HW 5, EE 420 Digital Filters
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1 Problem 1 (2.24 of text)

24. A causal linear shift-invariant system is described by the difference equation

$$y(n) = y(n-1) + y(n-2) + x(n-1)$$

(a) Find the system function $H(z) = Y(z)/X(z)$ for this system. Plot the poles and zeros of $H(z)$ and indicate the region of convergence.
(b) Find the unit-sample response of this system.
(c) You should have found this to be an unstable system. Find a stable (non-causal) unit-sample response that satisfies the difference equation.

1.1 Part (a)

$$y(n) = y(n-1) + y(n-2) + x(n-1)$$

Take the Z transform of the above, assuming zero initial conditions, we obtain

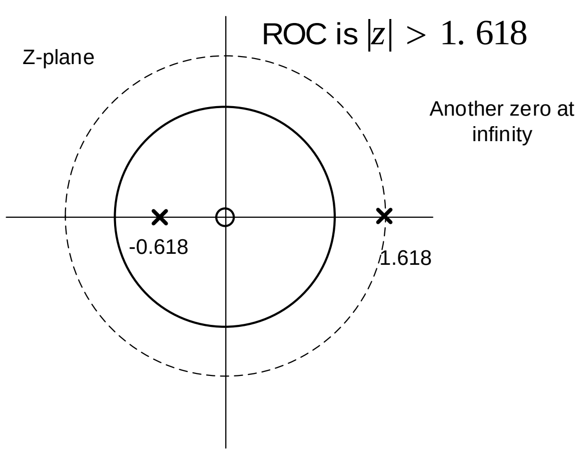
$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z)$$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}}$$

To find poles and zeros, easier to write in terms of z and not z^{-1}, hence the above becomes

$$\begin{aligned} H(z) &= \frac{z}{z^2 - z - 1} \\ &= \frac{z}{\left(z - \frac{(1-\sqrt{5})}{2}\right)\left(z - \frac{(1+\sqrt{5})}{2}\right)} \\ &\approx \frac{z}{(z + 0.61803)(z - 1.618)} \end{aligned}$$

So, a pole is at $z \approx -0.61803$, and at $z \approx 1.618$ and zero at $z = 0$. We need another zero. But $H(z) = \frac{1}{z-1-\frac{1}{z}}$ so at $z = \infty$ we have $H(\infty) \to 0$, hence another zero at $z = \infty$

Since this is a causal system, hence the ROC will extend to the outside of the largest pole, which is at $z = 1.618$, hence the ROC is $|z| > 1.618$



1.2 Part (b)

To find unit sample response. Let $X(z) = 1$, hence the Z transform of the unit sample response becomes

$$H(z) = Y(z) = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} = \frac{z}{z^2 - z - 1}$$

Using the second form above. Multiply both sides by z^{-1}

$$\begin{aligned} \frac{H(z)}{z} &= \frac{1}{z^2 - z - 1} \\ &= \frac{1}{\left(z - \frac{(1-\sqrt{5})}{2}\right)\left(z - \frac{(1+\sqrt{5})}{2}\right)} = \frac{A}{\left(z - \frac{(1-\sqrt{5})}{2}\right)} + \frac{B}{\left(z - \frac{(1+\sqrt{5})}{2}\right)} \end{aligned}$$

Hence

$$A = \lim_{z \to \frac{\left(1-\sqrt{5}\right)}{2}} \frac{1}{\left(z - \frac{\left(1+\sqrt{5}\right)}{2}\right)} = \frac{1}{\left(\frac{\left(1-\sqrt{5}\right)}{2} - \frac{\left(1+\sqrt{5}\right)}{2}\right)} = -\frac{1}{5}\sqrt{5}$$

and

$$B = \lim_{z \to \frac{\left(1+\sqrt{5}\right)}{2}} \frac{1}{\left(z - \frac{\left(1-\sqrt{5}\right)}{2}\right)} = \frac{1}{\left(\frac{\left(1+\sqrt{5}\right)}{2} - \frac{\left(1-\sqrt{5}\right)}{2}\right)} = \frac{1}{5}\sqrt{5}$$

Hence

$$\frac{H(z)}{z} = -\frac{1}{5}\sqrt{5}\frac{1}{\left(z - \frac{\left(1-\sqrt{5}\right)}{2}\right)} + \frac{1}{5}\sqrt{5}\frac{1}{\left(z - \frac{\left(1+\sqrt{5}\right)}{2}\right)}$$

$$\frac{H(z)}{z} = \frac{\sqrt{5}}{5}\left(\frac{-z^{-1}}{\left(1 - \frac{\left(1-\sqrt{5}\right)}{2}z^{-1}\right)} + \frac{z^{-1}}{\left(1 - \frac{\left(1+\sqrt{5}\right)}{2}z^{-1}\right)}\right)$$

