HW3. Math 499. Spring 2007, CSUF, Spring 2010)

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1 Problem 1

Question: By setting the derivative to zero, find the value of b_1 that minimizes

$$\|b_1 \sin x - \cos x\|^2 = \int_0^{2\pi} (b_1 \sin x - \cos x)^2 \, dx$$

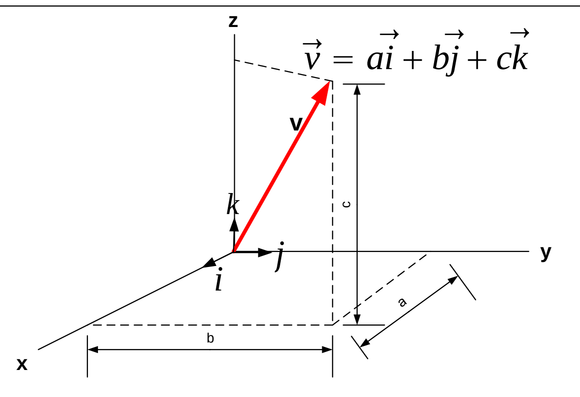
compare with the Fourier coefficient b_1

Answer:

First I thought it might be a good idea to refresh myself with Fourier series and how it comes about from geometrical perspective. Understanding how a function can be represented using Fourier series can be made easier by making an analogy of how a vector is represented using vector basis.

We know from basic Euclidean geometry, that a vector in the standard 3 dimensional space is written as the sum of its projections on the 3 basis vectors. When we write $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$, then in this case, a is the projection of the vector $\vec{v}$ onto the direction of the base vector $\vec{i}$, and similarly for the numbers b and c. The numbers a, b, c are called the *coordinates* of the vector $\vec{v}$ in this particular coordinate system.

The same vector $\vec{v}$ can then have different coordinate values depending on which coordinates system we are making our measurements in, but it the same exact vector. Hence a vector is invariant under coordinate transformation, but its representation (the coordinates) will change. This diagram below illustrates the above



Representing a vector as the sum of its projections on the coordinates system

Now that we know how a vector is represented by adding its projections along the direction of each base vector, we are ready to make the switch to a new and exciting world, where vectors become functions and the number of basis instead of being fixed at 3 become very large, in fact, it become infinitely large. This new vector space is called the Hilbert space.

Our goal is to express, or represent a function such as $f(x)$ using as basis the functions sin and cos. This leads to Fourier series representation of a function. One of the issues to consider right away, is what basis to use. There are many families of basis to select. Here we select the *sin* and *cos* functions as the basis.

As long as each base is orthogonal to each other (using a new definition of what it means to have two functions orthogonal to each others).

Hence by selecting $\sin(x), \sin(2x), \sin(3x), \cdots$, and $\cos(x), \cos(2x), \cdots$. I.e. $\sin(nx), \cos(nx)$ for n over all the integers from $0 \cdots \infty$. These basis work since any two different basis have zero as their dot product using the following definition of dot product, therefore they are orthogonal to each others.

