HW 1
EE 409 (Linear Systems), CSUF spring 2010
Spring 2010
CSUF

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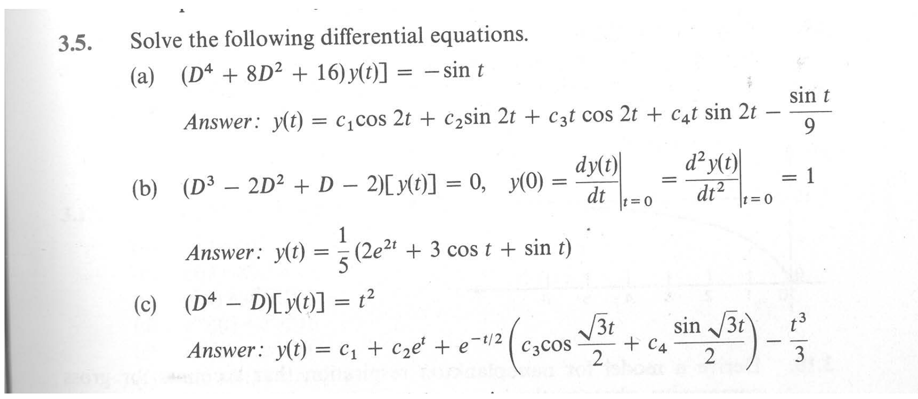
Spring 2010 Compiled on January 30, 2019 at 11:56pm

Contents

Date due and handed in Feb. 11,2010

1 Problem 3.5



3.5. Solve the following differential equations.

(a) $(D^4 + 8D^2 + 16)y(t)] = -\sin t$

Answer: $y(t) = c_1\cos 2t + c_2\sin 2t + c_3 t\cos 2t + c_4 t\sin 2t - \frac{\sin t}{9}$

(b) $(D^3 - 2D^2 + D - 2)[y(t)] = 0, \quad y(0) = \left.\frac{dy(t)}{dt}\right|_{t=0} = \left.\frac{d^2y(t)}{dt^2}\right|_{t=0} = 1$

Answer: $y(t) = \frac{1}{5}(2e^{2t} + 3\cos t + \sin t)$

(c) $(D^4 - D)[y(t)] = t^2$

Answer: $y(t) = c_1 + c_2e^t + e^{-t/2}\left(c_3\cos\frac{\sqrt{3}t}{2} + c_4\frac{\sin\sqrt{3}t}{2}\right) - \frac{t^3}{3}$

Figure 1: Problem description

1.1 Part a

Let $L \equiv D^4 + 8D^2 + 16$ and let $L_A \equiv D^2 + 1$. Since[1] $L_A[-\sin t] = 0$, then the differential equation can be written as

$$\begin{aligned} L_A[L[y(t)]] &= 0 \\ (D^2+1)(D^4+8D^2+16) &= 0 \\ (D^2+1)(D^2+4)(D^2+4) &= 0 \end{aligned}$$

Hence the characteristic equation is

$$(r^2+1)(r^2+4)(r^2+4) = 0$$

And the roots from the particular solution are $r_1 = j$ and $r_2 = -j$ and the roots from the homogeneous solution are $\pm 2j$ and $\pm 2j$, which we call $r_3 = 2j, r_4 = -2j$ and $r_5 = 2j$ and $r_6 = -2j$. Hence

$$y_p(t) = c_1e^{-r_1t} + c_2e^{-r_2t}$$

and

$$y_h(t) = c_3e^{-r_3t} + c_4e^{-r_4t} + c_5te^{-r_5t} + c_6te^{-r_6t}$$

Hence

$$\begin{aligned} y_p(t) &= c_1e^{-jt} + c_2e^{jt} \\ &= c_1(\cos t - j\sin t) + c_2(\cos t + j\sin t) \\ &= (c_1+c_2)\cos t + (jc_2 - jc_1)\sin t \\ &= C_1\cos t + C_2\sin t \end{aligned}$$

