

HW 10, EE 420 Digital Filters  
California State University, Fullerton  
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Nasser M. Abbasi

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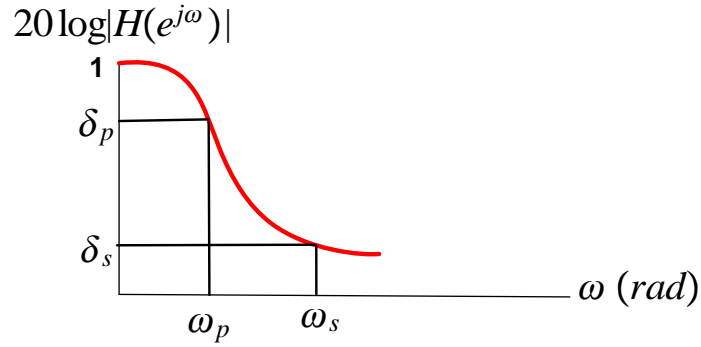
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## 1 Filter specifications

Filter specifications are 5 parameters. The frequency specifications are analog frequencies, while the attenuations for the passband and the stopband are given in db

$F_s$	The sampling frequency in Hz
$f_c$	The passband cutoff frequency in Hz
$f_s$	The stopband corner frequency in Hz
$\delta_p$	The attenuation in db at $f_c$
$\delta_s$	The attenuation in db at $f_s$

This diagram below illustrates these specifications



The frequency specifications are in Hz and they must be first converted to digital frequencies  $\omega$  where  $0 \leq |\omega| \leq \pi$  before using the attenuation specifications, The sampling frequency  $F_s$  is used to do this conversion since  $F_s$  corresponds to  $2\pi$  on the digital frequency scale.

## 2 Analytical derivation of the design steps

These are the design steps

1. Convert specifications from analog to digital frequencies
2. Based on design method (impulse invariance of bilinear) apply the attenuation criteria to determine  $\Omega_c$  and  $N$  (the filter order)
3. Using  $\Omega_c$  and  $N$  find the locations of the poles of  $H(s)$ , the Butterworth analog filter.
4. Find  $H(z)$  from  $H(s)$ . The method of doing this depends if we are using impulse invariance or bilinear. This step is much simpler for the bilinear method as it does not require performing partial fractions decomposition on  $H(s)$

Now we begin the analytical design procedure.

### 2.1 Finding the analog Butterworth filter $H(s)$

We first convert analog specifications to digital specifications:  $\frac{F_s}{2\pi} = \frac{f_p}{\omega_p}$ , hence  $\omega_p = 2\pi \frac{f_p}{F_s}$  and  $\omega_s = 2\pi \frac{f_s}{F_s}$

Convert the criteria relative to the digital normalized scale:

$$20 \log |H(e^{j\omega_p})| \geq \delta_p$$

$$20 \log |H(e^{j\omega_s})| \leq \delta_s$$

Hence

$$|H(e^{j\omega_p})| \geq 10^{\frac{\delta_p}{20}} \quad (\text{A})$$

$$|H(e^{j\omega_s})| \leq 10^{\frac{\delta_s}{20}} \quad (\text{B})$$

Butterworth analog filter squared magnitude Fourier transform is given by

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

hence equations (A) and (B) above are now written in terms of the analog Butterworth amplitude frequency response and become

$$\frac{1}{1 + \left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} \geq \left(10^{\frac{\delta_p}{20}}\right)^2 = 10^{\frac{\delta_p}{10}}$$

$$\frac{1}{1 + \left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} \leq \left(10^{\frac{\delta_s}{20}}\right)^2 = 10^{\frac{\delta_s}{10}}$$

Now we assign values for  $\Omega_p$  and  $\Omega_s$  as follows

	$\Omega_p$	$\Omega_s$
impulse invariance	$\frac{\omega_p}{T}$	$\frac{\omega_s}{T}$
bilinear transformation	$\frac{2}{T} \tan\left(\frac{\omega_p}{2}\right)$	$\frac{2}{T} \tan\left(\frac{\omega_s}{2}\right)$

