

Name (please print) KGF

1. Consider a sequence of days, and let R_i denote the event that it rains on day i . Let $P(R_0) = p$ (rain today), $P(R_i|R_{i-1}) = \alpha$, and $P(R_i^c|R_{i-1}^c) = \beta$. Suppose further that only today's weather is relevant to predicting tomorrow's; that is, $P(R_i|R_{i-1} \cap \dots \cap R_0) = P(R_i|R_{i-1})$. What is the probability that it rains n days from now? What happens as n approaches infinity?

$$\begin{aligned} P(R_n) &= P(R_n|R_{n-1})P(R_{n-1}) + P(R_n^c|R_{n-1}^c)P(R_{n-1}^c) \\ &= \cancel{P(R_n|R_{n-1})}P(R_{n-1}) + (1-\beta)[1 - P(R_{n-1})] \\ &= (\alpha + \beta - 1)P(R_{n-1}) + (1-\beta). \end{aligned}$$

$$\text{Let } a = \alpha + \beta - 1 \quad b = (1-\beta) \quad p_n = P(R_n).$$

$$\text{Then } p_n = a p_{n-1} + b = a(a p_{n-2} + b) + b$$

$$= a^2 p_{n-2} + b(1+a) = a^2(a p_{n-3} + b) + b(1+a)$$

$$= a^3 p_{n-3} + b(1+a+a^2)$$

$$= \dots = a^n p_0 + b(1+a+a^2+\dots+a^{n-1})$$

$$= a^n p_0 + b \frac{1-a^n}{1-a}$$

$$\text{as } n \rightarrow \infty \quad a^n \rightarrow 0 \Rightarrow p_n \rightarrow \frac{b}{1-a} = \frac{1-\beta}{-\alpha + \beta + 2} = \frac{P(R_i|R_{i-1}^c)}{P(R_i|R_{i-1}) + P(R_i^c|R_{i-1})}$$

2. Suppose that a rare disease has an incidence of 1 in 1000. Assuming that members of the population are affected independently, Find the probability that two individuals are affected in a population of 100,000. Give a numerical value.

2 Show that if the conditional probabilities exist, then

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \frac{P(A_2 | A_1 \cap A_2)}{P(A_2)} \dots P(A_n | A_1 \cap A_2 \cap \dots \cap A_{n-1}).$$

~~We have $P(A_1 \cap A_2) = P(A_1 | A_2) \cdot P(A_2)$~~

for $n=2$, the equality holds obviously.

Assume true for $n=k+1$, and let $B = A_1 \cap A_2 \cap \dots \cap A_{k+1}$

$$\text{Then } P(B) = P(A_1) \frac{P(A_2 | A_1 \cap A_2)}{P(A_2)} \dots P(A_{k+1} | A_1 \cap A_2 \cap \dots \cap A_{k+1})$$

$$P(A_1 \cap \dots \cap A_k) = P(B \cap A_k) = \cancel{P(B)} \frac{P(B | A_k)}{P(A_k | B) \cdot P(B)}$$

$$= \underbrace{P(A_1) \frac{P(A_2 | A_1 \cap A_2)}{P(A_2)} \dots P(A_{k+1} | A_1 \cap A_2 \cap \dots \cap A_{k+1})}_{P(B) \text{ by induction}} P(A_k | A_1 \cap A_2 \cap \dots \cap A_{k+1})$$

Q.E.D.

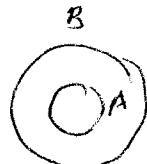
3. Let A and B be arbitrary events. Use the three axioms of probability to show that

$$P(A \cup B) \leq P(A) + P(B).$$

Identify the axiom(s) that you use at each step. You are not allowed to use any theorems.

