Non Computer
Problems

HW # 3

Math 501

Section 3.3 # 4

Question: if secant method applied to $f(x) = x^2 - 2$ with $x_0 = 0$, $x_1 = 1$, what is $x_2$?

Answer

$$x_{n+1} = x_n - f(x_n)\left[\frac{x_n - x_{n-1}}{f(x) - f(x_{n-1})}\right]$$

| $n$ | $x_n$ | $x_{n-1}$ | $f(x_n)$ | $f(x_{n-1})$ | $x_{n+1}$ |
|---|---|---|---|---|---|
| 1 | 1 | 0 | $-1$ | $-2$ | $\to 1-(-1)\left[\frac{1}{-1-(-2)}\right] = 1+\left[\frac{1}{-1+2}\right] = 1+\frac{1}{1} = \boxed{2}$ |
| 2 | 2 | 1 | 2 | $-1$ | $\to 2-2\left[\frac{2-1}{2-(-1)}\right] = 2-2\left[\frac{1}{3}\right] = 2-\frac{2}{3} = \frac{6-2}{3} = \frac{4}{3}$ |

so $x_2 = \boxed{\frac{4}{3}}$

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## Section 3.3 #5

what is $x_2$ if $x_0 = 1$, $x_1 = 2$, $f(x_0) = 2$, $f(x_1) = 1.5$ in an application of secant method?

### Answer

$$x_{n+1} = x_n - f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right]$$

so $n=1$, so

$$x_2 = 1 - 1.5\left[\frac{2-1}{1.5-2}\right]$$

$$= 1 - 1.5\left[\frac{1}{-0.5}\right] = 1 + \frac{1.5}{0.5} = 1 + 3 = \boxed{4}$$

$$\boxed{x_2 = \cancel{4}}$$

Section 3.3 #6

given $x_n \sim y_n$, $u_n \sim v_n$, $c \neq 0$ show that

(a) $cx_n \sim cy_n$

Since $x_n \sim y_n$, then $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = 1$

or $\lim_{n\to\infty}\left(\frac{cx_n}{cy_n}\right) = 1$ since $c$ is constant and not zero.

but this is by definition means that

$$\boxed{cx_n \sim cy_n}$$

(b) $x_n^c \sim y_n^c$.

Since $\lim_{n\to\infty} \frac{x_n}{y_n} = 1$, then

$$\left(\lim_{n\to\infty} \frac{x_n}{y_n}\right)^c = 1^c = 1$$

but $\left(\lim_{n\to\infty} \frac{x_n}{y_n}\right)^c = \lim_{n\to\infty}\left(\frac{x_n}{y_n}\right)^c = 1$

$= \lim_{n\to\infty} \frac{x_n^c}{y_n^c}. = 1$

therefor $\lim_{n\to\infty} \frac{x_n^c}{y_n^c} = 1$

but by definition this means

$$\boxed{x_n^c \sim y_n^c}$$

## Section 3.3 # 6

(c) $x_n u_n \sim y_n v_n$ :

since $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = 1$ and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = 1$ then

$$\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right)\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = 1 \times 1 = 1$$

but $\left(\lim_{n\to\infty} A\right)\left(\lim_{n\to\infty} B\right) = \lim_{n\to\infty}(AB)$.

therefore $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right)\left(\frac{u_n}{v_n}\right) = 1$

or $\lim_{n\to\infty}\frac{x_n u_n}{y_n v_n} = 1$

or $\boxed{x_n u_n \sim y_n v_n}$

(d) if $y_n \sim u_n$ then $x_n \sim v_n$.

given: ① $\lim_{n\to\infty}\frac{y_n}{u_n} = 1$

② $\lim_{n\to\infty}\frac{x_n}{y_n} = 1$ and $\lim_{n\to\infty}\frac{u_n}{v_n} = 1$

show: $\lim_{n\to\infty}\frac{x_n}{v_n} = 1$

$$\left(\lim_{n\to\infty}\frac{x_n}{y_n}\right)\left(\lim_{n\to\infty}\frac{u_n}{v_n}\right) = 1 \times 1 = 1$$

$$\lim_{n\to\infty}\left(\frac{x_n u_n}{y_n v_n}\right) = 1 \Rightarrow \left(\lim_{n\to\infty}\frac{x_n}{v_n}\frac{u_n}{y_n}\right) = 1$$

given

or $\left(\lim_{n\to\infty}\frac{x_n}{v_n}\right)\left(\lim_{n\to\infty}\frac{u_n}{y_n}\right) = 1$. but since $\left(\lim \frac{y_n}{u_n} = 1\right)$

the $\left(\lim_{n\to\infty}\frac{x_n}{v_n}\right) \times 1 = 1 \Rightarrow \lim_{n\to\infty}\frac{x_n}{v_n} = 1 \Rightarrow \boxed{x_n \sim v_n}$

section 3.3 # 6

(6) (e) show that $y_n \sim x_n$

Given $\lim_{n\to\infty} \frac{x_n}{y_n} = 1$ —— (1)

multiply both sides of (1) by $\lim_{n\to\infty} \left(\frac{y_n}{x_n}\right)$.

