HW 6
EE 409 (Linear Systems), CSUF spring 2010
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CSUF

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Contents

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1 Problem 3.25

Write state variable description of the following 2 systems. For what values of k will the system be stable?

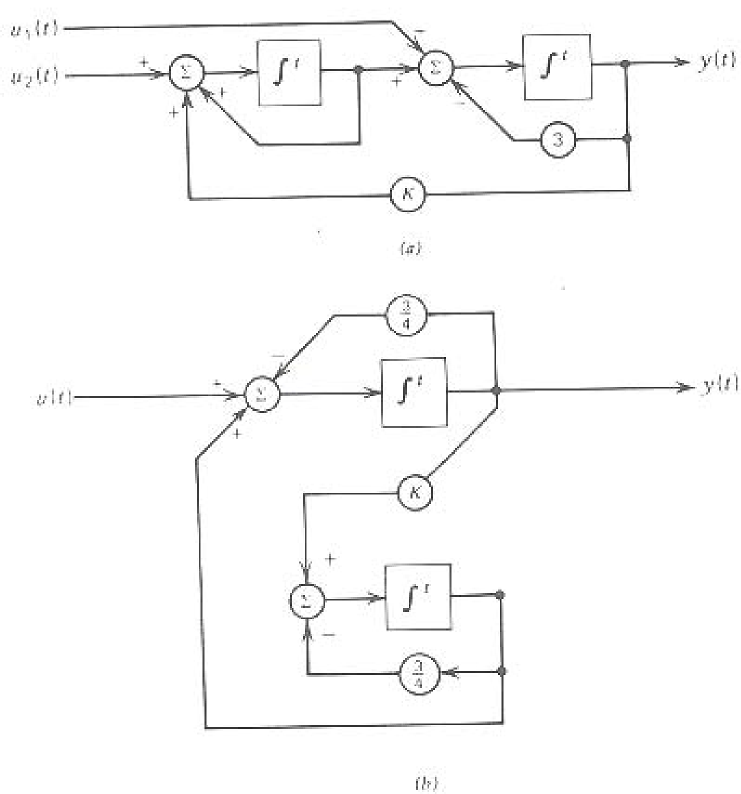


Figure 1: Problem description

1.1 part(a)

This system has 2 integrators, hence it is of order 2. Hence we need 2 state variables. Assign a state variable as the output of each integrator