Where $C_1 = (c_1 + c_2)$ and $C_2 = (jc_2 - jc_1)$

[1] $L_A[-\sin t] = (D^2+1)(-\sin t) = (D(D(-\sin t)) - \sin t) = (D(-\cos t) - \sin t) = (\sin t - \sin t) = 0$

and

$$\begin{aligned} y_h(t) &= c_3 e^{-2jt} + c_4 e^{2jt} + c_5 t e^{-2jt} + c_6 t e^{2jt} \\ &= c_3(\cos 2t - j\sin 2t) + c_4(\cos 2t + j\sin 2t) \\ &+ c_5 t(\cos 2t - j\sin 2t) + c_6 t(\cos 2t + j\sin 2t) \\ &= (c_3 + c_4)\cos 2t + (-jc_3 + jc_4)\sin 2t + (c_5 + c_6)\, t\cos 2t + (-jc_5 + jc_6)\, t\sin 2t \\ &= C_3\cos 2t + C_4\sin 2t + C_5 t\cos 2t + C_6 t\sin 2t \end{aligned}$$

Where $C_3 = (c_3 + c_4), C_4 = (-jc_3 + jc_4), C_5 = (c_5 + c_6), C_6 = (-jc_5 + jc_6)$

Hence we have

$$y(t) = \overbrace{C_1\cos t + C_2\sin t}^{y_p} + \overbrace{C_3\cos 2t + C_4\sin 2t + C_5 t\cos 2t + C_6 t\sin 2t}^{y_h} \tag{1}$$

To determine C_1 and C_2, we insert $y_p(t)$ into the ODE and obtain

$$\begin{aligned} \left(D^4 + 8D^2 + 16\right) y_p(t) &= -\sin t \\ \left(D^4 + 8D^2 + 16\right)(C_1\cos t + C_2\sin t) &= -\sin t \\ C_1\left(D^4 + 8D^2 + 16\right)\cos t + C_2\left(D^4 + 8D^2 + 16\right)\sin t &= -\sin t \end{aligned} \tag{2}$$

But $D^4(\cos t) = D^3(-\sin t) = D^2(-\cos t) = D(\sin t) = \cos t$ and $D^2(\cos t) = D(-\sin t) = -\cos t$ and $D^4(\sin t) = D^3(\cos t) = D^2(-\sin t) = D(-\cos t) = \sin t$ and $D^2(\sin t) = D(\cos t) = -\sin t$, hence (2) becomes

$$\begin{aligned} C_1(\cos t - 8\cos t + 16\cos t) + C_2(\sin t - 8\sin t + 16\sin t) &= -\sin t \\ (C_1 - 8C_1 + 16C_1)\cos t + (C_2 - 8C_2 + 16C_2)\sin t &= -\sin t \end{aligned}$$

Hence by comparing coefficients, we see that

$$\begin{aligned} C_2 - 8C_2 + 16C_2 &= -1 \\ C_1 - 8C_1 + 16C_1 &= 0 \end{aligned}$$

Or

$$\begin{aligned} 9C_2 &= -1 \\ 9C_1 &= 0 \end{aligned}$$

Hence $C_2 = \frac{-1}{9}$ and $C_1 = 0$, therefore the particular solution is

$$\begin{aligned} y_p(t) &= C_1\cos t + C_2\sin t \\ &= \frac{-1}{9}\sin t \end{aligned}$$

Substitute the above into (1), we obtain

$$y(t) = \frac{-\sin t}{9} + C_3\cos 2t + C_4\sin 2t + C_5 t\cos 2t + C_6 t\sin 2t$$

Which is what we are required to show. Book uses different names for the constants I used. This can be easily changed: Let $C_3 = C_1$, Let $C_4 = C_2$, Let $C_5 = C_3$ and let $C_6 = C_4$, the above can be written as

$$y(t) = C_1\cos 2t + C_2\sin 2t + C_3 t\cos 2t + C_4 t\sin 2t - \frac{\sin t}{9}$$

1.2 Part b

We need to solve $\left(D^3 - 2D^2 + D - 2\right) y(t) = 0$ subject to the initial conditions $y(0) = y'(0) = y''(0) = 1$. The characteristic equation is

$$r^3 - 2r^2 + r - 2 = 0$$

By trial and error, we see that

$$\begin{aligned}(r-2)(r-j)(r+j) &= (r-2)\left(r^2+1\right)\\ &= r^3-2r^2+r-2\end{aligned}$$

Therefore, the roots are $r_1 = 2, r_2 = j, r_3 = -j$, hence the solution can be written as

$$\begin{aligned}y(t) &= c_1 e^{r_1 t} + c_2 e^{r_2 t} + c_3 e^{r_3 t}\\ &= c_1 e^{2t} + c_2 e^{jt} + c_3 e^{-jt}\\ &= c_1 e^{2t} + c_2(\cos t + j\sin t) + c_3(\cos t - j\sin t)\\ &= c_1 e^{2t} + (c_2+c_3)\cos t + (jc_2 - jc_3)\sin t\end{aligned}$$