here we have

$$\left(\lim_{n\to\infty} \frac{x_n}{y_n}\right)\left(\lim_{n\to\infty} \frac{y_n}{x_n}\right) = 1 \times \lim_{n\to\infty} \left(\frac{y_n}{x_n}\right)$$

so $$\lim_{n\to\infty} \left(\frac{\cancel{x_n}}{\cancel{y_n}} \frac{\cancel{y_n}}{\cancel{x_n}}\right) = \lim_{n\to\infty} \left(\frac{y_n}{x_n}\right).$$

$$\lim_{n\to\infty} 1 = \lim_{n\to\infty} \left(\frac{y_n}{x_n}\right).$$

but $\lim_{n\to\infty} 1 = 1$ since does not depend on n.

hence $$1 = \lim_{n\to\infty} \left(\frac{y_n}{x_n}\right).$$

hence by definition

$$\boxed{y_n \sim x_n}$$

# Section 3.4 # 4

Show that these functions are contractive on selected intervals.
Determine best $\lambda$.

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Solution:

In all these problems need to show the following:

- $g(x)$ is $C^1$. (one time differentiable) over the domain
- $\max_{a \le x \le b} |g'(x)| \le \lambda$ where $\lambda < 1$.

this is equivalent to saying

$$|g(x) - g(y)| \le \lambda |x-y| \quad \text{For any } x, y \text{ in the interval. } \lambda < 1.$$

(a) $g(x) = \frac{1}{(1+x^2)}$, arbitrary interval.

$g(x)$ is differentiable once.

$g'(x) = \frac{-2x}{(1+x^2)^2}$. to find max, $g''(x) = \frac{-6x^2+2}{(1+x^2)^3} = 0 \Rightarrow \boxed{x = \sqrt{1/3}}$

So at $x = \sqrt{1/3}$, $g'(x) = \frac{-2\sqrt{1/3}}{(1+\frac{1}{3})^2} \Rightarrow \max |g'(x)| \doteq 0.6499$

So $\boxed{\lambda = 0.6499 < 1} \Rightarrow$ contractive.

(b) $F(x) = \frac{1}{2}x$. $1 \le x \le 5$.

$F(x)$ is one time differentiable. ok over domain.

$F'(x) = \frac{1}{2}$. hence $\max |F'(x)| \le \lambda < 1$. $\boxed{\lambda = \frac{1}{2}}$

(c) $F(x) = \arctan(x)$, arbitrary interval excluding 0.

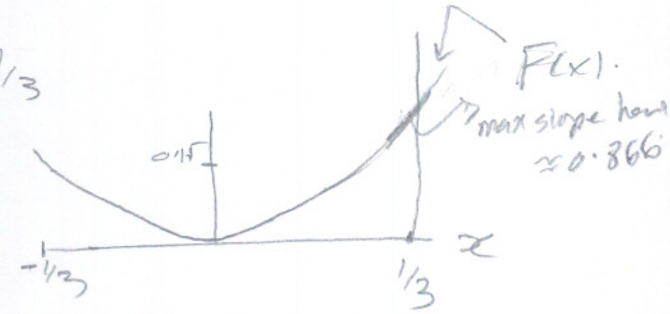
$F'(x)$ defined ok. over interval.

$F'(x) = \frac{1}{1+x^2}$. $|F'(x)| = \left|\frac{1}{1+x^2}\right|$. since $x \neq 0$, then

$\max |F'(x)| < 1 \Rightarrow \boxed{\lambda = 1}$ contractive.

# section 3.4 # 4

(d) $F(x) = |x|^{3/2}$ on $|x| \leq 1/3$




$F'(x)$ continuous on domain ok.

Consider positive range.

