Chapter 5

#1 $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \Rightarrow$ ~~$Var(\bar{X}_n) = \frac{1}{n^2} var(X_i) = \frac{\sigma_i^2}{n}$~~

$$Var(\bar{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2$$

By Chebyshev

$$P(|\bar{X}_n - \mu| > \varepsilon) \le \frac{Var(\bar{X}_n)}{\varepsilon^2} = \frac{\sum_{i=1}^{n}\sigma_i^2/n^2}{\varepsilon^2} \to 0 \text{ as } n\to\infty$$

#3 $N \sim \text{Poisson}(10{,}000) \approx N(10{,}000, 10{,}000)$

$$P(N > 10{,}200) \cong P\left(\frac{N-10{,}000}{\sqrt{10{,}000}} > \frac{10{,}200-10{,}000}{100}\right)$$

$$= P(Z > 2) \cong .023$$

#5 $n \to \infty \quad p \to 0 \quad np \to \lambda$

$X \sim \text{Binomial}(n,p) \to \text{Poisson}(\lambda)$

$M_X(t) = (pe^t + 1 - p)^n$

$\log M_X(t) = n \log(1 + p(e^t - 1))$

$= n\{p(e^t-1) + O(p^2(e^t-1))\}$ by Taylor expansion

$= np(e^t-1) + n\,O(p^2(e^t-1))$ $\to 0$

$\to \lambda(e^t-1)$ Note that if $Y \sim \text{Poisson}(\lambda)$

then $M_Y(t) = \exp\{\lambda(e^t-1)\}$

#6 $X \sim \Gamma(\alpha, \lambda)$

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$$

Let $$Z = \frac{X - \alpha/\lambda}{\sqrt{\alpha}/\lambda} = \frac{\lambda X - \alpha}{\sqrt{\alpha}}$$

$$M_Z(t) = E\left\{ e^{\left(\frac{\lambda}{\sqrt{\alpha}} X - \sqrt{\alpha}\right)t} \right\}$$

$$= e^{-\sqrt{\alpha}\, t} \; M_X\left(\frac{\lambda}{\sqrt{\alpha}} t\right)$$

$$= e^{-\sqrt{\alpha}\, t} \left(\frac{\lambda}{\lambda - \frac{\lambda}{\sqrt{\alpha}} t}\right)^{\alpha}$$

$$\log M_Z(t) = -\sqrt{\alpha}\, t + \left[\log(\lambda) - \log\left(\lambda - \frac{\lambda}{\sqrt{\alpha}} t\right)\right]\alpha$$

$$= -\sqrt{\alpha}\, t + \left[\log \lambda - \log \lambda - \log\left(1 - \frac{1}{\sqrt{\alpha}} t\right)\right]\alpha$$

$$= -\left\{\sqrt{\alpha}\, t + \alpha \log\left(1 - \frac{1}{\sqrt{\alpha}} t\right)\right\}$$

$$= -\left\{\cancel{\sqrt{\alpha}\, t} + \alpha\left(\cancel{-\frac{1}{\sqrt{\alpha}} t} - \frac{1}{2\alpha} t^2 - \frac{1}{3}\left(\frac{t}{\sqrt{\alpha}}\right)^3 - \cdots\right)\right\}$$

$$\longrightarrow t^2/2 \quad \text{as} \quad \alpha \to \infty \qquad \blacksquare$$

#10

$X =$ # of sixes that appear $\sim$ Binomial $(100, \frac{1}{6})$

X is approximated by $\tilde{X} \sim N\left(\frac{100}{6}, \frac{500}{36}\right)$

$$P(15 \le X \le 20) \doteq P(15 \le \tilde{X} \le 20)$$

$$= P\left(\frac{15 - 100/6}{\sqrt{500/36}} \le Z \le \frac{20 - 100/6}{\sqrt{500/36}}\right)$$

$$= P(-.447 \le Z \le .894) = .487$$

(True value is 0.46)

~~$X_i \sim$ Binomial~~

Let $X_i = 1, 2, 3, 4, 5, 6$ w/ Prob $\frac{1}{6}$

$$P\left(\sum_{i=1}^{100} X_i < 300\right) = P\left(\frac{1}{100}\sum_{i=1}^{100} X_i \le 3\right)$$

$$\bar{X} = \frac{1}{100}\sum_{i=1}^{100} X_i \sim N\left(3.5, \frac{\sqrt{105/36}}{\sqrt{100}}\right) = N\left(3.5, (.1707)^2\right)$$

Note $Var(X_i) = E(X_i^2) - (E(X_i))^2$

$$= (1+4+9+16+25+36)/6 - (3.5)^2$$

$$= \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{105}{36}$$

$$P(\bar{X} \le 3) = P\left(Z \le \frac{3 - 3.5}{.1707}\right) = P(Z \le -2.929$$

$$= .002$$

#12 Let x_i^* be the true value $i=1,\ldots,100$ and x_i be x_i^* with the roundoffs. So

$$x_i^* = x_i + \varepsilon_i \qquad \varepsilon_i \sim \text{unif}[-\tfrac{1}{2},\tfrac{1}{2}]$$

a. $E(\varepsilon_i)=0 \quad Var(\varepsilon_i)=\frac{1}{12}$

$$P\left(\sum_{i=1}^{100}\varepsilon_i > 1\right) = P\left(\bar{\varepsilon} > \frac{1}{100}\right) = P\left(Z > \frac{.01-0}{\sqrt{1200}}\right)$$

$$= P(Z > .0003) \cong .50$$

$$\bar{\varepsilon} \sim N\left(0, \frac{1}{1200}\right)$$

b. $P\left(\bar{\varepsilon} > \frac{5}{100}\right) = P\left(Z > \frac{.05}{\sqrt{1200}}\right) = P(Z > .0014) = .499$

