EGEE 420 HW #3 Soln.

1 Text 1-20 Assume a: real and $|a|<1$ $a \neq b$

$y(n) - a\,y(n-1) = x(n) - b\,x(n-1)$ causal LSI system

$$\Rightarrow Y(z) - a z^{-1} Y(z) = X(z) - b z^{-1} X(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1-bz^{-1}}{1-az^{-1}}$$

$$\therefore H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{1-be^{-j\omega}}{1-ae^{-j\omega}}$$

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = \frac{1-be^{-j\omega}}{1-ae^{-j\omega}} \cdot \frac{1-b^*e^{j\omega}}{1-a^*e^{j\omega}}$$

$$= \frac{1+|b|^2 - b^*e^{j\omega} - be^{-j\omega}}{1+|a|^2 - a^*e^{j\omega} - ae^{-j\omega}} = |b|^2 \frac{1+\frac{1}{|b|^2} - \frac{1}{b}e^{j\omega} - \frac{1}{b^*}e^{-j\omega}}{1+|a|^2 - a^*e^{j\omega} - ae^{-j\omega}}$$

Thus if $\boxed{b = \frac{1}{a^*}}$ (or $b = \frac{1}{a}$ since a is real), then $|H(e^{j\omega})| = |b|$ is constant $\Rightarrow$ All pass system

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2 $h(n)$: impulse resp. of problem 1 : $h(n) * \tilde{h}(n) = ?$, $\tilde{h}(n) = h(-n)$

Let $y(n) = h(n) * \tilde{h}(n) = h(n) * h(-n)$

$$\Rightarrow Y(e^{j\omega}) = H(e^{j\omega})H(e^{-j\omega}) = \frac{1-\frac{1}{a}e^{-j\omega}}{1-ae^{-j\omega}} \cdot \frac{1-\frac{1}{a}e^{+j\omega}}{1-ae^{j\omega}} = \frac{1}{a^2} = b^2$$

simplify as in $\boxed{1}$

$$y(n) = \mathcal{F}^{-1}[Y(e^{j\omega})] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{a^2\pi n}\left(\frac{e^{jn\pi}-e^{-jn\pi}}{2j}\right)$$

$$= \frac{1}{a^2}\frac{\sin n\pi}{n\pi} \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\boxed{y(n) = \frac{1}{a^2}\delta(n)}$$

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3 Text 1-28 $h(n) = h_r(n) + j\,h_i(n)$

$$H(e^{j\omega}) = H_R(e^{j\omega}) + j\,H_I(e^{j\omega})$$
$$= H_{ER}(e^{j\omega}) + H_{OR}(e^{j\omega}) + j\left[H_{EI}(e^{j\omega}) + H_{OI}(e^{j\omega})\right]$$

$$h_r(n) \xrightarrow{\mathcal{F}} H_A(e^{j\omega}) + j\,H_B(e^{j\omega})$$
$$h_i(n) \longrightarrow H_C(e^{j\omega}) + j\,H_D(e^{j\omega})$$

$$\begin{cases} h_r(n) \xrightarrow{\mathcal{F}} H_E(e^{j\omega}) = H_{ER}(e^{j\omega}) + j\,H_{EI}(e^{j\omega}) \Rightarrow \begin{cases} H_A = H_{ER} \\ H_B = H_{EI} \end{cases} \\ j\,h_i(n) \xrightarrow{\mathcal{F}} H_D(e^{j\omega}) = H_{OR}(e^{j\omega}) + j\,H_{OI}(e^{j\omega}) \Rightarrow \begin{cases} H_C = H_{OI} \\ H_D = -H_{OR} \end{cases} \end{cases}$$