$F'(x) = \frac{3}{2} x^{1/2}$. This is max when $x$ is max. i.e $x = 1/3$.

so $\max F(x) = \frac{3}{2}(1/3)^{1/2} \cong 0.866\ldots$

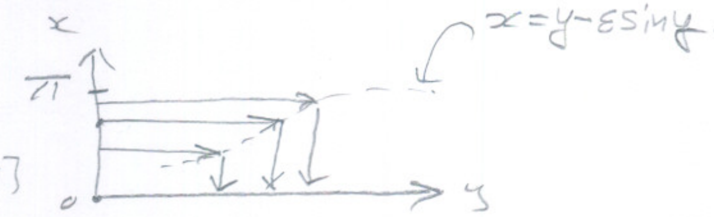
by symmetry of $F(x) \Rightarrow \max_{|x| \leq 1/3} |F'(x)| \leq \lambda < 1$ where $\boxed{\lambda \cong 0.866\ldots}$

$\rightarrow$ contraction.

section 3.4 #5

Kepler equation $x = y - \varepsilon \sin y$ $\quad 0 < \varepsilon < 1$.

show that for each $x \in [0,\pi]$ there is a $y$ satisfying the equation.

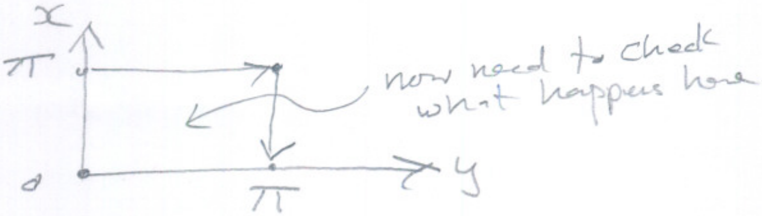



answer

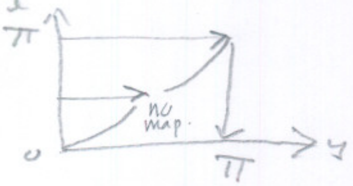
we need to show that if we pick any $x \in [0,\pi]$ value, then we can map that value to some $y$ value via function $y - \varepsilon \sin y$.

First, when $x = 0$, then $y = 0$ satisfy the equation.

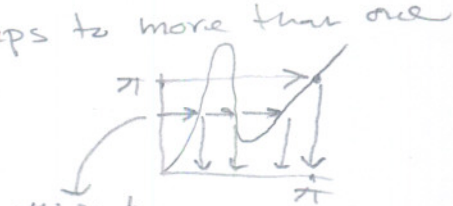
at $x = \pi$, then $y = \pi$ satisfy the equation.




now between $0,\pi$, for any $x$ to map to a $y$, then $y - \varepsilon \sin y$ must be continous over $[0,\pi]$ to avoid case such as




In addition to avoid case that $x$ maps to more than one $y$ value, need to avoid case such as




we see that $y - \varepsilon \sin y$ is continouse since its derivative is $1 - \varepsilon \cos y$ which defined over all $[0,\pi]$. to handle this second case need to show $y - \varepsilon \sin y$ has +ve derivative and $< 1$:

$g(y) = y - \varepsilon \sin y \Rightarrow g'(y) = 1 - \varepsilon \cos y \Rightarrow g'(y) = 1 - \varepsilon \cos y$. but $|\cos y| \le 1$

so $\boxed{0 < g'(y) < 1}$ since $|\varepsilon| < 1$ hence $g(y)$ has +ve slope $< 1$, hence satisfy second case. QED

## Section 3.4 #10

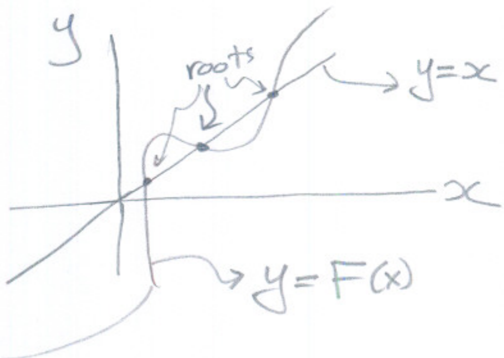
if we attempt to find a fixed point of $F$ by using Newton's method on the equation $F(x)-x=0$, what iteration formula results?

### Answer

Newton iteration formula is

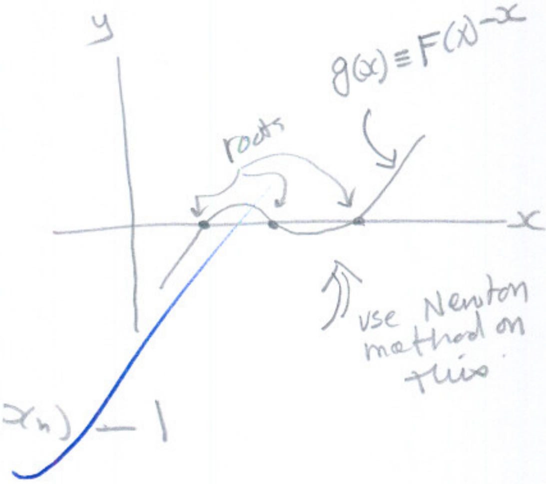
$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

in this problem we seek to find root for $F(x)-x=0$




hence $g(x) = F(x) - x$ will cross the x-axis at the place where $y=F(x)$ and $y=x$ intersect.

