

HW # 4

EG-EE 420

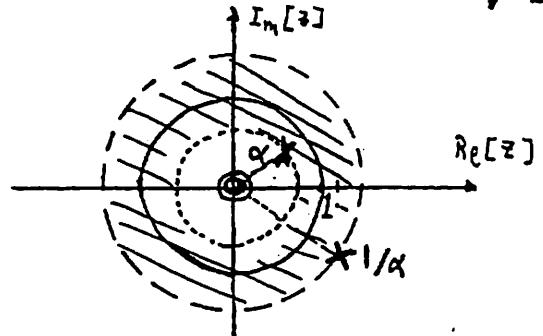
① 2-2 a,b Find $X(z)$, R.C., pole-zero pattern

a) $x(n) = \alpha^{|n|} \quad 0 < |\alpha| < 1$

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^{|n|} z^{-n} = \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

$$= \sum_{n=1}^{\infty} (\alpha z)^n + \frac{1}{1-\alpha z^{-1}} = \alpha z \frac{1}{1-\alpha z} + \frac{1}{1-\alpha z^{-1}} = \frac{z(1-\alpha^2)}{(1-\alpha z)(z-\alpha)}$$

R.C. : $|\alpha| < |z| < 1/|\alpha|$



b) $x(n) = A r^n \cos(\omega_0 n + \phi) u(n) \quad 0 < r < 1$

$$x(n) = \frac{A r^n}{2} [e^{j(\omega_0 n + \phi)} + e^{-j(\omega_0 n + \phi)}] u(n)$$

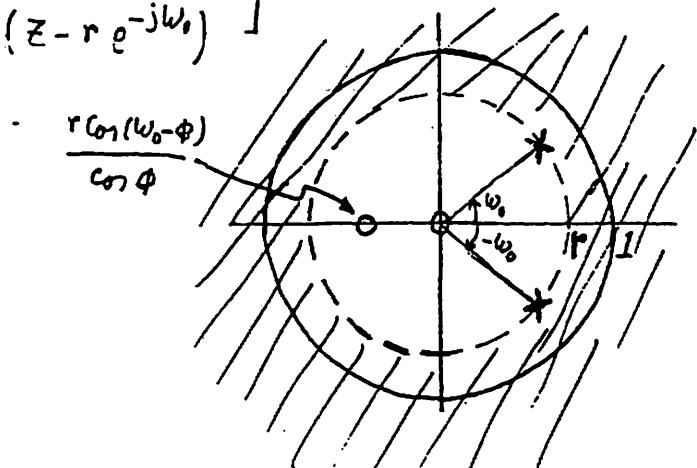
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} \frac{A r^n}{2} e^{j(\omega_0 n + \phi)} z^{-n} + \sum_{n=0}^{\infty} \frac{A r^n}{2} e^{-j(\omega_0 n + \phi)} z^{-n}$$

$$= \frac{A e^{j\phi}}{2} \sum_{n=0}^{\infty} (r e^{j\omega_0 z^{-1}})^n + \frac{A e^{-j\phi}}{2} \sum_{n=0}^{\infty} (r e^{-j\omega_0 z^{-1}})^n$$

$$X(z) = \frac{A e^{j\phi}}{2} \frac{1}{1 - r e^{j\omega_0 z^{-1}}} + \frac{A e^{-j\phi}}{2} \frac{1}{1 - r e^{-j\omega_0 z^{-1}}}$$

$$Z(z) = z A \left[\frac{z \cos \phi - r \cos(\omega_0 - \phi)}{(z - r e^{j\omega_0})(z - r e^{-j\omega_0})} \right]$$

