex. In the previous example we saw that : $x_3(n) \rightarrow$

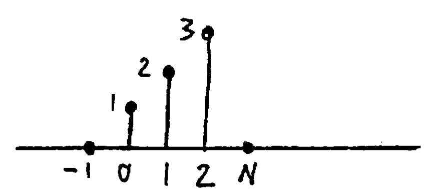


and $x_4(n) = N = 3 \qquad 0 \leq n \leq 2 = N-1$

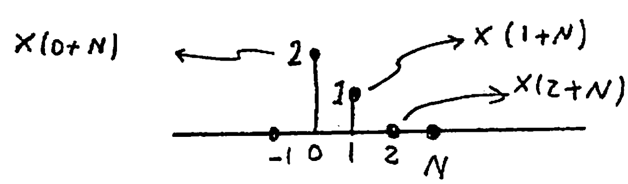
Try to find $x_4(n)$ using

$$x_4(n) = \left[\sum_{r=-\infty}^{\infty} x_3(n+rN)\right] R_N(n)$$

$r = 0 \quad \Rightarrow$

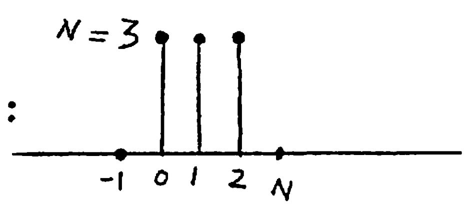


$r = 1 \quad \Rightarrow$

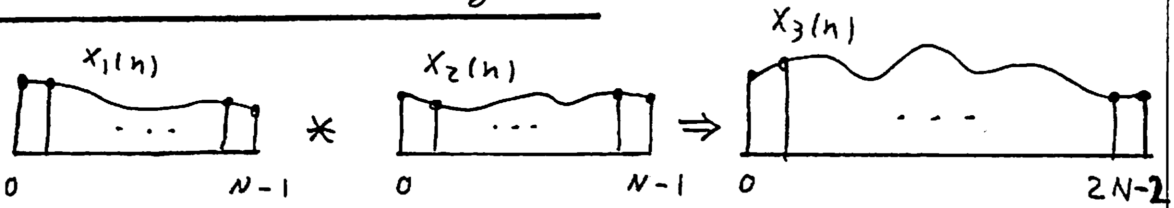


other values of $r \Rightarrow 0$

$\Sigma \Rightarrow x_4(n)$:



Linear convolution using DFT



$$x_3(n) = x_1(n) * x_2(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2(n-m)$$

$x_1(n)$ & $x_2(n)$ are N-pt. seq.'s $\Rightarrow$ $x_3(n)$ is of length $(2N-1)$
($2N-1$ nonzero points at most)

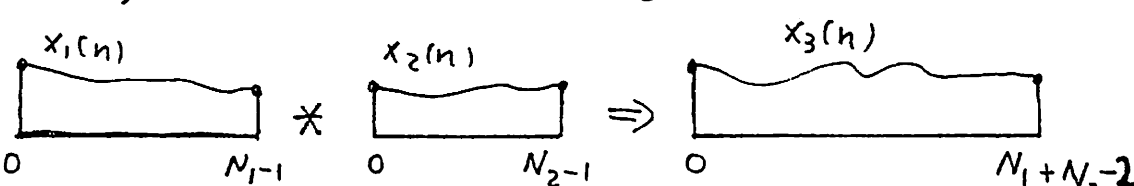
Use DFT to find $x_3(n)$: Compute $X_1(k)$ & $X_2(k)$ on the basis of $2N-1$ points

$$\begin{cases} X_1(k) = \displaystyle\sum_{n=0}^{2N-2} x_1(n)\, W_{2N-1}^{nk} \\ X_2(k) = \displaystyle\sum_{n=0}^{2N-2} x_2(n)\, W_{2N-1}^{nk} \end{cases}$$

Then $$x_3(n) = \frac{1}{2N-1}\left[\sum_{k=0}^{2N-2} \left[X_1(k)\, X_2(k)\right] W_{2N-1}^{-nk}\right] R_{2N-1}(n)$$