So when using Newton method, replace $f(x)$ in that formula by $g(x)$. This results in:




$$x_{n+1} = x_n - \frac{g(x)\big|_{x=x_n}}{g'(x)\big|_{x=x_n}}$$

but $g(x)\big|_{x=x_n} = F(x_n) - x_n$

$$g'(x)\big|_{x=x_n} = F'(x)\big|_{x=x_n} - 1 = F'(x_n) - 1$$

so

$$\boxed{x_{n+1} = x_n - \frac{F(x_n)-x_n}{F'(x_n)-1}}$$

## Section 3.4 # 12

$$x = \sqrt{P + \sqrt{P + \sqrt{P + \cdots}}}$$

Find $x$ given $P > 0$.

First need to show R.H.S. converges.
we see this is the sum of terms, each subsequent term is smaller than previouse term.

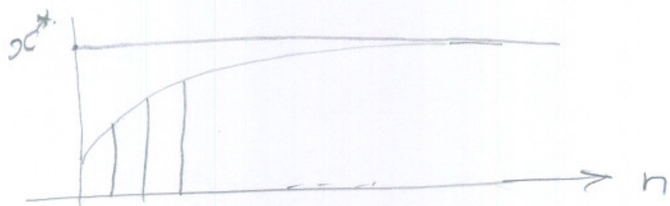
i.e

$$P + \sqrt{P + \sqrt{\cdots}} \qquad P + \sqrt{P + \sqrt{P + \cdots}}$$

smaller than $P$ smaller than $P$

hence we have a sum which is increasing but is converging to some upper fixed point, which is what we are trying to find.

(Can use ratio test to show convergence if needed.




Call This sum limit as $x^*$.

now we write the above as

$$\boxed{x_{n+1} = \sqrt{P + x_n}} \rightarrow g(x)$$

or $x_{n+1}^2 = P + x_n$

as $n \to \infty$ $\quad x_{n+1} \to x^*$ and $x_n \to x^*$

so $(x^*)^2 = P + x^*$

$(x^*)^2 - x - P = 0 \implies x^* = \dfrac{1 \pm \sqrt{1+4P}}{2}$

since $P > 0$, $x^* > 0 \implies \boxed{x^* = \frac{1}{2}\left(1 + \sqrt{1+4P}\right)}$

note: when $P = 1$ $\implies$ golden ratio.

## section 3.4 # 13

$P > 1$, what is the value of $x = \cfrac{1}{P + \cfrac{1}{P + \cfrac{1}{P + \cdots}}}$

First need to show that RHS converges.
Looking at denominator. we see this is the sum of terms each subsequent term is smaller than last term.

$$P + \frac{1}{P + \cdots}$$

smaller than P
since $P > 1$

Since we are adding terms $\{x_n\}$ s.t. $x_{n+1} < x_n$, then the sum will converge to some limit. Call this limit $x^*$.

now write the above as

$$x_{n+1} = \underbrace{\frac{1}{P + x_n}}_{g(x_n)}$$

so $\lim\limits_{n \to \infty}$ $\boxed{x^* = \frac{1}{P + x^*}}$ ✓

Since in the limit $n \to \infty$
$x_n \to x_{n+1} \to x^*$

Solving for $\boxed{x^* = \frac{-P + \sqrt{P^2 + 4}}{2}}$

(note $P = 1$ gives the inverse of the golden ratio)

# section 3.4 #29

Prove that $F(x) = 2 + x - \arctan(x)$ has property $|F'(x)| < 1$. Prove that $F(x)$ does not have a Fixed point.

Answer

$$F'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2}$$

hence $|F'(x)| < 1$

For a Function $F(x)$ which $|F'(x)| < 1$ not to have a Fixed point, then we need to show that

$F(x) - x = 0$ has No solution.

i.e $\boxed{g(x) = F(x) - x}$ is alway positive or always negative.

meaning it never cross the x-axis. hence no root.

