HW 4, EE 420 Digital Filters
California State University, Fullerton
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Contents

1 Problem 1 (chapter 2, 2, page 78)

2. Determine the z-transform of each of the following. Include with your answer the region of convergence in the z-plane and a sketch of the pole–zero pattern. Express all sums in closed form. α can be complex.

(a) $x(n) = \alpha^{|n|}$, $0 < |\alpha| < 1$.

(b) $x(n) = Ar^n \cos(\omega_0 n + \phi)u(n)$, $0 < r < 1$.

(c) $x(n) = \begin{cases} 1, & 0 \leq n \leq N-1, \\ 0, & N \leq n, \\ 0, & n < 0. \end{cases}$

(d) $x(n) = \begin{cases} n, & 0 \leq n \leq N, \\ 2N - n, & N+1 \leq n \leq 2N, \\ 0, & 2N \leq n, \\ 0, & 0 > n. \end{cases}$

[*Hint* (easy way): First express $x(n)$ in terms of the $x(n)$ in part (c).]

Part (a)

$$x(n) = \alpha^{|n|}, \qquad 0 < |\alpha| < 1$$

When $n < 0$, then $x(n) = \alpha^{-n}$ and when $n \geq 0$, then $x(n) = \alpha^n$, hence we split the sum

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \\
&= \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\
&= \sum_{1}^{\infty} \alpha^n z^n + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\
&= -1 + \sum_{0}^{\infty} \alpha^n z^n + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\
&= -1 + \sum_{0}^{\infty} (\alpha z)^n + \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n \\
&= -1 + \frac{1}{1-\alpha z} + \frac{1}{1-\alpha z^{-1}}
\end{aligned} \tag{1}$$

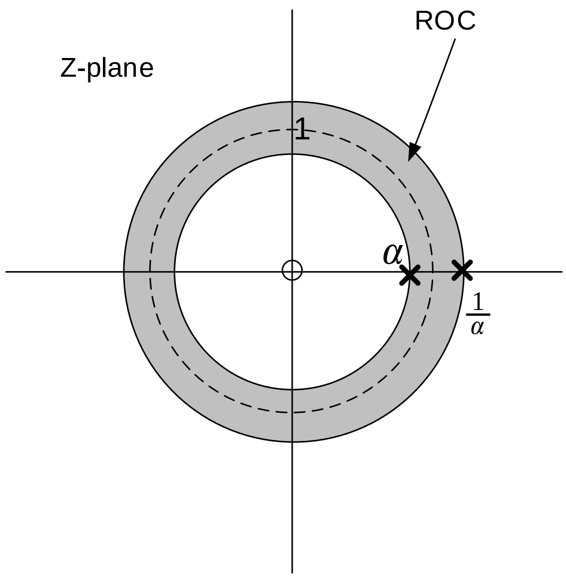
Where for the first sum $\sum_{0}^{\infty} (\alpha z)^n$, we need $|\alpha z| < 1$ or $|z| < \left|\frac{1}{\alpha}\right|$ for convergence, and for the second sum $\sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n$, we need $\left|\frac{\alpha}{z}\right| < 1$ for convergence, or $|z| > |\alpha|$ Hence, since $0 < |\alpha| < 1$, we have the ROC as

$$|\alpha| < |z| < \left|\frac{1}{\alpha}\right|$$

To help see where the poles and zeros are, expression (1) is simplfied to

$$X(z) = \frac{z\left(1-\alpha^2\right)}{(1-\alpha z)(z-\alpha)}$$

We now see that a pole exist at $z = \frac{1}{\alpha}$ and at $z = \alpha$ and a zero at $z = 0$



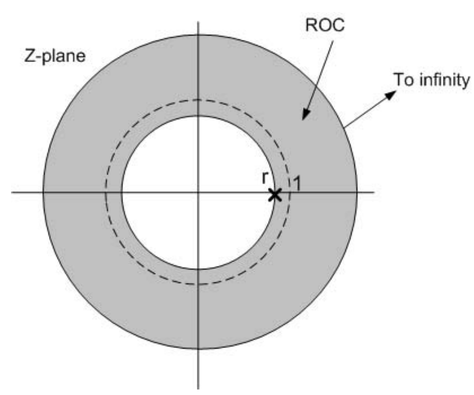
Part (2)