First we prove that

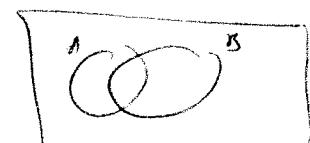
if $A \subset B$, then $P(A) \leq P(B)$



Proof: $B = A \cup [B \cap A^c]$

$$\Rightarrow P(B) = P\{A \cup [B \cap A^c]\} \\ = P(A) + P(B \cap A^c) \quad \text{By Axiom 3} \\ \geq P(A) \quad \text{by axiom (2)} \quad \square$$

Now $A \cup B = A \cup [B \cap A^c]$



$$P(A \cup B) = P(A) + P[B \cap A^c] \quad \text{1st Axiom} \\ P(B \cap A^c) \subset B \\ P(B \cap A^c) \leq P(B) \\ \text{Q.E.D.}$$

4. Let $X \sim \text{binomial}(n, p)$. Derive the mode of the probability mass function of X .

$$P_k = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\frac{P_k}{P_{k-1}} = \frac{n!}{k!(n-k)!} \cdot \frac{(k-1)!(n-k+1)!}{n!} \cdot \frac{p^k}{p^{k-1}} \cdot \frac{(1-p)^{n-k}}{(1-p)^{n-k+1}}$$

$$= \frac{(n-k+1)}{k} \frac{p}{(1-p)}$$

$$\frac{P_k}{P_{k-1}} \geq 1 \iff np - kp + p \geq k - kp$$

$$\iff k \leq (n+1)p$$

so the mode is at the greatest integer less than or equal to $(n+1)p$

5. Suppose that a rare disease has an incidence of 1 in 1000. Assuming that members of the population are affected independently, Find the probability that two individuals are affected in a population of 100,000. Give a numerical value.

$$P(D) = \frac{1}{1000}$$

$$X = \# \text{ of affected} \sim \text{Binomial}(100,000, \frac{1}{1000})$$

$$P(X=2) = \binom{100,000}{2} \left(\frac{1}{1000}\right)^2 \left(\frac{999,999}{1000}\right)^{100,000-2} = .03988694$$

$$\text{Poisson approximation, } \lambda = 100,000 \left(\frac{1}{1000}\right) = 100$$

$$P(X=2) = e^{-100} \frac{\frac{100}{2}}{100!} = .039861$$

Name (please print) KEY

1. Use the fact that $\Gamma(1/2) = \sqrt{\pi}$ to show that if n is an odd integer, then

$$\Gamma(n/2) = \frac{\sqrt{\pi}(n-1)!}{2^{n-1} \left(\frac{n-1}{2}\right)!}. \quad (*)$$

Obviously true for $n=1$.

Suppose (*) is true for $n=2k-1$ (i.e. $\Gamma\left(\frac{2k-1}{2}\right) = \frac{\sqrt{\pi}(2k-2)!}{2^{2k-2} \left(\frac{2k-2}{2}\right)!}$)

Then

$$\begin{aligned} \Gamma\left(\frac{2k+1}{2}\right) &= \Gamma\left(\frac{2k-1}{2} + 1\right) = \left(\frac{2k-1}{2}\right) \Gamma\left(\frac{2k-1}{2}\right) \stackrel{(*)}{=} \left(\frac{2k-1}{2}\right) \frac{\sqrt{\pi}(2k-2)!}{2^{2k-2} (k-1)!} \\ &= \frac{2k}{2k} \frac{(2k-1)(2k-2)! \sqrt{\pi}}{2^{2k-1} (k-1)!} = \frac{\sqrt{\pi}(2k)!}{2^{2k} k!} \quad \square \end{aligned}$$

2. If $U \sim \text{Uniform}[-1, 1]$, find the density of U^2 .

$$f_U(u) = \frac{1}{2} \quad -1 < u < 1 \quad \text{Let } Z = U^2$$

$$\text{for } 0 < z \leq 1 \quad F_Z(z) = P(Z \leq z) = P(U^2 \leq z)$$

$$= P(-\sqrt{z} \leq U \leq \sqrt{z})$$

$$= F_U(\sqrt{z}) - F_U(-\sqrt{z})$$

$$= \int_{-1}^{\sqrt{z}} \frac{1}{2} du - \int_{-1}^{-\sqrt{z}} \frac{1}{2} du$$

$$= \frac{1}{2} (\sqrt{z} + 1) - \frac{1}{2} (-\sqrt{z} + 1) = \sqrt{z}$$

$$\therefore f_Z(z) = \begin{cases} \frac{1}{2} z^{-1/2} & 0 \leq z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

3. The following ~~five~~^{four} numbers were randomly generated from the uniform random variable on $(0,1)$:

$$0.0153 \quad 0.7468 \quad 0.4451 \quad 0.9318 \quad \cancel{0.5555}$$

Using these numbers generate five random numbers from the geometric random variable with parameter $p = 1/3$. Very briefly explain how you obtain your solution.

