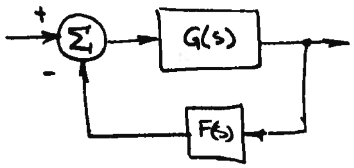
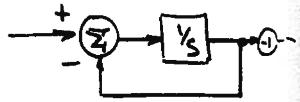
6.9



The transfer function of this system is $H(s) = \frac{G(s)}{1+G(s)H(s)}$

Now $G(s) = \frac{1}{s}$ (an integrator)

And $F(s) = \frac{-1/s}{1+1/s} = \frac{-1}{1+s}$



$$H(s) = \frac{1/s}{1-\frac{1}{s}\cdot\frac{1}{s+1}} = \frac{s+1}{s(s+1)-1}$$

$$= \frac{s+1}{s^2+s-1}\ ; \quad \text{Poles at } \frac{-1\pm\sqrt{1+4}}{2} = -\frac{1}{2}\pm\frac{\sqrt{5}}{2}$$

∴ system unstable because of pole at $\frac{\sqrt{5}-1}{2}$ in right hand half plane.

Check by state variables:

$x_1(t) \triangleq$ output of left integrator

$x_2(t) \triangleq$ output of right integrator

$$\underline{A} = \begin{bmatrix}-1 & 1\\ 1 & 0\end{bmatrix}\ ; \quad |A-\lambda I| = (-\lambda-1)(-\lambda)-1 = \lambda^2+\lambda-1$$

$\lambda_1, \lambda_2 = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. Again system is unstable.

$$H(s) = \frac{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)/\sqrt{5}}{s+\frac{1}{2}-\frac{\sqrt{5}}{2}} + \frac{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)/(-\sqrt{5})}{s+\frac{1}{2}+\frac{\sqrt{5}}{2}}$$

$$h(t) = \frac{\frac{1}{2}+\frac{\sqrt{5}}{2}}{\sqrt{5}} e^{\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)t} + \frac{\frac{1}{2}-\frac{\sqrt{5}}{2}}{-\sqrt{5}} e^{\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)t}$$

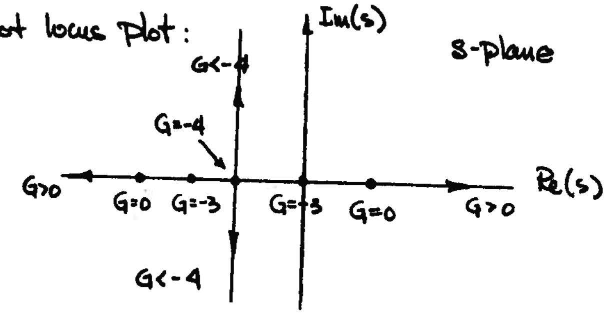
6.10 With $G = 0$, there is a pole at $s=1$ & so system is unstable.

$$H(s) = \frac{G\,G(s)}{1-G\,G(s)} = \frac{\frac{1}{(s-1)(s+3)}}{1-\frac{G}{(s-1)(s+3)}} = \frac{1}{s^2+2s-3-G}$$

Poles at $\frac{-2\pm\sqrt{4-4(-3-G)}}{2} = -1\pm\sqrt{4+G}$

$G=0$: $-1\pm2 = 1\,;-3$	$G=-3$: $-1\pm1 = 0\,;-2$
$G=5$: $-1\pm3 = 2\,;-4$	$G=-4$: $-1\pm0 = -1\,;-1$
$G=12$: $-1\pm4 = 3\,;-5$	$G=-5$: $-1\pm j$
⋮	⋮

Root locus plot:

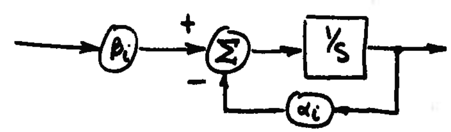


The system is stable for all $G < -3$.

6.11

Parallel form:

one section:



chapter 2

#2.1 (a) $e^{bk} = (e^b)^k$; $\quad L_A = s - e^b$

(b) $B\sinh ak = \frac{B}{2}(e^{ak}+e^{-ak})$; $\quad L_A = (s-e^a)(s-e^{-a})$
$= s^2 - 2\sinh a\, s + 1$

