HW 2
EE 409 (Linear Systems), CSUF spring 2010
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Contents

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1 Problem 3.8

Find the impulse response of the following systems defined by the following differential equations. Verify your answer

1.1 Part a

$\left(D^2+7D+12\right)y(t)=u(t)$

Answer

The impulse reponse $h(t)$ satisfies the homogenouse part of the differential equation under the initial conditions $h(0)=0, h'(0)=1$.

Hence we solve the following

$$\left(D^2+7D+12\right)h(t)=0 \tag{1}$$

The charateristic equation is $r^2+7r+12=0$ or $(r+4)(r+3)=0$, hence

$$h(t)=\left(c_1e^{-3t}+c_2e^{-4t}\right)\xi(t) \tag{2}$$

Where $\xi(t)$ is the unit step function. Now find c_1 and c_2 from initial conditions

$$h(0)=0=c+c_2 \tag{3}$$

and

$$\begin{aligned}h'(t)&=\left(-3c_1e^{-3t}-4c_2e^{-4t}\right)\xi(t)+\left(c_1e^{-3t}+c_2e^{-4t}\right)\delta(t)\\h'(0)&=1=(-3c_1-4c_2)+(c_1+c_2)\\1&=-2c_1-3c_2\end{aligned} \tag{4}$$

From (3) and (4), we solve for c_1, c_2

$$\begin{aligned}c_1&=1\\c_2&=-1\end{aligned}$$

Hence $h(t)$ from (2) becomes

$$h(t)=(e^-3t-e^-4t)\,\xi(t) \tag{5}$$

Now we verify this solution (note that $\xi'(t)=\delta(t)$)

$$\begin{aligned}h'(t)&=\left(-3e^{-3t}+4e^{-4t}\right)\xi(t)+\left(e^{-3t}-e^{-4t}\right)\delta(t)\\h'(t)&=\left(-3e^{-3t}+4e^{-4t}\right)\xi(t)\end{aligned} \tag{6}$$

And

$$\begin{aligned} h''(t) &= \left(9e^{-3t} - 16e^{-4t}\right)\xi(t) + \left(-3e^{-3t} + 4e^{-4t}\right)\delta(t) \\ &= \left(9e^{-3t} - 16e^{-4t}\right)\xi(t) + (-3+4)\,\delta(t) \\ &= \left(9e^{-3t} - 16e^{-4t}\right)\xi(t) + \delta(t) \end{aligned} \tag{7}$$

Substitute (5),(6) and (7) into LHS of (1) we obtain

$$\left(D^2 + 7D + 12\right) h(t) = h''(t) + 7h'(t) + 12h(t)$$

$$\begin{aligned} &= \left(9e^{-3t} - 16e^{-4t}\right)\xi(t) + \delta(t) + \\ &7\left(-3e^{-3t} + 4e^{-4t}\right)\xi(t) + \\ &12\left(e^{-3t} - e^{-4t}\right)\xi(t) \end{aligned}$$

$$\begin{aligned} &= \left(9e^{-3t} - 16e^{-4t} - 21e^{-3t} + 28e^{-4t} + 12e^{-3t} - 12e^{-4t}\right)\xi(t) + \delta(t) \\ &= \left[(9 - 21 + 12)\,e^{-3t} + (-16 + 28 - 12)\,e^{-4t}\right]\xi(t) + \delta(t) \\ &= \delta(t) \end{aligned}$$

Hence we see that when the input is $\delta(t)$, then the solution is $h(t)$, which is the definition of $h(t)$. Hence the solution is verified.

1.2 Part d

$\left(D^3 + 6D^2 + 12D + 8\right) y(t) = u(t)$

Answer

The impulse reponse $h(t)$ satisfies the homogenouse part of the differential equation under the initial conditions $h(0) = 0, h'(0) = 0, h''(0) = 1$

Hence we solve the following

$$\left(D^3 + 6D^2 + 12D + 8\right) h(t) = 0 \tag{1}$$

The charateristic equation is $r^3 + 6r^2 + 12r + 8 = 0$ or $(r+2)(r+2)(r+2) = 0$, hence

$$h(t) = \left(c_1 e^{-2t} + c_2 t e^{-2t} + c_3 t^2 e^{-2t}\right)\xi(t) \tag{2}$$

