HW # 7

Math 501

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CSUF

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section 4.4 # 7 (a), (c), 21, 37, 40 (a), (c), 48

Section 4.5 # 2, 5, 8, 12, 22, 24

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section 4.4 #7

Determine if these are Norms on $\mathbb{R}^n$

(a) $\max\{|x_2|, |x_3|, \cdots, |x_n|\}$

notice that $x_1$ cordinate of the Vector is not used.

hence $\|v\|$ where $v = \{c, 0, 0, \cdots, 0\}$

will give $\max\{0, 0, 0, \cdots, 0\} = 0$ ✓

so $\|v\| = 0$ but $v \neq 0$ since it has

one component not zero.

hence property (1) of norm is Violated which

says that $\|v\| > 0$ if $v \neq 0$

Not Norm definition

(b) $\left\{\sum_{i=1}^{n} |x_i|^{1/2}\right\}^2$

Property (1) is Clearly valid here since $\|v\|$ can be zero

only if $x_i = 0$ $i = 1 \cdots n$

property (11) which say that $\|\lambda v\| = \lambda \|v\|$ is also valid

since $\left\{\sum_{i=1}^{n} |\lambda x_i|^{1/2}\right\}^2 = \left\{\sum_{i=1}^{n} |\lambda|^{1/2} |x_i|^{1/2}\right\}^2 = \left\{|\lambda|^{1/2} \sum |x_i|^{1/2}\right\}^2$

$= \lambda \left\{\sum |x_i|^{1/2}\right\}^2 = \lambda \|v\|$

now Check property (3): $\|x+y\| \leq \|x\| + \|y\|$

Consider $v_1 = \{v_1, v_2\}$ and $w = \{w_1, w_2\}$. Then $x + w = \{v_1 + w_1, v_2 + w_2\}$

Then $\|v + w\| = \left(\sqrt{v_1 + w_1} + \sqrt{v_2 + w_2}\right)^2 = (v_1 + w_1) + (v_2 + w_2) + 2\sqrt{(v_1 + w_1)(v_2 + w_2)}$

but $\|v\| + \|w\| = \left(\sqrt{v_1} + \sqrt{v_2}\right)^2 + \left(\sqrt{w_1} + \sqrt{w_2}\right)^2 = v_1 + v_2 + 2\sqrt{v_1 v_2} + w_1 + w_2 + 2\sqrt{w_1 w_2}$

→

so we ask

is $||v+w|| \le ||v|| + ||w||$ ?

is $\left(\sqrt{v_1+w_1} + \sqrt{v_2+w_2}\right)^2 \le \left(\sqrt{v_1}+\sqrt{v_2}\right)^2 + \left(\sqrt{w_1}+\sqrt{w_2}\right)^2$

~~try $v=(10,0)$, $w=(0,10)$~~.

$\cancel{||v+w|| = \left(\sqrt{v_1+w_1} + \sqrt{v_2+w_2}\right)^2 = \left(\sqrt{10}+\sqrt{10}\right)^2}$

to make it simpler. try $v=(1,0)$, $w=(0,1)$

so $||v+w|| = \left(\sqrt{v_1+w_1} + \sqrt{v_2+w_2}\right)^2 = (1+1)^2 = 4$

$||v|| + ||w|| = \left(\sqrt{v_1}+\sqrt{v_2}\right)^2 + \left(\sqrt{w_1}+\sqrt{w_2}\right)^2$

$= 1^2 + 1^2 = 2$

so $\boxed{||v+w|| > ||v|| + ||w||}$

so property 3 NOT satisfied

so $\boxed{\text{NOT Norm}}$

Section 4.4 # 21

let $n = 3$ and let $A = \begin{bmatrix} 4 & -3 & 2 \\ -1 & 0 & 5 \\ 2 & 6 & -2 \end{bmatrix}$

among all the vectors $x$ satisfying $\|x\|_\infty \le 1$, find one for which $\|Ax\|_\infty$ is as large as possible. Also give numerical value of $\|A\|_\infty$.

Answer

$$\|Ax\|_\infty \le \|A\|_\infty \|x\|_\infty$$

but $\|A\|_\infty = \max$ abs. sum of rows of $A$

$= \max \begin{bmatrix} 9 \\ 6 \\ 10 \end{bmatrix} = \boxed{10}$

so $\|Ax\|_\infty \le 10 \|x\|_\infty$

so $\max \|Ax\|_\infty$ is when $\|x\|_\infty$ is max. which is 1

so $\|Ax\|_\infty \le 10 \longrightarrow$ max is $\boxed{10}$

the $x$ vector which satisfies this is when $\|x\|_\infty = 1$

so need a vector whose max coordinate is 1

so $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ will do.

