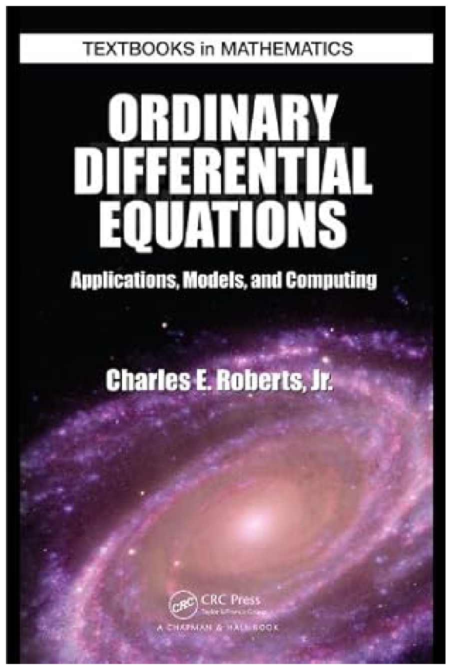
A Solution Manual For

Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010



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1.1 problem 15

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Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2y'' + y'x - y = 0}$$

1.1.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 1 = 0$$

Or

$$r^2 - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2 x$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + c_2 x \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + c_2 x$$

Verified OK.

1.1.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\left(\int p(x)dx\right)}dx \\ &= \int e^{-\left(\int \frac{1}{x}dx\right)}dx \\ &= \int e^{-\ln(x)}\,dx \\ &= \int \frac{1}{x}dx \\ &= \ln(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= -1 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - y(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} - e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +1 \\ \lambda_2 &= -1 \end{aligned}$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -1
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\
y(\tau) &= c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}
\end{aligned}
$$

Or

$$
y(\tau) = c_1 \mathrm{e}^{\tau} + c_2 \mathrm{e}^{-\tau}
$$

The above solution is now transformed back to y using (6) which results in

$$
y = \frac{c_1 x^2 + c_2}{x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x^2 + c_2}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x^2 + c_2}{x}
$$

Verified OK.

1.1.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$
x^2 y'' + y' x - y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -\frac{1}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-\frac{1}{x^2}}}{c} \\ \tau'' &= \frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\tau = \int \frac{1}{c}\sqrt{q}\, dx = \frac{\int \sqrt{-\frac{1}{x^2}}dx}{c} = \frac{\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c}$$

Substituting the above into the solution obtained gives

$$y = \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}$$

Verified OK.

1.1.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{2x^2} + c_2\right) x
$$

Verified OK.

1.1.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int \left(x^2 y'' + y'x - y\right) dx &= 0 \\
x^2 y' - yx &= c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right) \\ \mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{c_1}{x^3}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{c_1}{x^3}\,\mathrm{d}x \\ \frac{y}{x} &= -\frac{c_1}{2x^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

Summary
The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

1.1.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2 \\
B &= x \\
C &= -1 \\
F &= 0
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left(x^2\right)(0) + (x)(1) + (-1)(x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x^3v'' + (3x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$x^2(u'(x)\, x + 3u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{3u}{x}
\end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{3}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\
\ln (u) &= -3 \ln (x) + c_1 \\
u &= \mathrm{e}^{-3 \ln(x) + c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= \frac{c_1}{x^3}
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{c_1}{x^3}\, \mathrm{d}x \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y(x) &= Bv \\
&= (x) \left(-\frac{c_1}{2x^2} + c_2\right) \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}$$

Verification of solutions

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x$$

Verified OK.

1.1.7 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' + y'x - y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' + y'x - y)\, dx = 0$$
$$x^2y' - yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\
\frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right) \\
d\left(\frac{y}{x}\right) &= \left(\frac{c_1}{x^3}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int \frac{c_1}{x^3}\, dx \\
\frac{y}{x} &= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
y = -\frac{c_1}{2x} + c_2 x
$$

Summary

The solution(s) found are the following

$$
y = -\frac{c_1}{2x} + c_2 x \tag{1}
$$

Verification of solutions

$$
y = -\frac{c_1}{2x} + c_2 x
$$

Verified OK.

1.1.8 Solving using Kovacic algorithm

Writing the ode as

$$
x^2 y'' + y'x - y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= x \\
C &= -1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 1: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx$$
$$= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx$$
$$= y_1 \left(\frac{x^2}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^2}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x}{2}$$

Verified OK.

1.1.9 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$p(x) = x^2$$
$$q(x) = x$$
$$r(x) = -1$$
$$s(x) = 0$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 1
\end{aligned}
$$

Therefore (1) becomes

$$
2 - (1) + (-1) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
x^2 y' - yx = c_1
$$

We now have a first order ode to solve which is

$$
x^2 y' - yx = c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{y}{x} = \frac{c_1}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{c_1}{x^2}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right)$$
$$\mathrm{d}\left(\frac{y}{x}\right) = \left(\frac{c_1}{x^3}\right)\mathrm{d}x$$

Integrating gives

$$\frac{y}{x} = \int \frac{c_1}{x^3}\,\mathrm{d}x$$
$$\frac{y}{x} = -\frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

Summary
The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

1.1.10 Maple step by step solution

Let's solve

$y''x^2 + y'x - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 + y'x - y = 0$
- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$
  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$
  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^2 + \frac{d}{dt}y(t) - y(t) = 0$
- Simplify

  $\frac{d^2}{dt^2}y(t) - y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 - 1 = 0$
- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the ODE

  $y_1(t) = e^{-t}$
- 2nd solution of the ODE

  $y_2(t) = e^{t}$
- General solution of the ODE

$y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{-t} + c_2 e^t$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x} + c_2 x$