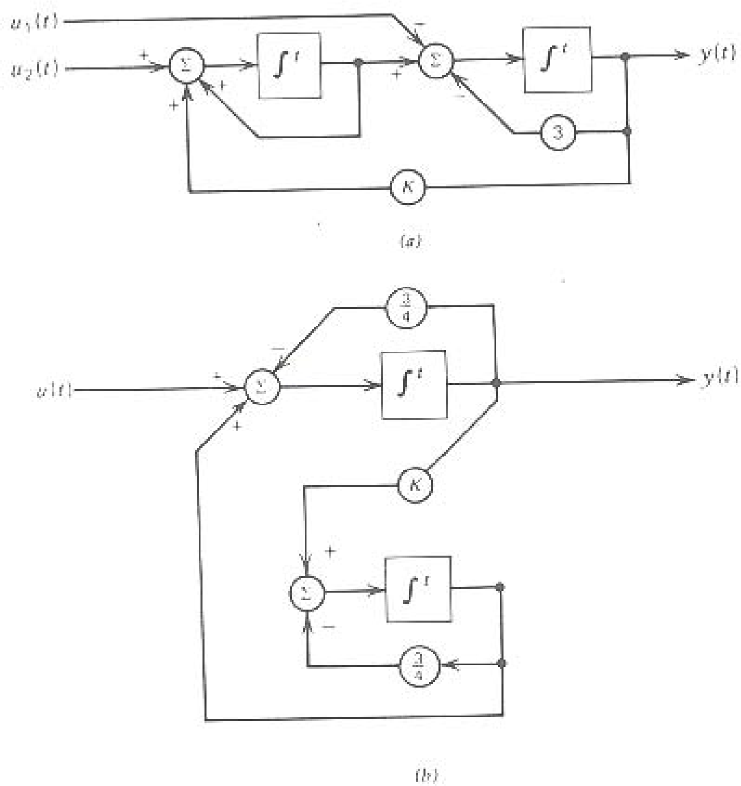


Figure 2: part(a) system with labels

Hence

$$
\begin{aligned}
x_1' &= -3x_1 + u_1 + x_2 \\
x_2' &= x_2 + kx_1 + u_2
\end{aligned}
$$

and $y = x_1$,Hence

$$
\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \overbrace{\begin{pmatrix} -3 & 1 \\ k & 1 \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{B} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
$$

$$
y = \overbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
$$

To find what values of k the system is stable, the eigenvalues of the A matrix are found and the K range which makes these values negative is the range of value needed.

$$|A - \lambda I| = \left| \begin{pmatrix} -3-\lambda & 1 \\ k & 1-\lambda \end{pmatrix} \right| = (1-\lambda)(-3-\lambda) - k$$

Hence the characteristic equation is

$$\lambda^2 + 2\lambda - k - 3 = 0$$

and the roots are

$$\begin{aligned} \lambda_1 &= -1 + \sqrt{k+4} \\ \lambda_2 &= -1 - \sqrt{k+4} \end{aligned}$$

consider λ_1. For this root to be stable, then $\sqrt{k+4} < 1$ or $k < -3$

consider λ_2. This root is stable for any value of k since when $k+4 < 0$ then it is stable since real part is already negative, and when $k+4 > 0$ then it is stable also.

Hence we conclude that the system is stable for $k < -3$

To find the ODE:

From $x_1' = -3x_1 + u_1 + x_2$ we obtain $x_1'' = -3x_1' + u_1' + x_2'$. Substitute the value of x_2' from above, we obtain $x_1'' = -3x_1' + u_1' + x_2 + kx_1 + u_2$, but $x_2 = .x_1' + 3x_1 - u_1$, hence

$$\begin{aligned} x_1'' &= -3x_1' + u_1' + x_1' + 3x_1 - u_1 + kx_1 + u_2 \\ &= -2x_1' + x_1(3+k) - u_1 + u_1' + u_2 \end{aligned}$$

since $x_1 = y$ we obtain

$$y'' = -2y' + y(3+k) - u_1 + u_1' + u_2$$

1.2 Part(b)

This system has 2 integrators, hence it is of order 2. Hence we need 2 state variables. Assign a state variable as the output of each integrator

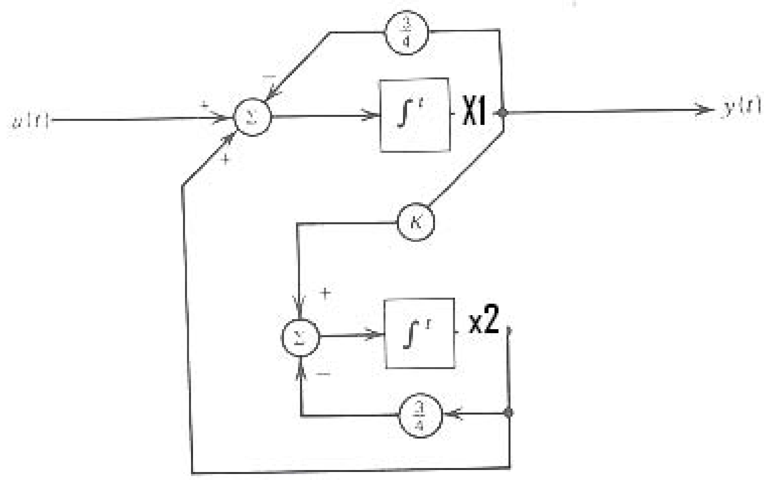


Figure 3: Part(b) system

Hence

$$x_1' = -\frac{3}{4}x_1 + u_1 + x_2$$
$$x_2' = -\frac{3}{4}x_2 + kx_1$$

and $y = x_1$,Hence

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \overbrace{\begin{pmatrix} -\frac{3}{4} & 1 \\ k & -\frac{3}{4} \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^{B} u_1$$
$$y = \overbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