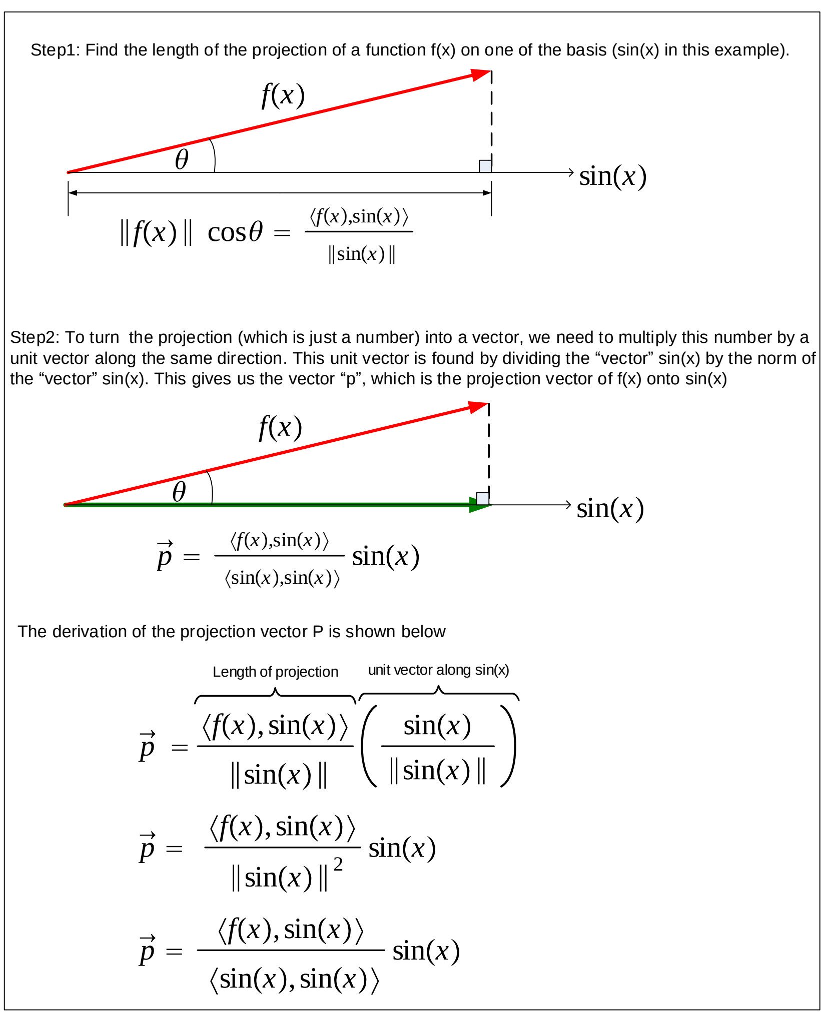
In Hilbert space, two functions are orthogonal to each others if their dot product is zero, defined as follows between the function $f(x), g(x)$

$$\langle f(x), g(x) \rangle = \int_0^{2\pi} f(x)\, g(x)\, dx$$

So, now when we are given a function $f(x)$ and asked for its representation with respect to the coordinate system called the fourier coordinates system, we follow the same idea as with normal vectors, and write

$$\begin{aligned} f(x) = {} & (\text{projection of } f(x) \text{ onto first basis}) \times \text{first basis} \\ & + (\text{projection of } f(x) \text{ onto second basis}) \times \text{second basis} \\ & + \cdots \end{aligned}$$

The above is the same as we did with Euclidean space. We now need to know how to find a projection of a function such as $f(x)$ onto a base function such as $\sin(x)$. This diagram shows how to do find one such projection of $f(x)$ onto one base function $\sin(x)$



The above tells us that the *coordinate* of $f(x)$ along $\sin(x)$ is given by

$$\frac{\langle f(x), \sin(x) \rangle}{\langle \sin(x), \sin(x) \rangle}$$

Let us express $f(x)$ using the first few coordinates. The first base is $\cos(0x) = 1$, the second base is $\cos(x)$, the third base is $\cos(2x)$, etc... and now for the sin basis, again we use $\sin(x), \sin(2x), \cdots$. Hence we have

$$f(x) = \frac{\langle f(x), \cos(0x)\rangle}{\langle \cos(0x), \cos(0x)\rangle}\cos(0x) + \frac{\langle f(x), \cos(x)\rangle}{\langle \cos(x), \cos(x)\rangle}\cos(x) + \frac{\langle f(x), \cos(2x)\rangle}{\langle \cos(2x), \cos(2x)\rangle}\cos(2x) + \cdots + \frac{\langle f(x), \mathrm{si}}{\langle \sin(x), }$$

