Attempt at Challenge problems for Math 504

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1 Problem 1

1. A stochastic process has states $0, 1, 2, \cdots, r$. If the process is in state $i \geq 1$, then during the next time step, it goes to a state that is selected at random from among the states $0, 1, 2, \cdots, i-1$. If the process is in state 0, it stays there. This process can be modelled as a Markov chain. (a) Specify the one-step transition probability matrix. (b) Let Y_i be the number of steps that the process takes to reach state 0, given that it starts in state i. Derive a linear system of equations that determines the expected values $E(Y_i)$, for $i = 1, 2, \cdots, r$. Write the linear system in the form $Ax = b$, where the components of the vector x are the unknown expected values. Specify the matrices A and b.

Problem 1 Solve explicitly for the expected values $E(Y_i)$, $i = 1, 2, \cdots, r$. Of particular interest is $E(Y_r)$, when r is large. What is an asymptotic estimate of this quantity?

Solution:

We start with the solution we already[1] obtained for $E(Y_i)$ which is

$$E(Y_i) = 1 + \sum_{k=1}^{i-1} E(Y_k) P_{ik}$$

Let $E(Y_i) = x_i$ hence the above can be written as

$$x_i = 1 + \sum_{k=1}^{i-1} x_k P_{ik}$$

[1] See my midterm solution for this problem

But $P_{ik} = \frac{1}{i}$ then the above becomes

$$x_i = 1 + \frac{1}{i}\sum_{k=1}^{i-1} x_k$$

Multiply by i the above becomes

$$i\, x_i = i + \sum_{k=1}^{i-1} x_k$$

Therefore, we obtain the following equations for $i = 1 \cdots r$

For $i = 1$

$$x_1 = 1 \tag{1}$$

For $i = 2$

$$2x_2 = 2 + x_1 \tag{2}$$

For $i = 3$

$$3x_3 = 3 + x_1 + x_2 \tag{3}$$

For $i = 4$

$$4x_4 = 4 + x_1 + x_2 + x_3 \tag{4}$$

etc...

Now evaluate (2)-(1) and (3)-(2) and (4)-(3), etc... we obtain the following equations

(2)-(1) gives

$$\begin{aligned} 2x_2 - x_1 &= 2 + x_1 - 1 \\ x_2 &= \frac{1 + 2x_1}{2} \end{aligned} \tag{5}$$

(3)-(2) gives

$$\begin{aligned} 3x_3 - 2x_2 &= 3 + x_1 + x_2 - 2 - x_1 \\ 3x_3 - 2x_2 &= 1 + x_2 \\ x_3 &= \frac{1 + 3x_2}{3} \end{aligned}$$

etc... Hence we see that for the r^{th} term we obtain

$$x_r = \frac{1 + r\, x_{r-1}}{r}$$

Hence

$$x_r = \frac{1}{r} + x_{r-1} \tag{6}$$

Now replace r by $r-1$ in the above we obtain

$$x_{r-1} = \frac{1}{r-1} + x_{r-2}$$

replace the above in (6) we obtain

$$x_r = \frac{1}{r} + \left(\frac{1}{r-1} + x_{r-2}\right) \tag{7}$$

And again, in the above, $x_{r-2} = \frac{1}{r-2} + x_{r-3}$ hence (7) becomes

$$x_r = \frac{1}{r} + \left(\frac{1}{r-1} + \left(\frac{1}{r-2} + x_{r-3}\right)\right) \tag{8}$$

and so on, until we get to $x_1 = 1$, hence we obtain

$$\boxed{x_r = \frac{1}{r} + \frac{1}{r-1} + \frac{1}{r-2} + \cdots + 1}$$

Hence

$$x_r = \sum_{k=1}^{r} \frac{1}{k}$$

Which is the harmonic series. Now, it is know that[2]

$$\boxed{\lim_{r\to\infty} x_r = \log(r) - \gamma}$$

Where γ is Euler Gamma constant given by

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In[47]:= N[EulerGamma, 16]

Out[47]= 0.5772156649015329
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[2] Do I have to proof this?
Please see http://en.wikipedia.org/wiki/Harmonic_number