HW 12

Math 501
Numerical analysis

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Contents

1 Section 7.1, Problem 6

Problem: Derive the following 2 formulas for approximation of derivatives and show they are both $O(h^4)$ by evaluating their error terms

$$
\begin{aligned}
f'(x) &= \frac{1}{12h}\left[-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)\right]\\
f''(x) &= \frac{1}{12h^2}\left[-f(x+2h)+16f(x+h)-30f(x)+16f(x-h)-f(x-2h)\right]
\end{aligned}
$$

Solution:

I could obtain the above results directly from applying Richardson interpolation formulas (which is a short approach), but I assumed the question wanted us to derive these from first principles. I first show how to do one using Richardson, then solve both from first principles.

To obtain the approximation for $f'(x)$ using Richardson, we do the following:

$$
\begin{aligned}
\varphi(h) &= \frac{1}{2h}\left[f(x+h)-f(x-h)\right]\\
L &= \varphi(h)+a_2h^2+a_4h^4+\cdots \qquad (1C)
\end{aligned}
$$

Replacing h by $2h$

$$
L = \varphi(2h)+a_2 4h^2+a_4 16h^4+\cdots \qquad (2C)
$$

Multiplying (1C) by 4 and subtract (2C) from result

$$
\begin{aligned}
3L &= \left(4\varphi(h)+4a_2h^2+4a_4h^4+\cdots\right)-\left(\varphi(2h)+a_2 4h^2+a_4 16h^4+\cdots\right)\\
&= 4\varphi(h)-\varphi(2h)-12a_4h^4-\cdots
\end{aligned}
$$

Hence

$$
\begin{aligned}
L &= \frac{1}{3}\left(\frac{2}{h}\left[f(x+h)-f(x-h)\right]-\frac{1}{4h}\left[f(x+2h)-f(x-2h)\right]-12a_4h^4-\cdots\right)\\
&= \frac{2}{3h}\left[f(x+h)-f(x-h)\right]-\frac{1}{12h}\left[f(x+2h)-f(x-2h)\right]-4a_4h^4-\cdots\\
&= \frac{1}{12h}\left(8\left[f(x+h)-f(x-h)\right]-\left[f(x+2h)-f(x-2h)\right]\right)-4a_4h^4-\cdots\\
&= \frac{1}{12h}\left[-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)\right]-4a_4h^4-\cdots
\end{aligned}
$$

Which is the same result obtained earlier using the long approach. We also see that the error term is $O\left(h^4\right)$

Now, solve it again, but using direct usage of Taylor (which I assume what the book wanted us to do)

From Taylor expansion, we write, by expanding around $x+h$ and $x-h$

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f^{(3)}(x) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(\xi_1) \\ f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f^{(3)}(x) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(\xi_2) \end{aligned}$$

Subtract the second from the first equation

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{3}f^{(3)}(x) + \frac{h^5}{60}\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right]$$

Solve for $f'(x)$ we obtain

$$f'(x) = \frac{1}{2h}\left[f(x+h) - f(x-h)\right] - \frac{1}{6}h^2 f^{(3)}(x) - \frac{1}{120}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right] \tag{1}$$

Now we do the same again, but by expanding around $x+2h$ and $x-2h$

$$\begin{aligned} f(x+2h) &= f(x) + 2hf'(x) + \frac{(2h)^2}{2}f''(x) + \frac{(2h)^3}{3!}f^{(3)}(x) + \frac{(2h)^4}{4!}f^{(4)}(x) + \frac{(2h)^5}{5!}f^{(5)}(\xi_1) \\ f(x-2h) &= f(x) - 2hf'(x) + \frac{(2h)^2}{2}f''(x) - \frac{(2h)^3}{3!}f^{(3)}(x) + \frac{(2h)^4}{4!}f^{(4)}(x) - \frac{(2h)^5}{5!}f^{(5)}(\xi_2) \end{aligned}$$

Subtract the second from the first equation

$$\begin{aligned} f(x+2h) - f(x-2h) &= 4hf'(x) + 2\frac{(2h)^3}{3!}f^{(3)}(x) + \frac{(2h)^5}{5!}\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right] \\ &= 4hf'(x) + \frac{8}{3}h^3 f^{(3)}(x) + \frac{4}{15}h^5\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right] \end{aligned}$$

Solve for $f'(x)$ we obtain

$$f'(x) = \frac{1}{4h}\left[f(x+2h) - f(x-2h)\right] - \frac{4}{6}h^2 f^{(3)}(x) - \frac{1}{15}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right] \tag{2}$$

