HW 8
EE 409 (Linear Systems), CSUF spring 2010
Spring 2010
CSUF

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Spring 2010 Compiled on January 31, 2019 at 1:35am

Contents

Date due and handed in April 29,2010

1 Problem 1 (problem 6.10 in text)

6.10. Is the feedback system shown below stable if the gain g is zero; that is, with no feedback? Plot the locus of poles in the s plane for the overall system for both positive and negative values of g. For what range of g is the feedback system stable?

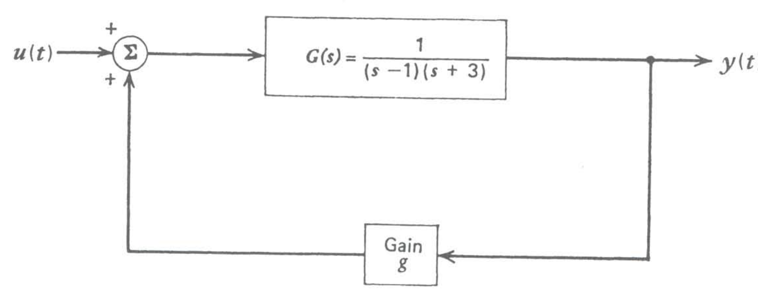

Figure 1: Problem description

Let $E(s)$ be the Laplace transform of the error signal, then we write

$$E(s) = u(s) + g\, y(s) \tag{1}$$

$$y(s) = E(s)\, G(s) \tag{2}$$

Substitute (1) into (2)

$$\begin{aligned} y(s) &= (u(s) + gy(s))\, G(s) \\ &= u(s)\, G(s) + gy(s)\, G(s) \\ y(s)\,[1 - gG(s)] &= u(s)\, G(s) \\ H(s) &= \frac{y(s)}{u(s)} = \frac{G(s)}{1 - gG(s)} \end{aligned}$$

But $G(s) = \frac{1}{(s-1)(s+3)}$, hence the above becomes

$$H(s) = \frac{1}{(s-1)(s+3) - g}$$

Pole of $H(s)$ is when denominator is zero. When $g = 0$, then the poles are $s = 1$ and $s = -3$. Since one of poles is in the RHS plane (pole $s = 1$), then the system is unstable when $g = 0$.

In other words, system stability is determined by the plant stability itself. Since the plant itself is unstable, then the overall system is unstable.

1.1 positive feedback

We found from the above what $H(s)$ is.

$$H(s)=\frac{1}{(s-1)(s+3)-g}=\frac{1}{s^2+2s-(3+g)}$$

The roots of the denominator of $H(s)$ are

$$s_{1,2}=\frac{-b}{2}\pm\frac{1}{2}\sqrt{b^2-4ac}=-1\pm\frac{1}{2}\sqrt{4+4(3+g)}=-1\pm\sqrt{4+g}$$

Hence

$$s_1=-1+\sqrt{4+g}$$
$$s_2=-1-\sqrt{4+g}$$

For s_1 to be stable, then $\sqrt{4+g}<1$ or $4+g<1$ or $g<-3$. For s_2, it is always stable for any value of g.

1.2 negative feedback

When using negative feedback, the overall system transfer function will come out to be

$$H(s)=\frac{1}{(s-1)(s+3)+g}=\frac{1}{s^2+2s+(g-3)}$$

Hence the roots of the denominator of $H(s)$ are

$$s_{1,2}=\frac{-b}{2}\pm\frac{1}{2}\sqrt{b^2-4ac}=-1\pm\frac{1}{2}\sqrt{4-4(g-3)}=-1\pm\sqrt{4-g}$$

Hence

$$s_1=-1+\sqrt{4-g}$$
$$s_2=-1-\sqrt{4-g}$$

For s_1 to be stable, then $\sqrt{4-g}<1$ or $4-g<1$ or $g>3$. For s_2, it is always stable for any value of g.

