H.W. #3 Sol.

① a) Autocorrelation sequence : $\phi_{xx}(n,m)$ is defined by

$$\phi_{xx}(n,m) = E\{x_n x_m^*\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_n x_m^* \, P_{x_n x_m}(x_n, n, x_m, m)\, dx_n\, dx_m$$

b) A random process $\{x_n\}$ is a stationary process if its statistics are not affected by a shift in the time origin. i.e., x_n and x_m have the same statistics for all n and m

c) A stationary random process in the wide sense mean

(i) The mean is constant

(ii) the autocorrelation (2^{nd} order statistics) depend only on the time difference between the random variables

d) Time average of a random process $\{x_n\}$ is defined as

$$\langle x_n \rangle = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x_n$$

Ensemble average of a random process $\{x_n\}$ is defined as

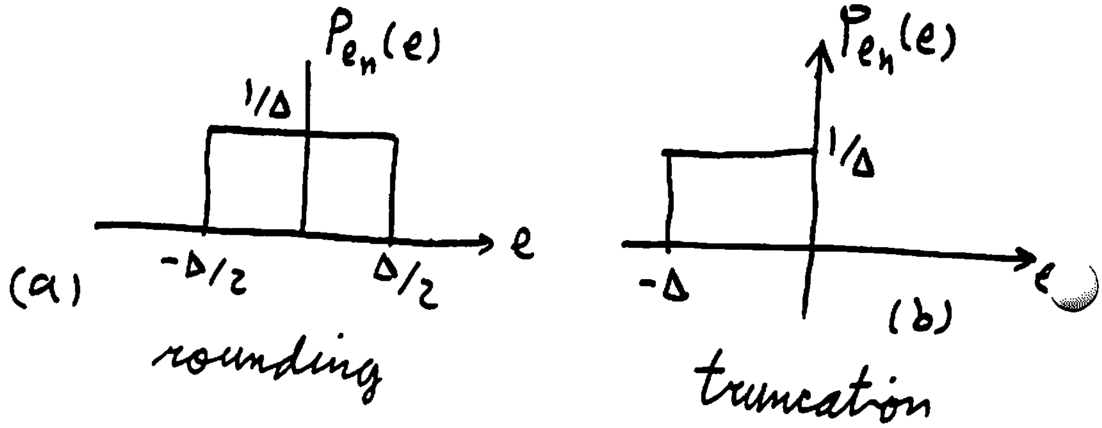
$$m_{x_n} = E\{x_n\} = \int_{-\infty}^{\infty} x \, P_{x_n}(x, n)\, dx$$

e) White noise is a random process in which all the random variables are independent with zero mean

$$\Phi_{xx}(m) = \sigma_x^2\ \delta(m)$$

f) A random process for which the time averages equal the ensemble averages is called an ergodic process.

② 8.3



Prob. distribution

a) Mean & variance, rounding

$$m_e = \int_{-\infty}^{\infty} e\, P_{e_n}(e)\, de = \int_{-\Delta/2}^{\Delta/2} e\, \frac{1}{\Delta}\, de = \frac{1}{\Delta}\ \frac{e^2}{2}\Bigg|_{-\Delta/2}^{\Delta/2} = 0$$

$$\sigma_e^2 = E\{e_n^2\} = \int_{-\infty}^{\infty} e^2 P_{e_n}(e)\, de = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} e^2 de$$

$$= \frac{e^3}{3\Delta}\Bigg|_{-\Delta/2}^{\Delta/2} = \frac{1}{3\Delta}\ 2\ \frac{\Delta^3}{8} = \Delta^2/12$$

b) For truncation

$$m_e = \frac{1}{\Delta}\int_{-\Delta}^{0} e\, de = \frac{1}{\Delta}\frac{e^2}{2}\Big|_{-\Delta}^{0} = -\Delta/2$$

$$\underline{\sigma_e^2} = E\left\{(e_n+\frac{\Delta}{2})^2\right\} = E\{e_n^2\} + \frac{\Delta^2}{4} + 2\frac{\Delta}{2}E\{e_n\}$$

$$= E\{e_n^2\} + \frac{\Delta^2}{4} - \frac{\Delta^2}{2} = \underline{E\{e_n^2\} - \frac{\Delta^2}{4}}$$

$$\sigma_e^2 = \frac{1}{\Delta}\int_{-\Delta}^{0} e^2\, de - \frac{\Delta^2}{4} = \frac{1}{\Delta}\frac{e^3}{3}\Big|_{-\Delta}^{0} - \frac{\Delta^2}{4} = \frac{\Delta^2}{12}$$

(3) 8.4 $e(n)$: white noise seq.
$s(n)$: uncorrelated with e_n

show $y(n) = s(n)\,e(n)$ is white ; i.e.

$$E\{y(n)\,y(n+m)\} = \underbrace{A}_{\text{const.}}\ \delta(m)$$

Sol.

