HW12, Math 501. CSUF. Spring 2007

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1 Section 7.1, Problem 6

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Problem: Derive the following 2 formulas for approximation of derivatives and show they are both $O(h^4)$ by evaluating their error terms

$$f'(x) = \frac{1}{12h}\left[-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)\right]$$
$$f''(x) = \frac{1}{12h^2}\left[-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)\right]$$

Solution:

I could obtain the above results directly from applying Richardson interpolation formulas (which is a short approach), but I assumed the question wanted us to derive these from first principles. I first show how to do one using Richardson, then solve both from first principles.

To obtain the approximation for $f'(x)$ using Richardson, we do the following:

$$\begin{aligned}\varphi(h) &= \frac{1}{2h}\left[f(x+h) - f(x-h)\right]\\ L &= \varphi(h) + a_2h^2 + a_4h^4 + \cdots\end{aligned} \tag{1C}$$

Replace h by $2h$

$$L = \varphi(2h) + a_2 4h^2 + a_4 16h^4 + \cdots \tag{2C}$$

Multiply (1C) by 4 and subtract (2C) from result

$$\begin{aligned}3L &= \left(4\varphi(h) + 4a_2h^2 + 4a_4h^4 + \cdots\right) - \left(\varphi(2h) + a_2 4h^2 + a_4 16h^4 + \cdots\right)\\ &= 4\varphi(h) - \varphi(2h) - 12a_4h^4 - \cdots\end{aligned}$$

Hence

$$\begin{aligned}L &= \frac{1}{3}\left(\frac{2}{h}\left[f(x+h) - f(x-h)\right] - \frac{1}{4h}\left[f(x+2h) - f(x-2h)\right] - 12a_4h^4 - \cdots\right)\\ &= \frac{2}{3h}\left[f(x+h) - f(x-h)\right] - \frac{1}{12h}\left[f(x+2h) - f(x-2h)\right] - 4a_4h^4 - \cdots\\ &= \frac{1}{12h}\left(8\left[f(x+h) - f(x-h)\right] - \left[f(x+2h) - f(x-2h)\right]\right) - 4a_4h^4 - \cdots\\ &= \frac{1}{12h}\left[-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)\right] - 4a_4h^4 - \cdots\end{aligned}$$

Which is the same result obtained earlier using the long approach. We also see that the error term is $O\left(h^4\right)$

Now, solve it again, but using direct usage of Taylor (which I assume what the book wanted us to do)

From Taylor expansion, we write, by expanding around $x+h$ and $x-h$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f^{(3)}(x) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(\xi_1)$$
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f^{(3)}(x) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(\xi_2)$$

$$
\begin{aligned}
3f'(x) = {} & \frac{2}{h}\left[f(x+h) - f(x-h)\right] - \frac{1}{30}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right] - \\
& \frac{1}{4h}\left[f(x+2h) - f(x-2h)\right] + \frac{1}{15}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right]
\end{aligned}
$$

$$
\begin{aligned}
f'(x) &= \frac{2}{3h}\left[f(x+h) - f(x-h)\right] - \frac{1}{12h}\left[f(x+2h) - f(x-2h)\right] + \frac{1}{90}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right] \\
&= \frac{1}{12h}\left[8f(x+h) - 8f(x-h) - f(x+2h) - f(x-2h)\right] + \frac{1}{90}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right]
\end{aligned}
$$

Rearrange terms to make it look as in the textbook

$$
f'(x) = \frac{1}{12h}\left[-f(x+2h) + 8f(x+h) - 8f(x-h) - f(x-2h)\right] + \frac{1}{90}h^4\left[f^{(5)}(\xi)\right] \tag{4}
$$

Where we replaced $\frac{1}{90}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right]$ by $\frac{1}{45}h^4\left[\frac{f^{(5)}(\xi_1)+f^{(5)}(\xi_2)}{2}\right] = \frac{1}{90}h^4\left[f^{(5)}(\xi)\right]$ with $f^{(5)}(\xi)$ being the mean value of $\frac{f^{(5)}(\xi_1)+f^{(5)}(\xi_2)}{2}$

Hence from equation (4) we see that the error is $O\left(h^4\right)$ as required to show.