In general : $\begin{cases} x_1(n) & \text{is of duration } N_1 \\ x_2(n) & \text{is of duration } N_2 \end{cases}$

then $x_3(n)$ is of duration $N_1 + N_2 - 1$



In this case DFT's are computed on basis of

$N \geq N_1 + N_2 - 1$

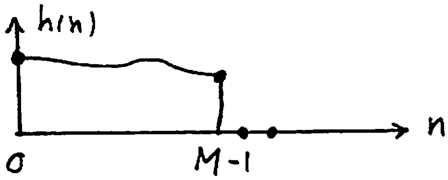
Using DFT to convolve a seq. of infinite duration, $x(n)$, with a finite-length seq., $h(n)$, $n=0,1,\dots,M-1$ (e.g. speech filtering)

Two solutions

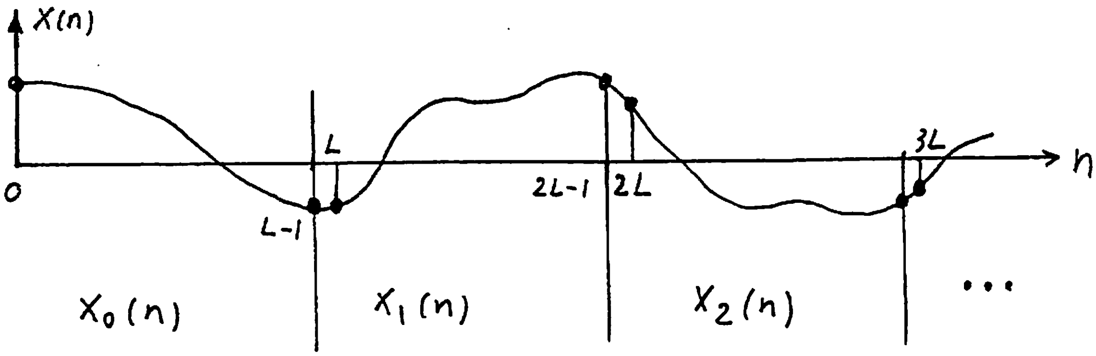
(1) collect $x(n)$, $n=0,\dots,N-1$, where N is very large. Then find $x(n) * h(n)$ using DFT on the basis of $N+M-1$ points.

{ DFT is usually too large to compute
{ There is a large delay in the processing

(2) Segment $x(n)$ into sections of length L, then convolve each section with $h(n)$, using DFT. The filtered sections are then fitted together in an appropriate manner.



both are discrete signals!



Decompose $x(n)$ into

$$x(n) = \sum_{k=0}^{\infty} X_k(n) \quad \text{where}$$

$$X_k(n) = \begin{cases} X(n) & kL \le n \le (k+1)L-1 \\ 0 & \text{otherwise} \end{cases}$$