$e(n)$ white $\Rightarrow E\{e(n)\,e(n+m)\} = \sigma_e^2\,\delta(m)$

uncorrelated $E\{e(n)\,y(m)\} = E\{e(n)\}E\{y(m)\}$

$$E\{y(n)\,y(n+m)\} = E\{s(n)\,e(n)\,s(n+m)\,e(n+m)\}$$

assume $s(n)$ is WSS or white noise

$$= E\{s(n)\,s(n+m)\,e(n)\,e(n+m)\}$$

$$= E\{s(n)\,s(n+m)\}\,E\{e(n)\,e(n+m)\}$$

$$= E\{s(n)\,s(n+m)\}\,\sigma_e^2\,\delta(m)$$

$$= \sigma_s^2\,\sigma_e^2\,\delta(m)$$

8.6 Consider the two real stationary random processes $\{x_n\}$ and $\{y_n\}$. with means m_x and m_y and variances σ_x^2 and σ_y^2.

Show the following

(a) $\gamma_{xx}(m) = \phi_{xx}(m) - m_x^2$ & $\gamma_{xy}(m) = \phi_{xy}(m) - m_x \cdot m_y$

$$\begin{aligned}
\gamma_{xx}(m) &= E[(x_n - m_x)(x_{n+m} - m_x)] \\
&= E[x_n x_{n+m}] - m_x E[x_{n+m}] - m_x E[x_n] + m_x m_x \\
&= \phi_{xx}(m) - m_x m_x - m_x m_x + m_x m_x \\
&= \phi_{xx}(m) - m_x^2
\end{aligned}$$

$$\begin{aligned}
\gamma_{xy}(m) &= E[(x_n - m_x)(y_{n+m} - m_y)] \\
&= E[x_n y_{n+m}] - m_x m_y - m_y m_x + m_x m_y \\
&= \phi_{xy}(m) - m_x m_y
\end{aligned}$$

(b) $\phi_{xx}(0) =$ mean square & $\gamma_{xx}(0) = \sigma_x^2$

$$\phi_{xx}(m) = E[x_n x_{n+m}] =$$

$$\phi_{xx}(0) = E[x_n x_n] = \text{mean square}$$

$$\gamma_{xx}(m) = E[(x_n - m_x)(x_{n+m} - m_x)]$$

$$\gamma_{xx}(0) = E[(x_n - m_x)^2] = \sigma_x^2$$

(c) $\phi_{xx}(m) = \phi_{xx}(-m)$

$$\phi_{xx}(-m) = (E[x_n x_{n-m}])$$

let $n' = n-m$

$$\begin{aligned}
\phi_{xx}(-m) &= (E[x_{n'+m} x_{n'}]) = E[x_{n'} x_{n'+m}] \\
&= \phi_{xx}(m)
\end{aligned}$$

$\gamma_{xx}(m) = \gamma_{xx}(-m)$

$$\begin{aligned}
\gamma_{xx}(-m) &= (E[(x_n - m_x)(x_{n-m} - m_x)]) \\
&= (E[(x_{n'+m} - m_x)(x_{n'} - m_x)]) \\
&= E[(x_{n'} - m_x)(x_{n'+m} - m_x)] \\
&= \gamma_{xx}(m)
\end{aligned}$$

$$\underline{\phi_{xy}(m) = \phi_{yx}(-m)}$$

$$\phi_{yx}(-m) = \left(E\left[(y_n - m_y)(x_{n-m} - m_x)\right]\right)$$

$$= \left(E\left[(y_{n'+m} - m_y)(x_{n'} - m_x)\right]\right)$$

$$= E\left[(x_{n'} - m_x)(y_{n'+m} - m_y)\right]$$

$$= \phi_{xy}(m)$$

$$\underline{\gamma_{xy}(m) = \gamma_{yx}(-m)}$$

$$\gamma_{yx}(-m) = \left(E\left[(y_n - m_y)(x_{y-m} - m_x)\right]\right)$$

$$= \left(E\left[(y_{n'+m} - m_y)(x_{n'} - m_x)\right]\right)$$

$$= E\left[(x_{n'} - m_x)(y_{n'+m} - m_y)\right]$$

$$= \gamma_{xy}(m).$$

(d) Consider the inequality

$$E\left\{\left(\frac{x_n}{(E[x_n^2])^{1/2}} - \frac{y_{n+m}}{(E[y^2_{n+m}])^{1/2}}\right)^2\right\} \geq 0$$

This is true since the quantity inside the brackets is >0 for all m and n.