The cdf for the geometric random variable

with parameter $\frac{1}{3}$ is given by

$$F(x) = \sum_{k=1}^x \left(\frac{2}{3}\right)^{k-1} \left(\frac{1}{3}\right) = \frac{1}{3} \sum_{k=0}^{x-1} \left(\frac{2}{3}\right)^k = \left(\frac{1}{3}\right) \frac{\left(1 - \left(\frac{2}{3}\right)^x\right)}{1 - \left(\frac{2}{3}\right)} = 1 - \left(\frac{2}{3}\right)^x \quad x=1,2,$$

$$\Rightarrow F^{-1}(x) = \frac{\log(1-x)}{\log(2/3)}.$$

If we apply F^{-1} to the above values we obtain (in order)

$$0.038, 3.3875, 1.4526, 6.6233$$

which implies the following generated values:

$$1, 4, 2, 7$$

4. Three players play 10 independent rounds of a game, and each player has probability $1/3$ of winning each round. (a) Find the joint distribution of the numbers of games won by each of the three players. (b) Identify the distribution of the number of games won by player one.

(a) $N_i = \# \text{ of games won by player } i \quad i=1, 2, 3$

$$P(n_1, n_2, n_3) = \binom{10}{n_1, n_2, n_3} \left(\frac{1}{3}\right)^{n_1} \left(\frac{1}{3}\right)^{n_2} \left(\frac{1}{3}\right)^{n_3}$$

$$n_1 + n_2 + n_3 = 10$$

(b) The distribution of $N_1 \sim \text{Binomial}(10, \frac{1}{3})$.

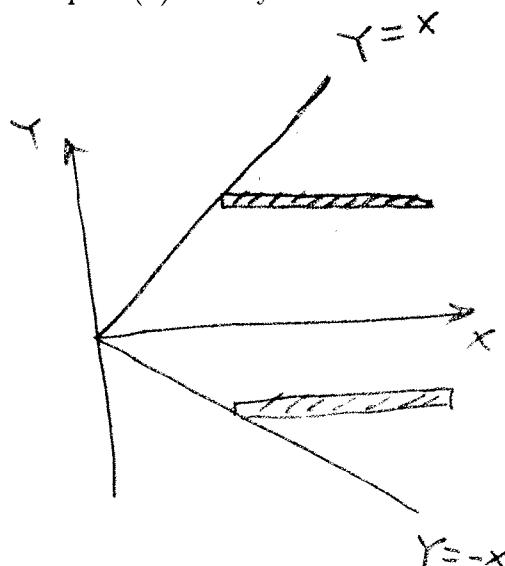
5. Let (X, Y) be jointly distributed random variables with pdf

$$f(x, y) = \frac{1}{8}(x^2 - y^2)e^{-x} \quad 0 \leq x \leq \infty \quad -x \leq y \leq x.$$

(a) Find the marginal density of Y . (b) Find $P(X + Y \leq 1)$. For part (b) leave your solution as integrals, and do not calculate the integrals.

a)

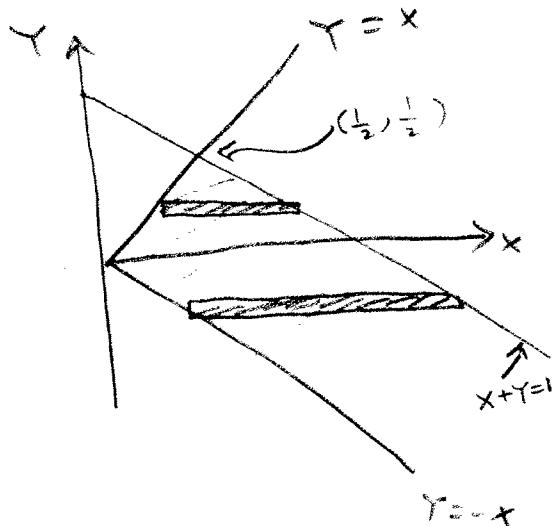
$$\text{If } y > 0 \quad f_Y(y) = \int_y^\infty \frac{1}{8}(x^2 - y^2)e^{-x} dx \\ = \frac{1}{4} e^{-y} (y+1)$$



$$\text{If } y < 0 \quad f_Y(y) = \int_{-y}^\infty \frac{1}{8}(x^2 - y^2)e^{-x} dx \\ = \frac{1}{4} e^y (1-y) \quad y < 0$$

b)

$$P(X + Y \leq 1) = \int_0^{\frac{1}{2}} \int_y^{1-y} \frac{1}{8}(x^2 - y^2)e^{-x} dx dy \\ + \int_{-\infty}^0 \int_{-y}^{1-y} \frac{1}{8}(x^2 - y^2)e^{-x} dx dy$$