Hence

$$H(z) = \frac{\sqrt{5}}{5}\left(\frac{-1}{\left(1 - \frac{\left(1-\sqrt{5}\right)}{2}z^{-}1\right)} + \frac{1}{\left(1 - \frac{\left(1+\sqrt{5}\right)}{2}z^{-}1\right)}\right)$$

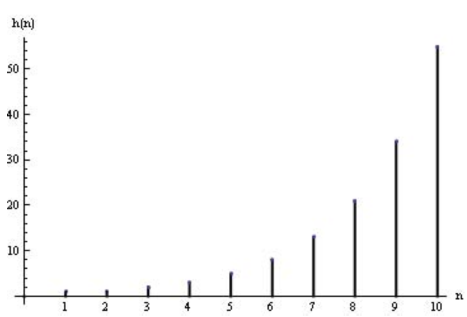
Hence

$$h(n) = \frac{\sqrt{5}}{5}\left(-\left[\frac{\left(1-\sqrt{5}\right)}{2}\right]^n u(n) + \left[\frac{\left(1+\sqrt{5}\right)}{2}\right]^n u(n)\right)$$

Therefore

$$h(n) = \frac{\sqrt{5}}{5}\left(0.618\,03^n + 1.618^n\right)u(n)$$

Here is a plot for $n = 0 \cdots 10$



Here is a table of few values

n	h(n)
0.	0.
1.	1.
2.	1.
3.	2.
4.	3.
5.	5.
6.	8.
7.	13.
8.	21.
9.	34.
10.	55.
11.	89.
12.	144.
13.	233.
14.	377.
15.	610.
16.	987.
17.	1597.
18.	2584.
19.	4181.
20.	6765.

We see that this is an unstable system as the response grows without bound with n. This can be seen also by noting that a pole exist outside the unit circle.

1.3 part (c)

To find a stable non-causal system, we know that its ROC must be inside some circle, and that the circle must include the unit circle (for it to be stable). The ROC can't include an poles. But we have a pole at $z = -0.681$ and at $z = 1.681$, so the ROC must be the annular region between $|z| = 0.681$ and between $|z| = 1.681$

Looking at the $H(z)$ found in part (b)

$$H(z) = \frac{\sqrt{5}}{5}\left(\frac{-1}{\left(1-\frac{\left(1-\sqrt{5}\right)}{2}z^{-1}\right)} + \frac{1}{\left(1-\frac{\left(1+\sqrt{5}\right)}{2}z^{-1}\right)}\right)$$
$$= \frac{\sqrt{5}}{5}\left(\overbrace{\frac{-1}{(1-(-0.618\,03)\,z^{-1})}}^{\text{stable}} + \overbrace{\frac{1}{(1-1.618z^{-1})}}^{\text{unstable}}\right)$$

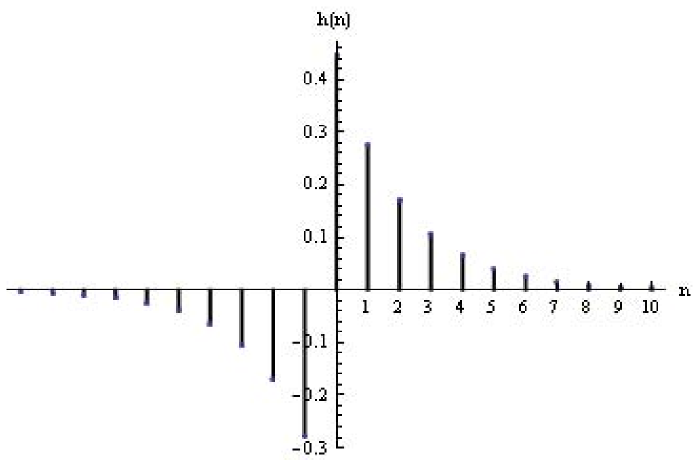
We see that the instability came from the part $\frac{1}{(1-1.618z^{-1})}$ because this generated the unstable sequence Hence, if we make this sequence anti causal, we change

$$h(n) = \frac{\sqrt{5}}{5}0.618\,03^{n}u(n) + \frac{\sqrt{5}}{5}1.618^{n}u(n)$$

To becomes

$$h(n) = \frac{\sqrt{5}}{5}0.618\,03^{n}u(n) - \frac{\sqrt{5}}{5}1.618^{n}u(-n-1)$$

