HW 5
EGEE 518 Digital Signal Processing I
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Contents

1 Problem 11.1

1. Let $X(e^{j\omega})$ be the Fourier transform of a real finite-length sequence $x(n)$ that is zero outside the interval $0 \leq n \leq N-1$. The periodogram $I_N(\omega)$ is defined in Eq. (11.24) as the Fourier transform of the $2N-1$ point autocorrelation estimate

$$c_{xx}(m) = \frac{1}{N}\sum_{n=0}^{N-|m|-1} x(n)x(n+m) \qquad |m| \leq N-1.$$

Show that the periodogram is related to the Fourier transform of the finite length sequence as follows:

$$I_N(\omega) = \frac{1}{N}|X(e^{j\omega})|^2.$$

Figure 1: the Problem statement

$$I_N(\omega) = \sum_{m=-(N-1)}^{N-1} c_{xx}(m)\, e^{-j\omega m}$$

$$\begin{aligned}
\left|X\left(e^{j\omega}\right)\right|^2 &= X\left(e^{j\omega}\right) X^*\left(e^{j\omega}\right) \\
&= \left(\sum_{m=0}^{N-1} x(m)\, e^{-j\omega m}\right)\left(\sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}\right)^* \\
&= \left(\sum_{m=0}^{N-1} x(m)\, e^{-j\omega m}\right)\left(\sum_{n=0}^{N-1} x^*(n)\, e^{j\omega n}\right) \\
&= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} x(m)\, x^*(n)\, e^{-j\omega m} e^{j\omega n}
\end{aligned}$$

But

$$e^{-j\omega m} e^{j\omega n} = e^{-j\omega(m-n)}$$

and

$$x(m)\, x^*(n) = x(m)\, x^*(m+(n-m))$$

So

$$\left|X\left(e^{j\omega}\right)\right|^2 = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} x(m)\, x^*(m+(n-m))\, e^{-j\omega(m-n)}$$

Let $n-m=\tau$ then above can be rewritten as

$$\left|X\left(e^{j\omega}\right)\right|^2 = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} x(m)\, x^*(m+\tau)\, e^{j\omega\tau}$$

When $n=0$,$m=-\tau$ and when $n=N-1$,$m=N-\tau-1$, hence the above becomes

$$\begin{aligned}
\left|X\left(e^{j\omega}\right)\right|^2 &= \sum_{m=0}^{N-1}\sum_{m=-\tau}^{N-\tau-1} x(m)\, x^*(m+\tau)\, e^{j\omega\tau} \\
&= \sum_{m=0}^{N-1}\left(\sum_{m=-\tau}^{-1} x(m)\, x^*(m+\tau)\, e^{j\omega\tau} + \sum_{m=0}^{N-|\tau|-1} x(m)\, x^*(m+\tau)\, e^{j\omega\tau}\right) \\
&= \sum_{m=0}^{N-1}\left(\sum_{m=-1}^{-\tau} x(m)\, x^*(m+\tau)\, e^{j\omega\tau} + N\, c_{xx}(m)\, e^{j\omega\tau}\right)
\end{aligned}$$

I made another attempt at the end,

2 Problem 11-2

2. The smoothed spectrum estimate $S_{xx}(\omega)$ is defined as

$$S_{xx}(\omega) = \sum_{m=-(M-1)}^{M-1} c_{xx}(m)w(m)e^{-j\omega m},$$

where $w(m)$ is a window sequence of length $2M-1$. Show that

$$E[S_{xx}(\omega)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} E[I_N(\theta)]W(e^{i(\omega-\theta)})\, d\theta,$$

where $W(e^{j\omega})$ is the Fourier transform of $w(n)$.

Figure 2: the Problem statement

We see that $S_{xx}(\omega)$ is the Fourier transform of $c_{xx}(m)\,w(m)$. i.e.

$$S_{xx}(\omega) = \mathcal{F}\left[c_{xx}(m)\,w(m)\right]$$

Where $\mathcal{F}$ is the Fourier transform operator. Using modulation property

$$S_{xx}(\omega) = \frac{1}{2\pi}\left(\mathcal{F}\left[c_{xx}(m)\right] \otimes \mathcal{F}\left[w(m)\right]\right)$$

But $I_N(\omega) = \mathcal{F}\left[c_{xx}(m)\right]$ and let $W(\omega) = \mathcal{F}\left[w(m)\right]$, then the above becomes

$$\begin{aligned} S_{xx}(\omega) &= \frac{1}{2\pi}\left(I_N(\omega) \otimes W(\omega)\right) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} I_N(\theta)\, W(\omega-\theta)\, d\theta \end{aligned}$$

Hence, taking expectation of LHS, and since only $I_N(\theta)$ is random, then the above becomes (after moving expectation inside the integral in the RHS)

$$E\left[S_{xx}(\omega)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} E\left[I_N(\theta)\right] W(\omega-\theta)\, d\theta$$