Hence
$$
\boxed{f'(x) \approx \frac{1}{12h}\left[-f(x+2h) + 8f(x+h) - 8f(x-h) - f(x-2h)\right]}
$$

Now we need to show the formula for $\boxed{f''(x)}$. We do the same as above, but instead of subtracting equations, we add them. We start from the top to show these again step by step.

From Taylor expansion, we write, by expanding around $x+h$ and $x-h$

$$
\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f^{(3)}(x) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(x) + \frac{h^6}{6!}f^{(6)}(\xi_1) \\
f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f^{(3)}(x) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(x) + \frac{h^6}{6!}f^{(6)}(\xi_2)
\end{aligned}
$$

Add the second to the first equation

$$
f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + \frac{h^4}{12}f^{(4)}(x) + \frac{h^6}{6!}\left[f^{(6)}(\xi_1) + f^{(6)}(\xi_2)\right]
$$

Solve for $f''(x)$ we obtain

$$
f''(x) = \frac{1}{h^2}\left[f(x+h) + f(x-h)\right] - \frac{2}{\cancel{h^2}}f(x) - \frac{h^2}{12}f^{(4)}(x) - \frac{1}{720}h^4\left[f^{(6)}(\xi_1) + f^{(6)}(\xi_2)\right] \tag{1A}
$$

Now we do the same again, but by expanding around $x+2h$ and $x-2h$

$$
\begin{aligned}
f(x+2h) &= f(x) + 2hf'(x) + \frac{(2h)^2}{2}f''(x) + \frac{(2h)^3}{3!}f^{(3)}(x) + \frac{(2h)^4}{4!}f^{(4)}(x) + \frac{(2h)^5}{5!}f^{(5)}(x) + \frac{(2h)^6}{6!}f^{(6)}(\xi_1) \\
f(x-2h) &= f(x) - 2hf'(x) + \frac{(2h)^2}{2}f''(x) - \frac{(2h)^3}{3!}f^{(3)}(x) + \frac{(2h)^4}{4!}f^{(4)}(x) - \frac{(2h)^5}{5!}f^{(5)}(x) + \frac{(2h)^6}{6!}f^{(6)}(\xi_1)
\end{aligned}
$$

Hence from equation (4A) we see that the error is $O\left(h^4\right)$ as required to show.

Hence

$$\boxed{f''(x) \approx \frac{1}{12h^2}\left(-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)\right)}$$

3 Section 7.1, Problem 14

problem: Using Taylor series, derive the error term for the approximation

$$f'(x) \approx \frac{1}{2h}\left[-3f(x) + 4f(x+h) - f(x+2h)\right]$$

answer:

expand around $x+h$

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(\xi_1) \\ f'(x) &= \frac{1}{h}f(x+h) - \frac{1}{h}f(x) - \frac{h}{2}f''(x) - \frac{h^2}{6}f'''(\xi_1) \qquad (1) \end{aligned}$$

Now expand around $x+2h$

$$\begin{aligned} f(x+2h) &= f(x) + 2hf'(x) + 2h^2f''(x) + \frac{8h^3}{6}f'''(\xi_2) \\ f'(x) &= \frac{1}{2h}f(x+2h) - \frac{1}{2h}f(x) - hf''(x) - \frac{4h^2}{6}f'''(\xi_2) \qquad (2) \end{aligned}$$

