

# HW11, Math 307. CSUF. Spring 2007

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## 1 Section 5.6, problem 1

problem: If  $B$  is similar to  $A$  and  $C$  is similar to  $B$ , show that  $C$  is similar to  $A$ . What matrices are similar to  $I$ ?

answer:

Since  $B$  is similar to  $A$  and  $C$  is similar to  $B$ , then we have the following

$$S_1^{-1}CS_1 = B \quad (1)$$

$$S_2^{-1}BS_2 = A \quad (2)$$

From (1) and (2)

$$\begin{aligned} S_2^{-1}BS_2 &= A \\ S_2^{-1}(S_1^{-1}CS_1)S_2 &= A \\ (S_2^{-1}S_1^{-1})C(S_1S_2) &= A \\ (S_1S_2)^{-1}C(S_1S_2) &= A \end{aligned}$$

Let  $S_1S_2 = S_3$ , hence the above becomes

$$S_3^{-1}CS_3 = A$$

Hence  $C$  is similar to  $A$ . Now for the second part. We write

$$\begin{aligned} S^{-1}AS &= I \\ S^{-1}A &= S \end{aligned}$$

So  $A$  must be  $I$ , hence only  $I$  is similar to  $I$ .

## 2 Section 5.6 problem 2

problem: Describe in words all the matrices that are similar to  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , and find 2 of them

answer:

Let  $A$  be the above matrix. The above matrix represents a reflection across the x-axis. Hence Reflection across the y axis will be similar to it. Any multiple of this reflection matrix will also be similar to  $A$ .

Since reflection across the y-axis is  $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  then this  $B$  matrix is similar to  $A$

Then any multiple of  $B$  is also similar to  $A$ , such as  $\begin{pmatrix} -10 & 0 \\ 0 & 10 \end{pmatrix}$  and  $\begin{pmatrix} -20 & 0 \\ 0 & 20 \end{pmatrix}$

## 3 Section 5.6, problem 5

Problem: show (if  $B$  is invertible) then  $BA$  is similar to  $AB$

answer: we want to show that  $M^{-1}(BA)M = AB$

Let  $M^{-1}(BA)M = H$ , i.e. let  $BA \sim H$ , and try to show that  $H = AB$

$$\begin{aligned} M^{-1}(BA)M &= H \\ (BA)M &= MH \\ BA &= MHM^{-1} \\ A &= B^{-1}MHM^{-1} \\ AB &= B^{-1}MHM^{-1}B \\ AB &= (B^{-1}M)H(M^{-1}B) \\ AB &= (M^{-1}B)^{-1}H(M^{-1}B) \end{aligned}$$

Let  $M^{-1}B = Z$ , hence the above becomes

$$AB = Z^{-1}HZ$$

Then  $H \sim AB$

But we started by stating that  $H \sim BA$ , and since if  $r_1 \sim r_2$  and  $r_2 \sim r_3$  then  $r_1 \sim r_3$  then we showed  $BA \sim AB$ .

## 4 Section 5.6 problem 18

problem: find normal matrix ( $NN^H = N^H N$ ) that is not Hermitian, skew symmetric, unitary, or diagonal. Show that all permutation matrices are normal

answer:

## 5 Section 6.1, problem 1

problem: quadratic  $f = x^2 + 4xy + 2y^2$  has saddle point at origin, despite that its coefficients are positive. Write  $f$  as difference of 2 squares

answer: Let  $f = (ax + by)^2 - (cx + dy)^2$ , hence

$$\begin{aligned} f &= (ax + by)^2 - (cx + dy)^2 \\ &= a^2x^2 + b^2y^2 + 2abxy - (c^2x^2 + d^2y^2 + 2cdxy) \\ &= a^2x^2 + b^2y^2 + 2abxy - c^2x^2 - d^2y^2 - 2cdxy \\ &= x^2(a^2 - c^2) + y^2(b^2 - d^2) + xy(2ab - 2cd) \end{aligned}$$

Hence, compare coefficients, we have  $a^2 - c^2 = 1, b^2 - d^2 = 2, 2ab - 2cd = 4$

so  $ab - cd = 2$ .

