HW 8, EE 420 Digital Filters
California State University, Fullerton
Spring 2010

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Spring 2010 Compiled on May 12, 2019 at 4:30pm

Contents

1 Problem 1 problem 4.1 in textbook (page 182)

Determine the system function of the 2 networks below and show that they have the same poles

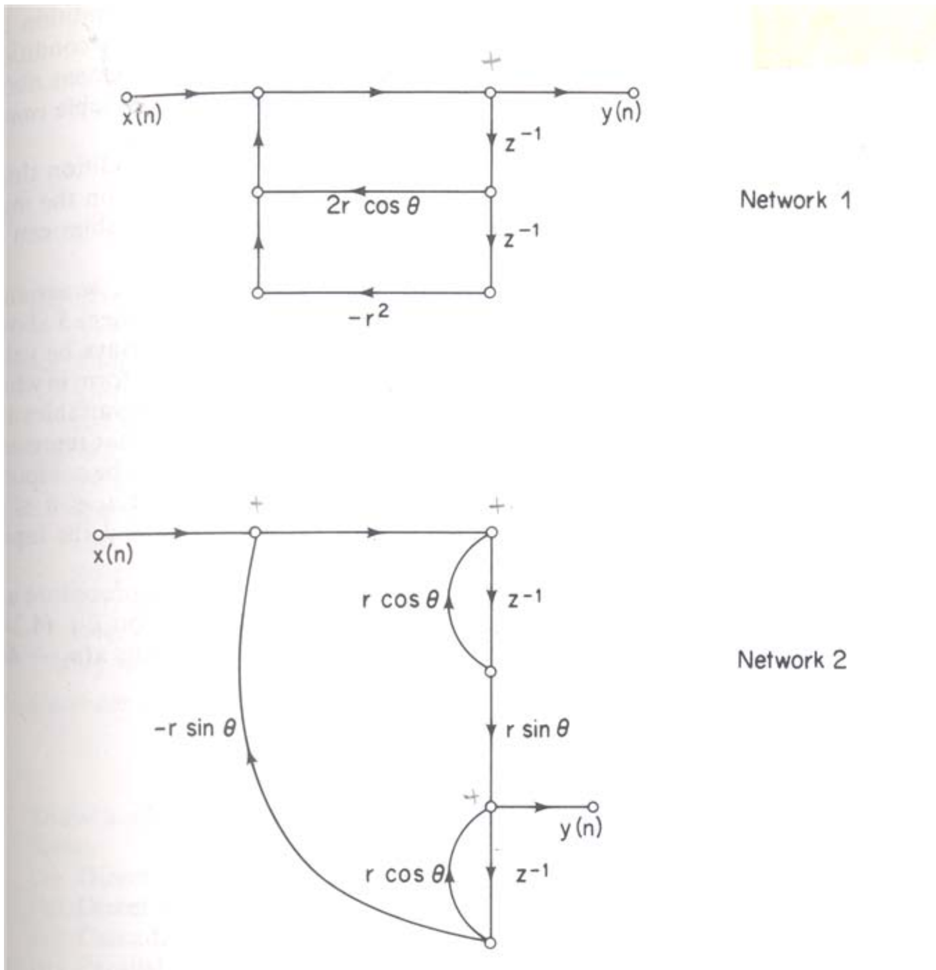


Fig. P4.1

For network 1),

$$y(n) = x(n) + 2r\cos\theta y(n-1) - r^2 y(n-2)$$

Hence

$$\begin{aligned} Y(z) &= X(z) + 2r\cos\theta z^{-1} Y(z) - r^2 z^{-2} Y(z) \\ Y(z)\left[1 - 2r\cos\theta z^{-1} + r^2 z^{-2}\right] &= X(z) \\ H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{1}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}} \end{aligned}$$