$$x(n) = Ar^n \cos(\omega_0 n + \phi)\, u(n), \qquad 0 < r < 1$$

$$\begin{aligned}
\Im(x(n)) = X(z) &= \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \\
&= \sum_{n=0}^{\infty} Ar^n \cos(\omega_0 n + \phi)\, z^{-n} \\
&= \frac{A}{2}\sum_{n=0}^{\infty} r^n \left(e^{i(\omega_0 n+\phi)} + e^{-i(\omega_0 n+\phi)}\right) z^{-n} \\
&= \frac{A}{2}\sum_{n=0}^{\infty} r^n \left(e^{i\omega_0 n} e^{i\phi} + e^{-i\omega_0 n} e^{-i\phi}\right) z^{-n} \\
&= \frac{A}{2}\sum_{n=0}^{\infty} r^n e^{i\omega_0 n} e^{i\phi} z^{-n} + \frac{A}{2}\sum_{n=0}^{\infty} r^n e^{-i\omega_0 n} e^{-i\phi} z^{-n} \\
&= \frac{A}{2} e^{i\phi} \sum_{n=0}^{\infty} \left(\frac{re^{i\omega_0}}{z}\right)^n + \frac{A}{2} e^{-i\phi} \sum_{n=0}^{\infty} \left(\frac{r}{e^{i\omega_0} z}\right)^n \\
&= \frac{A}{2} e^{i\phi} \frac{1}{1 - re^{i\omega_0} z^{-1}} + \frac{A}{2} e^{-i\phi} \frac{1}{1 - r\left(e^{i\omega_0} z\right)^{-1}}
\end{aligned}$$

Where the first sum $\sum_{n=0}^{\infty} \left(\frac{re^{i\omega_0}}{z}\right)^n$ requires that $\left|\frac{re^{i\omega_0}}{z}\right| < 1$ or $|z| > r$ (since $\left|e^{i\omega_0}\right| = 1$), and the second sum $\sum_{n=0}^{\infty} \left(\frac{r}{e^{i\omega_0} z}\right)^n$ requires that $\left|\frac{r}{e^{i\omega_0} z}\right| < 1$ or $|z| > |r|$, hence the ROC is $|z| > |r|$, and since $|r| < 1$, then the ROC contains the unit circle, i.e. the sequence $x(n)$ has a DTFT transform as well (it is BIBO stable).

To find poles and zeros: Since the ROC is surrounded by poles, we conclude that $z = r$ is a pole, and since this is a causal signal, the zero is at $z = \infty$



Part(3)

$$x(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & N \le n \\ 0 & n < 0 \end{cases}$$

$$\begin{aligned}X(z) &= \sum_{n=0}^{N-1} z^{-n}\\ &= \frac{1-z^{-N}}{1-z^{-1}}\end{aligned}$$

Where the only condition is that $\left|\frac{1}{z^n}\right| < \infty$ for any n in the above range. Since $n \geq 0$, then this implies that $z \neq 0$

Hence the ROC is the complete z plane, except for $z = 0$

To find poles and zero, easier to rewrite $X(z)$ in powers of z

$$X(z) = \frac{z^N - 1}{z^N - z^{N-1}} = \frac{z^N - 1}{z^{N-1}(z-1)}$$

So we have N zeros around the unit circle (roots of unity) and one pole at $z = 1$ and $N - 1$ poles at $z = 0$. notice pole at $z = 1$ would cancel the zero at $z = 0$

part (4)

$$x(n) = \begin{cases} 1 & 0 \leq n \leq N \\ 2N - n & N+1 \leq n \leq 2N \\ 0 & 2N \leq 0 \\ 0 & 0 > n \end{cases}$$

$$\begin{aligned}X(z) &= \sum_{n=0}^{N} z^{-n} + \sum_{n=N+1}^{2N} (2N-n) z^{-n}\\ &= \sum_{n=0}^{N} z^{-n} + 2N \sum_{n=N+1}^{2N} z^{-n} - \sum_{n=N+1}^{2N} n z^{-n}\\ &= \sum_{n=0}^{N} z^{-n} + 2N \left(\sum_{n=0}^{2N} z^{-n} - \sum_{n=0}^{N} z^{-n} \right) - \left(\sum_{n=0}^{2N} n z^{-n} - \sum_{n=0}^{N} n z^{-n} \right)\\ &= \frac{1-z^{-(N+1)}}{1-z} + 2N \left(\frac{1-z^{-(2N+1)}}{1-z^{-1}} - \frac{1-z^{-(N+1)}}{1-z^{-1}} \right) - \sum_{n=0}^{2N} n z^{-n} + \sum_{n=0}^{N} n z^{-n} \qquad (1)\end{aligned}$$