HW # 3 Cont.'d

$\cancel{2/3}$

4 Time domain approach

$$\begin{tikzcd} f(t) \arrow[r] & \otimes \arrow[r] & f_s(t) \\ & \sum_n \delta(t-nT) \arrow[u] & \end{tikzcd}$$

$$f_s(t) = f(t)\cdot \sum_n \delta(t-nT) \quad , \quad T = 2\pi/\Omega_s$$

a) $$f_{s_1} = \cos\left[\left(\frac{\Omega_s}{2}-\omega_0\right)t\right]\sum_n \delta(t-nT) = \sum_n \cos\left[\left(\frac{\Omega_s}{2}-\omega_0\right)t\right]\delta(t-nT)$$

$$= \sum_n \cos\left[\left(\frac{\Omega_s}{2}-\omega_0\right)nT\right] = \sum_n \cos\left[\left(\frac{\Omega_s}{2}-\omega_0\right)\frac{2\pi n}{\Omega_s}\right]$$

$$= \sum_n \cos\left(n\pi - \frac{2\pi n\omega_0}{\Omega_s}\right) = \sum_n (-1)^n \cos\frac{2\pi n\omega_0}{\Omega_s} \qquad (1)$$

$$f_{s2} = \cos\left[\left(\frac{\Omega_s}{2}+\omega_0\right)t\right]\sum_n \delta(t-nT) = \sum_n \cos\left[\left(\frac{\Omega_s}{2}+\omega_0\right)nT\right]$$

$$= \sum_n \cos\left(n\pi + \frac{2\pi n\omega_0}{\Omega_s}\right) = \sum_n (-1)^n \cos\frac{2\pi n\omega_0}{\Omega_s} \qquad (2)$$

comparing (1) & (2) $\Rightarrow$ $\boxed{f_{s_1}(t) = f_{s2}(t)}$ $\therefore$ identical samples

b) $$f_{s_1}(t) = \sin\left[\left(\frac{\Omega_s}{2}-\omega_0\right)t\right]\sum_n \delta(t-nT) = \sum_n \sin\left(\frac{\Omega_s}{2}-\omega_0\right)nT$$

$$= \sum_n \sin\left(\frac{\Omega_s}{2}-\omega_0\right)\frac{n2\pi}{\Omega_s} = \sum_n \sin\left(n\pi - \frac{2\pi n\omega_0}{\Omega_s}\right)$$

$$= \sum_n \left[\underbrace{\sin(n\pi)}_{0}\cos\frac{2\pi n\omega_0}{\Omega_s} - \cos(n\pi)\sin\frac{2\pi n\omega_0}{\Omega_s}\right]$$

$$= \sum_n (-1)^{n+1}\sin\frac{2\pi n\omega_0}{\Omega_s} \qquad (1)$$

$$f_{s2}(t) = \sin\left[\left(\frac{\Omega_s}{2}+\omega_0\right)t\right]\sum_n \delta(t-nT) = \sum_n \sin\left(\frac{\Omega_s}{2}+\omega_0\right)\frac{n2\pi}{\Omega_s}$$

$$= \sum_n \sin\left(n\pi + \frac{2\pi n\omega_0}{\Omega_s}\right)$$

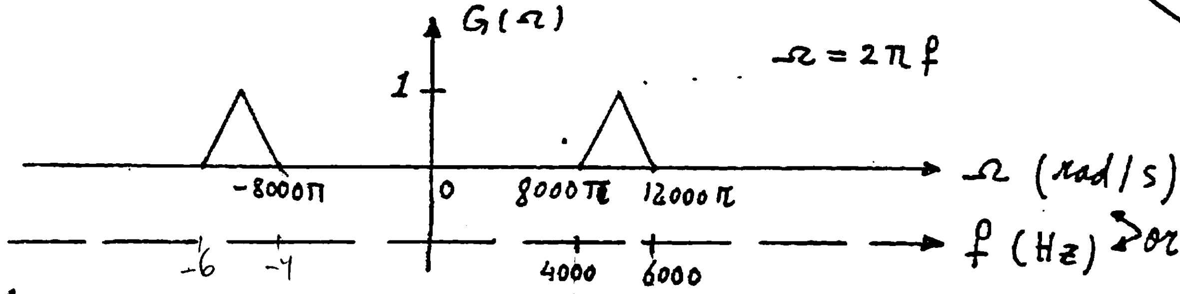
$$= \sum_n \left[\underbrace{\sin(n\pi)}_{0}\cos\frac{2\pi n\omega_0}{\Omega_s} + \cos(n\pi)\sin\frac{2\pi n\omega_0}{\Omega_s}\right]$$