Where  $\omega_p = 2\pi \frac{f_p}{F_s}$  and  $\omega_s = 2\pi \frac{f_s}{F_s}$  as was mentioned earlier. Let  $\alpha_p = \frac{1}{10^{\frac{\delta_p}{10}}}$  and  $\alpha_s = \frac{1}{10^{\frac{\delta_s}{10}}}$ , then the above inequalities can be written as

$$1 + \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} \leq \alpha_p \quad (1)$$

$$1 + \left(\frac{\Omega_s}{\Omega_c}\right)^{2N} \geq \alpha_s \quad (2)$$

Change inequalities to equalities and simplify

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \alpha_p - 1$$

$$\left(\frac{\Omega_s}{\Omega_c}\right)^{2N} = \alpha_s - 1$$

Divide the above 2 equations

$$\left(\frac{\Omega_p}{\Omega_s}\right)^{2N} = \frac{\alpha_p - 1}{\alpha_s - 1}$$

$$2N [\log(\Omega_p) - \log(\Omega_s)] = \log[\alpha_p - 1] - \log[\alpha_s - 1]$$

$$N = \frac{1}{2} \frac{\log[\alpha_p - 1] - \log[\alpha_s - 1]}{\log(\Omega_p) - \log(\Omega_s)}$$

We need to round the above to then nearest integer using the Ceiling function i.e.  $N = \lceil N \rceil$

Now for impulse invariance method, use equation (1) above to solve for  $\Omega_c$  and for bilinear transformation we used equation (2) above to find  $\Omega_c$

For impulse invariance method, let  $\bar{\Omega} = \Omega_p$  and let  $\bar{\alpha} = \alpha_p$ , and for bilinear method let  $\bar{\Omega} = \Omega_s$  and let  $\bar{\alpha} = \alpha_s$ . Hence we now solve for  $\Omega_c$

$$1 + \left(\frac{\bar{\Omega}}{\Omega_c}\right)^{2N} = \bar{\alpha}$$

$$2N \left( \log_{10} \frac{\bar{\Omega}}{\Omega_c} \right) = \log_{10} [\bar{\alpha} - 1]$$

$$\log_{10} \frac{\bar{\Omega}}{\Omega_c} = \frac{1}{2N} \log_{10} [\bar{\alpha} - 1]$$

$$\frac{\bar{\Omega}}{\Omega_c} = 10^{\left(\frac{1}{2N} \log_{10} [\bar{\alpha} - 1]\right)}$$

$$\Omega_c = \frac{\bar{\Omega}}{10^{\left(\frac{1}{2N} \log_{10} [\bar{\alpha} - 1]\right)}}$$

Hence for impulse invariance the butterworth magnitude square of the transfer function is

$$|H_a(s)|^2 = \frac{T^2}{1 + \left(\frac{s}{j\Omega_c}\right)^{2N}}$$

and for bilinear the magnitude square of the transfer function is

$$|H_a(s)|^2 = \frac{1}{1 + \left(\frac{s}{j\Omega_c}\right)^{2N}}$$

Hence  $H(s)$  poles are found by setting the denominator of the above to zero

$$\begin{aligned}
 1 + \left( \frac{s}{j\Omega_c} \right)^{2N} &= 0 \\
 \left( \frac{s}{j\Omega_c} \right)^{2N} &= -1 \\
 &= e^{j(\pi+2\pi k)} \quad k = 0, 1, 2, \dots, 2N-1 \\
 \frac{s}{j\Omega_c} &= e^{j\left(\frac{\pi+2\pi k}{2N}\right)} \\
 s &= j\Omega_c e^{j\left(\frac{\pi+2\pi k}{2N}\right)} \\
 &= \Omega_c e^{j\frac{\pi}{2}} e^{j\left(\frac{\pi+2\pi k}{2N}\right)} \\
 &= \Omega_c e^{j\left(\frac{\pi(1+2k+N)}{2N}\right)}
 \end{aligned}$$

We only need to find the LHS poles, which are located at  $i = 0 \dots N-1$ , because these are the stable poles. Hence the  $i^{th}$  pole is

$$s_i = \Omega_c e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)}$$

For example for  $i = 0$ ,  $N = 6$  we get

$$s_0 = \Omega_c e^{j\left(\frac{\pi(1+N)}{2N}\right)} = \Omega_c e^{j\left(\frac{\pi 7}{12}\right)}$$