$Ax = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

$\|Ax\|_\infty = \left\| \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\|_\infty = 4$ not 10.

section 4.5 # 37

Prove these properties

(a) $\|0\| = 0$

(b) $\|x+y\| \geq | \ \|x\| - \|y\| \ |$

(c) $\| \sum_{i=1}^{m} x^{(i)} \| \leq \sum_{i=1}^{m} \| x^{(i)} \|$ for vectors $x^{(1)}, x^{(2)}, \dots, x^{(m)}$

(a) from properties of Norms, we know that

$\| \lambda x \| = \lambda \| x \|$ for $\lambda$ any constant

so let $x = 0$, hence we write

$\| \lambda 0 \| = \lambda \| 0 \|$

but $\lambda(0) = 0$ since scalar multiplication.

so $\| 0 \| = \lambda \| 0 \|$

for this to be valid for any non zero $\lambda$, it must be that $\boxed{\|0\| = 0}$

This is like saying $a = 3a \rightarrow$ only true if $a = 0$

(b) $\|y\| = \| y + (x-x) \| = \| (y+x) + (-x) \|$

but since $\| A+B \| \leq \|A\| + \|B\|$, then above is

$\leq \|(y+x)\| + \|(-x)\|$

so $\|y\| \leq \|(y+x)\| + \|(-x)\|$

so $\|y\| \leq \|y+x\| + \|x\|$ since $\|-x\| = \|x\|$

so $\boxed{\|y\| - \|x\| \leq \|y+x\|}$ by moving $\|x\|$ to LHS.

QED

$\longrightarrow$

prove

(c) $\| \sum_{i=1}^{m} x^{(i)} \| \leq \sum_{i=1}^{m} \| x^{(i)} \|$  For vectors $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$.

LHS is $\| x^1 + x^2 + \;\; + x^m \| = \| x^1 + (x^2 + \cdots \;\; + x^m) \|$

$\leq \| x^1 \| + \| x^2 + x^3 + \cdots + x^m \|$

by triangle inequality.

Repeate the above on $\| x^2 + x^3 + \cdots + x^m \|$ we get

$\| x^1 + x^2 + \cdots + x^m \| \leq \| x^{(1)} \| + \| x^{(2)} + (x^{(3)} + x^{(4)} + \cdots + x^{(m)}) \|$

$\leq \| x^{(1)} \| + \| x^{(2)} \| + \| x^{(3)} + x^{(4)} + \cdots + x^{(m)} \|$

$\vdots$

etc

so we obtain

$\| x^{(1)} + x^{(2)} + \cdots + x^{(m)} \| \leq \| x^{(1)} \| + \| x^{(2)} \| + \cdots + \| x^{(m)} \|$

i.e $\boxed{\| \sum_{i=1}^{m} x^{(i)} \| \leq \sum_{i=1}^{m} \| x^{(i)} \|}$

## Section 4.4 # 40

compute condition numbers using norms $\|A\|_1$, $\|A\|_2$, $\|A\|_\infty$

(a) $\begin{bmatrix} a+1 & a \\ a & a-1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} \alpha & 1 \\ 1 & 1 \end{bmatrix}$

Answer

(a) The condition number of a matrix is defined as $K(A) = \|A\| \cdot \|A^{-1}\|$

Using $\|A\|_1$ norm

$K(A) = \|A\|_1 \cdot \|A^{-1}\|$

$\|A\|_1 = \max \sum_i$ over columns Abs. $\Rightarrow \max\left[\,|a+1| + |a|\,,\ |a| + |a-1|\,\right] = \boxed{2|a|+1}$

$$A^{-1} = \frac{1}{(a+1)(a-1)-a^2}\begin{bmatrix} a-1 & -a \\ -a & a+1 \end{bmatrix} = \frac{1}{a^2-a+a-1-a^2}\begin{bmatrix} a-1 & -a \\ -a & a+1 \end{bmatrix} = -\begin{bmatrix} a-1 & -a \\ -a & a+1 \end{bmatrix} = \begin{bmatrix} -a+1 & a \\ +a & -a-1 \end{bmatrix}$$