- Simplify

  $y = \frac{c_1}{x} + c_2 x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^2 + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 16

```
DSolve[x^2*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x} + c_2 x$$

1.2 problem 16

Internal problem ID [12575]
Internal file name [`OUTPUT/11227_Wednesday_October_18_2023_10_03_52_PM_6299146/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y'x - y = 0}$$

1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{x}\,dx \\
\ln(y) &= \ln(x) + c_1 \\
y &= e^{\ln(x)+c_1} \\
&= c_1 x
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x \tag{1}$$

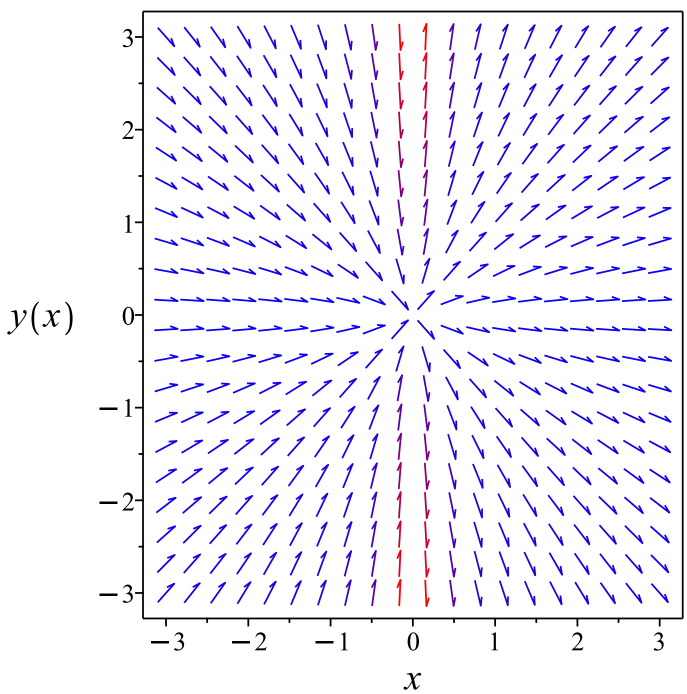


Figure 1: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

1.2.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x$$

Summary
The solution(s) found are the following

$$y = c_1 x \tag{1}$$

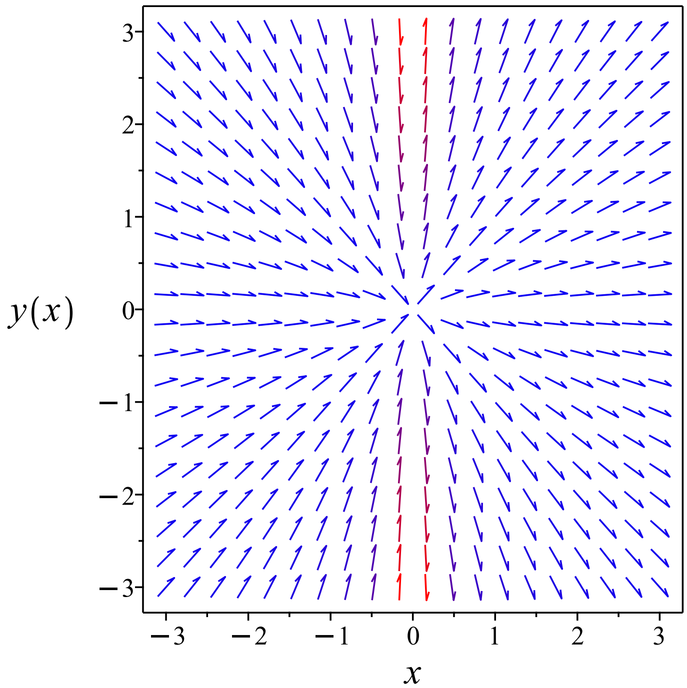


Figure 2: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

1.2.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x - u(x)\,x = 0$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0 \,\mathrm{d}x \\ &= c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= c_2 x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x \tag{1}$$

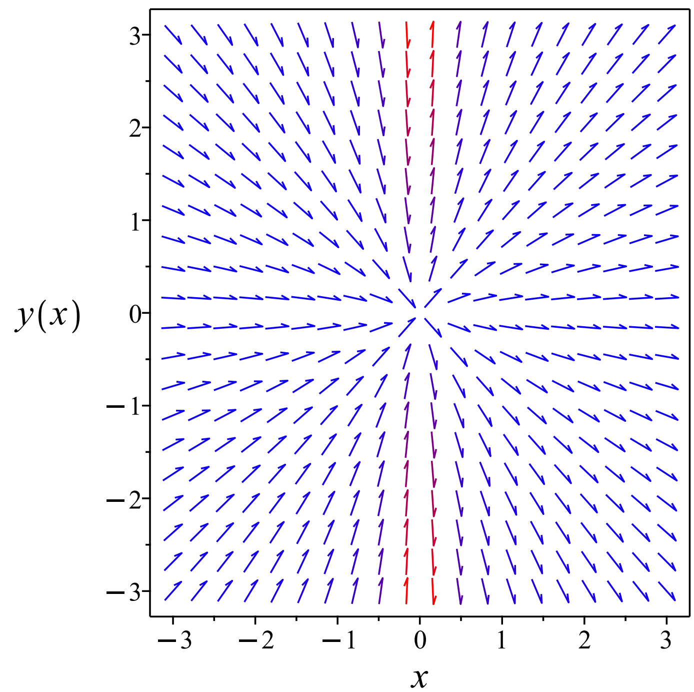


Figure 3: Slope field plot

Verification of solutions

$$y = c_2 x$$

Verified OK.

1.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 3: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = c_1$$

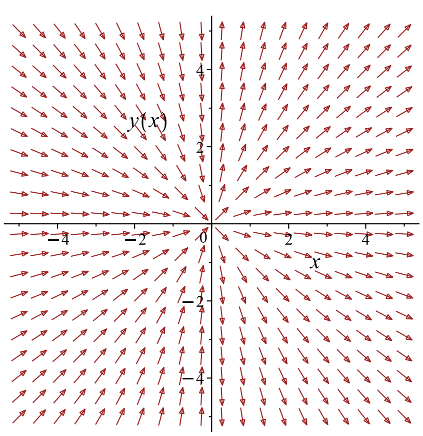
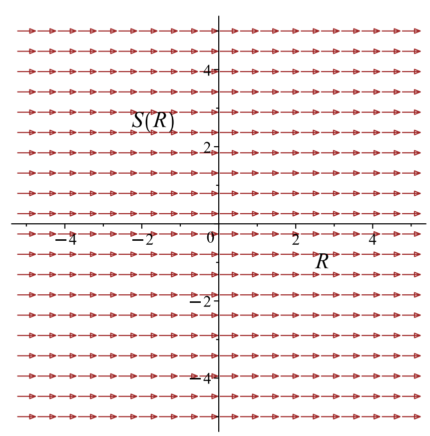
Which simplifies to

$$\frac{y}{x} = c_1$$

Which gives

$$y = c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 x \tag{1}$$

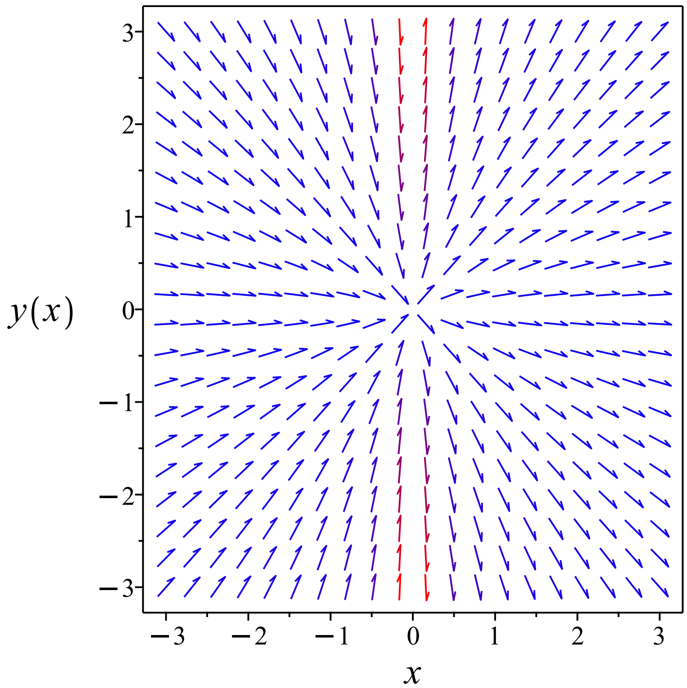


Figure 4: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 x$$

Verified OK.

1.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y}\right)\mathrm{d}y = \left(\frac{1}{x}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = \frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y)$$

The solution becomes

$$y = e^{c_1}x$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{c_1}x \tag{1}$$

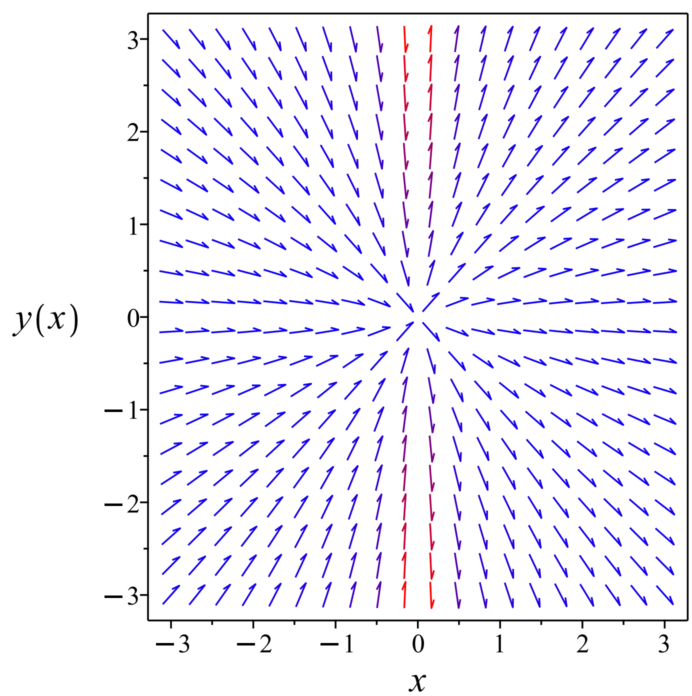


Figure 5: Slope field plot

<u>Verification of solutions</u>

$$y = e^{c_1}x$$

Verified OK.

1.2.6 Maple step by step solution

Let's solve

$y'x - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = e^{c_1} x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve(x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 14