The standard is to write the above in the order of increasing the frequency of each base, hence we write

$$\begin{aligned} f(x) &= \frac{\langle f(x), \cos(0x)\rangle}{\langle \cos(0x), \cos(0x)\rangle}\cos(0x) + \frac{\langle f(x), \cos(x)\rangle}{\langle \cos(x), \cos(x)\rangle}\cos(x) + \frac{\langle f(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle}\sin(x) + \frac{\langle f(x), \cos(2}{\langle \cos(2x), \cos(} \\ &= \frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle} + \frac{\langle f(x), \cos(x)\rangle}{\langle \cos(x), \cos(x)\rangle}\cos(x) + \frac{\langle f(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle}\sin(x) + \frac{\langle f(x), \cos(2x)\rangle}{\langle \cos(2x), \cos(2x)\rangle}\cos(2x) + \frac{\langle}{\langle \mathrm{s}} \end{aligned}$$

The coordinates are given standard names: the first one is called a_0, the second is called a_1 the third is called b_1, etc.. i.e. the coordinates of $f(x)$ on the cos basis are called $a_0, a_1, \cdots$ and the coordinates of $f(x)$ on the sin basis are called $b_1, b_2, \cdots$. Notice that b_0 does not exist, since $\sin(0x) = 0$.

So, we write the above as

$$\begin{aligned} f(x) &= \overbrace{\frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle}}^{a_0} + \overbrace{\frac{\langle f(x), \cos(x)\rangle}{\langle \cos(x), \cos(x)\rangle}}^{a_1}\cos(x) + \overbrace{\frac{\langle f(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle}}^{b_1}\sin(x) + \overbrace{\frac{\langle f(x), \cos(2x)\rangle}{\langle \cos(2x), \cos(2x)\rangle}}^{a_2}\cos(2x) + \frac{\langle}{\langle \mathrm{s}} \\ &= a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \cdots \end{aligned}$$

Using the above definition of an inner product, we know how to calculate each of the coordinates a_n, b_n :

$$\begin{aligned} a_0 &= \frac{\langle f(x), 1\rangle}{\langle 1, 1\rangle} = \frac{\int_0^{2\pi} f(x) \times 1 dx}{\int_0^{2\pi} 1 \times 1 dx} = \frac{\int_0^{2\pi} f(x)\, dx}{2\pi} \\ a_1 &= \frac{\langle f(x), \cos(x)\rangle}{\langle \cos(x), \cos(x)\rangle} = \frac{\int_0^{2\pi} f(x) \times \cos(x)\, dx}{\int_0^{2\pi} \cos(x) \times \cos(x)\, dx} = \frac{\int_0^{2\pi} f(x) \times \cos(x)\, dx}{\pi} \\ a_2 &= \frac{\langle f(x), \cos(2x)\rangle}{\langle \cos(2x), \cos(2x)\rangle} = \frac{\int_0^{2\pi} f(x) \times \cos(2x)\, dx}{\int_0^{2\pi} \cos(2x) \times \cos(2x)\, dx} = \frac{1}{\pi}\int_0^{2\pi} f(x) \times \cos(2x)\, dx \end{aligned}$$

Hence we see that

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx) dx \qquad n > 1 \end{aligned}$$

Similarly for the b_n coordinates, we obtain

$$\begin{aligned} b_1 &= \frac{\langle f(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle} = \frac{\int_0^{2\pi} f(x) \times \sin(x)\, dx}{\int_0^{2\pi} \sin(x) \times \sin(x)\, dx} = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(x)\, dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\, dx \end{aligned}$$

We know how to measure the norm of a vector in our standard Euclidean space, so we need to decide how to measure the norm of a function in Hilbert space. For this we use the following definition

$$\|f(x)\| = \sqrt{\int_0^{2\pi} \{f(x)\}^2\, dx}$$

I used the above range of integration because for fourier series, the basis used are the $\sin(x), \cos(x)$.

