Math 501

Section 6.1 # 13, 22, 26, 27, 37

Section 6.2 # 4, 9, 12, 23

due 4/30/2007

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and section 5.4 # 2a, 10, 11, 23, 34, 39 (part of my writeup)

2 Homework Solution for section 5.4

2.1 Problem section 5.4, 2(a)

question: Find the minimal solution for $x_1 x_2 = b$

answer:

First write the problem as

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [b]$$

Minimal solution is $\vec{x} = A^{+}b$, so we need to find A^{+}. Find $A = PDQ$, then $A^{+} = Q^{H} D^{+} P^{H}$

First find the set of $\vec{u}_i$ vectors to go to the Q matrix. I will use the economical SVD method.

$$\begin{aligned} A^{H}A &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

Hence $r = 1$

Hence $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \rightarrow (1-\lambda)^2 - 1 = 0 \rightarrow 1 + \lambda^2 - 2\lambda - 1 = 0 \rightarrow \lambda(\lambda - 2) = 0$

Hence $\lambda_1 = 2, \lambda_2 = 0 \rightarrow \sigma_1 = \sqrt{2}, \sigma_2 = 0$

Find eigenvectors $\vec{u}_1, \vec{u}_2$.

For $\lambda_1 = 2 \rightarrow \begin{bmatrix} 1-2 & 1 \\ 1 & 1-2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -y_1 + y_2 = 0 \Rightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{\text{normalize norm 2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

For $\lambda_2 = 0 \rightarrow \begin{bmatrix} 1-0 & 1 \\ 1 & 1-0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y_1 + y_2 = 0 \Rightarrow \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \xrightarrow{\text{normalize norm 2}} \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

Hence $Q = \begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}$

Not to find the P matrix. $AA^{H} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [2]$

The eigenvalue is $2 - \lambda = 0 \rightarrow \lambda = 2$. Hence the eigenvector is $2y_1 = 0 \rightarrow y_1 =$anything$\rightarrow [1]$

10/10

Hence the P matrix is $[1]$

The D matrix is $m \times n$, hence 1×2, then $D = \begin{bmatrix}\sigma_1 & 0\end{bmatrix}$

Hence this completes the SVD. We have that

$$\begin{aligned}
\begin{bmatrix}1 & 1\end{bmatrix} &= [1]\begin{bmatrix}\sigma_1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\\
&= [1]\begin{bmatrix}\sqrt{2} & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\\
&= \begin{bmatrix}\sqrt{2} & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\\
&= \begin{bmatrix}\sqrt{2} & \sqrt{2}\end{bmatrix}\frac{1}{\sqrt{2}}\\
&= \begin{bmatrix}1 & 1\end{bmatrix}
\end{aligned}$$

So the SVD is verified. Not find

$$\begin{aligned}
A^+ &= Q^H D^+ P^H\\
&= \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}^H \frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{2} & 0\end{bmatrix}^+ [1]^H\\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\end{bmatrix}[1]
\end{aligned}$$

Notice that D^+ mean we also take the conjugate transpose of D and then we take the reciprocal of each entry. Hence if D is $m \times n$ then D^+ is $n \times m$

$$\begin{aligned}
A^+ &= \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\\
&= \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\hat{x} &= A^+ b\\
&= \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix} b
\end{aligned}$$

So the minimal solution is $x_1 = \frac{b}{2}, x_2 = \frac{b}{2}$

2.2 Problem 5.4, 10

Problem: Prove the following properties of A^+

a)$A^{++} = A$

b)$A^{+H} = A^{H+}$

answer:

a)$(A^+)^+ = \left((PDQ)^+\right)^+ = \left(Q^H D^+ P^H\right)^+ = \left(P^H\right)^H (D^+)^+ \left(Q^H\right)^H$

But $\left(P^H\right)^H = P$ and $\left(Q^H\right)^H = Q$, and the reciprocal of a reciprocal gives us back the original value, hence $(D^+)^+ = D$