We want to eliminate $f^{(3)}(x)$ from the above. So we multiply eq(1) by 4 and subtract eq(2) from the result. So equation (1) becomes

$$\begin{aligned} 4f'(x) &= 4\left(\frac{1}{2h}\left[f(x+h) - f(x-h)\right] - \frac{1}{6}h^2 f^{(3)}(x) - \frac{1}{120}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right]\right) \\ &= \frac{2}{h}\left[f(x+h) - f(x-h)\right] - \frac{4}{6}h^2 f^{(3)}(x) - \frac{1}{30}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right] \end{aligned} \tag{3}$$

Now subtract (2) from (3) we obtain

$$\begin{aligned} 3f'(x) = {} & \frac{2}{h}\left[f(x+h) - f(x-h)\right] - \frac{4}{6}h^2 f^{(3)}(x) - \frac{1}{30}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right] - \\ & \left(\frac{1}{4h}\left[f(x+2h) - f(x-2h)\right] - \frac{4}{6}h^2 f^{(3)}(x) - \frac{1}{15}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right]\right) \end{aligned}$$

Hence

$$\begin{aligned} 3f'(x) = {} & \frac{2}{h}\left[f(x+h) - f(x-h)\right] - \frac{1}{30}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right] - \\ & \frac{1}{4h}\left[f(x+2h) - f(x-2h)\right] + \frac{1}{15}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right] \end{aligned}$$

$$\begin{aligned} f'(x) &= \frac{2}{3h}\left[f(x+h) - f(x-h)\right] - \frac{1}{12h}\left[f(x+2h) - f(x-2h)\right] + \frac{1}{90}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right] \\ &= \frac{1}{12h}\left[8f(x+h) - 8f(x-h) - f(x+2h) - f(x-2h)\right] + \frac{1}{90}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right] \end{aligned}$$

Rearrange terms to make it look as in the textbook

$$f'(x) = \frac{1}{12h}\left[-f(x+2h) + 8f(x+h) - 8f(x-h) - f(x-2h)\right] + \frac{1}{90}h^4\left[f^{(5)}(\xi)\right] \tag{4}$$

Where we replaced $\frac{1}{90}h^4\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right]$ by $\frac{1}{45}h^4\left[\frac{f^{(5)}(\xi_1)+f^{(5)}(\xi_2)}{2}\right] = \frac{1}{90}h^4\left[f^{(5)}(\xi)\right]$ with $f^{(5)}(\xi)$ being the mean value of $\frac{f^{(5)}(\xi_1)+f^{(5)}(\xi_2)}{2}$

Hence from equation (4) we see that the error is $O\left(h^4\right)$ as required to show.

Hence

$$\boxed{f'(x) \approx \frac{1}{12h}\left[-f(x+2h)+8f(x+h)-8f(x-h)-f(x-2h)\right]}$$

Now we need to show the formula for $f''(x)$. We do the same as above, but instead of subtracting equations, we add them. We start from the top to show these again step by step.

From Taylor expansion, we write, by expanding around $x+h$ and $x-h$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f^{(3)}(x) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(x) + \frac{h^6}{6!}f^{(6)}(\xi_1)$$
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f^{(3)}(x) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(x) + \frac{h^6}{6!}f^{(6)}(\xi_2)$$

Add the second to the first equation

$$f(x+h)+f(x-h) = 2f(x) + h^2 f''(x) + \frac{h^4}{12}f^{(4)}(x) + \frac{h^6}{6!}\left[f^{(6)}(\xi_1)+f^{(6)}(\xi_2)\right]$$

Solve for $f''(x)$ we obtain

$$f''(x) = \frac{1}{h^2}\left[f(x+h)+f(x-h)\right] - \frac{2}{h^2}f(x) - \frac{h^2}{12}f^{(4)}(x) - \frac{1}{720}h^4\left[f^{(6)}(\xi_1)+f^{(6)}(\xi_2)\right] \quad \text{(1A)}$$

Now we do the same again, but by expanding around $x+2h$ and $x-2h$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2}f''(x) + \frac{(2h)^3}{3!}f^{(3)}(x) + \frac{(2h)^4}{4!}f^{(4)}(x) + \frac{(2h)^5}{5!}f^{(5)}(x) + \frac{(2h)^6}{6!}f^{(6)}(\xi_1)$$
$$f(x-2h) = f(x) - 2hf'(x) + \frac{(2h)^2}{2}f''(x) - \frac{(2h)^3}{3!}f^{(3)}(x) + \frac{(2h)^4}{4!}f^{(4)}(x) - \frac{(2h)^5}{5!}f^{(5)}(x) + \frac{(2h)^6}{6!}f^{(6)}(\xi_1)$$

