Handout · 1st lecture 1/22/08

A Problem in Conditional Probability

A number is chosen at random from the interval $[0, 1]$. This value is placed in a box, and twice this value is placed in a second box. One of these boxes is selected at random and opened to reveal the number inside. Given this observed value, what is the probability that this number is the smaller of the two.

1. A Solution Let the random variable X denote the observed number, and let S denote the event that the selected box contains the smaller number. We seek $P(S|X = x)$ for $0 \le x \le 2$. The quantity $P(S|X = x)$ is undefined for other values of x. We will apply Bayes Theorem which gives us

$$P(S|X = x) = \frac{\overbrace{f_X(x|S)}^{U[0,1]}\overbrace{P(S)}^{\frac{1}{2}}}{f_X(x)} ,$$

where $f_X(x)$ is the density function of the random variable X, and $f_X(x|S)$ is the conditional density of X given the event S. If $1 < x \le 2$, then evidently we have the larger of the two numbers, and so $P(S|X = x) = 0$ when $1 < x \le 2$. Thus, we need consider only the case $0 \le x \le 1$. Since a box is selected at random, $P(S) = 1/2$. Next, the conditional density of X given the event S is just the uniform density on $[0, 1]$. Thus, $f_X(x|S) = 1$ for $0 \le x \le 1$, and $f_X(x|S) = 0$ otherwise. Finally, to determine the density function of the random variable X, we use

$$f_X(x) = \overbrace{f_X(x|S)}^{U[0,1]}\overbrace{P(S)}^{\frac{1}{2}} + \overbrace{f_X(x|\bar{S})}^{U[0,2]}\overbrace{P(\bar{S})}^{\frac{1}{2}} .$$

try to prove mathematically that $U[0,1] * 2 \to U[0,2]$

The conditional density of X given the event $\bar{S}$, is the uniform density on $[0, 2]$. Thus, $f_X(x|\bar{S}) = 1/2$ for $0 \le x \le 2$, and $f_X(x|\bar{S}) = 0$ otherwise. Hence, for $0 \le x \le 1$,

$$f_X(x) = f_X(x|S)P(S) + f_X(x|\bar{S})P(\bar{S}) = 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} ,$$

while for $1 < x \le 2$,

$$f_X(x) = f_X(x|S)P(S) + f_X(x|\bar{S})P(\bar{S}) = 0 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} ,$$

and otherwise, $f_X(x) = 0$. Returning now to the formula for $P(S|X = x)$ we have for $0 \le x \le 1$,

$$P(S|X = x) = \frac{(1)(1/2)}{3/4} = \frac{2}{3} .$$

* This result shows that when we select a box at random, and observe a value between 0 and 1, there is a 2/3 chance that the observed value is the smaller of the two.

use rand to generate original
Then double it
Then generate random # to select S or $\bar{S}$

Exercise

Y is final value even if we switch.

1. Suppose that when we select a box, and observe the value, we have an opportunity to switch to the other box. The result above suggests that if we observe a value between 0 and 1, then we should switch, and otherwise, hold the value we have. Let the random variable Y denote the reward using such a strategy. Write a simulation program (in MATLAB, say) to estimate the expected value of Y. Use a 95% confidence interval, and determine the sample size so that the relative accuracy of your estimate is about one percent. In your report, explain how you determined your sample size. Also, compare theory and practise; that is, did your confidence interval include the true value. → estimate S from data.

$\boxed{\bar{x} \pm 1.96\frac{S}{\sqrt{n}}}$ 95% C.I. so need $\leq 1\%\ \bar{x}$ ~~error~~ ← Error need to be within this.

2. **Expected Value of Y** Suppose now the strategy is to switch if the observed value is less than or equal to 1, and otherwise to hold. Let Y be the reward using this strategy. Then

$$E(Y) = E(Y \mid X \leq 1)P(X \leq 1) + E(Y \mid X > 1)P(X > 1)\ .$$

Consider first the events $\{X \leq 1\}$ and $\{X > 1\}$. In order for the event $\{X > 1\}$ to occur, we must select the box with the larger value, which occurs with probability 1/2, and also the original value must be in the interval $(1/2, 1)$, which occurs with probability 1/2. Since these two events are independent, it follows that $P(X > 1) = (1/2)(1/2) = 1/4$, and further, $P(X \leq 1) = 1 - 1/4 = 3/4$.

Next consider the expected value of Y given that the event $\{X > 1\}$ has occurred. Then Y is the observed value X. Given that the event $\{X > 1\}$ has occurred, the random variable X is uniformly distributed over the interval $(1, 2)$. Hence, $E(Y \mid X > 1) = 3/2$.

Consider now the expected value of Y given that the event $\{X \leq 1\}$ has occurred. Here, we will switch to the value in the other box. However, we will either (α) switch to the larger value, which occurs with probability $2/3$, or (β) switch to the smaller value, which occurs with probability $1 - 2/3 = 1/3$. In case (α), Y is the larger value, which is uniformly distributed over the interval $(0, 2)$. Hence its expected value is 1, and so $E(Y|\alpha) = 1$. In case (β), Y is the smaller value, which now, because the event $\{X \leq 1\}$ has taken place, is uniformly distributed over the interval $(0, 1/2)$. Hence, the expected value is $1/4$, and so $E(Y|\beta) = 1/4$. Thus,

$$E(Y \mid X \leq 1) = E(Y|\alpha)P(\alpha) + E(Y|\beta)P(\beta) = 1 \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{1}{3} = \frac{3}{4}.$$

We are ready finally to compute the expected value $E(Y)$. From the formula above, we get

$$E(Y) = E(Y|X \leq 1)P(X \leq 1) + E(Y|X > 1)P(X > 1) = \frac{3}{4} \cdot \frac{3}{4} + \frac{3}{2} \cdot \frac{1}{4} = \boxed{\frac{15}{16}}.$$

Exercise

1. Find the density function of the random variable Y.

trzy to do this $f_Y(y) = f_Y(y|S)P(S) + f_Y(y|\bar{S})P(\bar{S})$