Let  $c = 1$ , then we have

$$a^2 = 2, b^2 - d^2 = 2, 2ab - 2d = 4$$

3 equations in 3 unknown. Solve with computer for speed (running out of time!) I get one of the solutions as

$$d = 0, a = -\sqrt{2}, b = -\sqrt{2}$$

$$\text{So } f = (ax + by)^2 - (cx + dy)^2 = \boxed{(-\sqrt{2}x - \sqrt{2}y)^2 - (x)^2}$$

## 6 Section 6.1, problem 8

problem: decide for or against PD for these matrices, write out corresponding  $f = x^T A x$

Answer: I use  $a > 0$ , and  $ac > b^2$  test where  $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

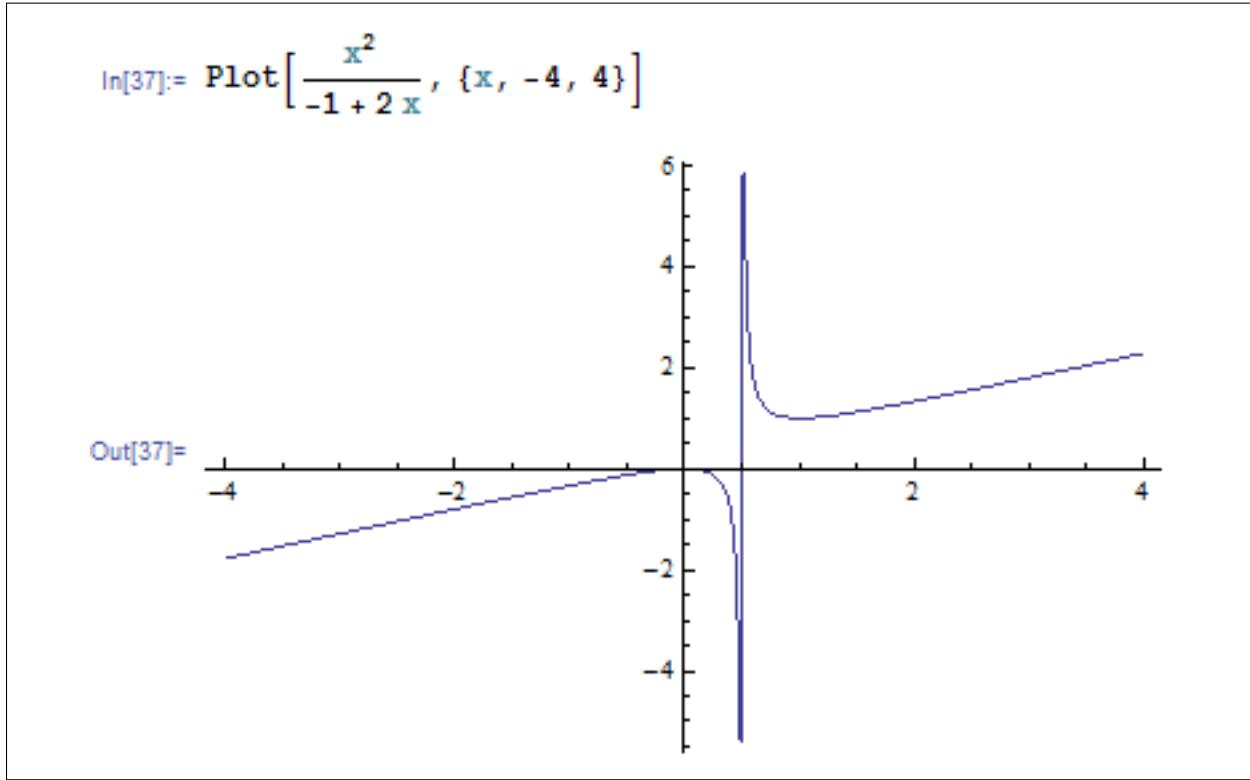
$$\begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} \rightarrow 1 > 0, 5 > 9 \text{ no, } \boxed{\text{Not PD}} \rightarrow f = ax^2 = 2bxy + cy^2 \rightarrow \boxed{f = x^2 + 6xy + 3y}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \rightarrow a > 0, 1 > 1, \text{no, [Not PD]} \rightarrow f = ax^2 = 2bxy + cy^2 \rightarrow [f = x^2 - 2xy + y]$$