Name (please print) KEY

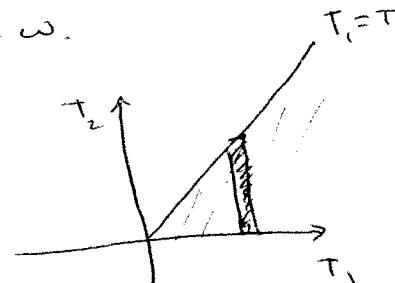
1. Suppose that two components have independent exponentially distributed lifetimes T_1 and T_2 , with parameters α and β , respectively. Find (a) $P(T_1 > T_2)$, (b) determine the distribution of $W = 2T_2$, and (c) use the results in parts (a) and (b) to obtain $P(T_1 > 2T_2)$.

$$f_{T_1, T_2}(t_1, t_2) = \begin{cases} \alpha\beta e^{-\alpha t_1 - \beta t_2} & t_1 \geq 0, t_2 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a)

$$P(T_1 > T_2) = \int_0^\infty \int_0^{t_1} \alpha\beta e^{-\alpha t_1 - \beta t_2} dt_2 dt_1$$

$$= \frac{\beta}{\alpha + \beta}$$



$$b) F_W(\omega) = P(2T_2 \leq \omega) = P(T_2 \leq \frac{\omega}{2})$$

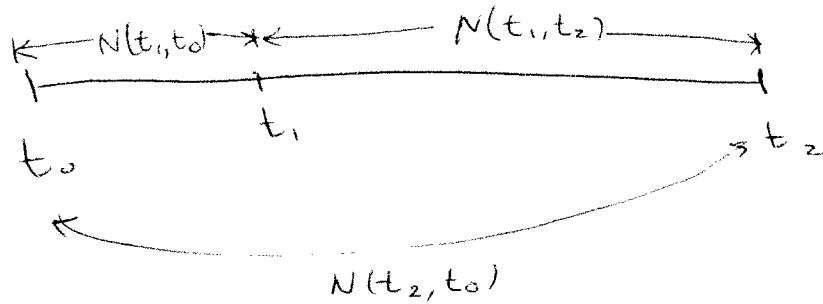
$$= \int_0^{\frac{\omega}{2}} \lambda_2 e^{-\lambda_2 t_2} dt_2 = 1 - e^{-\frac{\lambda_2}{2}\omega}$$

$$\therefore \omega \sim \exp\left(\frac{\lambda_2}{2}\right)$$

$$c) P(T_1 > T_2) = \frac{\beta}{\alpha + \beta} \quad \text{when } T_1 \sim \exp(\lambda_1) \text{ and } T_2 \sim \exp(\lambda_2)$$

$$P(T_1 > 2T_2) = P(T_1 > W) = \frac{(\lambda_2/2)}{\alpha + (\lambda_2/2)} = \frac{\lambda_2}{2\alpha + \lambda_2}$$

2. Consider a Poisson process on the real line, and denote by $N(t_1, t_2)$ the number of events in the interval (t_1, t_2) . If $t_0 < t_1 < t_2$, find the conditional distribution of $N(t_0, t_1)$ given that $N(t_0, t_2) = n$.