so $\|A^{-1}\|_1 = \max\left[\,1+|a|+|a|\,,\ |a|+|a|-1\,\right] = \boxed{2|a|+1}$

so $\boxed{K(A) = (2|a|+1)^2}$

Using $\|A\|_2$ :

$$A^TA = \begin{bmatrix} a+1 & a \\ a & a-1 \end{bmatrix}\begin{bmatrix} a+1 & a \\ a & a-1 \end{bmatrix} = \begin{bmatrix} (a+1)^2+a^2 & a(a+1)+a(a-1) \\ a(a+1)+a(a-1) & a^2+(a-1)^2 \end{bmatrix}$$

$$= \begin{bmatrix} a^2+2a+1+a^2 & a^2+a+a^2-a \\ a^2+a+a^2-a & a^2+a^2-2a+1 \end{bmatrix} = \begin{bmatrix} 2a^2+2a+1 & 2a^2 \\ 2a^2 & 2a^2-2a+1 \end{bmatrix}$$

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2a^2+2a+1-\lambda & 2a^2 \\ 2a^2 & 2a^2-2a+1-\lambda \end{vmatrix} = 0 \Rightarrow (2a^2+2a+1-\lambda)(2a^2-2a+1-\lambda) - 4a^4 = 0$$

$$4a^4 - 4a^3 + 2a^2 - 2a^2\lambda + 4a^3 - 4a^2 + 2a - 2a\lambda + 2a^2 - 2a + 1 - \lambda - 2a^2\lambda + 2a\lambda - \lambda + \lambda^2 - 4a^4 = 0$$

$$\cancel{\lambda^2(-1) + \lambda(-2a^2 - 2a - 1 + 2a^2 - 2a + 1) + 4a^2 - 4a^3 + 2a^2 + 4a^3 - 4a^2 + 2a + 2a^2 - 2a + 1 - 4a^4 = 0}$$

$$2a^2 - 2a^2\lambda - 4a^2 + 2a^2 + 1 - \lambda = 2a^2\lambda - \lambda + \lambda^2 = 0$$

$$\lambda^2 + \lambda(-2a^2 - 1 - 2a^2 - 1) + 4a^2 - 4a^2 + 1 = 0$$

$$\boxed{\lambda^2 + \lambda(-4a^2 - 2) + 1 = 0}$$

so $\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{4a^2 + 2 \pm \sqrt{(-4a^2-2)^2 - 4}}{2}$

$$= 2a^2 + 1 \pm \frac{1}{2}\sqrt{16a^4 + 4 + 16a^2 - 4} = 2a^2 + 1 \pm \frac{1}{2}\sqrt{16a^2(a^2+1)}$$

$$\boxed{\lambda = 2a^2 + 1 \pm 2a\sqrt{a^2+1}}$$

$\lambda_1 = 2a^2 + 1 + 2a\sqrt{a^2+1}$ $\qquad \lambda_2 = 2a^2 + 1 - 2a\sqrt{a^2+1}$

$\lambda_1$ is larger of the 2 eigenvalues

so $\|A\|_2 = \sqrt{\lambda_1} = \sqrt{2a^2 + 1 + 2a\sqrt{a^2+1}}$

now to find $\|A^{-1}\|_2$:

$$(A^{-1})^T A^{-1} = \begin{bmatrix} -a+1 & a \\ a & -a-1 \end{bmatrix}^T \begin{bmatrix} -a+1 & a \\ a & -a-1 \end{bmatrix} = \begin{bmatrix} -a+1 & a \\ a & -a-1 \end{bmatrix}\begin{bmatrix} -a+1 & a \\ a & -a-1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 - a - a + 1 + a^2 & -a^2 + a - a^2 - a \\ -a^2 + a - a^2 - a & a^2 + a^2 + a + a + 1 \end{bmatrix} = \begin{bmatrix} 2a^2 - 2a + 1 & -2a^2 \\ -2a^2 & 2a^2 + 2a + 1 \end{bmatrix}$$

$\downarrow$

Compare this to $A^TA$ before

$$\begin{bmatrix} 2a^2 + 2a + 1 & 2a^2 \\ 2a^2 & 2a^2 - 2a + 1 \end{bmatrix}$$

Negative signs on diagonal cancel out we see it will come out the same characteristic polynomial in $\lambda$.