Add the second to the first equation

$$\begin{aligned} f(x+2h)+f(x-2h) &= 2f(x) + (2h)^2 f''(x) + \frac{(2h)^4}{12}f^{(4)}(x) + \frac{(2h)^6}{6!}\left[f^{(6)}(\xi_1)+f^{(6)}(\xi_2)\right] \\ &= 2f(x) + 4h^2 f''(x) + \frac{4}{3}h^4 f^{(4)}(x) + \frac{4}{45}h^6\left[f^{(6)}(\xi_1)+f^{(6)}(\xi_2)\right] \end{aligned}$$

Solve for $f''(x)$ we obtain

$$f''(x) = \frac{1}{4h^2}\left[f(x+2h) + f(x-2h)\right] - \frac{1}{2h^2}f\ (x) - \frac{1}{3}h^2 f^{(4)}(x) - \frac{1}{45}h^4\left[f^{(6)}(\xi_1) + f^{(6)}(\xi_2)\right] \tag{2A}$$

We want to eliminate $f^{(4)}(x)$ from the above. So we multiply eq(1A) by 4 and subtract eq(2) from the result. So equation (1A) becomes

$$\begin{aligned} 4f''(x) &= 4\left(\frac{1}{h^2}\left[f(x+h) + f(x-h)\right] - \frac{2}{h^2}f\ (x) - \frac{h^2}{12}f^{(4)}(x) - \frac{1}{720}h^4\left[f^{(6)}(\xi_1) + f^{(6)}(\xi_2)\right]\right) \\ &= \frac{4}{h^2}\left[f(x+h) + f(x-h)\right] - \frac{8}{h^2}f\ (x) - \frac{1}{3}h^2 f^{(4)}(x) - \frac{1}{180}h^4\left[f^{(6)}(\xi_1) + f^{(6)}(\xi_2)\right] \end{aligned} \tag{3A}$$

Now subtract (2A) from (3A) we obtain

$$\begin{aligned} 3f''(x) = {} & \frac{4}{h^2}\left[f(x+h) + f(x-h)\right] - \frac{8}{h^2}f\ (x) - \frac{1}{3}h^2 f^{(4)}(x) - \frac{1}{180}h^4\left[f^{(6)}(\xi_1) + f^{(6)}(\xi_2)\right] - \\ & \left(\frac{1}{4h^2}\left[f(x+2h) + f(x-2h)\right] - \frac{1}{2h^2}f\ (x) - \frac{1}{3}h^2 f^{(4)}(x) - \frac{1}{45}h^4\left[f^{(6)}(\xi_1) + f^{(6)}(\xi_2)\right]\right) \end{aligned}$$

Hence

$$\begin{aligned} 3f''(x) = {} & \frac{4}{h^2}\left[f(x+h) + f(x-h)\right] - \frac{8}{h^2}f\ (x) - \frac{1}{180}h^4\left[f^{(6)}(\xi_1) + f^{(6)}(\xi_2)\right] - \\ & \frac{1}{4h^2}\left[f(x+2h) + f(x-2h)\right] + \frac{1}{2h^2}f\ (x) + \frac{1}{45}h^4\left[f^{(6)}(\xi_1) + f^{(6)}(\xi_2)\right] \end{aligned}$$

$$\begin{aligned} f''(x) = {} & \frac{4}{3h^2}\left[f(x+h) + f(x-h)\right] - \frac{1}{12h^2}\left[f(x+2h) + f(x-2h)\right] - \frac{8}{3h^2}f\ (x) + \frac{1}{6h^2}f\ (x) - \\ & \frac{1}{3\times 180}h^4\left[f^{(6)}(\xi_1) + f^{(6)}(\xi_2)\right] + \frac{1}{3\times 45}h^4\left[f^{(6)}(\xi_1) + f^{(6)}(\xi_2)\right] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{12h^2}\left(16\left[f(x+h) + f(x-h)\right] - \left[f(x+2h) + f(x-2h)\right] - 32f\ (x) + 2f\ (x)\right) + \frac{1}{180}h^4\left[f^{(6)}(\xi_1) + f^{(\right. \\ &= \frac{1}{12h^2}\left(16f(x+h) + 16f(x-h) - f(x+2h) - f(x-2h) - 30f\ (x)\right) + \frac{1}{180}h^4\left[f^{(6)}(\xi)\right] \end{aligned}$$

Rearrange terms to make it look as in the textbook

$$f''(x) = \frac{1}{12h^2}\left(-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)\right) + \frac{1}{180}h^4\left[f^{(6)}(\xi)\right] \tag{4A}$$

Hence from equation (4A) we see that the error is $O\left(h^4\right)$ as required to show.

