10.7 Nasser Abbasi

$$\exp\left(\frac{1}{2}x\left(h-\frac{1}{h}\right)\right) = \sum_{m=-\infty}^{\infty} J_m(x)\, h^m$$

using the above, derive the recursion relation

$$J_{m-1}(x) = \left(\frac{2m}{x}\right) J_m(x) - J_{m+1}(x).$$

note $\frac{d}{dh} e^{f(h)} = \frac{d}{dh} f(h)\, e^{f(h)}$

so differentiate left hand side w.r.t. h we set

$$= \frac{d}{dh}\left(\frac{1}{2}x\left(h-\frac{1}{h}\right)\right)\exp\left(\frac{1}{2}x\left(h-\frac{1}{h}\right)\right)$$

$$= \frac{x}{2}\left(1+\frac{1}{h^2}\right)\exp\left(\frac{1}{2}x\left(h-\frac{1}{h}\right)\right)$$

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now differentiate r.H.S. w.r.t h

$$\frac{d}{dh}\sum_{m=-\infty}^{\infty} J_m(x)\, h^m = \sum J_m(x)\frac{d}{dh}h^m = \sum J_m(x)\, m\, h^{m-1}$$

$$= \sum_{m=-\infty}^{\infty} J_m(x)\, m\, h^m h^{-1} = \frac{1}{h}\sum_{m=-\infty}^{\infty} J_m(x)\, h^m\, m$$

so $\frac{x}{2}\left(1+\frac{1}{h^2}\right)\exp\left(\frac{1}{2}x\left(h-\frac{1}{h}\right)\right) = \frac{1}{h}\sum_{m=-\infty}^{\infty} J_m h^m m$ ——— (1)

but $\exp\left(\frac{1}{2}x\left(h-\frac{1}{h}\right)\right) = \sum J_m h^m$ given, then (1) becomes

$$\boxed{\frac{x}{2}\left(1+\frac{1}{h^2}\right)\sum J_m h^m = \frac{1}{h}\sum J_m h^m\, m}$$ ——— (2)

Please see next attempt

hmm... I can't go more using this approach. trying different one →

now, I'll differentiate 2 time with respect to h.

Let $\frac{1}{2}x(h-\frac{1}{h}) \equiv A$

L.H.S.

$$\frac{d}{dh}(L.H.S.) = \frac{x}{2}\left(1+\frac{1}{h^2}\right)\exp\left(\frac{1}{2}x\left(h-\frac{1}{h}\right)\right) = \frac{x}{2}\left(1+\frac{1}{h^2}\right)\exp(A)$$

$$\frac{d^2}{dh^2}(L.H.S.) = -\frac{x}{h^3}\exp(A) + \frac{1}{4}x^2\left(1+\frac{1}{h^2}\right)^2\exp(A).$$

do the same for R.H.S.

$$\frac{d}{dh}(R.H.S.) = \sum J_m\, m\, h^{m-1}$$

$$\frac{d^2}{dh^2}(R.H.S) = \sum J_m\, m\,(m-1)\, h^{m-2}$$

so $$\boxed{-\frac{x}{h^3}\exp A + \frac{1}{4}x^2\left(1+\frac{1}{h^2}\right)^2\exp A = \sum J_m\, m\,(m-1)\, h^{m-2}}$$

but $\exp(A) = \sum J_m h^m$ (given)

so above becomes

$$-\frac{x}{h^3}\sum J_m h^m + \frac{1}{4}x^2\left(1+\frac{1}{h^2}\right)^2\sum J_m h^m = \frac{1}{h^2}\sum J_m\, m\,(m-1)\, h^m$$

$$-\frac{x}{h}\sum J_m h^m + \frac{h^2x^2}{4}\left(1+\frac{1}{h^2}\right)^2\sum J_m h^m = \sum J_m\, m\,(m-1)\, h^m$$

$$\left(\sum J_m h^m\right)\left[\frac{h^2x^2}{4}\left(1+\frac{1}{h^2}\right)^2 - \frac{x}{h}\right] = \sum J_m\, m\,(m-1)\, h^m$$

$$= \sum J_m\, m\,(mh^m - h^m)$$

$$= \sum J_m m^2 h^m - J_m m h^m$$

$$= \sum J_m m^2 h^m - \sum J_m m h^m$$

$\longrightarrow$

- 5a

$$\boxed{\left(\sum J_m h^m\right)\left[\frac{h^2x^2}{4}\left(1+\frac{1}{h^2}\right)^2-\frac{x}{h}\right] = \sum J_m m^2 h^m - \sum J_m m h^m}$$

Dr, can't go further.
need help in how to remove the $\sum$?

Thanks
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