(c) $k^2 a^k + Ae^{bk}$; $\quad L_A = (s-a)^3(s-e^b)$

(d) $ka^k + A\sin bk$; $\quad L_A = (s-a)^2(s-e^{jb})(s-e^{-jb})$

#2.2 (a) $y_{k+2} + 7y_{k+1} + 12y_k = 0$

Char eqn: $r^2+7r+12 = 0$

ie $(r+4)(r+3) = 0 \quad \begin{cases} r_1 = -3 \\ r_2 = -4 \end{cases}$

$$y_k = c_1(-4)^k + c_2(-3)^k$$

(b) $y_{k+2} + 2y_{k+1} + 2y_k = 0$

Char eqn: $r^2+2r+2 = 0 \quad \begin{cases} r_1 = 1+\sqrt{1-2} = 1+j \\ r_2 = 1-j = r_1^* \end{cases}$

$$\begin{aligned} y_k &= c_1(1+j)^k + c_2(1-j)^k \\ &= c_1\left(\sqrt{2}e^{j\pi/4}\right)^k + c_2\left(\sqrt{2}e^{-j\pi/4}\right)^k \\ &= c_1 2^{k/2} e^{j\pi k/4} + c_2 2^{k/2} e^{-j\pi k/4} \\ &= \hat{c}_1 2^{k/2}\cos \pi k/4 + \hat{c}_2 2^{k/2} \sin \pi k/4 \end{aligned}$$

(4 equivalent forms)

#2.2 (c) $y_{k+2} + y_k = \sin k$

Since $\sin k = \frac{e^{jk}+e^{-jk}}{j2}$,

Choose $L_A = (s-e^j)(s-e^{-j})$

Then $(s-e^j)(s-e^{-j})(s^2+1) = 0$

The characteristic equation is

$$(r-e^j)(r-e^{-j})(r-e^{j\pi/2})(r-e^{-j\pi/2}) = 0$$
$$(\text{where } e^{\pm j\pi/2} = \pm j)$$

Thus $y_k = c_1 \cos \frac{\pi k}{2} + c_2 \sin \frac{\pi k}{2} + c_3 \cos k + c_4 \sin k$

The constants c_3 and c_4 are found from

$(s^2+1)(c_3\cos k + c_4 \sin k) \equiv$

ie $c_3[\cos(k+2) + \cos k] + c_4[\sin(k+2) + \sin k] \equiv$

$c_3 \cos k \cos 2 - c_3 \sin k \sin 2 + c_3 \cos k$
$+ c_4 \sin k \cos 2 - \cos k \sin 2 + c_4 \sin k \equiv$

ie $\cos k\,(c_3\cos 2 + c_3 - c_4 \sin 2)$
$+ \sin k\,(-c_3 \sin 2 + c_4 \cos 2 + c_4) \equiv$

Thus $c_3(1+\cos 2) - c_4 \sin 2 = 0$
$c_3(-\sin 2) + c_4(1+\cos 2) = 1$

#2.2 (c) (cont)

$$c_3 = \frac{\begin{vmatrix} 0 & -\sin 2 \\ 1 & 1+\cos 2 \end{vmatrix}}{\begin{vmatrix} 1+\cos 2 & -\sin 2 \\ -\sin 2 & 1+\cos 2 \end{vmatrix}} = \frac{\sin 2}{1+2\cos 2+\cos^2 2-\sin^2 2}$$

$$= \frac{\sin 2}{2\cos 2(1+\cos 2)}$$

$$= \frac{\tan 2}{2(1+\cos 2)}$$

$$c_4 = \frac{\begin{vmatrix} 1+\cos 2 & 0 \\ -\sin 2 & 1 \end{vmatrix}}{2\cos 2(1+\cos 2)} = \frac{1}{2\cos 2}$$

soln:

$$y_k = c_1 \cos\frac{\pi k}{2} + c_2 \sin\frac{\pi k}{2} + \frac{\tan 2}{2(1+\cos 2)}\cos k + \frac{\sin k}{2\cos 2}$$

(d) $y_{k+2} - \frac{5}{2} y_{k+1} + y_k = 1$

use $L_A = S-1$ to annihilate the constant

Thus $(S^2 - \frac{5}{2}S + 1)(S-1) = 0$

Char eqn $(r^2 - \frac{5}{2}r + 1)(r-1) = 0$

$(r-\frac{1}{2})(r-2)(r-1) = 0$ $\left\{ \begin{array}{l} r_1 = 1/2 \\ r_2 = 2 \\ r_3 = 1 \end{array} \right.$

$$y_k = c_1(1/2)^k + c_2(2)^k + c_3$$

2.2(d) (cont)

c_3 is found from $L[y_k] = 1$:

$(S^2 - \frac{5}{2}S + 1)\, c_3 = c_3(1 - \frac{5}{2} + 1) = c_3(-1/2) \equiv 1$

$\Rightarrow c_3 = -2$

Thus $y_k = c_1(1/2)^k + c_2 2^k - 2$

with $y_0 = y_1 = 0$

$$\left.\begin{array}{l} y_0 = c_1 + c_2 - 2 \equiv 0 \\ y_1 = \frac{1}{2}c_1 + 2c_2 - 2 \equiv 0 \end{array}\right\} \quad \begin{array}{l} c_1 = 4/3 \\ c_2 = 2/3 \end{array}$$

Soln: $$y_k = \frac{4}{3}(1/2)^k + \frac{2}{3}2^k - 2$$

*2.3 $y_{k+2} - 2\tau y_{k+1} + y_k = 0$

Find solutions for above as τ varies. The auxiliary equation is: $r^2 - 2\tau r + 1 = 0$ with roots $r_1, r_2 = \tau \pm \sqrt{\tau^2 - 1}$.

(a) $\tau < -1$: The roots are distinct and real

$$y_k = c_1(\tau + \sqrt{\tau^2-1})^k + c_2(\tau - \sqrt{\tau^2-1})^k$$

(b) $\tau = -1$: The roots are repeated : $r_1 = r_2 = -1$

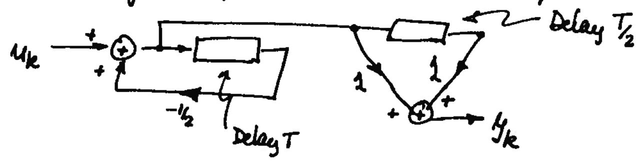
$$y_k = c_1(-1)^k + c_2 k(-1)^k$$
$$= (c_1 + c_2 k)(-1)^k$$

(2.12 cont.)

by convolving the sequences

$$\{1, 0, 1/2, 0, 1/4, 0, 1/8, 0, \cdots\} * \{1, 1, 0, 0, \cdots\}$$

We can preform this convolution by convolving $\{1, 1/2, 1/4, 1/8, \cdots\}$ with $\{1, 1, 0, 0, \cdots\}$ in which the "clock" of the sequence $\{1, 1, 0, 0, \cdots\}$ runs twice the rate as the clock of the sequence $(1/2)^k$. Thus a system is:



#2.13

$$y_{k+2} - y_{k+1} + \tfrac{1}{4} y_k = u_k$$

$$r^2 - r + 1/4 = 0 \Rightarrow r_1, r_2 = 1/2 \Rightarrow h_k = c_1(1/2)^k + c_2 k (1/2)^k$$

Initial Conditions: $h_{k+2} - h_{k+1} + \tfrac{1}{4} h_k = \delta_k$

$$\therefore h_0 = 0,\ h_1 = 0,\ h_2 = 1,\ h_3 = \delta_1 + h_2 - \tfrac{1}{4} h_1 = 1$$

Using h_2 and h_3 (h_0 and h_1 are special cases) we have

$$\left.\begin{aligned} h_2 &= c_1(1/2)^2 + c_2(2)(1/2)^2 = \tfrac{1}{4}c_1 + \tfrac{1}{2}c_2 = 1 \\ h_3 &= c_1(1/2)^3 + c_2(3)(1/2)^3 = \tfrac{1}{8}c_1 + \tfrac{3}{8}c_2 = 1 \end{aligned}\right\} \begin{aligned} c_1 &= -4 \\ c_2 &= 4 \end{aligned}$$

$$\therefore h_k = \begin{cases} -4(1/2)^k + 4k(1/2)^k, & k \geq 2 \\ 0, & k < 2 \end{cases} \quad \leftarrow$$

(2.13 cont.)

check: $(S^2 - S + 1/4)\, h_k = \left[4(k+2-1)\tfrac{1}{4}(1/2)^k\right] \xi_{k+1}$

$$- 4\left[(k+1-1)(1/2)(1/2)^k\right] \xi_k + \left[(k-1)(1/2)^k\right] \xi_{k-1}$$

$$= 0\cdot\delta_{k+1} + 1\cdot\delta_k + 0\cdot\delta_k + (k+1-2k+k-1)(1/2)^k \xi_{k-1}$$

$$= \delta_k \qquad \text{Recall: } \xi_{k-a} = \begin{cases} 1, & k \geq a \\ 0, & k < a \end{cases}.$$

(b) $(S^2 - \tfrac{1}{4}) y_k = u_k$

From $r^2 - 1/4 = (r - 1/2)(r + 1/2) = 0$

$$r_1 = 1/2, \quad r_2 = -1/2$$

Thus $h_k = c_1(1/2)^k + c_2(-1/2)^k$

The initial conditions are:

$$h_{k+2} - \tfrac{1}{4} h_k = \delta_k \Rightarrow \begin{cases} h_0 = 0 \\ h_1 = 0 \\ h_2 = 1 \end{cases}$$