Conclusion: For positive feedback, system is stable for $g<-3$ and for negative feedback, system is stable for $g>3$

2 Problem 2 (problem 2.2 part (c) in textbook)

Solve the following difference equation

$$y(k+2)+y(k)=\sin k \qquad k\geq 0 \tag{1}$$

$L_A=\left(1-e^jS^{-1}\right)\left(1-e^{-j}S^{-1}\right)$, hence

$$L_A\left[S^2+1\right]y(k)=0$$
$$\left(1-e^jS^{-1}\right)\left(1-e^{-j}S^{-1}\right)\left[S^2+1\right]y(k)=0$$

The roots for $y_p(k)$ are $r_3=e^j$ and $r_4=e^{-j}$, hence $y_p(k)=c_3e^{jk}+c_4e^{-jk}$. Substituting this into (1) gives

$$c_3e^{j(k+2)}+c_4e^{-j(k+2)}+c_3e^{jk}+c_4e^{-jk}=\sin k$$

But $\sin k=\frac{e^{jk}-e^{-jk}}{2j}$ hence

$$c_3e^{j(k+2)}+c_4e^{-j(k+2)}+c_3e^{jk}+c_4e^{-jk}=\frac{e^{jk}-e^{-jk}}{2j}$$
$$c_3e^{jk}e^{2j}+c_4e^{-jk}e^{-2j}+c_3e^{jk}+c_4e^{-jk}=\frac{1}{2j}e^{jk}-\frac{1}{2j}e^{-jk}$$
$$e^{jk}\left(c_3e^{2j}+c_3\right)+e^{-jk}\left(c_4e^{-2j}+c_4\right)=\frac{1}{2j}e^{jk}-\frac{1}{2j}e^{-jk}$$

Hence

$$\left(c_3 e^{2j} + c_3\right) = \frac{1}{2j}$$
$$\left(c_4 e^{-2j} + c_4\right) = -\frac{1}{2j}$$

or

$$c_3\left(1 + e^{2j}\right) = \frac{1}{2j}$$
$$c_4\left(1 + e^{-2j}\right) = -\frac{1}{2j}$$

or

$$c_3 = \frac{-j}{2\left(1+e^{2j}\right)}$$
$$c_4 = \frac{j}{2\left(1+e^{-2j}\right)}$$

Hence since $y_p(k) = c_3 e^{jk} + c_4 e^{-jk}$ we now obtain

$$y_p(k) = \frac{-je^{jk}}{2\left(1+e^{2j}\right)} + \frac{je^{-jk}}{2\left(1+e^{-2j}\right)}$$

Therefore

$$y(k) = y_p(k) + y_h(k)$$

But $y_h(k)$ has the auxiliary equation $r^2+1=0$, hence roots are $r = \pm j$ hence $y_h(k) = c_1 j^k - c_2 j^k$ hence

$$\begin{aligned} y(k) &= y_p(k) + y_h(k) \\ &= \frac{-je^{jk}}{2\left(1+e^{2j}\right)} + \frac{je^{-jk}}{2\left(1+e^{-2j}\right)} + c_1 j^k - c_2 j^k \end{aligned}$$

To find c_1 and c_2 we need initial conditions, which is not given. So we stop here. Hence

$$y(k) = \frac{j}{2}\left(\frac{e^{-jk}}{1+e^{-2j}} - \frac{e^{jk}}{1+e^{2j}}\right) + j^k\left(c_1 - c_2\right)$$

This can be simplified to

$$\begin{aligned} y(k) &= \frac{j}{2}\left(\frac{e^{-jk}\left(1+e^{2j}\right) - e^{jk}\left(1+e^{-2j}\right)}{\left(1+e^{-2j}\right)\left(1+e^{2j}\right)}\right) + j^k\left(c_1 - c_2\right) \\ &= \frac{j}{2}\left(\frac{e^{-jk} + e^{j(2-k)} - e^{jk} - e^{-j(2-k)}}{2+2\cos 2}\right) + j^k\left(c_1 - c_2\right) \\ &= \frac{j}{2}\left(\frac{\left(e^{-jk} - e^{jk}\right) + \left(e^{j(2-k)} - e^{-j(2-k)}\right)}{2+2\cos 2}\right) + j^k\left(c_1 - c_2\right) \\ &= \frac{j}{2}\left(\frac{-2j\sin k + 2j\sin(2-k)}{2+2\cos 2}\right) + j^k\left(c_1 - c_2\right) \\ &= \frac{j}{2}\left(\frac{-2j\sin k - 2j\sin(k-2)}{2+2\cos 2}\right) + j^k\left(c_1 - c_2\right) \\ &= \frac{-1}{2}\left(\frac{-2\sin k - 2\sin(k-2)}{2+2\cos 2}\right) + j^k\left(c_1 - c_2\right) \end{aligned}$$

Hence

$$y(k) = \frac{1}{2}\left(\frac{\sin k + \sin(k-2)}{1+\cos 2}\right) + j^k\left(c_1 - c_2\right)$$