To find what values of k the system is stable, the eigenvalues of the A matrix are found and the K range which makes these values negative is the range of value needed.

$$|A - \lambda I| = \left| \begin{pmatrix} -\frac{3}{4} - \lambda & 1 \\ k & -\frac{3}{4} - \lambda \end{pmatrix} \right| = \left(-\frac{3}{4} - \lambda\right)\left(-\frac{3}{4} - \lambda\right) - k$$

Hence the characteristic equation is

$$\lambda^2 + \frac{3}{2}\lambda - k + \frac{9}{16} = 0$$

and the roots are

$$\lambda_1 = -\frac{3}{4} - \sqrt{k}$$
$$\lambda_2 = -\frac{3}{4} + \sqrt{k}$$

For λ_1, all values of k will result in stable root. For λ_2, $\sqrt{k} < \frac{3}{4}$ or $k < \frac{9}{16}$ or $k < 0.5625$

Hence $k < \frac{9}{16}$ or $k < 0.5625$ is the range of k for stability.

To find the ODE: From $x_1' = -\frac{3}{4}x_1 + u_1 + x_2$, we obtain $x_1'' = -\frac{3}{4}x_1' + u_1' + x_2'$ Substitute the value of x_2' from above, we obtain $x_1'' = -\frac{3}{4}x_1' + u_1' - \frac{3}{4}x_2 + kx_1$ but $x_2 = x_1' + \frac{3}{4}x_1 - u_1$, hence

$$\begin{aligned}
x_1'' &= -\frac{3}{4}x_1' + u_1' - \frac{3}{4}\left(x_1' + \frac{3}{4}x_1 - u_1\right) + kx_1 \\
&= -\frac{3}{4}x_1' + u_1' - \frac{3}{4}x_1' - \frac{9}{16}x_1 + \frac{3}{4}u_1 + kx_1 \\
&= -\frac{3}{2}x_1' + x_1\left(k - \frac{9}{16}\right) + u_1' + \frac{3}{4}u_1
\end{aligned}$$

since $x_1 = y$ we obtain

$$y'' + \frac{3}{2}y' - y\left(k - \frac{9}{16}\right) = u_1' + \frac{3}{4}u_1$$

2 Problem 2

3.28. Consider the following state-variable system:

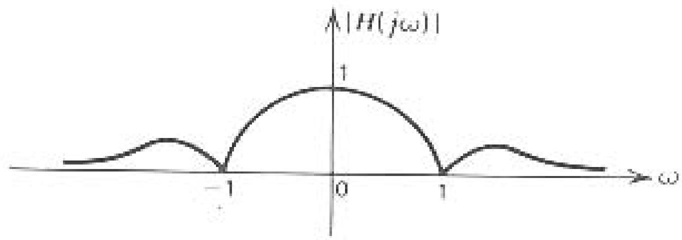
$$\begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [c_1 \quad c_2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [d]u(t)$$

(a) Find the matrix $(j\omega\mathbf{I} - \mathbf{A})^{-1}$.

(b) Find the matrix $e^{\mathbf{A}t}$.

(c) The amplitude-response function for the system is shown below. Determine c_1, c_2, and d.



(d) Find the impulse-response function $h(t)$.

(e) Is this system stable?

