HW1, EGME 431 (Mechanical Vibration)

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Contents

1 Problem 1.6

Figure P1.5

1.6. Find the equation of motion for the system of Figure P1.6 and compute the formula for the natural frequency. In particular, using static equilibrium along with Newton's law, determine what effect gravity has on the equation of motion and the system's natural frequency. Assume the block slides without friction.

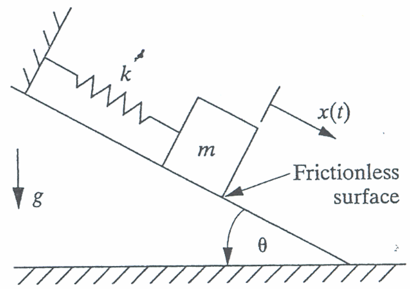


Figure P1.6

Taking displacement along the x-direction shown to be from the static equilibrium position, then applying $\sum F_x = m\ddot{x}$ along the shown x direction, we obtain

$$m\ddot{x} = -kx$$
$$\ddot{x} + \frac{k}{m}x = 0$$

which is the equation of motion. To obtain the natural frequency, we consider free vibration $\ddot{x} + \frac{k}{m}x = 0$, which implies that $\omega_n = \sqrt{\frac{k}{m}}$, hence we see that the $\boxed{\text{natural frequency is independent of } g}$

We see that gravity has no effect on the spring mass system, this is because we use x to be from the static equilibrium position of the spring.

2 Problem 1.16

1.16. A machine part is modeled as a pendulum connected to a spring as illustrated in Figure P1.16. Ignore the mass of pendulum's rod and derive the equation of motion. Then, following the procedure used in Example 1.1.1, linearize the equation of motion, and compute the formula for the natural frequency. Assume that the rotation is small enough so that the spring only deflects horizontally.

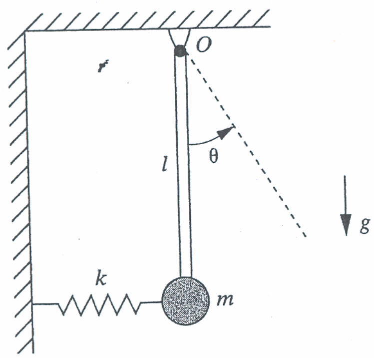
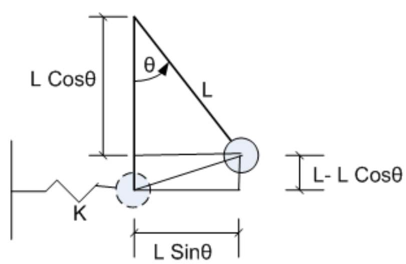


Figure P1.16

First we need to derive the equation of motion. Considering the following diagram



Using as generalized coordinates θ, we obtain

$$T = \frac{1}{2} m \left(L\dot{\theta} \right)^2$$
$$U = \frac{1}{2} k \left(L \sin\theta \right)^2 + mg \left(L - L\cos\theta \right)$$

Notice that in the calculation of U above, we assumed that the spring stretches by $L\sin\theta$ in the horizontal direction only, which we are allowed to do for small θ.

Now we can find Lagrangian

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2} m \left(L\dot{\theta} \right)^2 - \frac{1}{2} k L^2 \sin^2\theta - mgL \left(1 - \cos\theta \right) \end{aligned}$$

Hence the equation of motion is

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta} &= 0\\
\frac{d}{dt}\left(mL^2\dot{\theta}\right)-\left(-kL^2\sin\theta\cos\theta-mgL\sin\theta\right) &= 0\\
mL^2\ddot{\theta}+kL^2\sin\theta\cos\theta+mgL\sin\theta &= 0
\end{aligned}
$$

The above is nonlinear equation. Linearize around $\theta=0$ (equilibrium point) using Taylor series, and for small θ we obtain $\sin\theta\approx\theta$ and $\cos\theta\approx 1$, hence the above becomes

$$
\begin{aligned}
mL\ddot{\theta}+kL\theta+mg\theta &= 0\\
\ddot{\theta}+\left(\frac{mg+kL}{mL}\right)\theta &= 0
\end{aligned}
$$