3 check what is wrong version of solution and delete

Let $E(s)$ be the Laplace transform of the error signal, then we write

$$E(s) = u(s) + g \times y(s) \quad (1)$$
$$y(s) = E(s)\,G(s) \quad (2)$$

Substitute (1) into (2)

$$\begin{aligned} y(s) &= (u(s) + gy(s))\,G(s) \\ &= u(s)\,G(s) + gy(s)\,G(s) \\ y(s)\,[1 - gG(s)] &= u(s)\,G(s) \\ H(s) &= \frac{y(s)}{u(s)} = \frac{G(s)}{1 - gG(s)} \end{aligned}$$

But $G(s) = \frac{1}{(s-1)(s+3)}$, hence the above becomes

$$H(s) = \frac{1}{(s-1)(s+3) - g}$$

Pole of $H(s)$ is when denominator is zero. When $g = 0$, then the poles are $s = 1$ and $s = -3$. Since one of poles is in the RHS plane (pole $s = 1$), then the system is unstable when $g = 0$.

In other words, system stability is determined by the plant stability itself. Since the plant itself is unstable, then the overall system is unstable.

3.1 positive feedback

We found from the above what $H(s)$ is.

$$H(s) = \frac{1}{(s-1)(s+3) - g} = \frac{1}{s^2 + 2s - (3 + g)}$$

The roots of the denominator of $H(s)$ are

$$s_{1,2} = \frac{-b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4ac} = -1 \pm \frac{1}{2}\sqrt{4 + 4(3 + g)} = -1 \pm \sqrt{4 + g}$$

Hence

$$s_1 = -1 + \sqrt{4 + g}$$
$$s_2 = -1 - \sqrt{4 + g}$$

For s_1 to be stable, then $\sqrt{4 + g} < 1$ or $4 + g < 1$ or $g < -3$. For s_2, it is always stable for any value of g.

3.2 negative feedback

When using negative feedback, the overall system transfer function will come out to be

$$H(s) = \frac{1}{(s-1)(s+3) + g} = \frac{1}{s^2 + 2s + (g - 3)}$$

Hence the roots of the denominator of $H(s)$ are

$$s_{1,2} = \frac{-b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4ac} = -1 \pm \frac{1}{2}\sqrt{4 - 4(g - 3)} = -1 \pm \sqrt{4 - g}$$

Hence

$$s_1 = -1 + \sqrt{4 - g}$$
$$s_2 = -1 - \sqrt{4 - g}$$

For s_1 to be stable, then $\sqrt{4 - g} < 1$ or $4 - g < 1$ or $g > 3$. For s_2, it is always stable for any value of g.

Conclusion: For positive feedback, system is stable for $g < -3$ and for negative feedback, system is stable for $g > 3$

3.3 Problem 2 (problem 2.2 part (c) in textbook)

Solve the following difference equation

$$y(k+2) + y(k) = \sin k \qquad k \geq 0 \tag{1}$$

$L_A = \left(1 - e^j S^{-1}\right)\left(1 - e^{-j} S^{-1}\right)$, hence

$$\begin{aligned} L_A\left[S^2+1\right] y(k) &= 0 \\ \left(1 - e^j S^{-1}\right)\left(1 - e^{-j} S^{-1}\right)\left[S^2+1\right] y(k) &= 0 \end{aligned}$$

The roots for $y_p(k)$ are $r_3 = e^j$ and $r_4 = e^{-j}$, hence $y_p(k) = c_3 e^{jk} + c_4 e^{-jk}$ Substituting this into (1) gives

$$c_3 e^{j(k+2)} + c_4 e^{-j(k+2)} + c_3 e^{jk} + c_4 e^{-jk} = \sin k$$

But $\sin k = \frac{e^{jk} - e^{-jk}}{2j}$ hence

$$\begin{aligned} c_3 e^{j(k+2)} + c_4 e^{-j(k+2)} + c_3 e^{jk} + c_4 e^{-jk} &= \frac{e^{jk} - e^{-jk}}{2j} \\ c_3 e^{jk} e^{2j} + c_4 e^{-jk} e^{-2j} + c_3 e^{jk} + c_4 e^{-jk} &= \frac{1}{2j} e^{jk} - \frac{1}{2j} e^{-jk} \\ e^{jk}\left(c_3 e^{2j} + c_3\right) + e^{-jk}\left(c_4 e^{-2j} + c_4\right) &= \frac{1}{2j} e^{jk} - \frac{1}{2j} e^{-jk} \end{aligned}$$

Hence

$$\begin{aligned} \left(c_3 e^{2j} + c_3\right) &= \frac{1}{2j} \\ \left(c_4 e^{-2j} + c_4\right) &= -\frac{1}{2j} \end{aligned}$$