Hence we have $\left(P^H\right)^H (D^+)^+ \left(Q^H\right)^H = PDQ = A$

b)

$$\begin{aligned}
\left(A^+\right)^H &= \left(Q^H D^+ P^H\right)^H \\
&= \left(P^H\right)^H \left(D^+\right)^H \left(Q^H\right)^H \\
&= P \left(D^+\right)^H Q
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
A^{H+} &= \left((PDQ)^H\right)^+ \\
&= \left(Q^H D^H P^H\right)^+ \\
&= \left(P^H\right)^H \left(D^H\right)^+ \left(Q^H\right)^H \\
&= P \left(D^H\right)^+ Q
\end{aligned} \tag{2}$$

Hence (1)=(2) if $D^{H+} = D^{+H}$. But this is the case. Since $D^{H+} = D$ and $D^{+H} = D$

5.4 #23

$A = PDQ$

$\det(A - \lambda I) = 0$

$\det(PDQ - \lambda I) = 0$

$\det(P(DQ - \lambda P^{-1})) = 0$

$\det(P(D - \lambda P^{-1}Q^{-1})Q) = 0$

$\det(P) \det(D - \lambda P^{-1}Q^{-1}) \det(Q) = 0$

but $P^{-1} = P^H$

$Q^{-1} = Q^H$

so $\det(P) \det(D - \lambda P^H Q^H) \det(Q) = 0$

but $\left.\begin{array}{l}\det(P) \neq 0 \\ \det(Q) \neq 0\end{array}\right\}$ since unitary

so $\boxed{\det(D - \lambda P^H Q^H) = 0}$

since characteristic equation $= 0$ then $-(\text{char eq}) = 0$ also

so $\boxed{\pm \det(D - \lambda P^H Q^H) = 0}$

2.4 Problem 5.4, 34

problem: prove that if A is symmetric then so is A^+

answer:

Assuming complex matrix then symmetric means $A = A^H$ hence

$$\begin{aligned} PDQ &= (PDQ)^H \\ &= Q^H D^H P^H \\ &= Q^H D P^H \end{aligned} \tag{1}$$

Hence $PDQ = Q^H D P^H \rightarrow DQ = P^{-1} Q^H D P^H \rightarrow D = P^{-1} Q^H D P^H Q^{-1}$

But since $P^H = P^{-1}$ and $Q^H = Q^{-1}$ then the above becomes

$$D = P^H Q^H D P^H Q^H \tag{2}$$

now

$$A^+ = Q^H D^+ P^H$$

Sub (2) into the above equation we obtain

$$\begin{aligned} A^+ &= Q^H \left(P^H Q^H D P^H Q^H\right)^+ P^H \\ &= Q^H \left(\left(P^H Q^H\right)^H D^+ \left(P^H Q^H\right)^H\right) P^H \\ &= Q^H \left((QP) D^+ (QP)\right) P^H \\ &= Q^H Q P D^+ Q P P^H \end{aligned}$$

But $Q^H Q = I$ and $P P^H = I$

Hence the above becomes

$$A^+ = P D^+ Q \tag{3}$$

But

$$\begin{aligned} \left(A^+\right)^H &= \left(Q^H D^+ P^H\right)^H \\ &= P D^+ Q \end{aligned} \tag{4}$$

Compare (3) and (4), they are the same.

Hence A^+ is symmetric.

2.5 Problem 5.4, 39

problem: prove that eigenvalues of positive semi definite matrix are nonnegative

answer:

positive semi definite means $\vec{x}^T A\vec{x} \geq 0$ for all $\vec{x} \neq \vec{0}$

Hence $\vec{x}^T A\vec{x} = \vec{x}^T \lambda\vec{x} = \lambda\vec{x}^T\vec{x}$

But $\vec{x}^T\vec{x} = \|\vec{x}\|^2$

Hence $\vec{x}^T A\vec{x} = \lambda \|\vec{x}\|^2$

We are told the above is ≥ 0. Assume $\vec{x} \neq 0$, then we have $\lambda\times$ the norm, which is positive quantity ≥ 0, hence this is possible only if λ was zero (for the =0 case) or $\lambda > 0$ for the > 0 case. It is not possible to have λ negative and multiply it by positive quantity and obtain a positive quantity.