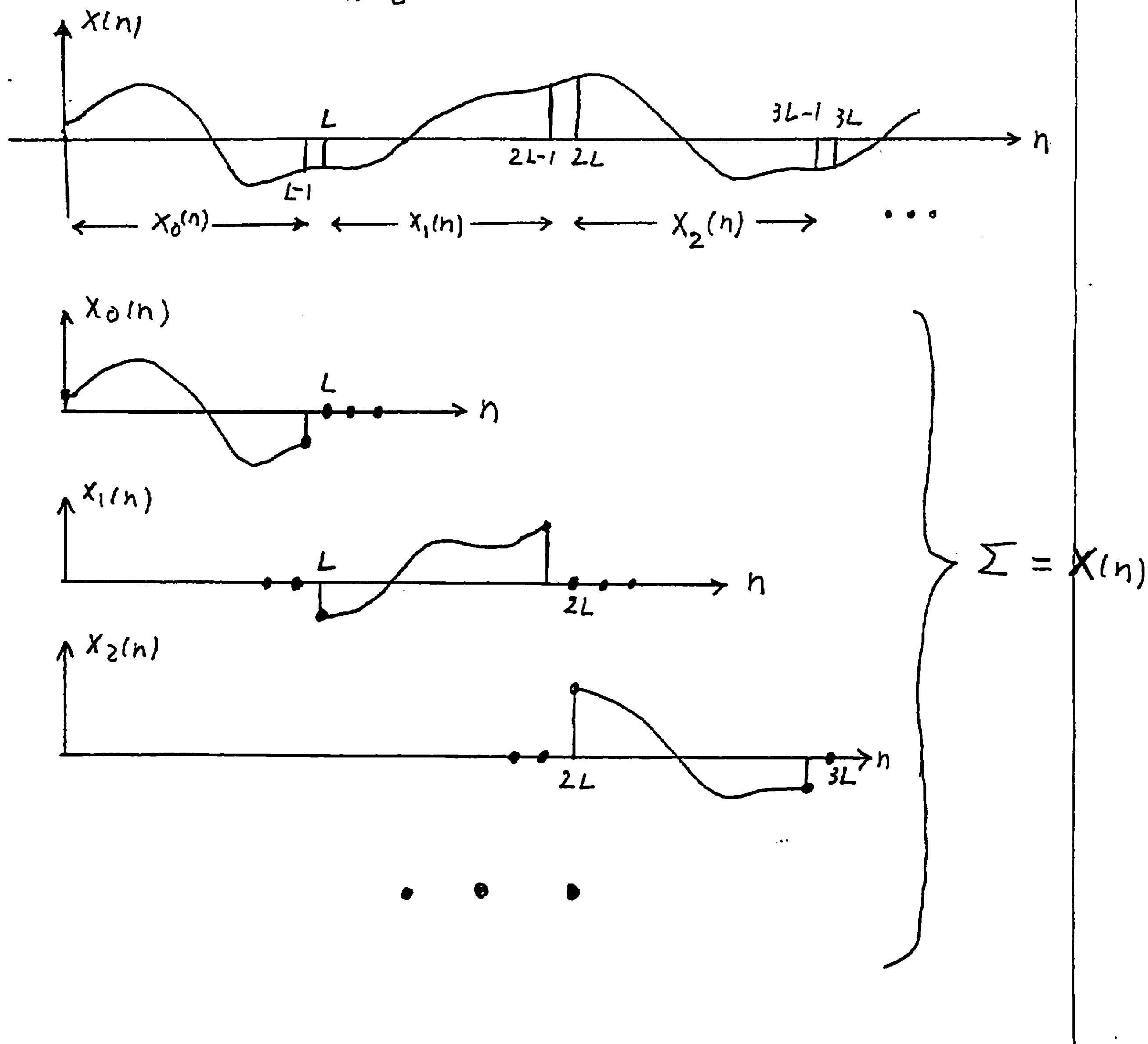
origin do not change for X_0, X_1, X_2, etc! difference is that X_1 has zeros from $0 \dots L-1$