Hence

$$\boxed{f''(x) \approx \frac{1}{12h^2}\left(-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)\right)}$$

2 Section 7.1, Problem 9

problem: Show that in Richardson extrapolation, $D(2,2) = \frac{16}{15}\psi\left(\frac{h}{2}\right) - \frac{1}{15}\psi(h)$

Solution:

$$D(n,k) = \frac{4^k}{4^k - 1}D(n,k-1) - \frac{1}{4^k - 1}D(n-1,k-1) \tag{1}$$

Now, when $n = 2, k = 2$, we obtain from (1)

$$\begin{aligned} D(2,2) &= \frac{4^2}{4^2 - 1}D(2,1) - \frac{1}{4^2 - 1}D(1,1) \\ &= \frac{16}{15}D(2,1) - \frac{1}{15}D(1,1) \end{aligned}$$

But since $D(1,1) = \psi(h), D(2,1) = \psi\left(\frac{h}{2}\right)$

$$D(2,2) = \frac{16}{15}\psi\left(\frac{h}{2}\right) - \frac{1}{15}\psi(h)$$

3 Section 7.1, Problem 14

problem: Using Taylor series, derive the error term for the approximation

$$f'(x) \approx \frac{1}{2h}\left[-3f(x) + 4f(x+h) - f(x+2h)\right]$$

answer:

expand around $x+h$

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(\xi_1) \\ f'(x) &= \frac{1}{h}f(x+h) - \frac{1}{h}f(x) - \frac{h}{2}f''(x) - \frac{h^2}{6}f'''(\xi_1) \qquad (1) \end{aligned}$$

Now expand around $x+2h$

$$\begin{aligned} f(x+2h) &= f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{8h^3}{6}f'''(\xi_2) \\ f'(x) &= \frac{1}{2h}f(x+2h) - \frac{1}{2h}f(x) - hf''(x) - \frac{4h^2}{6}f'''(\xi_2) \qquad (2) \end{aligned}$$

Multiply (2) by $-\frac{1}{2}$ and add result to (1) we obtain

$$\begin{aligned} -\frac{1}{2}f'(x) + f'(x) = -\frac{1}{2}&\left(\frac{1}{2h}f(x+2h) - \frac{1}{2h}f(x) - hf''(x) - \frac{4h^2}{6}f'''(\xi_2)\right) + \\ &\left(\frac{1}{h}f(x+h) - \frac{1}{h}f(x) - \frac{h}{2}f''(x) - \frac{h^2}{6}f'''(\xi_1)\right) \end{aligned}$$

$$\begin{aligned} \frac{1}{2}f'(x) &= \frac{-1}{4h}f(x+2h) + \frac{1}{4h}f(x) + \frac{h}{2}f''(x) + \frac{2h^2}{6}f'''(\xi_2) + \frac{1}{h}f(x+h) - \frac{1}{h}f(x) - \frac{h}{2}f''(x) - \frac{h^2}{6}f'''(\xi \\ f'(x) &= \frac{-1}{2h}f(x+2h) + \frac{1}{2h}f(x) + hf''(x) + \frac{4h^2}{6}f'''(\xi_2) + \frac{2}{h}f(x+h) - \frac{2}{h}f(x) - hf''(x) - \frac{2h^2}{6}f'''(\xi \\ &= \left[\frac{-1}{2h}f(x+2h) + \frac{1}{2h}f(x) + hf''(x) + \frac{2}{h}f(x+h) - \frac{2}{h}f(x) - hf''(x)\right] - \frac{2h^2}{6}f'''(\xi_1) + \frac{4h^2}{6}f''' \\ &= \frac{1}{2h}\left[-f(x+2h) + f(x) + 4f(x+h) - 4f(x)\right] - \frac{h^2}{3}f'''(\xi_1) + \frac{2h^2}{3}f'''(\xi_2) \\ &= \frac{1}{2h}\left[-f(x+2h) - 3f(x) + 4f(x+h)\right] - h^2\left(\frac{1}{3}f'''(\xi_1) + \frac{2}{3}f'''(\xi_2)\right) \end{aligned}$$

Which is the equation we are asked to show.

From the above we see that the error term is given by

$$h^2\left(\frac{1}{3}f'''(\xi_1)+\frac{2}{3}f'''(\xi_2)\right)$$

Hence the error is $O\left(h^2\right)$

4 Section 7.1, Problem 16

problem: Using Taylor series, derive the error term for the approximation

$$f''(x) \approx \frac{1}{h^2}\left[f(x) - 2f(x+h) + f(x+2h)\right]$$

Answer: expand around $x+h$

$$\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(\xi_1) \\
\frac{h^2}{2}f''(x) &= f(x+h) - f(x) - hf'(x) - \frac{h^3}{6}f'''(\xi_1) \\
f''(x) &= \frac{2}{h^2}f(x+h) - \frac{2}{h^2}f(x) - \frac{2}{h}f'(x) - \frac{h}{3}f'''(\xi_1) \qquad (1)
\end{aligned}$$