Now Assume $\vec{x} = 0$, hence the norm is zero. Hence $A\vec{x} = \vec{0}$ and so eigenvalues is zero. Hence eigenvalues can be either positive or zero. Hence nonnegative

6.1 # 22

x	-2	0	1
y	0	1	-1

#13 ?

$n=3$

5|10

Newton interpolation

since $n=3$, we need order 2 polynomial

we need P_0, P_1, P_2

$$P_0 = C_0$$
$$P_1 = P_0 + C_1(X-X_0) = C_0 + C_1(X-X_0)$$
$$P_2 = P_0 + P_1 + C_2(X-X_0)(X-X_1) = C_0 + C_1(X-X_0) + C_2(X-X_0)(X-X_1)$$
$$P_3 = P_0 + P_1 + P_2 + C_3(X-X_0)(X-X_1)(X-X_2)$$

so
$$P_0 = C_0$$
$$P_1 = C_0 + C_1(X-X_0)$$
$$P_2 = C_0 + C_1(X-X_0) + C_2(X-X_0)(X-X_1)$$

Now find C's.

$C_0 = y_0 = 0 \Rightarrow \boxed{P_0 = 0}$

Now use Formula $C_k = \dfrac{y_k - P_{k-1}(x_k)}{(X_k - X_0)(X_k - X_1)\cdots(X_k - X_{k-1})}$

$k=1, x_1=0, y_1=1$

$$C_1 = \frac{y_1 - P_0(y_1)}{(X_1 - X_0)} = \frac{1-0}{0+2} = \frac{1}{2}$$

so $P_1 = P_0 + C_1(X-X_0) = 0 + \frac{1}{2}(X+2) = \boxed{\frac{1}{2}(X+2)}$

$K=2, X_2=1, y_2=-1$

$$C_2 = \frac{y_2 - P_1(X_2)}{(X_2-X_0)(X_2-X_1)} = \frac{-1-\left(\frac{1}{2}(1+2)\right)}{(1+2)(1-0)} = \frac{-1-1.5}{3} = \frac{-2.5}{3}$$

so $P_2 = 0 + \frac{1}{2}(X+2) - \frac{2.5}{3}(X+2)(X)$ $\longrightarrow$

$$P_2 = \frac{1}{2}(x+2) - \frac{2.5}{3}(x^2+2x)$$

$$= \frac{1}{2}x + 1 - \frac{2.5}{3}x^2 - \frac{5}{3}x = -\frac{7}{6}x + 1 - \frac{2.5}{3}x^2$$

so $\boxed{P(x) = -\frac{2.5}{3}x^2 - \frac{7}{6}x + 1}$ or $\boxed{P(x) = 1 - 1.16667x - 0.8333x^2}$

Now solve using Lagrange interpolation

$P_0 = y_0 l_0(x)$

$P_1 = P_0 + y_1 l_1(x)$

$P_2 = P_0 + P_1 + y_2 l_2(x)$

$\boxed{n=2}$ degree of Poly

where $l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j} \qquad 0 \le i \le n$

$x_0=-2, x_1=0, x_2=1$

so for $i=0$

$$l_0 = \prod_{\substack{j=0 \\ j\neq i}}^{2} \frac{x-x_j}{x_i-x_j} = \frac{x-x_1}{x_0-x_1}\,\frac{x-x_2}{x_0-x_2} = \frac{x}{-2}\,\frac{(x-1)}{(-2-1)} = \frac{x(x-1)}{6}$$

