HW 2, EE 420 Digital Filters
California State University, Fullerton
Spring 2010

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Spring 2010 Compiled on May 12, 2019 at 4:24pm

Contents

1 Problem 1 (1.12)

For each of the following systems determine if system is (1) stable (2) causal (3) linear (4) shift invariant

1.1 part (a)

$T[x(n)] = g(n)\,x(n)$

(1) This system is not stable. Let the input be bounded, say $x(n) = M$, some constant, then if $g(n) = n$, then the output will be nM, and we see that as $n \longrightarrow \infty$, the output will grow with no limit, i.e. unbounded.

(2) This system is causal**.** We see that the output contain no index in n which is larger than in the input. Hence output can't occur before the input.

(3)To check for linearity. Let $T[x_1(n)] = g(n)\,x_1(n)$ and $T[x_2(n)] = g(n)\,x_2(n)$. Now let

$$x_3(n) = ax_1(n) + bx_2(n)$$

hence

$$\begin{aligned} T[x_3(n)] &= g(x)\,(ax_1(n) + bx_2(n)) \\ &= ag(x)\,x_1(n) + bg(x)\,x_2(n) \\ &= aT[x_1(n)] + bT[x_2(n)] \end{aligned}$$

Hence linear

(4)First delay the input, we obtain $T[x(n-N)] = g(n-N)\,x(n-N)$. Next obtain the output from non delayed input, which is $T[x(n)] = g(n)\,x(n)$, now delay this output by same about N and we obtain $g(n-N)\,x(n-N)$, hence it is shift invariant

1.2 part (c)

$T[x(n)] = \sum_{n-n_0}^{n+n_0} x(k)$

(1)This system is clearly stable. If the input is bounded, then the output must be bounded since the summation is over a limited range. To proof, let the maximum value that can occur in the input be M. Hence we write

$$\begin{aligned} T[x(n)] &\leq \sum_{n-n_0}^{n+n_0} M \\ &= M(2n_0 + 1) \end{aligned}$$

We see that the output does not contain n, hence for given n_0 the above sum is bounded (I am assuming that n_0 is a given parameter and so have some fixed value).

(2)This system is clearly not causal. We see that output contains values of an input which has not occurred yet. For example, for $n = 0$ and $n_0 = 1$, we obtain the output $x(-1) + x(0) + x(1)$, but $x(1)$ is a future value relative to the input which is $x(0)$.

(3)To check for linearity

$$
\begin{aligned}
T\left[x_1(n)\right] &= \sum_{n-n_0}^{n+n_0} x_1(k) \\
T\left[x_2(n)\right] &= \sum_{n-n_0}^{n+n_0} x_2(k) \\
x_3(n) &= ax_1(n) + bx_2(n) \\
T\left[x_3(n)\right] &= \sum_{n-n_0}^{n+n_0} ax_1(n) + bx_2(n) \\
&= a\sum_{n-n_0}^{n+n_0} x_1(n) + b\sum_{n-n_0}^{n+n_0} x_2(n) \\
&= ax_1(n) + bx_2(n)
\end{aligned}
$$

Hence system is linear

(4)A delayed input by some M gives

$$
T\left[x(n-M)\right] = \sum_{(n-M)-n_0}^{(n-M)+n_0} x(k)
$$

And a delayed output by same amount is

$$
\sum_{n-n_0}^{n+n_0} x(k-M)
$$

Let $k' = k - M$, hence when $k = n - n_0$ then $k' = n - n_0 - M$ and when $k = n + n_0$ then $k' = n + n_0^{,} - M$, then the above sums become

$$
T\left[x(n-M)\right] = \sum_{k'=n-n_0-M}^{n+n_0^{,}-M} x(k')
$$

Or, renaming the dummy summation index back to k we obtain

$$
T\left[x(n-M)\right] = \sum_{(n-M)-n_0}^{(n-M)+n_0^{,}} x(k)
$$

Hence delayed output is the same as the output from a delayed input. Hence shift invariant.

1.3 Part (e)

$T\left[x(n)\right] = e^{x(n)}$

(1)This is stable. Let maximum input be some bounded value, say M, hence the output $\leq e^{M}$, which is bounded.