Now we find unknown $c's$. We start from $h(0) = 0$ and obtain

$$h(0) = 0 = c_1$$

Hence the solution becomes

$$\begin{aligned} h(t) &= \left(c_2 t e^{-2t} + c_3 t^2 e^{-2t}\right)\xi(t) \\ h'(t) &= \left(c_2 t\left(-2e^{-2t}\right) + c_2 e^{-2t} + c_3 t^2\left(-2e^{-2t}\right) + 2c_3 t e^{-2t}\right)\xi(t) + \left(c_2 t e^{-2t} + c_3 t^2 e^{-2t}\right)\delta(t) \\ &= \left(-2c_2 t e^{-2t} + c_2 e^{-2t} - 2c_3 t^2 e^{-2t} + 2c_3 t e^{-2t}\right)\xi(t) \end{aligned}$$

And from $h'(0) = 0$ we obtain

$$0 = c_2$$

Hence the solution becomes

$$\begin{aligned}
h(t) &= \left(c_3 t^2 e^{-2t}\right) \xi(t) \\
h'(t) &= \left(2c_3 t e^{-2t} - 2c_3 t^2 e^{-2t}\right) \xi(t) + \left(c_3 t^2 e^{-2t}\right) \delta(t) \\
&= \left(2c_3 t e^{-2t} - 2c_3 t^2 e^{-2t}\right) \xi(t) \\
h''(t) &= \left(2c_3 e^{-2t} - 4c_3 t e^{-2t} - 4c_3 t e^{-2t} + 4c_3 t^2 e^{-2t}\right) \xi(t) + \left(2c_3 t e^{-2t} - 2c_3 t^2 e^{-2t}\right) \delta(t) \\
&= \left(2c_3 e^{-2t} - 4c_3 t e^{-2t} - 4c_3 t e^{-2t} + 4c_3 t^2 e^{-2t}\right) \xi(t)
\end{aligned}$$

And from $h''(0) = 1$ we find that

$$\begin{aligned}
h'' &= 1 = 2c_3 \\
c_3 &= \frac{1}{2}
\end{aligned}$$

Hence the final solution is

$$h(t) = \left(\frac{1}{2} t^2 e^- 2t\right) \xi(t)$$

To verify, we need to evaluate $h'''(t) + 6h''(t) + 12h'(t) + 8h(t)$ and see if we obtain $\delta(t)$ as the result.

$$\begin{aligned}
h'(t) &= \left(t e^{-2t} - t^2 e^{-2t}\right) \xi(t) + \left(\frac{1}{2} t^2 e^{-2t}\right) \delta(t) \\
&= \left(t e^{-2t} - t^2 e^{-2t}\right) \xi(t)
\end{aligned}$$

And

$$\begin{aligned}
h''(t) &= \left(e^{-2t} - 2t e^{-2t} - 2t e^{-2t} + 2t^2 e^{-2t}\right) \xi(t) + \left(t e^{-2t} - t^2 e^{-2t}\right) \delta(t) \\
&= \left(e^{-2t} - 4t e^{-2t} + 2t^2 e^{-2t}\right) \xi(t)
\end{aligned}$$

And

$$\begin{aligned}
h'''(t) &= \left(-2e^{-2t} - 4e^{-2t} + 8t e^{-2t} + 4t e^{-2t} - 4t^2 e^{-2t}\right) \xi(t) + \left(e^{-2t} - 4t e^{-2t} + 2t^2 e^{-2t}\right) \delta(t) \\
&= \left(-6e^{-2t} + 12t e^{-2t} - 4t^2 e^{-2t}\right) \xi(t) + \delta(t)
\end{aligned}$$

Therefore, $LHS = h'''(t) + 6h''(t) + 12h'(t) + 8h(t)$ becomes

$$\begin{aligned}
LHS &= \left(-6e^{-2t} + 12t e^{-2t} - 4t^2 e^{-2t}\right) \xi(t) + \delta(t) \\
&+ 6\left(\left(e^{-2t} - 4t e^{-2t} + 2t^2 e^{-2t}\right) \xi(t)\right) \\
&+ 12\left(\left(t e^{-2t} - t^2 e^{-2t}\right) \xi(t)\right) \\
&+ 8\left(\left(\frac{1}{2} t^2 e^{-2t}\right) \xi(t)\right) \\
\\
&= e^{-2t}(-6 + 6) + t e^{-2t}(12 - 24 + 12) + t^2 e^{-2t}(-4 + 12 - 12 + 4) + \delta(t) \\
&= \delta(t)
\end{aligned}$$

Hence we see that when the input is $\delta(t)$, then the solution is $h(t)$, which is the definition of $h(t)$. Hence the solution is verified

1.3 Part e

$$\left(D^3 + 6D^2 + 12D + 8\right) y(t) = (D - 1)\, u(t)$$

Note: There is a typo in the textbook. The problem as shown in the text had the number 4 in the above equation when it should be 6. I confirmend this with our course instructor. I am sloving the correct version of the problem statment as shown above.