Hence effective $\omega_n$can be found from

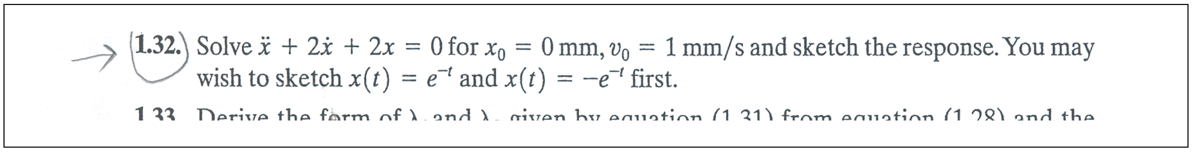
$$\omega_n^2=\frac{mg+kL}{mL}$$

Hence

$$\boxed{\omega_n=\sqrt{\frac{g}{L}+\frac{k}{m}}}$$

Compare the above to the natural frequency of pendulum with no spring attached which is $\omega_n=\sqrt{\frac{g}{L}}$, we can see the effect of adding a spring on the natural frequency: The more stiff the spring is, in other words, the larger k is, the larger ω_n will become, and the smaller the period of oscillation will be. We conclude that a pendulum with a spring attached to it will always oscillate with a period which is smaller than the same pendulum without the spring attached. This makes sense as a mass with spring alone has $\omega_n=\sqrt{\frac{k}{m}}$

3 Problem 1.32


1.32. Solve $\ddot{x} + 2\dot{x} + 2x = 0$ for $x_0 = 0$ mm, $v_0 = 1$ mm/s and sketch the response. You may wish to sketch $x(t) = e^{-t}$ and $x(t) = -e^{-t}$ first.

1.33 Derive the form of λ_1 and λ_2 given by equation (1.31) from equation (1.28) and the

We need to solve $\ddot{x}+2\dot{x}+2x=0$ for $x_0=0mm$ and $v_0=1mm/s$

The characteristic equation is $\lambda^2+2\lambda+2=0$ which has roots $\lambda_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{4-8}}{2}=-1\pm j$

Hence the solution is

$$x_h=e^{-t}\left(A\cos t+B\sin t\right)$$

is the general solution. Now we use I.C. to find A,B. When $t=0$

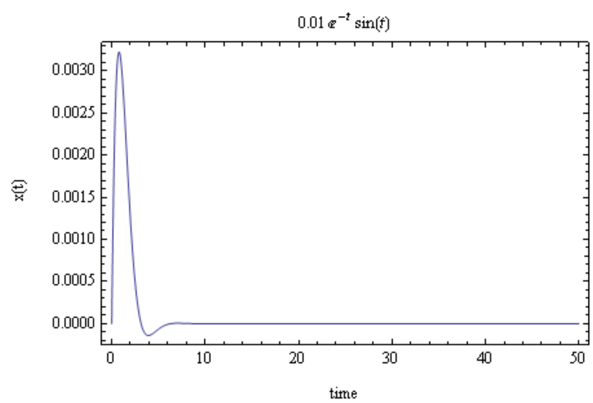
$$0=A$$

Hence $x_h=Be^{-t}\sin t$, and $\dot{x}_h=Be^{-t}\cos t-Be^{-t}\sin t$ and at $t=0$, we obtain $0.01=B$

Then

$$\boxed{x_h=0.01e^{-t}\sin t}$$

This is a plot of the solution for t up to 50 seconds



4 Problem 1.43

> $x_0 = 100$ mm.
> → **1.43.** Solve $\ddot{x} - \dot{x} + x = 0$ with $x_0 = 1$ and $v_0 = 0$ for $x(t)$ and sketch the response.
> **1.44.** A spring–mass–damper system has mass of 100 kg, stiffness of 3000 N/m, and damping coefficient of 300 kg/s. Calculate the undamped natural frequency, the damping ratio, and

We need to solve $\ddot{x} - \dot{x} + x = 0$ for $x_0 = 1$ and $v_0 = 0$

The characteristic equation is $\lambda^2 - \lambda + 1 = 0$ which has roots $\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

Hence the solution is

$$x_h = e^{\frac{1}{2}t}\left(A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right)$$

is the general solution. Now we use I.C. to find A, B. When $t = 0$

$$1 = A$$

Hence $x_h = e^{\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right)$, and

$$\dot{x}_h = \frac{1}{2}e^{\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right) + e^{\frac{1}{2}t}\left(-\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t + B\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t\right)$$

