Well-Ordering Axiom Worksheet

In this worksheet, we will pick apart the statement of the axiom to see what parts are necessary, and which are not. We will do this by looking at examples. This is an example of a process that you should do with all definitions, axioms and theorems. We end with a slightly different version of the axiom that has a bit more information (punch?) than the one in the book.

Well-Ordering Axiom: *Every nonempty subset of the set of nonnegative integers has a least element*.

1. Why must the set be nonempty? An exercise in reasoning: An empty set has no elements, so it cannot have a least element.
2. Must the set of integers be nonnegative? The answer has several parts:
a. Let X = $\{2n \mid n \text{ is an integer}\}$. X has negative elements (such as -2, -8, -24), and has no least element.
b. Let Y = {Integers larger than -37/2}. Although this set has negative integers in it, it has a least element, namely -18.
We come up with a modified version of the Well-Ordering Axiom: *Every nonempty subset of the set of the integers that is bounded below has a least element*.
So if there is some number smaller than every element in the set, then the set has a least element.

So try the following exercises:
i. Make up an example of a set of integers to which the Axiom applies.
ii. Make up an example of a set of integers to which the Axiom does not apply.

In the hypothesis, we have checked out the conditions "*nonempty*" "*nonnegative*" and found that they either were critical or could be modified. How about "*integers*"?

3. Must the set consist of integers? Let's make up a set of non-integers that is bounded below: F = $\left\{\frac{1}{n} \mid n \text{ is an integer}\right\}$. This is in fact a set of nonnegative rational numbers, and it has no least element. We will see very soon that this axiom is the basis for the tremendous difference between integers and rational numbers in terms of their arithmetic properties.

4. One last note. If our set of numbers is nonempty, but finite, then of course it always has a least element. Thus, we could modify our axiom to rule out the trivial finite case to say:

Every infinite subset of the set of the integers that is bounded below has a least element.