To find $\sum_{n=0}^{2N} n z^{-n}$, I will use the relation that if the z-transform of $x(n)$ is $X(z)$, then the z-transform of $nx(n)$ is $-zX'(z)$. Hence, let

$$x(n) = \begin{cases} 1 & 0 \leq n \leq 2N \\ 0 & N \leq n \\ 0 & n < 0 \end{cases}$$

Then

$$\begin{aligned}X(z) &= \sum_{n=0}^{2N} z^{-n}\\ &= \frac{1-\left(z^{-1}\right)^{2N+1}}{1-z^{-1}} = \frac{1-z^{-2N-1}}{1-z^{-1}}\end{aligned}$$

Hence

$$\begin{aligned}-zX'(z) &= -z\frac{d}{dz}\left(\frac{1-z^{-2N-1}}{1-z^{-1}}\right)\\ &= -z\left[\frac{-(-2N-1)z^{-2N-2}}{(1-z^{-1})} - \frac{\left(1-z^{-2N-1}\right)}{(1-z^{-1})^2 z^2}\right]\\ &= \left[\frac{z^{-2N}\left(-2N(z-1)+z\left(z^{2N}-1\right)\right)}{(z-1)^2}\right]\end{aligned}$$

Hence

$$\sum_{n=0}^{2N} n z^{-n} = \left[\frac{z^{-2N}\left(-2N(z-1)+z\left(z^{2N}-1\right)\right)}{(z-1)^2}\right]$$

Similarly, for $\sum_{n=0}^{N} n z^{-n}$, Let

$$x(n) = \begin{cases} 1 & 0 \leq n \leq N \\ 0 & N \leq n \\ 0 & n < 0 \end{cases}$$

then

$$X(z) = \sum_{n=0}^{N} z^{-n}$$
$$= \frac{1-\left(z^{-1}\right)^{N+1}}{1-z^{-1}} = \frac{1-z^{-N-1}}{1-z^{-1}}$$

Hence

$$-zX'(z) = -z\frac{d}{dz}\left(\frac{1-z^{-N-1}}{1-z^{-1}}\right)$$
$$= \frac{z^{-N}\left(N-Nz+z\left(z^{N}-1\right)\right)}{(z-1)^2}$$

Therefore

$$\sum_{n=0}^{N} nz^{-n} = \frac{z^{-N}\left(N-Nz+z\left(z^{N}-1\right)\right)}{(z-1)^2}$$

Going back to (1) and substitute the above results, we obtain

$$X(z) = \frac{1-z^{-(N+1)}}{1-z} + 2N\left(\frac{1-z^{-(2N+1)}}{1-z^{-1}} - \frac{1-z^{-(N+1)}}{1-z^{-1}}\right) - \sum_{n=0}^{2N} nz^{-n} + \sum_{n=0}^{N} nz^{-n}$$
$$= \frac{1-z^{-(N+1)}}{1-z} + 2N\left(\frac{1-z^{-(2N+1)}}{1-z^{-1}} - \frac{1-z^{-(N+1)}}{1-z^{-1}}\right)$$
$$- \left[\frac{z^{-2N}\left(-2N(z-1)+z\left(z^{2N}-1\right)\right)}{(z-1)^2}\right]$$
$$+ \frac{z^{-N}\left(N-Nz+z\left(z^{N}-1\right)\right)}{(z-1)^2}$$

This can be simplified little more to be

$$X(z) = \frac{1-z^{-(N+1)}}{1-z} + 2N\left(\frac{1-z^{-(2N+1)}}{1-z^{-1}} - \frac{1-z^{-(N+1)}}{1-z^{-1}}\right)$$
$$+ \frac{-z^{-2N}\left(-2N(z-1)+z\left(z^{2N}-1\right)\right) + z^{-N}\left(N-Nz+z\left(z^{N}-1\right)\right)}{(z-1)^2}$$

That is as closed form as I can get it to be.

Since this is finite sequence, and $n > 0$, then the sum converges as long as $z \neq 0$. Hence the ROC is all the z-plane except for $z = 0$.

To find poles and zero, easier to rewrite $X(z)$ in powers of z, but too complicated to do. I think my solution here is not what I should have done, but I am not sure now what else to do.