Here is a plot for $n = -10\cdots 10$



n	h(n)
-10.	-0.00363689
-9.	-0.00588448
-8.	-0.00952109
-7.	-0.0154051
-6.	-0.0249255
-5.	-0.0403295
-4.	-0.0652531
-3.	-0.105579
-2.	-0.170828
-1.	-0.276399
0.	0.447214
1.	0.276391
2.	0.170818
3.	0.105571
4.	0.0652459
5.	0.0403239
6.	0.0249214
7.	0.0154022
8.	0.009519
9.	0.00588303
10.	0.00363589

2 Problem 2 (2.27 of text)

27. Consider a linear discrete-time shift-invariant system with input $x(n)$ and output $y(n)$ for which

$$y(n-1) - \tfrac{10}{3}y(n) + y(n+1) = x(n)$$

The system is stable. Determine the unit-sample response.

$$y(n-1) - \frac{10}{3}y(n) + y(n+1) = x(n)$$

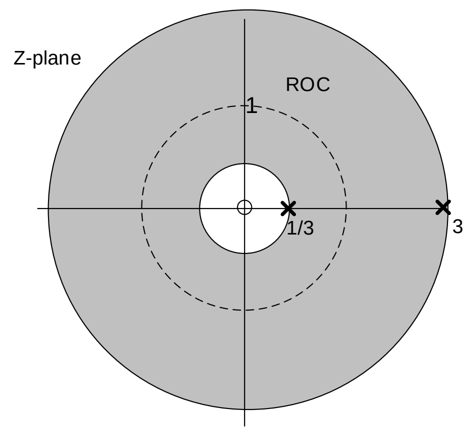
Assuming zero initial conditions, take the z transform we obtain

$$z^{-1}Y(z) - \frac{10}{3}Y(z) + zY(z) = X(z)$$
$$Y(z)\left(z^{-1} - \frac{10}{3} + z\right) = X(z)$$

Let the input $x(n) = \delta(n)$, hence $X(z) = 1$ and we obtain the z transform of the impulse response

$$H(z) = Y(z) = \frac{1}{z^{-1} - \frac{10}{3} + z} = \frac{z}{1 - \frac{10}{3}z + z^2} = \frac{z}{(z-3)\left(z - \frac{1}{3}\right)}$$

There is a pole at $z=3$ and a pole at $z=\frac{1}{3}$ and a zero at $z=0$ and a zero at $z=\infty$. Since we are told the system is stable, then it must contain the unit circle. Hence the ROC must be the annular region between $|z|=\frac{1}{3}$ and between $|z|=3$



Multiply both sides by z^{-1}

$$\frac{H(z)}{z}=\frac{1}{(z-3)\left(z-\frac{1}{3}\right)}=\frac{A}{(z-3)}+\frac{B}{\left(z-\frac{1}{3}\right)}$$

$$A=\lim_{z\to 3}\frac{1}{\left(z-\frac{1}{3}\right)}=\frac{1}{\left(3-\frac{1}{3}\right)}=\frac{3}{8}$$

$$B=\lim_{z\to \frac{1}{3}}\frac{1}{(z-3)}=\frac{1}{\left(\frac{1}{3}-3\right)}=-\frac{3}{8}$$

Hence

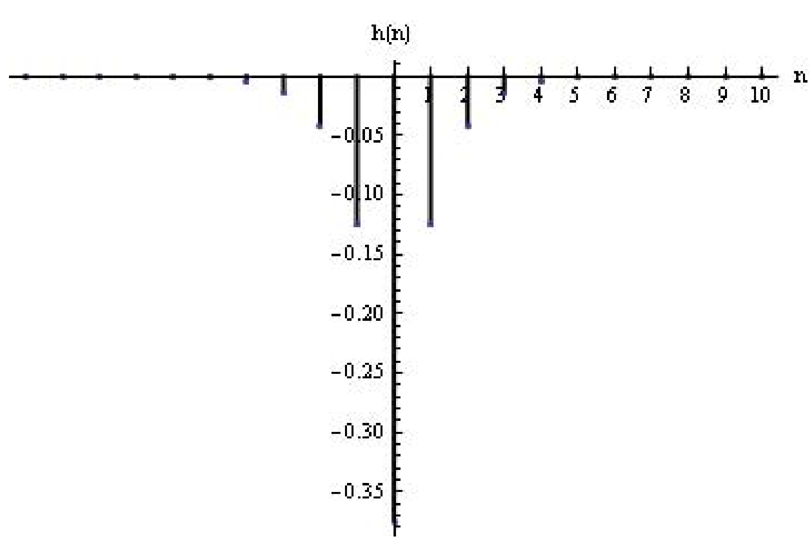
$$\frac{H(z)}{z}=\frac{3}{8}\frac{1}{(z-3)}-\frac{3}{8}\frac{1}{\left(z-\frac{1}{3}\right)}$$