$\Rightarrow \lambda_{max}$ as before. $\Rightarrow$

so $\|A\|_2 \, \|A^{-1}\|_2 = \sqrt{\lambda}\,\sqrt{\lambda} = \lambda$

$$\boxed{K(A) = 2a^2+1+2a\sqrt{a^2+1}}$$

<u>Now do $\|A\|_\infty$</u>

$K(A) = \|A\|_\infty \, \|A^{-1}\|_\infty$

but $\|A\|_\infty = \max\limits_{\text{rows}} \{ |a|+1+|a| , |a|+|a|-1\} = 2|a|+1$

$\|A^{-1}\|_\infty = \left\| \begin{bmatrix} -a+1 & a \\ a & -a-1 \end{bmatrix} \right\|_\infty = \max\{ |a|+1+|a| , |a|+|a|-1\}$

$= \max\{2|a|+1 , 2|a|-1\}$

$= 2|a|+1$

so $K(A) = \|A\|_\infty \, \|A^{-1}\|_\infty$

$$\boxed{K(A) = (2|a|+1)^2}$$

Notice $\underset{\text{norm}}{K(A)_\infty} = \underset{\text{norm}}{K(A)_1}$

$\longrightarrow$

(c)

using $\|A\|_1$ norm

$$\|A\|_1 = \max_{col.} \{ |\alpha|+1 \ , \ 2 \}$$

$$\|A^{-1}\|_1 = \left\| \frac{\begin{bmatrix} 1 & -1 \\ -1 & \alpha \end{bmatrix}}{(\alpha-1)} \right\|_1 = \frac{1}{|\alpha-1|} \max \{ 1+|-1| \ , \ |-1|+|\alpha| \}$$

$$= \frac{1}{|\alpha-1|} \max \{ 2, \ 1+|\alpha| \}$$

So it depends on value of $\alpha$ what to do

for $\|A\|_1 = \max\{1+|\alpha|, 2\}$. if $|\alpha|>1$, then max is $1+|\alpha|$

if $|\alpha|\leq 1$, then max is $2$

So need to do these 2 cases.

$|\alpha|>1$

$$\|A\|_1 = 1+|\alpha|$$

$$\|A^{-1}\|_1 = \max\left\{ \frac{2}{|\alpha-1|}, \frac{1+|\alpha|}{|\alpha-1|} \right\} = \frac{1+|\alpha|}{|\alpha-1|}$$

so $$K(A) = (1+|\alpha|)\left(\frac{1+|\alpha|}{|\alpha-1|}\right) = \boxed{\frac{(1+|\alpha|)^2}{|\alpha-1|}}$$

if $|\alpha|\leq 1$

$$\|A\|_1 = 2$$

$$\|A^{-1}\|_1 = \frac{2}{|\alpha-1|}$$

so $$K(A) = (2)\left(\frac{2}{|\alpha-1|}\right) = \boxed{\frac{4}{|\alpha-1|}}$$

$\longrightarrow$ For $\|A\|_2$

(the Hard one!)

$\|A\|_2$

$K(A) = \|A\|_2 \cdot \|A^{-1}\|_2$

Find $\|A\|_2$: $A^TA = \begin{bmatrix} \alpha & 1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} \alpha & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2+1 & \alpha+1 \\ \alpha+1 & 2 \end{bmatrix}$

Set up characteristic equation: $\begin{vmatrix} \alpha^2+1-\lambda & \alpha+1 \\ \alpha+1 & 2-\lambda \end{vmatrix} = (\alpha^2+1-\lambda)(2-\lambda) - (\alpha+1)^2 = 0$

$2\alpha^2 - \lambda\alpha^2 + 2 - \lambda - 2\lambda + \lambda^2 - \alpha^2 - 2\alpha - 1 = 0$

$\lambda^2 + \lambda(-\alpha^2 - 1 - 2) + (2\alpha^2 + 2 - \alpha^2 - 2\alpha - 1) = 0$

$\lambda^2 + \lambda(-\alpha^2 - 3) + (\alpha^2 - 2\alpha + 1) = 0$

$\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-(-\alpha^2-3)}{2} \pm \dfrac{1}{2}\sqrt{(-\alpha^2-3)^2 - 4(\alpha^2 - 2\alpha + 1)}$

