Definitions

(1) auto correlation $R_{xx}(n, n+m)$: Measures the similarity of R.P. $X(t)$ at time n and $X(t)$ at later time $n+m$.

$$R_{xx}(n,n+m) = E\{X(n)\, X^*(n+m)\}$$

(2) stationary process.

This is a random process whose statistics do not change with shift in time origin.

(3) Wide Sense Stationary process:

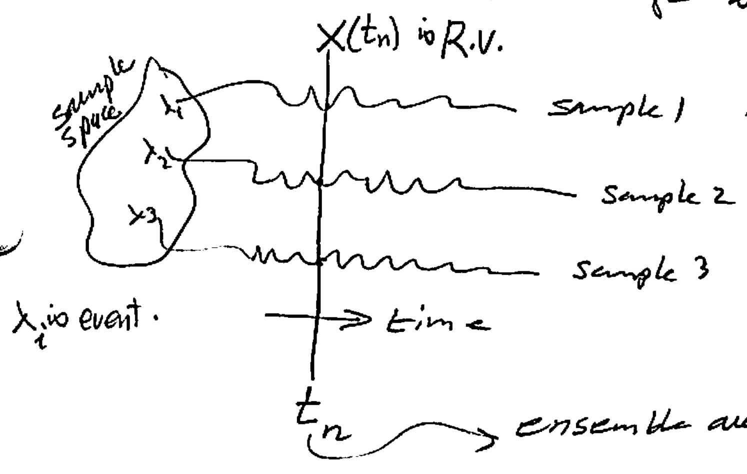
This is a random process $X(t)$ which satisfies the following conditions:

1. its mean is constant. i.e $E[X] =$ constant.
2. autocorrelation depends only on time interval m. i.e $R_{xx}(n, n+m) = R_{xx}(m)$.

notice that stationary process is WSS, but WSS is not neccessary stationary i.e

(4) Time averages, Ensemble averages

Time averages is the average of the sample sequence. while Ensemble average is statistical mean.



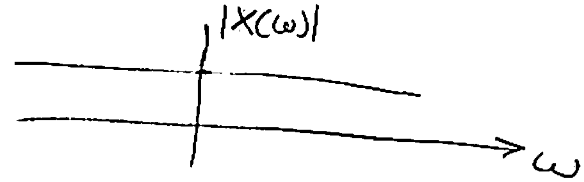
time average is <Sample1>

ensemble average is $E[X(t_n)]$

5 white Noise;

this is a R.P. whose power spectral density is constant. i.e. power contained in a frequency bandwidth B is the same regardless of where this bandwidth is centered.

"flat" spectrum implies $X(t)$ is white Noise process.



the above is a description in the frequency domain. in the time domain, $\Phi_{XX}(m) = \delta(m)$ i.e the autocorrelation is nonzero only if time interval is zero. i.e $X(t)$ only correlates with itself at zero time delay. so all R.V. that belong to a white noise process are uncorrelated with each other if time interval is nonzero.

6 Erogdic Process;

this is a R.P. whose statistics taken from the time samples are the same as statistics taken from Ensembles.

for example. we say a process is Ergodic in the mean, then $E\{X(t)\} = \langle X(t) \rangle$

statistical sample. expected value of R.V.

time average.. mean of a sample (or time series)

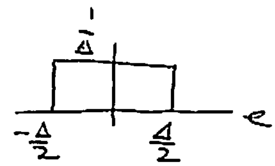
the above equality is in the <u>limit</u>, i.e as the time series length increases. and the statistical mean is when the Number of time series increases as well.

#3

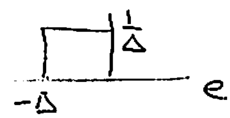
$$y(n) = Q[x(n)] = x(n) + \underbrace{e(n)}_{\text{quantization Error}}$$

$e(n)$ is white Noise.

Pdf for rounding is uniform



Pdf for truncation is



a) Find mean and Variance due to rounding
b) " " " " " truncation.