```
DSolve[x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x$$
$$y(x) \to 0$$

1.3 problem 17

Internal problem ID [12576]
Internal file name [`OUTPUT/11228_Wednesday_October_18_2023_10_03_53_PM_49814442/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{2x^2y'' + 3y'x - y = 0}$$

1.3.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$2x^2(r(r-1))x^{r-2} + 3xrx^{r-1} - x^r = 0$$

Simplifying gives

$$2r(r-1)\,x^r + 3r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$2r(r-1) + 3r - 1 = 0$$

Or

$$2r^2 + r - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = \frac{1}{2}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + \sqrt{x}\, c_2$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + \sqrt{x}\, c_2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \sqrt{x}\, c_2$$

Verified OK.

1.3.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$2x^2 y'' + 3y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{3}{2x}$$
$$q(x) = -\frac{1}{2x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{3}{2x}dx\right)}dx \\
&= \int e^{-\frac{3\ln(x)}{2}}\,dx \\
&= \int \frac{1}{x^{\frac{3}{2}}}dx \\
&= -\frac{2}{\sqrt{x}}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{2x^2}}{\frac{1}{x^3}} \\
&= -\frac{x}{2}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{xy(\tau)}{2} &= 0
\end{aligned}$$

But in terms of τ

$$-\frac{x}{2} = -\frac{2}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{2y(\tau)}{\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 2\tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r - 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau} + c_2\tau^2$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{-x^{\frac{3}{2}}c_1 + 8c_2}{2x}$$

Summary
The solution(s) found are the following

$$y = \frac{-x^{\frac{3}{2}} c_1 + 8c_2}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{-x^{\frac{3}{2}} c_1 + 8c_2}{2x}$$

Verified OK.

1.3.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$2x^2y'' + 3y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{2x} \\ q(x) &= -\frac{1}{2x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{3n}{2x^2} - \frac{1}{2x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = \frac{1}{2} \tag{6}$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \frac{5v'(x)}{2x} &= 0 \\
v''(x) + \frac{5v'(x)}{2x} &= 0 \qquad (7)
\end{aligned}
$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{2x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{5u}{2x}
\end{aligned}
$$

Where $f(x) = -\frac{5}{2x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{5}{2x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{5}{2x}\, dx \\
\ln(u) &= -\frac{5\ln(x)}{2} + c_1 \\
u &= e^{-\frac{5\ln(x)}{2} + c_1} \\
&= \frac{c_1}{x^{\frac{5}{2}}}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{2c_1}{3x^{\frac{3}{2}}} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)\sqrt{x} \\
&= \frac{3x^{\frac{3}{2}} c_2 - 2c_1}{3x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)\sqrt{x} \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)\sqrt{x}
$$

Verified OK.