$$\begin{aligned}
 & P(N(t_0, t_1) = k \mid N(t_0, t_2) = n) = \\
 &= \frac{P\{N(t_0, t_1) = k \cap N(t_0, t_2) = n\}}{P\{N(t_0, t_2) = n\}} \\
 &= \frac{P\{N(t_0, t_1) = k \cap N(t_1, t_2) = n - k\}}{P\{N(t_0, t_2) = n\}} \\
 &= \frac{P\{N(t_0, t_1) = k\} P\{N(t_1, t_2) = n - k\}}{P\{N(t_0, t_2) = n\}} \\
 &\quad \bullet \frac{[(t_1 - t_0)\lambda]^k e^{-(t_1 - t_0)\lambda}}{k!} \cdot \frac{[(t_2 - t_1)\lambda]^{n-k} e^{-(t_2 - t_1)\lambda}}{(n-k)!} \\
 &= \frac{[(t_2 - t_0)\lambda]^n e^{-(t_2 - t_0)\lambda}}{n!} \\
 &= \binom{n}{k} \left(\frac{t_1 - t_0}{t_2 - t_0}\right)^k \left(\frac{t_2 - t_1}{t_2 - t_0}\right)^{n-k} \quad k = 0, 1, \dots, n \\
 &\quad \sim \text{Binom}(n, \frac{t_1 - t_0}{t_2 - t_0})
 \end{aligned}$$

3. Suppose that the probability Θ of getting heads for a coin is unknown, and let the prior opinion about Θ be represented by the uniform distribution on $[0,1]$. You spin the coin repeatedly and record the number of times N until a heads comes up. (a) Find the posterior density of Θ given N . (b) Use Matlab or any other software to plot the posterior for cases where $N = 1$, $N = 2$, and $N = 6$. Using your plots, explain what you infer about the probability of heads in each circumstance.

$$\Theta \sim \text{unif}[0,1]$$

$$N | \Theta \sim \text{geometric}(\theta)$$

$$f_{N,\Theta}(n, \theta) = (1-\theta)^{n-1} \theta; \quad n=1, 2, \dots, 0 \leq \theta \leq 1$$

$$f_{\Theta|N}(\theta|n) = \frac{f_{\Theta, N}(\theta, n)}{f_N(n)} = \frac{\theta^{(1-\theta)^{n-1}}}{\int_0^1 \theta^{(1-\theta)^{n-1}} d\theta}$$

$$= \frac{\theta^{(1-\theta)^{n-1}}}{1/[n(n+1)]} = n(n+1) \theta^{(1-\theta)^{n-1}}$$

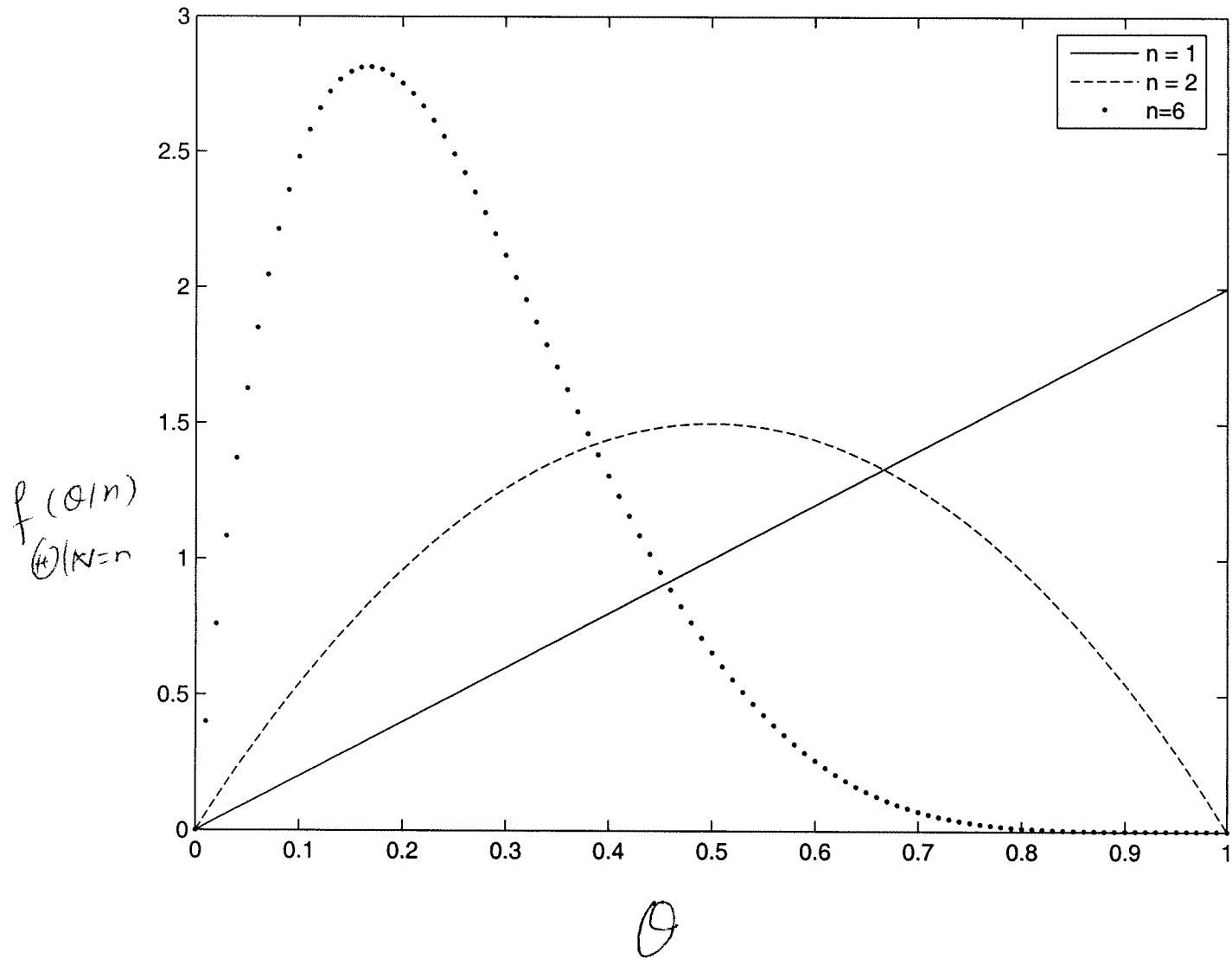
$$\Theta | N=n \sim \text{Beta}(\alpha=2, \beta=n)$$

