# HW 9, EE 420 Digital Filters
# California State University, Fullerton
# Spring 2010

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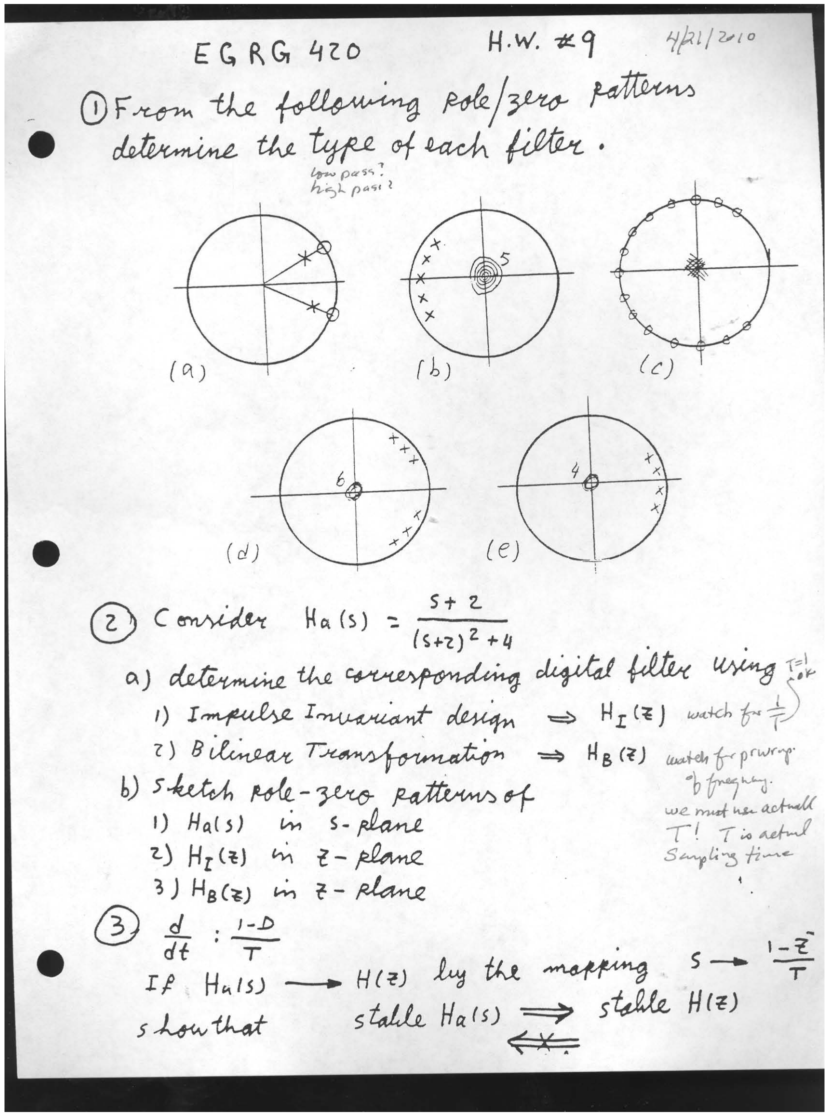
# 1 Problems



EGRG 420 H.W. #9 4/21/2010

(1) From the following pole/zero patterns determine the type of each filter.

low pass? high pass?



(2) Consider $H_a(s) = \dfrac{s+2}{(s+2)^2+4}$

a) determine the corresponding digital filter using (T=1 ok)

1) Impulse Invariant design $\Rightarrow H_I(z)$ watch for $\frac{1}{T}$

2) Bilinear Transformation $\Rightarrow H_B(z)$ watch for prewarp. of frequency.

b) sketch pole-zero patterns of

1) $H_a(s)$ in s-plane

2) $H_I(z)$ in z-plane

3) $H_B(z)$ in z-plane

we must use actual T! T is actual Sampling time.