1.3.4 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int \left(2x^2y'' + 3y'x - y\right) dx &= 0 \\
-yx + 2x^2y' &= c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{2x} \\
q(x) &= \frac{c_1}{2x^2}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{y}{2x} = \frac{c_1}{2x^2}
$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{2x}dx}$$
$$= \frac{1}{\sqrt{x}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{2x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{\sqrt{x}}\right) = \left(\frac{1}{\sqrt{x}}\right)\left(\frac{c_1}{2x^2}\right)$$
$$d\left(\frac{y}{\sqrt{x}}\right) = \left(\frac{c_1}{2x^{\frac{5}{2}}}\right) dx$$

Integrating gives

$$\frac{y}{\sqrt{x}} = \int \frac{c_1}{2x^{\frac{5}{2}}} dx$$
$$\frac{y}{\sqrt{x}} = -\frac{c_1}{3x^{\frac{3}{2}}} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x}}$ results in

$$y = -\frac{c_1}{3x} + \sqrt{x}\, c_2$$

Summary
The solution(s) found are the following

$$y = -\frac{c_1}{3x} + \sqrt{x}\, c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{3x} + \sqrt{x}\, c_2$$

Verified OK.

1.3.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$2x^2y'' + 3y'x - y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned}\int \left(2x^2y'' + 3y'x - y\right) dx &= 0\\ -yx + 2x^2y' &= c_1\end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{1}{2x}\\ q(x) &= \frac{c_1}{2x^2}\end{aligned}$$

Hence the ode is

$$y' - \frac{y}{2x} = \frac{c_1}{2x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{2x}dx}\\ &= \frac{1}{\sqrt{x}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{2x^2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{x}}\right) &= \left(\frac{1}{\sqrt{x}}\right)\left(\frac{c_1}{2x^2}\right)\\ \mathrm{d}\left(\frac{y}{\sqrt{x}}\right) &= \left(\frac{c_1}{2x^{\frac{5}{2}}}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\frac{y}{\sqrt{x}} = \int \frac{c_1}{2x^{\frac{5}{2}}}\, \mathrm{d}x$$
$$\frac{y}{\sqrt{x}} = -\frac{c_1}{3x^{\frac{3}{2}}} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x}}$ results in

$$y = -\frac{c_1}{3x} + \sqrt{x}\, c_2$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{3x} + \sqrt{x}\, c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{3x} + \sqrt{x}\, c_2$$

Verified OK.

1.3.6 Solving using Kovacic algorithm

Writing the ode as

$$2x^2y'' + 3y'x - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 \\ B &= 3x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5}{16x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5$$
$$t = 16x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{5}{16x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 6: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{5}{16x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{5}{16x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{4x} + (-)\,(0) \\ &= -\frac{1}{4x} \\ &= -\frac{1}{4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{4x}\right)(0) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x}\right)^2 - \left(\frac{5}{16x^2}\right)\right) &= 0 \\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{4x} dx} \\ &= \frac{1}{x^{\frac{1}{4}}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3x}{2x^2} \, dx} \\ &= z_1 e^{-\frac{3\ln(x)}{4}} \\ &= z_1\left(\frac{1}{x^{\frac{3}{4}}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{2x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{3\ln(x)}{2}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{2x^{\frac{3}{2}}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{2x^{\frac{3}{2}}}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} + \frac{2\sqrt{x}\, c_2}{3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x} + \frac{2\sqrt{x}\, c_2}{3}
$$

Verified OK.

1.3.7 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= 2x^2 \\
q(x) &= 3x \\
r(x) &= -1 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 4 \\
q'(x) &= 3
\end{aligned}
$$

Therefore (1) becomes

$$
4 - (3) + (-1) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
-yx + 2x^2 y' = c_1
$$

We now have a first order ode to solve which is

$$
-yx + 2x^2 y' = c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{2x} \\
q(x) &= \frac{c_1}{2x^2}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{y}{2x} = \frac{c_1}{2x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{1}{2x}dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{2x^2}\right) \\
\frac{d}{dx}\left(\frac{y}{\sqrt{x}}\right) &= \left(\frac{1}{\sqrt{x}}\right)\left(\frac{c_1}{2x^2}\right) \\
d\left(\frac{y}{\sqrt{x}}\right) &= \left(\frac{c_1}{2x^{\frac{5}{2}}}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{\sqrt{x}} &= \int \frac{c_1}{2x^{\frac{5}{2}}}\, dx \\
\frac{y}{\sqrt{x}} &= -\frac{c_1}{3x^{\frac{3}{2}}} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x}}$ results in

$$
y = -\frac{c_1}{3x} + \sqrt{x}\, c_2
$$

Summary

The solution(s) found are the following

$$
y = -\frac{c_1}{3x} + \sqrt{x}\, c_2 \tag{1}
$$

Verification of solutions

$$
y = -\frac{c_1}{3x} + \sqrt{x}\, c_2
$$

Verified OK.

1.3.8 Maple step by step solution

Let's solve

$2y''x^2 + 3y'x - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{2x} + \frac{y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{3y'}{2x} - \frac{y}{2x^2} = 0$

- Multiply by denominators of the ODE

  $2y''x^2 + 3y'x - y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^2 + 3\frac{d}{dt}y(t) - y(t) = 0$

- Simplify

  $2\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) - y(t) = 0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{\frac{d}{dt}y(t)}{2} + \frac{y(t)}{2}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) + \frac{\frac{d}{dt}y(t)}{2} - \frac{y(t)}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{1}{2}r - \frac{1}{2} = 0$

- Factor the characteristic polynomial

  $\frac{(r+1)(2r-1)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(-1, \frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(t) = e^{-t}$
- 2nd solution of the ODE

  $y_2(t) = e^{\frac{t}{2}}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 e^{-t} + c_2 e^{\frac{t}{2}}$
- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x} + \sqrt{x}\, c_2$
- Simplify