~~The larger n , our prediction of probability~~

Larger n 's indicate lower probabilities

for getting heads (i.e. smaller θ).

It is clear from the graph (next page) that as n increases, the density gets more concentrated about values closer to 0° .



Name (please print) Ket

1. Let X be a continuous random variable with a pdf that is symmetric about a point ξ . Provided that $E(X)$ exists, show that $E(X) = \xi$.

Since $\xi = \int_{-\infty}^{\infty} xf(x)dx$, it is sufficient to show that $\int_{-\infty}^{\infty} (x-\xi) f(x) dx = 0$

$$\begin{aligned} \text{let } u &= x - \xi \\ x &= u + \xi \end{aligned}$$

$$\int_{-\infty}^{\infty} (x-\xi) f(x) dx = \int_{-\infty}^{\infty} u f(u+\xi) du = \int_{-\infty}^0 u f(u+\xi) du$$

$$+ \int_0^{\infty} u f(u+\xi) du = \int_0^{\infty} u f(u+\xi) du - \int_0^{\infty} u f(\xi-u) du$$

But since $f(u+\xi) = f(\xi-u)$ by symmetry, the last expression equals zero.

2. Let X be an exponential random variable with parameter λ . Find

$$P\left[|X - E(X)| > \frac{2}{\lambda}\right]$$

and compare your result to the Chebyshev's bound.

$$X \sim \exp(\lambda) \quad E(X) = \frac{1}{\lambda}$$

$$P\left(|X - \frac{1}{\lambda}| > \frac{2}{\lambda}\right) = 1 - P\left(|X - \frac{1}{\lambda}| \leq \frac{2}{\lambda}\right)$$

$$= 1 - P\left(-\frac{1}{\lambda} \leq X \leq \frac{3}{\lambda}\right) = \int_{-\frac{1}{\lambda}}^{\frac{3}{\lambda}} \lambda e^{-\lambda x} dx = e^{-3} = .049787$$

$$P\left(|X - E(X)| > \frac{2}{\lambda}\right) \approx .95$$

By C.C. it is at least 0.75

3. If X is a discrete random variable, taking values on the positive integers, then show that $E(X) = \sum_{k=1}^{\infty} P(X \geq k)$.

$$P(X=1) + P(X=2) + P(X=3) + \dots$$

~~$P(X=1)$~~ $P(X=2) + P(X=3) + \dots$

~~$P(X=1)$~~

$$\overline{P(X=1) + 2P(X=2) + 3P(X=3) + \dots}$$

$$E(X) = \sum_{x \in \mathbb{N}} x P(X=x)$$

4. Find the mean of a negative binomial random variable X with parameters r and p , by expressing X as sum of indicator variables.