$i=1$

$$l_1 = \prod_{\substack{j=0 \\ j\neq i}}^{2} \frac{x-x_j}{x_i-x_j} = \frac{x-x_0}{x_1-x_0}\,\frac{x-x_2}{x_1-x_2} = \frac{(x+2)}{2}\,\frac{(x-1)}{-1} = \frac{(x+2)(x-1)}{-2}$$

$i=2$

$$l_2 = \prod_{\substack{j=0 \\ j\neq i}}^{2} \frac{x-x_j}{x_i-x_j} = \frac{x-x_0}{x_2-x_0}\,\frac{x-x_1}{x_2-x_1} = \frac{(x+2)}{(1+2)}\,\frac{(x)}{1} = \frac{(x+2)x}{3}$$

$y_0=0, y_1=1, y_2=-1$

so $P_2(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x)$

$$= 0 + 1\,\frac{(x+2)(x-1)}{-2} - 1\,\frac{(x+2)x}{3} = -\frac{1}{2}(x^2-x+2x-2) - \frac{1}{3}(x^2+2x)$$

$$= -\frac{1}{2}x^2 - \frac{x}{2} + 1 - \frac{x^2}{3} - \frac{2}{3}x = 1 - \frac{7}{6}x - \frac{3x^2+2x^2}{6}$$

$$= \boxed{1 - 1.16667x - 0.8333x^2}$$

same as Newton

#

6.1 #26

$x - 9^{-x} = 0 \Rightarrow$

x	0	0.5	1
y	$f(x)$	$f(x)$	$f(x)$

when $f(x) = x - 9^{-x}$

$\Rightarrow$

x	0	0.5	1
y	-1	0.166667	0.888889

Use Newton

$P = C_0 + C_1(X - X_0) + C_2(X - X_0)(X - X_1)$

$C_0 = y_0 = -1$

$C_1 = \dfrac{y_1 - P_0(X_1)}{X_1 - X_0} = \dfrac{0.166667 + 1}{0.5} = 2.3333$

$\Rightarrow P_1 = C_1(X - X_0)$

$C_2 = \dfrac{y_2 - P_1(X_2)}{(X_2 - X_0)(X_2 - X_1)} = \dfrac{0.88889 - 2.3333(1)}{(1-0)(1-0.5)} = 2.8888$

$\Rightarrow P_2(x) = 2.8888(X - 0)(X - 0.5) = 2.8888(X)(X - 0.5)$

So $P(x) = -1 + 2.3333(X) + 2.88888(X)(X - 0.5)$

$$\boxed{P(X) = -1 + 2.89x + 2.888X^2}$$

Note (using exact rational values is more accurate numerically, but by hand faster to use Calculator and use decimal points)

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2.89 \pm \sqrt{2.89^2 + 4(2.888)}}{2 \times 2.888}$$

$$\Rightarrow X = 0.413$$

$$\text{or } X = 2.71$$

so $\boxed{x \approx 0.413}$ since in the domain

6.1 #27

if we interpolate $f(x) = e^{x-1}$ with polynomial p of degree 12 using 13 nodes in $[-1,1]$ what is a good upper bound for $|f(x)-p(x)|$ on $[-1,1]$?

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using theorem 2, page 315 on interpolation error:

$$f(x)-p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x-x_i)$$

$$\prod_{i=0}^{12} |x-x_i| \leq 2^{12}$$

$f(x) = e^{x-1}$
$f^{(1)}(x) = e^{x-1}$
$f^{(2)}(x) = e^{x-1}$. so $f^{(13)}(x) = e^{x-1}$

so $|f^{(13)}(\xi_x)| \leq e^{1}$

so $|f(x)-p(x)| \leq \dfrac{(e^{1})(2^{12})}{(13)!} \leq \boxed{1.78803 \times 10^{-6}}$

6.1

37

Cost	2	3	2	3	4	5	6	8	10
year	1885	1917	1919	1932	1958	1963	1968	1971	1974

15	18	20	22	25	29	32	33	34.
1978	1981	1981	1985	1988	1991	1995	1999	2001

find Newton polynomial. when will it cost $1 to mail a letter? when will it cost $10?

solution back.

