HW 1
EGEE 443, Electronic Communication Systems
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California State University, Fullerson

Nasser M. Abbasi

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Contents

1 Questions

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due Thursday.

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Problem 2.1 ⟹ 2.19 in new Book.

(a) Find the Fourier transform of the half-cosine pulse shown in Fig. P2.4(*a*).
(b) Apply the time-shifting property to the result obtained in part (a) to evaluate the spectrum of the half-sine pulse shown in Fig. P2.4(*b*).
(c) What is the spectrum of a half-sine pulse having a duration equal to aT?
(d) What is the spectrum of the negative half-sine pulse shown in Fig. P2.4(*c*)?
(e) Find the spectrum of the single sine pulse shown in Fig. P2.4(*d*).

Hint: $g(t) = A\cos\left(\frac{\pi t}{T}\right) \cdot \mathrm{rect}\left(\frac{t}{T}\right)$

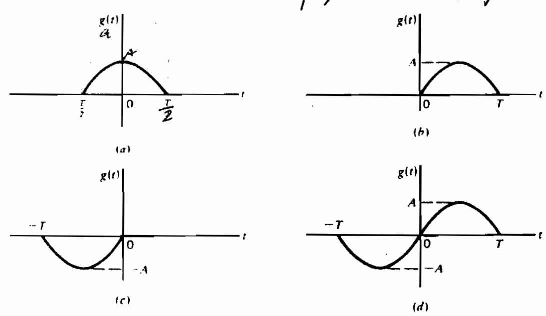

Figure P2.4

Prob. # 2.2

Given $g(t) = \exp(-t)\sin(2\pi f_c t)\,u(t)$. Find the Fourier Transform of $g(t)$. $F.T[g(t)] = ?$

⟹ 2.20 in new Book.

Problem 2.3 Any function $g(t)$ can be split unambiguously into an *even part* and an *odd part*, as shown by

$$g(t) = g_e(t) + g_o(t) \quad \Rightarrow \quad g(t) = g_e(t) + g_o(t)$$

The even part is defined by

$$g_e(t) = \tfrac{1}{2}[g(t) + g(-t)]$$

and the odd part is defined by

$$g_o(t) = \tfrac{1}{2}[g(t) - g(-t)]$$

(a) Evaluate the even and odd parts of a rectangular pulse defined by

$$g(t) = A\,\mathrm{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$$

(That is find F.T. of $g_e(t)$ & $g_o(t)$)

(b) What are the Fourier transforms of these two parts of the pulse?

2.4

Problem Determine the inverse Fourier transform of the frequency function $G(f)$ defined by the amplitude and phase spectra shown in Fig. P .

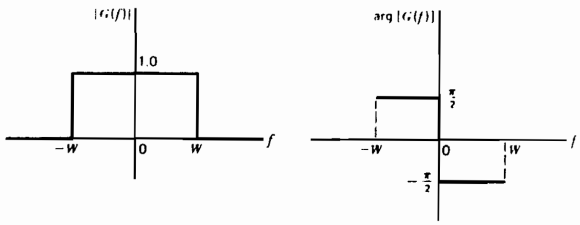

Figure P2.5

2 Problem 2.1

2.1 part(a)

Let $\mathcal{F}\left(g\left(t\right)\right)$ be the Fourier Transform of $g\left(t\right)$, i.e. $\mathcal{F}\left(g\left(t\right)\right)=G\left(f\right)$. First we use the given hint and note that $g\left(t\right)$ can be written as follows

$$g\left(t\right)=A\cos\left(\frac{\pi t}{T}\right)\ rect\left(\frac{t}{T}\right)$$

Start by writing $\frac{\pi t}{T}$as $2\pi f_0 t$, where $f_0=\frac{1}{2T}$. Now using the property that multiplication in time domain is the same as convolution in frequency domain, we obtain

$$G\left(f\right)=\mathcal{F}\left(A\cos\left(2\pi f_0 t\right)\right)\otimes\mathcal{F}\left(rect\left(\frac{t}{T}\right)\right) \tag{1}$$

But

$$\begin{aligned}\mathcal{F}\left(A\cos\left(2\pi f_0 t\right)\right)&=A\ \mathcal{F}\left(\cos\left(2\pi f_0 t\right)\right)\\&=A\ \mathcal{F}\left(\frac{e^{j2\pi f_0 t}+e^{-j2\pi f_0 t}}{2}\right)\\&=\frac{A}{2}\ \mathcal{F}\left(e^{j2\pi f_0 t}+e^{-j2\pi f_0 t}\right)\\&=\frac{A}{2}\left[\ \mathcal{F}\left(e^{j2\pi f_0 t}\right)+\ \mathcal{F}\left(e^{-j2\pi f_0 t}\right)\right]\end{aligned}$$

But $\mathcal{F}\left(e^{j2\pi f_0 t}\right)=\delta\left(f-f_0\right)$ and $\mathcal{F}\left(e^{-j2\pi f_0 t}\right)=\delta\left(f+f_0\right)$hence the above becomes

$$\mathcal{F}\left(A\cos\left(2\pi f_0 t\right)\right)=\frac{A}{2}\left[\ \delta\left(f-f_0\right)+\ \delta\left(f+f_0\right)\right] \tag{2}$$

Substitute (2) into (1) we obtain

$$G\left(f\right)=\frac{A}{2}\left[\ \delta\left(f-f_0\right)+\ \delta\left(f+f_0\right)\right]\otimes\mathcal{F}\left(rect\left(\frac{t}{T}\right)\right)$$

But $\mathcal{F}\left(rect\left(\frac{t}{T}\right)\right)=T\operatorname{sinc}\left(fT\right)$, hence the above becomes

$$\mathcal{F}\left(g\left(t\right)\right)=\frac{A}{2}\left[\ \delta\left(f-f_0\right)+\ \delta\left(f+f_0\right)\right]\otimes T\operatorname{sinc}\left(fT\right)$$

Now using the property of convolution with a delta, we obtain

$$\boxed{G\left(f\right)=\frac{AT}{2}\left[\ \operatorname{sinc}\left(\left(f-f_0\right)T\right)+\ \operatorname{sinc}\left(\left(f+f_0\right)T\right)\right]}$$

note: by doing more trigonometric manipulations, the above can be written as

$$\boxed{G\left(f\right)=\frac{2AT\cos(\pi fT)}{\pi(1-4f^2T^2)}}$$

2.2 part(b)