or

$$\begin{aligned}H(z)&=\frac{3}{8}\frac{z}{(z-3)}-\frac{3}{8}\frac{z}{\left(z-\frac{1}{3}\right)}\\&=\frac{3}{8}\overbrace{\frac{1}{(1-3z^{-1})}}^{\text{ROC }|z|<3\text{, anticausal}}-\frac{3}{8}\overbrace{\frac{1}{\left(1-\frac{1}{3}z^{-1}\right)}}^{\text{ROC }|z|>1/3\text{, causal}}\end{aligned}$$

with ROC $\frac{1}{3}<|z|<3$, Hence $h(n)$ is

$$h(n)=-\frac{3}{8}(3)^n u(-n-1)-\frac{3}{8}\left(\frac{1}{3}\right)^n u(n)$$



n	h(n)
-10.	-6.35066×10^{-6}
-9.	-0.000019052
-8.	-0.0000571559
-7.	-0.000171468
-6.	-0.000514403
-5.	-0.00154321
-4.	-0.00462963
-3.	-0.0138889
-2.	-0.0416667
-1.	-0.125
0.	-0.375
1.	-0.125
2.	-0.0416667
3.	-0.0138889
4.	-0.00462963
5.	-0.00154321
6.	-0.000514403
7.	-0.000171468
8.	-0.0000571559
9.	-0.000019052
10.	-6.35066×10^{-6}

3 Problem 3

Find $Z^{-1}\left[X(z)Y\left(\frac{1}{z}\right)\right]$

Solution:

Let $X(z)=\Im[x(n)]$ and let $Y(z)=\Im[y(n)]$

From properties of Z transform, we note that $Y\left(\frac{1}{z}\right)=\Im(y(-n))$

Let

$$g(n)=Z^{-1}\left[X(z)Y\left(\frac{1}{z}\right)\right]$$
$$Z[g(n)]=X(z)Y\left(\frac{1}{z}\right)$$
$$Z[x(n)\circledast y(-n)]=X(z)Y\left(\frac{1}{z}\right)$$

But

$$x(n)\circledast y(-n)=\sum_{k=-\infty}^{\infty}x(k)y(n-(-(k)))$$
$$=\sum_{k=-\infty}^{\infty}x(k)y(n+k)$$

Hence $Z^{-1}\left[X(z)Y\left(\frac{1}{z}\right)\right]$ is the cross correlation between $x(n)$ and $y(n)$

4 Problem 4

Problem: Given $g(n)=a^n u(n)$ where $|a|<1$ and $f(n)=\sum_{k=n}^{\infty}g(k)$, find

(a)$\frac{F(z)}{G(z)}$,(b) the ROC of $F(z)$

4.1 Part (a)

$$G(z)=\Im(g(n))=\sum_{n=0}^{\infty}a^n z^{-n}=\frac{1}{1-az^{-1}}$$

And

$$
\begin{aligned}
f(n) &= \sum_{k=n}^{\infty} g(k) \\
&= g(n) + g(n+1) + \cdots \\
&= a^n u(n) + a^{n+1} u(n+1) + a^{n+2} u(n+2) + \cdots \\
&= a^n u(n) \left(1 + a + a^2 + \cdots\right) \\
&= a^n u(n) \sum_{l=0}^{\infty} a^l \\
&= a^n u(n) \frac{1}{1-a}
\end{aligned}
$$

Hence

$$
\begin{aligned}
F(z) = \Im\left(f(n)\right) &= \Im\left(a^n u(n) \frac{1}{1-a}\right) \\
&= \left(\frac{1}{1-a}\right)\left(\frac{1}{1-az^{-1}}\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{F(z)}{G(z)} &= \frac{\left(\frac{1}{1-a}\right)\left(\frac{1}{1-az^{-1}}\right)}{\left(\frac{1}{1-az^{-1}}\right)} \\
&= \left(\frac{1}{1-a}\right)
\end{aligned}
$$

4.2 Part (b)

The ROC of $F(z)$ is the same as the ROC of $g(z)$, which is $|z| > a$

5 Problem 5

Given

$$
H(z) = \frac{a_3^* + a_2^* z^{-1} + a_1^* z^{-2} + z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}
$$

Show if $H(z)$ is an all-pass system or not.