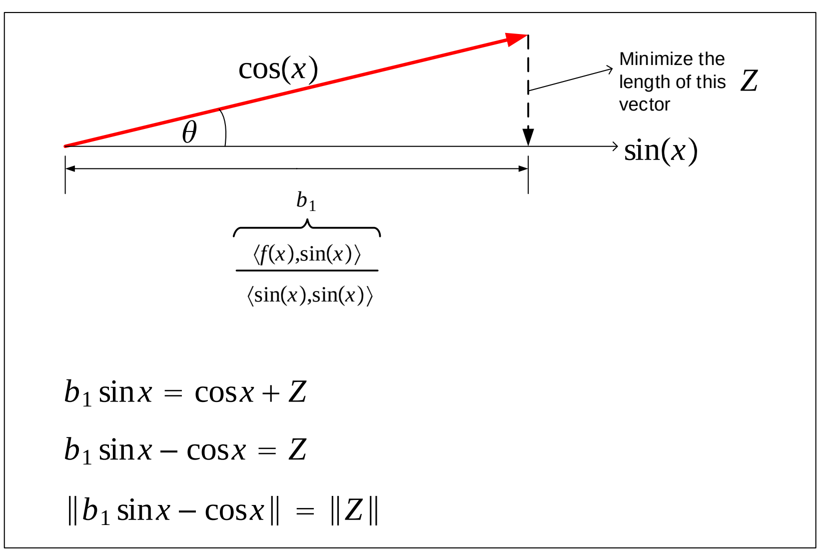
Now that we have reviewed the fourier series expansion, let us try to answer the actual question.

First, use calculus to answer the question itself:

$$
\begin{aligned}
\frac{\partial}{\partial b_1}\left(\|b_1 \sin x - \cos x\|^2\right) &= \frac{\partial}{\partial b_1}\left(\int_0^{2\pi} (b_1 \sin x - \cos x)^2\, dx\right) \\
&= \frac{\partial}{\partial b_1}\int_0^{2\pi} \left(b_1^2 \sin^2 x + \cos^2 x - 2b_1 \sin x \cos x\right) dx \\
&= \frac{\partial}{\partial b_1}\left(b_1^2 \int_0^{2\pi} \sin^2 x dx + \int_0^{2\pi} \cos^2 x dx - 2b_1 \int_0^{2\pi} \sin x \cos x dx\right) \\
&= \frac{\partial}{\partial b_1}\left(b_1^2 \left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{2\pi} + \left[\frac{x}{2} + \frac{1}{4}\sin 2x\right]_0^{2\pi} - 2b_1\left[\frac{-1}{2}\cos^2 x\right]_0^{2\pi}\right) \\
&= \frac{\partial}{\partial b_1}\left(b_1^2 [\pi] + [\pi] + b_1 \left[\cos^2 2\pi - \cos^2 0\right]\right) \\
&= \frac{\partial}{\partial b_1}\left(\pi b_1^2 + \pi\right) \\
&= 2b_1
\end{aligned}
$$

Hence for minimum, $b_1 = 0$.

Now the question is asking to compare this to the fourier coefficient b_1, i.e. with the coordinate b_1 of the function being expanded. The question did not tell us what is $f(x)$ itself. But from geometry we deduce that the problem is to minimize the distance between the function $f(x)$ and the basis, which is $\sin(x)$ in this case. Hence $b_1 \sin x - \cos x$ is the vector between the function being expressed and the basis $\sin(x)$. Hence $f(x) = \cos(x)$ in this example, as shown in this diagram



Hence, we now need to find b_1 given that $f(x)$ is $\cos(x)$ in this example:

$$
\begin{aligned}
b_1 &= \frac{\langle f(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle} \\
&= \frac{\langle \cos(x), \sin(x)\rangle}{\langle \sin(x), \sin(x)\rangle} \\
&= \frac{\int_0^{2\pi} \cos(x) \times \sin(x)\, dx}{\int_0^{2\pi} \sin(x) \times \sin(x)\, dx} \\
&= \frac{0}{\pi} \\
&= 0
\end{aligned}
$$

Hence confirmed to be the same.