2 Problem 2 (problem 7 in text book, chapter 2, page 79)

Determine whether or not the function $F(z) = z^*$ can correspond to the z transform of a sequence.

Answer:

I assume we are only to consider a real sequence $x(n)$ and not complex sequence.

Consider z at the unit circle, hence $z = e^{j\omega}$, therefore $z^* = e^{-j\omega} = z^{-1}$, So, we want to find a sequence $x(n)$, such that

$$\sum_n x(n)\, z^{-n} = z^{-1}$$

But if $x(n) = \delta(n-1)$, then

$$\sum_n \delta(n-1)\, z^{-n} = \sum_n \delta(n-1)\, z^{-1} = z^{-1}\sum_n \delta(n-1) = z^{-1}$$

Hence, we found such a sequence.

3 Problem (3) (problem 10, chapter 2, page 80)

Show that if $X(z)$ is the z transform of a sequence $x(n)$, then

(a) $z^{n_0}X(z)$ is the z-transform of $x(n+n_0)$

(b) $X\left(a^{-1}z\right)$ is the z-transform of $a^n x(n)$

(c)$-zX'(z)$ is the z-transform of $nx(n)$

Part(a)

$$\Im(x(n)) = X(z) = \sum_n x(n) z^{-n}$$

$$z^{n_0}X(z) = z^{n_0}\sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n) z^{-n+n_0}$$

Let $m = n - n_0$, hence $n = m + n_0$ and when $n = -\infty, = -\infty$, and then $n = \infty$, $m = \infty$, then the above becomes

$$z^{n_0}X(z) = \sum_{m=-\infty}^{\infty} x(m+n_0) z^{-m}$$

But m is a dummy variable, rename it back to n, we have

$$z^{n_0}X(z) = \sum_{n=-\infty}^{\infty} x(n+n_0) z^{-n}$$

But RHS above is just the z-transform of $x(n+n_0)$. QED Part(b)

$$\Im(x(n)) = X(z) = \sum_n x(n) z^{-n}$$

$$\begin{aligned} X\left(\frac{z}{a}\right) &= \sum_{n=-\infty}^{\infty} x(n)\left(\frac{z}{a}\right)^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n) a^n z^{-n} \\ &= \sum_{n=-\infty}^{\infty} [a^n x(n)] z^{-n} \end{aligned}$$

Hence, RHS is $\Im(a^n x(n))$, hence

$$\Im(a^n x(n)) = X\left(\frac{z}{a}\right)$$

Part(c)

$$\Im(x(n)) = X(z) = \sum_n x(n) z^{-n}$$

$$\begin{aligned} X'(z) &= \sum_n x(n)\left(-nz^{-n-1}\right) \\ &= -\sum_n x(n) n \frac{z^{-n}}{z} \\ &= -\frac{1}{z}\sum_n [nx(n)] z^{-n} \end{aligned}$$

Hence

$$-zX'(z) = \sum_n [nx(n)] z^{-n}$$

Therefore

$$\Im(nx(n)) = -zX'(z)$$

QED

4 Problem (4) (Problem 11, chapter 2, page 80)

→ **11.** With $X(z)$ denoting the z-transform of $x(n)$, show that

$$\mathfrak{Z}[x^*(n)] = X^*(z^*).$$

$$\mathfrak{Z}[x(-n)] = X\left(\frac{1}{z}\right).$$

$$\mathfrak{Z}[\text{Re } x(n)] = \tfrac{1}{2}[X(z) + X^*(z^*)].$$

$$\mathfrak{Z}[\text{Im } x(n)] = \frac{1}{2j}[X(z) - X^*(z^*)].$$

part (1) show that $\mathfrak{I}(x^*(n)) = X^*(z^*)$. By definition,

$$\mathfrak{I}(x(n)) = X(z) = \sum_n x(n) z^{-n}$$

For the sequence $x^*(n)$

$$\mathfrak{I}(x^*(n)) = \sum_n x^*(n) z^{-n} \tag{1}$$

But

$$\begin{aligned} X^*(z^*) &= \left(\sum_n x(n)(z^*)^{-n}\right)^* \\ &= \left(\sum_n x(n)(z^{-n})^*\right)^* \end{aligned}$$

Move the outside conjugate operation to inside the sum results in

$$\begin{aligned} X^*(z^*) &= \sum_n x^*(n)\left((z^{-n})^*\right)^* \\ &= \sum_n x^*(n) z^{-n} \end{aligned} \tag{2}$$