Now we can write the analog filter generated based on the above selected poles, which is, for impulse invariance

$$H_a(s) = \frac{T K}{\prod_{i=0}^{N-1} (s - s_i)} \quad (3)$$

and for bilinear

$$H_a(s) = \frac{K}{\prod_{i=0}^{N-1} (s - s_i)} \quad (3)$$

$K$  is found by solving  $H_a(0) = T$  for impulse invariance, and by setting  $H_a(0) = 1$  for bilinear, hence for impulse invariance, we obtain

$$k = \prod_{i=0}^{N-1} (-s_i)$$

and for bilinear we obtain

$$k = \prod_{i=0}^{N-1} (-s_i)$$

We see that the same expression results for  $k$  for both cases.

Now we need to write poles in non-polar form and plug them into (3)

$$s_i = \Omega_c e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)} = \Omega_c \left( \cos \frac{\pi(1+2i+N)}{2N} + j \sin \frac{\pi(1+2i+N)}{2N} \right) \quad i = 0 \dots N-1$$

Hence, for impulse invariance

$$H_a(s) = \frac{T K}{\prod_{i=0}^{N-1} \left( s - \Omega_c \left( \cos \frac{\pi(1+2i+N)}{2N} + j \sin \frac{\pi(1+2i+N)}{2N} \right) \right)} \quad (4)$$

and for bilinear

$$H_a(s) = \frac{K}{\prod_{i=0}^{N-1} \left( s - \Omega_c \left( \cos \frac{\pi(1+2i+N)}{2N} + j \sin \frac{\pi(1+2i+N)}{2N} \right) \right)} \quad (4)$$

Where

$$\Omega_c = \frac{\bar{\Omega}}{10^{\left(\frac{1}{2N} \log_{10}[\bar{\alpha}-1]\right)}}$$

and

$$N = \left\lceil \frac{1}{2} \frac{\log[\alpha_p - 1] - \log[\alpha_s - 1]}{\log(\Omega_1) - \log(\Omega_2)} \right\rceil$$

Now that we have found  $H(s)$  we need to convert it to  $H(z)$

2.2 Analog to digital conversion or finding  $H(z)$  from  $H(s)$

**impulse invariance method** We need to make sure that we multiply poles of complex conjugates with each others to make the result simple to see.

Now that we have  $H_a(s)$ , we do the  $A \rightarrow D$  conversion. I.e. obtain  $H(z)$  from the above  $H(s)$ . When using impulse invariance, we need to perform partial fraction decomposition on (4) above in order to write  $H(s)$  in this form

$$H(s) = \sum_{i=0}^{N-1} \frac{A_i}{s - s_i}$$

For example, to obtain  $A_j$ , we write

$$A_j = \lim_{s \rightarrow s_j} H_a(s) = \frac{T k}{\prod_{\substack{i=0 \\ i \neq j}}^{N-1} (s - s_i)}$$

Once we find all the  $A$ 's, we now write  $H(z)$  as follows

$$H(z) = \sum_{i=0}^{N-1} \frac{A_i}{1 - \exp(Ts_i)z^{-1}}$$

This completes the design. We can try to convert the above form of  $H(z)$  to a rational expression as  $H(z) = \frac{N(z)}{D(z)}$

**bilinear transformation method** After finding  $H(s)$  as shown above, we simply replace  $s$  by  $\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}$ . This is much simpler than the impulse invariance method. Before doing this substitution, multiply poles which are complex conjugate of each others in the denominator of  $H(s)$ . After this, then do the above substitution

2.3 Summary of analytical derivation method

We will now make a table with the derivation equations to follow to design in either bilinear or impulse invariance. Note that the same steps are used in both designs except for step 5,6,8,13. This table make it easy to develop a program.