Answer

a) $$m_e = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e \cdot f(e)\, de = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{e}{\Delta} de = \frac{1}{\Delta}\left(\frac{e^2}{2}\right)_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}$$

$$= \frac{1}{2\Delta}\left[\left(\frac{\Delta}{2}\right)^2 - \left(-\frac{\Delta}{2}\right)^2\right] = \frac{1}{2\Delta}(0) = \boxed{0}$$

$$E[e^2] = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 f(e)\, de = \frac{1}{\Delta}\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 de = \frac{1}{\Delta}\left(\frac{e^3}{3}\right)_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}$$

$$= \frac{1}{3\Delta}\left[\left(\frac{\Delta}{2}\right)^3 - \left(-\frac{\Delta}{2}\right)^3\right] = \frac{1}{3\Delta}\left[\frac{\Delta^3}{8} + \frac{\Delta^3}{8}\right] = \frac{1}{3\Delta}\left[\frac{\Delta^3}{4}\right]$$

$$= \boxed{\frac{\Delta^2}{12}}$$

so $$\sigma^2 = E[e^2] - (E[e])^2 = \frac{\Delta^2}{12} - 0^2 = \boxed{\frac{\Delta^2}{12}}$$

b) $$m_e = \int_{-\Delta}^{0} e\, f(e)\, de = \int_{-\Delta}^{0} e \frac{1}{\Delta} de = \frac{1}{\Delta}\left(\frac{e^2}{2}\right)_{-\Delta}^{0} = \frac{1}{2\Delta}\left(0^2 - (-\Delta)^2\right)$$

$$= \frac{1}{2\Delta}(0 - \Delta^2) = \boxed{-\frac{\Delta}{2}}$$

$$E[e^2] = \int_{-\Delta}^{0} e^2 f(e)\, de = \int_{-\Delta}^{0} e^2 \frac{1}{\Delta} de = \frac{1}{\Delta}\left[\frac{e^3}{3}\right]_{-\Delta}^{0}$$

$$= \frac{1}{3\Delta}\left[0^3 - (-\Delta)^3\right] = \boxed{\frac{\Delta^2}{3}}$$

so $$\sigma^2 = E[e^2] - (E[e])^2 = \frac{\Delta^2}{3} - \left(-\frac{\Delta}{2}\right)^2 = \frac{\Delta^2}{3} - \frac{\Delta^2}{4} = \frac{4\Delta^2 - 3\Delta^2}{12} = \boxed{\frac{\Delta^2}{12}}$$

\# 4 let $e(n)$ white Noise Sequence. Let $s(n)$ uncorrelated sequence to $e(n)$. Show that $y(n) = s(n)\,e(n)$ is white. i.e $E[\,y(n)\;y(n+m)\,] = A\,\delta(m)$.

Answer

$$E[\,y(n)\;y(n+m)\,] = E[\,s(n)e(n)\;\;s(n+m)e(n+m)\,]$$

$$= E[\,s(n)\,s(n+m)\;\;e(n)\,e(n+m)\,]$$

Since $e(n)$ and $s(n)$ are uncorrelated, hence independent, then we can write the above as

$$= E[\,s(n)\,s(n+m)\,]\;\;E[\,e(n)\,e(n+m)\,]$$

but $e(n)$ is white. hence $\phi_{ee}(n,m) = E[\,e(n)\;e(n+m)\,] = \boxed{\delta(m)}$ by definition of white signal.

hence $\phi_{yy}(n,m) = E[\,s(n)\;s(n+m)\,]\;\delta(m)$.

Now, when $m=0$, $\phi_{yy}(n,m) = E[\,s(n)\;s(n)\,]\;.\;1.$

Since $s(n)$ is uncorrelated with white Noise, then $m_s = 0$
Since $s(n)$ is also white.

hence $E[\,s^2(n)\,] =$ Total average power in $s(n)$
$= A$ some constant.

hence when $m=0$, $\phi_{yy}(n,m) = A$

when $m \neq 0$ $\phi_{yy}(n,m) = E[\,s(n)\;s(n+m)\,]\,.\,0$
$= 0$

Therefor $\boxed{\phi_{yy}(n,m) = A\,\delta(m)}$

Since $\phi_{yy}(n,m)$ is function of only m, it is White signal.