Let $c_2 + c_3 = C_2$ and let $jc_2 - jc_3 = C_3$, the above can be written as

$$y(t) = C_1 e^2 t + C_2\cos t + C_3 \sin t \tag{1}$$

Now to find the constants C_i we apply the boundary conditions. The first boundary condition $y(0) = 1$ yields

$$y(0) = 1 = C_1 + C_2 \tag{2}$$

Now

$$y'(t) = 2C_1 e^{2t} - C_2\sin t + C_3\cos t$$

And the second boundary condition $y'(0) = 1$ yields

$$y'(0) = 1 = 2C_1 + C_3 \tag{3}$$

and

$$y''(t) = 4C_1 e^{2t} - C_2\cos - C_3\sin t$$

and the third boundary condition $y''(0) = 1$ yields

$$y''(0) = 1 = 4C_1 - C_2 \tag{4}$$

So we have 3 equations to solve for C_1, C_2, C_3. Add (2) and (4), we obtain $2 = 5C_1$, hence

$$C_1 = \frac{2}{5}$$

Hence from (2) we obtain $C_2 = 1 - \frac{2}{5}$

$$C_2 = \frac{3}{5}$$

and from (3) we obtain

$C_3 = 1 - 2C_1 = 1 - \frac{4}{5}$, hence

$$C_3 = \frac{1}{5}$$

Hence the solution is from (1) is found to be

$$\begin{aligned}y(t) &= C_1 e^{2t} + C_2\cos t + C_3\sin t\\ &= \frac{2}{5}e^{2t} + \frac{3}{5}\cos t + \frac{1}{5}\sin t\\ &= \frac{1}{5}\left(2e^{2t} + 3\cos t + \sin t\right)\end{aligned}$$

Which is the answer we are asked to show.

1.3 Part(c)

The ODE is

$$\left(D^4 - D\right) y(t) = t^2$$

Hence $L \equiv D^4 - D$ and $L_A = D^3$ since $D^3\left(t^2\right) = D^2(2t) = D(2) = 0$, then the above ODE can be written as

$$D^3\left(D^4 - D\right) y(t) = 0$$

And the characteristic equation is

$$\begin{aligned} r^3\left(r^4-r\right) &= 0 \\ r^3 r\left(r^3-1\right) &= 0 \end{aligned}$$

Hence, for the roots that are related to the particular solution are $r_1 = r_2 = r_3 = 0$.

And the roots that are related to the homogenous solution are $r_4 = 0$ (notice now that this root is repeated 4 times now), and the roots of $\left(r^3-1\right) = 0$ which are the cubic roots of unity and can be found as follows

$$\begin{aligned} r^3 &= 1 \\ r^3 &= e^{2\pi j} \\ r &= e^{\frac{2\pi}{3}j} \end{aligned}$$

Hence the 3 roots of unity are $1, e^{\frac{2\pi}{3}j}, e^{\frac{4\pi}{3}j}$, therefore the first root of unity 1, and the second root of unity is $e^{\frac{2\pi}{3}j} = \cos\left(\frac{2}{3}\pi\right) + j\sin\left(\frac{2}{3}\pi\right) = -\frac{1}{2} + j\frac{1}{2}\sqrt{3}$and the third root of unity is $e^{\frac{4\pi}{3}j} = \cos\left(\frac{4}{3}\pi\right) + j\sin\left(\frac{4}{3}\pi\right) = -\frac{1}{2} - j\frac{1}{2}\sqrt{3}$

Hence $r_5 = 1, r_6 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}, r_7 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$, in otherwords, the solution is

$$y(t) = \overbrace{c_1 e^{r_1 t} + c_2 t e^{r_2 t} + c_3 t^2 e^{r_3 t}}^{y_p(t)} + \overbrace{c_4 t^3 e^{r_4 t} + c_5 e^{r_5 t} + c_6 e^{r_6 t} + c_7 e^{r_7 t}}^{y_h(t)}$$