$X_i = \# \text{ of trials until a success after } i^{\text{th}} \text{ success has been attained.}$

$$X = X_1 + X_2 + \dots + X_r$$

$$X_i \sim \text{geometric}(p)$$

$$E(X_i) = \frac{1}{p}$$

$$E(X) = \frac{r}{p}$$

5. If $U = a + bX$ and $V = c + dY$, show that $|\rho_{UV}| = |\rho_{XY}|$.

$$\rho_{UV} = \frac{\text{Cov}(U, V)}{\sqrt{\text{Var}(U) \text{Var}(V)}}$$

$$\begin{aligned}\text{Cov}(U, V) &= \text{Cov}(a + bX, c + dY) \\ &= bd \text{Cov}(X, Y)\end{aligned}$$

$$\text{Var}(U) = b^2 \text{Var}(X)$$

$$\text{Var}(V) = d^2 \text{Var}(Y)$$

Plug-in these values and the result follows.

Name (please print) _____

1. The moment generating function for a random variable X having a χ^2 distribution with degrees of freedom $n \geq 1$ is $M_X(t) = (1 - 2t)^{-n/2}$. Let W have a χ^2 distribution with degrees of freedom $n > 1$, and let V have a χ^2 distribution with degrees of freedom 1. (a). If $W = U + V$, and U and V are independent, determine the distribution of U ? (b). What are the mean and variance of W ?

$$\text{a) } M_W(t) = M_U(t) \cdot M_V(t) \Rightarrow M_V(t) = \frac{M_W(t)}{M_U(t)}$$

$$\Rightarrow M_V(t) = \left(\frac{1}{1-2t}\right)^{1/2} / \left(\frac{1}{1-2t}\right)^{1/2} = \left(\frac{1}{1-2t}\right)^{\frac{n-1}{2}}$$

$$\Rightarrow V \sim \chi_{(n-1)}^2$$

$$\left. \begin{array}{l} E(W) = n \\ \text{Var}(W) = 2n \end{array} \right\} \text{Can be obtained from the table, using the } \chi_{(n)}^2 \sim \text{gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$$

2. Find the approximate variance of $Y = \sqrt{X}$, where X is a Poisson random variable with parameter λ .

$$\text{Var}(\sqrt{X}) = E(\sqrt{X}) - [E(\sqrt{X})]^2 = \lambda - [E(\sqrt{X})]^2$$

and $\text{Var}(x) = \lambda$

since $E(x) = \lambda$, we have

$$E(\sqrt{x}) \approx \sqrt{\lambda} + \frac{1}{2} \lambda g''(\lambda)$$

where $g(x) = \sqrt{x} \Rightarrow g'(x) = \frac{1}{2} x^{-1/2} \Rightarrow g''(x) = -\frac{1}{4} x^{-3/2}$

$$\Rightarrow E(\sqrt{x}) \approx \sqrt{\lambda} + \frac{1}{2} \lambda \left[-\frac{1}{4} \lambda^{-3/2}\right] = \lambda^{1/2} - \frac{1}{8} \lambda^{-1/2}$$

$$\text{Var}(\sqrt{x}) \approx \lambda - (\lambda^{1/2} - \frac{1}{8} \lambda^{-1/2})^2$$

3. The random variable Y has a Gamma distribution with parameters α and λ . Furthermore, assume that X given Y has a Poisson distribution with parameter Y^2 . (a) Obtain $E(X)$.
 (b) Obtain $Var(X)$.