Problem 37, section 6.1, Mathematics 501. CSUF,spring 2007. by NasserAbbasi

In[162]:=

```
points = {{2, 1885}, {3, 1917}, {2, 1919}, {3, 1932}, {4, 1958},
   {5, 1963}, {6, 1968}, {8, 1971}, {10, 1974}, {15, 1978}, {18, 1981},
   {20, 1981}, {22, 1985}, {29, 1991}, {32, 1995}, {33, 1999}, {34, 2001}};

points2 = {{1885, 2}, {1917, 3}, {1919, 2}, {1932, 3}, {1958, 4},
   {1963, 5}, {1968, 6}, {1971, 8}, {1974, 10}, {1978, 15}, {1981, 18},
   {1981, 20}, {1985, 22}, {1991, 29}, {1995, 32}, {1999, 33}, {2001, 34}};

MatrixForm[points2]
```

Out[164]//MatrixForm=

$$\begin{pmatrix} 1885 & 2 \\ 1917 & 3 \\ 1919 & 2 \\ 1932 & 3 \\ 1958 & 4 \\ 1963 & 5 \\ 1968 & 6 \\ 1971 & 8 \\ 1974 & 10 \\ 1978 & 15 \\ 1981 & 18 \\ 1981 & 20 \\ 1985 & 22 \\ 1991 & 29 \\ 1995 & 32 \\ 1999 & 33 \\ 2001 & 34 \end{pmatrix}$$

In[165]:=

```
nPoints = Length[points]
```

Out[165]=

17

In[166]:=

```
basis = Table[x^i, {i, 0, nPoints - 1}]
```

Out[166]=

$\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}, x^{12}, x^{13}, x^{14}, x^{15}, x^{16}\}$

In[167]:=

```
poly = Fit[points2, basis, x]
```

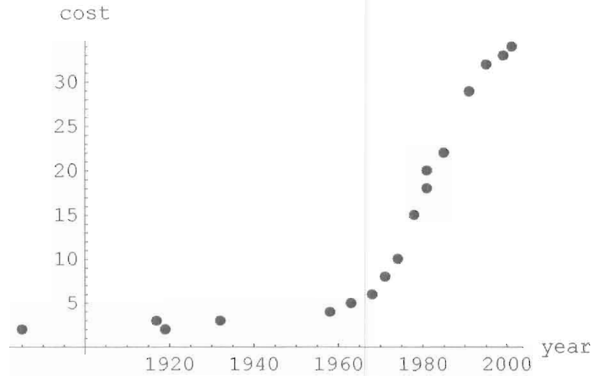
Polynomial

Out[167]=

$$-1.00778\times10^{11} + 1.31165\times10^{8}\,x - 10246.6\,x^2 - 26.1148\,x^3 - 0.00737334\,x^4 + 2.38338\times10^{-6}\,x^5 + 3.00171\times10^{-9}\,x^6 + 1.15284\times10^{-12}\,x^7 - 5.49\times10^{-17}\,x^8 - 3.30197\times10^{-19}\,x^9 - 1.91043\times10^{-22}\,x^{10} - 2.76573\times10^{-26}\,x^{11} + 3.74687\times10^{-29}\,x^{12} + 2.87737\times10^{-32}\,x^{13} + 1.52668\times10^{-36}\,x^{14} - 9.8697\times10^{-39}\,x^{15} + 2.05966\times10^{-42}\,x^{16}$$

In[168]:=

```
p1 = ListPlot[points2, PlotStyle → PointSize[0.02], AxesLabel → {"year", "cost"}]
```