  $y = \frac{c_1}{x} + \sqrt{x}\, c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(2*x^2*diff(y(x),x$2)+3*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^{\frac{3}{2}} + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 20

```
DSolve[2*x^2*y''[x]+3*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^{3/2} + c_1}{x}$$

1.4 problem 18

Internal problem ID [12577]
Internal file name [`OUTPUT/11229_Thursday_October_19_2023_04_43_29_PM_53032459/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 3y' + 2y = 0}$$

1.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\lambda_{1,2} = \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-3^2 - (4)(1)(2)}$$
$$= \frac{3}{2} \pm \frac{1}{2}$$

Hence

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x} \tag{1}$$

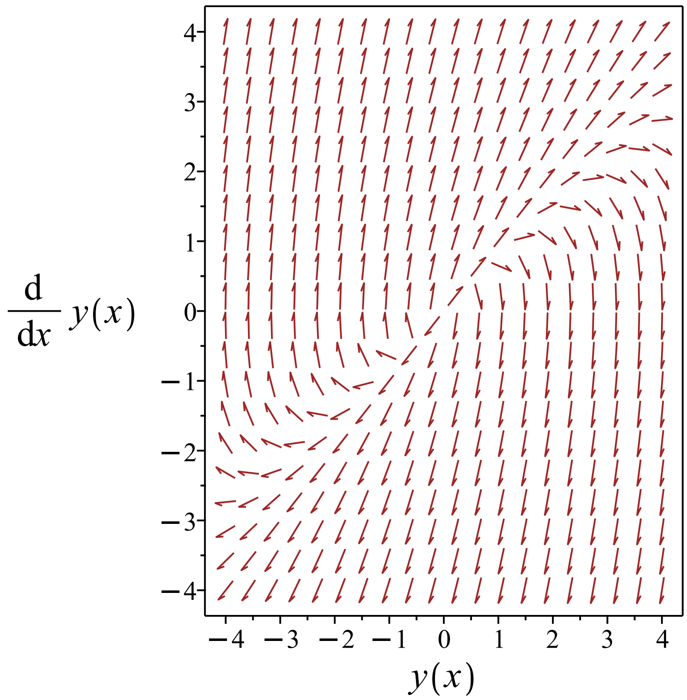


Figure 6: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{2x} + c_2 e^{x}$$

Verified OK.

1.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 8: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\mathrm{e}^x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^x c_1 + c_2 \mathrm{e}^{2x} \tag{1}
$$

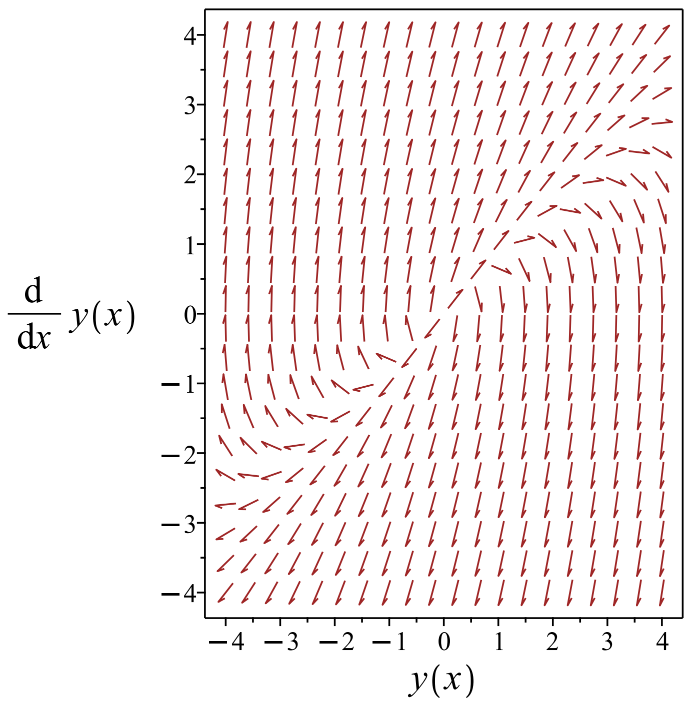


Figure 7: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^x c_1 + c_2 \mathrm{e}^{2x}
$$

Verified OK.

1.4.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the ODE

  $y_1(x) = e^x$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = e^x c_1 + c_2 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + c_2 e^{2x}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 18

```
DSolve[y''[x]-3*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(c_2 e^x + c_1)$$

1.5 problem 19

Internal problem ID [12578]
Internal file name [`OUTPUT/11230_Thursday_October_19_2023_04_43_30_PM_79343540/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2y'' - 2y = 0}$$

1.5.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 0rx^{r-1} - 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 0\,x^r - 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2 x^2$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1}{x} + c_2 x^2 \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1}{x} + c_2 x^2$$

Verified OK.

1.5.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2 y'' - 2y) \, dx = 0$$
$$x^2 y' - 2yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{c_1}{x^2}\right) \\ \mathrm{d}\left(\frac{y}{x^2}\right) &= \left(\frac{c_1}{x^4}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2} &= \int \frac{c_1}{x^4}\,\mathrm{d}x \\ \frac{y}{x^2} &= -\frac{c_1}{3x^3} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = -\frac{c_1}{3x} + c_2 x^2$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{c_1}{3x} + c_2 x^2 \tag{1}$$

<u>Verification of solutions</u>

$$y = -\frac{c_1}{3x} + c_2 x^2$$

Verified OK.

1.5.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' - 2y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' - 2y)\, dx = 0$$
$$x^2y' - 2yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x} dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \left(\frac{1}{x^2}\right)\left(\frac{c_1}{x^2}\right)$$
$$d\left(\frac{y}{x^2}\right) = \left(\frac{c_1}{x^4}\right) dx$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{c_1}{x^4}\, \mathrm{d}x$$
$$\frac{y}{x^2} = -\frac{c_1}{3x^3} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = -\frac{c_1}{3x} + c_2 x^2$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{3x} + c_2 x^2 \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{3x} + c_2 x^2$$

Verified OK.

1.5.4 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2$$
$$t = x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 10: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \frac{1}{x} \int \frac{1}{\frac{1}{x^2}} \, dx \\ &= \frac{1}{x}\left(\frac{x^3}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x}\right) + c_2\left(\frac{1}{x}\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x} + \frac{c_2 x^2}{3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x} + \frac{c_2 x^2}{3}
$$

Verified OK.

1.5.5 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x^2 \\
q(x) &= 0 \\
r(x) &= -2 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
2 - (0) + (-2) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$x^2 y' - 2yx = c_1$$

We now have a first order ode to solve which is

$$x^2 y' - 2yx = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{c_1}{x^2} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{2}{x} dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\ \frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{c_1}{x^2}\right) \\ d\left(\frac{y}{x^2}\right) &= \left(\frac{c_1}{x^4}\right) dx \end{aligned}$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{c_1}{x^4}\, dx$$
$$\frac{y}{x^2} = -\frac{c_1}{3x^3} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = -\frac{c_1}{3x} + c_2 x^2$$

Summary
The solution(s) found are the following

$$y = -\frac{c_1}{3x} + c_2 x^2 \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{3x} + c_2 x^2$$

Verified OK.