$$= \sum_n (-1)^n \sin\frac{2\pi n\omega_0}{\Omega_s} \qquad (2)$$

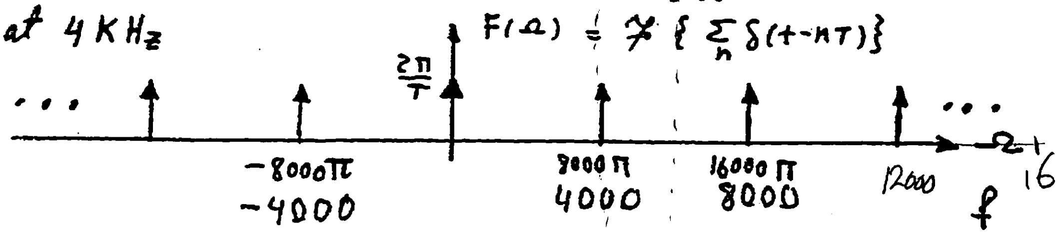
comparing (1) & (2) $\Rightarrow$ $\boxed{f_{s_1} \neq f_{s2}}$ They have the same magnitude but differ by 180° in phase

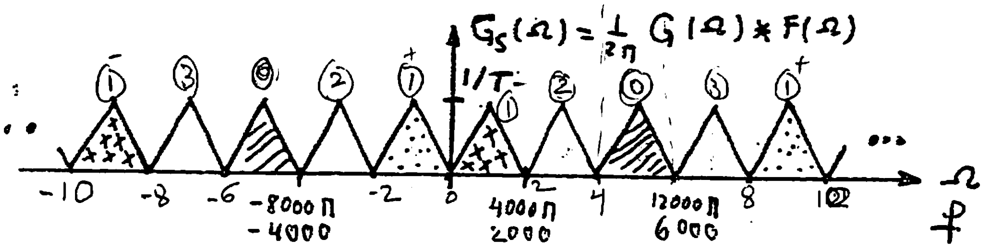
HW #3 Cont'd

5



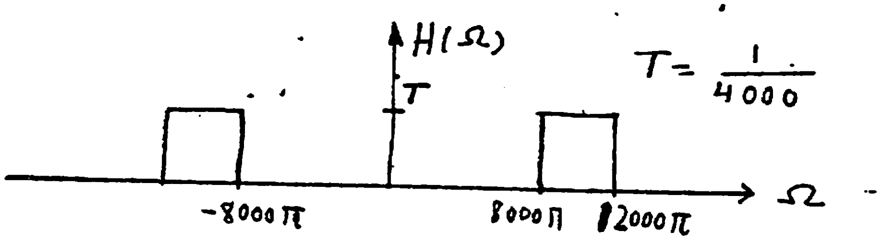
Sampling at 4 KHz





a) From the above plots it is concluded that the minimum sampling rate is $\boxed{4\text{ KHz}}$.

b) The required interpolation filter is a bandpass filter with the following frequency response.



$$h(t) = T\frac{1}{2\pi}\int_{-12000\pi}^{-8000\pi} e^{j\Omega t}\,d\Omega + T\frac{1}{2\pi}\int_{8000\pi}^{12000\pi} e^{j\Omega t}\,d\Omega$$

$$= \frac{T}{2\pi}\left\{\frac{e^{-j8000\pi t} - e^{-j12000\pi t}}{jt} + \frac{e^{j12000t} - e^{j8000t}}{jt}\right\}$$

$$= \frac{T}{\pi t}\left\{\sin 12000\pi t - \sin 8000\pi t\right\}$$

$$h(t) = 3\,\frac{\sin 12000\pi t}{12000\pi t} - 2\,\frac{\sin 8000\pi t}{8000\pi t}$$

Convolution Example **(Boldface number represents the value at the origin)**

x = { **1** 2 -2 1 4 -1 1 2} **BLUE**
h = {0.25 **0.5** 0.25}

y = conv(x,h) = 0.25 **1.0** 0.75 -0.25 1.0 2.0 0.75 0.75 1.25 0.5 **GREEN**

z = conv(y,h) = 0.0625 0.375 **0.75** 0.5625 0.3125 0.9375 1.4375 1.0625 0.875 0.9375 0.5625 0.125 **RED**