We start by finding the impluse response for the system $\left(D^3 + 6D^2 + 12D + 8\right) y(t) = u(t)$, which we call $\hat{h}(t)$, then find the required impulse reponse using

$$h(t) = (D - 1)\, \hat{h}(t)$$

However, the impulse reponse of the above was found in part (d), and it is

$$\hat{h}(t) = \left(\frac{1}{2} t^2 e^{-2t}\right) \xi(t)$$

Therefore the required reponse reponse is

$$\begin{aligned} h(t) &= (D - 1)\left(\frac{1}{2} t^2 e^{-2t}\right) \xi(t) \\ &= \left(te^{-2t} - t^2 e^{-2t}\right) \xi(t) + \left(\frac{1}{2} t^2 e^{-2t}\right) \delta(t) - \left(\frac{1}{2} t^2 e^{-2t}\right) \xi(t) \\ &= \left(te^{-2t} - \frac{3}{2} t^2 e^{-2t}\right) \xi(t) \end{aligned}$$

Therefore

$$h(t) = \left(te^{-}2t - \frac{3}{2} t^2 e^{-}2t\right) \xi(t)$$

Now we need to verify this solution.

$$\begin{aligned} h'(t) &= \left(e^{-2t} - 2te^{-2t} - 3te^{-2t} + 3t^2 e^{-2t}\right) \xi(t) + \left(te^{-2t} - \frac{3}{2} t^2 e^{-2t}\right) \delta(t) \\ &= \left(e^{-2t} - 5te^{-2t} + 3t^2 e^{-2t}\right) \xi(t) \end{aligned}$$

And

$$\begin{aligned} h''(t) &= \left(-2e^{-2t} - 5e^{-2t} + 10te^{-2t} + 6te^{-2t} - 6t^2 e^{-2t}\right) \xi(t) + \left(e^{-2t} - 5te^{-2t} + 3t^2 e^{-2t}\right) \delta(t) \\ &= \left(-7e^{-2t} + 16te^{-2t} - 6t^2 e^{-2t}\right) \xi(t) + \delta(t) \end{aligned}$$

And

$$\begin{aligned} h'''(t) &= \left(14e^{-2t} + 16e^{-2t} - 32te^{-2t} - 12te^{-2t} + 12t^2 e^{-2t}\right) \xi(t) + \left(-7e^{-2t} + 16te^{-2t} - 6t^2 e^{-2t}\right) \delta(t) + \delta'(t) \\ &= \left(30e^{-2t} - 44te^{-2t} + 12t^2 e^{-2t}\right) \xi(t) - 7\delta(t) + \delta'(t) \end{aligned}$$

Now using the above, we evaluate the LHS of the ODE, we obtain

$$
\begin{aligned}
LHS &= \left(D^3 + 6D^2 + 12D + 8\right) h(t) \\
&= h'''(t) + 6h''(t) + 12h'(t) + 8h(t) \\
\\
&= \left(30e^{-2t} - 44te^{-2t} + 12t^2e^{-2t}\right) \xi(t) - 7\delta(t) + \delta'(t) \\
&+ 6\left[\left(-7e^{-2t} + 16te^{-2t} - 6t^2e^{-2t}\right) \xi(t) + \delta(t)\right] \\
&+ 12\left[\left(e^{-2t} - 5te^{-2t} + 3t^2e^{-2t}\right) \xi(t)\right] \\
&+ 8\left[\left(te^{-2t} - \frac{3}{2}t^2e^{-2t}\right) \xi(t)\right] \\
\\
&= e^{-2t}(30 - 42 + 12)\xi(t) \\
&+ te^{-2t}(-44 + 96 - 60 + 8)\xi(t) \\
&+ t^2e^{-2t}(12 - 36 + 36 - 12)\xi(t) \\
&- \delta(t) + \delta'(t) \\
\\
&= e^{-2t}(0) + te^{-2t}(0) + t^2e^{-2t}(0) - \delta(t) + \delta'(t) \\
\\
&= \delta'(t) - \delta(t)
\end{aligned}
$$

But the RHS is $(D-1)\delta(t)$ which is $\delta'(t) - \delta(t)$. Hence LHS=RHS, hence verified.