(3) $\dfrac{d}{dt} : \dfrac{1-D}{T}$

If $H_a(s) \longrightarrow H(z)$ by the mapping $s \longrightarrow \dfrac{1-z^{-1}}{T}$

show that stable $H_a(s) \Longrightarrow$ stable $H(z)$

$\cancel{\Longleftarrow}$

# 2 problem 1

We will find the magnitude spectrum $\left|H\left(e^{j\omega}\right)\right|$ as the digital frequency $\omega$ is changed from 0 radians to $\pi$ radians. At each different value of $\omega$, the magnitude of the frequency response is $\left|H\left(e^{j\omega}\right)\right| = \frac{\prod_{i=1}^{M}|\omega - z_i|}{\prod_{i=1}^{N}|\omega - p_i|}$ where $|\omega - z_i|$ is the length of the vector from the point $\omega$ (which is the point on the unit circle) to the point where the $i^{th}$ zero is located. And similarly, $|\omega - p_i|$ is the length of the vector from the point $\omega$ to the point where the $i^{th}$ pole is located. So, by estimating these products, one can estimate a value for $\left|H\left(e^{j\omega}\right)\right|$ as $\omega$ is moved around the unit circle.

### 2.1 Part (a)

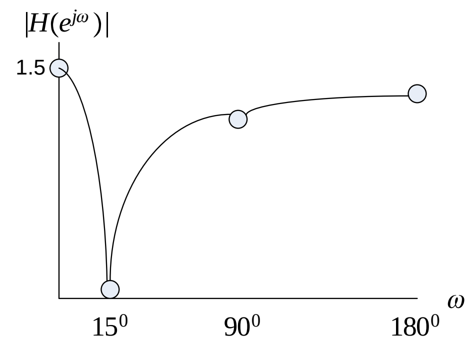
At $\omega = 0^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{.25\times.25}{.2\times.2} \approx 1.5$

At $\omega = 15^0$ where the zero is located, $\left|H\left(e^{j\omega}\right)\right| = 0$

At $\omega = 90^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{.7\times1.2}{.65\times1.1} \approx 1.1$

At $\omega = 180^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{1.9\times1.9}{1.7\times1.7} \approx 1.3$

Hence this is a sketch



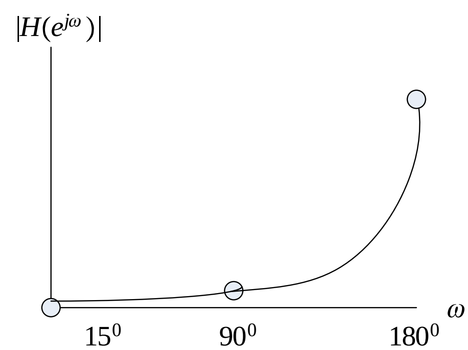

So this is a $\boxed{\text{notch filter}}$

### 2.2 Part (b)

At $\omega = 0^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{1}{2^5} \approx 0.03$

At $\omega = 90^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{1}{.8\times1\times1.4\times1.6\times1.7} \approx 0.328\,26$

At $\omega = 180^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{1}{small\ values} \approx \text{large}$




So this allows frequencies very close to $\pi$ to pass. So high pass filter

### 2.3 Part (c)

At $\omega = 0^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{.7\times1\times1.4\times1.6\cdots\times2\times1.8\times1.6\cdots}{1} \approx 20$

At $\omega = 90^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{\text{smaller values than above since vector is smaller now}}{1} \approx 10$

At $\omega = 180^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{\text{much smaller values than above since close to zeros}}{1} \approx 0$

So, this is low pass filter

### 2.4 part (d)

At $\omega = 0^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{1}{.3\times.5\times.7\times.3\times.5\times.7} \approx$ large value

At $\omega = 30^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{1}{\text{very small values due to being close to poles}}$ $\approx$much larger value the above

At $\omega = 90^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{1}{\text{larger values than the above due to vectors below x-axis being further away}}$ $\approx$smaller than where at $\omega = 0^0$

At $\omega = 180^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{1}{\text{much larger values than the above}} \approx 0$

So, this is band pass filter

### 2.5 Part (e)

At $\omega = 0^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{1}{\text{very small values due to being close to poles}} \approx$ large value

At $\omega = 90^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{1}{1.3\times1.4\times1.5\times1.6} \approx .2$

At $\omega = 180^0$, $\left|H\left(e^{j\omega}\right)\right| \approx \frac{1}{1.8\times1.8\times1.8\times1.8} \approx$ smaller values than above

So, low pass filter

## 3 Problem 2

$H(s) = \frac{s+2}{(s+2)^2+4}$

### 3.1 part(a)

Using impulse invariance, $H(z) = \sum_{i=1}^{N} \frac{T\ A_i}{1-e^{p_i T} z^{-1}}$ where $p_i$ are the poles of $H(s)$ and $A_i$ is the partial fraction result of expressing $H(s)$ as $\sum_{i=1}^{N} \frac{A_i}{s-p_i}$. Notice that this method works only for distinct poles in $H(s)$. So the first step is to express $H(s)$ is partial fraction form to determine $A_i$. The poles of $H(s)$ are roots of the denominator $(s+2)^2 + 4$ hence poles are roots of $s^2 + 4s + 8$ or $-\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4ac} = -1 \pm \frac{1}{2}\sqrt{16 - 4 \times 8} = -1 \pm 2j$, hence

$$
\begin{aligned}
p_1 &= -1 + 2j \\
p_2 &= -1 - 2j
\end{aligned}
$$

Then $H(s) = \frac{s+2}{(s-p_1)(s-p_2)} = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2}$, then

$$A_1 = \lim_{s\to p_1} \frac{s+2}{(s-p_2)} = \lim_{s\to p_1} \frac{-1+2j+2}{((-1+2j)-(-1-2j))} = \frac{1+2j}{4j}$$

and

$$A_2 = \lim_{s\to p_2} \frac{s+2}{(s-p_1)} = \lim_{s\to p_2} \frac{-1-2j+2}{((-1-2j)-(-1+2j))} = \frac{1-2j}{-4j}$$

Hence

$$H(s) = \frac{\frac{1+2j}{4j}}{s-(-1+2j)} + \frac{\frac{1-2j}{-4j}}{s-(-1-2j)}$$

And

$$H(z) = \frac{T\frac{1+2j}{4j}}{1-z^{-1}\exp(-1+2j)T} + \frac{T\frac{1-2j}{-4j}}{1-z^{-1}\exp(-1-2j)T}$$

We can take $T=1$ and the above becomes

$$H(z) = \frac{\frac{1+2j}{4j}}{1-z^-1\exp(-1+2j)} + \frac{\frac{1-2j}{-4j}}{1-z^-1\exp(-1-2j)}$$

This can be simplified to

$$H(z) = \frac{z^2+0.32035z}{z^2+0.30618z+0.13535}$$

The poles are

$$z_1 = -0.153 - 0.3345\,j$$
$$z_2 = -0.153 + 0.3345\,i$$

So, they are both inside the unit circle.

### 3.2 part (2)

Using bilinear transformation, $H(z) = H(s)\big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$ Since $H(s) = \frac{s+2}{(s+2)^2+4}$, then

$$\begin{aligned} H(z) &= \frac{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}+2}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}+2\right)^2+4} \\ &= \frac{T(1+z)(z-1+T+Tz)}{2\left((z-1)^2+2T^2(1+z)^2+2T(z^2-1)\right)} \end{aligned}$$

For $T=1$, the above simplifies to

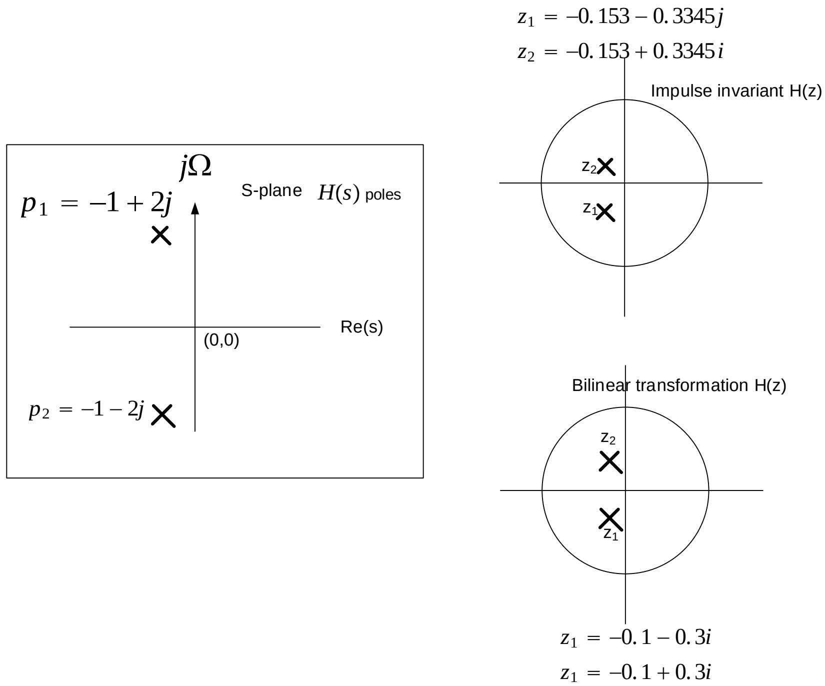
$$H(z) = \frac{z+z^2}{1+2z+10z^2}$$

The poles are located at roots of $1+2z+10z^2$, which are

$$z_1 = -0.1 - 0.3i$$
$$z_1 = -0.1 + 0.3i$$

So, they are both inside the unit circle.

### 3.3 part(B)




## 4 Problem 3

Consider some $H(s) = \frac{N(s)}{D(s)}$. Let $D(s)$ be written in factored form $\prod_i^N (s - p_i)$ where $N$ is number of $H(s)$ poles and $p_i$ is the pole. For the purpose of this solution, we can assume there is one pole only. The same idea applied for all others. Hence, we have

$$H(s) = \frac{N(s)}{s - p} \tag{1}$$

And now we want to show that if $p < 0$, then the transformation results in $H(z)$ with a pole inside the unit circle. Let

$$s = \frac{1 - z^{-1}}{T}$$

then (1) becomes

$$H(z) = \frac{N(z)}{\frac{1-z^{-1}}{T} - p} = \frac{TN(z)}{1 - z^{-1} - Tp} = \frac{zTN(z)}{z - 1 - zTp} = \frac{zTN(z)}{z(1 - Tp) - 1} = \frac{\frac{zTN(z)}{1-Tp}}{z - \frac{1}{1-Tp}}$$

Hence pole of the $H(z)$ is

$$q = \frac{1}{1 - Tp}$$

Since $p < 0$ then the numerator of $q$ is larger than one. Hence $q < 1$, hence a stable pole of $H(z)$. Therefore, a stable pole of $H(s)$ maps to a stable pole of $H(z)$ Now we need to show that a stable pole of $H(z)$ will not map to a stable pole of $H(s)$. First we need to find the inverse transformation. Since $s = \frac{1-z^{-1}}{T}$ then

$$
\begin{aligned}
sT &= 1 - z^{-1} \\
sT &= \frac{z-1}{z} \\
zsT &= z - 1 \\
zsT - z &= -1 \\
z - zsT &= 1 \\
z(1 - sT) &= 1
\end{aligned}
$$

Hence

$$z = \frac{1}{1 - sT}$$

Now, given $H(z) = \frac{N(z)}{D(z)} = \frac{N(z)}{z-q}$ where $q$ is a pole of $H(z)$ where $q$ is stable. Hence $|q| < 1$, i.e. pole is inside the unit circle. Now apply the above transformation

$$
\begin{aligned}
H(s) &= \frac{N(z)}{z-q} \\
&= \frac{N(s)}{\frac{1}{1-sT} - q} = \frac{N(s)(1-sT)}{1 - q(1-sT)} = \frac{N(s)(1-sT)}{1 - q + qsT} = \frac{N(s)\frac{(1-sT)}{qT}}{s + \frac{1-q}{qT}} = \frac{N(s)\frac{(1-sT)}{qT}}{s - \left(\frac{q-1}{qT}\right)}
\end{aligned}
$$

Hence $H(s)$ pole is at

$$\frac{q-1}{qT}$$

this pole will be stable only if the real part of it is less than zero. Let $q = \frac{j}{2}$ a stable pole in the z plane. Then the above pole size becomes $\frac{\frac{j}{2}-1}{\frac{j}{2}T} = \frac{-j\left(\frac{j}{2}-1\right)}{\frac{1}{2}T} = \frac{\left(\frac{-j^2}{2}-j\right)}{\frac{1}{2}T} = \frac{\left(\frac{1}{2}-j\right)}{\frac{1}{2}T}$. Hence the real part of this pole is $\frac{1}{T}$, which is $> 0$ since $T$ is positive. Hence $H(s)$ is unstable. Hence, starting with stable $H(z)$, using this transformation, the resulting $H(s)$ is not always stable. (it depends on the location of the z pole), sometimes we get stable $H(s)$ and sometimes unstable $H(s)$. For example, if we have used $q = \frac{1}{2}$, then doing the above results in $\frac{\frac{1}{2}-1}{\frac{1}{2}T} = -\frac{1}{T}$ which is $< 0$ since $T$ is positive. Hence we see that depending on the z pole, the resulting $H(s)$ can be stable or not.