Compare (1) and (2), they are the same. Hence $\mathfrak{I}(x^*(n)) = X^*(z^*)$ Part (2)

Show that $\mathfrak{I}(x(-n)) = X\left(\frac{1}{z}\right)$

By definition,

$$\mathfrak{I}(x(n)) = X(z) = \sum_n x(n) z^{-n} \tag{1}$$

For the sequence $x(-n)$, we have

$$\mathfrak{I}(x(-n)) = \sum_n x(-n) z^{-n}$$

Let $m = -n$, hence when $n = -\infty$, $m = \infty$, and when $n = \infty$, $m = \infty$, so the above becomes

$$\mathfrak{I}(x(-n)) = \sum_{m=\infty}^{-\infty} x(m) z^m$$

But m is a dummy variable, hence rename to n, we have

$$\mathfrak{I}(x(-n)) = \sum_{n=-\infty}^{\infty} x(n) z^n \tag{2}$$

Now, in (1), Let $z = \frac{1}{z}$on both sides, we obtain

$$\begin{aligned} X\left(\frac{1}{z}\right) &= \sum_n x(n)\left(\frac{1}{z}\right)^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n) z^n \end{aligned} \tag{3}$$

Compare (2) and (3), we see they are the same, hence

$$\Im\left(x\left(-n\right)\right) = X\left(\frac{1}{z}\right)$$

Part (3)

Show that $\Im\left(\mathrm{Re}\left[x\left(n\right)\right]\right) = \frac{1}{2}\left[X\left(z\right) + X^{*}\left(z^{*}\right)\right]$

By definition,

$$\Im\left(x\left(n\right)\right) = X\left(z\right) = \sum_{n} x\left(n\right) z^{-n}$$

Let

$$x\left(n\right) = \mathrm{Re}\left(x\left(n\right)\right) + j\,\mathrm{Im}\left(x(n)\right)$$

Now

$$X^{*}\left(z\right) = \left(\sum_{n} x\left(n\right) z^{-n}\right)^{*} = \sum_{n} x^{*}\left(n\right)\left(z^{-n}\right)^{*}$$

Then

$$X^{*}\left(z^{*}\right) = \sum_{n} x^{*}\left(n\right) z^{-n}$$

Hence

$$\begin{aligned}
X\left(z\right) + X^{*}\left(z^{*}\right) &= \sum_{n} x\left(n\right) z^{-n} + \sum_{n} x^{*}\left(n\right) z^{-n} \\
&= \sum_{n}\left[x\left(n\right) + x^{*}\left(n\right)\right] z^{-n} \\
&= \sum_{n}\left[\mathrm{Re}\left(x\left(n\right)\right) + j\,\mathrm{Im}\left(x(n)\right) + \mathrm{Re}\left(x\left(n\right)\right) - j\,\mathrm{Im}\left(x(n)\right)\right] z^{-n} \\
&= \sum_{n} 2\,\mathrm{Re}\left(x\left(n\right)\right)\; z^{-n}
\end{aligned}$$

Hence

$$\frac{1}{2}\left(X\left(z\right) + X^{*}\left(z^{*}\right)\right) = \sum_{n} \mathrm{Re}\left(x\left(n\right)\right)\; z^{-n}$$

But RHS above is $\Im\left(\mathrm{Re}\left(x\left(n\right)\right)\right)$, hence

$$\Im\left(\mathrm{Re}\left(x\left(n\right)\right)\right) = \frac{1}{2}\left(X\left(z\right) + X^{*}\left(z^{*}\right)\right)$$

Part (4)

Show that $\Im\left(\mathrm{Im}\left[x\left(n\right)\right]\right) = \frac{1}{2j}\left[X\left(z\right) - X^{*}\left(z^{*}\right)\right]$

By definition,

$$\Im\left(x\left(n\right)\right) = X\left(z\right) = \sum_{n} x\left(n\right) z^{-n}$$

Let

$$x\left(n\right) = \mathrm{Re}\left(x\left(n\right)\right) + j\,\mathrm{Im}\left(x(n)\right)$$

