The Discrete Fourier Transform (DFT)

3/18/10

I.) continuous F. T.

I. C. F. T.

$$\begin{cases} X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t)\, e^{-j\Omega t}\, dt \\ x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)\, e^{j\Omega t}\, d\Omega \end{cases}$$

cont. aperiodic

rad/sec.

II.) Cont. F. series

$$\begin{cases} c_k = \frac{1}{T}\int_0^T f(t)\, e^{-jk\Omega_0 t}\, dt \\ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\Omega_0 t} \end{cases}$$

periodic (T)

III.) (DTFT) Sequence F. T.

I. S. F. T.

Aperiodic discrete

$$\begin{cases} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \\ x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \end{cases}$$

radians

used for finite length

IV.) Discrete F. T.

I. D. F. T.

$$\begin{cases} X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\left(\frac{2\pi}{N}\right)nk} \\ x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\left(\frac{2\pi}{N}\right)nk} \end{cases}$$

$n = 0 \ldots N-1$

used for finite duration seq.'s

discrete Fourier Series

Representation of periodic sequences – (The DFSeries)

$\tilde{x}(n)$: periodic with period N ($\sim \Leftrightarrow$ periodic)

i.e., $\tilde{x}(n) = \tilde{x}(n+kN)$ k : integer

2 sided

$\mathcal{Z}[\tilde{x}(n)]$ does not exist.

Represent $\tilde{x}(n)$ by Fourier series, i.e., sum of sine & cosine seq.'s (or complex exp. seq.'s) with freq.'s that are integer multiples of the fundamental freq. $2\pi/N$ associated with $\tilde{x}(n)$.

We only have N of such freq.'s. Consider

$$\boxed{e_k(n) = e^{j(2\pi/N)nk}}$$

periodic in k with period N

$e_0(n) = e_N(n)$, $e_1(n) = e_{N+1}(n)$, $\cdots$ $\Rightarrow$ N complex exp.'s with $k = 0, 1, \ldots, N-1$

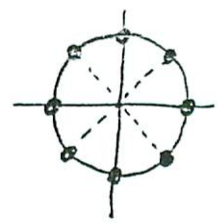
$\therefore$ F. series for $\tilde{x}(n)$:

$$\boxed{\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j\left(\frac{2\pi}{N}\right)nk}} \qquad (\mathrm{I})$$

$\frac{1}{N}$ $\rightarrow$ for convenience

Using

$$\underbrace{\frac{1}{N} \sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)nr}}_{\text{acts like } \delta(\cdot)} = \begin{cases} 1, & \text{for } r = mN \\ 0, & \text{otherwise} \end{cases} \qquad m: \text{integer}$$



using this

derive this

we can show

$$\boxed{\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\left(\frac{2\pi}{N}\right)nk}} \qquad (\mathrm{II})$$

$\hookrightarrow$ periodic with period N.

Eqn.'s (I) & (II) : Discrete Fourier Series (DFS) representation of a periodic seq.

Let $\boxed{W_N = e^{-j(2\pi/N)}}$

Then

$$(\mathrm{I}) \Rightarrow \quad \tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k)\, W_N^{-kn} \qquad (1)$$

$$(\mathrm{II}) \Rightarrow \quad \tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, W_N^{kn} \qquad (2)$$

Let $x(n)$ be one period of $\tilde{x}(n)$

i.e., $x(n) = \begin{cases} \tilde{x}(n) & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$

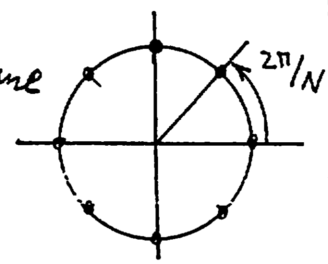
Then $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{N-1} x(n) z^{-n}$ (3)

Compare (3) & (2)

$$\Rightarrow \boxed{\tilde{X}(k) = X(z)\Big|_{z = e^{j\left(\frac{2\pi}{N}\right)k}} = W_N^{-k}}$$

$\therefore$ <u>$\tilde{X}(k)$ corresponds to sampling $X(z)$ at N points equally spaced in angle around the unit circle</u>

z - plane



<u>Ex. 1</u>

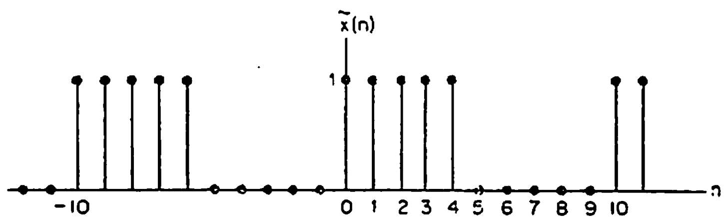


Fig. 3.3 Periodic sequence for which the Fourier series representation is to be computed.

