HW 7, EE 420 Digital Filters
California State University, Fullerton
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Nasser M. Abbasi

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Contents

1 Problem 1

Given an N point sequence $h(n)$, let $H(z) = Z(h(n))$ and $H(k) = DFT(h(n))$

1. Find $H(z)$ in terms of $H(k)$
2. For z on the unit circle ($z = e^{j\omega}, \omega \in [0, 2\pi]$), compute result in (1) using equation (d) in D-5 using $\beta = 1, \Omega = \frac{\omega N}{2\pi}$

Solution: Part (1)

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \tag{1}$$

and

$$\begin{aligned} H(k) &= \sum_{n=0}^{N-1} h(n) W_N^{nk} \qquad\qquad k = 0 \cdots N-1 \\ &= \sum_{n=0}^{N-1} h(n) e^{-j\frac{2\pi}{N}nk} \end{aligned} \tag{2}$$

Compare (1) and (2) we see that

$$H(k) = H(z)|_{z=e^{j\frac{2\pi}{N}k}}$$

2 Problem 2 (3.18 in text book, page 125)

18. A finite-duration sequence $x(n)$ of length 8 has the eight-point DFT $X(k)$ shown in Fig. P3.18-1. A new sequence $y(n)$ of length 16 is defined by

$$y(n) = \begin{cases} x\left(\frac{n}{2}\right), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

From the list in Fig. P3.18-2, choose the sketch corresponding to the 16-point DFT of $y(n)$.

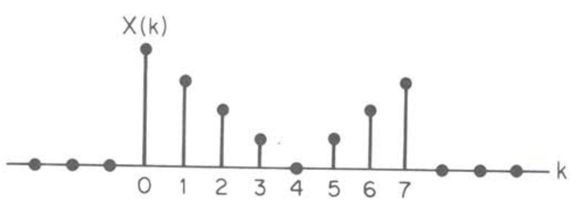


Fig. P3.18-1

Answer:

2.1 First solution

$$y(n)=\begin{cases} x\left(\frac{n}{2}\right) & n \text{ } even \\ 0 & n \text{ } odd \end{cases}$$

We are asked to D-5 in handout E. Using equation (c) in the D-5 handout, let $\beta=2$, using equation (c) we obtain

$$Y_N(k)=X_M(k \mod M)$$

In this case $N=16$ and $M=8$, Hence

$$Y_{16}(k)=X_8((k))_8$$

Hence based on the above, we select $Y(k)$ as the one shown in diagram (c) in the book on page 126

2.2 second solution

This solution is longer, but this is how I first solved the problem, before using the above method. Start by writing $y(n)$ in terms of $x(n)$

$$\begin{aligned} y_n &= \{y_0, y_1, y_2, \cdots, y_{15}\} \\ &= \{x_0, 0, x_1, 0, x_2, 0, x_3, \cdots, x_7, 0\} \end{aligned} \tag{1}$$

Now

$$\begin{aligned} Y(k) &= \sum_{n=0}^{15} y(n) W_{16}^{nk} \qquad k=0\cdots 15 \\ &= y_0 + y_1 W_{16}^{k} + y_2 W_{16}^{2k} + \cdots + y_{15} W_{16}^{15k} \end{aligned}$$

But odd values of y are zero, so the above becomes (using (1))

$$\begin{aligned} Y(k) &= x_0 + 0 + x_1 W_{16}^{2k} + 0 + x_2 W_{16}^{4k} + \cdots + x_7 W_{16}^{14k} + 0 \\ &= x_0 + x_1 W_{16}^{2k} + x_2 W_{16}^{4k} + \cdots + x_7 W_{16}^{14k} \qquad k=0\cdots 15 \end{aligned}$$

We can simplify W_{16}^{nk} above by dividing by 2 to obtain $W_8^{\frac{n}{2}k}$, so the above becomes

$$Y(k) = x_0 + x_1 W_8^{k} + x_2 W_8^{2k} + \cdots + x_7 W_8^{7k} \qquad k=0\cdots 15 \tag{2}$$

But if we compare the above to $X(k)$, which is

$$X(k) = x_0 + x_1 W_8^{k} + x_2 W_8^{2k} + \cdots + x_7 W_8^{7k} \qquad k=0\cdots 7 \tag{3}$$