(2)This is causal. We see that the output contain no index in n which is larger than that in the input.

(3)To check for linearity

$$
\begin{aligned}
T[x_1(n)] &= e^{x_1(n)} \\
T[x_2(n)] &= e^{x_2(n)} \\
x_3(n) &= ax_1(n) + bx_2(n) \\
T[x_3(n)] &= e^{ax_1(n)+bx_2(n)} \\
&= e^{ax_1(n)}e^{bx_2(n)} \neq ae^{x_1(n)} + be^{x_2(n)}
\end{aligned}
$$

Hence not linear

(4)A delayed input by some M gives

$$T[x(n-M)] = e^{x(n-M)}$$

and a delayed output by same amount is

$$e^{x(n-M)}$$

Hence shift invariant

2 Problem 2 (1.19)

Let $x(n)$ and $X(\omega)$ denote a sequence and its DTFT transform. Show that

$$\sum_{n=-\infty}^{\infty} x(n)\, x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, X^*(\omega)\, d\omega$$

Answer:

(Please note, when writing $X(\omega)$, it meant as the same as $X\left(e^{j\omega}\right)$ all the time. This is just short notation to make it easier to type).

First, note that

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n} d\omega$$

Hence

$$x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)\, e^{-j\omega n} d\omega$$

Then

$$\sum_{n=-\infty}^{\infty} x(n)\, x^*(n) = \sum_{n=-\infty}^{\infty} x(n)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)\, e^{-j\omega n} d\omega\right)$$

Where I have replaced $x^*(n)$ by its definition in terms of its own DTFT, hence now by interchanging the order of integration and summation, we obtain

$$\sum_{n=-\infty}^{\infty} x(n)\, x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)\left(\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\right) d\omega$$

But $\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$ is just $X(\omega)$, hence the above becomes

$$\sum_{n=-\infty}^{\infty} x(n) x^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) X(\omega) d\omega$$

QED

3 Problem (3) 1.22

$f(n)$ and $g(n)$ are real, causal and stable sequences with DTFT $F(\omega)$ and $G(\omega)$ respectively, show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) G(\omega) d\omega = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) d\omega \right) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} G(\omega) d\omega \right) \tag{1}$$

Answer:

First we note that for a real sequence, the DTFT is conjugate symmetric. And for a causal sequence it means it is zero for $n < 0$.

Before starting the proof, I show 2 common "tricks" that are used: $\int \left(\sum_n f(n) e^{-j\omega n} \right) d\omega$ can be written as $\sum_n \left(\int f(n) e^{-j\omega n} d\omega \right)$ which can be written as $\sum_n \left(f(n) \int e^{-j\omega n} d\omega \right)$, this allows one to do the integration on the complex exponential. The second "trick" is in handling multiplication of summations: $\left(\sum_n f(n) \right) \left(\sum_n g(n) \right)$ can be written as $\left(\sum_n \sum_k f(n) g(k) \right)$

Now we start the proof.

Consider the RHS in (1), this can be written as

$$\begin{aligned} RHS &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega) d\omega \right) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} G(\omega) d\omega \right) \\ &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=0}^{\infty} f(n) e^{-j\omega n} \right) d\omega \right) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=0}^{\infty} g(n) e^{-j\omega n} \right) d\omega \right) \\ &= \frac{1}{4\pi} \left(\sum_{n=0}^{\infty} \int_{-\pi}^{\pi} f(n) e^{-j\omega n} d\omega \right) \left(\sum_{n=0}^{\infty} \int_{-\pi}^{\pi} g(n) e^{-j\omega n} d\omega \right) \end{aligned}$$

Now $f(n)$ and $g(n)$ can be removed from inside the integral, but kept inside the summation, as they depend on n and not ω, and since the sequence is stable and real, we know they are both are some finite real quantities. Hence the above becomes

$$\begin{aligned} &= \frac{1}{4\pi} \left(\sum_{n=0}^{\infty} f(n) \int_{-\pi}^{\pi} e^{-j\omega n} d\omega \right) \left(\sum_{n=0}^{\infty} g(n) \int_{-\pi}^{\pi} e^{-j\omega n} d\omega \right) \\ &= \frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left(f(n) g(k) \int_{-\pi}^{\pi} e^{-j\omega n} d\omega \int_{-\pi}^{\pi} e^{-j\omega k} d\omega \right) \end{aligned}$$