Now

$$E\left[\frac{x_n^2}{E[x_n^2]}\right] + E\left[\frac{y_{n+m}^2}{E[y^2_{n+m}]}\right] - 2\frac{E[x_n y_{n+m}]}{(E[x_n^2])^{1/2}(E[y^2_{n+m}])^{1/2}} \geq 0$$

This can be written as

$$\frac{\phi_{xx}(0)}{\phi_{xx}(0)} + \frac{\phi_{yy}(0)}{\phi_{yy}(0)} - \frac{2\,\phi_{xy}(m)}{\phi_{xx}^{1/2}(0)\,\phi_{yy}^{1/2}(0)} \geq 0$$

$$\frac{\phi_{xy}(m)}{\phi_{xx}^{1/2}(0)\,\phi_{yy}^{1/2}(0)} \leq 1 \quad \Rightarrow \quad \boxed{[\phi_{xx}(0)\,\phi_{yy}(0)]^{1/2} \geq \phi_{xy}(m)}$$

Now if we replace x_n by $(x_n - m_x)$ and y_{n+m} by $(y_{n+m} - m_y)$ in the inequality we can manipulate it in the same way to get

$$\boxed{[\gamma_{xx}(0)\,\gamma_{yy}(0)]^{1/2} \geq \gamma_{xy}(m)}$$

Letting $y_n = x_n$ we can specialize these inequalities to

$$\boxed{\begin{array}{c} \phi_{xx}(0) \geq \phi_{xx}(m) \\ \hline \gamma_{xx}(0) \geq \gamma_{xx}(m) \end{array}}$$

(e) Let $y_n = x_{n-n_0}$

$$\underline{\phi_{yy}(m)} = E[y_n\, y_{n+m}]$$
$$= E[x_{n-n_0}\, x_{n+m-n_0}]$$
$$= \underline{\phi_{xx}(m)}$$

Obviously $\underline{\gamma_{yy}(m) = \gamma_{xx}(m)}$ for the same reasons.

(f) Let $\gamma_{xx}(m) \longleftrightarrow \Gamma_{xx}(z)$

$$\gamma_{xy}(m) \longleftrightarrow \Gamma_{xy}(z)$$

$\Gamma_{xx}(z) \triangleq \sum_m \gamma_{xx}(m)\, z^{-m} \Rightarrow$ **(1)** $\quad \gamma_{xx}(m) = \frac{1}{2\pi j}\oint_c \Gamma_{xx}(z)\, z^{m-1}\,dz$

$$\underline{\gamma_{xx}(0) =} \boxed{\sigma_x^2 = \frac{1}{2\pi j}\oint_c \Gamma_{xx}(z)\, z^{-1}\,dz}$$

(2) We have shown that $\gamma_{xx}(m) = \gamma_{xx}(-m)$

Therefore $\quad \Gamma_{xx}(z) = \sum_{m=-\infty}^{\infty} \gamma_{xx}(m)\, z^{-m}$

$$\underline{\Gamma_{xx}(z^{-1})} = \sum_{m=-\infty}^{\infty} \gamma_{xx}(m)\, z^{m} = \sum_{p=-\infty}^{\infty} \gamma_{xx}(-p)\, z^{-p}$$

$$= \sum_{m=-\infty}^{\infty} \gamma_{xx}(m)\, z^{-m} = \underline{\Gamma_{xx}(z)}$$

$p \to m \Rightarrow$

use $\gamma_{xy}(m) = \gamma^{*}_{yx}(-m)$

$$\Gamma_{xy}(z) = \sum_{m=-\infty}^{\infty} \gamma_{xy}(m)\, z^{-m} = \sum_{m=-\infty}^{\infty} \gamma^{*}_{yx}(-m)\, z^{-m}$$

$$= \left(\sum_{\ell=-\infty}^{\infty} \gamma_{yx}(\ell)\, z^{*\,\ell} \right)^{*}$$

$$= \left(\sum_{\ell=-\infty}^{\infty} \gamma_{yx}(\ell) \left(z^{*^{-1}} \right)^{-\ell} \right)^{*} = \Gamma^{*}_{yx}(1/z^{*}$$