$$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \rightarrow a > 0, 10 > 9, \text{yes, [PD]} \rightarrow f = ax^2 = 2bxy + cy^2 \rightarrow [f = 2x^2 + 6xy + 5y]$$

$$\begin{pmatrix} -1 & 2 \\ 2 & -8 \end{pmatrix} \rightarrow -1 > 0, \text{no [Not PD]} \rightarrow f = ax^2 + 2bxy + cy^2 \rightarrow [f = -x^2 + 4xy - 8y]$$

For (b) we have  $f = x^2 - 2xy + y$ , if  $y = \frac{x^2}{2x-1}$  then  $f = x^2 - 2x \frac{x^2}{2x-1} + \frac{x^2}{2x-1} = 0$ , hence I plot this:



And along the lines shown is  $f = 0$

## 7 Section 6.1, problem 3

problem: if  $A$  is  $2 \times 2$  symmetric matrix, passes test that  $a > 0$ ,  $ac > b^2$  solve equation  $\det(A - \lambda I) = 0$  and show that eigenvalues are  $> 0$

answer:

Matrix is PD, then

$$\begin{aligned} \det\left(\begin{pmatrix} a & b \\ b & c \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= 0 \\ \left| \begin{pmatrix} a - \lambda & b \\ b & c - \lambda \end{pmatrix} \right| &= 0 \\ (a - \lambda)(c - \lambda) - b^2 &= 0 \\ ac - a\lambda - c\lambda + \lambda^2 &= 0 \\ \lambda^2 + \lambda(-a - c) + ac &= 0 \end{aligned}$$

Hence  $\lambda_1 = a$ ,  $\lambda_2 = c$

But  $a > 0$ , so  $\lambda_1 > 0$ , and given  $ac >$  positive quantity  $b^2$ , then  $\lambda_2 = c \rightarrow \lambda_2 > 0$

## 8 Section 6.1 problem 5

(a) For which numbers  $b$  is  $\begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix}$  PD?

$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$  is PD is  $a > 0$  and  $ac > b^2$

for PD need  $ac > b^2$ , hence need  $9 > b^2$  ie.  $b < 3$  and  $b > -3$ , so  $-3 < b < 3$

(b) Factor  $A = LDL^T$  when  $b$  is in the range above

$$\begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix} \rightarrow l_{21} = b \rightarrow U = \begin{pmatrix} 1 & b \\ 0 & 9 - b^2 \end{pmatrix}$$

$$\text{So } L = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 9 - b^2 \end{pmatrix}, L^T = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

(c) What is the minimum of  $f(x, y) = \frac{1}{2}(x^2 + 2bxy + 9y^2) - y$  when in this range

$$\text{when } f(x, y) = \frac{1}{2}(x^2 + 2bxy + 9y^2) - y = \frac{1}{2}x^2 + bxy + \frac{9}{2}y^2 - y$$

$$\frac{\partial f}{\partial x} = x + by = 0, \frac{\partial f}{\partial y} = bx + 9y - 1 = 0$$

$$\text{Hence } \begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & b \\ 0 & 9-b^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{Hence } \boxed{y = \frac{1}{9-b^2}}, \text{ and } x + by = 0 \rightarrow \boxed{x = -\frac{b}{9-b^2}}$$

$$\text{So } f(x, y) = \frac{1}{2}(x^2 + 2bxy + 9y^2) - y$$

$$\text{Hence } f(x, y) \rightarrow \frac{1}{2} \left( \left( -\frac{b}{9-b^2} \right)^2 + 2b \left( -\frac{b}{9-b^2} \right) \left( \frac{1}{9-b^2} \right) + 9 \left( \frac{1}{9-b^2} \right)^2 \right) - \left( \frac{1}{9-b^2} \right) = \frac{1}{2(b^2-9)}$$