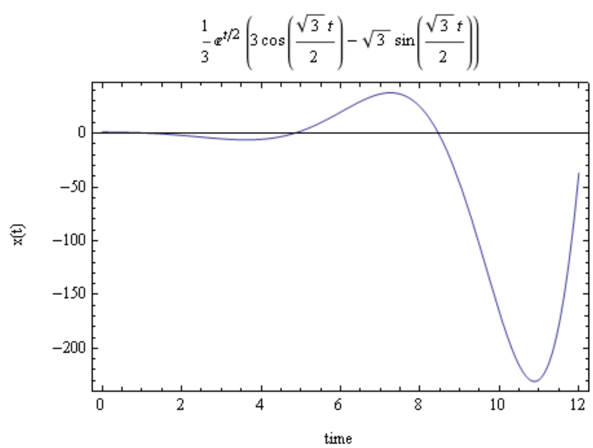
and at $t = 0$, we obtain

$$0 = \frac{1}{2} + B\frac{\sqrt{3}}{2}$$
$$B = \frac{-1}{\sqrt{3}}$$

Hence

$$\boxed{x_h = e^{\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)}$$

This is a plot of the solution for t up to 12 seconds



5 Problem 1.62

How much effect does the rotational inertia have on the undamped natural frequency?

1.62. Use Lagrange's formulation to calculate the equation of motion and the natural frequency of the system of Figure P1.62. Model each of the brackets as a spring of stiffness k, and assume the inertial of the pulleys is negligible.

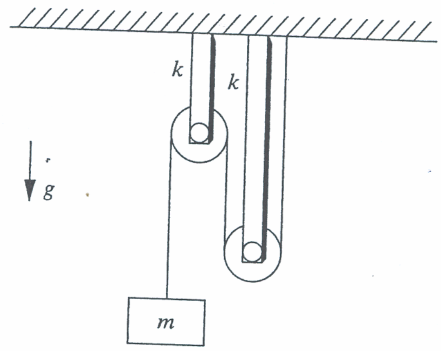


Figure P1.62

This is a single degree of freedom linear system. Assume x from static equilibrium, then (using parallel springs) we obtain

$$T = \frac{1}{2}m\dot{x}^2$$
$$U = \frac{1}{2}kx^2 + \frac{1}{2}kx^2 = kx^2$$

Hence $L = T - U = \frac{1}{2}m\dot{x}^2 - kx^2$ and the Lagrangian equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$
$$\frac{d}{dt}(m\dot{x}) - (-2kx) = 0$$

Hence equation of motion is

$$\boxed{m\ddot{x} + 2kx = 0}$$

And $\omega_n = \sqrt{\frac{2k}{m}}$

6 Problem 1.90

Section 1.8 (see also Problem 1.43)

1.90. Consider the system of Figure P1.90. (a) Write the equations of motion in terms of the angle, θ, the bar makes with the vertical. Assume linear deflections of the springs and linearize the equations of motion. (b) Discuss the stability of the linear system's solutions in terms of the physical constants, m, k, and l. Assume the mass of the rod acts at the center as indicated in the figure.

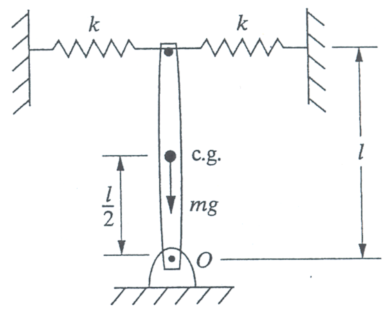
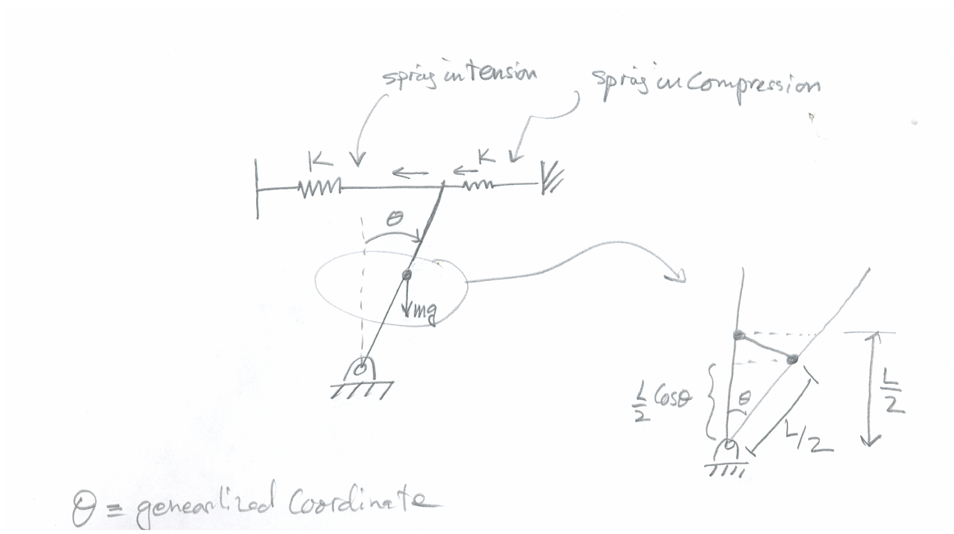