But $\int_{-\pi}^{\pi} e^{-j\omega n} d\omega = \frac{\left[e^{-j\omega n}\right]_{-\pi}^{\pi}}{-jn} = \frac{\left[e^{-j\pi n} - e^{j\pi n}\right]}{-jn} = -2\sin n\pi$ and $\int_{-\pi}^{\pi} e^{-j\omega k} d\omega = -2\sin k\pi$, hence the above becomes

$$RHS = \frac{1}{\pi}\sum_{n}\sum_{k} f(n)\, g(k) \sin n\pi \sin k\pi \tag{2}$$

But $\sin n\pi = \sin k\pi = 0$ because n and k are integers. Hence

$$RHS = 0$$

Now consider the LHS

$$\begin{aligned}
LHS &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\omega)\, G(\omega)\, d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{n=0}^{\infty} f(n)\, e^{-j\omega n}\right)\left(\sum_{n=0}^{\infty} g(n)\, e^{-j\omega n}\right) d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f(n)\, e^{-j\omega n} g(k)\, e^{-j\omega k}\right) d\omega \\
&= \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(f(n)\, g(k) \int_{-\pi}^{\pi} e^{-j\omega n} e^{-j\omega k} d\omega\right) \qquad (3)
\end{aligned}$$

But $\int_{-\pi}^{\pi} e^{-j\omega n} e^{-j\omega k} d\omega = \int_{-\pi}^{\pi} e^{-j\omega(n+k)} d\omega = \frac{\left[e^{-j\omega(n+k)}\right]_{-\pi}^{\pi}}{-j(n+k)} = \frac{\left[e^{-j\pi(n+k)} - e^{j\pi(n+k)}\right]}{-j(n+k)} = -2\sin\pi(n+k)$, hence (3) becomes

$$LHS = \frac{-1}{\pi}\sum_{n}\sum_{k} f(n)\, g(k) \sin\pi(n+k) \tag{5}$$

Now, since n and k run over the integers, then $\sin\pi(n+k) = \sin\pi k = \sin\pi n = 0$, hence

$$LHS = 0$$

Hence, RHS=LHS because both are zero.

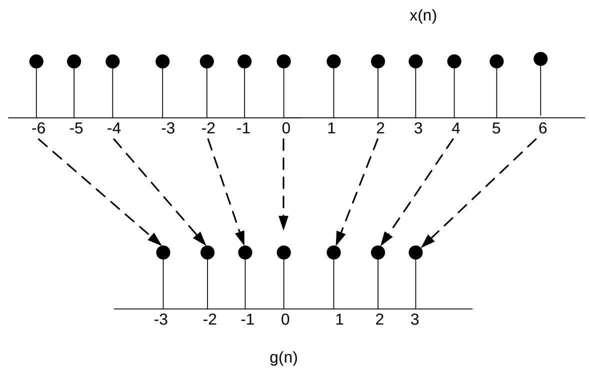
4 Problem (4) (1.24)

Let $x(n)$ and $X(\omega)$ represent a sequence and its DTFT. Do not assume that $x(n)$ is real nor that $x(n)$ is zero for $n < 0$, determine in terms of $X(\omega)$ the transforms of of the following sequences

4.1 part(c)

$$g(n) = x(2n)$$

Answer:



showing effect of g(n)=x(2n) in time domain

To solve this, I found it easier to write out the few terms and compare.

