A CLASS ASSIGNED PROBLEM FOR PHYSICS 555ASPRING 2008. CSUF
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1 Problem

Show that the recurence formula

$$C_q = -\frac{2(k-q)}{q(q+2l+1)(k+l)} C_{q-1} \tag{1}$$

can be written as

$$C_q = (-1)^q \left(\frac{2}{k+l}\right)^q \frac{(k-1)!}{(k-q-1)!} \frac{(2l+1)!}{q!\,(q+2l+1)!} C_0 \tag{2}$$

2 Solution

Proof by induction on q. For $q=1$, equation (1) becomes

$$C_1 = -\frac{2(k-1)}{(2l+2)(k+l)} C_0$$

and equation (2) becomes

$$\begin{aligned} C_1 &= (-1)\left(\frac{2}{k+l}\right) \frac{(k-1)!}{(k-2)!} \frac{(2l+1)!}{(2l+2)!} C_0 \\ &= -\frac{2(k-1)}{(k+l)(2l+2)!} C_0 \end{aligned}$$

Hence it is true for $q=1$. Now *assume* it is true for $q=n$, in otherwords, assume that

$$C_n = -\frac{2(k-n)}{n(n+2l+1)(k+l)} C_{n-1} \tag{3}$$

implies

$$C_n = (-1)^n \left(\frac{2}{k+l}\right)^n \frac{(k-1)!}{(k-n-1)!} \frac{(2l+1)!}{n!\,(n+2l+1)!} C_0 \tag{4}$$

Now for the induction step. we need to show that it is true for $n+1$, i.e. given (4) is true, we need to show that, by replacing n by $n+1$ in the above, that

$$C_{n+1} = -\frac{2(k-(n+1))}{(n+1)((n+1)+2l+1)(k+l)} C_n \tag{5}$$

implies

$$\begin{aligned} C_{n+1} &= (-1)^{n+1} \left(\frac{2}{k+l}\right)^{n+1} \frac{(k-1)!}{(k-(n+1)-1)!} \frac{(2l+1)!}{(n+1)!\,((n+1)+2l+1)!} C_0 \\ &= (-1)^{n+1} \left(\frac{2}{k+l}\right)^{n+1} \frac{(k-1)!}{(k-n-2)!} \frac{(2l+1)!}{(n+1)!\,(n+2l+2)!} C_0 \end{aligned} \tag{6}$$

We start with (5), and replace the C_n term with what we assumed to be true from (4), hence (5) can be rewritten as

$$C_{n+1} = -\frac{2\left(k-(n+1)\right)}{(n+1)\left((n+1)+2l+1\right)(k+l)} \overbrace{\left[(-1)^n \left(\frac{2}{k+l}\right)^n \frac{(k-1)!}{(k-n-1)!} \frac{(2l+1)!}{n!\,(n+2l+1)!} C_0\right]}^{C_n \text{ from (4)}}$$

Simplify the above leads to

$$C_{n+1} = (-1)^{n+1} \left(\frac{2}{k+l}\right)^{n+1} \frac{(k-1)!}{(k-n-2)!} \frac{(2l+1)!}{n!\,(n+2l+2)!} C_0$$

Which is (6). Therefore, the relationship is true for any n. QED