#6 Consider 2 real stationary random processes $\{X_n\}$ and $\{y_n\}$, with mean m_x, m_y, and variance σ_x^2, σ_y^2. (5)

(a) $\gamma_{xx}(m)$ this is autocovariance.

$$\gamma_{xx}(m) = E\{(x(n)-m_x)(x^*(n+m) - m_x^*)\}$$
$$= E\{x(n)x^*(n+m) - m_x x(n) - m_x x^*(n+m) + m_x^2\}$$
$$= E\{x(n)\, x^*(n+m)\} - m_x E\{x(n)\} - m_x E\{x^*(n+m)\} + m_x^2.$$
$$= \phi_{xx}(n,n+m) - m_x^2 - m_x E\{x^*(n+m)\} + m_x^2$$
$$= \phi_{xx}(n,n+m) - m_x E\{x^*(n+m)\}.$$

but $\{x_n\}$ is stationary, so its statistics do not change with shift of time origin. hence $E\{x^*(n+m)\} = E\{x^*(n)\} = m_x$.
so above becomes

$$\gamma_{xx}(m) = \phi_{xx}(n,n+m) - m_x^2.$$

but $\phi_{xx}(n,n+m) = \phi_{xx}(m)$ since stationary hence

$$\boxed{\gamma_{xx}(m) = \phi_{xx}(m) - m_x^2}$$

$$\gamma_{xy}(m) = E[(x(n)-m_x)(y^*(n+m) - m_y^*)]$$
$$= E[x(n)\, y^*(n+m) - m_y^* x(n) - m_x y^*(n+m) + m_y^* m_x]$$
$$= E\{x(n)\, y^*(n+m)\} - m_y^* E\{x(n)\} - m_x E\{y^*(n+m)\} + m_y^* m_x$$

but due to stationarity, $E\{y^*(n+m)\} = m_y$. so above becomes

$$= E\{x(n)\, y^*(n+m)\} - m_y m_x - m_x m_y + m_y m_x$$
$$= E\{x(n)\, y^*(n+m)\} - m_y m_x.$$

but $E\{x(n)\, y^*(n+m)\} = \phi_{xy}(m)$ since stationary

so $\boxed{\gamma_{xy}(m) = \phi_{xy}(m) - m_y m_x}$ →

(b) $\phi_{xx}(0) = E\{x(n)\, x^*(n+m)\}$

but $m=0$. hence

$$\phi_{xx}(0) = E\{x(n)\, x^*(n)\} = E\{x^2(n)\}.$$

$=$ mean square.

$$\gamma_{xx}(0) = E\{(x(n)-m_x)(x^*(n+m)-m^*_x)\}$$

but $m=0$ so

$$\gamma_{xx}(0) = E\{(x(n)-m_x)(x^*(n)-m^*_x)\}$$

$$= E\{x^2(n) - x(n)m_x - m_x x^*(n) + m_x^2\}$$

$$= E\{x^2(n)\} - m_x E\{x(n)\} - m_x E\{x^*(n)\} + m_x^2$$

$$= E\{x^2(n)\} - m_x^2 - \cancel{m_x^2} + \cancel{m_x^2}$$

$$= E\{x^2(n)\} - m_x^2$$

but this is the definition of σ_x^2. hence

$$\boxed{\gamma_{xx}(0) = \sigma_x^2}$$

$\longrightarrow$

(c) $\phi_{xx}(m) = E\{x(n)\, x^*(n+m)\} = E\{x^*_{n+m}\, x_n\} = \left(E\{x_{n+m} x^*_n\}\right)^*$ 7

$= \Phi^*_{xx}(-m)$

if process is real, then $\Phi^*_{xx}(-m) = \Phi_{xx}(-m)$.

i.e $\Phi_{xx}(m) = \Phi_{xx}(-m)$

$\gamma_{xx}(m) = E\{(x(n)-m_x)\,(x^*(n+m)-m^*_x)\}$

$= \Phi_{xx}(m) - m_x m^*_x$ (from part (a)). 1

$= \Phi^*_{xx}(-m) - m_x m^*_x$. (using result above).