Hence

$$\begin{aligned}
X\left(z\right) - X^{*}\left(z^{*}\right) &= \sum_{n} x\left(n\right) z^{-n} - \sum_{n} x^{*}\left(n\right) z^{-n} \\
&= \sum_{n}\left(x\left(n\right) - x^{*}\left(n\right)\right) z^{-n} \\
&= \sum_{n}\left(\mathrm{Re}\left(x\left(n\right)\right) + j\,\mathrm{Im}\left(x(n)\right) - \left[\mathrm{Re}\left(x\left(n\right)\right) - j\,\mathrm{Im}\left(x(n)\right)\right]\right) z^{-n} \\
&= \sum_{n} 2j\,\mathrm{Im}\left(x(n)\right) z^{-n}
\end{aligned}$$

Hence

$$\frac{1}{2j}\left[X\left(z\right) - X^{*}\left(z^{*}\right)\right] = \sum_{n} \mathrm{Im}\left(x(n)\right)\; z^{-n}$$

But RHS above is $\Im\left(\mathrm{Im}\left(x(n)\right)\right)$, hence

$$\Im\left(\mathrm{Im}\left(x(n)\right)\right) = \frac{1}{2j}\left[X\left(z\right) - X^{*}\left(z^{*}\right)\right]$$

5 Problem (5) problem 15, chapter 2, page 81

Consider a finite impulse response filter with unit-sample response $h(n)$ of length $(2N+1)$. If $h(n)$ is real and even, show that the zeros of the system function occur in mirror image pairs about the unit circle. i.e. if $H(z)=0$ for $z=\rho e^{j\theta}$ then $H(z)=0$ also for $z=\left(\frac{1}{\rho}\right)e^{j\theta}$

Answer:

Since $h(n)$ is real and even, then $\Im(h(n))$ is real and even. This can be seen as follows

$$\begin{aligned}
\Im(h(n)) = H(z) &= \sum_{n=-N}^{N} h(n) z^{-n} \\
&= -h(0) + \sum_{n=-N}^{0} h(n) z^{-n} + \sum_{n=0}^{N} h(n) z^{-n} \\
&= -h(0) + \sum_{n=0}^{N} h(-n) z^{n} + \sum_{n=0}^{N} h(n) z^{-n}
\end{aligned}$$

Since $h(n)$ is real and even, then $h(-n)=h(n)$, so the above becomes

$$\begin{aligned}
H(z) &= -h(0) + \sum_{n=0}^{N} h(n) z^{n} + \sum_{n=0}^{N} h(n) z^{-n} \\
&= -h(0) + \sum_{n=0}^{N} h(n)\left(z^{n}+z^{-n}\right) \qquad (1)
\end{aligned}$$

Now, let $z=\rho e^{j\theta}$, so

$$\begin{aligned}
H(z) &= -h(0) + \sum_{n=0}^{N} h(n)\left(\rho e^{j\theta} + \frac{1}{\rho} e^{-j\theta}\right)^{n} \\
&= -h(0) + \sum_{n=0}^{N} h(n)\left(\rho e^{j\theta} + \frac{1}{\rho} e^{-j\theta}\right)^{n} \qquad (2)
\end{aligned}$$

And let $z=\frac{1}{\rho}e^{j\theta}$ again in (1), we obtain

$$\begin{aligned}
H(z) &= -h(0) + \sum_{n=0}^{N} h(n)\left(\left(\frac{1}{\rho}e^{j\theta}\right)^{n} + \left(\frac{1}{\rho}e^{j\theta}\right)^{-n}\right) \\
&= -h(0) + \sum_{n=0}^{N} h(n)\left(\left(\frac{1}{\rho}e^{j\theta}\right)^{n} + \left(\rho e^{-j\theta}\right)^{n}\right) \\
&= -h(0) + \sum_{n=0}^{N} h(n)\left(\frac{1}{\rho}e^{j\theta} + \rho e^{-j\theta}\right)^{n} \qquad (3)
\end{aligned}$$

Compare (2) and (3), they are the same. Hence $H(z)$ is the same at $z=\frac{1}{\rho}e^{j\theta}$ and at $z=\rho e^{j\theta}$, therefore $H(z)$ is even function w.r.t. unit circle. Hence if a zero occurs outside a unit circle, there will be a mirror image of this zero inside the unit circle (since $H(z)$ will have the same value, which is zero in this case) at both location.

This problem is similar to looking at the DTFT $F\left(e^{j\omega}\right)$, which is real and even when $x(n)$ is real and even. The difference is that $F\left(e^{j\omega}\right)$ will be even about the $y-axis$ while $H(z)$ is even w.r.t. unit circle.