

HW 2
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Contents

1	Questions	2
2	Problem 2.30	3
2.1	part(b)	3
2.2	part(c)	6
3	Problem 2.32	9
4	Problem 2.33	11
5	Problem 2.35	12
6	extra Problem	14
7	Key solution	15
8	my graded HW	24

1 Questions

EE 443

Chapt. # 2

HW #2

page 1



Problem 2.30 Determine and sketch the autocorrelation functions of the following exponential pulses:

- (a) $g(t) = \exp(-at)u(t)$
- ✓ (b) $g(t) = \exp(-a|t|)$
- ✓ (c) $g(t) = \exp(-at)u(t) - \exp(at)u(-t)$

✓ **Problem 2.32** Determine the autocorrelation function of the sinc pulse $A \text{sinc}(2Wt)$, and sketch it.

✓ **Problem 2.33** The Fourier transform of a signal is defined by $|\text{sinc}(f)|$. Show that the autocorrelation function of this signal is triangular in form.

(Hint: Find $|G(f)|^2$, then find $R_g(\tau)$)

✓ **Problem 2.35** Consider a signal $g(t)$ defined by

$$g(t) = A_0 + A_1 \cos(2\pi f_1 t + \theta) + A_2 \cos(2\pi f_2 t + \theta)$$

- (a) Determine the autocorrelation function $R_g(\tau)$ of this signal.
- (b) What is the value of $R_g(0)$?
- (c) Has any information about $g(t)$ been lost in obtaining the autocorrelation function?

(Hint Use Freq. domain approach.)

Extra problem. do:

a) $\Xi(t) \otimes \Xi(t)$ where $\Xi(t)$ is unit step function

b) $y(t) = t \Xi(t) \otimes e^{at} \Xi(t)$ $a < 0$

c) $y(t) = u(t) \otimes h(t)$ where $u(t)$ is a rectangular pulse of height 1 and width $4T$, and $h(t) = e^{-3t} u(t)$ is an exponential decay starting at $t=0$.

2 Problem 2.30

Problem

Determine and sketch the autocorrelation function of the following

(b) $g(t) = e^{-a|t|}$

(c) $g(t) = e^{-at}u(t) - e^{at}u(-t)$

2.1 part(b)

$$g(t) = \begin{cases} e^{-at} & t > 0 \\ 1 & t = 0 \\ e^{at} & t < 0 \end{cases}$$

Assume $a > 0$ for the integral to be defined. From definition, autocorrelation of a function $g(t)$ is

$$R(\tau) = \int_{-\infty}^{\infty} g(t) g^*(t - \tau) dt$$

Since $g(t)$ in this case is real, then $g^*(t - \tau) = g(t - \tau)$, hence

$$R(\tau) = \int_{-\infty}^{\infty} g(t) g(t - \tau) dt$$

Consider the 3 cases, $\tau < 0$ and $\tau > 0$ and when $\tau = 0$

case $\tau > 0$

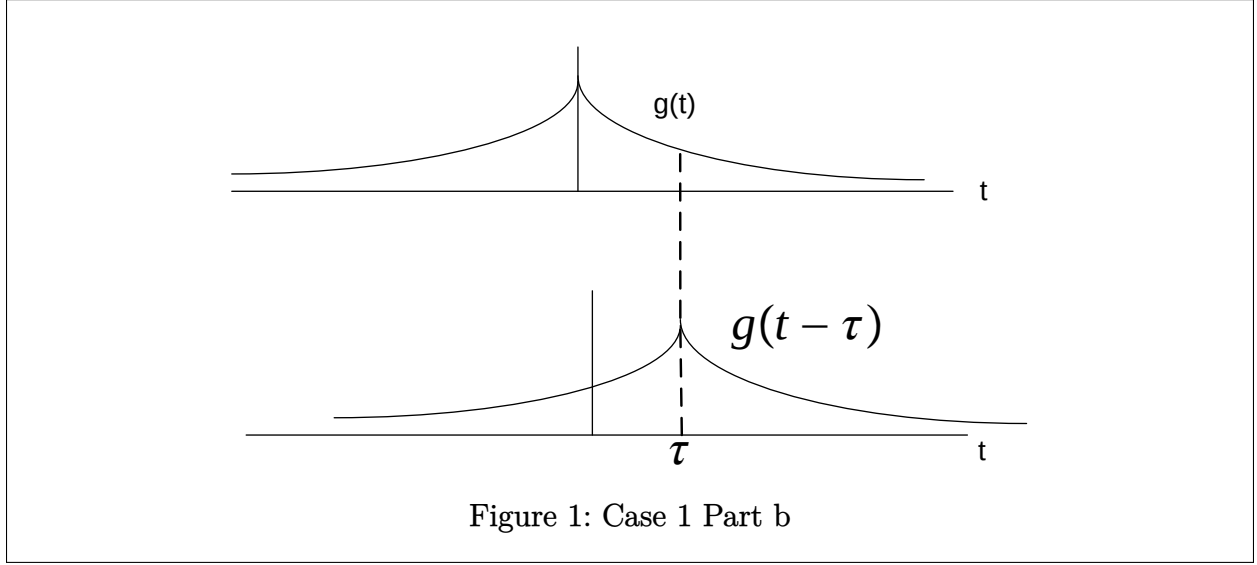


Figure 1: Case 1 Part b

Break the integral over the 3 regions, $\{-\infty, 0\}$, $\{0, \tau\}$, $\{\tau, \infty\}$

$$R(\tau) = \int_{-\infty}^0 e^{at} e^{a(t-\tau)} dt + \int_0^{\tau} e^{-at} e^{a(t-\tau)} dt + \int_{\tau}^{\infty} e^{-at} e^{-a(t-\tau)} dt$$

But $\int_{-\infty}^0 e^{at} e^{a(t-\tau)} dt = e^{-a\tau} \int_{-\infty}^0 e^{2at} dt = e^{-a\tau} \frac{[e^{2at}]_{-\infty}^0}{2a} = e^{-a\tau} \frac{[1-0]}{2a} = \frac{e^{-a\tau}}{2a}$

and $\int_0^{\tau} e^{-at} e^{a(t-\tau)} dt = e^{-a\tau} \int_0^{\tau} 1 dt = \tau e^{-a\tau}$

and $\int_{\tau}^{\infty} e^{-at} e^{-a(t-\tau)} dt = e^{a\tau} \int_{\tau}^{\infty} e^{-2at} dt = e^{a\tau} \frac{[e^{-2at}]_{\tau}^{\infty}}{-2a} = e^{a\tau} \frac{[0-e^{-2a\tau}]}{-2a} = \frac{e^{-a\tau}}{2a}$

Hence for $\tau > 0$ we obtain

$$\begin{aligned} R(\tau) &= \frac{e^{-a\tau}}{2a} + \tau e^{-a\tau} + \frac{e^{-a\tau}}{2a} \\ &= \frac{e^{-a\tau}}{a} + \tau e^{-a\tau} \\ &= \boxed{e^{-a\tau} \left(\frac{1}{a} + \tau \right)} \end{aligned}$$

case $\tau < 0$

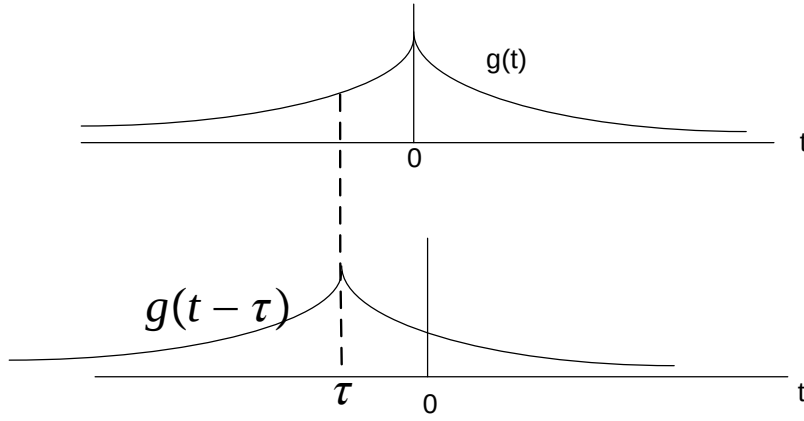


Figure 2: Case 2 Part b

Break the integral over the 3 regions, $\{-\infty, \tau\}, \{\tau, 0\}, \{0, \infty\}$

$$R(\tau) = \int_{-\infty}^{\tau} e^{at} e^{a(t-\tau)} dt + \int_{\tau}^0 e^{-at} e^{a(t-\tau)} dt + \int_0^{\infty} e^{-at} e^{-a(t-\tau)} dt$$

$$\text{Now } \int_{-\infty}^{\tau} e^{at} e^{a(t-\tau)} dt = e^{-a\tau} \int_{-\infty}^{\tau} e^{2at} dt = e^{-a\tau} \left[\frac{e^{2at}}{2a} \right]_{-\infty}^{\tau} = e^{-a\tau} \left[\frac{e^{2a\tau} - 0}{2a} \right] = \frac{e^{a\tau}}{2a}$$

$$\text{and } \int_{\tau}^0 e^{-at} e^{a(t-\tau)} dt = e^{-a\tau} \int_{\tau}^0 1 dt = -\tau e^{-a\tau}$$

$$\text{and } \int_0^{\infty} e^{-at} e^{-a(t-\tau)} dt = e^{a\tau} \left[\frac{e^{-2at}}{-2a} \right]_0^{\infty} = \frac{e^{a\tau}}{-2a} (0 - 1) = \frac{e^{a\tau}}{2a}$$

Hence

$$\begin{aligned} R(\tau) &= \frac{e^{a\tau}}{2a} - \tau e^{-a\tau} + \frac{e^{a\tau}}{2a} \\ &= \boxed{e^{a\tau} \left(\frac{1}{a} - \tau \right)} \end{aligned}$$

When $\tau = 0$

$R(0)$ gives the the maximum power in the signal $g(t)$. Now evaluate this

$$\begin{aligned} R(\tau) &= \int_{-\infty}^0 e^{at} e^{at} dt + \int_0^{\infty} e^{-at} e^{-at} dt \\ &= \frac{[e^{2at}]_{-\infty}^0}{2a} + \frac{[e^{-2at}]_0^{\infty}}{-2a} \\ &= \frac{1}{a} \end{aligned}$$

Hence

$$R(\tau) = \begin{cases} e^{-a\tau} \left(\frac{1}{a} + \tau \right) & \tau > 0 \\ \frac{1}{a} & \tau = 0 \\ e^{a\tau} \left(\frac{1}{a} - \tau \right) & \tau < 0 \end{cases}$$

Or we could write

$$R(\tau) = e^{-|\tau|a} \left(\frac{1}{a} - (-|\tau|) \right)$$

This is a plot of $R(\tau)$, first plot is for $a = 1$ and the second for $a = 4$

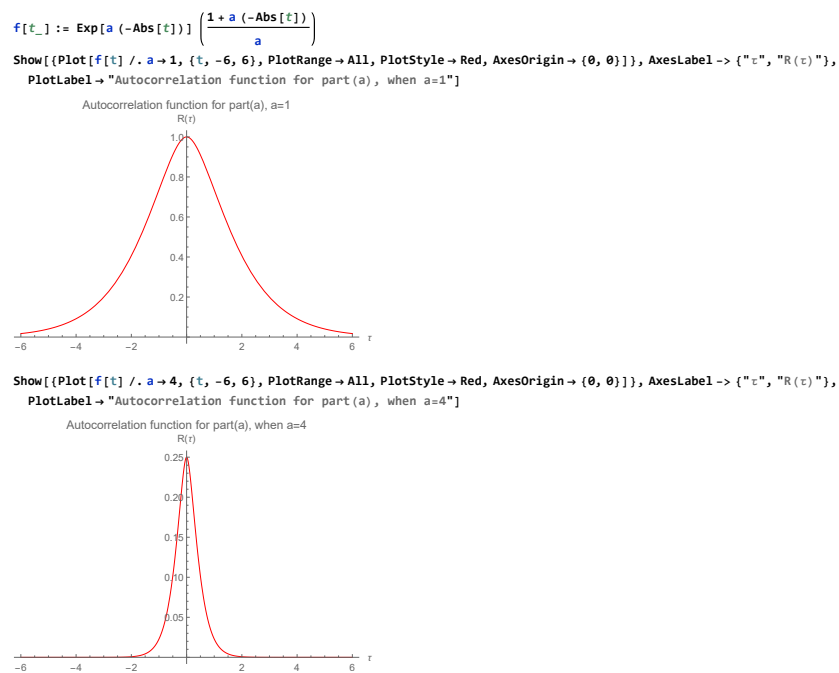


Figure 3: final part

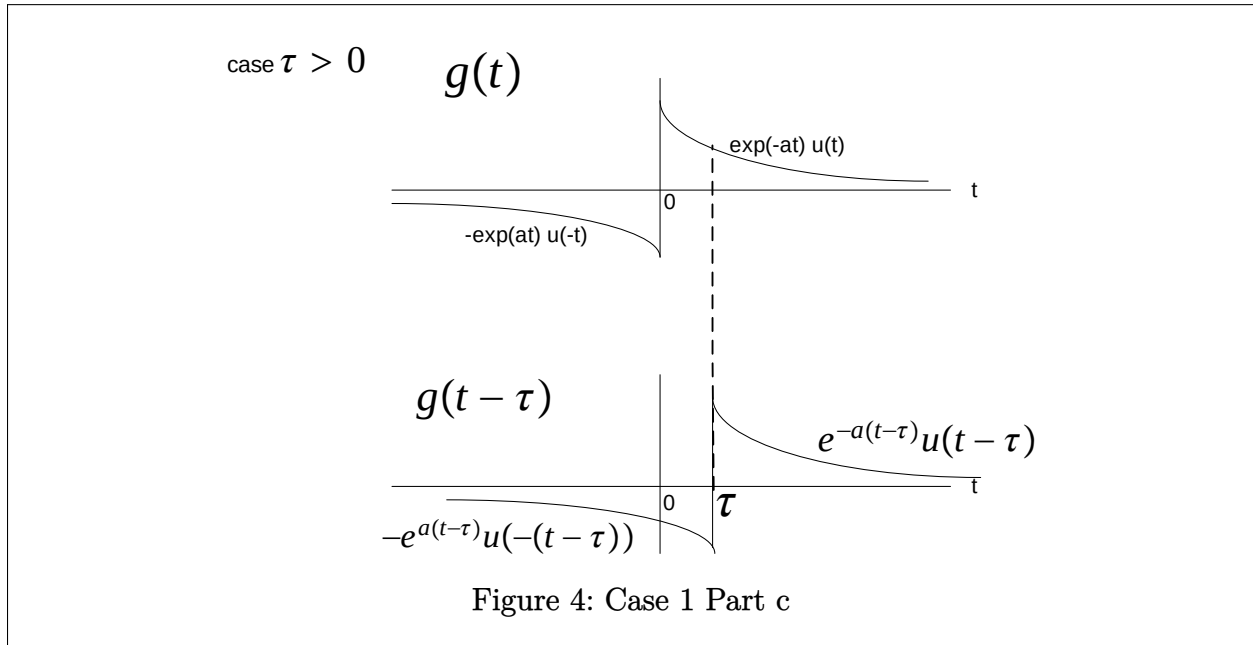
2.2 part(c)

$$g(t) = e^{-at}u(t) - e^{at}u(-t)$$

Assume $a > 0$.

Consider the 3 cases, $\tau < 0$ and $\tau > 0$ and when $\tau = 0$

case $\tau > 0$



Break the integral into 3 parts, $\{-\infty, 0\}, \{0, \tau\}, \{\tau, \infty\}$

$$\begin{aligned}
 R(\tau) &= \int_{-\infty}^0 g(t) g(t-\tau) dt + \int_0^{\tau} g(t) g(t-\tau) dt + \int_{\tau}^{\infty} g(t) g(t-\tau) dt \\
 &= \int_{-\infty}^0 -e^{at} (-e^{a(t-\tau)}) dt + \int_0^{\tau} e^{-at} (-e^{a(t-\tau)}) dt + \int_{\tau}^{\infty} e^{-at} (e^{-a(t-\tau)}) dt \\
 &= e^{-a\tau} \int_{-\infty}^0 e^{2at} dt - e^{-a\tau} \int_0^{\tau} 1 dt + e^{a\tau} \int_{\tau}^{\infty} e^{-2at} dt \\
 &= e^{-a\tau} \frac{[e^{2at}]_{-\infty}^0}{2a} - \tau e^{-a\tau} + e^{a\tau} \frac{[e^{-2at}]_{\tau}^{\infty}}{-2a} \\
 &= e^{-a\tau} \frac{[1-0]}{2a} - \tau e^{-a\tau} + e^{a\tau} \frac{[0-e^{-2a\tau}]}{-2a} \\
 &= \frac{e^{-a\tau}}{2a} - \tau e^{-a\tau} + \frac{e^{-a\tau}}{2a} \\
 &= e^{-a\tau} \left(\frac{1}{2a} - \tau + \frac{1}{2a} \right) \\
 &= e^{-a\tau} \left(\frac{1}{a} - \tau \right)
 \end{aligned}$$

case $\tau < 0$

case $\tau < 0$

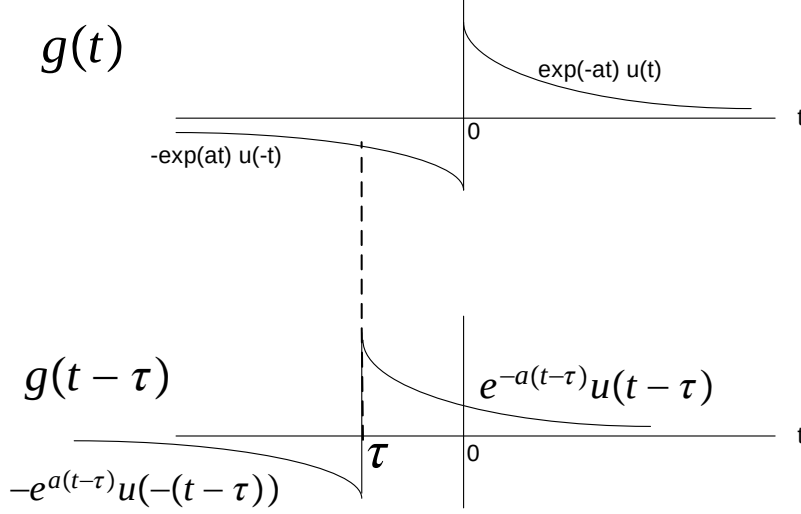


Figure 5: Case 2 Part c

Break the integral into 3 parts, $\{-\infty, \tau\}, \{\tau, 0\}, \{0, \infty\}$

$$\begin{aligned}
 R(\tau) &= \int_{-\infty}^{\tau} g(t) g(t - \tau) dt + \int_{\tau}^0 g(t) g(t - \tau) dt + \int_0^{\infty} g(t) g(t - \tau) dt \\
 &= \int_{-\infty}^{\tau} -e^{at} (-e^{a(t-\tau)}) dt + \int_{\tau}^0 -e^{at} e^{-a(t-\tau)} dt + \int_0^{\infty} e^{-at} e^{-a(t-\tau)} dt \\
 &= e^{-a\tau} \int_{-\infty}^{\tau} e^{2at} dt - e^{a\tau} \int_{\tau}^0 1 dt + e^{a\tau} \int_0^{\infty} e^{-2at} dt \\
 &= e^{-a\tau} \frac{[e^{2at}]_{-\infty}^{\tau}}{2a} + \tau e^{a\tau} + e^{a\tau} \frac{[e^{-2at}]_0^{\infty}}{-2a} \\
 &= e^{-a\tau} \frac{[e^{2a\tau} - 0]}{2a} + \tau e^{a\tau} + e^{a\tau} \frac{[0 - 1]}{-2a} \\
 &= \frac{e^{a\tau}}{2a} + \tau e^{a\tau} + \frac{e^{a\tau}}{2a} \\
 &= e^{a\tau} \left(\frac{1}{a} + \tau \right)
 \end{aligned}$$

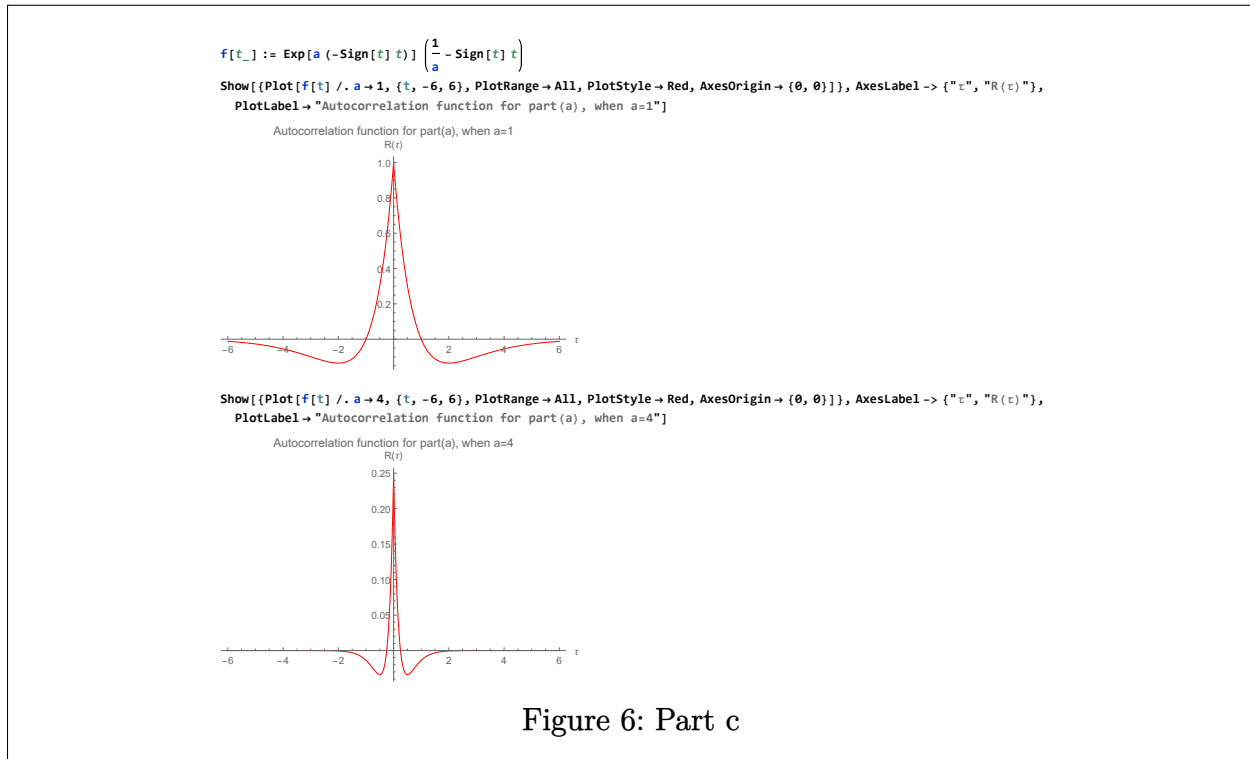
At $\tau = 0$, we see that $R(0) = \frac{1}{a}$, hence the final answer is

$$R(\tau) = \begin{cases} e^{-a\tau} \left(\frac{1}{a} - \tau \right) & \tau > 0 \\ \frac{1}{a} & \tau = 0 \\ e^{a\tau} \left(\frac{1}{a} + \tau \right) & \tau < 0 \end{cases}$$

Or we could write

$$R(\tau) = e^{-|\tau|^a} \left(\frac{1}{a} - |\tau| \right)$$

This is a plot of $R(\tau)$, first plot is for $a = 1$ and the second for $a = 4$



3 Problem 2.32

problem: Determine the autocorrelation function of $g(t) = A \text{sinc}(2Wt)$ and sketch it

solution:

$$R(\tau) = \int_{-\infty}^{\infty} g(t) g^*(t - \tau) dt$$

The above is difficult to do directly, hence we use the second method.

Since the function $g(t)$ is an energy function, hence $R(\tau)$ and the energy spectrum density $\Psi_g(f)$ of $g(t)$ make a Fourier transform pairs.

$$R(\tau) \Leftrightarrow \Psi_g(f)$$

Therefore, to find $R(\tau)$, we first find $\Psi_g(f)$, then find the Inverse Fourier Transform of $\Psi_g(f)$, i.e.

$$R(\tau) = F^{-1}(\Psi_g(f)) \quad (1)$$

But

$$\Psi_g(f) = |G(f)|^2 \quad (2)$$

and we know that

$$A \operatorname{sinc}(2Wt) \Leftrightarrow \frac{A}{2W} \operatorname{rect}\left(\frac{f}{2W}\right)$$

Hence

$$G(f) = \frac{A}{2W} \operatorname{rect}\left(\frac{f}{2W}\right)$$

The (2) becomes

$$\begin{aligned} \Psi_g(f) &= \left| \frac{A}{2W} \operatorname{rect}\left(\frac{f}{2W}\right) \right|^2 \\ &= \left(\frac{A}{2W} \right)^2 \left| \operatorname{rect}\left(\frac{f}{2W}\right) \right|^2 \end{aligned}$$

But $\left| \operatorname{rect}\left(\frac{f}{2W}\right) \right|^2 = \operatorname{rect}\left(\frac{f}{2W}\right)$, since it has height of 1, so

$$\boxed{\Psi_g(f) = \left(\frac{A}{2W} \right)^2 \operatorname{rect}\left(\frac{f}{2W}\right)}$$

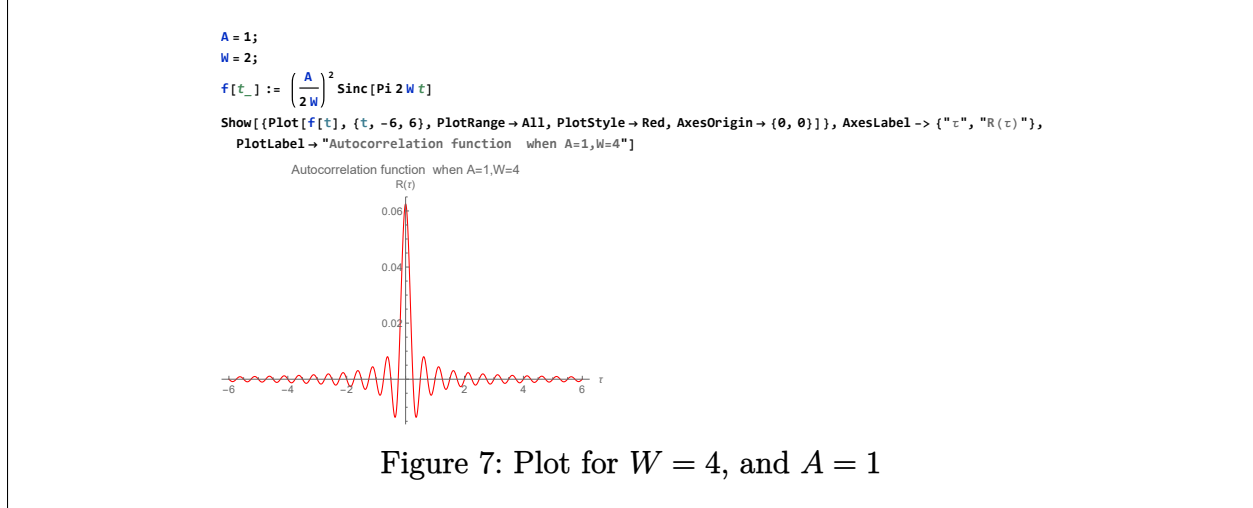
Hence from (1)

$$\begin{aligned} R(\tau) &= F^{-1}\left(\left(\frac{A}{2W}\right)^2 \operatorname{rect}\left(\frac{f}{2W}\right)\right) \\ &= \left(\frac{A}{2W}\right)^2 F^{-1}\left[\operatorname{rect}\left(\frac{f}{2W}\right)\right] \end{aligned}$$

Hence

$$R(\tau) = \left(\frac{A}{2W}\right)^2 \text{sinc}(2W\tau)$$

This is a plot of the above function, for $W = 4$, and $A = 1$



4 Problem 2.33

The Fourier transform of a signal is defined by $|\text{sinc}(f)|$. Show that $R(\tau)$ of the signal is triangular in form.

Answer:

Since

$$R(\tau) \Leftrightarrow |G(f)|^2$$

Then

$$\begin{aligned} R(\tau) &\Leftrightarrow |\text{sinc}(f)|^2 \\ &\Leftrightarrow \text{sinc}^2(f) \end{aligned}$$

Hence to find $R(\tau)$ we need to find the inverse Fourier transform of $\text{sinc}^2(f)$

But

$$\begin{aligned} F^{-1}(\text{sinc}^2(f)) &= F^{-1}(\text{sinc}(f) \times \text{sinc}(f)) \\ &= F^{-1}\{\text{sinc}(f)\} \otimes F^{-1}\{\text{sinc}(f)\} \end{aligned}$$

But $F^{-1}\{\text{sinc}(f)\} = \text{rect}(t)$, hence

$$\begin{aligned} F^{-1}(\text{sinc}^2(f)) &= \text{rect}(t) \otimes \text{rect}(t) \\ &= \int_{-\infty}^{\infty} \text{rect}(\tau) \text{rect}(t - \tau) d\tau \end{aligned}$$

This integral has the value of $\text{tri}(t)$ (we also did this in class) Hence

$$\text{tri}(\tau) \Leftrightarrow \text{sinc}^2(f)$$

Hence

$$R(\tau) = \text{tri}(\tau)$$

Where $\text{tri}(\tau)$ is the triangle function, defined as

$$\text{tri}(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

5 Problem 2.35

Consider the signal $g(t)$ defined by

$$g(t) = A_0 + A_1 \cos(2\pi f_1 t + \theta) + A_2 \cos(2\pi f_2 t + \theta)$$

- (a) determine $R(\tau)$
- (b) what is $R(0)$
- (c) has any information been lost in obtaining $R(\tau)$?

Answer:

- (a)

Take the Fourier transform of $g(t)$ we obtain

$$G(f) = A_0 \delta(f) + \frac{A_1}{2} [e^{j\theta} \delta(f - f_1) + e^{-j\theta} \delta(f + f_1)] + \frac{A_2}{2} [e^{j\theta} \delta(f - f_2) + e^{-j\theta} \delta(f + f_2)]$$

Hence $|G(f)|^2 = G(f) G^*(f)$, so we need to find $G^*(f)$

$$G^*(f) = A_0\delta(f) + \frac{A_1}{2} [e^{-j\theta}\delta(f - f_1) + e^{j\theta}\delta(f + f_1)] + \frac{A_2}{2} [e^{-j\theta}\delta(f - f_2) + e^{j\theta}\delta(f + f_2)]$$

So

$$G(f) G^*(f) = A_0^2\delta(f) + \frac{A_1^2}{4} [\delta(f - f_1) + \delta(f + f_1)] + \frac{A_2^2}{4} [\delta(f - f_2) + \delta(f + f_2)]$$

So

$$S_g(f) = A_0^2\delta(f) + \frac{A_1^2}{4} [\delta(f - f_1) + \delta(f + f_1)] + \frac{A_2^2}{4} [\delta(f - f_2) + \delta(f + f_2)]$$

So

$$\begin{aligned} R(\tau) &= F^{-1}(S_g(f)) \\ &= F^{-1}(A_0^2\delta(f)) + \frac{A_1^2}{4} F^{-1}[\delta(f - f_1) + \delta(f + f_1)] + \frac{A_2^2}{4} F^{-1}[\delta(f - f_2) + \delta(f + f_2)] \end{aligned}$$

Hence

$$\boxed{R(\tau) = A_0^2 + \frac{A_1^2}{2} \cos 2\pi f_1 \tau + \frac{A_2^2}{2} \cos 2\pi f_2 \tau} \quad (1)$$

Part (b)

$$\begin{aligned} R(0) &= A_0^2 + \frac{A_1^2}{2} + \frac{A_2^2}{2} \\ &= \frac{1}{2} (2A_0^2 + A_1^2 + A_2^2) \end{aligned}$$

part(c)

In obtaining $R(\tau)$ we have lost the phase information in the original signal as can be seen from (1) above

6 extra Problem

(a) find $\xi(t) \otimes \xi(t)$ where $\xi(t)$ is unit step function

(b) Find $t\xi(t) \otimes e^{at}\xi(t)$ where $a > 0$

(c) find $u(t) \otimes h(t)$ where $h(t) = e^{-3t}u(t)$ and $u(t)$ is as shown

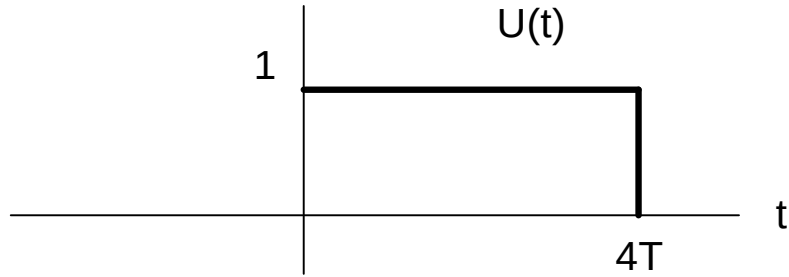


Figure 8: Extra problem

To DO

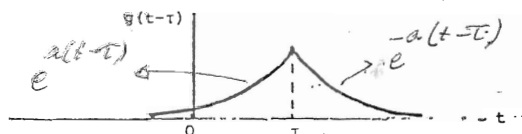
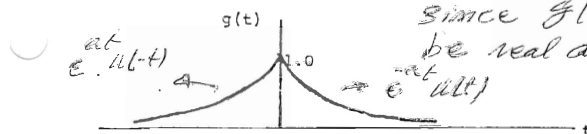
7 Key solution

EE 443 chap 2 HW#2 key page 1

30 a) see handout page (28)

30 b) $g = \exp(-a|t|) = e^{-at} u(t) + e^{at} u(-t)$ with $a > 0$

Since $g(t)$ is real the $R_g(\tau)$ will be real and even $\Rightarrow R_g(-\tau) = R_g(\tau)$



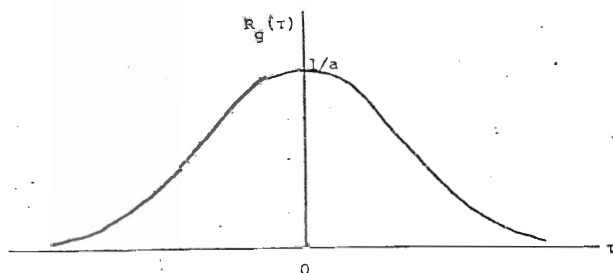
Therefore, for $\tau > 0$, $\Rightarrow R_g(\tau) = \int_{-\infty}^{+\infty} g(t) g(t-\tau) dt$

$$\begin{aligned} R_g(\tau) &= \int_{-\infty}^0 \exp(at) \exp[a(t-\tau)] dt \\ &\quad + \int_0^{\tau} \exp(-at) \exp[a(t-\tau)] dt \\ &\quad + \int_{\tau}^{\infty} \exp(-at) \exp[-a(t-\tau)] dt \\ &= \frac{1}{2a} \exp(-a\tau) + \tau \exp(-a\tau) + \frac{1}{2a} \exp(-a\tau) \\ &= \left(\frac{1}{a} + \tau\right) \exp(-a\tau) \end{aligned}$$

Since $R_g(-\tau) = R_g(\tau)$, we may express $R_g(\tau)$ for all τ as follows:

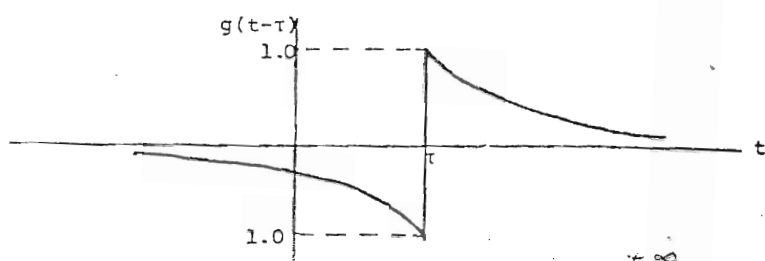
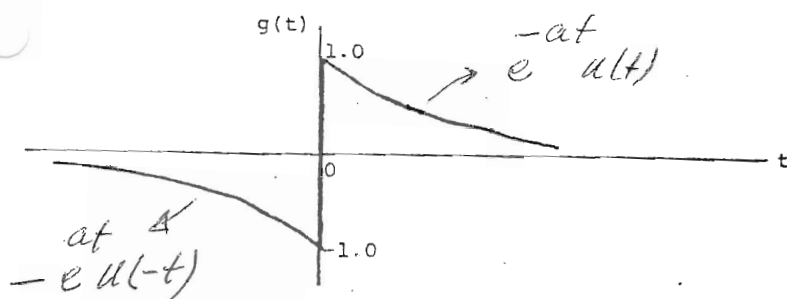
$$R_g(\tau) = \left(\frac{1}{a} + |\tau|\right) \exp(-a|\tau|)$$

which is illustrated below:



(c) $g(t) = \exp(-at)u(t) - \exp(at)u(-t)$, $a > 0$, $g(t)$ is real.

For $\tau > 0$, we have



Therefore, for $\tau > 0$,

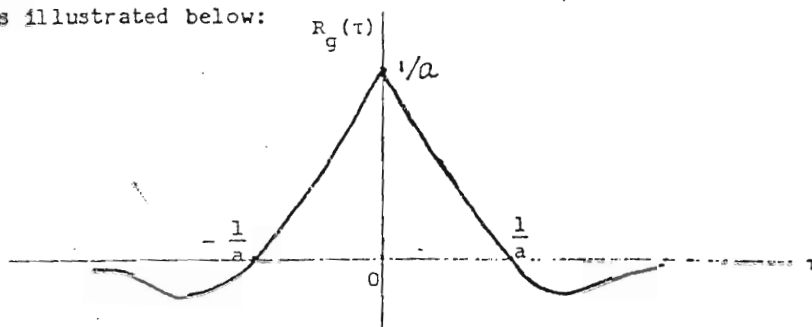
$$R_g(\tau) = \int_{-\infty}^{\infty} g(t) g(t-\tau) dt$$

$$\begin{aligned}
 R_g(\tau) &= \int_{-\infty}^0 \exp(at) \exp[a(t-\tau)] dt \\
 &\quad - \int_0^{\tau} \exp(-at) \exp[a(t-\tau)] dt \\
 &\quad + \int_{\tau}^{\infty} \exp(-at) \exp[-a(t-\tau)] dt \\
 &= \frac{1}{2a} \exp(-a\tau) - \tau \exp(-a\tau) + \frac{1}{2a} \exp(-a\tau) \\
 &= \left(\frac{1}{a} - \tau\right) \exp(-a\tau)
 \end{aligned}$$

Since $R_g(-\tau) = R_g(\tau)$, we may express $R_g(\tau)$ for all τ as follows:

$$R_g(\tau) = \left(\frac{1}{a} - |\tau|\right) \exp(-a|\tau|)$$

which is illustrated below:



2.32)

$$A \operatorname{sinc}(2Wt) \Leftrightarrow \frac{A}{2W} \operatorname{rect}\left(\frac{f}{2W}\right) = G(f)$$

Since,

$$R_g(\tau) \Leftrightarrow |G(f)|^2,$$

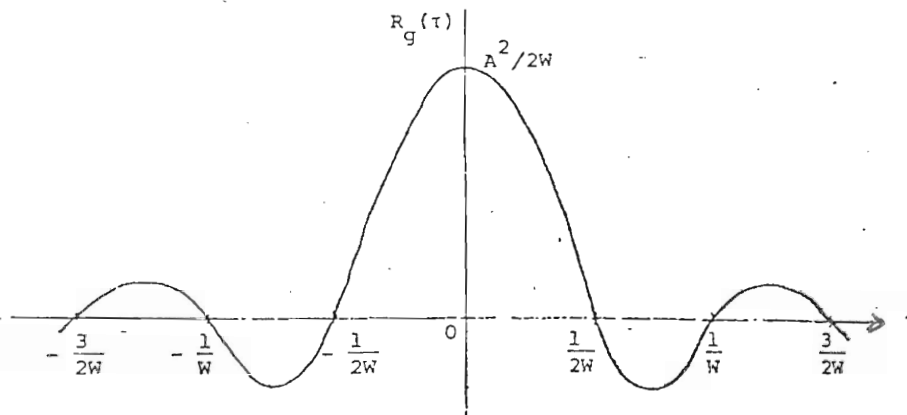
it follows that for the given sinc pulse

$$R_g(\tau) \Leftrightarrow \frac{A^2}{4W^2} \operatorname{rect}\left(\frac{f}{2W}\right)$$

Therefore,

$$R_g(\tau) = \frac{A^2}{2W} \operatorname{sinc}(2W\tau)$$

which is shown illustrated below:



Problem 2.33

→ see page (4)
also

$$G(f) = |\operatorname{sinc}(f)|$$

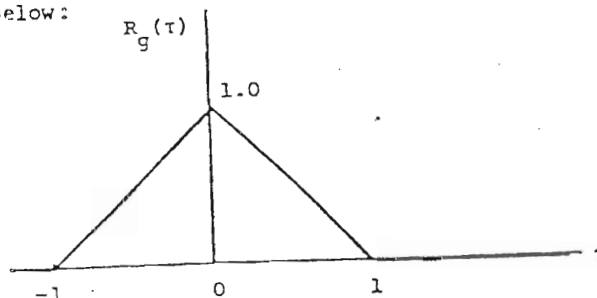
Therefore,

$$|G(f)|^2 = \operatorname{sinc}^2(f) \xleftrightarrow{F.T} R_g(\tau)$$

The function $\operatorname{sinc}^2(f)$ represents the Fourier transform of a triangular pulse of unit amplitude and width 2 seconds, centered at the origin. Therefore,

$$R_g(\tau) = \begin{cases} 1-|\tau|, & |\tau| < 1 \\ 0, & |\tau| > 1 \end{cases}$$

which is illustrated below:



2.33 :
(second method)

$$G(f) = |\text{sinc}(f)|$$

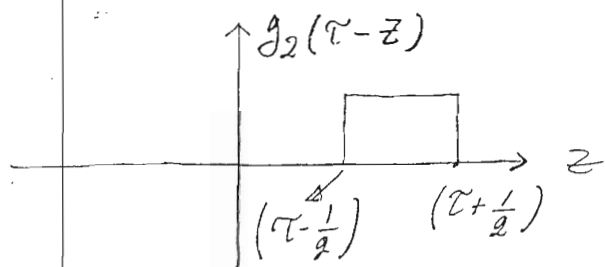
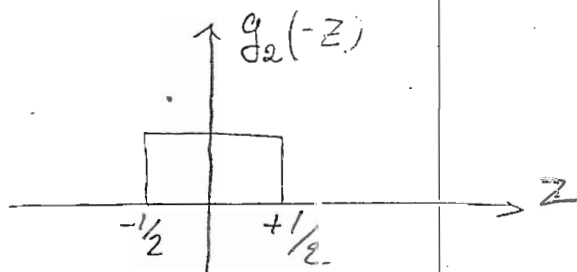
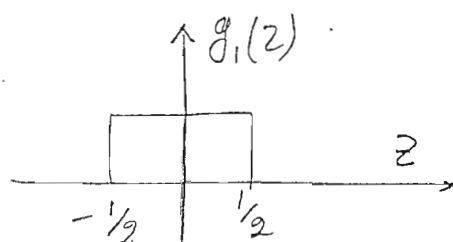
$$R_g(\tau) \xleftrightarrow{\text{F.T.}} |G(f)|^2 = \text{sinc}^2(f)$$

$$= \underbrace{\text{sinc}(f)}_{G_1(f)} \cdot \underbrace{\text{sinc}(f)}_{G_2(f)}$$

Therefore $R_g(\tau) = g_1(\tau) \oplus g_2(\tau)$

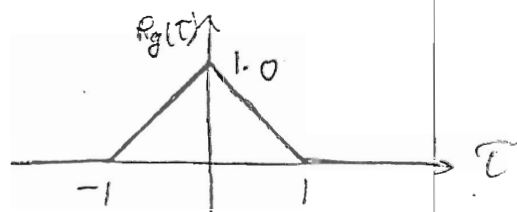
where $g_1(\tau) = g_2(\tau) = \mathcal{F}^{-1}[\text{sinc}(f)] = \text{rect}(\tau)$

$$R_g(\tau) = \int_{-\infty}^{+\infty} g_1(z) g_2(\tau - z) dz$$



After computing the convolution we have

$$R_g(\tau) = \begin{cases} 0 & \tau < -1 \\ 1 + \tau & -1 \leq \tau < 0 \\ 1 - \tau & 0 \leq \tau < 1 \\ 0 & \tau \geq 1 \end{cases} \Rightarrow$$



2.35

$$(a) \quad g(t) = A_0 + A_1 \cos(2\pi f_1 t + \theta) + A_2 \cos(2\pi f_2 t + \theta)$$

Therefore,

$$G(f) = A_0 \delta(f) + \frac{A_1}{2} [\delta(f-f_1) \exp(j\theta) + \delta(f+f_1) \exp(-j\theta)] \\ + \frac{A_2}{2} [\delta(f-f_2) \exp(j\theta) + \delta(f+f_2) \exp(-j\theta)]$$

and

$$|G(f)|^2 = A_0^2 \delta(f) + \frac{A_1^2}{4} [\delta(f-f_1) + \delta(f+f_1)] + \frac{A_2^2}{4} [\delta(f-f_2) + \delta(f+f_2)]$$

$$\text{Since } R_g(\tau) \rightleftharpoons |G(f)|^2$$

it follows that

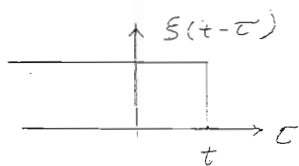
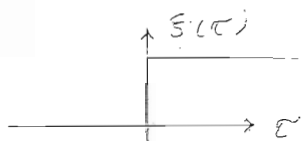
$$R_g(\tau) = A_0^2 + \frac{A_1^2}{2} \cos(2\pi f_1 \tau) + \frac{A_2^2}{2} \cos(2\pi f_2 \tau)$$

$$(b) \quad R_g(0) = A_0^2 + \frac{A_1^2}{2} + \frac{A_2^2}{2}$$

(c) We see that $R_g(\tau)$ depends only on the dc component A_0 , the amplitudes A_1 and A_2 of the two sinusoidal components and their frequencies f_1 and f_2 . The phase information contained in the phase angles of the two sinusoidal components is completely lost when evaluating $R_g(\tau)$.

Extra prob #2) Evaluate the following convolutions:

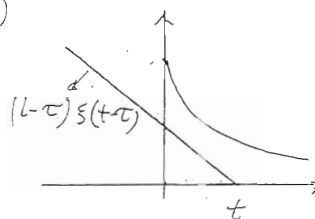
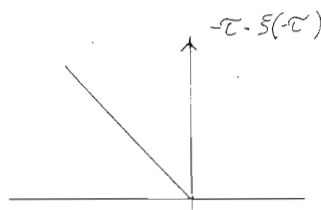
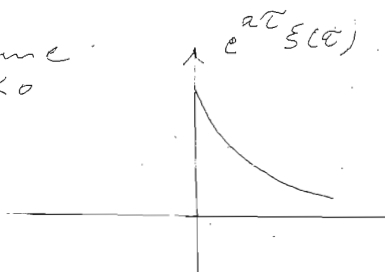
$$a) \quad \xi(t) * \xi(t) = \int_{-\infty}^{+\infty} \xi(\tau) \xi(t-\tau) d\tau$$



$$= \int_0^t 1 \cdot d\tau = \begin{cases} t, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases}$$

$$b) \quad y(t) = t \xi(t) * e^{at} \xi(t) = \int_{-\infty}^{+\infty} e^{a\tau} \xi(\tau) \cdot (t-\tau) \xi(t-\tau) d\tau$$

Assume $a < 0$



$$\begin{aligned} y(t) &= \int_0^t e^{a\tau} (t-\tau) d\tau = \frac{1}{a} e^{a\tau} (t-\tau) \Big|_0^t + \frac{1}{a} \int_0^t e^{a\tau} d\tau \\ &= \frac{1}{a^2} (e^{at} - 1) - \frac{t}{a} \end{aligned}$$

$$c) \quad e^{at} \xi(t) * e^{at} \xi(t) =$$

$$= \int_0^t e^{a\tau} \cdot e^{a(t-\tau)} d\tau = \int_0^t e^{at} d\tau$$

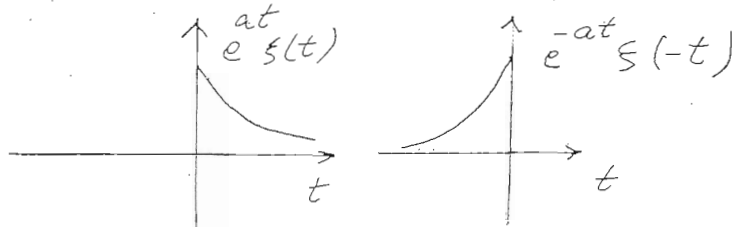
$$= \begin{cases} t e^{at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

d) Find the following convolution

(Extra problem:)
 $e^{at} \xi(t) * e^{-at} \xi(t)$

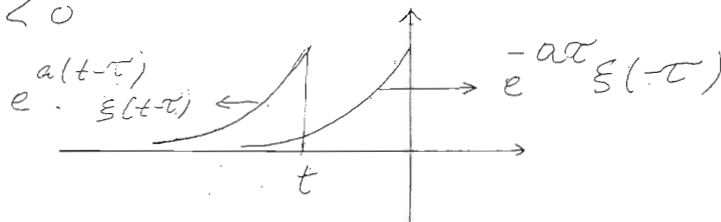
Note: The parameter a must be negative otherwise the convolution integral will not converge.

$a < 0$



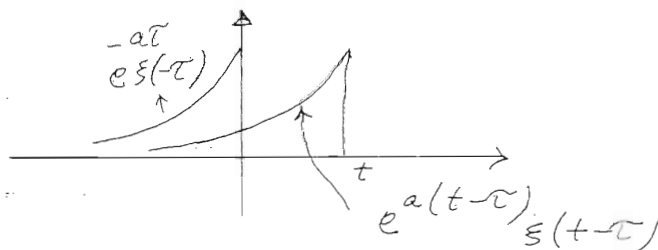
$$y(t) = e^{at} \xi(t) * e^{-at} \xi(-t) = \int_{-\infty}^{+\infty} e^{a(t-\tau)} \xi(t-\tau) \cdot e^{-a\tau} \xi(-\tau) d\tau$$

1) for $t < 0$



$$y(t) = \int_{-\infty}^t e^{a(t-\tau)} e^{-a\tau} d\tau = \frac{-e^{-at}}{2a}$$

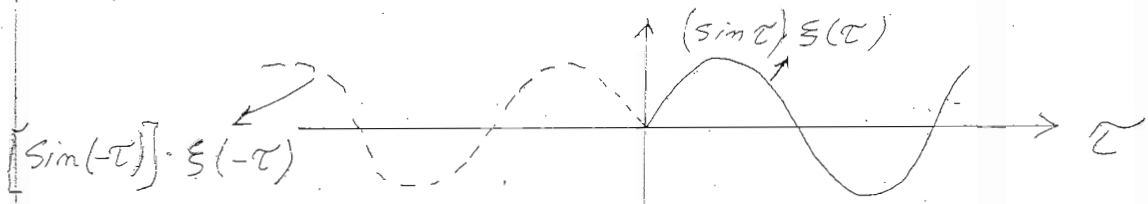
2) for $t \geq 0$



$$y(t) = \int_{-\infty}^0 e^{a(t-\tau)} e^{-a\tau} d\tau = -\frac{e^{at}}{2a}$$

Thus: $y(t) = \begin{cases} -\frac{e^{-at}}{2a} & t < 0 \\ -\frac{e^{at}}{2a} & t \geq 0 \end{cases} \Rightarrow y(t) = -\frac{e^{a|t|}}{2a}$

c) $y(t) = (\sin t) \cdot \xi(t) * \sin t \xi(t)$



$$y(t) = 0 \quad \text{for } t \leq 0$$

$$y(t) = \int_0^t \sin \tau \cdot \sin(t-\tau) d\tau \quad \text{for } t > 0$$

$$= \frac{1}{2} \sin t - \frac{1}{2} t \cos t$$

$$y(t) = \int_0^t \frac{1}{2} [\cos(t-\tau-\tau) - \cos(t-\tau+\tau)] d\tau$$

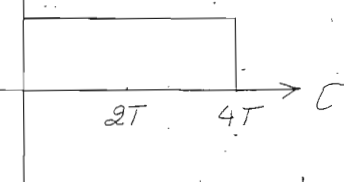
$$= \frac{1}{2} \int_0^t [\cos(t-2\tau) - \cos t] d\tau = \frac{1}{2} \left[-\frac{1}{2} \sin(t-2\tau) - \tau \cos t \right]_0^t$$

$$= \frac{1}{2} \left[-\frac{1}{2} \sin(t-2t) + \frac{1}{2} \sin(t) - t \cos t \right] = \frac{1}{2} [\sin t - t \cos t]$$

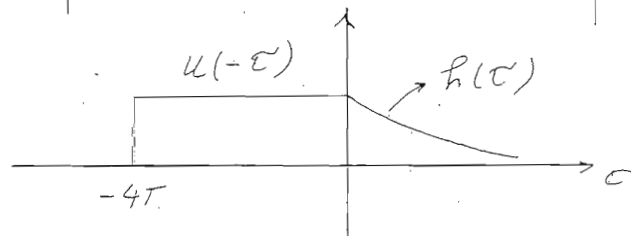
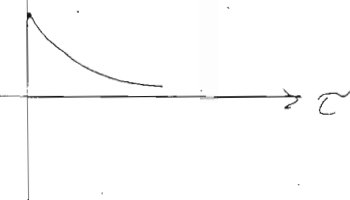
f.)
Given $u(t)$ and
 $h(t)$ find

$$y(t) = u(t) \otimes h(t)$$

$$u(\tau) = \text{rect}\left(\frac{\tau - 2T}{4T}\right)$$



$$h(\tau) = e^{-3\tau} \zeta(\tau)$$

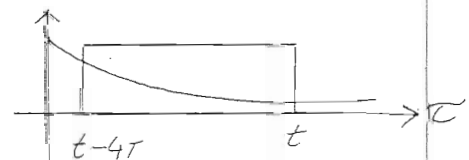


a) For $t < 0 \Rightarrow y(t) = 0$

b) For $0 \leq t < 4T$

$$y(t) = \int_0^t 1 \cdot e^{-3\tau} d\tau = -\frac{e^{-3\tau}}{3} \Big|_0^t = \frac{1 - e^{-3t}}{3}$$

c) For $t \geq 4T$



$$y(t) = \int_{t-4T}^t e^{-3\tau} d\tau = \frac{1}{3} \left(e^{-3t+12T} - e^{-3t} \right)$$

$$= \frac{e^{-3t}}{3} (e^{12T} - 1)$$

From a, b, c we have

8 my graded HW

HW2, EGEE 443. CSUF, Fall 2008

Nasser Abbasi

September 18, 2008

1 Problem 2.30

19
20

Problem

Determine and sketch the autocorrelation function of the following

(b) $g(t) = e^{-a|t|}$

(c) $g(t) = e^{-at}u(t) - e^{at}u(-t)$

1.1 part(b)

$$g(t) = \begin{cases} e^{-at} & t > 0 \\ 1 & t = 0 \\ e^{at} & t < 0 \end{cases}$$

Assume $a > 0$ for the integral to be defined. From definition, autocorrelation of a function $g(t)$ is

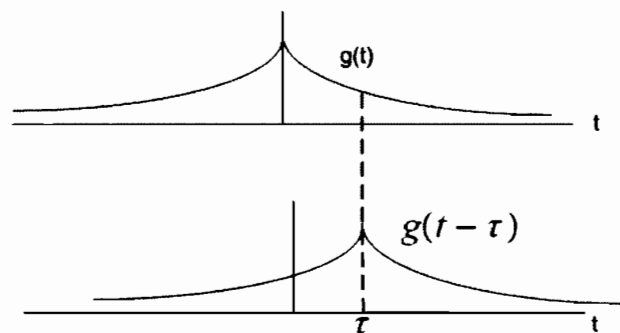
$$R(\tau) = \int_{-\infty}^{\infty} g(t) g^*(t - \tau) dt$$

Since $g(t)$ in this case is real, then $g^*(t - \tau) = g(t - \tau)$, hence

$$R(\tau) = \int_{-\infty}^{\infty} g(t) g(t - \tau) dt$$

Consider the 3 cases, $\tau < 0$ and $\tau > 0$ and when $\tau = 0$

case $\tau > 0$



Break the integral over the 3 regions, $\{-\infty, 0\}$, $\{0, \tau\}$, $\{\tau, \infty\}$

$$R(\tau) = \int_{-\infty}^0 e^{at} e^{a(t-\tau)} dt + \int_0^{\tau} e^{-at} e^{a(t-\tau)} dt + \int_{\tau}^{\infty} e^{-at} e^{-a(t-\tau)} dt$$

But $\int_{-\infty}^0 e^{at} e^{a(t-\tau)} dt = e^{-a\tau} \int_{-\infty}^0 e^{2at} dt = e^{-a\tau} \left[\frac{e^{2at}}{2a} \right]_{-\infty}^0 = e^{-a\tau} \frac{[1-0]}{2a} = \frac{e^{-a\tau}}{2a}$

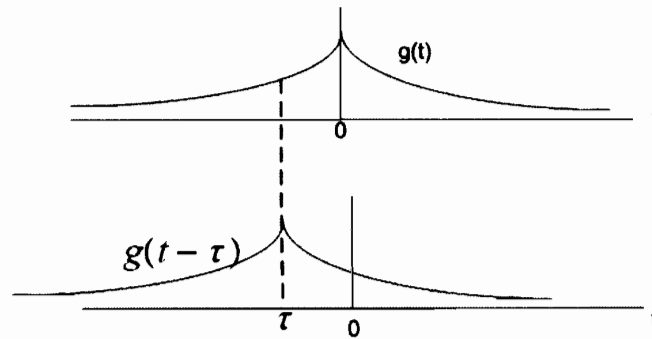
$$\text{and } \int_0^\tau e^{-at} e^{a(t-\tau)} dt = e^{-a\tau} \int_0^\tau 1 dt = \tau e^{-a\tau}$$

$$\text{and } \int_\tau^\infty e^{-at} e^{-a(t-\tau)} dt = e^{a\tau} \int_\tau^\infty e^{-2at} dt = e^{a\tau} \left[\frac{e^{-2at}}{-2a} \right]_\tau^\infty = e^{a\tau} \left[\frac{0 - e^{-2a\tau}}{-2a} \right] = \frac{e^{-a\tau}}{2a}$$

Hence for $\tau > 0$ we obtain

$$\begin{aligned} R(\tau) &= \frac{e^{-a\tau}}{2a} + \tau e^{-a\tau} + \frac{e^{-a\tau}}{2a} \\ &= \frac{e^{-a\tau}}{a} + \tau e^{-a\tau} \\ &= e^{-a\tau} \left(\frac{1}{a} + \tau \right) \end{aligned}$$

case $\tau < 0$



Break the integral over the 3 regions, $\{-\infty, \tau\}, \{\tau, 0\}, \{0, \infty\}$

$$R(\tau) = \int_{-\infty}^\tau e^{at} e^{a(t-\tau)} dt + \int_\tau^0 e^{-at} e^{a(t-\tau)} dt + \int_0^\infty e^{-at} e^{-a(t-\tau)} dt$$

$$\text{Now } \int_{-\infty}^\tau e^{at} e^{a(t-\tau)} dt = e^{-a\tau} \int_{-\infty}^\tau e^{2at} dt = e^{-a\tau} \left[\frac{e^{2at}}{2a} \right]_{-\infty}^\tau = e^{-a\tau} \left[\frac{e^{2a\tau} - 0}{2a} \right] = \frac{e^{a\tau}}{2a}$$

$$\text{and } \int_\tau^0 e^{-at} e^{a(t-\tau)} dt = e^{-a\tau} \int_\tau^0 1 dt = -\tau e^{-a\tau}$$

$$\text{and } \int_0^\infty e^{-at} e^{-a(t-\tau)} dt = e^{a\tau} \left[\frac{e^{-2at}}{-2a} \right]_0^\infty = \frac{e^{a\tau}}{-2a} (0 - 1) = \frac{e^{a\tau}}{2a}$$

Hence

$$\begin{aligned} R(\tau) &= \frac{e^{a\tau}}{2a} - \tau e^{-a\tau} + \frac{e^{a\tau}}{2a} \\ &= e^{a\tau} \left(\frac{1}{a} - \tau \right) \end{aligned}$$

When $\tau = 0$

$R(0)$ gives the the maximum power in the signal $g(t)$. Now evaluate this

$$\begin{aligned} R(\tau) &= \int_{-\infty}^0 e^{at} e^{at} dt + \int_0^{\infty} e^{-at} e^{-at} dt \\ &= \frac{[e^{2at}]_{-\infty}^0}{2a} + \frac{[e^{-2at}]_0^{\infty}}{-2a} \\ &= \frac{1}{a} \end{aligned}$$

Hence

$$R(\tau) = \begin{cases} e^{-a\tau} \left(\frac{1}{a} + \tau \right) & \tau > 0 \\ \frac{1}{a} & \tau = 0 \\ e^{a\tau} \left(\frac{1}{a} - \tau \right) & \tau < 0 \end{cases}$$

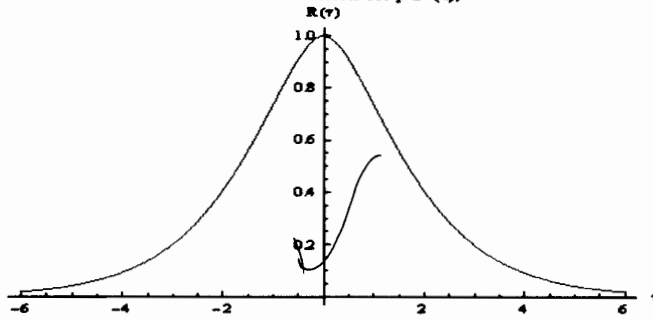
Or we could write

$R(\tau) = e^{-| \tau | a} \left(\frac{1}{a} - (-|\tau|) \right)$

This is a plot of $R(\tau)$, first plot is for $a = 1$ and the second for $a = 4$

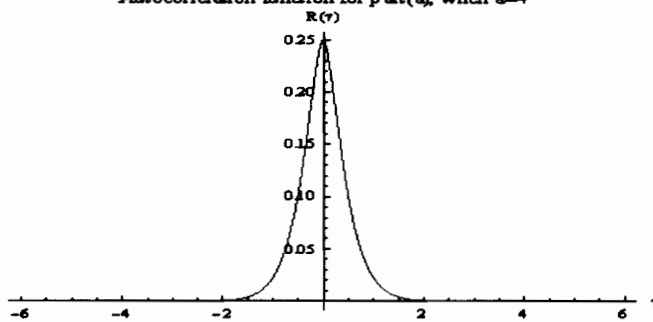
```
f[t_] := Exp[a (-Abs[t])]  $\left( \frac{1 + a (-Abs[t])}{a} \right)$ 
Show[{Plot[f[t] /. a -> 1, {t, -6, 6}, PlotRange -> All, PlotStyle -> Red,
  AxesOrigin -> {0, 0}], AxesLabel -> {"x", "R(x)"},
  PlotLabel -> "Autocorrelation function for part(a), when a=1"}]
```

Out[18]=



```
In[21]:= Show[{Plot[f[t] /. a -> 4, {t, -6, 6}, PlotRange -> All, PlotStyle -> Red,
  AxesOrigin -> {0, 0}], AxesLabel -> {"x", "R(x)"},
  PlotLabel -> "Autocorrelation function for part(a), when a=4"}]
```

Out[21]=



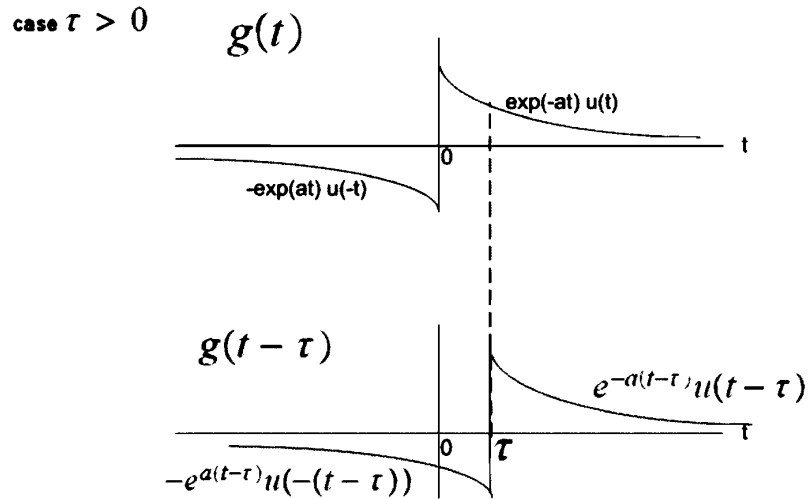
1.2 part(c)

$$g(t) = e^{-at}u(t) - e^{at}u(-t)$$

Assume $a > 0$.

Consider the 3 cases, $\tau < 0$ and $\tau > 0$ and when $\tau = 0$

case $\tau > 0$

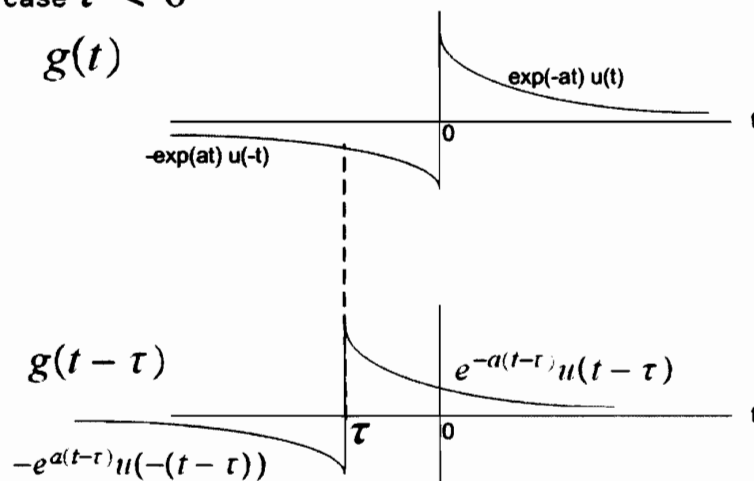


Break the integral into 3 parts, $\{-\infty, 0\}$, $\{0, \tau\}$, $\{\tau, \infty\}$

$$\begin{aligned}
 R(\tau) &= \int_{-\infty}^0 g(t)g(t-\tau)dt + \int_0^{\tau} g(t)g(t-\tau)dt + \int_{\tau}^{\infty} g(t)g(t-\tau)dt \\
 &= \int_{-\infty}^0 -e^{at}(-e^{a(t-\tau)})dt + \int_0^{\tau} e^{-at}(-e^{a(t-\tau)})dt + \int_{\tau}^{\infty} e^{-at}(e^{-a(t-\tau)})dt \\
 &= e^{-a\tau} \int_{-\infty}^0 e^{2at}dt - e^{-a\tau} \int_0^{\tau} 1dt + e^{a\tau} \int_{\tau}^{\infty} e^{-2at}dt \\
 &= e^{-a\tau} \frac{[e^{2at}]_{-\infty}^0}{2a} - \tau e^{-a\tau} + e^{a\tau} \frac{[e^{-2at}]_{\tau}^{\infty}}{-2a} \\
 &= e^{-a\tau} \frac{[1-0]}{2a} - \tau e^{-a\tau} + e^{a\tau} \frac{[0-e^{-2a\tau}]}{-2a} \\
 &= \frac{e^{-a\tau}}{2a} - \tau e^{-a\tau} + \frac{e^{-a\tau}}{2a} \\
 &= e^{-a\tau} \left(\frac{1}{2a} - \tau + \frac{1}{2a} \right) \\
 &= e^{-a\tau} \left(\frac{1}{a} - \tau \right)
 \end{aligned}$$

case $\tau < 0$

case $\tau < 0$



Break the integral into 3 parts, $\{-\infty, \tau\}, \{\tau, 0\}, \{0, \infty\}$

$$\begin{aligned}
 R(\tau) &= \int_{-\infty}^{\tau} g(t) g(t-\tau) dt + \int_{\tau}^0 g(t) g(t-\tau) dt + \int_0^{\infty} g(t) g(t-\tau) dt \\
 &= \int_{-\infty}^{\tau} -e^{at} (-e^{a(t-\tau)}) dt + \int_{\tau}^0 -e^{at} e^{-a(t-\tau)} dt + \int_0^{\infty} e^{-at} e^{-a(t-\tau)} dt \\
 &= e^{-a\tau} \int_{-\infty}^{\tau} e^{2at} dt - e^{a\tau} \int_{\tau}^0 1 dt + e^{a\tau} \int_0^{\infty} e^{-2at} dt \\
 &= e^{-a\tau} \frac{[e^{2at}]_{-\infty}^{\tau}}{2a} + \tau e^{a\tau} + e^{a\tau} \frac{[e^{-2at}]_0^{\infty}}{-2a} \\
 &= e^{-a\tau} \frac{[e^{2a\tau} - 0]}{2a} + \tau e^{a\tau} + e^{a\tau} \frac{[0 - 1]}{-2a} \\
 &= \frac{e^{a\tau}}{2a} + \tau e^{a\tau} + \frac{e^{a\tau}}{2a} \\
 &= e^{a\tau} \left(\frac{1}{a} + \tau \right)
 \end{aligned}$$

At $\tau = 0$, we see that $R(0) = \frac{1}{a}$, hence the final answer is

$$R(\tau) = \begin{cases} e^{-a\tau} \left(\frac{1}{a} - \tau \right) & \tau > 0 \\ \frac{1}{a} & \tau = 0 \\ e^{a\tau} \left(\frac{1}{a} + \tau \right) & \tau < 0 \end{cases}$$

Or we could write

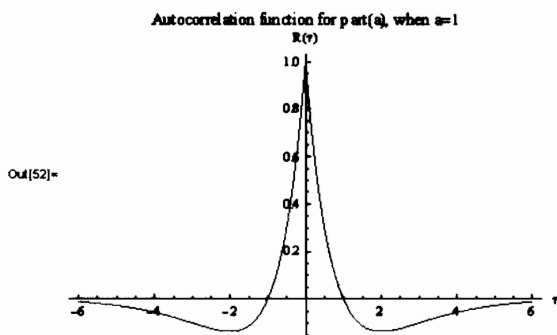
$$R(\tau) = e^{-|\tau|a} \left(\frac{1}{a} - |\tau| \right)$$

This is a plot of $R(\tau)$, first plot is for $a = 1$ and the second for $a = 4$

```

In[51]:= f[t_] := Exp[a (-Sign[t] t)] (1/a - Sign[t] t)
Show[(Plot[f[t] /. a -> 1, {t, -6, 6}, PlotRange -> All, PlotStyle -> Red, AxesOrigin -> {0, 0}]),
  AxesLabel -> {"τ", "R(τ)"}, PlotLabel -> "Autocorrelation function for part(a), when a=1"]

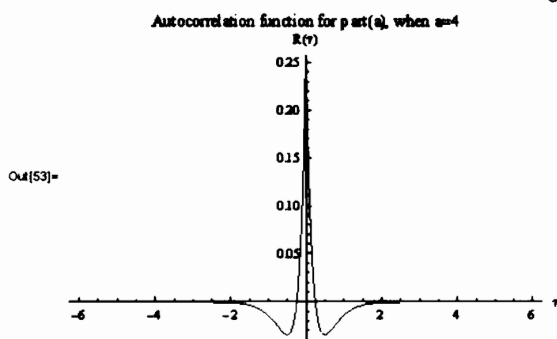
```



```

In[53]:= Show[(Plot[f[t] /. a -> 4, {t, -6, 6}, PlotRange -> All, PlotStyle -> Red, AxesOrigin -> {0, 0}]),
  AxesLabel -> {"τ", "R(τ)"}, PlotLabel -> "Autocorrelation function for part(a), when a=4"]

```



2 Problem 2.32

problem: Determine the autocorrelation function of $g(t) = A \operatorname{sinc}(2Wt)$ and sketch it

solution:

$$R(\tau) = \int_{-\infty}^{\infty} g(t) g^*(t - \tau) dt$$

The above is difficult to do directly, hence we use the second method.

Since the function $g(t)$ is an energy function, hence $R(\tau)$ and the energy spectrum density $\Psi_g(f)$ of $g(t)$ make a Fourier transform pairs.

$$R(\tau) \Leftrightarrow \Psi_g(f)$$

Therefore, to find $R(\tau)$, we first find $\Psi_g(f)$, then find the Inverse Fourier Transform of $\Psi_g(f)$, i.e.

$$R(\tau) = F^{-1}(\Psi_g(f)) \quad (1)$$

But

$$\Psi_g(f) = |G(f)|^2 \quad (2)$$

and we know that

$$A \operatorname{sinc}(2Wt) \Leftrightarrow \frac{A}{2W} \operatorname{rect}\left(\frac{f}{2W}\right)$$

Hence

$$G(f) = \frac{A}{2W} \operatorname{rect}\left(\frac{f}{2W}\right)$$

The (2) becomes

$$\begin{aligned} \Psi_g(f) &= \left| \frac{A}{2W} \operatorname{rect}\left(\frac{f}{2W}\right) \right|^2 \\ &= \left(\frac{A}{2W} \right)^2 \left| \operatorname{rect}\left(\frac{f}{2W}\right) \right|^2 \end{aligned}$$

But $\left| \operatorname{rect}\left(\frac{f}{2W}\right) \right|^2 = \operatorname{rect}\left(\frac{f}{2W}\right)$, since it has height of 1, so

$$\boxed{\Psi_g(f) = \left(\frac{A}{2W} \right)^2 \operatorname{rect}\left(\frac{f}{2W}\right)}$$

Hence from (1)

$$R(\tau) = F^{-1} \left(\left(\frac{A}{2W} \right)^2 \text{rect} \left(\frac{f}{2W} \right) \right)$$

$$= \left(\frac{A}{2W} \right)^2 F^{-1} \left[\text{rect} \left(\frac{f}{2W} \right) \right]$$

Hence

$$R(\tau) = \left(\frac{A}{2W} \right)^2 \text{sinc}(2W\tau)$$

This is a plot of the above function, for $W = 4$, and $A = 1$

In[9]:=

A = 1.

W = 2

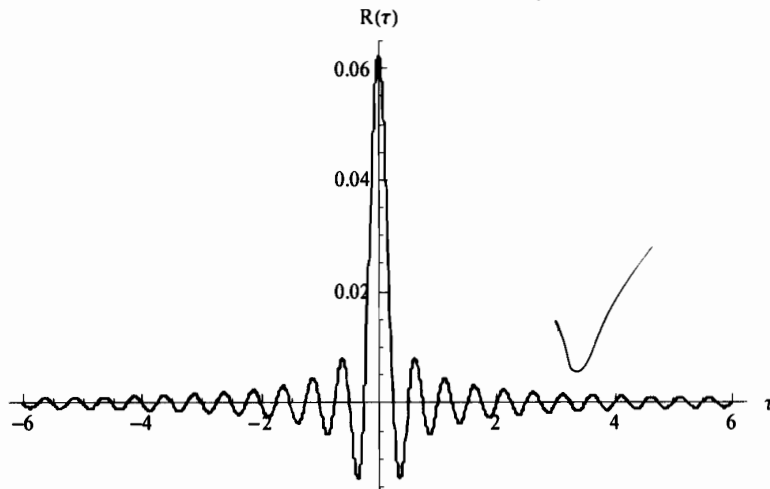
f[t_] := $\left(\frac{A}{2W} \right)^2 \text{Sinc}[\text{Pi } 2W t]$

**Show[{Plot[f[t], {t, -6, 6}, PlotRange -> All, PlotStyle -> Red, AxesOrigin -> {0, 0}},
AxesLabel -> {"τ", "R(τ)"}, PlotLabel -> "Autocorrelation function when A=1,W=4"]**

Out[9]= 1.

Out[10]= 2

Autocorrelation function when A=1,W=4



Out[12]=

3 Problem 2.33

The Fourier transform of a signal is defined by $|\text{sinc}(f)|$. Show that $R(\tau)$ of the signal is triangular in form.

Answer:

Since

$$R(\tau) \Leftrightarrow |G(f)|^2$$

Then

$$\begin{aligned} R(\tau) &\Leftrightarrow |\text{sinc}(f)|^2 \\ &\Leftrightarrow \text{sinc}^2(f) \end{aligned}$$

Hence to find $R(\tau)$ we need to find the inverse Fourier transform of $\text{sinc}^2(f)$

But

$$\begin{aligned} F^{-1}(\text{sinc}^2(f)) &= F^{-1}(\text{sinc}(f) \times \text{sinc}(f)) \\ &= F^{-1}\{\text{sinc}(f)\} \otimes F^{-1}\{\text{sinc}(f)\} \end{aligned}$$

But $F^{-1}\{\text{sinc}(f)\} = \text{rect}(t)$, hence

$$\begin{aligned} F^{-1}(\text{sinc}^2(f)) &= \text{rect}(t) \otimes \text{rect}(t) \\ &= \int_{-\infty}^{\infty} \text{rect}(\tau) \text{rect}(t - \tau) d\tau \end{aligned}$$

This integral has the value of $\text{tri}(t)$ (we also did this in class) Hence

$$\text{tri}(\tau) \Leftrightarrow \text{sinc}^2(f)$$

Hence

$$R(\tau) = \text{tri}(\tau)$$

Where $\text{tri}(\tau)$ is the triangle function, defined as

$$\text{tri}(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

4 Problem 2.35

Consider the signal $g(t)$ defined by

$$g(t) = A_0 + A_1 \cos(2\pi f_1 t + \theta) + A_2 \cos(2\pi f_2 t + \theta)$$

- (a) determine $R(\tau)$
- (b) what is $R(0)$
- (c) has any information been lost in obtaining $R(\tau)$?

Answer:

- (a)

Take the Fourier transform of $g(t)$ we obtain

$$G(f) = A_0 \delta(f) + \frac{A_1}{2} [e^{j\theta} \delta(f - f_1) + e^{-j\theta} \delta(f + f_1)] + \frac{A_2}{2} [e^{j\theta} \delta(f - f_2) + e^{-j\theta} \delta(f + f_2)]$$

Hence $|G(f)|^2 = G(f) G^*(f)$, so we need to find $G^*(f)$

$$G^*(f) = A_0 \delta(f) + \frac{A_1}{2} [e^{-j\theta} \delta(f - f_1) + e^{j\theta} \delta(f + f_1)] + \frac{A_2}{2} [e^{-j\theta} \delta(f - f_2) + e^{j\theta} \delta(f + f_2)]$$

So

$$G(f) G^*(f) = A_0^2 \delta(f) + \frac{A_1^2}{4} [\delta(f - f_1) + \delta(f + f_1)] + \frac{A_2^2}{4} [\delta(f - f_2) + \delta(f + f_2)]$$

So

$$S_g(f) = A_0^2 \delta(f) + \frac{A_1^2}{4} [\delta(f - f_1) + \delta(f + f_1)] + \frac{A_2^2}{4} [\delta(f - f_2) + \delta(f + f_2)]$$

So

$$\begin{aligned} R(\tau) &= F^{-1}(S_g(f)) \\ &= F^{-1}(A_0^2 \delta(f)) + \frac{A_1^2}{4} F^{-1}[\delta(f - f_1) + \delta(f + f_1)] + \frac{A_2^2}{4} F^{-1}[\delta(f - f_2) + \delta(f + f_2)] \end{aligned}$$

Hence

$$\boxed{R(\tau) = A_0^2 + \frac{A_1^2}{2} \cos 2\pi f_1 \tau + \frac{A_2^2}{2} \cos 2\pi f_2 \tau} \quad (1)$$

Part (b)

$$\begin{aligned} R(0) &= A_0^2 + \frac{A_1^2}{2} + \frac{A_2^2}{2} \\ &= \frac{1}{2} (2A_0^2 + A_1^2 + A_2^2) \end{aligned}$$

part(c)

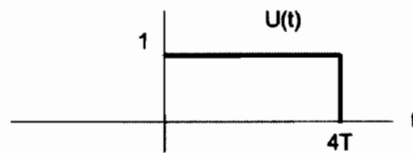
In obtaining $R(\tau)$ we have lost the phase information in the original signal as can be seen from (1) above

5 extra Problem

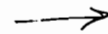
(a) find $\xi(t) \otimes \xi(t)$ where $\xi(t)$ is unit step function

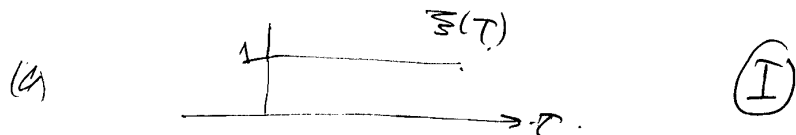
(b) Find $t\xi(t) \otimes e^{at}\xi(t)$ where $a > 0$

(c) find $u(t) \otimes h(t)$ where $h(t) = e^{-3t}u(t)$ and $u(t)$ is as shown



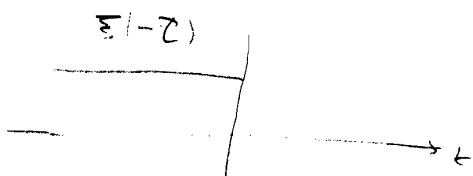
Answer



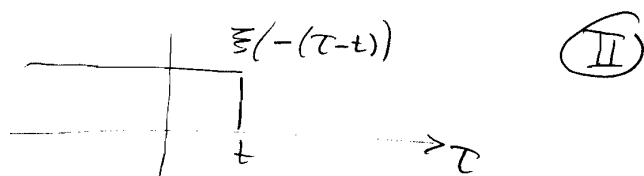


$$\xi(t) \otimes \xi(\tau) = \int_{-\infty}^{\infty} \xi(\tau) \xi(t-\tau) d\tau$$

first flip $\xi(\tau)$ to be $\xi(-\tau)$

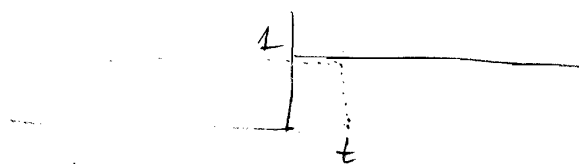


now shift to right



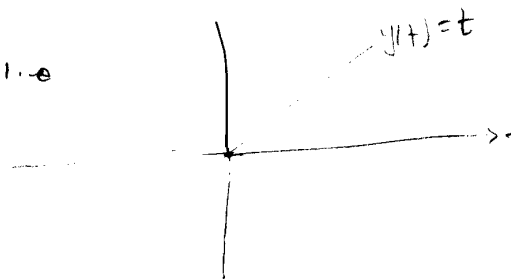
now integrate (I) with (II). we see that we just need to integrate the following.

for $t < 0$, product is ~~zero~~



$$\int_0^t \xi(\tau) \xi(t-\tau) d\tau = \int_0^t 1 \cdot d\tau = \boxed{t}$$

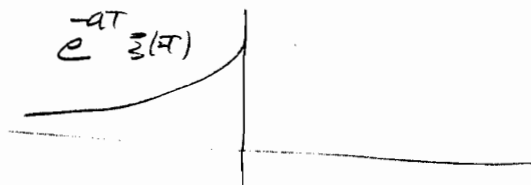
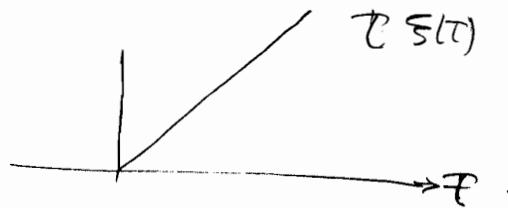
so $y(t) = t$ i.e.



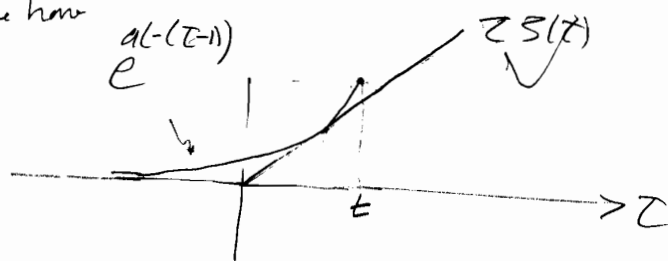
$t > 0$
only

(b) $t \mathcal{L}\{f(t)\} \otimes e^{at} \mathcal{L}\{f(t)\}$.

Q. 40



here we have



only need to do $t > 0$

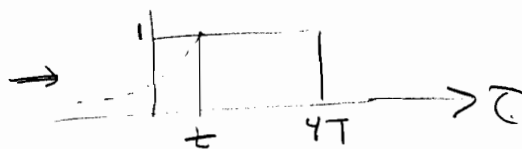
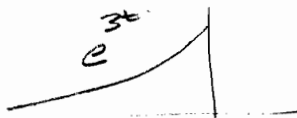
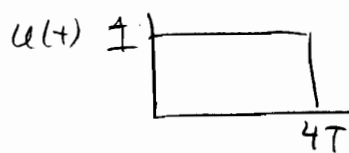
$$\text{so } \int_t^\infty \tau e^{a(t-\tau)} d\tau = \int_t^\infty \tau e^{at} e^{-a\tau} d\tau.$$

$$= e^{at} \int_t^\infty \tau e^{-a\tau} d\tau \quad \text{integrate by parts.}$$

$$\begin{aligned} \int_t^\infty \tau e^{-a\tau} d\tau &= \frac{1}{-a} [\tau e^{-a\tau}]_t^\infty - \int_t^\infty \frac{e^{-a\tau}}{-a} d\tau = \left[\frac{0 - te^{-at}}{-a} \right] + \frac{1}{a} \left[\frac{e^{-a\tau}}{-a} \right]_t^\infty \\ &= \frac{te^{-at}}{a} - \frac{1}{a^2} [0 - e^{-at}] = \frac{te^{-at}}{a} - \frac{1}{a^2} [-e^{-at}] = \frac{te^{-at}}{a} + \frac{e^{-at}}{a^2} \\ &= e^{-at} \left(\frac{t}{a} + \frac{1}{a^2} \right) = e^{-at} \left(\frac{at+1}{a^2} \right) \end{aligned}$$

$$\text{so } y(t) = e^{at} \left[e^{-at} \left(\frac{at+1}{a^2} \right) \right] = \boxed{\frac{at+1}{a^2}} \times \text{see sol, } t > 0$$

∴ (c) $u(t) \otimes h(t)$



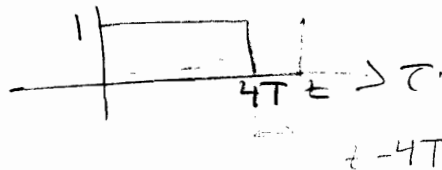
For $t < 0$, no product

for $0 \leq t \leq 4T$

$$\int_0^t 1 \cdot e^{-3(t-\tau)} d\tau$$

$$\begin{aligned} &= e^{-3t} \int_0^t e^{3\tau} d\tau = e^{-3t} \left[\frac{e^{3\tau}}{3} \right]_0^t \\ &= \left(\frac{e^{3t} - e^0}{3} \right) e^{-3t} = \left(\frac{e^{3t} - 1}{3} \right) e^{-3t} \\ &= \boxed{\frac{1 - e^{-3t}}{3}} \quad 0 \leq t \leq 4T \end{aligned}$$

Case $t > 4T$



$$y(t) = \int_0^{4T} 1 \cdot e^{-3(t-\tau)} d\tau$$

$$= e^{-3t} \int_0^{4T} e^{3\tau} d\tau = e^{-3t} \left[\frac{e^{3\tau}}{3} \right]_0^{4T}$$

$$= \frac{e^{-3t}}{3} [e^{12T} - e^0] = \boxed{\frac{e^{-3t}}{3} (e^{12T} - 1)} \quad t > 4T$$