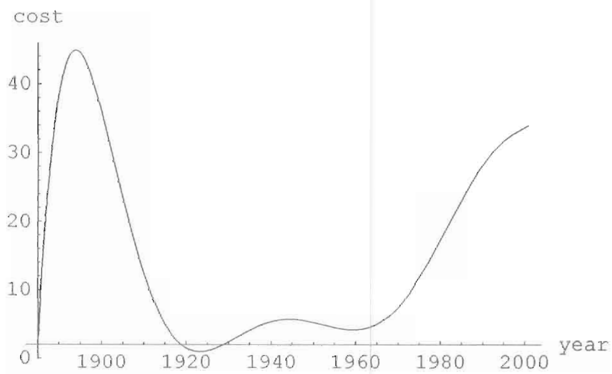


Out[168]=

- Graphics -

In[171]:=

```
p2 = Plot[poly, {x, 1885, 2001}, PlotRange → All,
  AxesLabel → {"year", "cost"}, AxesOrigin → {1885, 2}]
```

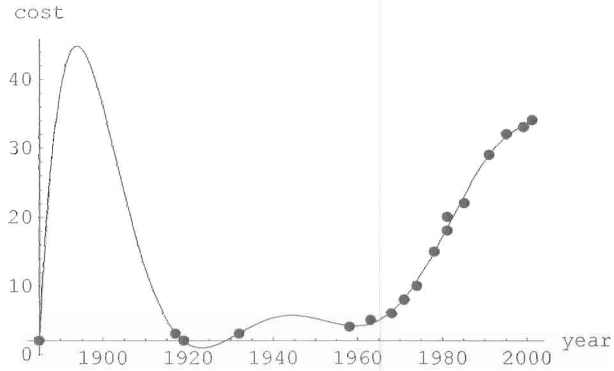


Out[171]=

- Graphics -

In[174]:=

```
Show[{p1, p2}, AxesLabel → {"year", "cost"}, AxesOrigin → {1885, 2}, PlotRange → All]
```



Out[174]=

- Graphics -

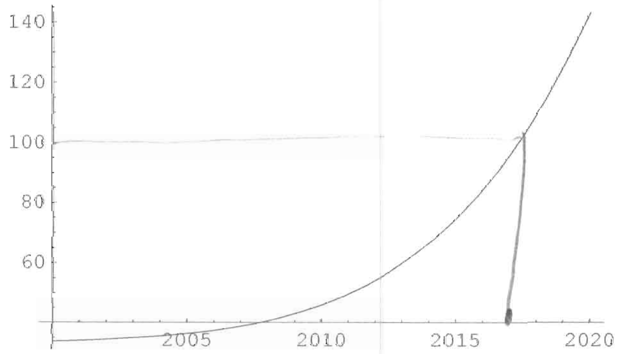
Out[173]=

-8.77912×10^{10}

In[178]:=

```
Plot[poly, {x, 2000, 2020}]
```

cost



year

cost will be $1

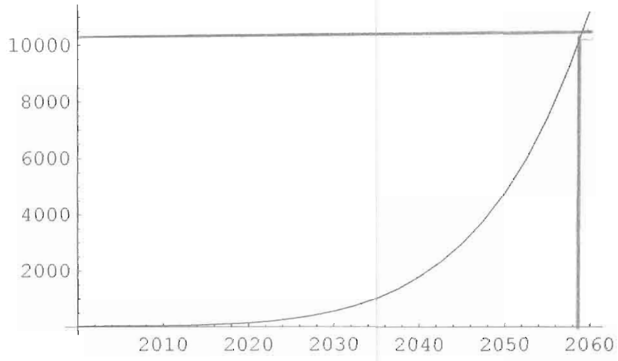
Out[178]=

- Graphics -

year $\boxed{2017}$

In[180]:=

```
Plot[poly, {x, 2000, 2060}]
```



Out[180]=

- Graphics -

year $\boxed{2058}$ cost $B

Dr Lee
I ran out of time to
complet 6.2
sorry.
Was working on the write up
for lecture 2 ?