Solution:

Let $a_3 = r_3 e^{j\theta_3}, a_2 = r_2 e^{j\theta_2}, a_1 = r_1 e^{j\theta_1}$, hence the DTFT of the system can be found be setting $z = e^{j\omega}$ and also noting that $a_3^* = r_3 e^{-j\theta_3}$ and similarly for a_2^* and a_1^*, hence

$$
H\left(e^{j\omega}\right) = \frac{r_3 e^{-j\theta_3} + r_2 e^{-j\theta_2} e^{-j\omega} + r_1 e^{-j\theta_1} e^{-2j\omega} + e^{-j3\omega}}{1 + r_1 e^{j\theta_1} e^{-j\omega} + r_2 e^{j\theta_2} e^{-2j\omega} + r_3 e^{j\theta_3} e^{-j3\omega}}
$$

Hence

$$
\begin{aligned}
\left|H\left(e^{j\omega}\right)\right|^2 &= H\left(e^{j\omega}\right) H^*\left(e^{j\omega}\right) \\
&= \left(\frac{r_3 e^{-j\theta_3} + r_2 e^{-j\theta_2} e^{-j\omega} + r_1 e^{-j\theta_1} e^{-2j\omega} + e^{-j3\omega}}{1 + r_1 e^{j\theta_1} e^{-j\omega} + r_2 e^{j\theta_2} e^{-2j\omega} + r_3 e^{j\theta_3} e^{-j3\omega}}\right)\left(\frac{r_3 e^{-j\theta_3} + r_2 e^{-j\theta_2} e^{-j\omega} + r_1 e^{-j\theta_1} e^{-2j\omega} + e^{-j3\omega}}{1 + r_1 e^{j\theta_1} e^{-j\omega} + r_2 e^{j\theta_2} e^{-2j\omega} + r_3 e^{j\theta_3} e^{-j3\omega}}\right)^* \\
&= \left(\frac{r_3 e^{-j\theta_3} + r_2 e^{-j\theta_2} e^{-j\omega} + r_1 e^{-j\theta_1} e^{-2j\omega} + e^{-j3\omega}}{1 + r_1 e^{j\theta_1} e^{-j\omega} + r_2 e^{j\theta_2} e^{-2j\omega} + r_3 e^{j\theta_3} e^{-j3\omega}}\right)\left(\frac{r_3 e^{j\theta_3} + r_2 e^{j\theta_2} e^{j\omega} + r_1 e^{j\theta_1} e^{2j\omega} + e^{j3\omega}}{1 + r_1 e^{-j\theta_1} e^{j\omega} + r_2 e^{-j\theta_2} e^{2j\omega} + r_3 e^{-j\theta_3} e^{j3\omega}}\right)
\end{aligned}
$$

Which simplifies to 1 Hence $\left|H\left(e^{j\omega}\right)\right|^2 = 1$ or

$$
\left|H\left(e^{j\omega}\right)\right| = 1
$$

Hence this is an all-pass system.

For the simplification, it was tedious to work out completely, unless I am missing some obvious short cut, so I used Mathematica, and it reported that the above simplifies to one. Here is the command:

```
In[141]:= H = (r3 Exp[-I θ3] + r2 Exp[-I θ2] Exp[-I w] + r1 Exp[-I θ1] Exp[-2 I w] + Exp[-I 3 w]) /
            (1 + r1 Exp[I θ1] Exp[-I w] + r2 Exp[I θ2] Exp[-2 I w] + r3 Exp[I θ3] Exp[-I 3 w])
```

$$
\text{Out[141]=}\quad \frac{e^{-3 i w} + e^{-2 i w - i \theta 1}\, r1 + e^{-i w - i \theta 2}\, r2 + e^{-i \theta 3}\, r3}{1 + e^{-i w + i \theta 1}\, r1 + e^{-2 i w + i \theta 2}\, r2 + e^{-3 i w + i \theta 3}\, r3}
$$

```
In[142]:= ComplexExpand[H * Conjugate[H]] // Simplify

Out[142]= 1
```