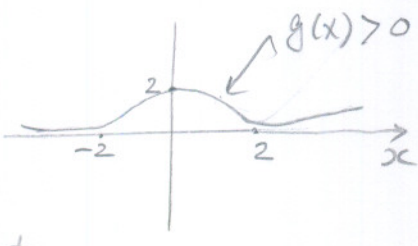
$$g(x) = 2 + x - \arctan(x) - x = 2 - \arctan(x)$$

Now $\arctan(x)$ will range between $[0, \frac{\pi}{2}]$ or $[0, -\frac{\pi}{2}]$.

but $\left|\frac{\pi}{2}\right| \simeq 1.57079\ldots$

hence $\min g(x) = 2 - \frac{\pi}{2} = 0.4292\ldots$

$\max g(x) = 2 + \frac{\pi}{2} = 3.5707$.

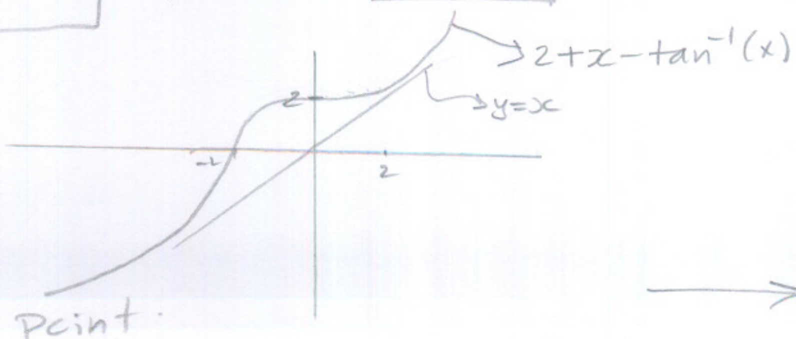



$\Rightarrow$ $\boxed{g(x) > 0 \quad \text{For all } x}$ ThereFor No root

graphically:

The line $y = x$ never intersects $F(x)$.

$\Rightarrow$ No Fixed point.



The contractive theorm says that if $g(x)$ is continouse on $[a,b]$ <u>AND</u> $a \le g(x) \le b$ for every $a \le x \le b$ Then $g(x)$ has at least one fixed point in $[a,b]$.

In this problem, g(x) violate the second condition above which says that $a \le g(x) \le b$ for every $a \le x \le b$.

since $g(x) > x$ For every $x$.

for example, take $a=0$, $b=1$. then

$$2 \le g(x) \le 1.214\ldots$$

$$\downarrow \qquad\qquad\quad \downarrow$$

$$0 \le x \le 1$$

So we see the condition is not satisfied.

Hence No contradiction with contractive theorm.

# Section 3.4 # 40

Show that the following method has $3^{rd}$ order convergence for computing $\sqrt{R}$

$$x_{n+1} = \frac{x_n(x_n^2 + 3R)}{3x_n^2 + R}$$

Solution

need to show $|e_{n+1}| \le C|e_n|^M$

where $M=3$, $C$ is constant $>0$.

Using theorm 3, Lecture notes, Wed 2/7/07 which says:

if $g'(x^*) = g''(x^*) = \cdots = g^{(m-1)}(x^*) = 0$
with $g^{(m)}(x^*) \ne 0$ Then $|e_{n+1}| \le C|e_n|^m$.

hence, here $g(x) = \dfrac{x(x^2+3R)}{3x^2+R}$

$x^* = \sqrt{R}$.

So need to check that $g'(x^*) = g''(x^*) = 0$, and to check that $g'''(x^*) \ne 0$ to proof:

$$g'(x) = \frac{3(x^4 - 2x^2R + R^2)}{(3x^2+R)^2}, \quad g'(x)\Big|_{x=x^*=\sqrt{R}} = \frac{3(R^2 - 2RR + R^2)}{(3R+R)^2} = 0$$

$$g''(x) = -\frac{48xR(-x^2+R)}{(3x^2+R)^3}, \quad g''(x)\Big|_{x=x^*=\sqrt{R}} = -\frac{48\sqrt{R}R(-R+R)}{(3R+R)^3} = 0$$

$$g'''(x) = -\frac{48R(9x^4 - 18x^2R + R^2)}{(3x^2+R)^4}, \quad g'''(x)\Big|_{x=x^*=\sqrt{R}} = -\frac{48R(9R^2 - 18R^2 + R^2)}{(3R+R)^4}$$

$$= -\frac{48R(-8R^2)}{(4R)^4} = \frac{-196R^3}{256R^4} = \boxed{-\frac{49}{64}\frac{1}{R} \ne 0} \implies \text{by above theorm order 3 convergence.}$$