Now expand around $x+2h$

$$\begin{aligned}
f(x+2h) &= f(x) + 2hf'(x) + 2h^2f''(x) + \frac{8h^3}{6}f'''(\xi_2) \\
2h^2f''(x) &= f(x+2h) - f(x) - 2hf'(x) - \frac{8h^3}{6}f'''(\xi_2) \\
f''(x) &= \frac{1}{2h^2}f(x+2h) - \frac{1}{2h^2}f(x) - \frac{1}{h}f'(x) - \frac{2h}{3}f'''(\xi_2) \qquad (2)
\end{aligned}$$

Multiply (2) by -2 and add result to (1) we obtain

$$\begin{aligned}
-2f''(x) + f''(x) &= -2\left(\frac{1}{2h^2}f(x+2h) - \frac{1}{2h^2}f(x) - \frac{1}{h}f'(x) - \frac{2h}{3}f'''(\xi_2)\right) \\
&\quad + \frac{2}{h^2}f(x+h) - \frac{2}{h^2}f(x) - \frac{2}{h}f'(x) - \frac{h}{3}f'''(\xi_1) \\
-f''(x) &= -\frac{1}{h^2}f(x+2h) + \frac{1}{h^2}f(x) + \frac{2}{h}f'(x) + \frac{4h}{3}f'''(\xi_2) + \frac{2}{h^2}f(x+h) - \frac{2}{h^2}f(x) - \frac{2}{h}f'(x) - \frac{h}{3}f'''(\xi \\
f''(x) &= \frac{1}{h^2}f(x+2h) - \frac{1}{h^2}f(x) - \frac{2}{h}f'(x) - \frac{4h}{3}f'''(\xi_2) - \frac{2}{h^2}f(x+h) + \frac{2}{h^2}f(x) + \frac{2}{h}f'(x) + \frac{h}{3}f'''(\xi_1 \\
&= \frac{1}{h^2}f(x+2h) - \frac{1}{h^2}f(x) - \frac{2}{h^2}f(x+h) + \frac{2}{h^2}f(x) + \frac{h}{3}f'''(\xi_1) - \frac{4h}{3}f'''(\xi_2) \\
&= \frac{1}{h^2}\left(f(x+2h) - f(x) - 2f(x+h) + 2f(x)\right) + h\left(\frac{1}{3}f'''(\xi_1) - \frac{4}{3}f'''(\xi_2)\right) \\
&= \frac{1}{h^2}\left(f(x+2h) + f(x) - 2f(x+h)\right) + h\left(\frac{1}{3}f'''(\xi_1) - \frac{4}{3}f'''(\xi_2)\right)
\end{aligned}$$

Hence

$$f''(x) \approx \frac{1}{h^2}\Big(f\,(x+2h) + f(x) - 2f\,(x+h)\Big)$$

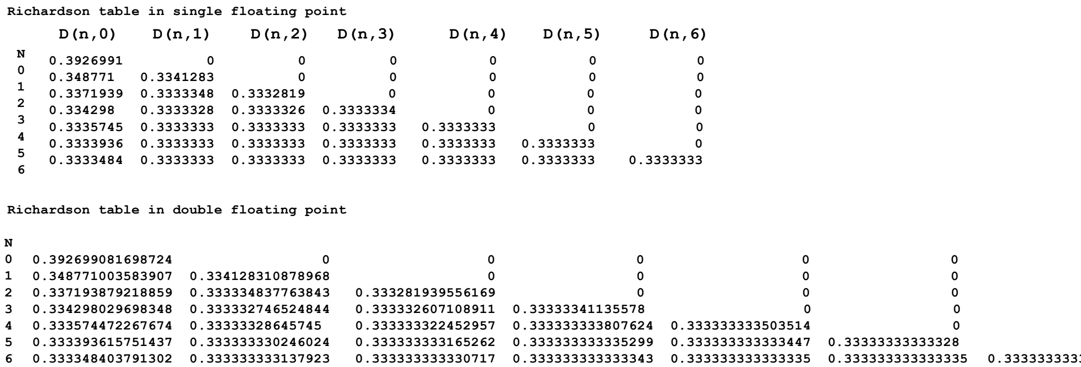
with the error term

$$h\left(\frac{1}{3}f'''\,(\xi_1) - \frac{4}{3}f'''\,(\xi_2)\right)$$

Hence the error is $O(h)$

5 Computer assignment 4/30/2007. Richardson Algorithm

This is the output



Richardson table in single floating point

N	D(n,0)	D(n,1)	D(n,2)	D(n,3)	D(n,4)	D(n,5)	D(n,6)
0	0.3926991	0	0	0	0	0	0
1	0.348771	0.3341283	0	0	0	0	0
2	0.3371939	0.3333348	0.3332819	0	0	0	0
3	0.334298	0.3333328	0.3333326	0.3333334	0	0	0
4	0.3335745	0.3333333	0.3333333	0.3333333	0.3333333	0	0
5	0.3333936	0.3333333	0.3333333	0.3333333	0.3333333	0.3333333	0
6	0.3333484	0.3333333	0.3333333	0.3333333	0.3333333	0.3333333	0.3333333