$= \dfrac{\alpha^2+3}{2} \pm \dfrac{1}{2}\sqrt{\alpha^4 + 9 + 6\alpha^2 - 4\alpha^2 + 8\alpha - 4} = \dfrac{\alpha^2+3}{2} \pm \dfrac{1}{2}\sqrt{\alpha^4 + 2\alpha^2 + 8\alpha + 5}$

so $\lambda_1 = \dfrac{\alpha^2+3}{2} + \dfrac{1}{2}\sqrt{\alpha^4 + 2\alpha^2 + 8\alpha + 5}$, $\lambda_2 = \dfrac{\alpha^2+3}{2} - \dfrac{1}{2}\sqrt{\alpha^4 + 2\alpha^2 + 8\alpha + 5}$

$\downarrow$

$\boxed{\lambda_{max} = \lambda_1}$ so $\sigma_1 = \sqrt{\lambda_1}$

so $\|A\|_2 = \sqrt{\lambda_1}$

now find $\|A^{-1}\|_2$: $A^{-1} = \dfrac{1}{|\alpha-1|}\begin{bmatrix} 1 & -1 \\ -1 & \alpha \end{bmatrix}$

so $(A^{-1})^T A^{-1} = \dfrac{1}{|\alpha-1|^2}\begin{bmatrix} 1 & -1 \\ -1 & \alpha \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & \alpha \end{bmatrix} = \dfrac{1}{|\alpha-1|^2}\begin{bmatrix} 2 & -1-\alpha \\ -1-\alpha & 1+\alpha^2 \end{bmatrix}$

Compare $\begin{bmatrix} \alpha^2+1 & \alpha+1 \\ \alpha+1 & 2 \end{bmatrix}$ with $\begin{bmatrix} 2 & -(1+\alpha) \\ -(1+\alpha) & 1+\alpha^2 \end{bmatrix}$ we see same $\lambda_{max}$.

so $\lambda_{max}$ for $(A^{-1})^T A^{-1}$ is $\dfrac{1}{|\alpha-1|^2}\lambda_1 \rightarrow$ found above

$\longrightarrow$

so $\|A^{-1}\|_2 = \sqrt{\lambda_{max}} = \sqrt{\frac{1}{|\alpha-1|^2}\lambda_1} = \frac{1}{|\alpha-1|}\sqrt{\lambda_1}$

so $K(A) = \|A\|_2 \cdot \|A^{-1}\|_2$

$$= \sqrt{\lambda_1}\,\frac{1}{|\alpha-1|}\sqrt{\lambda_1} = \boxed{\left(\frac{\lambda_1}{|\alpha-1|}\right)}$$

where $\lambda_1 = \frac{\alpha^2+3}{2} + \frac{1}{2}\sqrt{\alpha^4+2\alpha^2+8\alpha+5}$

Now do $\|A\|_\infty$

$\|A\|_\infty = \max_{row} \{ |\alpha|+1 , 2 \}$

$\|A^{-1}\|_\infty = \frac{1}{|\alpha-1|} \max \{ 2, 1+|\alpha| \}$

as befor for $\|A\|_1$ ; it depends on value for $\alpha$

for $|\alpha|>1$, we set $\|A\|_\infty = 1+|\alpha|$

$\|A^{-1}\|_\infty = \frac{1+|\alpha|}{|\alpha-1|}$

so $K(A) = (1+|\alpha|)\left(\frac{1+|\alpha|}{|\alpha-1|}\right) = \boxed{\frac{(1+|\alpha|)^2}{|\alpha-1|}}$

For $|\alpha| \leq 1$

$\|A\|_\infty = 2$

$\|A^{-1}\|_\infty = \frac{2}{|\alpha-1|}$

so $K(A) = \boxed{\frac{4}{|\alpha-1|}}$

Section 4.5 # 48

Prove condition number has the property

$$K(\lambda A) = K(A) \qquad \lambda \neq 0$$

solution

$$K(A) = \|A\| \cdot \|A^{-1}\|$$

so $K(\lambda A) = \|\lambda A\| \cdot \|(\lambda A)^{-1}\|$

but $(\lambda A)^{-1} = A^{-1}\lambda^{-1}$

so $K(\lambda A) = \|\lambda A\| \; \|A^{-1}\lambda^{-1}\|$

but $\|\lambda A\| = |\lambda| \; \|A\|$

and $\|A^{-1}\lambda^{-1}\| = |\lambda^{-1}| \; \|A^{-1}\|$

so $K(\lambda A) = |\lambda| \; \|A\| \; |\lambda^{-1}| \; \|A^{-1}\|$

$$K(\lambda A) = \underbrace{\|A\| \; \|A^{-1}\|}$$

$$K(\lambda A) = K(A)$$

QED

section 4.5 #2 15/15

Prove that if $A$ is invertible and $\|B-A\| < \frac{1}{\|A^{-1}\|}$, the $B$ is invertible.