Multiply (2) by $-\frac{1}{2}$ and add result to (1) we obtain

$$\begin{aligned} -\frac{1}{2}f'(x) + f'(x) = -\frac{1}{2}&\left(\frac{1}{2h}f(x+2h) - \frac{1}{2h}f(x) - hf''(x) - \frac{4h^2}{6}f'''(\xi_2)\right) + \\ &\left(\frac{1}{h}f(x+h) - \frac{1}{h}f(x) - \frac{h}{2}f''(x) - \frac{h^2}{6}f'''(\xi_1)\right) \end{aligned}$$

$$\begin{aligned} \frac{1}{2}f'(x) &= \frac{-1}{4h}f(x+2h) + \frac{1}{4h}f(x) + \frac{h}{2}f''(x) + \frac{2h^2}{6}f'''(\xi_2) + \frac{1}{h}f(x+h) - \frac{1}{h}f(x) - \frac{h}{2}f''(x) - \frac{h^2}{6}f'''(\xi_1) \\ f'(x) &= \frac{-1}{2h}f(x+2h) + \frac{1}{2h}f(x) + hf''(x) + \frac{4h^2}{6}f'''(\xi_2) + \frac{2}{h}f(x+h) - \frac{2}{h}f(x) - hf''(x) - \frac{2h^2}{6}f'''(\xi_1) \\ &= \left[\frac{-1}{2h}f(x+2h) + \frac{1}{2h}f(x) + hf''(x) + \frac{2}{h}f(x+h) - \frac{2}{h}f(x) - hf''(x)\right] - \frac{2h^2}{6}f'''(\xi_1) + \frac{4h^2}{6}f'''(\xi_2) \\ &= \frac{1}{2h}\left[-f(x+2h) + f(x) + 4f(x+h) - 4f(x)\right] - \frac{h^2}{3}f'''(\xi_1) + \frac{2h^2}{3}f'''(\xi_2) \\ &= \frac{1}{2h}\left[-f(x+2h) - 3f(x) + 4f(x+h)\right] - h^2\left(\frac{1}{3}f'''(\xi_1) + \frac{2}{3}f'''(\xi_2)\right) \end{aligned}$$

Which is the equation we are asked to show.

From the above we see that the error term is given by

$$h^2\left(\frac{1}{3}f'''(\xi_1) + \frac{2}{3}f'''(\xi_2)\right)$$

Hence the error is $O\left(h^2\right)$

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5 Computer assignment 4/30/2007. Richardson Algorithm

This is the output

Richardson table in single floating point

N	D(n,0)	D(n,1)	D(n,2)	D(n,3)	D(n,4)	D(n,5)	D(n,6)
0	0.3926991	0	0	0	0	0	0
1	0.348771	0.3341283	0	0	0	0	0
2	0.3371939	0.3333348	0.3332819	0	0	0	0
3	0.334298	0.3333328	0.3333326	0.3333334	0	0	0
4	0.3335745	0.3333333	0.3333333	0.3333333	0.3333333	0	0
5	0.3333936	0.3333333	0.3333333	0.3333333	0.3333333	0.3333333	0
6	0.3333484	0.3333333	0.3333333	0.3333333	0.3333333	0.3333333	0.3333333

Richardson table in double floating point

N							
0	0.392699081698724	0	0	0	0	0	0
1	0.348771003583907	0.334128310878968	0	0	0	0	0
2	0.337193879218859	0.333334837763843	0.333281939556169	0	0	0	0
3	0.334298029698348	0.333332746524844	0.333332607108911	0.33333341135578	0	0	0
4	0.333574472267674	0.33333328645745	0.333333322452957	0.333333333807624	0.333333333503514	0	0
5	0.333393615751437	0.333333330246024	0.333333333165262	0.333333333335299	0.333333333333447	0.33333333333328	0
6	0.333348403791302	0.333333333137923	0.333333333330717	0.333333333333343	0.333333333333335	0.333333333333335	0.333333333333335