Or

$$\begin{aligned} c_3\left(1 + e^{2j}\right) &= \frac{1}{2j} \\ c_4\left(1 + e^{-2j}\right) &= -\frac{1}{2j} \end{aligned}$$

Or

$$\begin{aligned} c_3 &= \frac{-j}{2\left(1 + e^{2j}\right)} \\ c_4 &= \frac{j}{2\left(1 + e^{-2j}\right)} \end{aligned}$$

Hence since $y_p(k) = c_3 e^{jk} + c_4 e^{-jk}$ then

$$y_p(k) = \frac{-j e^{jk}}{2\left(1 + e^{2j}\right)} + \frac{j e^{-jk}}{2\left(1 + e^{-2j}\right)}$$

Therefore

$$y(k) = y_p(k) + y_h(k)$$

But $y_h(k)$ has the auxiliary equation $r^2 + 1 = 0$, hence roots are $r = \pm j$ hence $y_h(k) = c_1 j^k - c_2 j^k$ and

$$\begin{aligned} y(k) &= y_p(k) + y_h(k) \\ &= \frac{-j e^{jk}}{2\left(1 + e^{2j}\right)} + \frac{j e^{-jk}}{2\left(1 + e^{-2j}\right)} + c_1 j^k - c_2 j^k \end{aligned}$$

To find c_1 and c_2 we need initial conditions, which is not given. So we stop here.

Using initial conditions. Assuming zero initial conditions, we have at $k = 0$ that $y(0) = 0$, hence

$$\begin{aligned}
0 &= \frac{-j}{2\left(1+e^{2j}\right)} + \frac{j}{2\left(1+e^{-2j}\right)} + c_1 - c_2 \\
&= \frac{1}{2}\frac{-j\left(1+e^{-2j}\right) + j\left(1+e^{2j}\right)}{\left(1+e^{2j}\right)\left(1+e^{-2j}\right)} + c_1 - c_2 \\
0 &= \frac{1}{2}\frac{-je^{-2j} + je^{2j}}{\left(2+e^{-2j}+e^{2j}\right)} + c_1 - c_2 \\
0 &= \frac{1}{2}\frac{2\sin 2}{\left(2+2\cos 2\right)} + c_1 - c_2 \\
0 &= \frac{1}{2}\frac{\sin 2}{1+\cos 2} + c_1 - c_2
\end{aligned}$$

Therefore

$$c_1 - c_2 = \frac{-1}{2}\frac{\sin 2}{1+\cos 2} \tag{2}$$

Now at $k = 1$, $y(k) = 0$, hence from $y(k) = \frac{-je^{jk}}{2(1+e^{2j})} + \frac{je^{-jk}}{2(1+e^{-2j})} + c_1 j^k - c_2 j^k$ we obtain

$$\begin{aligned}
0 &= \frac{-je^{j}}{2\left(1+e^{2j}\right)} + \frac{je^{-j}}{2\left(1+e^{-2j}\right)} + c_1 j - c_2 j \\
&= \frac{1}{2}\left(\frac{-e^{j}}{\left(1+e^{2j}\right)} + \frac{e^{-j}}{\left(1+e^{-2j}\right)}\right) + c_1 - c_2 \\
&= \frac{1}{2}\frac{\left(-e^{j} - e^{-j}\right) + \left(e^{-j} + e^{j}\right)}{\left(1+e^{2j}\right)\left(1+e^{-2j}\right)} + c_1 - c_2 \\
&= \frac{1}{2}\frac{0}{2+e^{-2j}+e^{2j}} + c_1 - c_2
\end{aligned}$$

Hence

$$c_1 = c_2 \tag{3}$$

(2)+(3) gives

$$\begin{aligned}
2c_1 &= \frac{1}{2}\frac{-\sin 2}{1+\cos 2} \\
c_1 &= \frac{-1}{4}\frac{\sin 2}{1+\cos 2}
\end{aligned}$$

And

$$c_2 = \frac{1}{4}\frac{\sin 2}{1+\cos 2}$$

Hence the final solution is

$$\begin{aligned}
y(k) &= \frac{-je^{jk}}{2\left(1+e^{2j}\right)} + \frac{je^{-jk}}{2\left(1+e^{-2j}\right)} + c_1 j^k - c_2 j^k \\
&= \frac{-je^{jk}}{2\left(1+e^{2j}\right)} + \frac{je^{-jk}}{2\left(1+e^{-2j}\right)} - \frac{1}{4}\frac{j^k \sin 2}{1+\cos 2} - \frac{1}{4}\frac{j^k \sin 2}{1+\cos 2}
\end{aligned}$$