step	Impulse invariance	common equation	bilinear
1		$\omega_p = 2\pi \frac{f_p}{F_s}$	
2		$\omega_s = 2\pi \frac{f_s}{F_s}$	
3		$\alpha_p = \frac{1}{\frac{\delta_p}{10^{\frac{10}{10}}}}$	
4		$\alpha_s = \frac{1}{\frac{\delta_s}{10^{\frac{10}{10}}}}$	
5	$\Omega_p = \frac{\omega_p}{T}$		$\frac{2}{T} \tan\left(\frac{\omega_p}{2}\right)$
6	$\Omega_s = \frac{\omega_s}{T}$		$\frac{2}{T} \tan\left(\frac{\omega_s}{2}\right)$
7		$N = \left\lceil \frac{\frac{1}{2} \log[a_p-1] - \log[\alpha_s-1]}{\log(\Omega_p) - \log(\Omega_s)} \right\rceil$	
8	$\Omega_c = \frac{\Omega_p}{10^{\left(\frac{1}{2N} \log_{10}[a_p-1]\right)}}$		$\Omega_c = \frac{\Omega_s}{10^{\left(\frac{1}{2N} \log_{10}[\alpha_s-1]\right)}}$
9		H(S) poles $s_i = \Omega_c e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)}$ $i=0 \cdots N-1$	
10		$k = \prod_{i=0}^{N-1} (-s_i)$	
11	$H_a(s) = \frac{T K}{\prod_{i=0}^{N-1} (s-s_i)}$		$H_a(s) = \frac{K}{\prod_{i=0}^{N-1} (s-s_i)}$
12	fraction $H_a(s) = \sum_{i=0}^{N-1} \frac{A_i}{s-s_i}$		
13	$H(z) = \sum_{i=0}^{N-1} \frac{A_i}{1 - \exp(Ts_i)z^{-1}}$		$H(z) = H_a(s) \Big _{s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}}$

### 3 Numerical design

Sampling frequency  $F_s = 10\text{khz}$ , passband corner frequency  $f_p = 1\text{khz}$ , stopband corner frequency  $f_s = 2\text{khz}$ , with criteria  $\delta_p \geq -3\text{db}$  and  $\delta_{stop} \leq -10\text{db}$

using impulse invariance method     $T = 1$

step	Impulse invariance
1	$\omega_p = 2\pi \frac{f_p}{F_s} \rightarrow \frac{2\pi(1000)}{10000} \rightarrow 0.2\pi$
2	$\omega_s = 2\pi \frac{f_s}{F_s} \rightarrow \frac{2\pi(2000)}{10000} \rightarrow 0.4\pi$
3	$\alpha_p = \frac{1}{\delta_p} \rightarrow \frac{1}{-3} \rightarrow 1.995\,3$
4	$\alpha_s = \frac{1}{\delta_s} \rightarrow \frac{1}{-10} \rightarrow 10.0$
5	$\Omega_p = \frac{\omega_p}{T} \rightarrow \frac{\omega_p}{1} \rightarrow 0.2\pi$
6	$\Omega_s = \frac{\omega_s}{T} \rightarrow \frac{0.4\pi}{1} \rightarrow 0.4\pi$
7	$N = \left\lceil \frac{\frac{1}{2} \frac{\log[\alpha_p-1]-\log[\alpha_s-1]}{\log(\Omega_p)-\log(\Omega_s)}}{\log(\Omega_p)-\log(\Omega_s)} \right\rceil \rightarrow \frac{1}{2} \frac{\log_{10}(1.995\,3-1)-\log_{10}(10.0-1)}{\log_{10}(0.2\pi)-\log_{10}(0.4\pi)} \rightarrow 1.588\,4 \rightarrow 2$
8	$\Omega_c = \frac{\Omega_p}{10^{\left(\frac{1}{2N} \log_{10}[\alpha_p-1]\right)}} \rightarrow \frac{0.2\pi}{10^{\left(\frac{1}{2 \times 2} \log_{10}(1.995\,3-1)\right)}} \rightarrow 0.629\,06$
9	poles of H(S) $s_i = \Omega_c \, e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)} \rightarrow s_i = 0.629\,06 \, e^{j\left(\frac{\pi(3+2i)}{4}\right)} \quad i = 0 \dots 1$
	$s_0 = -0.444\,81 + j0.444\,81, s_1 = -0.444\,81 - j0.444\,81$
10	$k = \prod_{i=0}^{N-1} (-s_i) = (0.444\,81 - j0.444\,81) (0.444\,81 + j0.444\,81) = 0.395\,71$
11	$H_a(s) = \frac{T \, K}{\prod_{i=0}^{N-1} (s-s_i)} = \frac{0.395\,71}{(s+0.444\,81-j0.444\,81)(s+0.444\,81+j0.444\,81)} = \frac{0.395\,71}{s^2+0.889\,62s+0.395\,71}$
12	partial fraction $H_a(s) = \sum_{i=0}^{N-1} \frac{A_i}{s-s_i} = \frac{0.444\,81j}{s+0.444\,81+0.444\,81j} - \frac{0.444\,81j}{s+0.444\,81-0.444\,81j}$
13	$H(z) = \sum_{i=0}^{N-1} \frac{A_i}{1-\exp(Ts_i)z^{-1}} = \frac{0.444\,81j}{1-\exp(-0.444\,81-j0.444\,81)z^{-1}} - \frac{0.444\,81j}{1-\exp(-0.444\,81+j0.444\,81)z^{-1}} = \frac{0.245\,35z}{z^2-1.572z+0.410\,81}$