This is the source code

6 Computer assignment 5/2/2007. Midpoint,Trapezoid and Simpson

6.1 Conclusion

This table summarizes the results of the 3 methods

Method	RESULTS
Simpson	Error term $\frac{1}{180}(b-a)h^4 \max\left\|f^{(4)}(\xi)\right\|$
	$I=\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(x_0)+2\sum_{i=2}^{N/2} f(x_{2i-2})+4\sum_{i=1}^{N/2} f(x_{2i-1})+f(x_N)\right)$
	Intervals needed: 900
	long format print of numerical integration: 90.379254649757272
Midpoint	Error term $\frac{1}{24}(b-a)h^2 \max\left\|f^{(2)}(\xi)\right\|$
	$\int_a^b f(x)\,dx \approx h\sum_{i=1}^{N-1} f\left(\frac{x_{i+1}+x_i}{2}\right)$ note: N here is number of points
	Intervals needed: $174,285$
	long format print of numerical integration: 90.379254649446878
Trapezoid	Error term $\frac{1}{12}(b-a)h^2 \max\left\|f^{(2)}(\xi)\right\|$
	$h\left(\frac{f(x_1)}{2}+\sum_{i=2}^{N-1} f(x_i)+\frac{f(x_N)}{2}\right)$ note: N here is number of points
	Intervals needed: $246,476$
	long format print of numerical integration: 90.379254649958952

Source code:

why?

```
function nma_simpson_math_501
%
%Math 501, CSUF, spring 2007
%Computer assignment 5/2/2007
%Nasser Abbasi

%For reference, this is exact answer for 60 decimal places
%NIntegrate[x*Log[x], {x, 1, 10}, WorkingPrecision -> 60]
%90.3792546497022842008995727342182103800550744314386488016663957747002355720573 1`60.
%

a = 1;
b = 10;
maxError = 10^-9;

%(2/x^3) is  d^4/dx^4 (x log(x))
%so max error will be when x is smallest, i.e. at a=1
I4      = abs(2/a^3);
errTerm = 1/180 * (b-a) * I4;
h       = maxError /errTerm;
h       = h^(1/4);
N       = ceil((b-a)/h);  % N is number of intervals

%N isnumber of intervals it needs to be EVEN number of intervals
if mod(N,2)==1
   N = N+1;
end

h = (b-a)/N;  %update h since we rounded up above.
fprintf('Simposon: Number of intervals needed is %d\n',N);

x = linspace(a,b,N+1);
f = @(x) x.*log(x);   %the function to integrate

%vectorized solution
I = f(x(1)) + 2*sum(f(x(3:2:end-2))) + 4*sum(f(x(2:2:end-1))) + f(x(end));
I = (h/3)*I;

fprintf('answer is'); format long;  I
```

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6.4 Trapezoid numerical integration

The error term is $\frac{1}{12}(b-a)h^2 \max\left|f^{(2)}(\xi)\right|$ for some ξ between b, a. Trapezoid is evaluated as follows

$$h\left(\frac{f(x_1)}{2} + \sum_{i=2}^{n-1} f(x_i) + \frac{f(x_n)}{2}\right)$$

Where n is number of points, and I am using the Matlab indexing where x_1 is the first point, and not x_0, hence the last point is x_n

The following is the source code and the output

```
function nma_trap_math_501
%
%Math 501, CSUF, spring 2007
%Computer assignment 5/2/2007
%Nasser Abbasi

%For reference, this is exact answer for 60 decimal places
%NIntegrate[x*Log[x], {x, 1, 10}, WorkingPrecision -> 60]
%90.3792546497022842008995727342182103800550744314386488016663957747002355720573１`60.
%

a = 1;
b = 10;
maxError = 10^-9;

%d^2/dx^2 (x log(x)) is (1/x)
%so max error will be when x is smallest, i.e. at a=1
I2      = abs(1/a);
errTerm = 1/12 * (b-a) * I2;
h       = maxError /errTerm;
h       = sqrt(h);
N       = ceil((b-a)/h);  % Number of intervals
h       = (b-a)/N;
fprintf('Trapezoid: Number of intervals needed is %d\n',N);

x     = linspace(a,b,N+1);
f     = @(x) x.*log(x);   %the function to integrate
fbar  = sum(f(x(2:end-1)));

%vectorized solution
I = h * ( f(x(1))/2 + fbar + f(x(end))/2 );

fprintf('answer is'); format long;  I
```

output

```
Trapezoid: Number of intervals needed is 246476
answer is
I =

 90.379254649958952
```