Richardson table in double floating point

N							
0	0.392699081698724	0	0	0	0	0	
1	0.348771003583907	0.334128310878968	0	0	0	0	
2	0.337193879218859	0.333334837763843	0.333281939556169	0	0	0	
3	0.334298029698348	0.333332746524844	0.333332607108911	0.33333341135578	0	0	
4	0.333574472267674	0.33333328645745	0.333333322452957	0.333333333807624	0.333333333503514	0	
5	0.333393615751437	0.333333330246024	0.333333333165262	0.333333333335299	0.333333333333447	0.33333333333328	
6	0.333348403791302	0.333333333137923	0.333333333330717	0.333333333333343	0.333333333333335	0.333333333333335	0.333333333

Figure 1: Table output

This is the source code

```
%script to test nma_richardson
%Nasser Abbasi

h=1
x=sqrt(2);
f=@(x)atan(x);
M=6;

%first compute in single prcesion
 D=nma_richardson(h,x,f,M,0);
 format long g;
 fprintf('Richardson table in single floating point\n');
 D

 %Now do it in double prcesion
 D=nma_richardson(h,x,f,M,1);
 format long g;
 fprintf('Richardson table in double floating point\n');
 D
```

```
function D=nma_richardson(h,x,f,M,doubleFlag)
%function D=nma_richardson(h,x,f,M,doubleFlag)
%
%INPUT:
% h:  spacing for numerical differentiation
% x:  where to find diff
% f:  the function whos derivative we are finding
```

```
% M:  how big a richardson table to make
% doubleFlag: 0 to do it in single floating point
%             or 1 to do it in double floating


% MATH 501, CSUF, spring 2007
% computer assignment 4/30/2007
% Richardson extrapolation
%
%Nasser Abbasi, May 5,2007

if doubleFlag
    D=zeros(M+1,M+1);
else
    D=zeros(M+1,M+1,'single');
end

for n=1:M+1
    D(n,1)=phi(h/(2^(n-1)),x,f);
end

for k=2:M+1
    for n=k:M+1
        D(n,k)=D(n,k-1)+(D(n,k-1)-D(n-1,k-1))/(4^(k-1)-1);
    end
end
end


function r=phi(h,x,f)
r=1/(2*h)*(f(x+h)-f(x-h));
end
```

6 Computer assignment 5/2/2007. Midpoint,Trapezoid and Simpson

6.1 Conclusion

This table summarizes the results of the 3 methods

Method	RESULTS
Simpson	Error term $\frac{1}{180}(b-a)h^4 \max\left\|f^{(4)}(\xi)\right\|$
	$I=\int_a^b f(x)dx \approx \frac{h}{3}\left(f(x_0)+2\sum_{i=2}^{N/2} f(x_{2i-2})+4\sum_{i=1}^{N/2} f(x_{2i-1})+f(x_N)\right)$
	Intervals needed: 900
	long format print of numerical integration: 90.379254649757272
Midpoint	Error term $\frac{1}{24}(b-a)h^2 \max\left\|f^{(2)}(\xi)\right\|$
	$\int_a^b f(x)dx \approx h\sum_{i=1}^{N-1} f\left(\frac{x_{i+1}+x_i}{2}\right)$ note: N here is number of points
	Intervals needed: $174,285$
	long format print of numerical integration: 90.379254649446878
Trapezoid	Error term $\frac{1}{12}(b-a)h^2 \max\left\|f^{(2)}(\xi)\right\|$
	$h\left(\frac{f(x_1)}{2}+\sum_{i=2}^{N-1} f(x_i)+\frac{f(x_N)}{2}\right)$ note: N here is number of points
	Intervals needed: $246,476$
	long format print of numerical integration: 90.379254649958952

6.2 Simpson

The error term in simpson is $\frac{1}{180}(b-a)h^4 \ \max\left|f^{(4)}(\xi)\right|$ for some ξ between b, a. Since we want to limit the maximum error, we look to find where $f(\xi)$ is Max.

The function is $x\ln(x)$, hence $f^{(4)}(x) = \frac{2}{x^3}$ and this is maximum when x is smallest. Hence the maximum will occur at the lower end of the range, which is $x = 1$ in this example.

Now we find the number of intervals N from solving $\frac{1}{180}(b-a)h^4 \ \max\left|f^{(4)}(\xi)\right| < 10^{-9}$ where 10^{-9} is the error we are asked to limit our computation error to be below.