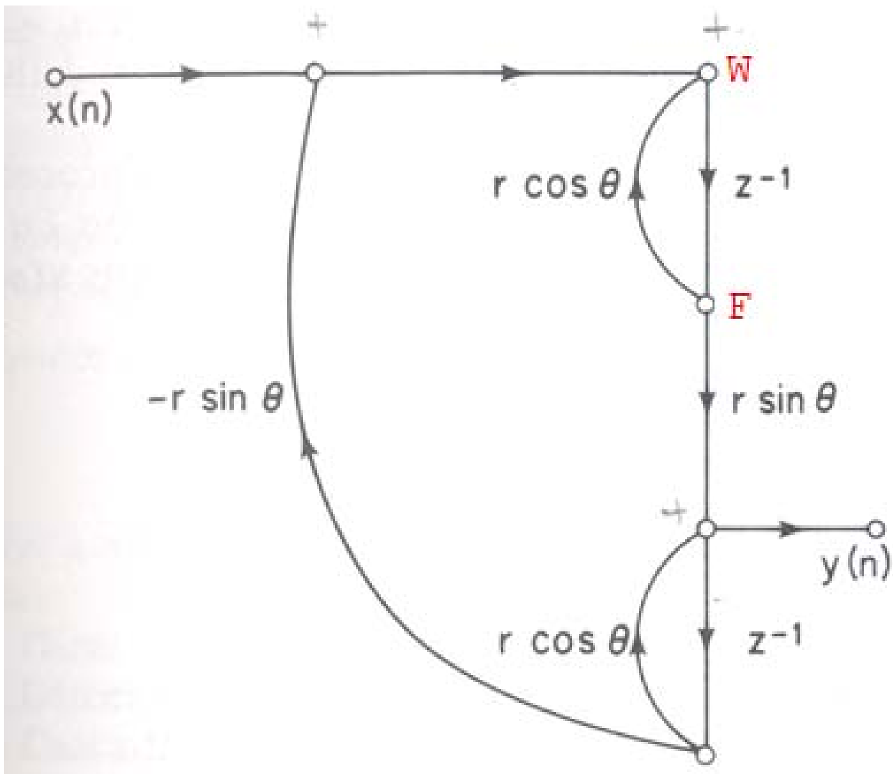
To see the poles and zeros more easily, mutiply the above by z^2/z^2

$$H(z) = \frac{z^2}{z^2 - 2r\cos\theta z + r^2}$$

Hence poles of system are at $z = \frac{-b\pm\sqrt{b^2-4ac}}{2} = \frac{2r\cos\theta}{2} \pm \frac{1}{2}\sqrt{(-2r\cos\theta)^2 - 4r^2} = \frac{2r\cos\theta}{2} \pm \frac{1}{2}\sqrt{4r^2\cos^2\theta - 4r^2}$
Hence

$$\begin{aligned} z &= r\cos\theta \pm r\sqrt{\cos^2\theta - 1} = r\cos\theta \pm rj\sin\theta \\ &= r(\cos\theta \pm j\sin\theta) \end{aligned} \tag{1}$$

For network 2, we start by labeling the corner point at W



$$y(n) = r\cos\theta\, y(n-1) + r\sin\theta\, W(n-1) \tag{1}$$

Now find $W(n)$

$$W(n) = r\cos\theta\, W(n-1) - r\sin\theta\, y(n-1) + x(n) \tag{2}$$

Hence from the above equation (2) we need to find $W(n-1)$, but we do this by delaying n one unit time, hence

$$W(n-1) = r\cos\theta\, W(n-2) - r\sin\theta\, y(n-2) + x(n-1)$$

now substitute this back into (1) we obtain

$$y(n) = r\cos\theta\, y(n-1) + r\sin\theta\, [\, r\cos\theta\, W(n-2) - r\sin\theta\, y(n-2) + x(n-1)] \tag{3}$$

We need to eliminate W from these equation to be able to obtain relation between $y(n)$ and $x(n)$ only. This is the tricky part.