2 Problem 2

Show that the complex exponential $\phi(x) = a_0 e^{inx}$ [1] are eigen functions of the convolution operator

$$g(x) = (k * f)(x) = \int_{-\infty}^{\infty} k(x-\tau) f(\tau)\, d\tau$$

For $k \in L^2(-\infty, \infty)$ and how representing $f(x)$ as a linear combination of complex exponential greatly simplifies this equation

Answer: We need to show that by applying the convolution operator on $\phi(x)$, we obtain a scaled version of $\phi(x)$, i.e. need to show that

$$g(x)\rfloor_{f(x)=\phi(x)} = \lambda \phi(x)$$

Where λ is a scalar. From the above definition, we obtain

$$\begin{aligned} g(x) &= (k * \phi)(x) \\ &= \int_{-\infty}^{\infty} k(x-\tau)\, \phi(\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} k(x-\tau)\, a_0 e^{in\tau} d\tau \end{aligned}$$

Using the commutative property of convolution, where $(k * \phi)(x) = (\phi * k)(x)$, we can write the above as

$$\begin{aligned} g(x) &= (\phi * k)(x) \\ &= \int_{-\infty}^{\infty} k(\tau)\, a_0 e^{in(x-\tau)} d\tau \\ &= a_0 \int_{-\infty}^{\infty} k(\tau)\, e^{inx} e^{-in\tau} d\tau \\ &= a_0 e^{inx} \int_{-\infty}^{\infty} k(\tau)\, e^{-in\tau} d\tau \end{aligned}$$

But $\int_{-\infty}^{\infty} k(\tau)\, e^{-in\tau} d\tau$ is the fourier transform of the function $k(x)$, Call this fourier transform $F(k(\tau)) = X(n)$. Hence

$$\begin{aligned} g(x) &= (\phi * k)(x) \\ &= a_0 X(n)\ e^{i\omega x} \\ &= \lambda e^{i\omega x} \end{aligned}$$

Where I called $a_0 X(n)$ as the parameter λ since $a_0 X(n)$ does not depend on x but depends on n, I.e. given the function $k(\tau)$, we can determine its Fourier transform for the specific n provided, and this Fourier transform integral, which will evaluate to some value, is multiplied with a_0 to obtain the scaling factor by which we scale e^{inx} which is $\phi(x)$ with. Hence we showed that $\phi(x)$ is an eigenfunction of $g(x)$.

Now for the second part. If $f(x)$ can be written as linear combination of complex exponential

[1] note: I renamed e^{ikx} in the original question to e^{inx} so as not to confuse with the k function used in the definition of the convolution operator

functions as in $f(x) = \sum_{n=1}^{N} a_n e^{inx}$, then we write

$$
\begin{aligned}
g(x) &= (k * f)(x) \\
&= \int_{-\infty}^{\infty} k(x-\tau) f(\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} k(\tau) f(x-\tau)\, d\tau \\
&= \int_{-\infty}^{\infty} k(\tau) \left(\sum_{n=1}^{N} a_n e^{in(x-\tau)} \right) d\tau \\
&= \int_{-\infty}^{\infty} k(\tau) \left(\sum_{n=1}^{N} a_n e^{inx} e^{-in\tau} \right) d\tau \\
&= \int_{-\infty}^{\infty} \left(\sum_{n=1}^{N} k(\tau)\, a_n e^{inx} e^{-in\tau} \right) d\tau \\
&= \sum_{n=1}^{N} \int_{-\infty}^{\infty} k(\tau)\, a_n e^{inx} e^{-in\tau}\, d\tau \\
&= \sum_{n=1}^{N} a_n e^{inx} \int_{-\infty}^{\infty} k(\tau)\, e^{-in\tau}\, d\tau
\end{aligned}
$$

But $\int_{-\infty}^{\infty} k(\tau) e^{-in\tau} d\tau$ is the Fourier transform of $k(\tau)$, call it $X(n)$, hence the above becomes

$$g(x) = \sum_{n=1}^{N} a_n e^{inx} X(n)$$

Hence we have replaced the integration operation with a summation operation and we have simplified this equation.