$$Y \sim P(\alpha, \lambda)$$

$$X|Y \sim \text{Poisson}(Y^2)$$

$$E(X) = E\{E(X|Y)\}$$

$$= E(Y^2)$$

$$= Var(Y) + (E(Y))^2 = \frac{\alpha}{\lambda} + \left(\frac{\alpha}{\lambda}\right)^2$$

$$Var(X) = E[Var(X|Y)] + Var[E(X|Y)]$$

$$= E(Y^2) + Var(Y^2)$$

$$= E(Y^2) + E(Y^4) - [E(Y^2)]^2$$

$$E(Y^2) = \frac{\alpha}{\lambda} + \left(\frac{\alpha}{\lambda}\right)^2$$

$$E(Y^4) = \int_0^\infty y^4 \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y} dy$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+3} e^{-\lambda y} dy$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+4)}{\lambda^{\alpha+4}} = \frac{(\alpha+3)(\alpha+2)(\alpha+1)\alpha}{\lambda^4}$$

$$Var(X) = \frac{(\alpha+3)(\alpha+2)(\alpha+1)\alpha}{\lambda^4} + \frac{\alpha}{\lambda} + \left(\frac{\alpha}{\lambda}\right)^2$$

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In solving the problems below, you can use all the results that we have derived in class. You do not need to re-derive results. Make sure to cite the results that you use.

1. Let X_1, \dots, X_n be iid random variables from a $\mathcal{N}(\mu, \sigma^2)$, and S^2 be the sample variance. What is $\text{Var}(S^2)$?

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{(n-1)}^2 \Rightarrow \text{Var}\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n-1)$$

$$\Rightarrow \frac{(n-1)^2}{\sigma^4} \text{Var}(S^2) = 2(n-1)$$

$$\Rightarrow \text{Var}(S^2) = \frac{2(n-1)\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{(n-1)}$$

2. Let X_1, \dots, X_n be iid random variables from a $\mathcal{N}(0, 1)$. Determine the asymptotic distribution of

$$(1/n) \sum_{i=1}^n |X_i|.$$

$$E(|X_i|) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x| e^{-\frac{1}{2}x^2} dx = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x e^{-\frac{1}{2}x^2} dx = \sqrt{\frac{2}{\pi}}$$

$$E(|X_i|^2) = E(X_i^2) = \text{Var}(X_i) + (E(X_i))^2 = 1$$

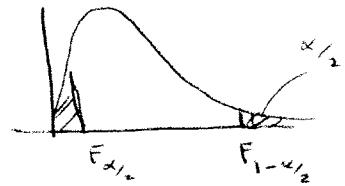
$$\text{Var}(|X_i|) = 1 - \frac{2}{\pi} = \frac{\pi-2}{\pi}$$

By central limit theorem

$$\frac{\frac{1}{n} \sum_{i=1}^n |X_i| - \sqrt{\frac{2}{\pi}}}{\sqrt{\frac{\pi-2}{n\pi}}} \rightarrow N(0, 1)$$

3.. Let X_1, \dots, X_n be iid random variables from a $\mathcal{N}(0, \sigma_1^2)$ and Y_1, \dots, Y_n be iid random variables from a $\mathcal{N}(0, \sigma_2^2)$. Write a 95% confidence interval for σ_1^2/σ_2^2 .

$$\frac{(n-1)S_x^2}{\sigma_1^2} \sim \chi_{(n-1)}^2 \quad \frac{(n-1)S_y^2}{\sigma_2^2} \sim \chi_{(n-1)}^2$$



$$\Rightarrow \frac{\sigma_1^2 S_x^2}{\sigma_2^2 S_y^2} \sim F_{(n,n)} \Rightarrow P \left[F_{\alpha/2(n,n)} < \frac{\sigma_1^2 S_x^2}{\sigma_2^2 S_y^2} < F_{1-\alpha/2(n,n)} \right] = 1 - \alpha$$

$$\Rightarrow 95\% \text{ CI} : \left[F_{\alpha/2(n,n)} \cdot \frac{S_x^2}{S_y^2}, F_{1-\alpha/2(n,n)} \cdot \frac{S_x^2}{S_y^2} \right]$$

4. Let $X \sim N(0, 2)$ and $Y \sim \text{exponential}(1)$. Provided that X is independent of Y , identify the distribution of X/\sqrt{Y} .

$$X/\sqrt{2} \sim N(0, 1)$$

$$Y/2 \sim \chi^2(2)$$

$$\frac{(X/\sqrt{2})}{\sqrt{(Y)/2}} \sim t(2)$$

$$\sqrt{2} \frac{X}{\sqrt{Y}} \sim t(2)$$

$\therefore \frac{X}{\sqrt{Y}}$ is a multiple of t with 2 degrees of freedom.