1.5.6 Maple step by step solution

Let's solve

$y''x^2 - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 - 2y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

$y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^2 - 2y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - 2y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the ODE

  $y_1(t) = e^{-t}$

- 2nd solution of the ODE

  $y_2(t) = e^{2t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{-t} + c_2 e^{2t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x} + c_2 x^2$

- Simplify

$y = \frac{c_1}{x} + c_2 x^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^3 + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^3 + c_1}{x}$$

1.6 problem 20

Internal problem ID [12579]
Internal file name [`OUTPUT/11231_Thursday_October_19_2023_04_43_31_PM_58456927/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' + \frac{1}{2y} = 0}$$

1.6.1 Solving as quadrature ode

Integrating both sides gives

$$\int -2y dy = x + c_1$$
$$-y^2 = x + c_1$$

Solving for y gives these solutions

$$y_1 = \sqrt{-x - c_1}$$
$$y_2 = -\sqrt{-x - c_1}$$

Summary
The solution(s) found are the following

$$y = \sqrt{-x - c_1} \tag{1}$$
$$y = -\sqrt{-x - c_1} \tag{2}$$

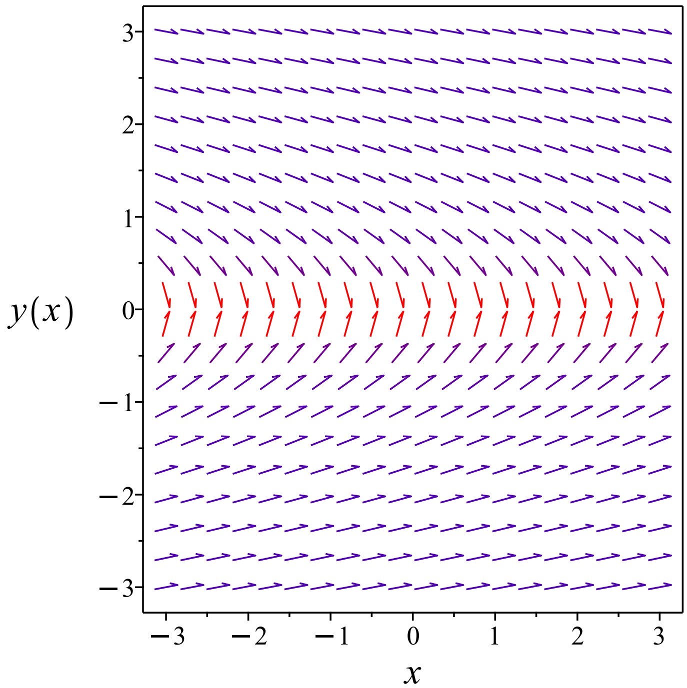


Figure 8: Slope field plot

Verification of solutions

$$y = \sqrt{-x - c_1}$$

Verified OK.

$$y = -\sqrt{-x - c_1}$$

Verified OK.

1.6.2 Maple step by step solution

Let's solve

$y' + \frac{1}{2y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = -\frac{1}{2}$

- Integrate both sides with respect to x

$$\int yy'dx = \int -\frac{1}{2}dx + c_1$$

- Evaluate integral

$$\frac{y^2}{2} = -\frac{x}{2} + c_1$$

- Solve for y

$$\left\{y = \sqrt{-x + 2c_1}, y = -\sqrt{-x + 2c_1}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)+1/(2*y(x))=0,y(x), singsol=all)
```

$$y(x) = \sqrt{c_1 - x}$$
$$y(x) = -\sqrt{c_1 - x}$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 35

```
DSolve[y'[x]+1/(2*y[x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x + 2c_1}$$
$$y(x) \to \sqrt{-x + 2c_1}$$

1.7 problem 21

Internal problem ID [12580]
Internal file name [`OUTPUT/11232_Thursday_October_19_2023_04_43_32_PM_85705207/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y}{x} = 1}$$

1.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 1$$

Hence the ode is

$$y' - \frac{y}{x} = 1$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \frac{1}{x} \\
\mathrm{d}\left(\frac{y}{x}\right) &= \frac{1}{x}\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int \frac{1}{x}\,\mathrm{d}x \\
\frac{y}{x} &= \ln (x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
y = x \ln (x) + c_1 x
$$

which simplifies to

$$
y = x(\ln (x) + c_1)
$$

Summary

The solution(s) found are the following

$$
y = x(\ln (x) + c_1) \tag{1}
$$

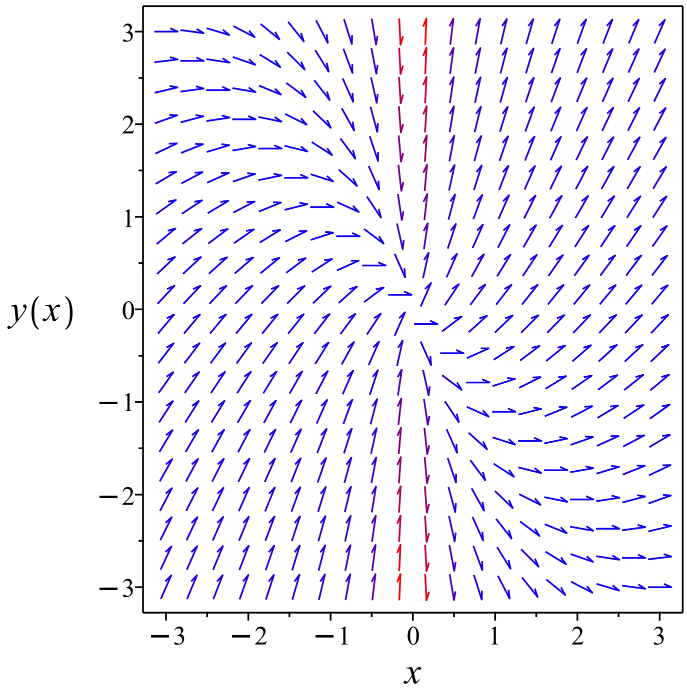


Figure 9: Slope field plot

<u>Verification of solutions</u>

$$y = x(\ln(x) + c_1)$$

Verified OK.

