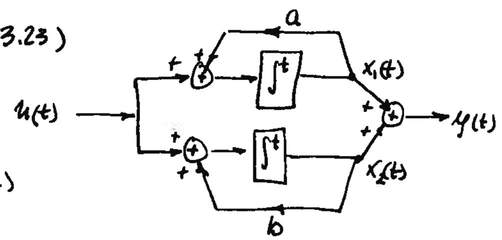
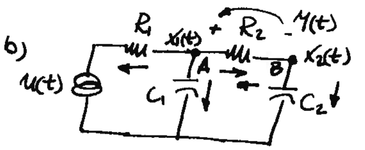
(3.23)

(a)



$$x_1'(t) = a x_1(t) + u(t)$$
$$x_2'(t) = b x_2(t) + u(t) \quad \Rightarrow \quad A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$y(t) = x_1(t) + x_2(t) \qquad C = [1 \quad 1], \quad D = 0.$$

(b)



$x_1(t)$ and $x_2(t)$ are the voltages across the capacitors C_1 and C_2, respectively.

Thus writing node eqs we obtain:

$$\frac{x_1(t) - u(t)}{R_1} + \frac{x_1(t) - x_2(t)}{R_2} + x_1'(t) C_1 = 0$$

$$\frac{x_2(t) - x_1(t)}{R_2} + x_2'(t) C_2 = 0$$

$$y(t) = x_1(t) - x_2(t)$$

Thus

$$x_1'(t) = \frac{1}{C_1}\left(\frac{-1}{R_1} - \frac{1}{R_2}\right) x_1(t) + \frac{1}{C_1 R_2} x_2(t) + \frac{1}{C_1 R_1} u(t)$$

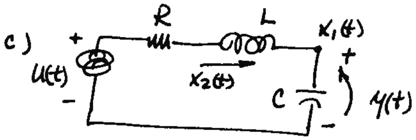
$$x_2'(t) = \frac{1}{C_2} \frac{1}{R_2} x_1(t) - \frac{1}{C_2 R_2} x_2(t)$$

$$y(t) = x_1(t) - x_2(t)$$

(3.23 cont.)

$$A = \begin{bmatrix} -\frac{1}{C_1 R_1} - \frac{1}{C_1 R_2} & \frac{1}{C_1 R_2} \\ \frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{C_1 R_1} \\ 0 \end{bmatrix} \quad C = [1 \quad -1]$$
$$D = 0$$

(c)



$x_2(t)$ is the current through L and $x_1(t)$ is the voltage across C.

Thus we have

$$R x_2(t) + L x_2'(t) + x_1(t) = u(t) \quad \text{(loop equation)}$$

$$C x_1'(t) = x_2(t) \quad \text{(node equation)}$$

$$y(t) = x_1(t)$$

Thus

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}, \quad C = [1 \quad 0], \quad D = 0.$$

(3.24) For a system with system matrix A, the system is stable iff $g(\lambda) = \det(A - \lambda I) = 0$ has roots λ_i all with real parts < 0 (negative).

(a) $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \Rightarrow \lambda_1 = a, \ \lambda_2 = b$

$\therefore$ system stable iff $a < 0$, $b < 0$ (assuming a, b real)

(3.27 cont.)

$$x_1'(t) = 2x_1(t) + 9x_2(t)$$
$$x_2'(t) = -x_1(t) - 4x_2(t) + u(t)$$
$$x_3'(t) = -2x_3(t) + u(t)$$

$$A = \begin{bmatrix} 2 & 9 & 0 \\ -1 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = [0\ 1\ 1], \quad D = 0$$

$$g(\lambda) = \det(A - \lambda I) = 0 \Rightarrow \lambda_1 = -2,\ \lambda_2 = -1,\ \lambda_3 = -1.$$

Because we have one repeated root A is of the form:

$$A = \lambda_1 E_1 + \lambda_2 E_2 + N_2$$

There are several methods one can use to find E_1, E_2, N_2. We shall use a method which is not covered in the text but is straightforward and useful to know.

Consider the function of a matrix $f(A) = (sI - A)^{-1}$. Then we have that, from (2.85),

$$f(A) = f(\lambda_1)E_1 + f(\lambda_2)E_2 + f'(\lambda_2)N_2$$

where $f(\lambda_i) = \frac{1}{s-\lambda_i} \Rightarrow f'(\lambda_i) = \frac{1}{(s-\lambda_i)^2}$

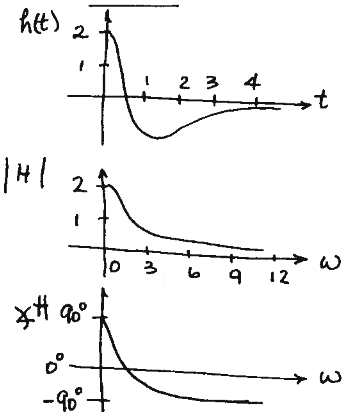
Thus

$$(sI-A)^{-1} = \frac{1}{s-\lambda_1}E_1 + \frac{1}{s-\lambda_2}E_2 + \frac{1}{(s-\lambda_2)^2}N_2$$

Now $(sI-A)^{-1} = \begin{bmatrix} s-2 & -9 & 0 \\ 1 & s+4 & 0 \\ 0 & 0 & s+2 \end{bmatrix}^{-1}$

(3.27 cont.)

Sketches:



(b) choose $x_1(t)$ as the output of right most integrator and $x_2(t)$ as the output of the left most integrator. Then we have

$$A = \begin{bmatrix} 0 & 1 \\ -26 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [29\ \ 0], \quad D = 0$$

Again compute $(sI-A)^{-1}$ and expand in partial fractions. The "coefficients" of the partial fraction terms are matrices, E_1 & E_2.

$$g(\lambda) = \det(A - \lambda I) = 0 \Rightarrow \lambda_1 = -1-5j,\ \lambda_2 = -1+5j$$

$$(sI-A)^{-1} = \begin{bmatrix} s & -1 \\ 26 & s+2 \end{bmatrix}^{-1} = \frac{1}{s^2+2s+26}\begin{bmatrix} s+2 & 1 \\ -26 & s \end{bmatrix}$$

$$= \frac{1}{s-(1+5j)}\begin{bmatrix} \frac{5-j}{10} & \frac{-j}{10} \\ \frac{26j}{10} & \frac{5+j}{10} \end{bmatrix} + \frac{1}{s-(1-5j)}\begin{bmatrix} \frac{5+j}{10} & j/10 \\ \frac{-26j}{10} & \frac{5-j}{10} \end{bmatrix}$$

$$= \frac{1}{s-\lambda_1}E_1 + \frac{1}{s-\lambda_2}E_2.$$

(3.27 cont)

Note: $E_1 = E_2^*$, $E_1 + E_2 = I$, $E_1 E_2 = 0$

Thus $e^{At} = e^{\lambda_1 t}E_1 + e^{\lambda_2 t}E_2 = e^{-t}\begin{bmatrix} \cos 5t + \frac{\sin 5t}{5} & \frac{\sin 5t}{5} \\ -\frac{26 \sin 5t}{5} & \cos 5t - \frac{\sin 5t}{5} \end{bmatrix}$

Impulse Response:

$$h(t) = Ce^{At}B = [29 \;\; 0]\, e^{At}\begin{bmatrix}0\\1\end{bmatrix} = \frac{29\, e^{-t}\sin 5t}{5}\; \xi(t)$$

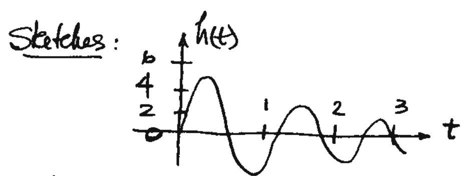
Transfer function:

$$H(j\omega) = D + C(j\omega I - A)^{-1}B = \frac{29}{(j\omega)^2 + 2j\omega + 26}$$

$\begin{cases} \text{poles: } 1 \pm 5j \\ \text{zeros: } 2 \text{ at } \infty \end{cases}$

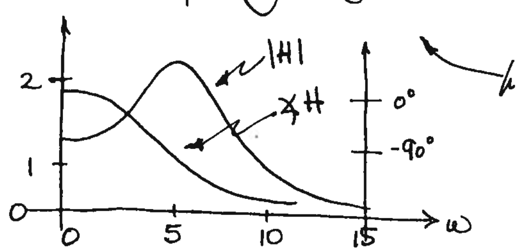
Frequency response: $|H(j\omega)| = \dfrac{29}{\sqrt{26-\omega^2+4\omega^2}}$

Sketches:



t	$\frac{29}{5}e^{-t}\sin 5t$
.1	2.52
.2	3.99
.3	4.28
.4	3.54
.5	2.11
.7	−1.01
1.0	−2.05
1.2	−.488
1.4	.94
1.6	1.16

etc.



(3.28) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [c_1, c_2]$ $D = 0$

Note: In the first printing there is an error in the sign of the a_{22} entry in A. It should be −3, not 3. If 3 is used the eigen values are 1 and 2 implying the system is unstable. If −3 is used, we obtain

$$g(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = 3\lambda + \lambda^2 + 2 = 0$$

$\Rightarrow \lambda_1 = -1,\; \lambda_2 = -2.$

Then $A = -E_1 - 2E_2$ where we can obtain E_1 & E_2 via

$$(sI - A)^{-1} = \frac{1}{s+1}E_1 + \frac{1}{s+2}E_2 = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

or $E_1 = I - E_2$ and $E_2 = -(I + A) = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$

$\Rightarrow E_1 = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$. Thus

$$e^{At} = e^{-t}\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + e^{-2t}\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

Now $H(j\omega) = D + C(j\omega I - A)^{-1}B$

$= D + [c_1 \;\; c_2]\begin{bmatrix} 1 \\ j\omega \end{bmatrix}\dfrac{1}{(j\omega+1)(j\omega+2)}$

$$= \frac{d\left((j\omega)^2 + 3j\omega + 2\right) + c_1 + c_2 j\omega}{(j\omega+1)(j\omega+2)} \qquad (1)$$

(3.28 cont.)

From the graph we have a DC gain of unity and two zeros at $\omega = \pm 1$. Thus

$$H(j\omega) = \frac{2\left((j\omega)^2 + 1\right)}{(j\omega+1)(j\omega+2)} \qquad (2)$$

At $j\omega = 0$, $H(0) = 1$ giving the DC gain. The term $(j\omega)^2 + 1$ gives us the zeros at $\omega = \pm 1$. Equating coefficients of like powers of $(j\omega)$ in (1) and (2) in the numerators gives:

$$\left.\begin{array}{ll} (j\omega)^0 : & 2d + c_1 = 2 \\ (j\omega)^1 : & 3d + c_2 = 0 \\ (j\omega)^2 : & d = 2 \end{array}\right\} \Rightarrow \begin{array}{l} d = 2 \\ c_1 = -2 \\ c_2 = -6 \end{array}$$

(3.29) There are (at least) three possible approaches:

(a) Use state variable methods or classical methods to solve for $w(t)$ in

$$b_n \frac{d^n}{dt^n} w(t) + \cdots + b_1 \frac{dw(t)}{dt} + b_0 w(t) = u(t)$$

Then operate on $w(t)$ to obtain $y(t)$ using superposition

$$y(t) = L_D[w(t)] = [a_0 + a_1 D + \cdots + a_m D^m]\, w(t)$$

$$= a_0 w(t) + a_1 \frac{d}{dt} w(t) + \cdots + a_m \frac{d^m}{dt^m} w(t)$$

(3.29 cont)

(b) If $m \le n$, we can integrate both sides of the given equation n times to obtain an equation for $y(t)$ in terms of integrals of $y(t)$ and $u(t)$, with no derivatives present:

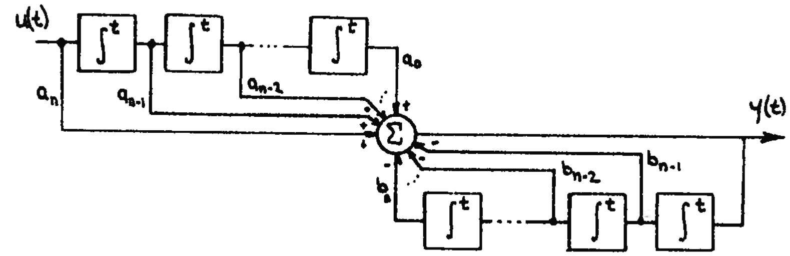
$$b_n y(t) + b_{n-1} \int^t y(t')\,dt' + \cdots + b_0 \int^t\!\!\int^{t_{n-1}}\!\!\cdots\int^{t_2} y(t_1)\,dt_1 \ldots dt$$

$$= a_0 \int^t\!\!\int^{t_{n-1}}\!\!\cdots\int^{t_2} u(t_1)\,dt_1 \ldots dt_n + \cdots + a_m \underbrace{\int^t \!\!\cdots \int^{t_2}}_{n-m \text{ fold}} u(t_1)\,dt_1 \cdots dt$$

This system is sketched below, assuming that $n = m$ and with b_n normalized to 1 by dividing through the equation by b_n. If $m < n$, the coefficients a_k below with $k > m$ will be zero.

$$\left[1 + b_{n-1} D^{-1} + \cdots + b_1 D^{-n+1} + b_0 D^{-n}\right] y(t)$$

$$= \left[a_0 D^{-n} + a_1 D^{-n+1} + \cdots + a_n\right] u(t)$$



This system can now be solved using state variable methods. Note that the A matrix has dimension $2n \times 2n$.

(c) An equivalent n-integral system is shown in the block diagram below. Again, state variables may be used to solve for the output $y(t)$, here with only an $n \times n$ A-matrix. Using Laplace transforms, one can readily establish the equivalence of these two block diagrams.

(3.24 cont)

$$A = \begin{bmatrix} -\frac{1}{C_1R_1} - \frac{1}{C_2R_2} & \frac{1}{C_1R_2} \\ \frac{1}{C_2R_2} & -\frac{1}{C_2R_2} \end{bmatrix}$$

$$g(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & \frac{1}{C_1R_2} \\ \frac{1}{C_2R_2} & b-\lambda \end{bmatrix} = (a-\lambda)(b-\lambda) + c = 0$$

where $a = -\frac{1}{C_1R_1} - \frac{1}{C_2R_2}$, $b = -\frac{1}{C_2R_2}$, $c = -\frac{1}{C_1C_2R_1R_2}$

and we know that a, b, c are all < 0.

Now $g(\lambda) = \lambda^2 - (a+b)\lambda + ab + c = 0$

Further let $ab + c = \beta$. Then $\beta = \left(-\frac{1}{C_2R_2}\right)\left(-\frac{1}{C_1R_1} - \frac{1}{C_2R_2}\right) - \frac{1}{C_1C_2R_1R_2}$

$$= \left(\frac{1}{C_2R_2}\right)^2 > 0$$

Let $-(a+b) = \alpha$; then $\alpha > 0$.

Thus

$$\lambda_1, \lambda_2 = -\frac{\alpha}{2} \pm \frac{\sqrt{\alpha^2 - 4\beta}}{2}$$

But $\beta > 0 \Rightarrow -4\beta < 0 \Rightarrow (\alpha^2 - 4\beta)^{1/2} < \alpha$

∴ System is always stable for any values of R_1, R_2, C_1, C_2 which are > 0. (This result is, of course, clear from the structure of the circuit.)

(c)

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ -\beta & -\gamma \end{bmatrix} \text{ with } \alpha, \beta, \gamma > 0.$$

(3.24 cont)

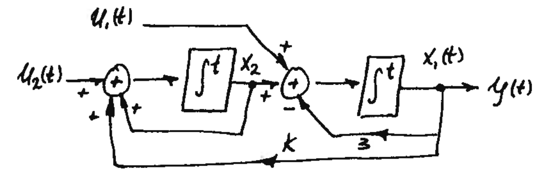
$$g(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & \alpha \\ -\beta & -\lambda-\gamma \end{pmatrix} = \lambda^2 + \lambda\gamma + \alpha\beta = 0$$

$$\therefore \quad \lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\alpha\beta}}{2}$$

But $\alpha\beta > 0 \Rightarrow -4\alpha\beta < 0 \Rightarrow (\gamma^2 - 4\alpha\beta)^{1/2} < \gamma$

∴ This system is always stable for any values of $R, L, C > 0$.

(3.25)



(a)

$$x_1'(t) = 3x_1(t) + x_2(t) + u_1(t)$$
$$x_2'(t) = k x_1(t) + x_2(t) + u_2(t)$$
$$y(t) = x_1(t)$$

$$A = \begin{bmatrix} 3 & 1 \\ k & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For stability the real part of the eigenvalues of A must be negative.

$$g(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ k & 1-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + (3-k) = 0$$

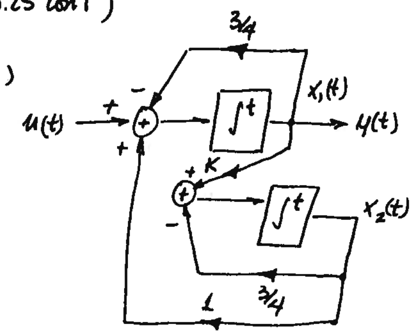
$$\lambda_1, \lambda_2 = \frac{4 \pm \sqrt{16 - 4(3-k)}}{2} = 2 \pm \sqrt{1+k}$$

$$\left.\begin{aligned} \lambda_1 &= 2 + \sqrt{1+k} \\ \lambda_2 &= 2 - \sqrt{1+k} \end{aligned}\right\} \Rightarrow \lambda_1 \text{ can never have a real part} <$$

System is never stable.

(3.25 cont)

(b)



$$\left.\begin{aligned} x_1'(t) &= -\tfrac{3}{4}x_1(t) + x_2(t) + u(t) \\ x_2'(t) &= K x_1(t) - \tfrac{3}{4}x_2(t) \end{aligned}\right\} \quad A = \begin{bmatrix} -3/4 & 1 \\ K & -3/4 \end{bmatrix}$$

$$g(\lambda) = \det\begin{pmatrix} -\tfrac{3}{4}-\lambda & 1 \\ K & -\tfrac{3}{4}-\lambda \end{pmatrix} = \lambda^2 + \tfrac{3}{2}\lambda + \tfrac{9}{16} - K = 0$$

$$\lambda_1, \lambda_2 = \frac{-3/2 \pm \sqrt{(3/2)^2 - 4(\frac{9}{16}-K)}}{2} = -\tfrac{3}{4} \pm \sqrt{K}$$

$$\left.\begin{aligned} \lambda_1 &= -3/4 + \sqrt{K} \\ \lambda_2 &= -3/4 - \sqrt{K} \end{aligned}\right\} \quad \therefore \text{ For stability } \sqrt{K} - 3/4 < 0$$

$$\Rightarrow K < \tfrac{9}{16}.$$

(3.26) In general, we have:

$$A = \sum_{i=1}^{n} \lambda_i E_i \quad \& \quad f(A) = \sum_{i=1}^{n} f(\lambda_i) E_i \quad \text{for distinct } \lambda_i.$$

Thus $f(A) = e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} E_i$

where E_i can be obtained via: $E_1 = \dfrac{A-\lambda_2 I}{\lambda_1-\lambda_2}$, $E_2 = \dfrac{A-\lambda_1 I}{\lambda_2-\lambda_1}$

(for $n = 2$)

(3.26 cont)

(a) $A = \begin{bmatrix} 3/4 & 0 \\ 1/2 & 1/2 \end{bmatrix}$ $g(\lambda) = \det(A-\lambda I) = 0 \Rightarrow \lambda_1 = 1/2,\ \lambda_2 = 3/4$

Thus $E_1 = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ (Note: $E_1 + E_2 = I$)

$$\therefore \quad e^{At} = e^{t/2}\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} + e^{\frac{3}{4}t}\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} e^{3t/4} & 0 \\ 2e^{3t/4} - 2e^{t/2} & e^{t/2} \end{bmatrix}.$$

(b) $A = \begin{bmatrix} 1/2 & 1/4 \\ \frac{1}{16} & 1/2 \end{bmatrix}$, $g(\lambda) = 0 \Rightarrow \lambda_1 = 3/8,\ \lambda_2 = 5/8$

$E_1 = \begin{bmatrix} 1/2 & -1 \\ -1/4 & 1/2 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1/2 & 1 \\ 1/4 & 1/2 \end{bmatrix}$

$$\therefore \ e^{At} = \begin{bmatrix} \frac{1}{2}(e^{3t/8} + e^{5t/8}) & e^{\frac{5t}{8}} - e^{3t/8} \\ \frac{1}{4}(e^{5t/8} - e^{3t/8}) & \frac{1}{2}(e^{3t/8} + e^{5t/8}) \end{bmatrix}$$

(c) $A = \begin{bmatrix} 1/2 & 1/4 \\ 1 & 1/2 \end{bmatrix}$, $g(\lambda) = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = 1$

Thus $E_1 = \begin{bmatrix} 1/2 & 1/4 \\ -1 & 1/2 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1/2 & -1/4 \\ 1 & 1/2 \end{bmatrix}$

$$\therefore \ e^{At} = \begin{bmatrix} \frac{1}{2}(1+e^t) & \frac{1}{4}(1-e^t) \\ e^t - 1 & \frac{1}{2}(1+e^t) \end{bmatrix}$$

(d) $A = \begin{bmatrix} 1/2 & 0 \\ 1 & 1/2 \end{bmatrix}$, $g(\lambda) = 0 \Rightarrow \lambda_1 = 1/2,\ \lambda_2 = 1/2$

In the case of repeated roots $A = \lambda E_1 + N_1$