Next, we solve for h from the above. Knowing h, we find N which is the number of intervals. Next, we make sure N is even number by adjusting it if needed. We need to have even number of intervals Next we apply the simpson integration formula which is

$$I = \int_a^b f(x)dx \approx \frac{h}{3}\left(f(x_0) + 2\sum_{i=2}^{N/2} f(x_{2i-2}) + 4\sum_{i=1}^{N/2} f(x_{2i-1}) + f(x_N)\right)$$

In the above N is the number of intervals. Not to be confused with the following 2 algorithms below, where I used N to be number of points. For simpson, it was easier to stick with N being number of intervals.

The matlab implementation uses a vectorized version for speed.

To verify that the correct answer is obtain, it was compared with the output from a computer algebra system which uses an arbitrary large number of correct decimal points. The Matlab output was aligned against the CAS output and the digits verified to be correct to 9 decimal places are required.

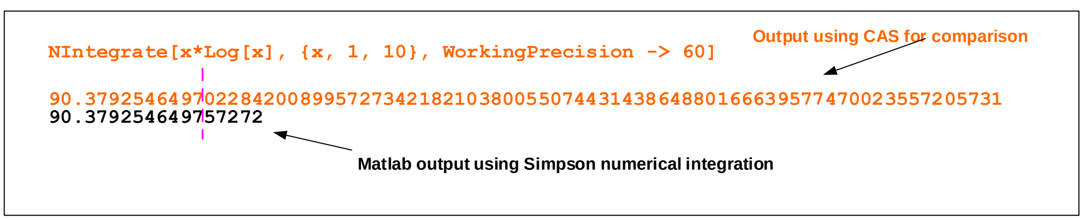


Figure 2: Result

Source code:

```
function nma_simpson_math_501
%
%Math 501, CSUF, spring 2007
%Computer assignment 5/2/2007
%Nasser Abbasi
```

```
%For reference, this is exact answer for 60 decimal places
%NIntegrate[x*Log[x], {x, 1, 10}, WorkingPrecision -> 60]
%90.3792546497022842008995727342182103800550744314386488016663957747002355720573 1`60.

%

a = 1;
b = 10;
%maxError = 10^(-9);

%(2/x^3) is  d^4/dx^4 (x log(x))
%so max error will be when x is smallest, i.e. at a=1
I4      = abs(2/a^3);
errTerm = 1/180 * (b-a) * I4;
h       = maxError /errTerm;
h       = h^(1/4);
N       = ceil((b-a)/h);  % N is number of intervals

%N isnumber of intervals it needs to be EVEN number of intervals
if mod(N,2)==1
   N = N+1;
end

h = (b-a)/N;  %update h since we rounded up above.
fprintf('Simposon: Number of intervals needed is %d\n',N);

x = linspace(a,b,N+1);
f = @(x) x.*log(x);   %the function to integrate

%vectorized solution
I = f(x(1)) + 2*sum(f(x(3:2:end-2))) + 4*sum(f(x(2:2:end-1))) + f(x(end));
I = (h/3)*I;

fprintf('answer is'); format long;  I
```

6.3 Midpoint

The error term is $\frac{1}{24}(b-a)h^2 \max\left|f^{(2)}(\xi)\right|$ for some ξ between b, a. Midterm is evaluated as follows

$$I = \int_a^b f(x)dx \approx h \sum_{i=1}^{N-1} f\left(\frac{x_{i+1}+x_i}{2}\right)$$

where N is the number of points. And I am using the Matlab convention for indexing, where the first point is x_1 and not x_0

We start by finding the number of intervals by solving for h from $\frac{1}{24}(b-a)h^2 \max\left|f^{(2)}(\xi)\right| < 10^{-9}$ where 10^{-9} is the error we are asked to limit our computation error to be below.

The function is $x\ln(x)$, hence $f^{(2)}(x) = \frac{1}{x}$ which is maximum at $x = a$.

The matlab implementation is below with the output.

```
function nma_midpoint_math_501
%
%Math 501, CSUF, spring 2007
%Computer assignment 5/2/2007
%Nasser Abbasi

%For reference, this is exact answer for 60 decimal places
%NIntegrate[x*Log[x], {x, 1, 10}, WorkingPrecision -> 60]
%90.3792546497022842008995727342182103800550744314386488016663957747002355720573l`60.

%

a = 1;
b = 10;
maxError = 10^-9;

%d^2/dx^2 (x log(x)) is (1/x)
%so max error will be when x is smallest, i.e. at a=1
I2      = abs(1/a);
errTerm = 1/24 * (b-a) * I2;
h       = maxError /errTerm;
h       = sqrt(h);
N       = ceil((b-a)/h);
h       = (b-a)/N;  %update h since we rounded up above.
fprintf('Midpoint: Number of intervals needed is %d\n',N);

x     = linspace(a,b,N+1);
xbar  = (x(1:end-1)+x(2:end))/2;
f     = @(x) x.*log(x);   %the function to integrate
```

```
%vectorized solution
I = h*sum(f(xbar));

fprintf('answer is'); format long;  I
```