1.7.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x = 1$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{1}{x}\,\mathrm{d}x \\ &= \ln(x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x(\ln(x) + c_2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x(\ln (x) + c_2) \tag{1}$$

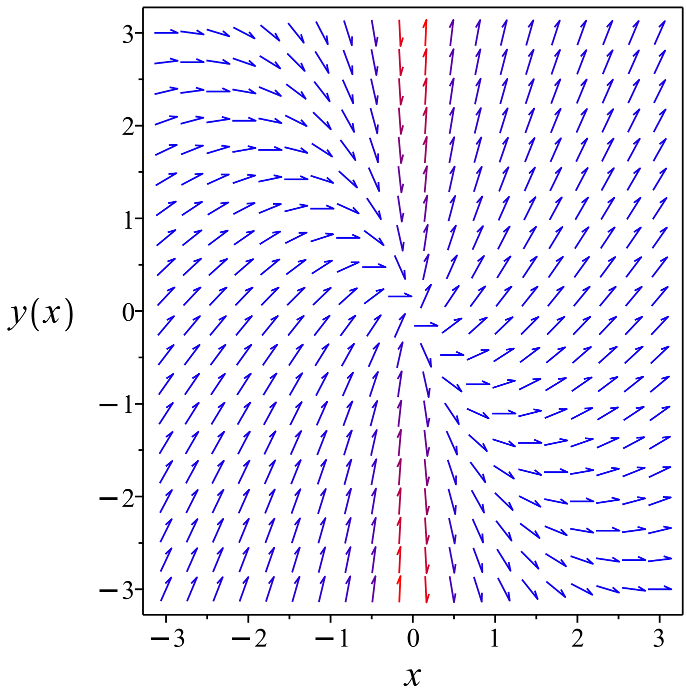


Figure 10: Slope field plot

Verification of solutions

$$y = x(\ln (x) + c_2)$$

Verified OK.

1.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + x}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 13: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + x}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \ln(x) + c_1$$

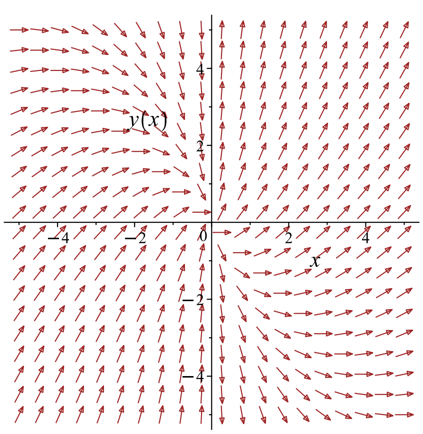
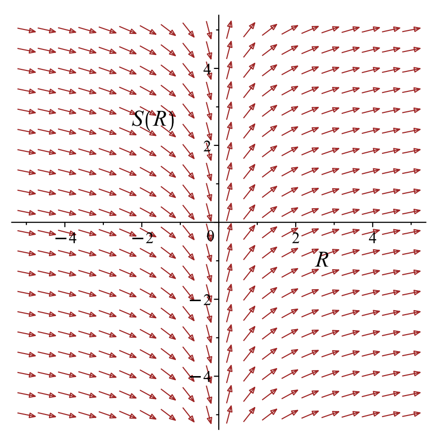
Which simplifies to

$$\frac{y}{x} = \ln(x) + c_1$$

Which gives

$$y = x(\ln(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+x}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary
The solution(s) found are the following

$$y = x(\ln(x) + c_1) \tag{1}$$

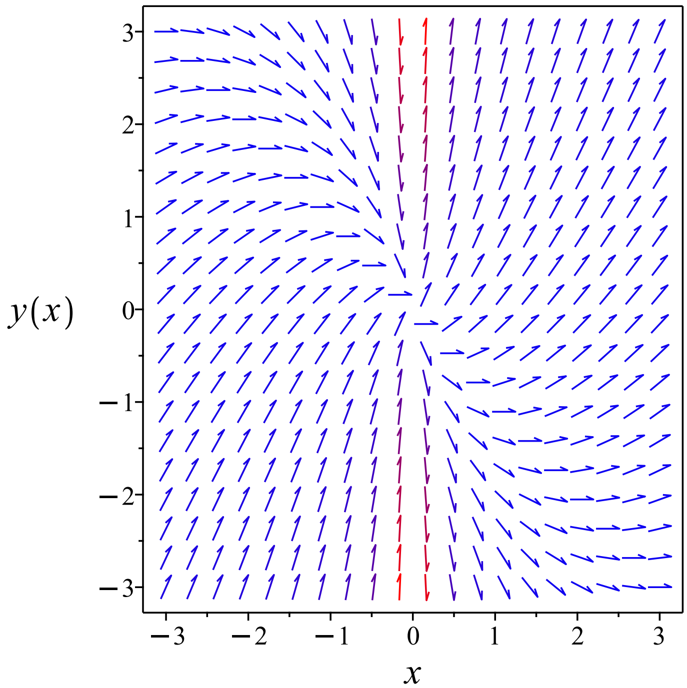


Figure 11: Slope field plot

Verification of solutions

$$y = x(\ln(x) + c_1)$$

Verified OK.

1.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(1 + \frac{y}{x}\right) \mathrm{d}x \\
\left(-1 - \frac{y}{x}\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -1 - \frac{y}{x} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-1 - \frac{y}{x}\right) \\
&= -\frac{1}{x}
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(-\frac{1}{x}\right) - (0)\right)\\ &= -\frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(x)}\\ &= \frac{1}{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x}\left(-1 - \frac{y}{x}\right)\\ &= \frac{-y - x}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x}(1)\\ &= \frac{1}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-y-x}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-y-x}{x^2}\,\mathrm{d}x$$
$$\phi = \frac{y}{x} - \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y}{x} - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y}{x} - \ln(x)$$

The solution becomes

$$y = x(\ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = x(\ln(x) + c_1) \tag{1}$$

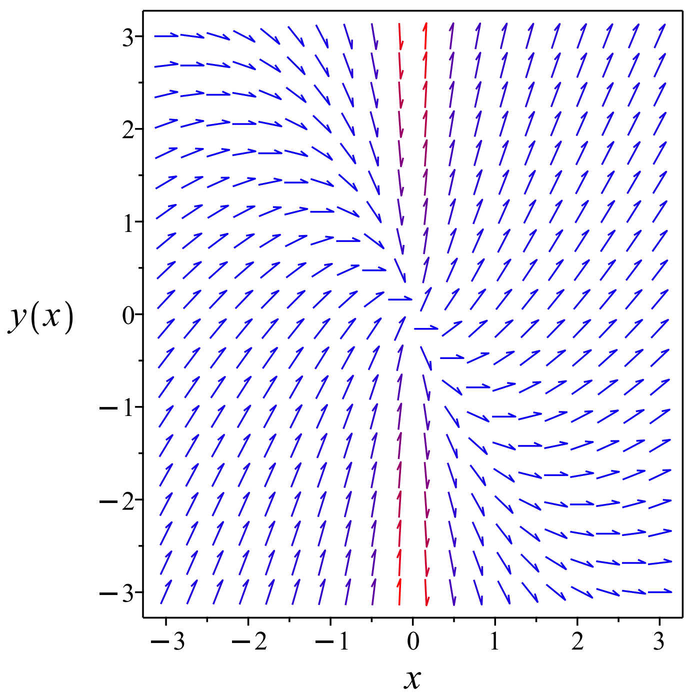


Figure 12: Slope field plot

Verification of solutions

$$y = x(\ln(x) + c_1)$$

Verified OK.