R.C. : $|z| > r$



② 2-2 c; d

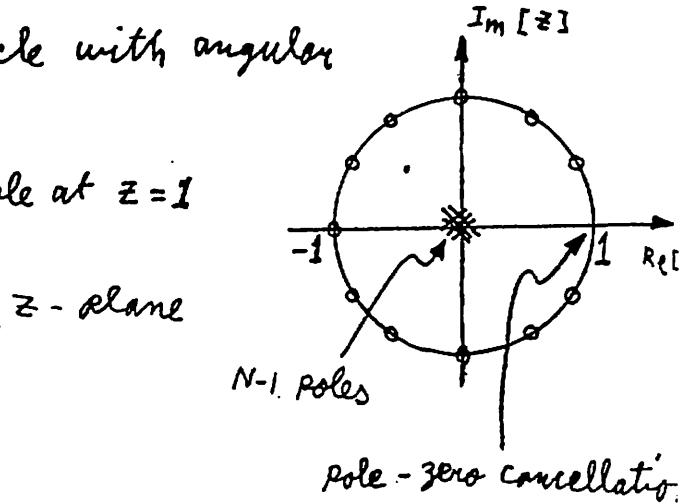
$$c) x(n) = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & N \leq n \\ 0 & n < 0 \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}} = \frac{z^N - 1}{z^{N-1}(z-1)}$$

R.C. : all z except $z=0$

Zeros : N zeros on the unit circle with angular spacing $\frac{2\pi}{N}$

Poles : $N-1$ poles at $z=0$, one pole at $z=1$



$$d) x(n) = \begin{cases} n, & 0 \leq n \leq N \\ 2N-n, & N+1 \leq n \leq 2N \\ 0, & n \geq 2N \\ 0, & n < 0 \end{cases}$$

Hint : express $x(n)$ in terms of $x(n)$ in part (c)

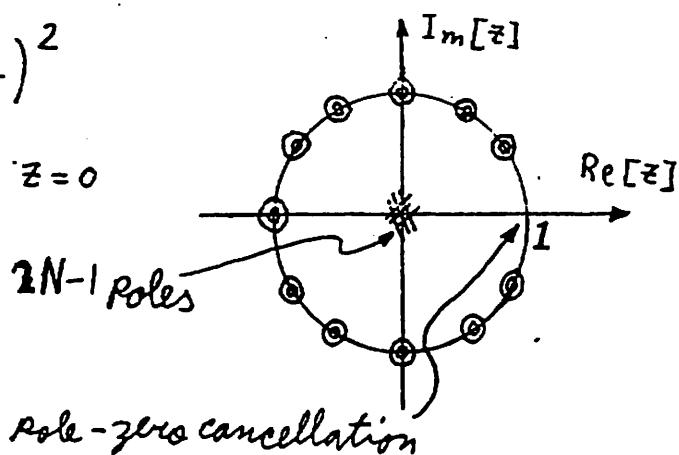
$x(n)$ is a finite duration sequence and we have

$x(n+1) = x_1(n) * x_1(n)$, where $x_1(n)$ is the sequence in part (c)

$$\therefore X(z) = z^{-1} X_1(z) X_1(z) = z^{-1} \left(\frac{z^N - 1}{z^{N-1}(z-1)} \right)^2$$

$$X(z) = \frac{1}{z^{2N-1}} \left(\frac{z^N - 1}{z-1} \right)^2$$

R.C. : all z -plane except $z=0$



③ 2-7 Does $F(z) = z^*$ correspond to \mathcal{Z} -transform of a seq.?

If $F(z)$ correspond to \mathcal{Z} -transform of $f(n)$, it should be an analytical function in the neighborhood of $z=0$.

Let $F(z) = F(x+jy) = u(x,y) + jv(x,y)$.

For F to be analytic, check the Cauchy Riemann condition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$z = x+jy, \quad z^* = x-jy \Rightarrow \begin{cases} u = x \\ v = -y \end{cases} \quad \frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$$

So $F(z)$ is not analytic (In other words $F'(z)$ exists everywhere except at $z=0$). Hence by residue theorem, inverse \mathcal{Z} -transform of $F(z)$ does not exist.