Figure 4: Problem description

2.1 part(a)

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{B} u_1$$

$$y = \overbrace{\begin{pmatrix} c_1 & c_2 \end{pmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + [d] \; u_1$$

$$(j\omega I - A) = j\omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} j\omega & -1 \\ 2 & j\omega + 3 \end{pmatrix}$$

Hence

$$\begin{aligned}(j\omega I - A)^{-1} &= \begin{pmatrix} j\omega & -1 \\ 2 & j\omega + 3 \end{pmatrix}^{-1} = \frac{1}{(j\omega)(j\omega+3)+2} \begin{pmatrix} j\omega + 3 & 1 \\ -2 & j\omega \end{pmatrix} \\ &= \frac{1}{-\omega^2 + 3j\omega + 2} \begin{pmatrix} j\omega + 3 & 1 \\ -2 & j\omega \end{pmatrix}\end{aligned}$$

2.2 part(b)

To find e^{At} use the eigenvalue method.

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2$$

Hence the roots of $\lambda^2 + 3\lambda + 2 = 0$ are found to be $\lambda_1 = -1$ and $\lambda_2 = -2$. Hence the 2 equations to solve are

$$\begin{aligned} e^{\lambda_1 t} &= \beta_0 + \beta_1 \lambda_1 \\ e^{\lambda_2 t} &= \beta_0 + \beta_1 \lambda_2 \end{aligned}$$

or

$$\begin{aligned} e^{-t} &= \beta_0 - \beta_1 \\ e^{-2t} &= \beta_0 - 2\beta_1 \end{aligned}$$

Solving we obtain

$$\begin{aligned} \beta_0 &= 2e^{-t} - e^{-2t} \\ \beta_1 &= e^{-t} - e^{-2t} \end{aligned}$$

Hence

$$\begin{aligned} e^{At} &= \beta_0 I + \beta_1 A \\ &= \left(2e^{-t} - e^{-2t}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(e^{-t} - e^{-2t}\right) \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \\ &= \end{aligned}$$

Hence

$$e^{At} = \begin{pmatrix} \left(2e^{-t} - e^{-2t}\right) & \left(e^{-t} - e^{-2t}\right) \\ -2\left(e^{-t} - e^{-2t}\right) & -e^{-t} + 2e^{-2t} \end{pmatrix}$$

2.3 part (c)

First need to find $H(j\omega)$. We start from the system equations

$$x' = Ax + Bu \tag{1}$$
$$y = Cx + Du \tag{2}$$

Let $u = e^{j\omega t}$, hence the state particular solution is

$$x_p(t) = X(j\omega)e^{j\omega t} \tag{3}$$

And

$$y_p(t) = H(j\omega)e^{j\omega t} \tag{4}$$

From (1) and (3) , we obtain

$$\begin{aligned} j\omega X(j\omega)e^{j\omega t} &= AX(j\omega)e^{j\omega t} + Be^{j\omega t} \\ j\omega X(j\omega) &= AX(j\omega) + B \\ (j\omega I - A)X(j\omega) &= B \\ X(j\omega) &= (j\omega I - A)^{-1}B \end{aligned} \tag{5}$$

and from (2) and (4) we obtain

$$\begin{aligned} H(j\omega)e^{j\omega t} &= CX(j\omega)e^{j\omega t} + De^{j\omega t} \\ H(j\omega) &= CX(j\omega) + D \end{aligned}$$

Substitute (5) into the above

$$H(j\omega) = C(j\omega I - A)^{-1}B + D$$

From part(a) we found $(j\omega I - A)^{-1}$, hence the above becomes

$$\begin{aligned} H(j\omega) &= \begin{pmatrix} c_1 & c_2 \end{pmatrix} \frac{1}{-\omega^2 + 3j\omega + 2} \begin{pmatrix} j\omega + 3 & 1 \\ -2 & j\omega \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + d \\ &= \frac{1}{-\omega^2 + 3j\omega + 2} \begin{pmatrix} (j\omega + 3)c_1 - 2c_2 & c_1 + c_2 j\omega \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + d \\ &= \frac{(c_1 + c_2 j\omega)}{-\omega^2 + 3j\omega + 2} + d \\ &= \frac{(c_1 + c_2 j\omega) + d(-\omega^2 + 3j\omega + 2)}{-\omega^2 + 3j\omega + 2} \\ &= \frac{(c_1 + 2d - d\omega^2) + j(c_2\omega + 3d\omega)}{(-\omega^2 + 2) + 3j\omega} \end{aligned}$$