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ &= \cdots + x(0) + x(1) e^{-j\omega} + x(2) e^{-2j\omega} + x(3) e^{-3j\omega} + x(4) e^{-4j\omega} + \cdots \end{aligned} \tag{1}$$

But $G(\omega) = \sum_{n=-\infty}^{\infty} g(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(2n) e^{-j\omega n}$, hence if write $G(\omega)$ as above we see

$$G(\omega) = \cdots + x(0) + x(2) e^{-j\omega} + x(4) e^{-2j\omega} + x(4) e^{-3j\omega} + x(6) e^{-4j\omega} + \cdots \tag{2}$$

To make the terms in $X(\omega)$ match those in $G(\omega)$, lets start by expressing $X\left(\frac{\omega}{2}\right)$, we obtain

$$\begin{aligned} X\left(\frac{\omega}{2}\right) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{\omega}{2}n} \\ &= \cdots + x(0) + x(1) e^{-j\frac{\omega}{2}} + x(2) e^{-2j\frac{\omega}{2}} + x(3) e^{-3j\frac{\omega}{2}} + x(4) e^{-4j\frac{\omega}{2}} + \cdots \\ &= \cdots + x(0) + x(1) e^{-j\frac{\omega}{2}} + x(2) e^{-j\omega} + x(3) e^{-3j\frac{\omega}{2}} + x(4) e^{-2j\omega} + \cdots \end{aligned} \tag{3}$$

Now looking at (2) and (3), we see that the terms match up when n is even. We need to get rid, in (4), of those terms that are odd in n then this will make it match (2), which is what we are after.

Consider $X\left(\frac{\omega-2\pi}{2}\right)$, this has series

$$\begin{aligned} X\left(\frac{\omega-2\pi}{2}\right) &= \cdots + x(0) + x(1) e^{-j\left(\frac{\omega-2\pi}{2}\right)} + x(2) e^{-2j\left(\frac{\omega-2\pi}{2}\right)} + x(3) e^{-3j\left(\frac{\omega-2\pi}{2}\right)} + x(4) e^{-4j\left(\frac{\omega-2\pi}{2}\right)} + \cdots \\ &= \cdots + x(0) + x(1) e^{-j\frac{\omega}{2}} e^{\pi} + x(2) e^{-j\omega} e^{2\pi} + x(3) e^{-3j\frac{\omega}{2}} e^{3\pi} + x(4) e^{-2j\omega} e^{4\pi} + \cdots \end{aligned}$$

But $e^{m\pi} = -1$ for m odd, and $e^{m\pi} = +1$ for m even, so the above becomes

$$X\left(\frac{\omega-2\pi}{2}\right) = \cdots + x(0) - x(1) e^{-j\frac{\omega}{2}} + x(2) e^{-j\omega} - x(3) e^{-3j\frac{\omega}{2}} + x(4) e^{-2j\omega} e^{4\pi} - \cdots \tag{4}$$

We are getting close. Add(4) to (3), terms cancel, and we obtain

$$\begin{aligned}
X\left(\frac{\omega}{2}\right)+X\left(\frac{\omega-2\pi}{2}\right) &= \cdots + x(0) + x(1)\,e^{-j\frac{\omega}{2}} + x(2)\,e^{-j\omega} + x(3)\,e^{-3j\frac{\omega}{2}} + x(4)\,e^{-2j\omega} + \cdots \\
&\quad + \left(\cdots + x(0) - x(1)\,e^{-j\frac{\omega}{2}} + x(2)\,e^{-j\omega} - x(3)\,e^{-3j\frac{\omega}{2}} + x(4)\,e^{-2j\omega}e^{4\pi} - \cdots\right) \\
&= \cdots + 2x(0) + 2x(2)\,e^{-j\omega} + 2x(4)\,e^{-2j\omega} + 2x(6)\,e^{-4j\omega} + \cdots \\
&= 2\left(\cdots + x(0) + x(2)\,e^{-j\omega} + x(4)\,e^{-2j\omega} + x(6)\,e^{-4j\omega} + \cdots\right)
\end{aligned}$$

But the RHS of the above is just (2), which is twice $G(\omega)$, hence

$$2G(\omega) = X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega-2\pi}{2}\right)$$