Output is

```
Midpoint: Number of intervals needed is 174285
answer is
I =

  90.379254649446878
```

6.4 Trapezoid numerical integration

The error term is $\frac{1}{12}(b-a)h^2 \max\left|f^{(2)}(\xi)\right|$ for some ξ between b, a. Trapezoid is evaluated as follows

$$h\left(\frac{f(x_1)}{2} + \sum_{i=2}^{n-1} f(x_i) + \frac{f(x_n)}{2}\right)$$

Where n is number of points, and I am using the Matlab indexing where x_1 is the first point, and not x_0, hence the last point is x_n

The following is the source code and the output

```
function nma_trap_math_501
%
%Math 501, CSUF, spring 2007
%Computer assignment 5/2/2007
%Nasser Abbasi

%For reference, this is exact answer for 60 decimal places
%NIntegrate[x*Log[x], {x, 1, 10}, WorkingPrecision -> 60]
%90.3792546497022842008995727342182103800550744314386488016663957747002355720573 1`60.

%

a = 1;
b = 10;
maxError = 10^-9;

%d^2/dx^2 (x log(x)) is (1/x)
%so max error will be when x is smallest, i.e. at a=1
I2      = abs(1/a);
errTerm = 1/12 * (b-a) * I2;
h       = maxError /errTerm;
h       = sqrt(h);
N       = ceil((b-a)/h);  % Number of intervals
h       = (b-a)/N;
fprintf('Trapezoid: Number of intervals needed is %d\n',N);

x     = linspace(a,b,N+1);
f     = @(x) x.*log(x);   %the function to integrate
fbar  = sum(f(x(2:end-1)));

%vectorized solution
I = h * ( f(x(1))/2 + fbar + f(x(end))/2 );

fprintf('answer is'); format long;  I
```

Output

```
Trapezoid: Number of intervals needed is 246476
answer is
I =

  90.379254649958952
```

7 source code

7.1 nma_compare.m

```
% Matlab code to illustrate the how the error changes in
% computing the derivative of arctan(x) at x=SQRT(2) as a function
% of changing h in Taylor approximation. Forcing Matlab to do the
% computation using 32 bits
% by Nasser Abbasi

h=single(1);
M=26;
X=single(sqrt(2));
f=@(x) single(atan(x));

F1=f(X);
S = zeros(26,6,'single');

for k=1:M
    F2=f(X+h);
    d=single(F2-F1);
    r=single(d/h);
    S(k,1)=k; S(k,2)=h; S(k,3)=F2; S(k,4)=F1; S(k,5)=d; S(k,6)=r;
    h=single(h/2);
end
format long g
S


% Matlab code to illustrate the how the error changes in
% computing the derivative of arctan(x) at x=SQRT(2) as a function
% of changing h in Taylor approximation. using Matlab default double
% floating point
% by Nasser Abbasi
clear all

h=1;
M=60;
X=sqrt(2);
f=@(x) atan(x);

F1=f(X);
S = zeros(26,6);

for k=1:M
    F2=f(X+h);
```

```
    d=F2-F1;
    r=d/h;
    S(k,1)=k; S(k,2)=h; S(k,3)=F2; S(k,4)=F1; S(k,5)=d; S(k,6)=r;
    h=h/2;
end
format long g
S
```

7.2 nma_trapezoidal.m

```
function I=nma_trapezoidal(func,from,to,nStrips)
%function r=nma_trapezoidal(f,from,to,nStrips)
%
% integrates a function using trapezoidal rule using
% specific number of strips.
%
% INPUT:
%   func : a string that repesents the function itself
%       for example 'x*sin(x)'. The independent variable
%       used in the string must be 'x' and no other letter.
%
%  from: lower limit
%  to  : upper limit
%  nStrips: number of strips to use
%
% OUTPUT
%   I : The integral.
%
% Author: Nasser Abbasi
% May 3, 2003

I=0;

if(nStrips<=0)
    error('number of strips must be > 0');
end

nPoints=nStrips+1;
X=linspace(from,to,nPoints);
h=X(2)-X(1);

for(i=1:length(X))
    x=X(i);
    f(i)=eval(func);
    if(i==1 | i==length(X) )
        I=I+f(i);
    else
```

```
        I=I+2*f(i);
    end
end

I=I/2;
I=I*h;
```