$= \left(E\{x_{n+m} X^*_n\}\right)^* - m_x m^*_x$

$= E\{x^*_{n+m}\, x_n\} - m_x m^*_x$

$= \left(E\{x_{n+m}\, x^*_n\} - m^*_x\, m_x\right)^*$

$= \gamma^*_{xx}(-m)$

if Real process, then $\gamma^*_{xx}(-m) = \gamma_{xx}(-m)$. $\Rightarrow$ $\boxed{\gamma_{xx}(m) = \gamma_{xx}(-m)}$

$\cancel{\Phi_{xy}(m) = E\{(x(n)-m_x)(y(n+m)-m_y)\}}$

$= \cancel{E\{x(n)\,y(n+m) - m_y x(n) - m_x y(n+m) + m_x m_y\}}$

$= \cancel{E\{x(n)\,y(n+m)\} - m_y m_x - m_x m_y + m_x m_y}$

$= \cancel{E\{x(n)\,y(n+m)\} - m_y m_x}$

But $\cancel{\Phi_{yx}(-m) = E\{(y(n)-m_y)\,(x(n-m)-m_x)\}}$

$= \cancel{E\{y(n)\,x(n-m)\} - m_x m_y - m_y m_x + m_y m_x}$

$= \cancel{E\{y(n)\,x(n-m)\} - m_x m_y}$

Since sequence $x(n)$ and $y(n)$ are real, ~~then~~ $\cancel{\Phi^*_{yx}(-m) = \Phi_{yx}(-m)}$.

so $\cancel{\Phi^*_{yx}(-m) = E\{x(n-m)\,y(n)\} - m_x m_y}$ $\longrightarrow$

part (c) cont.

show that $\Phi_{xy}(m) = \Phi^*_{yx}(-m)$.

$$\Phi_{xy}(m) = E\{X_n y^*_{n+m}\} = E\{y^*_{n+m} x_n\} = \left(E\{y_{n+m} x^*_n\}\right)^*$$

$$= \Phi^*_{yx}(-m)$$

---

show that $\gamma_{xy}(m) = \gamma^*_{yx}(-m)$

$$\gamma_{xy}(m) = E\{(X_n - m_x)(y^*_{n+m} - m^*_y)\}$$

$$= E\{(y^*_{n+m} - m^*_y)(X_n - m_x)\}$$

$$= \left(E\{(y_{n+m} - m_y)(X^*_n - m^*_x)\}\right)^*$$

$$= \gamma^*_{yx}(-m)$$

part (d)

show that $|\Phi_{xy}(m)| \leq \sqrt{\Phi_{xx}(0)\,\Phi_{yy}(0)}$

$$\Phi_{xy}(m) = E\{x_n\, y^*_{n+m}\}$$
$$\Phi_{xx}(0) = E\{x_n^2\}$$
$$\Phi_{yy}(0) = E\{y_n^2\}.$$

we did this in class as follows:

$$0 \leq E\{(x_n + a\,y_{n+m})^2\} = E\{x_n^2 + a^2 y^2_{n+m} + 2a\,x_n\,y_{n+m}\}$$
$$= E(x_n^2) + a^2 E(y^2_{n+m}) + 2a\,E(x_n\,y_{n+m})$$
$$= \Phi_{xx}(0) + a^2\,\Phi_{yy}(0) + 2a\,\Phi_{xy}(m) \quad = Ax^2+Bx+C$$

this is a quadratic equation that is ≥ 0 always. hence can't have 2 real roots. i.e discriminant ≤ 0. i.e

where $A = \Phi_{yy}(0)$, $B = 2\Phi_{xy}(m)$, $C = \Phi_{xx}(0)$.

but discriminant is $B^2 - 4AC$

so
$$4\Phi^2_{xy}(m) - 4\,\Phi_{yy}(0)\,\Phi_{xx}(0) \leq 0.$$

i.e
$$\Phi^2_{xy}(m) \leq \Phi_{yy}(0)\,\Phi_{xx}(0)$$

i.e
$$\boxed{|\Phi_{xy}(m)| \leq \sqrt{\Phi_{yy}(0)\,\Phi_{xx}(0)}}$$