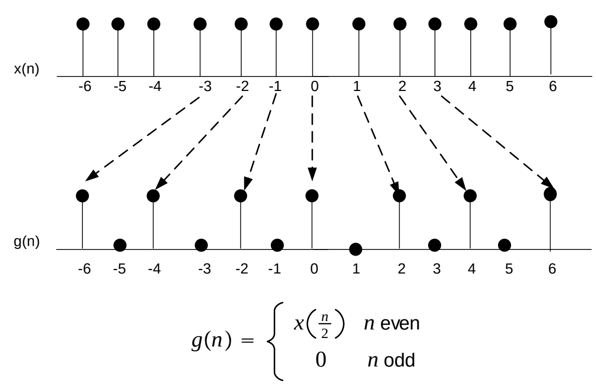
Therefore

$$\boxed{G(\omega) = \frac{1}{2}\left[X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega}{2}-\pi\right)\right]}$$

QED

4.2 Part (d)

$$g(n) = \begin{cases} x\left(\frac{n}{2}\right) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$



Write out few terms to see the pattern, we obtain

$$\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n} \\
&= \cdots + x(0) + x(1)\,e^{-j\omega} + x(2)\,e^{-2j\omega} + x(3)\,e^{-3j\omega} + x(4)\,e^{-4j\omega} + \cdots \qquad (1)
\end{aligned}$$

But $G(\omega) = \sum_{n=-\infty}^{\infty} g(n)\,e^{-j\omega n} = \sum_{\substack{n=-\infty \\ even}}^{\infty} x\left(\frac{n}{2}\right)\,e^{-j\omega n}$, hence if write $G(\omega)$ as above we see

$$\begin{aligned}
G(\omega) &= \cdots + g(0) + 0 + g(2)\,e^{-2j\omega} + 0 + g(4)\,e^{-4j\omega} + 0 + g(6)\,e^{-6j\omega} + \cdots \\
&= \cdots + x(0) + 0 + x(1)\,e^{-2j\omega} + 0 + x(2)\,e^{-4j\omega} + 0 + x(3)\,e^{-6j\omega} + \cdots \\
&= \cdots + x(0) + x(1)\,e^{-2j\omega} + x(2)\,e^{-4j\omega} + x(3)\,e^{-6j\omega} + x(4)\,e^{-8j\omega} + \cdots \qquad (2)
\end{aligned}$$

To make (2) match (1), consider $X\left(\frac{\omega}{2}\right)$

$$
\begin{aligned}
X(2\omega) &= \cdots + x(0) + x(1)\,e^{-j2\omega} + x(2)\,e^{-2j(2\omega)} + x(3)\,e^{-3j(2\omega)} + x(4)\,e^{-4j(2\omega)} + \cdots \\
&= \cdots + x(0) + x(1)\,e^{-j2\omega} + x(2)\,e^{-4j\omega} + x(3)\,e^{-6j\omega} + x(4)\,e^{-8j\omega} + \cdots \qquad (3)
\end{aligned}
$$

Compare (3) and (2), we see they are the same, hence

$$G(\omega) = X(2\omega)$$

5 Problem (5) 1.25 parts 1,2

5.1 Part (1)

Proof the following property

$$x^*(n) \Longleftrightarrow X^*\left(e^{-j\omega}\right)$$

Solution

Note, I will write $X(\omega)$ as short hand notation for $X\left(e^{j\omega}\right)$ everywhere.

Let

$$X(\omega) \equiv F(x(n)) = \sum_{k=-\infty}^{\infty} x(k)\,e^{-j\omega k} \tag{1}$$

and let

$$x(n) \equiv F^{-1}(X(\omega)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\,e^{j\omega n}d\omega \tag{2}$$

We need to show that $F(x^*(n)) = X^*(-\omega)$

From definition (1)

$$F(x^*(n)) = \sum_{k=-\infty}^{\infty} x^*(k)\,e^{-j\omega k} \tag{3}$$

But complex conjugate of $X(\omega)$ is

$$
\begin{aligned}
X^*(\omega) &= \left(\sum_{k=-\infty}^{\infty} x(k)\,e^{-j\omega k}\right)^* \\
&= \sum_{k=-\infty}^{\infty} x^*(k)\,e^{j\omega k}
\end{aligned}
$$