$$\text{So minimum is } \boxed{\frac{1}{2(b^2-9)}}$$

(d) When  $b = 3$ , we see that we get  $\frac{1}{0} = \infty$  so no minimum

## 9 Section 6.1 problem 17

Problem: If  $A$  has independent columns then  $A^T A$  is square and symmetric and invertible. Rewrite  $\vec{x}^T A^T A \vec{x}$  to show why it is positive except when  $\vec{x} = 0$ , then  $A^T A$  is PD

answer:  $\vec{x}^T (A^T A) \vec{x} = (A\vec{x})^T A\vec{x}$

Let  $A\vec{x} = \vec{b}$ , then the above is  $\vec{b}^T \vec{b} = \|\vec{b}\|^2$ , which is positive quantity except when  $\vec{b} = \vec{0}$ , which occurs when  $A\vec{x} = \vec{b} = \vec{0}$  which happens only when  $\vec{x} = \vec{0}$ , since  $A$  is invertible.

Hence  $A^T A$  is positive definite except when  $\vec{x} = 0$

## 10 Section 6.2, problem 7

**problem:** If  $A = Q\Lambda Q^T$  is P.D. then  $R = Q\sqrt{\Lambda}Q^T$  is its S.P.D. square root. Why does  $R$  have positive eigenvalues? Compute  $R$  and verify  $R^2 = A$  for  $A = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$ ,  $A = \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix}$

**answer:**

$$\text{For } A = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$$

Given  $R$  is P.D. (problem said so), Hence  $\vec{x}^T R \vec{x} > 0$  for all  $\vec{x} \neq 0$

Now (assuming in all that follows that  $x \neq 0$ )

$$\begin{aligned} R\vec{x} &= \lambda\vec{x} \\ \vec{x}^T R \vec{x} &= x^T \lambda \vec{x} \\ \vec{x}^T R \vec{x} &= \lambda \|\vec{x}\|^2 \end{aligned}$$

Since  $\vec{x}^T R \vec{x} > 0$  then  $\lambda \|\vec{x}\|^2 > 0$ , and since  $\|\vec{x}\|^2 > 0$  hence  $\boxed{\lambda > 0}$

To compute  $R$  we first need to find  $Q$ .

$$A = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} \rightarrow l_{21} = \frac{6}{10} \rightarrow \begin{pmatrix} 10 & 6 \\ 6 - \frac{6}{10} \times 10 & 10 - \frac{6}{10} \times 6 \end{pmatrix} \rightarrow \begin{pmatrix} 10 & 6 \\ 0 & \frac{32}{5} \end{pmatrix}$$

$$\text{Hence } L = \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix}, U = \begin{pmatrix} 10 & 6 \\ 0 & \frac{32}{5} \end{pmatrix}$$

Then

$$\begin{aligned} LDU &= \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix} \begin{pmatrix} 1 & \frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix}^T \end{aligned}$$

$$\text{Hence we see that } Q = L = \begin{pmatrix} 1 & 0 \\ \frac{6}{10} & 1 \end{pmatrix}, \Lambda = D = \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix}, Q^T = L^T$$

Since  $A$  is SPD, then  $A = R^T R$  and  $A = Q \Lambda Q^T$ , hence we can take  $R = \sqrt{\Lambda} Q^T$

$$\begin{aligned} R &= \sqrt{\Lambda} Q^T = \sqrt{\begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix}} \begin{pmatrix} 1 & \frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{\frac{32}{5}} \end{pmatrix} \begin{pmatrix} 1 & \frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{10} & \frac{3}{5}\sqrt{2}\sqrt{5} \\ 0 & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \end{aligned}$$

Verify that  $R^T R = A$

$$\begin{aligned} R^T R &= \begin{pmatrix} \sqrt{10} & 0 \\ \frac{3}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \begin{pmatrix} \sqrt{10} & \frac{3}{5}\sqrt{2}\sqrt{5} \\ 0 & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \\ &= \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} \end{aligned}$$

verified oK.

$$\text{Now do the same for } A = \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix}$$

$$A = \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix} \rightarrow l_{21} = \frac{-6}{10} \rightarrow U = \begin{pmatrix} 10 & -6 \\ -6 - \frac{-6}{10} \times 10 & 10 - \frac{-6}{10} \times -6 \end{pmatrix} \rightarrow \begin{pmatrix} 10 & -6 \\ 0 & \frac{32}{5} \end{pmatrix} :$$

$$\text{Hence } L = \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix}, U = \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix}$$