Apply the time shifting property $g(t) \Longleftrightarrow G(f)$, hence $g(t-t_0) \Longleftrightarrow e^{-j2\pi f t_0} G(f)$

From part(a) we found that $F(g(t)) = \frac{AT}{2}[\text{ sinc}((f-f_0)T) + \text{ sinc}((f+f_0)T)]$, so in this part, the function in part(a) is shifted in time to the right by amount $\frac{T}{2}$, let the new function be $h(t)$,hence we need to multiply $G(f)$ by $e^{-j2\pi f\frac{T}{2}}$,hence

$$\begin{aligned} F\left(g\left(t-\frac{T}{2}\right)\right) &= F(h(t)) \\ &= H(f) \\ &= e^{-j\pi f T}\left(\frac{AT}{2}[\text{ sinc}((f-f_0)T) + \text{ sinc}((f+f_0)T)]\right) \end{aligned}$$

2.3 part(c)

Using the time scaling property $g(t) \Longleftrightarrow G(f)$, hence $g(at) \Longleftrightarrow \frac{1}{|a|}G\left(\frac{f}{a}\right)$, and since we found in part(b) that $H(f) = e^{-j\pi f T}\left(\frac{AT}{2}[\text{ sinc}((f-f_0)T) + \text{ sinc}((f+f_0)T)]\right)$, hence

$$\boxed{F\{h(at)\} = \frac{1}{|a|}e^{-j\pi\frac{f}{a}T}\left(\frac{AT}{2}\left[\text{ sinc}\left(\left(\frac{f}{a}-f_0\right)T\right) + \text{ sinc}\left(\left(\frac{f}{a}+f_0\right)T\right)\right]\right)}$$

2.4 part(d)

Let $f(t)$ be the function which is shown in figure 2.4c, we see that

$$f(t) = -h(-t)$$

where $h(t)$ is the function shown in figure 2.4(b). We found in part(b) that

$$H(f) = e^{-j\pi f T}\left(\frac{AT}{2}[\text{ sinc}((f-f_0)T) + \text{ sinc}((f+f_0)T)]\right)$$

Now using the property that $h(t) \Longleftrightarrow H(f)$ then $h(-t) \Longleftrightarrow \frac{1}{|-1|}H(-f) = H(-f)$, hence

$$\boxed{F\{f(t)\} = -e^{j\pi f T}\left(\frac{AT}{2}[\text{ sinc}((-f-f_0)T) + \text{ sinc}((-f+f_0)T)]\right)}$$

2.5 part(e)

This function, call it $g_1(t)$, is the sum of the functions shown in figure 2.4(b) and figure 2.4(c), then the Fourier transform of $g_1(t)$ is the sum of the Fourier transforms of the functions in these two figures (using the linearity of the Fourier transforms). Hence

$$\begin{aligned} F(g_1(t)) = e^{-j\pi f T}&\left(\frac{AT}{2}[\text{ sinc}((f-f_0)T) + \text{ sinc}((f+f_0)T)]\right) \\ &- e^{j\pi f T}\left(\frac{AT}{2}[\text{ sinc}((-f-f_0)T) + \text{ sinc}((-f+f_0)T)]\right) \end{aligned}$$

The above can be simplified to

$$\begin{aligned}\mathcal{F}(g_1(t)) &= \frac{AT}{2}\left(\operatorname{sinc}((f+f_0)T)\left[e^{j\pi fT}+\ e^{-j\pi fT}\right]+\operatorname{sinc}((f-f_0)T)\left[e^{j\pi fT}+e^{-j\pi fT}\right]\right)\\ &= \frac{AT}{2}\left(\operatorname{sinc}((f+f_0)T)\left[2\cos(\pi fT)\right]+\operatorname{sinc}((f-f_0)T)\left[2\cos(\pi fT)\right]\right)\end{aligned}$$

Hence

$$\boxed{\mathcal{F}(g_1(t)) = AT\cos(\pi fT)\left[\operatorname{sinc}((f+f_0)T)+\operatorname{sinc}((f-f_0)T)\right]}$$

3 Problem 2.2

Given $g(t) = e^{-t}\sin(2\pi f_c t)\,u(t)$ find $\mathcal{F}(g(t))$ Answer:

$$\mathcal{F}(g(t)) = \mathcal{F}\left(e^{-t}u(t)\right)\otimes\mathcal{F}(\sin(2\pi f_c t)) \tag{1}$$

But

$$\mathcal{F}(\sin(2\pi f_0 t)) = \frac{1}{2j}\left[\delta(f-f_c)-\delta(f+f_c)\right] \tag{2}$$

and

$$\begin{aligned}\mathcal{F}\left(e^{-t}u(t)\right) &= \int_0^\infty e^{-t}e^{-j2\pi ft}dt = \int_0^\infty e^{-t(1+j2\pi f)}dt\\ &= \frac{\left[e^{-t(1+j2\pi f)}\right]_0^\infty}{-(1+j2\pi f)} = \frac{0-1}{-(1+j2\pi f)}\\ &= \frac{1}{1+j2\pi f}\end{aligned} \tag{3}$$

Substitute (2) and (3) into (1) we obtain

$$\begin{aligned}\mathcal{F}(g(t)) &= \frac{1}{2j}\left[\delta(f-f_c)-\delta(f+f_c)\right]\otimes\frac{1}{1+j2\pi f}\\ &= \frac{1}{2j}\left[\frac{1}{1+j2\pi(f-f_c)}-\frac{1}{1+j2\pi(f+f_c)}\right]\end{aligned}$$

4 Problem 2.3

4.1 part(a)

$$\begin{aligned}g(t) &= A\ rect\left(\frac{t}{T}-\frac{1}{2}\right)\\ &= A\ rect\left(\frac{t-\frac{T}{2}}{T}\right)\end{aligned}$$

hence it is a rect function with duration T and centered at $\frac{T}{2}$ and it has height A

$$g_e = \frac{g(t) + g(-t)}{2} \qquad (1)$$
$$g_o = \frac{g(t) - g(-t)}{2}$$

Hence $g_e = \frac{1}{2}\left[A\ rect\left(\frac{t}{T} - \frac{1}{2}\right) + A\ rect\left(\frac{-t}{T} - \frac{1}{2}\right)\right]$ which is a rectangular pulse of duration $2T$ and centered at zero and height A

$g_o = \frac{1}{2}\left[A\ rect\left(\frac{t}{T} - \frac{1}{2}\right) - A\ rect\left(\frac{-t}{T} - \frac{1}{2}\right)\right]$ which is shown in the figure below

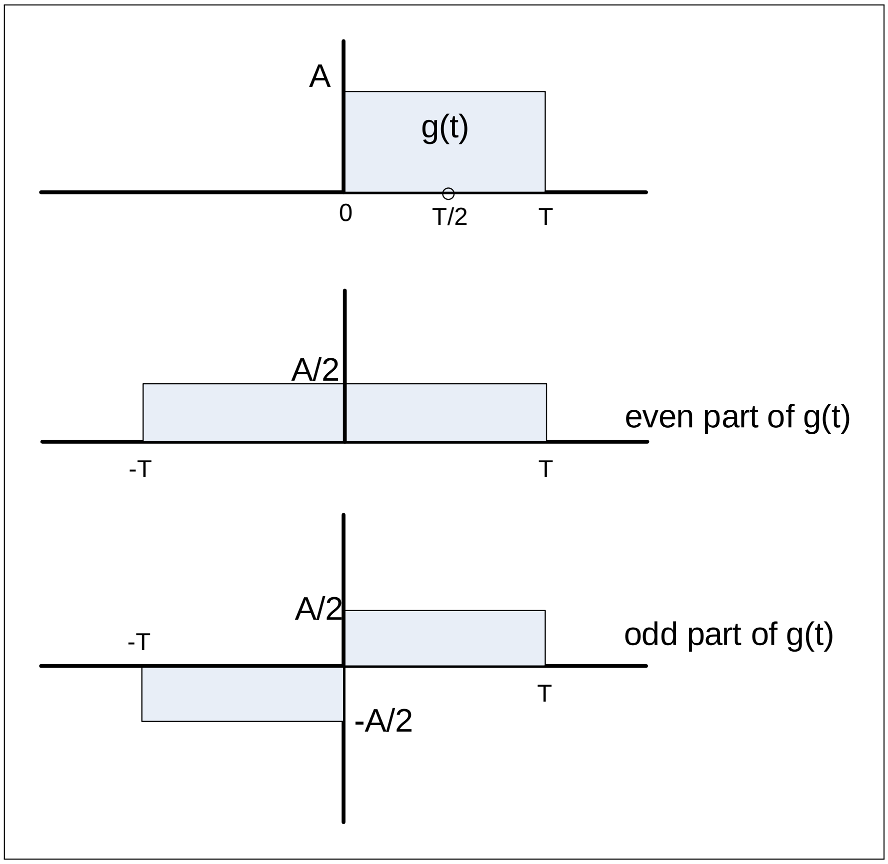

Figure 1: rectangular pulse

4.2 part(b)

$$\begin{aligned} F\left(g(t)\right) &= F\left(A\ rect\left(\frac{t - \frac{T}{2}}{T}\right)\right) \\ &= AT\ \ \text{sinc}\left(fT\right)\ \ e^{-j2\pi f \frac{T}{2}} \\ &= AT\ \ \text{sinc}\left(fT\right)\ \ e^{-j\pi f T} \end{aligned} \qquad (2)$$

Now using the property that $g(t) \Leftrightarrow G(f)$, then $g(-t) \Leftrightarrow G(-f)$, then we write

$$\begin{aligned} \mathcal{F}(g(-t)) &= G(-f) \\ &= AT\ \operatorname{sinc}(-fT)\ e^{j\pi fT} \end{aligned} \tag{3}$$

Now, using linearity of Fourier transform, then from (1) we obtain

$$\begin{aligned} \mathcal{F}(g_e(t)) &= \mathcal{F}\left(\frac{g(t)+g(-t)}{2}\right) \\ &= \frac{1}{2}\left[\mathcal{F}(g(t)) + \mathcal{F}(g(-t))\right] \\ &= \frac{1}{2}\left[AT\ \operatorname{sinc}(fT)\ e^{-j\pi fT} + AT\ \operatorname{sinc}(-fT)\ e^{j\pi fT}\right] \\ &= \frac{AT}{2}\left[\operatorname{sinc}(fT)\ e^{-j\pi fT} + \operatorname{sinc}(-fT)\ e^{j\pi fT}\right] \end{aligned}$$

now $\operatorname{sinc}(-fT) = \frac{\sin(-\pi fT)}{-\pi fT} = \frac{-\sin(\pi fT)}{-\pi fT} = \operatorname{sinc}(fT)$, hence the above becomes

$$\begin{aligned} \mathcal{F}(g_e(t)) &= \frac{AT\operatorname{sinc}(fT)}{2}\left[\ e^{-j\pi fT} +\ e^{j\pi fT}\right] \\ &= \frac{AT\operatorname{sinc}(fT)}{2}\left[\ 2\cos(\pi fT)\right] \end{aligned}$$

Hence

$$\boxed{\mathcal{F}(g_e(t)) = AT\operatorname{sinc}(fT)\cos(\pi fT)}$$

Now to find the Fourier transform of the odd part

$$g_o = \frac{g(t)-g(-t)}{2}$$

Hence

$$\begin{aligned} \mathcal{F}(g_o(t)) &= \mathcal{F}\left(\frac{g(t)-g(-t)}{2}\right) \\ &= \frac{1}{2}\left[\mathcal{F}(g(t)) - \mathcal{F}(g(-t))\right] \\ &= \frac{1}{2}\left[AT\ \operatorname{sinc}(fT)\ e^{-j\pi fT} - AT\ \operatorname{sinc}(-fT)\ e^{j\pi fT}\right] \\ &= \frac{AT}{2}\left[\operatorname{sinc}(fT)\ e^{-j\pi fT} - \operatorname{sinc}(fT)\ e^{j\pi fT}\right] \\ &= \frac{AT\operatorname{sinc}(fT)}{2}\left[\ e^{-j\pi fT} -\ e^{j\pi fT}\right] \\ &= \frac{-AT\operatorname{sinc}(fT)}{2}\left[\ e^{j\pi fT} - e^{-j\pi fT}\right] \\ &= \frac{-AT\operatorname{sinc}(fT)}{2}\left[\ 2j\sin(\pi fT)\right] \end{aligned}$$

Hence

$$\boxed{F\left(g_o\left(t\right)\right) = -jAT\,\text{sinc}\left(fT\right)\sin\left(\pi fT\right)}$$

5 Problem 2.4

$$G\left(f\right) = \left|G\left(f\right)\right| e^{j\arg(G(f))}$$

Hence from the diagram given, we write

$$G\left(f\right) = \begin{cases} 1\times e^{j\frac{\pi}{2}} & -W \leq f < 0 \\ 1\times e^{-j\frac{\pi}{2}} & 0 \leq f \leq W \end{cases}$$

Therefore, we can use a rect function now to express $G\left(f\right)$ over the whole f range as follows

$$G\left(f\right) = e^{j\frac{\pi}{2}}\ rect\left(\frac{f+\frac{W}{2}}{W}\right) - e^{-j\frac{\pi}{2}} rect\left(\frac{f-\frac{W}{2}}{W}\right)$$

Now, noting that $\delta\left(t-t_0\right) \Leftrightarrow e^{-j2\pi t_0}$ and $\delta\left(t+t_0\right) \Leftrightarrow e^{j2\pi t_0}$ and $W\,\text{sinc}\left(tW\right) \Leftrightarrow rect\left(\frac{f}{W}\right)$ and noting that shift in frequency by $\frac{W}{2}$becomes multiplication by $e^{-j2\pi t\frac{W}{2}}$, then now we write

$$\begin{aligned} g\left(t\right) &= F^{-1}\left(e^{j\frac{\pi}{2}}\ rect\left(\frac{f+\frac{W}{2}}{W}\right)\right) - F^{-1}\left(e^{-j\frac{\pi}{2}} rect\left(\frac{f-\frac{W}{2}}{W}\right)\right) \\ &= F^{-1}\left(e^{j\frac{\pi}{2}}\right)\ \otimes F^{-1}\left(rect\left(\frac{f+\frac{W}{2}}{W}\right)\right) - F^{-1}\left(e^{-j\frac{\pi}{2}}\right)\ \otimes F^{-1}\left(rect\left(\frac{f-\frac{W}{2}}{W}\right)\right) \end{aligned}$$

Hence

$$\begin{aligned} g\left(t\right) &= \left[\delta\left(t+\frac{\pi}{2}\right)\otimes W\,\text{sinc}\left(tW\right)e^{-j2\pi t\frac{W}{2}}\right] - \left[\delta\left(t-\frac{\pi}{2}\right)\otimes W\,\text{sinc}\left(tW\right)e^{j2\pi t\frac{W}{2}}\right] \\ &= W\,\text{sinc}\left(\left(t+\frac{\pi}{2}\right)W\right)e^{-j2\pi\left(t+\frac{\pi}{2}\right)\frac{W}{2}} - W\,\text{sinc}\left(\left(t-\frac{\pi}{2}\right)W\right)e^{j2\pi\left(t-\frac{\pi}{2}\right)\frac{W}{2}} \\ &= W\,\text{sinc}\left(\left(t+\frac{\pi}{2}\right)W\right)e^{-j\pi Wt-j\pi W\frac{\pi}{2}} - W\,\text{sinc}\left(\left(t-\frac{\pi}{2}\right)W\right)e^{j\pi Wt-j\pi W\frac{\pi}{2}} \end{aligned}$$

Hence

$$\boxed{g\left(t\right) = We^{-\frac{j\pi^2 W}{2}}\left(\text{sinc}\left(\left(t+\tfrac{\pi}{2}\right)W\right)e^{-j\pi Wt} - \text{sinc}\left(\left(t-\tfrac{\pi}{2}\right)W\right)e^{j\pi Wt}\right)}$$

6 Key solution

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CHAPTER 2

Representation of Signals and Systems

Problem 2.1

(a) The half-cosine pulse g(t) of Fig. P2.1(a) may be considered as the product of the rectangular function rect(t/T) and the sinusoidal wave A cos(πt/T). Since

$$\text{rect}(\frac{t}{T}) \rightleftharpoons T\,\text{sinc}(fT)$$

$$A\cos(\frac{\pi t}{T}) \rightleftharpoons \frac{A}{2}[\delta(f-\frac{1}{2T}) + \delta(f+\frac{1}{2T})]$$

and multiplication in the time domain is transformed into convolution in the frequency domain, it follows that

$$G(f) = [T\,\text{sinc}(fT)] \star \{\frac{A}{2}[\delta(f-\frac{1}{2T}) + \delta(f+\frac{1}{2T})]\}$$

where $\star$ denotes convolution. Therefore, noting that

$$\text{sinc}(fT) \star \delta(f-\frac{1}{2T}) = \text{sinc}[T(f-\frac{1}{2T})]$$

$$\text{sinc}(fT) \star \delta(f+\frac{1}{2T}) = \text{sinc}[T(f+\frac{1}{2T})]$$

we obtain the desired result

$$G(f) = \frac{AT}{2}[\text{sinc}(fT-\frac{1}{2}) + \text{sinc}(fT+\frac{1}{2})]$$

(b) The half-sine pulse of Fig. P2.1(b) may be obtained by shifting the half-cosine pulse to the right by T/2 seconds. Since a time shift of T/2 seconds is equivalent to multiplication by exp(-jπfT) in the frequency domain, it follows that the Fourier transform of the half-sine pulse is

$$G(f) = \frac{AT}{2}[\text{sinc}(fT-\frac{1}{2}) + \text{sinc}(fT+\frac{1}{2})]\exp(-j\pi fT)$$

(c) The Fourier transform of a half-sine pulse of duration aT is equal to

$$\frac{|a|AT}{2}[\text{sinc}(afT-\frac{1}{2}) + \text{sinc}(afT+\frac{1}{2})]\exp(-j\pi afT)$$

(d) The Fourier transform of the negative half-sine pulse shown in Fig. P2.1(c) is obtained from the result of part (c) by putting a= -1, and multiplying the result by -1, and so we find that its Fourier transform is equal to

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$$-\frac{AT}{2}[\text{sinc}(fT+\frac{1}{2}) + \text{sinc}(fT-\frac{1}{2})]\exp(j\pi fT)$$

(e) The full-sine pulse of Fig. P2.1(d) may be considered as the superposition of the half-sine pulses shown in parts (b) and (c) of the figure. The Fourier transform of this pulse is therefore

$$G(f) = \frac{AT}{2}[\text{sinc}(fT-\frac{1}{2}) + \text{sinc}(fT+\frac{1}{2})][\exp(-j\pi fT)-\exp(j\pi fT)]$$

$$= -jAT[\text{sinc}(fT-\frac{1}{2}) + \text{sinc}(fT+\frac{1}{2})]\sin(\pi fT)$$

$$= -jAT\left[\frac{\sin(\pi fT-\frac{\pi}{2})}{\pi fT-\frac{\pi}{2}} + \frac{\sin(\pi fT+\frac{\pi}{2})}{\pi fT+\frac{\pi}{2}}\right]\sin(\pi fT)$$

$$= -jAT[-\frac{\cos(\pi fT)}{\pi fT-\frac{\pi}{2}} + \frac{\cos(\pi fT)}{\pi fT+\frac{\pi}{2}}]\sin(\pi fT)$$

$$= jAT[\frac{\sin(2\pi fT)}{2\pi fT-\pi} - \frac{\sin(2\pi fT)}{2\pi fT+\pi}]$$

$$= jAT[-\frac{\sin(2\pi fT-\pi)}{2\pi fT-\pi} + \frac{\sin(2\pi fT+\pi)}{2\pi fT+\pi}]$$

$$= jAT[\text{sinc}(2fT+1) - \text{sinc}(2fT-1)]$$

Problem 2.2

Consider next an exponentially damped sinusoidal wave defined by (see Fig. 1):

$$g(t) = \exp(-t)\sin(2\pi f_c t)u(t)$$

In this case, we note that

$$\sin(2\pi f_c t) = \frac{1}{2j}[\exp(j2\pi f_c t) - \exp(-j2\pi f_c t)]$$

Therefore, applying the frequency-shifting property to the Fourier transform pair we find that the Fourier transform of the damped sinusoidal wave of Fig. 1 is

$$G(f) = \frac{1}{2j}\left[\frac{1}{1+j2\pi(f-f_c)} - \frac{1}{1+j2\pi(f+f_c)}\right]$$

$$= \frac{2\pi f_c}{(1+j2\pi f)^2 + (2\pi f_c)^2}$$

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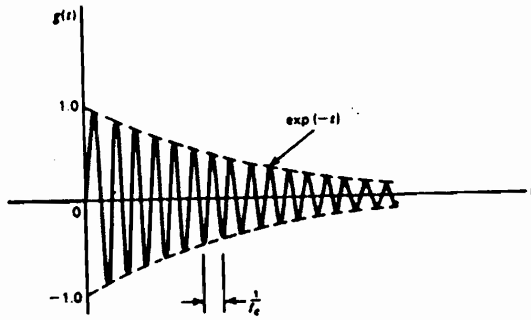


Figure 1 Damped sinusoidal wave.

Problem 2.3

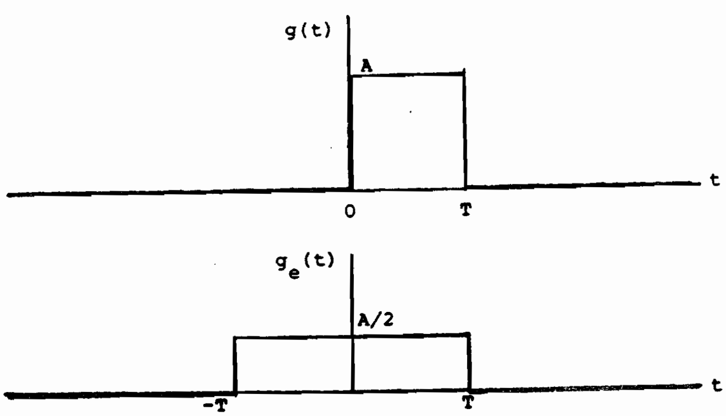
(a) The even part $g_e(t)$ of a pulse $g(t)$ is given by

$$g_e(t) = \frac{1}{2}[g(t) + g(-t)]$$

Therefore, for $g(t) = A\ \mathrm{rect}(\frac{t}{T} - \frac{1}{2})$, we obtain

$$g_e(t) = \frac{A}{2}[\mathrm{rect}(\frac{t}{T} - \frac{1}{2}) + \mathrm{rect}(-\frac{t}{T} - \frac{1}{2})]$$
$$= \frac{A}{2}[\mathrm{rect}(\frac{t}{2T})]$$

which is shown illustrated below:



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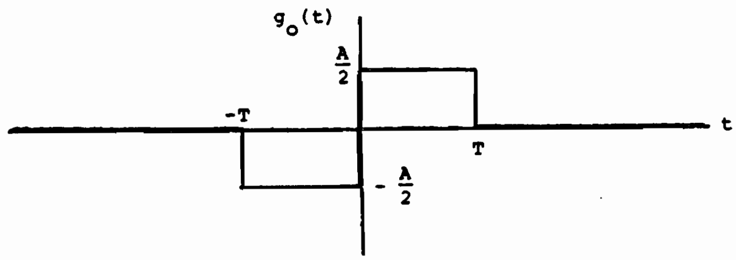
ıe odd part of g(t) is defined by

$$g_o(t) = \frac{1}{2}[g(t) - g(-t)]$$
$$= \frac{A}{2}[\text{rect}(\frac{t}{T} - \frac{1}{2}) - \text{rect}\,(-\frac{t}{T} - \frac{1}{2})]$$

which is illustrated below:



(b) The Fourier transform of the even part is

$$G_e(f) = AT\ \text{sinc}(2fT)$$

The Fourier transform of the odd part is

$$G_o(f) = \frac{AT}{2}\,\text{sinc}(fT)\,\exp(-j\pi fT)$$
$$- \frac{AT}{2}\,\text{sinc}(fT)\,\exp(j\pi fT)$$
$$= \frac{AT}{j}\,\text{sinc}(fT)\,\sin(\pi fT)$$

Problem 2.4

$$G(f) = \begin{cases} \exp(j\,\frac{\pi}{2}), & -W \le f \le 0 \\ \exp(-j\,\frac{\pi}{2}), & 0 \le f \le W \\ 0, & \text{otherwise} \end{cases}$$

Therefore, applying the formula for the inverse Fourier transform, we get

$$g(t) = \int_{-W}^{0} \exp(j\,\frac{\pi}{2})\exp(j2\pi ft)df + \int_{0}^{W} \exp(-j\,\frac{\pi}{2})\exp(j2\pi ft)dt$$

Replacing f with -f in the first integral and then interchanging the limits of integration:

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$$g(t) = \int_0^W \exp(-j2\pi ft + j\tfrac{\pi}{2}) + \exp(j2\pi ft - j\tfrac{\pi}{2})]df$$

$$= 2\int_0^W \cos(2\pi ft - \tfrac{\pi}{2})df$$

$$= 2\int_0^W \sin(2\pi ft)df$$

$$= \left[-\frac{\cos(2\pi ft)}{\pi t}\right]_0^W$$

$$= \frac{1}{\pi t}[1-\cos(2\pi Wt)]$$

$$= \frac{2}{\pi t}\sin^2(\pi Wt)$$

7 my graded HW

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HW1, EGEE 443. CSUF, Fall 2008

Nasser Abbasi

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1 Problem 2.1

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1.1 part(a)

Let $F(g(t))$ be the Fourier Transform of $g(t)$, i.e. $F(g(t)) = G(f)$. First we use the given hint and note that $g(t)$ can be written as follows

$$g(t) = A\cos\left(\frac{\pi t}{T}\right)\ rect\left(\frac{t}{T}\right)$$

Start by writing $\frac{\pi t}{T}$as $2\pi f_0 t$, where $f_0 = \frac{1}{2T}$. Now using the property that multiplication in time domain is the same as convolution in frequency domain, we obtain

$$G(f) = F(A\cos(2\pi f_0 t)) \otimes F\left(rect\left(\frac{t}{T}\right)\right) \tag{1}$$

But

$$\begin{aligned} F(A\cos(2\pi f_0 t)) &= A\ F(\cos(2\pi f_0 t)) \\ &= A\ F\left(\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right) \\ &= \frac{A}{2}\ F\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right) \\ &= \frac{A}{2}\left[\ F\left(e^{j2\pi f_0 t}\right) + \ F\left(e^{-j2\pi f_0 t}\right)\right] \end{aligned}$$

But $F\left(e^{j2\pi f_0 t}\right) = \delta(f - f_0)$ and $F\left(e^{-j2\pi f_0 t}\right) = \delta(f + f_0)$hence the above becomes

$$F(A\cos(2\pi f_0 t)) = \frac{A}{2}\left[\ \delta(f - f_0) + \ \delta(f + f_0)\right] \tag{2}$$

Substitute (2) into (1) we obtain

$$G(f) = \frac{A}{2}\left[\ \delta(f - f_0) + \ \delta(f + f_0)\right] \otimes F\left(rect\left(\frac{t}{T}\right)\right)$$

But $F\left(rect\left(\frac{t}{T}\right)\right) = T\,\text{sinc}(fT)$, hence the above becomes

$$F(g(t)) = \frac{A}{2}\left[\ \delta(f - f_0) + \ \delta(f + f_0)\right] \otimes T\,\text{sinc}(fT)$$

Now using the property of convolution with a delta, we obtain

$$\boxed{G(f) = \tfrac{AT}{2}\left[\ \text{sinc}((f - f_0)T) + \ \text{sinc}((f + f_0)T)\right]}$$

, $f_0 = \frac{1}{2T} \Rightarrow G(f) = \ldots$

note: by doing more trigonometric manipulations, the above can be written as

$$\boxed{G(f) = \frac{2AT\cos(\pi f T)}{\pi(1 - 4f^2T^2)}}$$

1.2 part(b)

Apply the time shifting property $g(t) \Longleftrightarrow G(f)$, hence $g(t-t_0) \Longleftrightarrow e^{-j2\pi f t_0} G(f)$

From part(a) we found that $F(g(t)) = \frac{AT}{2}[\operatorname{sinc}((f-f_0)T) + \operatorname{sinc}((f+f_0)T)]$, so in this part, the function in part(a) is shifted in time to the right by amount $\frac{T}{2}$, let the new function be $h(t)$,hence we need to multiply $G(f)$ by $e^{-j2\pi f \frac{T}{2}}$,hence

$$
\begin{aligned}
F\left(g\left(t-\frac{T}{2}\right)\right) &= F(h(t)) \\
&= H(f) \\
&= e^{-j\pi f T}\left(\frac{AT}{2}[\operatorname{sinc}((f-f_0)T) + \operatorname{sinc}((f+f_0)T)]\right)
\end{aligned}
$$

1.3 part(c)

Using the time scaling property $g(t) \Longleftrightarrow G(f)$, hence $g(at) \Longleftrightarrow \frac{1}{|a|}G\left(\frac{f}{a}\right)$, and since we found in part(b) that $H(f) = e^{-j\pi f T}\left(\frac{AT}{2}[\operatorname{sinc}((f-f_0)T) + \operatorname{sinc}((f+f_0)T)]\right)$, hence

Why h(at)? T→aT

$$\boxed{F\{h(at)\} = \frac{1}{|a|}e^{-j\pi \frac{f}{a} T}\left(\frac{AT}{2}\left[\operatorname{sinc}\left(\left(\frac{f}{a}-f_0\right)T\right) + \operatorname{sinc}\left(\left(\frac{f}{a}+f_0\right)T\right)\right]\right)}$$

See Solution.

should replace $f_0 = \frac{1}{2T}$

1.4 part(d)

Let $f(t)$ be the function which is shown in figure 2.4c, we see that

$$f(t) = -h(-t)$$

where $h(t)$ is the function shown in figure 2.4(b). We found in part(b) that

$$H(f) = e^{-j\pi f T}\left(\frac{AT}{2}[\operatorname{sinc}((f-f_0)T) + \operatorname{sinc}((f+f_0)T)]\right)$$

Now using the property that $h(t) \Longleftrightarrow H(f)$ then $h(-t) \Longleftrightarrow \frac{1}{|-1|}H(-f) = H(-f)$, hence

$$\boxed{F\{f(t)\} = -e^{j\pi f T}\left(\frac{AT}{2}[\operatorname{sinc}((-f-f_0)T) + \operatorname{sinc}((-f+f_0)T)]\right)}$$

1.5 part(e)

This function, call it $g_1(t)$, is the sum of the functions shown in figure 2.4(b) and figure 2.4(c), then the Fourier transform of $g_1(t)$ is the sum of the Fourier transforms of the functions in these two figures (using the linearity of the Fourier transforms). Hence

$$\begin{aligned} F(g_1(t)) &= e^{-j\pi fT}\left(\frac{AT}{2}\left[\operatorname{sinc}((f-f_0)T)+\operatorname{sinc}((f+f_0)T)\right]\right) \\ &\quad - e^{j\pi fT}\left(\frac{AT}{2}\left[\operatorname{sinc}((-f-f_0)T)+\operatorname{sinc}((-f+f_0)T)\right]\right) \end{aligned}$$

The above can be simplified to

$$\begin{aligned} F(g_1(t)) &= \frac{AT}{2}\left(\operatorname{sinc}((f+f_0)T)\left[e^{j\pi fT}+e^{-j\pi fT}\right]+\operatorname{sinc}((f-f_0)T)\left[e^{j\pi fT}+e^{-j\pi fT}\right]\right) \\ &= \frac{AT}{2}\left(\operatorname{sinc}((f+f_0)T)\left[2\cos(\pi fT)\right]+\operatorname{sinc}((f-f_0)T)\left[2\cos(\pi fT)\right]\right) \end{aligned}$$

Hence

$$\boxed{F(g_1(t)) = AT\cos(\pi fT)\left[\operatorname{sinc}((f+f_0)T)+\operatorname{sinc}((f-f_0)T)\right]}$$

2 Problem 2.2

Given $g(t) = e^{-t} \sin(2\pi f_c t)\, u(t)$ find $F(g(t))$ Answer:

$$F(g(t)) = F\left(e^{-t}u(t)\right) \otimes F(\sin(2\pi f_c t)) \tag{1}$$

But

$$F(\sin(2\pi f_0 t)) = \frac{1}{2j}\left[\delta(f - f_c) - \delta(f + f_c)\right] \tag{2}$$

and

$$\begin{aligned} F\left(e^{-t}u(t)\right) &= \int_0^\infty e^{-t}e^{-j2\pi f t}dt = \int_0^\infty e^{-t(1+j2\pi f)}dt \\ &= \frac{\left[e^{-t(1+j2\pi f)}\right]_0^\infty}{-(1+j2\pi f)} = \frac{0-1}{-(1+j2\pi f)} \\ &= \frac{1}{1+j2\pi f} \end{aligned} \tag{3}$$

Substitute (2) and (3) into (1) we obtain

$$\begin{aligned} F(g(t)) &= \frac{1}{2j}\left[\delta(f - f_c) - \delta(f + f_c)\right] \otimes \frac{1}{1+j2\pi f} \\ &= \frac{1}{2j}\left[\frac{1}{1+j2\pi(f - f_c)} - \frac{1}{1+j2\pi(f + f_c)}\right] \end{aligned}$$

3 Problem 2.3

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3.1 part(a)

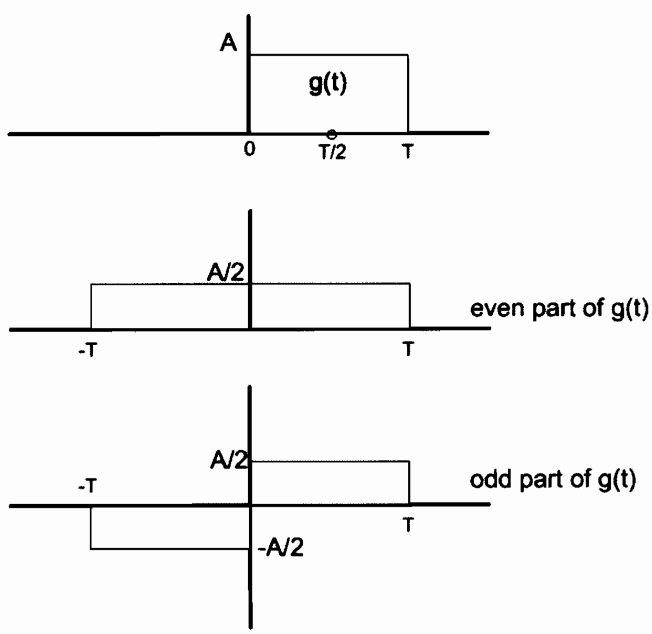
$$g(t) = A\ rect\left(\frac{t}{T} - \frac{1}{2}\right) = A\ rect\left(\frac{t - \frac{T}{2}}{T}\right)$$

hence it is a rect function with duration T and centered at $\frac{T}{2}$ and it has height A

$$g_e = \frac{g(t) + g(-t)}{2} \qquad g_o = \frac{g(t) - g(-t)}{2} \tag{1}$$

Hence $g_e = \frac{1}{2}\left[A\ rect\left(\frac{t}{T} - \frac{1}{2}\right) + A\ rect\left(\frac{-t}{T} - \frac{1}{2}\right)\right]$ which is a rectangular pulse of duration $2T$ and centered at zero and height A

$g_o = \frac{1}{2}\left[A\ rect\left(\frac{t}{T} - \frac{1}{2}\right) - A\ rect\left(\frac{-t}{T} - \frac{1}{2}\right)\right]$ which is shown in the figure below



3.2 part(b)

$$
\begin{aligned}
F\left(g\left(t\right)\right) &= F\left(A\ rect\left(\frac{t-\frac{T}{2}}{T}\right)\right)\\
&= AT\ \operatorname{sinc}\left(fT\right)\ e^{-j2\pi f\frac{T}{2}}\\
&= AT\ \operatorname{sinc}\left(fT\right)\ e^{-j\pi fT} \qquad (2)
\end{aligned}
$$

Now using the property that $g\left(t\right)\Leftrightarrow G\left(f\right)$, then $g\left(-t\right)\Leftrightarrow G\left(-f\right)$, then we write

$$
\begin{aligned}
F\left(g\left(-t\right)\right) &= G\left(-f\right)\\
&= AT\ \operatorname{sinc}\left(-fT\right)\ e^{j\pi fT} \qquad (3)
\end{aligned}
$$