We see that $Y(k)=X(k)$ at least for the range of $k=0\cdots 7$. So now let us look at the second half of values, i.e. for $k=8\cdots 15$. If we write down few terms we see

$$\begin{aligned} Y(8) &= x_0 + x_1 W_8^{8} + x_2 W_8^{2\times 8} + \cdots + x_7 W_8^{7\times 8} \\ &= x_0 + x_1 W + x_2 W^2 + \cdots + x_7 W^7 \\ &= x_0 = X(0) \end{aligned}$$

and

$$
\begin{aligned}
Y(9) &= x_0 + x_1 W_8^9 + x_2 W_8^{2\times 9} + \cdots + x_7 W_8^{7\times 9} \\
&= x_0 + x_1 W_8^9 + x_2 W_8^{18} + \cdots + x_7 W_8^{63} \\
&= x_0 + x_1 W_8^1 + x_2 W_8^2 + \cdots + x_7 W_8^7 \\
&= X(1)
\end{aligned}
$$

and so on. i.e. by taking advantage of the relation that $\boxed{W_N^M = W_N^{((M))_N}}$, where $((M))_N$ means M module N, ie. the reminder when M is divided by N, we see that the second half of $Y(k)$ also is the same as $X(k)$. Hence the correct result is the one shown in diagram (c) in the book.

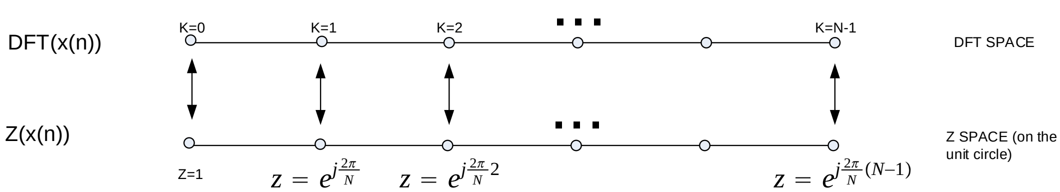
3 Problem 3 (problem 3.25 in textbook, page 130)

25. Consider a finite-duration sequence $x(n)$ of length N so that $x(n) = 0$ for $n < 0$ and for $n > N - 1$. We want to compute samples of its z-transform $X(z)$ at M equally spaced points around the unit circle. One of the samples is to be at $z = 1$. The number of samples M is *less than* the duration of the sequence N; i.e., $M < N$. Determine and justify a procedure for obtaining the M samples of $X(z)$ by computing *only once* the M-point DFT of an M-point sequence obtained from $x(n)$.

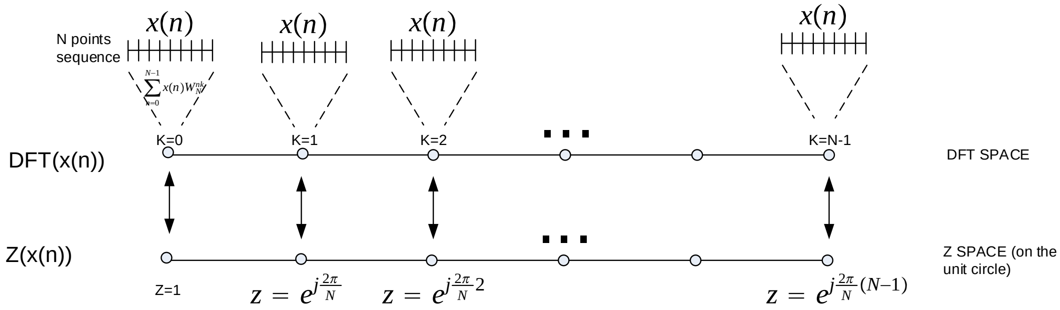
We know that

$$
DFT\left(x\left(n\right)\right) = X_N\left(k\right) = \sum_{n=0}^{N-1} x\left(n\right) W_N^{nk} = Z\left(x\left(n\right)\right)\Big|_{z=W_N^k}
$$

So, to find the DFT of $x(n)$ at some specific k value, instead of performing the sum above over the length of $x(n)$, we can just evaluate Z transform of $x(n)$ at just one point, $z = W_N^k$. So the sum operation is replaced by one evaluation operation (sampling) of the Z transform. So the idea is to compute the Z transform once, and then evaluate it at specific locations on the unit circle to obtain specific values of the DFT. I start by ilustrating the relation above on a diagram



Now, each point in the DFT space is found by performin the sum over N points in the sequence $x(n)$. (i.e the DFT calculation). Here is another diagram to illustrate



Now, we are asked to find M samples (equally spaced) in the Z space by using only M samples from $x(n)$, which is of length N. So we need to pick some values of x (M of them) and use them to find M samples in Z space. How to find these M values of x? Looking at the DFT sum itself

$$X_N(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

Break this into sums of M elements at a time (since $M < N$). Last segment can be less than M points if M do not divide N exactly.

$$X_N(k) = \sum_{n=0}^{M-1} x(n) W_N^{nk} + \sum_{n=M}^{2M-1} x(n) W_N^{nk} + \sum_{n=2M}^{3M-1} x(n) W_N^{nk} \cdots + \sum_{n=\beta M}^{(\beta+1)M-1} x(n) W_N^{nk}$$

Where in the above, we assumed $N = \beta M$ QED.