Then

$$\begin{aligned} LDU &= \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix} \begin{pmatrix} 1 & -\frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix}^T \end{aligned}$$

$$\text{Hence we see that } Q = L = \begin{pmatrix} 1 & 0 \\ -\frac{6}{10} & 1 \end{pmatrix}, \Lambda = D = \begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix}, Q^T = L^T$$

Then now we find  $R$

Since  $A$  is SPD, then  $A = R^T R$  and  $A = Q \Lambda Q^T$ , hence we can take  $R = \sqrt{\Lambda} Q^T$

$$\begin{aligned} R = \sqrt{\Lambda} Q^T &= \sqrt{\begin{pmatrix} 10 & 0 \\ 0 & \frac{32}{5} \end{pmatrix}} \begin{pmatrix} 1 & -\frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{\frac{32}{5}} \end{pmatrix} \begin{pmatrix} 1 & -\frac{6}{10} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{10} & -\frac{3}{5}\sqrt{2}\sqrt{5} \\ 0 & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \end{aligned}$$

Verify that  $R^T R = A$

$$\begin{aligned} R^T R &= \begin{pmatrix} \sqrt{10} & 0 \\ -\frac{3}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \begin{pmatrix} \sqrt{10} & -\frac{3}{5}\sqrt{2}\sqrt{5} \\ 0 & \frac{4}{5}\sqrt{2}\sqrt{5} \end{pmatrix} \\ &= \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix} \end{aligned}$$

verified oK.

## 11 Section 6.2, problem 4

Show from the eigenvalues that if  $A$  is P.D. so is  $A^2$  and so is  $A^{-1}$

answer:

Given  $A$  is PD. Hence Eigenvalues of  $A$  are positive.

Let eigenvalue of  $A$  be  $\lambda_A$

Let  $B = A^2$

Let eigenvalue of  $B$  be  $\lambda_B$

We need to show that  $\lambda_B > 0$

Now

$$\begin{aligned} Bx &= \lambda_B x \\ A^2x &= \lambda_B x \\ AAx &= \lambda_B x \\ A\lambda_A x &= \lambda_B x \\ \lambda_A A x &= \lambda_B x \\ \lambda_A \lambda_A x &= \lambda_B x \end{aligned}$$

From the last statement above we can now say

$$\lambda_A^2 = \lambda_B$$

Hence  $\lambda_B > 0$ , hence by theorem 6B which says that if all eigenvalues are positive then the matrix is PD, then in this case the matrix  $B$  which is  $A^2$  is PD. QED

Now for  $A^{-1}$

$$Ax = \lambda_A x$$

pre multiply both sides by  $A^{-1}$

$$\begin{aligned} \overbrace{A^{-1}Ax}^I &= A^{-1}\lambda_A x \\ x &= A^{-1}\lambda_A x \\ \frac{1}{\lambda_A}x &= A^{-1}x \end{aligned}$$

i.e.

$$A^{-1}x = \frac{1}{\lambda_A}x$$

Hence eigenvalue of  $A^{-1}$  is  $\frac{1}{\lambda_A}$ . And since  $\lambda_A > 0$ , then so is  $\frac{1}{\lambda_A}$ , and by theorem 6B again, since all eigenvalues are positive then  $A^{-1}$  is P.D.

## 12 Section 6.2, problem 6

From the pivots, eigenvalues, eigenvectors of  $A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ , write  $A$  as  $R^T R$  in 3 ways

1.  $(L\sqrt{D}) (\sqrt{D}L^T)$
2.  $(Q\sqrt{\Lambda}) (\sqrt{\Lambda}Q^T)$
3.  $(Q\sqrt{\Lambda}Q^T) (Q\sqrt{\Lambda}Q^T)$

Answer:

First find if  $A$  is PD or not. Since this is a 2 by 2 matrix, a simple test is to look at the quantity  $a^2 - bc$  and if it is positive, and if  $a$  is also positive, then the matrix is PD

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$a = 5 > 0$$

$$a^2 - bc = 25 - 16$$

$$= 9 > 0$$

hence  $A$  is P.D.