From (1), we solve for $W(n-1)$

$$W(n-1) = \frac{y(n) - r\cos\theta\, y(n-1)}{r\sin\theta}$$

Hence delay it one more time to find $W(n-2)$

$$W(n-2) = \frac{y(n-1) - r\cos\theta\, y(n-2)}{r\sin\theta}$$

Substitute the above into (3) to finally remove W from the equation, we obtain

$$\begin{aligned}
y(n) &= r\cos\theta\, y(n-1) + r\sin\theta \left[r\cos\theta \overbrace{\left[\frac{y(n-1) - r\cos\theta\, y(n-2)}{r\sin\theta}\right]}^{W(n-2)} - r\sin\theta\, y(n-2) + x(n-1) \right] \\
&= r\cos\theta\, y(n-1) + \ r\cos\theta\ [y(n-1) - r\cos\theta\, y(n-2)] - r^2\sin^2\theta\, y(n-2) + r\sin\theta x(n-1) \\
&= r\cos\theta\, y(n-1) + r\cos\theta\, y(n-1) - r^2\cos^2\theta\, y(n-2) - r^2\sin^2\theta\, y(n-2) + r\sin\theta x(n-1) \\
&= 2r\cos\theta\, y(n-1) - r^2\, y(n-2)\overbrace{\left(\cos^2\theta + \sin^2\theta\right)}^{1} + r\sin\theta x(n-1)
\end{aligned}$$

Take Z transform

$$\begin{aligned}
Y(z) &= 2r\cos\theta z^{-1}Y(z) - r^2z^{-2}Y(z) + r\sin\theta z^{-1}X(z) \\
Y(z)\left[1 - 2r\cos\theta z^{-1} + r^2z^{-2}\right] &= r\sin\theta z^{-1}X(z)
\end{aligned}$$

Hence

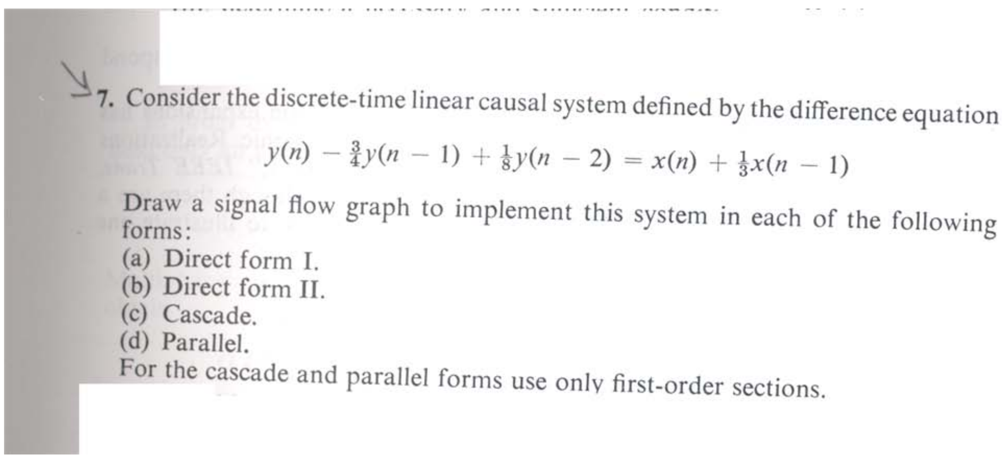
$$H(z) = \frac{Y(z)}{X(z)} = \frac{r\sin\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$
$$= \frac{r\sin\theta z}{z^2 - 2r\cos\theta z + r^2}$$

Compare the above to network (1) which was

$$H(z) = \frac{z^2}{z^2 - 2r\cos\theta z + r^2}$$

Same denominator, hence same poles.

2 Second problem (problem 4.7, textbook, page 185)



7. Consider the discrete-time linear causal system defined by the difference equation

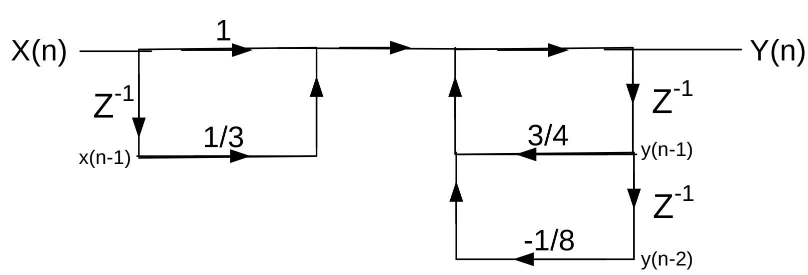
$$y(n) - \tfrac{3}{4}y(n-1) + \tfrac{1}{8}y(n-2) = x(n) + \tfrac{1}{3}x(n-1)$$

Draw a signal flow graph to implement this system in each of the following forms:
(a) Direct form I.
(b) Direct form II.
(c) Cascade.
(d) Parallel.
For the cascade and parallel forms use only first-order sections.