Solution

$\|B-A\| < \frac{1}{\|A^{-1}\|}$

i.e $\|B-A\| \, \|A^{-1}\| < 1$

from eq (10) page 190, we have $\|XY\| \leq \|X\| \, \|Y\|$

$\|(B-A)(A^{-1})\| \leq (B-A) \; (A^{-1})$

i.e $\|B-A\| \, \|A^{-1}\| \geq \|(B-A)(A^{-1})\|$

So $\|(B-A)(A^{-1})\| < 1$ since $\|B-A\| \, \|A^{-1}\| < 1$

So $\|BA^{-1} - AA^{-1}\| < 1$

so $\|BA^{-1} - I\| < 1$ ——— (1)

But from theorem 2, page 200, theorm on invertible matrices, it says that if $\|I - XY\| < 1$ then $X, Y$ are invertible.

So compare this to (1), it means that $\underset{X}{B}$ and $\underset{Y}{A^{-1}}$ are both invertible.

$\Longrightarrow$ $\boxed{B \text{ is invertible}}$

section 4.5 # 5

Prove that if $\| AB - I \| < 1$ then

$$A^{-1} = B - BE + BE^2 - BE^3 + \cdots \qquad \text{where } E = AB - I.$$

Solution

we know that if $\| I - AB \| < 1$ then $A^{-1} = B \sum_{k=0}^{\infty} (I-AB)^k$

From theorem 2, page 200

But we are given $\| AB - I \|$ and not $\| I - AB \|$

But $\| AB - I \|$ is same as $\| I - AB \|$ since it is Norm

But $\boxed{(I-AB) = -(AB-I)}$

so replace these into above theorem, we get

$$\boxed{A^{-1} = B \sum_{k=0}^{\infty} \left[ -(AB-I) \right]^k}$$

so $A^{-1} = B \sum_{k=0}^{\infty} (-1)^k (AB-I)^k$

$$= B \left[ -1^0 (AB-I)^0 + (-1)^1 (AB-I)^1 + (-1)^2 (AB-I)^2 + \cdots \right]$$

let $E = AB - I$.

So $A^{-1} = B \left[ I - E^1 + E^2 - E^3 + \cdots \right]$

$$\boxed{A^{-1} = B - BE + BE^2 - BE^3 + \cdots}$$

Section 4.5 #8

Prove that if $\|A\| < 1$, then

$$(I+A)^{-1} = I - A + A^2 - A^3 + \cdots$$

Solution

Theorem 1, theorem on Neumann series, on page 198 of book says

if $A$ $n\times n$ s.t. $\|A\| < 1$, then $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$

let $B = -A$. so $A = -B$. replace $A$ by $-B$ in the above, we obtain

if $\|-B\| < 1$ Then $(I+B)^{-1} = \sum_{k=0}^{\infty} (-B)^k$

But $\|-B\| < 1$ since $\|-B\| = \|-A\| = \|A\| < 1$

Therfor $\boxed{(I+B)^{-1} = \sum_{k=0}^{\infty} (-B)^k}$

i.e $(I+B)^{-1} = \sum_{k=0}^{\infty} (-1)^k (B)^k$

$\boxed{\text{if } \|B\| < 1 \text{ Then } (I+B)^{-1} = I - B + B^2 - B^3 +}$

QED

Section 4.5 # 12

For any $n \times n$ matrix, prove that

$$A^m = I - (I-A)\sum_{k=0}^{m-1} A^k$$

Answer

$$I-(I-A)\sum_{k=0}^{m-1} A^k = I - \left( I\sum_{k=0}^{m-1} A^k - A\sum_{k=0}^{m-1} A^k \right)$$

$$= I - \sum_{k=0}^{m-1} I(A^k) + A\sum_{k=0}^{m-1} A^k$$

$$= I - \sum_{k=0}^{m-1} A^k + \sum_{k=0}^{m-1} A(A^k)$$

$$= I - \sum_{k=0}^{m-1} A^k + \sum_{k=0}^{m-1} A^{k+1} \qquad (1)$$

$\sum_{k=0}^{m-1} A^{k+1}$ Let $\boxed{k+1=z}$, when $k=0$, $z=1$, when $k=m-1$, $z=m-1+1=m$

So this can be written as $\sum_{z=1}^{m} A^z$ since $z$ is free variable, call it $k$.