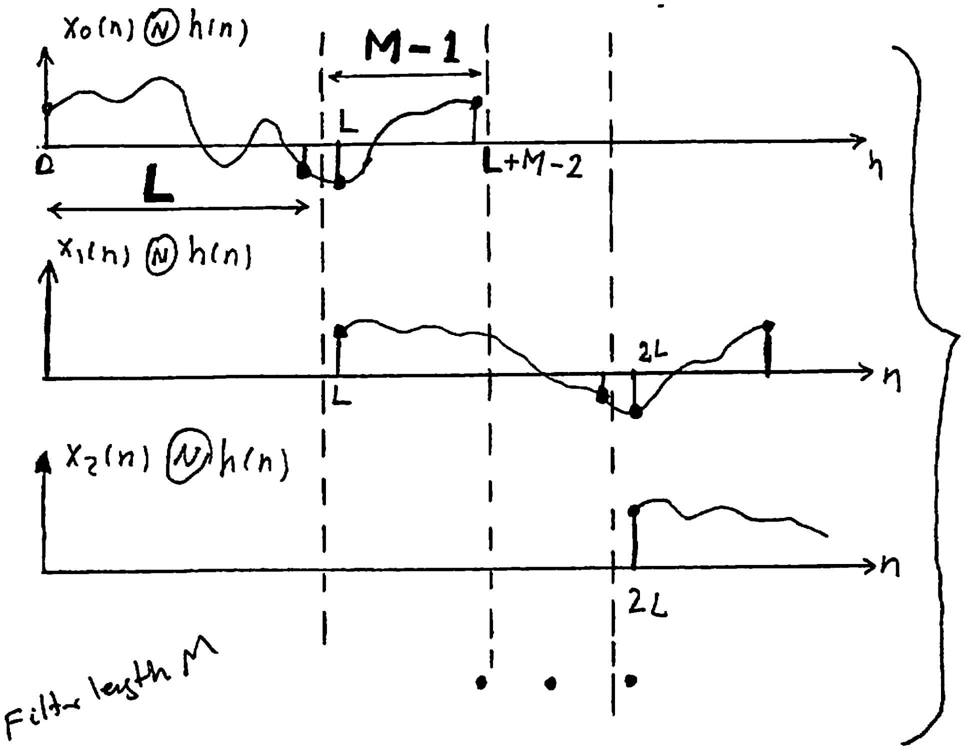
Now $\quad x(n) * h(n) = \sum_{k=0}^{\infty} x_k(n) * h(n)$

$$\left\{\begin{array}{l} x_k(n) \text{ has } L \text{ nonzero points} \\ h(n) \text{ has } M \quad \text{"} \quad \text{"} \end{array}\right. \Rightarrow \left\{\begin{array}{l} \text{each term } [x_k(n) * h(n)] \\ \text{is of length } M+L-1 \end{array}\right.$$

$\therefore$ Linear conv. $x_k(n) * h(n)$ must be obtained using a $(L+M-1)$-point DFT.

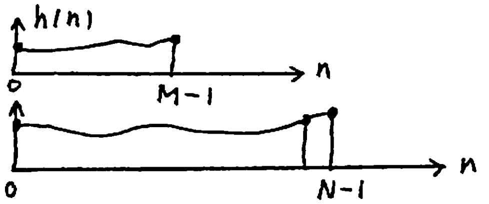
<u>Overlap-add method</u> : Each section of length L results in a filtered section of $(L+M-1)$ points. The nonzero points in the filtered section will overlap by $(M-1)$ points in carrying out $\sum_{k=0}^{\infty} x_k(n) * h(n)$.





$$\rightarrow \sum_{k=0}^{\infty} x_k(n) * h(n)$$

<u>Overlap-save method</u> : compute $x_k(n)$ (N) $h(n)$ and identify that part of it which corresponds to a linear convolution



(N) $\Rightarrow$ The first M-1 points are incorrect, because of overlapping

The remaining points are identical to those that would be obtained using a linear convolution (*).

<u>Procedure</u> : Section $x(n)$ into sections of length N, such that input sections overlap the preceding section by M-1 points. i.e.,

$$\boxed{x_k(n) = x(n + k(N-M+1))} \quad , \; 0 \leqslant n \leqslant N-1$$

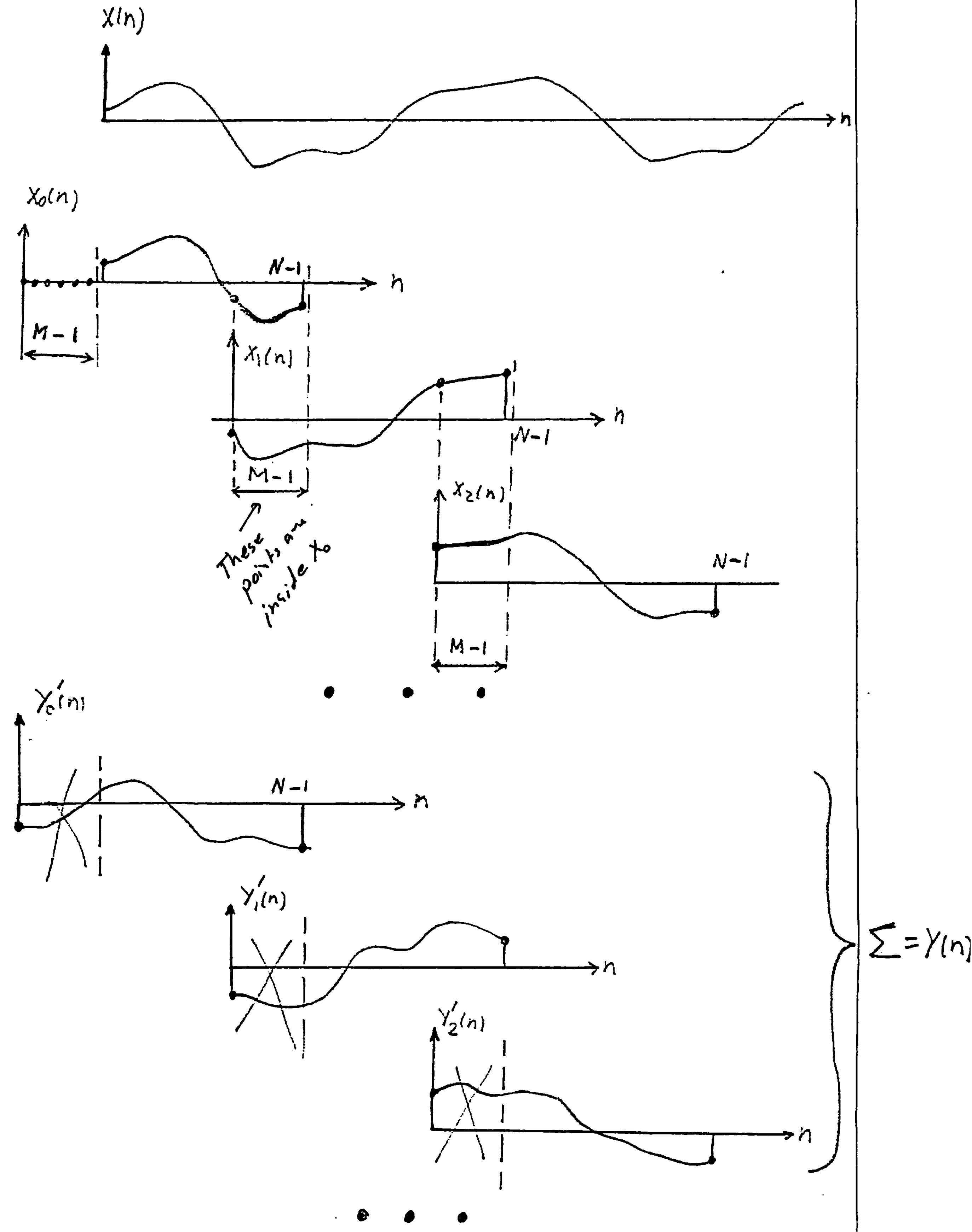
<u>Note</u> : time origin for each section is the begining of that section.

$$x_k(n) \text{ (N) } h(n) = y_k'(n)$$

$$y(n) = \sum_{k=0}^{\infty} y_k(n - k(N+M-1))$$

where
$$y_k(n) = \begin{cases} y_k'(n) & M-1 \leqslant n \leqslant N-1 \\ 0 & \text{otherwise} \end{cases}$$

Not on exam.


X(n)
n
$X_0(n)$
N-1
M-1
$X_1(n)$
N-1
M-1
These points are inside X_0
$X_2(n)$
N-1
M-1
$Y_0'(n)$
N-1
$Y_1'(n)$
$Y_2'(n)$
$\sum = Y(n)$

Properties of DFT cont.'d

D-3 Circular convolution

$$\mathcal{F}_D\left[f(n) \; \textcircled{N} \; g(n)\right] = F(k)\, G(k)$$

D-4 Product

$$\mathcal{F}_D\left[f(n) \cdot g(n)\right] = \frac{1}{N} F(k) \; \textcircled{N} \; G(k)$$

D-5 Scaling

(See page 17)

D-6 Parseval's theorem

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

$$\sum_{n=0}^{N-1} f(n)\, g^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} F(k)\, G^*(k)$$

Scaling (Sampling) effects D-5

(a) Let $N = \beta M$, β & M integers

Let $h(n)$ be an M-sequence defined as

$$h(n) = x(\beta n) \quad , n = 0, 1, \ldots, M-1$$

Denote $H_M(k)$ as the M-DFT of $h(n)$, and

$X(k) \equiv X_N(k)$ as the N-DFT of $x(n)$.

then

$$H_M(k) = \frac{1}{\beta} \sum_{\alpha=0}^{\beta-1} X(k+\alpha M) \quad \text{.................. (1)}$$

(b) Let $g(n)$ be an N-sequence

$$g(n) = \begin{cases} x(n) & , n = \beta m \\ 0 & , \text{otherwise} \end{cases}$$

then

$$G_N(k) = \frac{1}{\beta} \sum_{\alpha=0}^{\beta} X((k+\alpha M) \bmod . N) \quad \text{.............. (2)}$$

(c) Given an M-sequence $h(n)$, and its M-DFT as $H_M(k)$, let

$$g(n) = \begin{cases} h(n/\beta) & n = \beta m \\ 0 & \text{otherwise} \end{cases}$$

Then

$$G_N(k) = H_M(k \bmod . M) \quad \text{.......................... (3)}$$

Proof

Not in exams

Part (b) Note that

$$\frac{1}{\beta} \sum_{\alpha=0}^{\beta-1} (W_N)^{\alpha M n} = \begin{cases} 1 & n = \beta m \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad g(n) = \frac{1}{\beta} \left(\sum_{\alpha=0}^{\beta-1} (W_N)^{\alpha M n} \right) x(n)$$

By D-1, it is clear that

$$G_N(k) = \frac{1}{\beta}\sum_{\alpha=0}^{\beta-1} \delta((k+\alpha M) \bmod. N) \Rightarrow \quad (2)$$

Part (c) $\quad G_N(k) = \sum_{n=0,\beta,2\beta,\ldots}^{N-\beta} h(n/\beta)(W_N)^{nk}$

$$= \sum_{m=0}^{M-1} h(m)(W_M)^{mk}$$

$(W_M)^{mM}=1 \Rightarrow G_N(k) = \sum_{m=0}^{M-1} h(m)(W_M)^{m(k \bmod. M)}$

$$= H_M(k \bmod. M) \Rightarrow \quad (3)$$

Part (a) : Note that when $0 \leqslant k \leqslant M-1$, eqn.'s (2) and (3) lead trivially to (1)

(d) Given $h(n)$ an M-seq., let $f(n)$ be an N-seq.,

$$f(n) = \begin{cases} h(n), & 0 \leqslant n \leqslant M-1 \\ 0, & M \leqslant n \leqslant N-1 \end{cases}$$

Then

$$F_N(\beta k) = H_M(k) \quad \text{-------------------------} \quad (4)$$

use this defin'n to solve example of HW 1

Pf. $\quad F_N(\beta k) = \sum_{n=0}^{N-1} f(n)(W_N)^{\beta k n}$

$$= \sum_{n=0}^{M-1} h(n)(W_M)^{kn} \qquad (W_N)^\beta = W_M$$

$$= H_M(k) \qquad \blacksquare$$

Remark : If $\Omega \neq \beta k$, then there is no simple form for $F_N(\Omega)$. A general formula (by interpolation) : Define

$$\tilde{F}_N(\Omega) = \begin{cases} F_N(\Omega), & \text{when } \Omega = \beta k \\ 0, & \text{otherwise} \end{cases}, \text{ then}$$

$$F_N(\Omega) = \frac{1}{M}\tilde{F}_N(\Omega) \; (N) \left[\frac{1-W_N^{M\Omega}}{1-W_N^{\Omega}}\right] \quad \cdot \quad \text{Prove it}$$