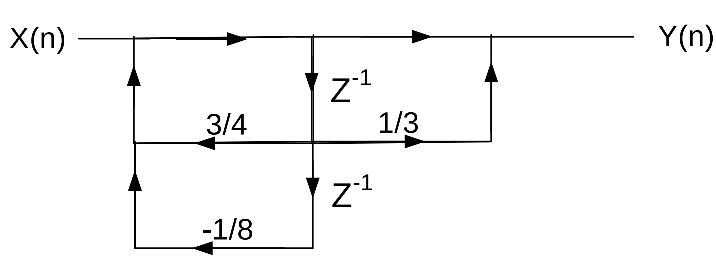
$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + \frac{1}{3}x(n-1)$$
$$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n) + \frac{1}{3}x(n-1)$$
$$y(n) = a_1 y(n-1) + a_2 y(n-2) + a_0 x(n) + a_1 x(n-1)$$

Direct form I and II:

Direct form I

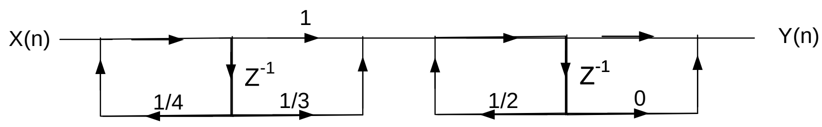


Direct form II



To find cascade form:

$$Y(z) = \frac{3}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z) + X(z) + \frac{1}{3}z^{-1}X(z)$$
$$Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) = X(z)\left(1 + \frac{1}{3}z^{-1}\right)$$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{1 + \frac{1}{3}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}$$
$$= \frac{1 + \frac{1}{3}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)} \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)}$$



3 Problem 3 (problem 4.10, page 106)

10. A class of digital filter structures based on continued fraction expansions has been proposed (S. K. Mitra and R. J. Sherwood, "Canonic Realizations of Digital Filters Using the Continued Fraction Expansion," *IEEE Trans. Audio Electroacoust.*, Vol. AU-20, 1972, pp. 185–194). Although there are a variety of forms of such structures, we wish in this problem to illustrate one particular form.

Consider a system function $H(z)$ in the form

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 - a_1 z^{-1} - \ldots - a_N z^{-N}}$$

where we assume that $b_0 \neq 0$, $a_N \neq 0$, and $M \leq N$.

Multiplying numerator and denominator of $H(z)$ by z^N it can be expressed as

$$H(z) = \frac{b_0 z^N + b_1 z^{N-1} + \ldots + b_M z^{N-M}}{z^N - a_1 z^{N-1} - \ldots - a_N}$$

If we divide denominator into numerator, we obtain

$$H(z) = A_0 + G_0(z)$$

where $A_0 = b_0$ and $G_0(z)$ would in general have the form

$$G_0(z) = \frac{c_1 z^{N-1} + \ldots + c_M}{z^N - a_1 z^{N-1} - \ldots - a_N}$$

Now if $c_1 \neq 0$ and we divide numerator into denominator, we can express $G_0(z)$ as

$$G_0(z) = \frac{1}{A_1 + B_1 z + G_1(z)}$$

where $G_1(z)$ will have the form

$$G_1(z) = \frac{d_2 z^{N-2} + \ldots + d_N}{c_1 z^{N-1} + \ldots + c_M}$$

We can repeat the process of dividing numerator into denominator to obtain

$$G_1(z) = \frac{1}{A_2 + B_2 z + G_2(z)}$$

Thus, assuming that the set of rational functions $\{G_k(z)\}$, $k = 0, 1, \ldots, N$, obtained by the above process is such that the numerator is of degree $N - k - 1$ and the denominator is of degree $N - k$, then $H(z)$ can be expressed as

$$H(z) = A_0 + \cfrac{1}{A_1 + B_1 z + \cfrac{1}{A_2 + B_2 z + \cfrac{1}{\ddots + \cfrac{1}{A_N + B_N z}}}} \qquad \text{(P4.10-1)}$$

In order to implement a network realization based on Eq. (P4.10-1), we need only an implementation of the system function

$$G_k(z) = \frac{1}{A_{k+1} + B_{k+1} z + G_{k+1}(z)} \qquad \text{(P4.10-2)}$$

Multiplying numerator and denominator of Eq. (P4.10-2) by $(1/B_{k+1})z^{-1}$, we obtain

$$G_k(z) = \frac{(1/B_{k+1})z^{-1}}{1 + (A_{k+1}/B_{k+1})z^{-1} + (1/B_{k+1})z^{-1}G_{k+1}(z)} \qquad \text{(P4.10-3)}$$

A network realization of Eq. (P4.10-3) is shown in Fig. P4.10.