#13 $X_i = \begin{cases} 50 & \text{if he steps north at } i^{th} \text{ minute} \\ -50 & \text{" " " south " "} \end{cases}$

L = location after 1 hour $= \sum_{i=1}^{60} X_i$

$E(X_i)=0 \qquad Var(X_i) = 5000$

$L \sim \cancel{N(0,(5000)60)} \quad N(0,(5000)60)$

#14 $E(X_i) = 50(\tfrac{2}{3}) - 50(\tfrac{1}{3}) = \frac{50}{3}$

$Var(X_i) = 5000$

$L \sim N\left(60\left(\frac{50}{3}\right), 5000(60)\right)$

#21

a) Let $Z_i = \frac{f(X_i)}{g(X_i)}$

$$E(Z_i) = \int_a^b \frac{f(x)}{g(x)} g(x)\, dx = \int_a^b f(x)\, dx$$

$$\Rightarrow E\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right) = \int_a^b f(x)\, dx = I(f)$$

b) $Var(Z_i) = E(Z_i^2) - (E(Z_i))^2$

$$= \int_a^b \frac{f^2(x)}{(g(x))^2} g(x)\, dx - (I(f))^2$$

$$= \int_a^b f^2(x)/g(x)\, dx - (I(f))^2$$

$$\therefore Var(\hat{I}(f)) = \frac{1}{n}\left[\int_a^b \frac{f^2(x)}{g(x)}\, dx - (I(f))^2\right]$$

Let $[a,b] = [-1, 1]$. Take $g(x) = \frac{3}{2}x^2$ $-1 \le x \le 1$

and take $f(x) = 1$. This gives an example with infinite variance.

Take $g(x) \sim Unif(0,1)$ and $f(x) = 1$. This is a finite example.

#21c

That it is the same as Exmple A is obvious.

If $g(x)=1$, then

$$Var(\hat{I}(f)) = \frac{1}{n}\left[\int_0^1 f^2(x)\,dx - [I(f)]^2\right]$$

Choosing a $g(x) \neq 1$ would be more efficient if

$$\int_0^1 f^2(x)\,dx > \int_0^1 \frac{f^2(x)}{g(x)}\,dx$$

Chapter 6

#2.

$$F = \frac{U/m}{V/n} = \frac{n}{m}\left(\frac{U}{V}\right) \quad W = V \qquad U = \frac{m}{n} F W$$

$$\frac{\partial F}{\partial U} = \frac{n}{mV} \qquad \frac{\partial F}{\partial V} = -\frac{n}{m}\frac{U}{V^2}$$

$$\frac{\partial W}{\partial U} = 0 \qquad \frac{\partial W}{\partial V} = 1$$

$$|J|^{-1} = \frac{mV}{n}$$

$$F_{UV}(u,v) = \frac{1}{2^{\frac{m+n}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} v^{\frac{n}{2}-1} u^{\frac{m}{2}-1} e^{-v/2} e^{-u/2}$$

$$F_{F,W}(f,w) = \frac{1}{2^{\frac{m+n}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \cdot w^{\frac{n}{2}-1}\left(\frac{m}{n} f w\right)^{\frac{m}{2}-1} e^{-w/2} e^{-\frac{mfw}{2n}}$$

$$\int_0^\infty F_{F,W}(f,w)\,dw = \frac{1}{2^{\frac{m+n}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \cdot \left(\frac{m}{n}\right)^{\frac{m}{2}-1} f^{\frac{m}{2}-1}$$

Work out the remaining Algebra

$$\int_0^\infty w^{\frac{m+n}{2}-2} e^{-w\left[\frac{1}{2}+\frac{mf}{2n}\right]}\,dw$$

$$\alpha = \frac{m+n}{2} - 1 \qquad \lambda = \frac{1}{2} + \frac{mf}{2n}$$

$$= \frac{\left(\frac{m}{n}\right)^{\frac{m}{2}-1} f^{\frac{m}{2}-1}}{2^{\frac{m+n}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \cdot \left(\frac{1}{2}+\frac{mf}{n}\right)^{-1-\frac{m+n}{2}} \Gamma\left(\frac{m+n}{2}+1\right)$$

$$= \frac{\left(\frac{m}{n}\right)^{-1} f^{\frac{m}{2}-1}}{2^{\frac{m+n}{2}}\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \left(\frac{1}{2}\right)^{-1-\frac{m+n}{2}} \left(\frac{m}{n}\right)^{-1}\left(1+\frac{mf}{n}\right)^{-\frac{m+n}{2}} \left(\frac{1}{2}+\frac{mf}{2n}\right)^{-\frac{m+n}{2}} \Gamma\left(\frac{m+n}{2}\right)$$

work the remaining detail

#8 Let $X \sim exp(1)$

$Y \sim exp(1)$

Consider $U = \frac{X}{2}$

$M_U(t) = M_X(\frac{t}{2}) = \left(\frac{1}{1-2t}\right)$

Note that if $W \sim \chi^2(2)$

then

$$M_W(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right) = \frac{1}{1-2t}$$

So $U \sim \chi^2(2)$

$\Rightarrow X/2 \sim \chi^2(2)$, $Y/2 \sim \chi^2(2)$

$$\frac{[X/2]/2}{[Y/2]/2} \sim F(2,2)$$

$\Rightarrow \frac{X}{Y} \sim F(2,2)$