10.14

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$$\int_a^b f(x)\,dx \cong \frac{h}{3}\Big[f(a) + 4f(a+h) + 2f(a+2h) + \cdots + 2f(b-2h) + 4f(b-h) + f(b) \Big]$$

show that this rule is equivalent to second colum of romberg integration

that is $R_{i,2}$.

From 10.52

$$R_{i+1,j+1} = R_{i+1,j} + \frac{1}{4^j - 1}\Big[R_{i+1,j} - R_{i,j} \Big].$$

For second column, $j=1$.

$$\begin{array}{llll} R_{11} & & & \\ R_{21} & R_{22} & & \\ R_{31} & R_{32} & R_{33} & \\ R_{41} & R_{42} & R_{43} & R_{44} \end{array}$$

so $$R_{i+1,2} = R_{i+1,1} + \frac{1}{3}\Big[R_{i+1,1} - R_{i,1} \Big]$$

$i=1$ $$R_{2,2} = R_{2,1} + \frac{1}{3}\Big[R_{2,1} - R_{1,1} \Big]$$

$i=2$ $$R_{3,2} = R_{3,1} + \frac{1}{3}\Big[R_{3,1} - R_{2,1} \Big]$$

$i=3$ $$R_{4,2} = R_{4,1} + \frac{1}{3}\Big[R_{4,1} - R_{3,1} \Big]$$

etc.

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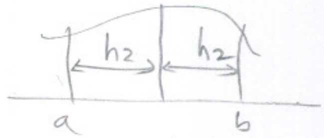
now $R_{i,1} = I_T(h_i)$, so,
$R_{1,1} = I_T(h_1)$
$R_{2,1} = I_T(h_2)$
$R_{3,1} = I_T(h_3)$
etc...

so $\quad R_{2,2} = I_T(h_2) + \frac{1}{3}\left[I_T(h_2) - I_T(h_1) \right]$

$R_{2,2}$ = 2 panels.



$$\cancel{= \frac{1}{2} h_2 \left[f(a) + f(b) \right] + h_2 f(a+h_2) + \frac{1}{3}\Big[}$$

$$= \frac{4}{3} I_T(h_2) - \frac{1}{3} I_T(h_1)$$

$$= \frac{4}{3}\left[\frac{1}{2} h_2 \left[f(a)+f(b) \right] + h_2 f(a+h_2) \right] - \frac{1}{3}\frac{1}{2} h_1 \left[f(a)+f(b) \right]$$

but $h_1 = 2 h_2$

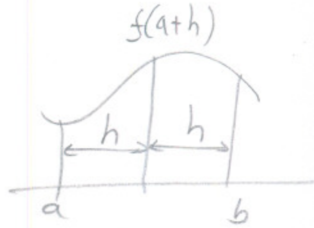
$$R_{2,2} = \frac{4}{3}\left(\frac{1}{2} h_2 \left(f(a)+f(b) \right) + h_2 f(a+h_2) \right) - \frac{1}{3} h_2 \left[f(a)+f(b) \right]$$

$$= \frac{2}{3} h_2 \left(f(a)+f(b) \right) + \frac{4}{3} h_2 f(a+h_2) - \frac{1}{3} h_2 \left(f(a)+f(b) \right)$$

$$= \frac{1}{3} h_2 \left(f(a)+f(b) \right) + \frac{4}{3} h_2 f(a+h_2) \quad \text{—— (1)}$$

$\because$, simpson rule for 2 panels is

$$\frac{h}{3}\left[f(a) + 4 f(a+h) + f(b) \right] \quad \text{—— (2)}$$



Compare (1) and (2), They are the same. when $h_2 = h$.
so for two panels, second column $R_{2,2}$ is the same as simpson Rule.
I can continue like this showing for each $R_{i,2}$, it is the same as simpson rule by adding one more panel each time.

will show for $R_{3,2}$ ⟶

and stop. if correct for $R_{2,2}$ and $R_{3,2}$ then must be correct for all others sine recursive equation was used to derive each time.

$n=3$

$$R_{3,2} = R_{3,1} + \frac{1}{3}\left[R_{3,1} - R_{2,1}\right]$$

$$= \frac{4}{3} R_{3,1} - \frac{1}{3} R_{2,1}$$

$$= \frac{4}{3} I_T(h_3) - \frac{1}{3} I_T(h_2)$$

$$= \frac{4}{3}\left[\frac{1}{2} I_T(h_2) + h_3 \sum_{i=1}^{2^{3-1}} f\left[a+(2i-1)h_3\right]\right] - \frac{1}{3} I_T(h_2)$$

$$= \frac{4}{3}\left[\frac{1}{2} I_T(h_2) + h_3\Big(f(a+h_3) + f(a+3h_3)\Big)\right] - \frac{1}{3} I_T(h_2)$$

$$= \frac{2}{3} I_T(h_2) + \frac{4}{3} h_3 \Big(f(a+h_3) + f(a+3h_3)\Big) - \frac{1}{3} I_T(h_2)$$

$$= \frac{1}{3} I_T(h_2) + \underbrace{\frac{4}{3} h_3 \Big(f(a+h_3) + f(a+3h_3)\Big)}$$

$$= \frac{1}{3}\left[\frac{1}{2} h_2 \left[f(a)+f(b)\right] + h_2 f(a+h_2)\right] +$$

but $h_2 = 2h_3$

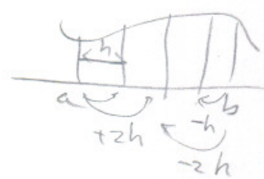
so $$R_{3,2} = \frac{1}{3}\left[\frac{1}{2} 2h_3 \big(f(a)+f(b)\big) + 2h_3 f(a+2h_3)\right] + \frac{4}{3} h_3 \big(f(a+h_3) + f(a+3h_3)\big)$$

$$= \frac{1}{3} h_3 \big(f(a)+f(b)\big) + \frac{2}{3} h_3 \big(f(a+2h_3)\big) + \frac{4}{3} h_3 \big(f(a+h_3) + f(a+3h_3)\big)$$

$$= \frac{h_3}{3}\left[f(a) + f(b) + 2f(a+2h_3) + 4 f(a+h_3) + 4 f(a+3h_3)\right] \quad (1)$$

now simpson rule for 4 panels is

I'd prefer a more general proof but OK



$$\frac{h}{3}\Big[f(a) + 4f(a+h) + 2f(a+2h) + 4f(b-h) + f(b)\Big] \quad (2)$$

Compare (1) with (2) They are the same when $h_3 = h$. note $a+3h_3 = b-h$

QED.