We now substitute the values of r_i we found and obtain

$$\begin{aligned} y(t) &= \overbrace{c_1 + c_2 t + c_3 t^2}^{y_p(t)} + \overbrace{c_4 t^3 + c_5 e^t + c_6 e^{\left(-\frac{1}{2}+j\frac{1}{2}\sqrt{3}\right)t} + c_7 e^{\left(-\frac{1}{2}-j\frac{1}{2}\sqrt{3}\right)t}}^{y_h(t)} \\ &= c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^t + c_6 e^{-\frac{1}{2}t} e^{j\frac{\sqrt{3}}{2}t} + c_7 e^{-\frac{1}{2}t} e^{-j\frac{\sqrt{3}}{2}t} \\ &= c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^t + e^{-\frac{1}{2}t}\left(c_6 e^{j\frac{\sqrt{3}}{2}t} + c_7 e^{-j\frac{\sqrt{3}}{2}t}\right) \\ &= c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^t + e^{-\frac{1}{2}t}\left(c_6\left[\cos\frac{\sqrt{3}}{2}t + j\sin\frac{\sqrt{3}}{2}t\right] + c_7\left[\cos\frac{\sqrt{3}}{2}t - j\sin\frac{\sqrt{3}}{2}t\right]\right) \end{aligned}$$

Hence

$$y(t) = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^t + e^{-\frac{1}{2}t}\left([c_6 + c_7]\cos\frac{\sqrt{3}}{2}t + [jc_6 - jc_7]\sin\frac{\sqrt{3}}{2}t\right)$$

Let $[c_6 + c_7] = C_6$ and let $jc_6 - jc_7 = C_7$ the above becomes

$$y(t) = \overbrace{c_1 + c_2 t + c_3 t^2}^{y_p(t)} + \overbrace{c_4 t^3 + c_5 e^t + e^{-\frac{t}{2}}\left(C_6\cos\frac{\sqrt{3}}{2}t + C_7\sin\frac{\sqrt{3}}{2}t\right)}^{y_h(t)} \tag{1}$$

Now plug $y_p(t)$ back in the original ODE we obtain

$$\begin{aligned} \left(D^4 - D\right) y_p(t) &= t^2 \\ \left(D^4 - D\right)\left(c_1 + c_2 t + c_3 t^2\right) &= t^2 \\ D^4\left(c_1 + c_2 t + c_3 t^2\right) - D\left(c_1 + c_2 t + c_3 t^2\right) &= t^2 \\ D^3\left(c_2 + 2c_3 t\right) - \left(c_2 + 2c_3 t\right) &= t^2 \\ D^2\left(2c_3\right) - \left(c_2 + 2c_3 t\right) &= t^2 \\ -\left(c_2 + 2c_3 t\right) &= t^2 \end{aligned}$$

Hence we see that $c_2 = 0$ and $c_3 = 0$, then (1) simplifies to

$$y(t) = c_1 + c_4 t^3 + c_5 e^t + e^{-\frac{t}{2}}\left(C_6\cos\frac{\sqrt{3}}{2}t + C_7\sin\frac{\sqrt{3}}{2}t\right) \tag{2}$$

To find c_4 , we substitute $y(t)$ found above, into the ode, hence

$$\left(D^4 - D\right) y(t) = t^2$$
$$\left(D^4 - D\right)\left[c_1 + c_4 t^3 + c_5 e^t + e^{-\frac{t}{2}}\left(C_6 \cos\frac{\sqrt{3}}{2}t + C_7 \sin\frac{\sqrt{3}}{2}t\right)\right] = 0$$

Now, since we only care about finding c_4, we can just apply D on that, hence

$$\begin{aligned}
D^4\left[\cdots + c_4 t^3 + \cdots\right] - D\left[\cdots + c_4 t^3 + \cdots\right] &= t^2 \\
D^3\left[\cdots + 3c_4 t^2 + \cdots\right] - \left[\cdots + 3c_4 t^2 + \cdots\right] &= t^2 \\
D^2\left[\cdots + 6c_4 t + \cdots\right] - \left[\cdots + 3c_4 t^2 + \cdots\right] &= t^2 \\
D\left[\cdots + 6c_4 + \cdots\right] - \left[\cdots + 3c_4 t^2 + \cdots\right] &= t^2 \\
-\left[\cdots + 3c_4 t^2 + \cdots\right] &= t^2
\end{aligned}$$

By comparing coefficients, we see that $c_4 = -\frac{1}{3}$ then (1) becomes

$$y(t) = c_1 + c_5 e^t + e^{-\frac{t}{2}}\left(C_6 \cos\frac{\sqrt{3}}{2}t + C_7 \sin\frac{\sqrt{3}}{2}t\right) - \frac{1}{3}t^3$$

Which is what we are asked to show.