Now, using linearity of Fourier transform, then from (1) we obtain

$$
\begin{aligned}
F\left(g_e\left(t\right)\right) &= F\left(\frac{g\left(t\right)+g\left(-t\right)}{2}\right)\\
&= \frac{1}{2}\left[F\left(g\left(t\right)\right)+F\left(g\left(-t\right)\right)\right]\\
&= \frac{1}{2}\left[AT\ \operatorname{sinc}\left(fT\right)\ e^{-j\pi fT}+AT\ \operatorname{sinc}\left(-fT\right)\ e^{j\pi fT}\right]\\
&= \frac{AT}{2}\left[\operatorname{sinc}\left(fT\right)\ e^{-j\pi fT}+\operatorname{sinc}\left(-fT\right)\ e^{j\pi fT}\right]
\end{aligned}
$$

now $\operatorname{sinc}\left(-fT\right)=\frac{\sin(-\pi fT)}{-\pi fT}=\frac{-\sin(\pi fT)}{-\pi fT}=\operatorname{sinc}\left(fT\right)$, hence the above becomes

$$
\begin{aligned}
F\left(g_e\left(t\right)\right) &= \frac{AT\operatorname{sinc}\left(fT\right)}{2}\left[\ e^{-j\pi fT}+e^{j\pi fT}\right]\\
&= \frac{AT\operatorname{sinc}\left(fT\right)}{2}\left[\ 2\cos\left(\pi fT\right)\right]
\end{aligned}
$$

Hence

$$\boxed{F\left(g_e\left(t\right)\right)=AT\operatorname{sinc}\left(fT\right)\cos\left(\pi fT\right)}$$

See Solution.

Now to find the Fourier transform of the odd part

$$g_o=\frac{g\left(t\right)-g\left(-t\right)}{2}$$

Hence

$$\begin{aligned}
F\left(g_o(t)\right) &= F\left(\frac{g(t)-g(-t)}{2}\right) \\
&= \frac{1}{2}\left[F\left(g(t)\right) - F\left(g(-t)\right)\right] \\
&= \frac{1}{2}\left[AT\ \operatorname{sinc}(fT)\ e^{-j\pi fT} - AT\ \operatorname{sinc}(-fT)\ e^{j\pi fT}\right] \\
&= \frac{AT}{2}\left[\operatorname{sinc}(fT)\ e^{-j\pi fT} - \operatorname{sinc}(fT)\ e^{j\pi fT}\right] \\
&= \frac{AT\operatorname{sinc}(fT)}{2}\left[\ e^{-j\pi fT} - \ e^{j\pi fT}\right] \\
&= \frac{-AT\operatorname{sinc}(fT)}{2}\left[\ e^{j\pi fT} - e^{-j\pi fT}\right] \\
&= \frac{-AT\operatorname{sinc}(fT)}{2}\left[\ 2j\sin(\pi fT)\right]
\end{aligned}$$

Hence

$$\boxed{F\left(g_o(t)\right) = -jAT\operatorname{sinc}(fT)\sin(\pi fT)}$$

4 Problem 2.4

$$G(f) = |G(f)|\, e^{j \arg(G(f))}$$

Hence from the diagram given, we write

$$G(f) = \begin{cases} 1 \times e^{j\frac{\pi}{2}} & -W \leq f < 0 \\ 1 \times e^{-j\frac{\pi}{2}} & 0 \leq f \leq W \end{cases}$$

Therefore, we can use a rect function now to express $G(f)$ over the whole f range as follows

$$G(f) = e^{j\frac{\pi}{2}}\ rect\left(\frac{f + \frac{W}{2}}{W}\right) - e^{-j\frac{\pi}{2}} rect\left(\frac{f - \frac{W}{2}}{W}\right)$$

Now, noting that $\delta(t - t_0) \Leftrightarrow e^{-j2\pi t_0}$ and $\delta(t + t_0) \Leftrightarrow e^{j2\pi t_0}$ and $W \operatorname{sinc}(tW) \Leftrightarrow rect\left(\frac{f}{W}\right)$ and noting that shift in frequency by $\frac{W}{2}$becomes multiplication by $e^{-j2\pi t\frac{W}{2}}$, then now we write

$$\begin{aligned} g(t) &= F^{-1}\left(e^{j\frac{\pi}{2}}\ rect\left(\frac{f+\frac{W}{2}}{W}\right)\right) - F^{-1}\left(e^{-j\frac{\pi}{2}} rect\left(\frac{f-\frac{W}{2}}{W}\right)\right) \\ &= F^{-1}\left(e^{j\frac{\pi}{2}}\right)\ \otimes F^{-1}\left(rect\left(\frac{f+\frac{W}{2}}{W}\right)\right) - F^{-1}\left(e^{-j\frac{\pi}{2}}\right)\ \otimes F^{-1}\left(rect\left(\frac{f-\frac{W}{2}}{W}\right)\right) \end{aligned}$$

Hence

$$\begin{aligned} g(t) &= \left[\delta\left(t + \frac{\pi}{2}\right) \otimes W \operatorname{sinc}(tW)\, e^{-j2\pi t\frac{W}{2}}\right] - \left[\delta\left(t - \frac{\pi}{2}\right) \otimes W \operatorname{sinc}(tW)\, e^{j2\pi t\frac{W}{2}}\right] \\ &= W \operatorname{sinc}\left(\left(t + \frac{\pi}{2}\right) W\right) e^{-j2\pi\left(t+\frac{\pi}{2}\right)\frac{W}{2}} - W \operatorname{sinc}\left(\left(t - \frac{\pi}{2}\right) W\right) e^{j2\pi\left(t-\frac{\pi}{2}\right)\frac{W}{2}} \\ &= W \operatorname{sinc}\left(\left(t + \frac{\pi}{2}\right) W\right) e^{-j\pi Wt - j\pi W\frac{\pi}{2}} - W \operatorname{sinc}\left(\left(t - \frac{\pi}{2}\right) W\right) e^{j\pi Wt - j\pi W\frac{\pi}{2}} \end{aligned}$$

Hence

$$\boxed{g(t) = We^{-\frac{j\pi^2 W}{2}}\left(\operatorname{sinc}\left(\left(t + \frac{\pi}{2}\right) W\right) e^{-j\pi Wt} - \operatorname{sinc}\left(\left(t - \frac{\pi}{2}\right) W\right) e^{j\pi Wt}\right)}$$