Hence, replacing ω by $-\omega$ above we obtain

$$X^*(-\omega) = \sum_{k=-\infty}^{\infty} x^*(k)\,e^{-j\omega k} \tag{4}$$

Compare (3) and (4) we see that $F(x^*(n)) = X^*(-\omega)$

5.2 Part (2)

Show that

$$x^*(-n) \Longleftrightarrow X^*\left(e^{j\omega}\right)$$

From definition (1)

$$F(x^*(n)) = \sum_{k=-\infty}^{\infty} x^*(k)\, e^{-j\omega k} \tag{5}$$

Hence,

$$F(x^*(-n)) = \sum_{k=-\infty}^{\infty} x^*(-k)\, e^{-j\omega k}$$

In the RHS, let $m = -k$, we we obtain

$$F(x^*(-n)) = \sum_{m=\infty}^{\infty} x^*(m)\, e^{j\omega m}$$

m is a dummy variable, so we can rename it back to k, and order of summation is not important, so

$$F(x^*(-n)) = \sum_{k=-\infty}^{\infty} x^*(k)\, e^{j\omega k} \tag{6}$$

But complex conjugate of $X(\omega)$ is

$$\begin{aligned} X^*(\omega) &= \left(\sum_{k=-\infty}^{\infty} x(k)\, e^{-j\omega k}\right)^* \\ &= \sum_{k=-\infty}^{\infty} x^*(k)\, e^{j\omega k} \end{aligned} \tag{7}$$

Compare (6) and (7), we see they are the same hence

$$F(x^*(-n)) = X^*(\omega)$$

6 Problem 6

Find $H\left(e^{j\omega}\right)$ and plot magnitude and phase diagrams. This is $h(n)$ over the first 6 points, it is zero everywhere else.

$$h(n) = \left\{ \underset{origin}{0}, \frac{1}{3}, \frac{2}{3}, 1, \frac{2}{3}, \frac{1}{3}, 0 \right\}$$

Solution:

First we note that the sequence is real, hence the $H(\omega)$ will be conjugate symmetric, and the $|H(\omega)|$

will be even and phase of $H(\omega)$ will be odd. Now we find $H(\omega)$

$$
\begin{aligned}
H(\omega) &= \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n} \\
&= \sum_{n=0}^{6} h(n)\, e^{-j\omega n} \\
&= 0\left(e^{-j\omega 0}\right) + \frac{1}{3}e^{-j\omega} + \frac{2}{3}e^{-j2\omega} + e^{-j3\omega} + \frac{2}{3}e^{-j4\omega} + \frac{1}{3}e^{-j5\omega} + 0\left(e^{-j6\omega}\right) \\
&= \frac{1}{3}e^{-j\omega} + \frac{2}{3}e^{-j2\omega} + e^{-j3\omega} + \frac{2}{3}e^{-j4\omega} + \frac{1}{3}e^{-j5\omega} \\
&= \frac{1}{3}\left(e^{-j\omega} + e^{-j5\omega}\right) + \frac{2}{3}\left(e^{-j2\omega} + e^{-j4\omega}\right) + e^{-j3\omega} \qquad (1)
\end{aligned}
$$

But $\frac{1}{3}\left(e^{-j\omega} + e^{-j5\omega}\right) = \frac{1}{3}e^{-j\omega}\left(1 + e^{-j4\omega}\right) = \frac{1}{3}e^{-j\omega}e^{-j2\omega}\left(e^{j2\omega} + e^{-j2\omega}\right) = \frac{2}{3}e^{-3j\omega}(\cos 2\omega)$. Similarly, $\frac{2}{3}\left(e^{-j2\omega} + e^{-j4\omega}\right) = \frac{2}{3}e^{-j2\omega}\left(1 + e^{-j2\omega}\right) = \frac{2}{3}e^{-j2\omega}e^{-j\omega}\left(e^{j\omega} + e^{-j\omega}\right) = \frac{4}{3}e^{-j3\omega}(\cos\omega)$. Hence (1) becomes

$$
\begin{aligned}
H(\omega) &= \frac{2}{3}e^{-3j\omega}(\cos 2\omega) + \frac{4}{3}e^{-j3\omega}(\cos\omega) + e^{-j3\omega} \\
&= e^{-3j\omega}\left[\frac{2}{3}\cos 2\omega + \frac{4}{3}\cos\omega + 1\right]
\end{aligned}
$$