Figure P1.90

Solution

Part(a)



$$T = \frac{1}{2}m\left(\frac{l}{2}\dot{\theta}\right)^2$$
$$= \frac{1}{8}ml^2\dot{\theta}^2$$

$$U_{springs} = \frac{1}{2}k(l\sin\theta)^2 + \frac{1}{2}k(l\sin\theta)^2$$

Assuming small angle oscillation, $\sin\theta \simeq \theta$, hence

$$U_{springs} = kl^2\theta^2$$

and for the mass, since it losses potential, we have

$$U_{mass} = -mg\left(\frac{l}{2} - \frac{l}{2}\cos\theta\right)$$

Hence Lagrangian L is

$$\begin{aligned}
L &= T - (U_{springs} + U_{mass}) \\
&= \frac{1}{8}ml^2\dot{\theta}^2 - \left(kl^2\theta^2 - mg\frac{l}{2}(1-\cos\theta)\right) \\
&= \frac{1}{8}ml^2\dot{\theta}^2 - kl^2\theta^2 + mg\frac{l}{2} - mg\frac{l}{2}\cos\theta
\end{aligned}$$

Now find the Lagrangian equation

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} &= \frac{1}{4}ml^2\dot{\theta} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \frac{1}{4}ml^2\ddot{\theta} \\
\frac{\partial L}{\partial \theta} &= -2kl^2\theta + mg\frac{l}{2}\sin\theta
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= \frac{1}{4}ml^2\ddot{\theta} - \left(-2kl^2\theta + mg\frac{l}{2}\sin\theta\right) \\
&= \frac{1}{4}ml^2\ddot{\theta} + 2kl^2\theta - mg\frac{l}{2}\sin\theta
\end{aligned}$$

And the equation of motion is

$$\begin{aligned}
\frac{1}{4}ml^2\ddot{\theta} + 2kl^2\theta - mg\frac{l}{2}\sin\theta &= 0 \\
\ddot{\theta} + \frac{8k}{m}\theta - 2\frac{g}{l}\sin\theta &= 0
\end{aligned}$$

Linearize by setting $\sin\theta \simeq \theta$ we obtain equation of motion

$$\ddot{\theta} + \theta\left(\frac{8k}{m} - 2\frac{g}{l}\right) = 0 \tag{1}$$

Hence

$$\boxed{\omega_n = \sqrt{2\left(4\frac{k}{m} - \frac{g}{l}\right)}}$$

Part (b)

To discuss stability, we need to determine the location of the roots of the characteristic equation of the homogeneous EQM, hence from equation (1), we see that

$$\ddot{\theta} + \omega_n^2\theta = 0$$

And assuming solution $\theta(t) = e^{\lambda t}$ leads to the characteristic equation

$$\begin{aligned}\lambda^2 + \omega_n^2 &= 0\\ \lambda^2 &= -\omega_n^2\\ \lambda &= \pm\sqrt{-\omega_n^2}\\ &= \pm j\sqrt{\omega_n^2}\end{aligned}$$

Since $\omega_n^2 > 0$, then

$$\boxed{\lambda = \pm j\omega_n}$$

Since roots of the characteristic equation on the imaginary axis, this is a **marginally stable system** regardless of the values of m, l, k.

Since we are looking at the linearized system, there is only one equilibrium point, and the system is either stable or not. Here we found it is marginally stable. The effect of changing k, l, m is to change the period of oscillation around the equilibrium point.