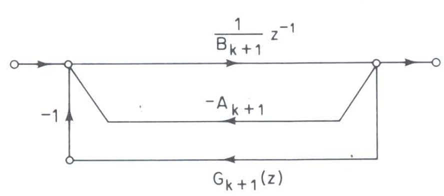


Fig. P4.10

(a) Since each successive $G_k(z)$ can be realized by a similar network, this suggests a complete structure for $H(z)$ expressed in the form of Eq. (P4.10-1). Assuming that N is odd, draw the network for such a structure. Each branch in this network must have a transmittance that is a constant or a constant times z^{-1}.

(b) For the second-order system with transfer function

$$H(z) = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

express $H(z)$ in the form of Eq. (P4.10-1).

(c) Draw the network realization of the system in part (b) in the form that determined in part (a).

3.1 Part(a)

Solution: To better understand this representation, this is a small example. Assume $H(z) = \frac{3+4z^{-1}}{1-5z^{-1}-6z^{-2}}$, where $N = 2, M = 1$, start by multiplying numerator and denominator by z^2 we obtain

$$H(z) = \frac{3z^2 + 4z}{z^2 - 5z - 6} = \overbrace{3}^{A_0} + \overbrace{\frac{19z - 18}{z^2 - 5z - 6}}^{G_0(z)}$$

Now, for $G_0(z)$, divide numerator and denominator by $19z - 18$, we obtain

$$H(z) = 3 + \frac{1}{\frac{z^2-5z-6}{19z-18}} = 3 + \overbrace{\frac{1}{\underbrace{\frac{1}{3}}_{A_1} + \underbrace{\frac{1}{9}}_{B_1} z + \underbrace{\frac{-12}{9z-18}}_{G_1(z)}}}^{G_0(z)}$$

Now, for $G_1(z)$, divide numerator and denominator by -12, we obtain

$$H(z) = 3 + \overbrace{\frac{1}{\frac{1}{3} + \frac{1}{9}z + \overbrace{\frac{1}{\underbrace{\frac{18}{12}}_{A_2} - \underbrace{\frac{9}{12}}_{B_2} z}}^{G_1(z)}}}^{G_0(z)}$$

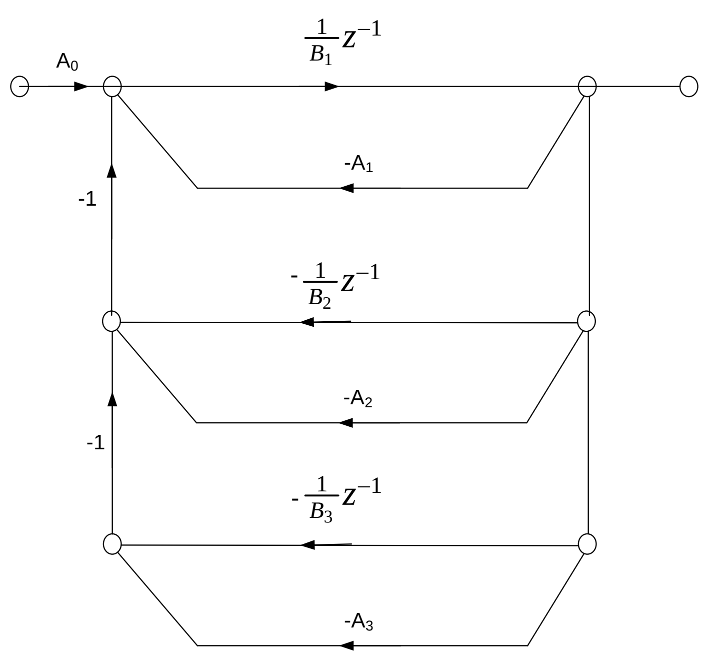
Now we need to draw a network diagram when N is odd. For example, if $N = 3$, then equation P4.10-1 will be

$$H(z) = A_0 + \frac{1}{A_1 + B_1 z + \frac{1}{A_2 + B_2 z + \frac{1}{A_3 + B_3 z}}}$$