$N = 10$

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{kn} = \sum_{n=0}^{4} W_{10}^{nk} = \sum_{n=0}^{4} e^{-j\left(\frac{2\pi}{10}\right)nk}$$

But $$\sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}} = \frac{\sin(\omega M/2)}{\sin(\omega/2)} e^{-j(M-1)\frac{\omega}{2}}$$

Let $M = 5$ & $\omega = \frac{2\pi}{10} k$

$$\Rightarrow \tilde{X}(k) = e^{-j(4\pi k/10)} \frac{\sin(\pi k/2)}{\sin(\pi k/10)}$$

X means we do not know phase.
if magnitude is zero, we can't talk about phase

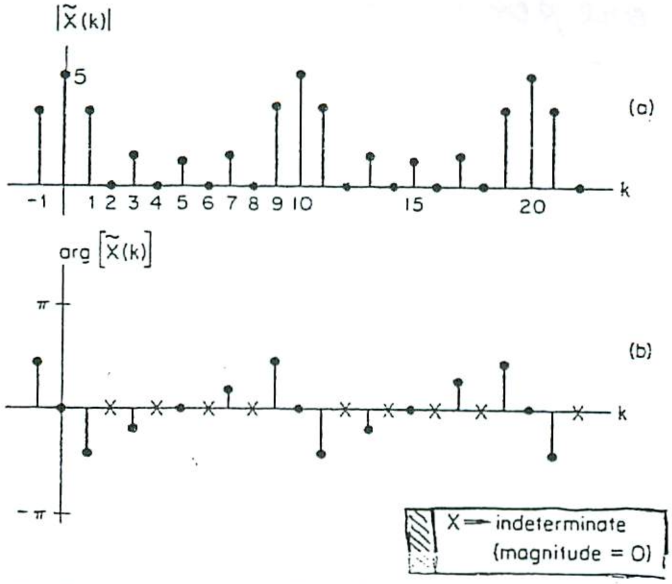
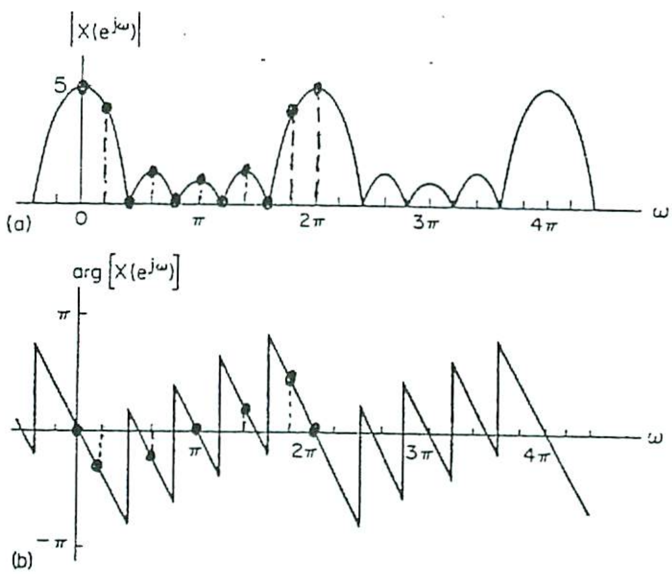


Fig. 3.4 Magnitude and phase of the Fourier series coefficients of the sequence of Fig. 3.3.

The Z-transform evaluated on the unit circle for one period of $\tilde{x}(n)$ is

$$X(e^{j\omega}) = e^{-j2\omega}\,\frac{\sin(5\omega/2)}{\sin(\omega/2)}$$



So ~~DFT~~ DFT is samples of DTFT.

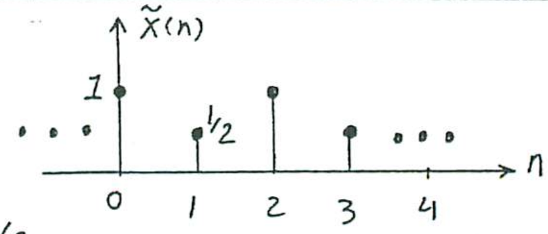
Fig. 3.5 Magnitude and phase of the z-transform evaluated on the unit circle of one period of the sequence in Fig. 3.3.

Notice that $|\tilde{X}(k)|$ and $\arg[\tilde{X}(k)]$ are sampled versions of $|X(e^{j\omega})|$ and $\arg[X(e^{j\omega})]$.

$\left(\frac{2\pi}{N} = \frac{2\pi}{10}\right)$

exam!