④ 2-10 $x(n) \xrightarrow{\mathcal{Z}} X(z)$. Show that

a) $x(n+n_0) \xrightarrow{\mathcal{Z}} z^{n_0} X(z)$

$$\begin{aligned} \mathcal{Z}[x(n+n_0)] &= \sum_{n=-\infty}^{\infty} x(n+n_0) z^{-n} = \sum_{n'=-\infty}^{\infty} x(n') z^{-(n'-n_0)} \\ &= \sum_{n'} x(n') z^{-n'} z^{n_0} = z^{n_0} X(z) \end{aligned}$$

b) $a^n x(n) \xrightarrow{\mathcal{Z}} X(a^{-1}z)$

$$\mathcal{Z}[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n} = \sum_n x(n) (a^{-1}z)^{-n} = X(a^{-1}z)$$

c) $n x(n) \xrightarrow{\mathcal{Z}} -z X'(z)$

$$X(z) = \sum_n x(n) z^{-n}$$

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} -n x(n) z^{-n-1}$$

$$\Rightarrow -X'(z) = \bar{z}^{-1} \sum_n n x(n) z^{-n}$$

$$\Rightarrow -z X'(z) = \sum_n (n x(n)) z^{-n} = \mathcal{Z}[n x(n)]$$

⑤ 2-11 $x(n) \leftrightarrow X(z)$ show that

a) $\mathcal{Z}[x^*(n)] = X^*(\bar{z}^*)$

$$\mathcal{Z}[x^*(n)] = \sum_n x^*(n) z^{-n} = \left[\sum_n x(n) (\bar{z}^*)^{-n} \right]^* = X^*(\bar{z}^*)$$

b) $\mathcal{Z}[x(-n)] = X(1/\bar{z})$

$$\mathcal{Z}[x(-n)] = \sum_n x(-n) z^{-n} = \sum_{n'} x(n') (\bar{z}^1)^{-n'} = X(\bar{z}^1) = X(1/\bar{z})$$

c) $\mathcal{Z}[\operatorname{Re} x(n)] = \frac{1}{2} [X(z) + X^*(\bar{z}^*)]$

$$\operatorname{Re} x(n) = \frac{1}{2} [x(n) + x^*(n)]$$

$$\mathcal{Z}[\operatorname{Re} x(n)] = \frac{1}{2} [X(z) + X^*(\bar{z}^*)] \text{ using (a)}$$

d) $\mathcal{Z}[\operatorname{Im} x(n)] = \frac{1}{2j} [X(z) - X^*(\bar{z}^*)]$

$$\operatorname{Im} x(n) = \frac{1}{2j} [x(n) - x^*(n)]$$

$$\mathcal{Z}[\operatorname{Im} x(n)] = \frac{1}{2j} [X(z) - X^*(\bar{z}^*)] \text{ using (a)}$$

⑥ 2-15 FIR filter $h(n)$: length $2N+1$, real & even

Show that zeros of $H(z)$ occur in mirror-image about the unit circle

i.e., $\rho e^{j\theta}$ & $(1/\rho) e^{j\theta}$.

$$h(n) : \text{real & even} \Rightarrow \begin{cases} h(n) = h(-n) \\ h^*(n) = h(n) \end{cases}$$

using results of problem ⑤ a & b

$$\begin{cases} H(z) = H(1/z) \\ H(z) = H^*(\bar{z}^*) \Rightarrow H^*(z) = H(z^*) \end{cases}$$

If z_0 is a zero of $H(z)$, then $H(z_0) = 0 \Rightarrow H^*(z_0) = 0$

$$\Rightarrow H(z_0^*) = 0 \Rightarrow H(1/z_0^*) = 0$$

$$\text{i.e. } H(z_0) = 0 \Rightarrow H(1/z_0^*) = 0$$

Hence zero at $z_0 = \boxed{\rho e^{j\theta}}$

$$\Rightarrow \text{zero at } \frac{1}{z_0^*} = \frac{1}{\rho e^{-j\theta}} = \boxed{\frac{1}{\rho} e^{j\theta}}$$