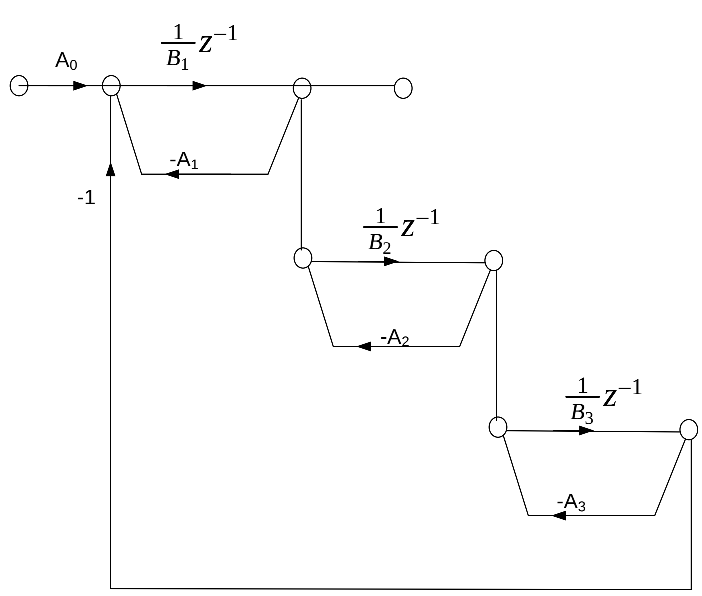
or

$$H(z) = A_0 + \frac{\frac{1}{B_1}z^{-1}}{1 + \frac{A_1}{B_1}z^{-1} + \frac{\frac{1}{B_2}z^{-1}}{1 + \frac{A_2}{B_2}z^{-1} \frac{\frac{1}{B_3}z^{-1}}{1 + \frac{A_3}{B_3}z^{-1}}}}$$

So, network will look like



After doing the above, I started having doubts about it. Here is another interpretation of the network:



3.2 part(b)

$$H(z) = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$
$$= \frac{z^2}{z^2 - 2r\cos\theta z + r^2}$$

Divide polynomials, we obtain

$$H(z) = 1 + \frac{2r\cos\theta z - r^2}{z^2 - 2r\cos\theta z + r^2}$$
$$= 1 + \frac{1}{\frac{z^2 - 2r\cos\theta z + r^2}{2r\cos\theta z - r^2}}$$

Divide the polynomials, we obtain

$$H(z) = 1 + \frac{1}{\frac{1-4\cos^2\theta}{4\cos^2\theta} + \frac{z}{2r\cos\theta} + \frac{r^2\left(1+\frac{1-4\cos^2\theta}{4\cos^2\theta}\right)}{2r\cos\theta z - r^2}}$$
$$= 1 + \frac{1}{\frac{1-4\cos^2\theta}{4\cos^2\theta} + \frac{z}{2r\cos\theta} + \frac{1}{\frac{2r\cos\theta z - r^2}{r^2\left(1+\frac{1-4\cos^2\theta}{4\cos^2\theta}\right)}}}$$
$$= 1 + \frac{1}{\frac{1-4\cos^2\theta}{4\cos^2\theta} + \frac{z}{2r\cos\theta} + \frac{1}{-\frac{1}{\left(1+\frac{1-4\cos^2\theta}{4\cos^2\theta}\right)} + \frac{2\cos\theta z}{r\left(1+\frac{1-4\cos^2\theta}{4\cos^2\theta}\right)}}}$$

Let $\frac{1-4\cos^2\theta}{4\cos^2\theta} = \beta$, then above can be written as

$$H(z) = 1 + \frac{1}{\beta + \frac{z}{2r\cos\theta} + \frac{1}{-\frac{1}{\beta} + \frac{2\cos\theta z}{r(1+\beta)}}}$$

Part(c): Network diagram using part(a) for the above is the following: