(2.19 cont)

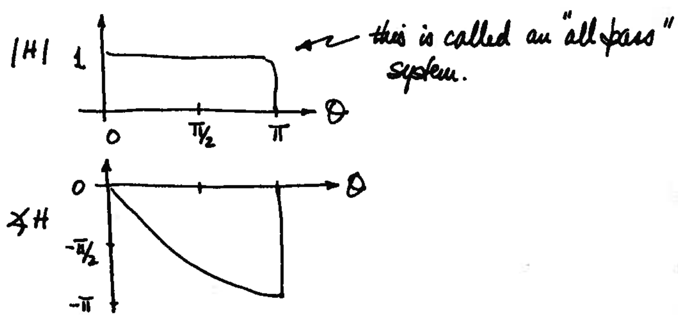
$$H(e^{j\theta}) = \frac{-\frac{1}{2} + e^{-j\theta}}{1 - \frac{1}{2}e^{-j\theta}} = \frac{-\frac{1}{2}e^{j\theta} + 1}{e^{j\theta} - \frac{1}{2}}, \quad \theta = \omega T$$

Thus
$$|H(e^{j\theta})| = \frac{|-\frac{1}{2}e^{j\theta} + 1|}{|e^{j\theta} - \frac{1}{2}|} = \frac{\left[(-\frac{1}{2}\cos\theta + 1)^2 + (-\frac{1}{2}\sin\theta)^2\right]^{1/2}}{\left[(\cos\theta - \frac{1}{2})^2 + \sin^2\theta\right]^{1/2}}$$

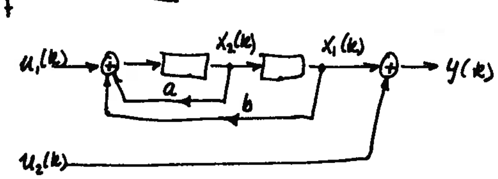
$$= \frac{\left[\frac{1}{4}\cos^2\theta - \cos\theta + 1 + \frac{1}{4}\sin^2\theta\right]^{1/2}}{\left[\cos^2\theta - \cos\theta + \frac{1}{4} + \sin^2\theta\right]^{1/2}} = \frac{(\frac{5}{4} - \cos\theta)^{1/2}}{(\frac{5}{4} - \cos\theta)^{1/2}} = 1, \quad \forall\theta$$

And
$$\measuredangle H(e^{j\theta}) = \measuredangle(-\tfrac{1}{2}e^{j\theta} + 1) - \measuredangle(e^{j\theta} - \tfrac{1}{2})$$

Sketch:



#2.20



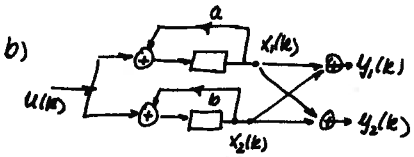
(2.20 cont)

From the block diagram we have:

$$\left.\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= b x_1(k) + a x_2(k) + u_1(k) \\ y(k) &= x_1(k) + u_2(k) \end{aligned}\right\} \quad \underline{x}(k+1) = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}\underline{x}(k) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\underline{u}$$

$$y(k) = [1 \; 0]\,\underline{x}(k) + [0 \; 1]\,\underline{u}(k)$$

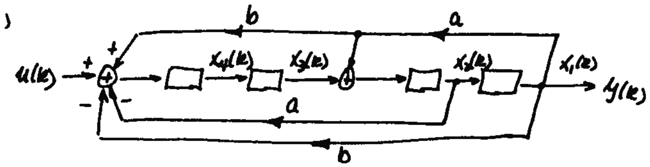
(b)



$$\left\{\begin{aligned} x_1(k+1) &= a x_1(k) + u(k) \\ x_2(k+1) &= b x_2(k) + u(k) \\ y_1(k) &= x_1(k) + x_2(k) \\ y_2(k) &= x_1(k) + x_2(k) \end{aligned}\right.$$

And so,
$$\underline{x}(k+1) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\underline{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(k)$$
$$y(k) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\underline{x}(k)$$

(c)



$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= a x_1(k) + x_3(k) \\ x_3(k+1) &= x_4(k) \\ x_4(k+1) &= -b x_1(k) - a x_2(k) + ab x_1(k) + u(k) \\ y(k) &= x_1(k) \end{aligned}$$

And so,
$$\underline{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ ab-b & -a & 0 & 0 \end{bmatrix}\underline{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}u(k)$$
$$y(k) = [1 \; 0 \; 0 \; 0]\,\underline{x}(k)$$

(2.23 cont)

Thus $E_1 = \begin{bmatrix} \frac{a_{11}-\alpha+\beta}{2\beta} & \frac{a_{12}}{2\beta} \\ \frac{a_{21}}{2\beta} & \frac{a_{22}-\alpha+\beta}{2\beta} \end{bmatrix}$, $E_2 = \begin{bmatrix} \frac{a_{11}-\alpha-\beta}{-2\beta} & \frac{a_{12}}{-2\beta} \\ \frac{a_{21}}{-2\beta} & \frac{a_{22}-\alpha-\beta}{-2\beta} \end{bmatrix}$

Thus $A^k = (\alpha+\beta)^k E_1 + (\alpha-\beta)^k E_2$ (in general)

(a) $A = \begin{bmatrix} 3/4 & 0 \\ 1/2 & 1/2 \end{bmatrix}$, $g(\lambda) = \det\begin{bmatrix} 3/4-\lambda & 0 \\ 1/2 & 1/2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda_1 = 3/4,\ \lambda_2 = 1/2$

$A = 3/4\, E_1 + 1/2\, E_2 \qquad E_1 = \frac{A-\lambda_2 I}{\lambda_1-\lambda_2} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \qquad E_2 = \frac{A-\lambda_1 I}{\lambda_2-\lambda_1} = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$

$\therefore A^k = (3/4)^k \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + (1/2)^k \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1/2 & 1/4 \\ 1/16 & 1/2 \end{bmatrix}$, $g(\lambda) = \det\begin{bmatrix} 1/2-\lambda & 1/4 \\ \frac{1}{16} & 1/2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda_1 = \frac{1}{2} + \frac{1}{\sqrt{48}},\ \lambda_2 = \frac{1}{2} - \frac{1}{\sqrt{48}}$

Let $\alpha = 1/2$, $\beta = \frac{1}{\sqrt{48}}$

Using general result at top of page we have:

$A^k = \left(\frac{1}{2} + \frac{1}{\sqrt{48}}\right)^k \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \frac{\sqrt{3}}{4} & 1 \end{bmatrix} + \left(\frac{1}{2} - \frac{1}{\sqrt{48}}\right)^k \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ -\frac{\sqrt{3}}{4} & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 1/2 \end{bmatrix}$, $g(\lambda) = \det\begin{bmatrix} 1/2-\lambda & 1/2 \\ 1 & 1/2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda_1 = \frac{1}{2} + \frac{\sqrt{2}}{2},\ \lambda_2 = \frac{1}{2} - \frac{\sqrt{2}}{2}$

$\Rightarrow \alpha = 1/2 \quad \beta = \frac{\sqrt{2}}{2}$

then $A^k = \left(\frac{1}{2} + \frac{\sqrt{2}}{2}\right)^k \begin{bmatrix} 1/2 & \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1/2 \end{bmatrix} + \left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right)^k \begin{bmatrix} 1/2 & \frac{-1}{2\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 1/2 \end{bmatrix}$

(2.23 cont)

(d) $A = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1/2 \end{bmatrix}$, $g(\lambda) = \det\begin{bmatrix} 1/2-\lambda & 0 \\ 1/2 & 1/2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda_1 = \lambda_2 = 1/2$

$A = \lambda_1 E_1 + N_1$; $A^k = \lambda_1^k E_1 + k(\lambda_1)^{k-1} N_1$

$A^0 = I = E_1$; $A = \frac{1}{2} I + N_1$, $N_1 = A - 1/2\, I = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}$

$\therefore A^k = \left(\frac{1}{2}\right)^k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k\left(\frac{1}{2}\right)^{k-1} \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}$.

(e) $A = \begin{bmatrix} 3/4 & -1/2 \\ -15/32 & 1/2 \end{bmatrix}$, $g(\lambda) = \det(A - \lambda I) = 0 \Rightarrow \lambda_1 = 9/8,\ \lambda_2 = 1/8$

$E_1 = \frac{A-\lambda_2 I}{\lambda_1-\lambda_2} = \frac{1}{2}\begin{bmatrix} 5/4 & -1 \\ \frac{15}{16} & 3/4 \end{bmatrix}$, $E_2 = \frac{1}{2}\begin{bmatrix} 3/4 & 1 \\ \frac{15}{16} & 5/4 \end{bmatrix}$

$\therefore A^k = \left(\frac{9}{8}\right)^k E_1 + \left(\frac{1}{8}\right)^k E_2$.

$I = E_1 + E_2$

$A = \lambda_1 E_1 + \lambda_2 E_2$

#2.24

$x(n+1) = 3x(n) + 5y(n) + 2z(n)$

$y(n+1) = x(n) - y(n) + z(n)$

$z(n+1) = 2x(n) + y(n) + 3z(n)$

then $\underline{v}(n+1) = \begin{bmatrix} 3 & 5 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \underline{v}(n)$ with $\underline{v}(n) = \begin{bmatrix} x(n) \\ y(n) \\ z(n) \end{bmatrix}$

Our solution is thus $\underline{v}(n) = A^n \underline{v}(0)$. We need A^n.

$g(\lambda) = \det[A - \lambda I] = \lambda^3 - 5\lambda^2 - 7\lambda + 11 = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2+\sqrt{15},\ \lambda_3 = 2-\sqrt{15}$