3 Problem 3

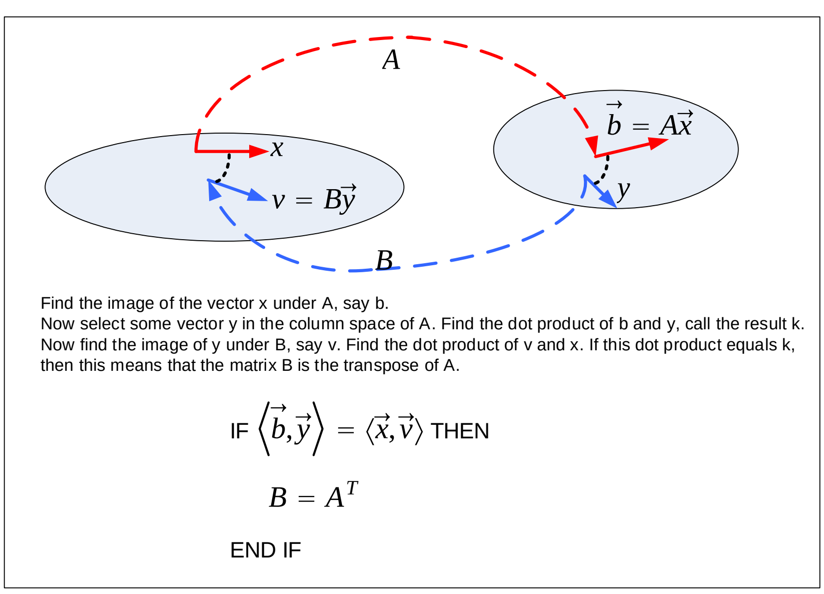
The transpose of a matrix can be defined as the matrix A^T such that $\langle Ax, y \rangle = \langle x, A^T y \rangle$

This definition generalizes to function operators like the fourier transform $g(\xi) = F\{f\} = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\xi x} dx$

Find the adjoint $F^T\{g\}$ using the definition above.

Answer:

First, a geometric view of a matrix transpose can be illustrated in this diagram



Now let us try to apply the above diagram to find the adjoint operator we need. Instead of using the Matrix notation of A and A^T, we now use the notation of L and L^* , where here L^* is the adjoint operator of L. Hence we seek to find an operator L^* such that $\langle Lf, q\rangle = \langle f, L^*q\rangle$

We are given what L is, it is the fourier transform, it takes the function $f(x)$ and generates $g(\xi)$ according to this operation

$$g(\xi) = F\{f\} = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\xi x} dx$$

For the inner product operation on the space of complex functions over the infinite domain, I will use the following definition

$$\langle f, g\rangle = \int_{-\infty}^{\infty} \overline{f}\, g\, dx$$

Hence, applying $\langle Lf, q\rangle = \langle f, L^*q\rangle$

$$\begin{aligned}
\langle Lf, q\rangle &= \langle f, L^*q\rangle \\
\langle g(\xi), q\rangle &= \langle f, L^*q\rangle \\
\left\langle \overbrace{\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\xi x} dx}^{g(\xi)},\ q \right\rangle &= \langle f(x), (L^*q)\rangle \\
\int_{-\infty}^{\infty} \left(\overline{\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\xi x} dx} \right) q\, d\xi &= \int_{-\infty}^{\infty} \overline{f(x)}\, (L^*q)\, dx \\
\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \overline{f(x)} e^{i2\pi\xi x} dx \right) q\, d\xi &= \int_{-\infty}^{\infty} \overline{f(x)}\, (L^*q)\, dx
\end{aligned}$$

Exchanging the order of integration gives

$$\int_{-\infty}^{\infty} \overline{f(x)} \left(\int_{-\infty}^{\infty} q\, e^{i2\pi\xi x} d\xi \right) dx = \int_{-\infty}^{\infty} \overline{f(x)}\, (L^*q)\, dx$$

Hence we see that

$$\int_{-\infty}^{\infty} q\, e^{i2\pi\xi x} d\xi = L^*q$$

So, the adjoint operator is the *inverse fourier transform.*