Using bilinear     $T = \frac{1}{10000}$

step	Impulse invariance
1	$\omega_p = 2\pi \frac{f_p}{F_s} = \frac{2\pi(1000)}{10000} = 0.2\pi$
2	$\omega_s = 2\pi \frac{f_s}{F_s} = \frac{2\pi(2000)}{10000} = 0.4\pi$
3	$\alpha_p = \frac{1}{\frac{\delta_p}{10^{\frac{10}{10}}}} = \frac{1}{\frac{-3}{10^{\frac{10}{10}}}} = 1.995\ 3$
4	$\alpha_s = \frac{1}{\frac{\delta_s}{10^{\frac{10}{10}}}} = \frac{1}{\frac{-10}{10^{\frac{10}{10}}}} = 10.0$
5	$\Omega_p = \frac{2}{T} \tan\left(\frac{\omega_p}{2}\right) = 2 \times 10000 \tan\left(\frac{0.2\pi}{2}\right) \rightarrow 6498.4$
6	$\Omega_s = \frac{2}{T} \tan\left(\frac{\omega_s}{2}\right) = 2 \times 10000 \tan\left(\frac{0.4\pi}{2}\right) \rightarrow 14531.$
7	$N = \left\lceil \frac{1}{2} \frac{\log[\alpha_p-1]-\log[\alpha_s-1]}{\log(\Omega_p)-\log(\Omega_s)} \right\rceil = \frac{1}{2} \frac{\log_{10}(1.995\ 3-1)-\log_{10}(10.0-1)}{\log_{10}(6498.4)-\log_{10}(14531.)} = 1.368\ 1 \rightarrow 2$
8	$\Omega_c = \frac{\Omega_s}{10^{\left(\frac{1}{2N} \log_{10}[\alpha_s-1]\right)}} = \frac{14531}{10^{\left(\frac{1}{2 \times 2} \log_{10}(10.0-1)\right)}} = 8389.5$
9	poles of H(S) $s_i = \Omega_c\ e^{j\left(\frac{\pi(1+2i+N)}{2N}\right)} = s_i = 8389.5\ e^{j\left(\frac{\pi(3+2i)}{4}\right)}\ i = 0 \cdots 1$
	$s_0 = -5932.3 + j5932.3, s_1 = -5932.3 - j5932.3$
10	$k = \prod_{i=0}^{N-1} (-s_i) \rightarrow (5932.3 - j5932.3)(5932.3 + j5932.3) \rightarrow 7.038\ 4 \times 10^7$
11	$H_a(s) = \frac{K}{\prod_{i=0}^{N-1} (s-s_i)} = \frac{7.038\ 4 \times 10^7}{(s+5932.3-i5932.3)(s+5932.3+i5932.3)} = \frac{7.038\ 4 \times 10^7}{s^2+11865.s+7.038\ 4 \times 10^7}$
12	
13	$H(z) = H_a(s) \Big _{s=\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}} = \frac{7.038\ 4 \times 10^7}{\left(\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 11865\left(\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}\right) + 7.038\ 4 \times 10^7}$
	$= \frac{7.038\ 4 \times 10^7 (T)^2 (1+z^{-1})^2}{2(1-z^{-1})^2 + 2 \times 11865((1-z^{-1})(1+z^{-1})T + 7.038\ 4 \times 10^7 (T^2)(1+z^{-1})^2)}$
	$= \frac{0.09945\ 9 z^2 + 0.198\ 92 z + 0.09945\ 9}{z^2 - 0.931\ 56 z + 0.329\ 38}$

4    **References**

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