Then it can be written as  $R^T R$  where  $R$  is full rank square matrix.

1) Since  $A$  is symmetric P.D., then it has choleskly decomposition  $CC^T$  where  $C = L\sqrt{D}$ , and  $C^T = \sqrt{D}L^T$  (the pivots are positive in the  $D$  matrix diagonal, so we can take their square root)

Then we write  $A = R^T R = (L\sqrt{D}) (\sqrt{D}L^T)$  where  $R = (\sqrt{D}L^T)$

$$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \rightarrow l_{21} = \frac{4}{5} \rightarrow U = \begin{pmatrix} 5 & 4 \\ 0 & 5 - \frac{4}{5} \times 4 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 0 & \frac{9}{5} \end{pmatrix}$$

$$\text{Hence } L = \begin{pmatrix} 1 & 0 \\ \frac{4}{5} & 1 \end{pmatrix}, U = \begin{pmatrix} 5 & 4 \\ 0 & \frac{9}{5} \end{pmatrix} \rightarrow LDU = \begin{pmatrix} 1 & 0 \\ \frac{4}{5} & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & \frac{9}{5} \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{pmatrix}$$

$$\text{Hence } R = \sqrt{\begin{pmatrix} 5 & 0 \\ 0 & \frac{9}{5} \end{pmatrix}} \begin{pmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{\frac{9}{5}} \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{5} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{5} & \frac{4}{5}\sqrt{5} \\ 0 & \frac{3}{5}\sqrt{5} \end{pmatrix}$$

Hence

$$A = \overbrace{\begin{pmatrix} \sqrt{5} & 0 \\ \frac{4}{5}\sqrt{5} & \frac{3}{5}\sqrt{5} \end{pmatrix}}^{L\sqrt{D}} \overbrace{\begin{pmatrix} \sqrt{5} & \frac{4}{5}\sqrt{5} \\ 0 & \frac{3}{5}\sqrt{5} \end{pmatrix}}^{\sqrt{D}L^T}$$

2) From  $A = Q\Lambda Q^T$  where  $Q$  is the matrix which contains as its columns the normalized eigenvectors of  $A$  and  $\Lambda$  contains in its diagonal the eigenvalues of  $A$ . First start by finding eigenvalues and eigenvectors of  $A$

$$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \rightarrow \text{eigenvectors: } \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \leftrightarrow 1, \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \leftrightarrow 9$$

$$\text{Hence } Q = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \text{normalize columns} \rightarrow Q = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$$

So, verify first that the above is correct:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} = \frac{1}{2} \times \begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

$$\text{Correct. So we write } R = (\sqrt{\Lambda}Q^T) = \sqrt{\begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} = \begin{pmatrix} -1 & 1 \\ 3 & 3 \end{pmatrix} \frac{1}{\sqrt{2}}$$

Hence

$$\begin{aligned} A &= R^T R \\ &= \overbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 3 \\ 1 & 3 \end{pmatrix}}^{R^T = Q\sqrt{\Lambda}} \overbrace{\begin{pmatrix} -1 & 1 \\ 3 & 3 \end{pmatrix} \frac{1}{\sqrt{2}}}_{R = \sqrt{\Lambda}Q^T} \end{aligned}$$

$$3) \text{ now find } R = (Q\sqrt{\Lambda}Q^T)$$

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \sqrt{\begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Hence

$$\begin{aligned} A &= R^T R \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^T \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \overbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}^{R^T = Q\sqrt{\Lambda}Q^T} \overbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}^{R = Q\sqrt{\Lambda}Q^T} \end{aligned}$$

## 13 Section 6.2 problem 8

problem: if  $A$  is SPD and  $C$  is nonsingular, prove that  $B = C^T AC$  is also SPD

solution: Since  $A$  is SPD, then it has positive eigenvalues.

Since  $B$  is similar to  $A$  (given), then  $B$  has the same eigenvalues as  $A$ , Hence  $B$  also has all its eigenvalues positive.

Hence by theorem 6B,  $B$  is symmetric positive definite.