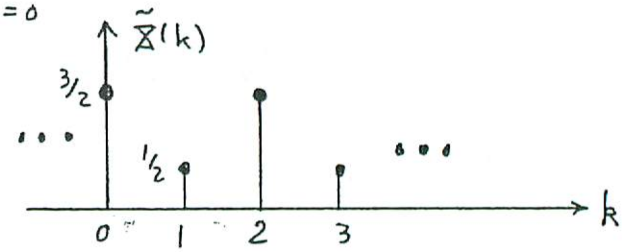
Ex. 2



$N = 2$

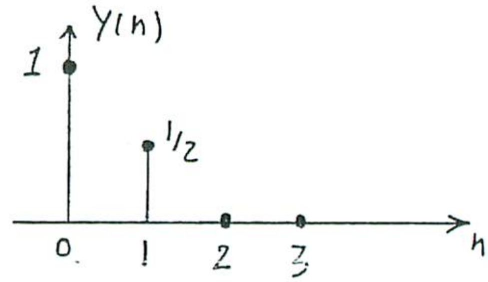
$W_N = W_2 = e^{-j2\pi/2} = -1$

$\tilde{X}(k) = \sum_{n=0}^{1} \tilde{x}(n)\, W_2^{kn} = \sum_{n=0}^{1} \tilde{x}(n)\,(-1)^{nk}$



$$\begin{cases} \tilde{X}(0) = \tilde{x}(0) + \tilde{x}(1) = 3/2 \\ \tilde{X}(1) = \tilde{x}(0) - \tilde{x}(1) = 1/2 \\ \tilde{X}(2) = \tilde{x}(0) - \tilde{x}(1) = 3/2 \\ \quad \cdots \end{cases}$$

Ex. 3



$\tilde{y}(n) \Rightarrow$ repeat $y(n)$

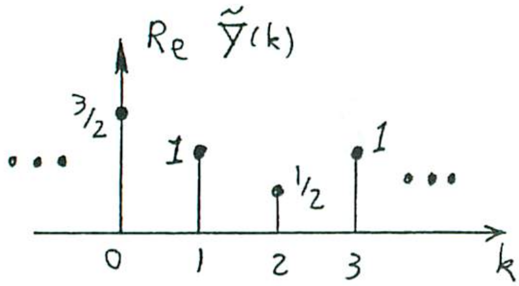
$N = 4$

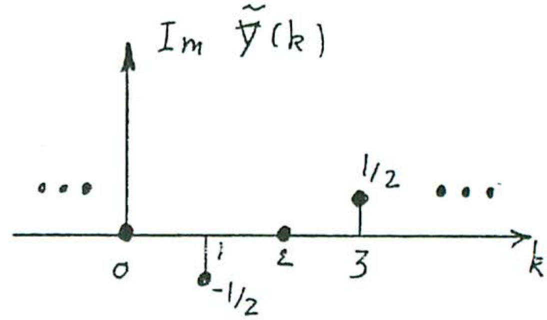
$W_N = e^{-j\frac{2\pi}{N}} = e^{-j\frac{\pi}{2}} = -j$

$\tilde{Y}(k) = \sum_{n=0}^{N-1} \tilde{y}(n)\, W_N^{kn} = \sum_{n=0}^{3} \tilde{y}(n)\,(-j)^{kn}$

Need 4 points of it

$$\begin{cases} \tilde{Y}(0) = \tilde{y}(0) + \tilde{y}(1) + \tilde{y}(2) + \tilde{y}(3) = 1 + \frac{1}{2} + 0 + 0 = 3/2 \\ \tilde{Y}(1) = \sum_{n=0}^{3} \tilde{y}(n)(-j)^n = 1(-j)^0 + \frac{1}{2}(-j)^1 + 0 + 0 = 1 - \frac{1}{2}j \\ \tilde{Y}(2) = 1(-j)^0 + \frac{1}{2}(-j)^{2\times 1} = 1 - 1/2 = 1/2 \\ \tilde{Y}(3) = 1(-j)^0 + \frac{1}{2}(-j)^{3\times 1} = 1 + 1/2\, j \end{cases}$$





Recall : Discrete Fourier series

$$\begin{cases} \tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j\frac{2\pi}{N}nk} = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k)\, W_N^{-kn} \\ \tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{N-1} \tilde{x}(n)\, W_N^{kn} \end{cases}$$

where $W_N = e^{-j\frac{2\pi}{N}}$

$$\tilde{X}(k) = X(z)\Big|_{z = e^{j\frac{2\pi}{N}k} = W_N^{-k}}$$

Properties of DFS: but we are more interested in properties of DFT!