Hence

$$\begin{aligned} |H(j\omega)|^2 &= \frac{(c_1 + 2d - d\omega^2)^2 + (c_2\omega + 3d\omega)^2}{(-\omega^2 + 2)^2 + 9\omega^2} \\ &= \frac{d^2\omega^4 + 5d^2\omega^2 + 4d^2 - 2d\omega^2 c_1 + 6d\omega^2 c_2 + 4dc_1 + \omega^2 c_2^2 + c_1^2}{\omega^4 + 5\omega^2 + 4} \end{aligned}$$

Now, from diagram, at $\omega = 0$ we have $|H(j\omega)|^2 = 1$, hence

$$1 = d^2 + dc_1 + \frac{1}{4}c_1^2 \tag{6}$$

And at $\omega = 1$ we have $|H(j\omega)|^2 = 0$ hence

$$0 = \frac{10d^2 + 2dc_1 + 6dc_2 + c_2^2 + c_1^2}{10}$$

Or

$$0 = 10d^2 + 2dc_1 + 6dc_2 + c_2^2 + c_1^2 \tag{7}$$

And at $\omega = -1$ we have $|H(j\omega)|^2 = 0$ but this will not add new equation. So need to look at the limit as $\omega \to \infty$

$$|H(j\omega)|^2 = \frac{d^2 + \frac{5d^2}{\omega^2} + \frac{4d^2}{\omega^4} - \frac{2dc_1}{\omega^2} + \frac{6dc_2}{\omega^2} + \frac{4dc_1}{\omega^4} + \frac{c_2^2}{\omega^2} + \frac{c_1^2}{\omega^4}}{1 + \frac{5}{\omega^2} + \frac{4}{\omega^4}}$$

Hence we see that as $\omega \to \infty$, $|H(j\omega)|^2 \to d^2$, hence $d = 0$ since $|H(j\omega)| \to 0$ in the limit. So now we know d, we have 2 equations and 2 unknowns to solve for from (6) and (7). Re write (6) and (7) again by setting $d = 0$ we obtain

$$1 = \frac{1}{4}c_1^2 \tag{6}$$

$$0 = c_2^2 + c_1^2 \tag{7}$$

Hence $c_1 = 2$ and $c_2 = 2j$ therefore, the system now looks like

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{B} u_1$$

$$y = \overbrace{\begin{pmatrix} 2 & 2j \end{pmatrix}}^{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2.4 Part(d)

To find $h(t)$, Let the input be $\delta(t)$, and find $y(t)$. From the system equation

$$y_p(t) = \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau$$

Let $u(\tau) = \delta(t)$, so the above becomes

$$\begin{aligned} h(t) &= \int_{t_0}^{t} Ce^{A(t-\tau)}B\delta(\tau)\,d\tau \\ &= Ce^{A(t)}B \qquad t \geq 0 \end{aligned}$$

But we found $e^{A(t)}$ in part (b), hence

$$\begin{aligned} h(t) &= \begin{pmatrix} 2 & 2j \end{pmatrix} \begin{pmatrix} \left(2e^{-t} - e^{-2t}\right) & \left(e^{-t} - e^{-2t}\right) \\ -2\left(e^{-t} - e^{-2t}\right) & -e^{-t} + 2e^{-2t} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= 2e^{-t} - 2e^{-2t} - 2j\left(e^{-t} - 2e^{-2t}\right) \end{aligned}$$

2.5 part(e)

To check for stability

$$|A - \lambda I| = \left| \begin{pmatrix} -\lambda & 1 \\ -2 & -3 - \lambda \end{pmatrix} \right| = (-\lambda)(-3 - \lambda) + 2$$

Hence

$$\lambda^2 + 3\lambda + 2 = 0$$

The roots are $-1, -2$ and since they are both negative, hence the system is stable.