Thus

$$\left.\begin{aligned} c_1(1/2) + c_2(-1/2) &= 0 \\ c_1(1/4) + c_2(1/4) &= 1 \end{aligned}\right\} c_1 = c_2 = 2$$

And so, $h_k = \begin{cases} [1 + (-1)^k](1/2)^{k-1}, & k \geq 1 \\ 0, & k < 1 \end{cases}$

(2.13 cont.)

check: $(S^2 - 1/4)\, h_k = [1+(-1)^{k+2}](1/2)^{k+1}\, \xi_{k+1}$

$$- [1+(-1)^{k}](1/2)^{k-1}\, \xi_{k-1}$$

$$= 0\cdot\delta_{k+1} + 1\cdot\delta_k + 0 \quad , k \ge 1$$

$$= \delta_k .$$

(c) $y_k = u_k + 3y_{k-1} - 3y_{k-2} + y_{k-3}$

From $r^3 - 3r^2 + 3r - 1 = 0$

$(r-1)^3 = 0 \Rightarrow r_1 = r_2 = r_3 = 1$

$\Rightarrow h_k = c_1 + c_2 k + c_3 k^2$

with initial conditions: $h_{-2} = 0,\ h_{-1} = 0,\ h_0 = 1$

Thus
$$\left.\begin{aligned} c_1 - 2c_2 + 4c_3 &= 0 \\ c_1 - c_2 + c_3 &= 0 \\ c_1 &= 1 \end{aligned}\right\} \Rightarrow c_1 = 1,\ c_2 = 3/2,\ c_3 = 1/2$$

$\Rightarrow h_k = 1 + \frac{3}{2}k + \frac{1}{2}k^2, \quad k \ge 0$ (also for $-1,-2$)

check: $(1 - 3S^{-1} + 3S^{-2} - S^{-3})\, h_k = (1 + \frac{3}{2}k + \frac{1}{2}k^2)\, \xi_{k+2}$

$$+ \left[-3 - \frac{9}{2}(k-1) - \frac{3}{2}(k-1)^2\right] \xi_{k+1}$$

(cont)

(2.13 cont)

$$+ \left[+3 + \frac{9}{2}(k-2) + \frac{3}{2}(k-2)^2\right] \xi_k$$

$$+ \left[-1 - \frac{3}{2}(k-3) - \frac{1}{2}(k-3)^2\right] \xi_{k-1}$$

Thus $(1 - 3S^{-1} + 3S^{-2} - S^{-3})\, h_k =$

$$\left[0\cdot\delta_{k-2} + 0\cdot\delta_{k-1} + \delta_k + (1 + \tfrac{3}{2}k + \tfrac{1}{2}k^2)\right] \xi_{k-1}$$

$$+ \left[0\cdot\delta_{k-1} + 0\cdot\delta_k + (-3 - \tfrac{9}{2}k + \tfrac{9}{2} - \tfrac{3}{2}k^2 + 3k - \tfrac{3}{2})\right] \xi_{k-1}$$

$$+ \left[0\cdot\delta_k + (3 + \tfrac{9}{2}k - 9 + \tfrac{3}{2}k^2 - 6k + 6)\right] \xi_{k-1}$$

$$+ \left[-1 - \tfrac{3}{2}k + \tfrac{9}{2} - \tfrac{1}{2}k^2 + 3k - \tfrac{9}{2}\right] \xi_{k-1}$$

$$= \delta_k + \Big\{ (1 - 3 + \tfrac{9}{2} - \tfrac{3}{2} + 3 - 9 + 6 - 1 + \tfrac{9}{2} - \tfrac{9}{2})$$

$$+ k\,(\tfrac{3}{2} - \tfrac{9}{2} + 3 + \tfrac{9}{2} - 6 - \tfrac{3}{2} + 3)$$

$$+ k^2\,(\tfrac{1}{2} - \tfrac{3}{2} + \tfrac{3}{2} - \tfrac{1}{2}) \Big\}$$

$$= \delta_k$$

(d) $(1 - 3S^{-1} + 3S^{-2} - S^{-3})\, y_k = S^{-3} u_k$

From (c) $h_k = L_0(\hat{h}_k) = S^{-3}(1 + \frac{3}{2}k + \frac{1}{2}k^2) = (1 - \frac{3}{2}k + \frac{1}{2}k^2)\cdot \xi_{k-3}$