1) Linearity : $\tilde{x}_1(n)$ & $\tilde{x}_2(n)$ both with period N

$$\tilde{x}_3(n) = a\,\tilde{x}_1(n) + b\,\tilde{x}_2(n)$$

$$\Rightarrow \tilde{X}_3(k) = a\,\tilde{X}_1(k) + b\,\tilde{X}_2(k)$$

2) Shift of a sequence

$$\begin{cases} \tilde{x}(n) \xrightarrow{\text{F. Coeff.}} \tilde{X}(k) \\ \tilde{x}(n+m) \longrightarrow W_N^{-km}\,\tilde{X}(k) \end{cases} \qquad \text{Period} = N$$

Can shift be > N ?

Note : any shift $m \geq N$ cannot be distinguished in the time-domain from a shorter shift $m' = m$ modulo N

$$W_N^{n\ell}\,\tilde{x}(n) \xrightarrow{\text{F. Coeff.}} \tilde{X}(k+\ell) \qquad \ell : \text{integer}$$

3) Symmetry

complex seq. $\tilde{x}(n) \xrightarrow{\text{F. Coeff.}} \tilde{X}(k)$

$$\tilde{x}^*(n) \longrightarrow \tilde{X}^*(-k)$$

$$\tilde{x}^*(-n) \longrightarrow \tilde{X}^*(k)$$

$\mathrm{Re}[\tilde{x}(n)] \longrightarrow \tilde{X}_e(k)$ conj. symmetric part

$j\,\mathrm{Im}[\tilde{x}(n)] \longrightarrow \tilde{X}_o(k)$ " antisym. "

$$\tilde{x}_e(n) \longrightarrow \mathrm{Re}[\tilde{X}(k)]$$

$$\tilde{x}_o(n) \longrightarrow j\,\mathrm{Im}[\tilde{X}(k)]$$

For real seq. $x(n)$

$$\tilde{X}(k) = \tilde{X}^*(-k)$$

$\Rightarrow$ $Re[\tilde{X}(k)]$: even

$Im[\tilde{X}(k)]$: odd

$|\tilde{X}(k)|$: even

phase $[\tilde{X}(k)]$: odd

$\tilde{x}_e(n) \longrightarrow Re[\tilde{X}(k)]$

$\tilde{x}_o(n) \longrightarrow j\, Im[\tilde{X}(k)]$

4) Periodic convolution : $\tilde{x}_1(n)$ & $\tilde{x}_2(n)$: period N

must have same period.

$$\begin{cases} \tilde{x}_1(n) \xrightarrow{DFS} \tilde{X}_1(k) \\ \tilde{x}_2(n) \xrightarrow{DFS} \tilde{X}_2(k) \end{cases}$$

If $\tilde{x}_3(n) \longrightarrow \tilde{X}_1(k) \cdot \tilde{X}_2(k)$ then

$$\boxed{\tilde{x}_3(n) = \sum_{m=0}^{N-1} \tilde{x}_1(m)\, \tilde{x}_2(n-m)}$$

$$\boxed{= \sum_{m=0}^{N-1} \tilde{x}_2(m)\, \tilde{x}_1(n-m)}$$

Periodic convolution

$\tilde{x}_3(n)$: periodic, period $= N$

sum go over one period

conv. is carried out only over one period

For

$$\begin{cases} \tilde{x}_3(n) = \tilde{x}_1(n) \cdot \tilde{x}_2(n) \\ \tilde{X}_3(k) = \dfrac{1}{N} \displaystyle\sum_{\ell=0}^{N-1} \tilde{X}_1(\ell)\, \tilde{X}_2(k-\ell) \end{cases}$$

Sampling the Z-transform

$x(n)$: aperiodic seq.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$ R.C. includes the unit circle assume

Evaluate $X(z)$ at N equally spaced points around the unit circle

$$\tilde{X}(k) = X(z)\Big|_{z=W_N^{-k}} = \sum_{n=-\infty}^{\infty} x(n) W_n^{kn} \qquad W_N = e^{-j\frac{2\pi}{N}}$$

Now

$$\tilde{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k) W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty} x(m) W_N^{km} W_N^{-kn}$$

$$= \sum_{m=-\infty}^{\infty} x(m) \underbrace{\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)}\right]}_{= \begin{cases} 1 & m=n+rN \\ 0 & \text{otherwise}\end{cases}}$$

$$\therefore \boxed{\tilde{x}(n) = \sum_{r=-\infty}^{\infty} x(n+rN)}$$

If duration of $x(n)$ is $\leq N$, each period of $\tilde{x}(n)$ is a replica of $x(n)$. $x(n)$ can be recovered exactly from $\tilde{x}(n)$.

If duration of $x(n) > N$, aliasing occurs. $x(n)$ can't be exactly recovered from $\tilde{x}(n)$.

Sampling in z domain, repeats sequence in time domain