We have now $H(\omega)$ in the form of

$$H(\omega) = |H(\omega)|\, e^{j\Theta(\omega)}$$

Where

$$|H(\omega)| = 1 + \frac{2}{3}\cos 2\omega + \frac{4}{3}\cos\omega$$

And phase

$$\Theta(\omega) = -3\omega$$

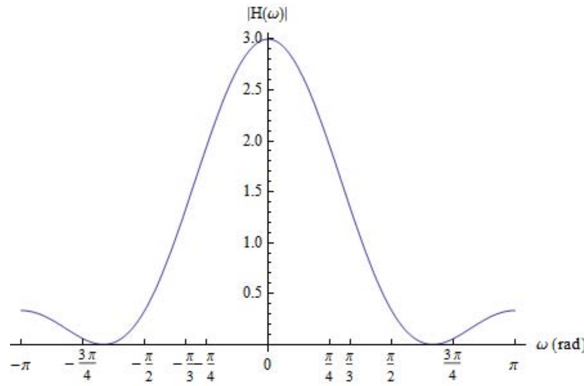
Few points to plot (keep ω between $-\pi$ and π) since $|H(\omega)|$ is even and phase is odd, I only need to look at some points from 0 to π. Then use the even and odd property to know the values from 0 to $-\pi$

$\omega(rad)$	$\lvert H(\omega)\rvert$	$\Theta(\omega)$
0	$\frac{2}{3} + \frac{4}{3} + 1 = 3$	0
$\frac{\pi}{4}$	$\frac{2}{3}\cos 90^0 + \frac{4}{3}\cos 45^0 + 1 = 1.9428$	-135^0
$\frac{\pi}{2}$	$\frac{2}{3}\cos 180^0 + \frac{4}{3}\cos 90^0 + 1 = 1.6667$	$-270^0 = +90^0$
π	$\frac{2}{3}\cos 360^0 + \frac{4}{3}\cos 180^0 + 1 = 0.33333$	$-180^0 = 0$

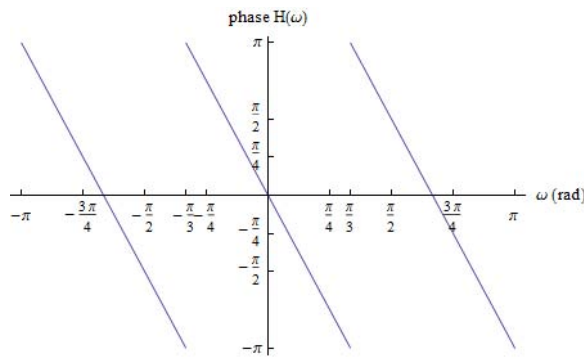
Hence

$\omega(rad)$	$\lvert H(\omega)\rvert$	$\Theta(\omega)$
$-\frac{\pi}{4}$	1.942	$+135^0$
$-\frac{\pi}{2}$	1.666	-90^0
$-\pi$	0.333	180^0

So, now we can sketch this by hand, but need a little bit more point to get a good plot, here is a quick plot done using the computer




```
h[w_] := Exp[(-I)*3*w]*(1 + (2/3)*Cos[2*w] + (4/3)*Cos[w])
Plot[Abs[h[w]], {w, -Pi, Pi},
Ticks -> {{-Pi, (-3/4)*Pi, -Pi/2, -Pi/3, -Pi/4, 0, Pi/4, Pi/3,
Pi/2, (3/4)*Pi, Pi}, Automatic},
AxesLabel -> {"\[Omega] (rad)", "|H(\[Omega])|"}]
```




```
Plot[Arg[h[w]], {w, -Pi, Pi},
Ticks -> {{-Pi, (-3/4)*Pi, -Pi/2, -Pi/3, -Pi/4, 0, Pi/4, Pi/3,
Pi/2, (3/4)*Pi, Pi}, {-Pi, -Pi/2, -Pi/4, 0, Pi/4, Pi/2, Pi}},
AxesLabel -> {"\[Omega] (rad)", "phase H(\[Omega])"}]
```