So this is $\boxed{\sum_{k=1}^{m} A^k}$

Plug back to (1) we get

$$= I - \sum_{k=0}^{m-1} A^k + \sum_{k=1}^{m} A^k$$

$$= I - (A^0 + A^1 + \cdots + A^{m-1}) + (A^1 + A^2 + \cdots + A^{m-1} + A^m)$$

$$= I - (A^0) + (A^m) = I - I + A^m = \boxed{A^m}$$

QED

section 4.5 # 22

let $B_K = \sum_{j=0}^{K} A^j$. show that $[B_K]$ can be computed recursively by the formula $B_0 = I$, $B_{K+1} = I + AB_K$

solution

For $K=0$ $\quad B_0 = A^0 = I$ $\quad$ so $\boxed{B_0 = I}$

For $K=1$

$$B_1 = A^0 + A$$

$$B_1 = I + A$$

$$= I + AI \quad \text{but since } I = B_0$$

$$\boxed{B_1 = I + AB_0}$$

For $K=2$

$$B_2 = A^0 + A^1 + A^2$$

$$= I + A(I + A) \quad \text{but } I + A = B_1 \text{ From above}$$

so

$$\boxed{B_2 = I + AB_1}$$

For $K=3$

$$B_3 = A^0 + A^1 + A^2 + A^3$$

$$= I + A(I + A + A^2)$$

$$= I + A(I + A(I + A)) \quad \text{but } I + A = B_1 \text{ From above}$$

so

$$= I + A(I + AB_1) \quad \text{but } I + AB_1 = B_2 \text{ From above}$$

$$\boxed{B_3 = I + AB_2}$$

Therfor we see that the general recursive eqation builds up as follows

$$B_0 = I$$

$$B_1 = I + AB_0$$

$$B_2 = I + AB_1$$

$$B_3 = I + AB_2$$

$$\vdots \quad \vdots$$

$$\boxed{B_{K+1} = I + AB_K}$$

QED

Can also use proof by induction.

This is proof by Construction to show general Pattern.

section 4.5 # 24

in Normed Vector Space, prove that if sequence of vectors converges, then it must also satisfy Cauchy criterion.

Solution Cauchy Criterion says

If a sequence satisfies $\lim_{k\to\infty} \sup_{i,j\geq k} \|v^i - v^j\| = 0$, then

such sequence converges to some $v^*$

here we are told that the sequence already converges to some $v^*$, and we need to show that it satisfies Cauchy Criterion

i.e we are told $\boxed{\lim_{k\to\infty} \|v^k - v^*\| = 0}$ ← This is Convergence definition. Page 197 and we

need to show that $\boxed{\lim_{k\to\infty} \sup_{i,j\geq k} \|v^i - v^j\| = 0}$ ← This is Cauchy Convergence Criterion.

we can write our Convergence definition as $\lim_{k\to\infty} \sup_{i\geq k} \|v^i - v^*\| = 0$

Now, add and subtract $v^j$ inside, will not effect the result, so get

$$\lim_{k\to\infty} \sup_{i,j\geq k} \| v^i - v^* + (v^j - v^j) \| = 0$$

$$\lim_{k\to\infty} \sup_{i,j\geq k} \| v^i - v^j + v^j - v^* \| = 0$$

$$\lim_{k\to\infty} \sup_{i,j\geq k} \| (v^i - v^j) + (v^j - v^*) \| = 0$$

but $\|A+B\| \leq \|A\| + \|B\|$

$$\lim_{k\to\infty} \sup_{i,j\geq k} \| v^i - v^j \| + \underbrace{\lim_{k\to\infty} \sup_{j\geq k} \| v^j - v^* \|}_{\text{but this is Zero since}} \geq 0$$

This is same as $\lim_{k\to\infty} \|v^k - v^*\|$ which is Zero since seq. converges.

So $\boxed{\lim_{k\to\infty} \sup_{i,j\geq k} \|v^i - v^j\| = 0}$

but this is Cauchy Convergence. hence QED