1.7.5 Maple step by step solution

Let's solve

$y' - \frac{y}{x} = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \frac{1}{x}dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(\ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)-y(x)/x=1,y(x), singsol=all)
```

$$y(x) = (\ln(x) + c_1)\,x$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 12

```
DSolve[y'[x]-y[x]/x==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(\log(x) + c_1)$$

1.8 problem 22

Internal problem ID [12581]
Internal file name [`OUTPUT/11233_Thursday_October_19_2023_04_43_32_PM_53349013/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 2\sqrt{|y|} = 0}$$

1.8.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{2\sqrt{|y|}} dy = \int dx$$

$$\frac{\left(\begin{cases} -2\sqrt{-y} & y \le 0 \\ 2\sqrt{y} & 0 < y \end{cases}\right)}{2} = x + c_1$$

Summary
The solution(s) found are the following

$$\frac{\left(\begin{cases} -2\sqrt{-y} & y \le 0 \\ 2\sqrt{y} & 0 < y \end{cases}\right)}{2} = x + c_1 \tag{1}$$

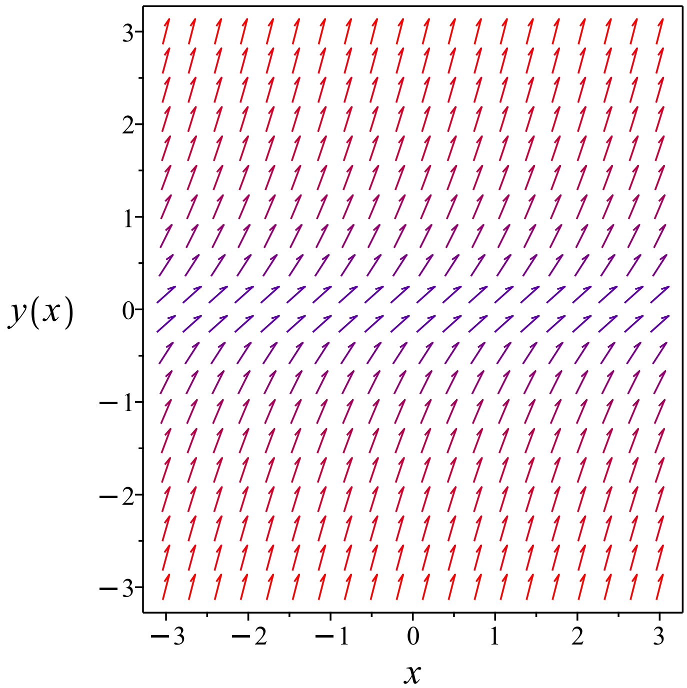


Figure 13: Slope field plot

Verification of solutions

$$\frac{\left(\begin{cases} -2\sqrt{-y} & y \leq 0 \\ 2\sqrt{y} & 0 < y \end{cases}\right)}{2} = x + c_1$$

Verified OK.

1.8.2 Maple step by step solution

Let's solve

$y' - 2\sqrt{|y|} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{|y|}} = 2$

- Integrate both sides with respect to x

$$\int \frac{y'}{\sqrt{|y|}}dx = \int 2dx + c_1$$

- Evaluate integral

$$\left\{\begin{array}{cc} -2\sqrt{-y} & y \leq 0 \\ 2\sqrt{y} & 0 < y \end{array}\right. = 2x + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve(diff(y(x),x)-2*sqrt(abs(y(x)))=0,y(x), singsol=all)
```

$$x + \left(\left\{\begin{array}{cc} \sqrt{-y(x)} & y(x) \leq 0 \\ -\sqrt{y(x)} & 0 < y(x) \end{array}\right.\right) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.291 (sec). Leaf size: 31

```
DSolve[y'[x]-Sqrt[Abs[y[x]]]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{1}{\sqrt{|K[1]|}}dK[1]\&\right][x + c_1]$$
$$y(x) \to 0$$

1.9 problem 23

Internal problem ID [12582]
Internal file name [`OUTPUT/11234_Thursday_October_19_2023_04_43_34_PM_68847940/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x^2y' + 2yx = 0}$$

1.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{2y}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= -\frac{2}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{2}{x}\,dx \\
\ln(y) &= -2\ln(x) + c_1 \\
y &= \mathrm{e}^{-2\ln(x)+c_1} \\
&= \frac{c_1}{x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^2} \tag{1}
$$

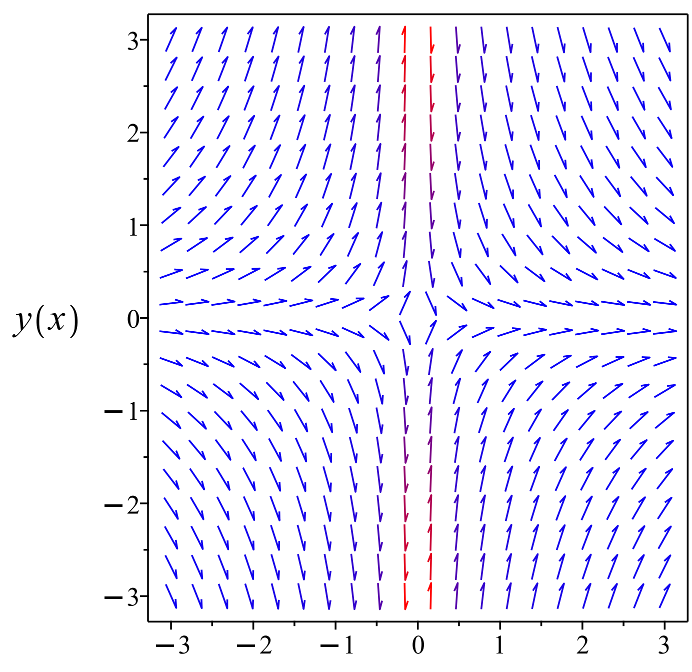


Figure 14: Slope field plot

Verification of solutions

$$
y = \frac{c_1}{x^2}
$$

Verified OK.

1.9.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{2y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{2}{x}dx} \\ &= x^2 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}(x^2 y) &= 0 \end{aligned}$$

Integrating gives

$$x^2 y = c_1$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{c_1}{x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x^2} \tag{1}$$

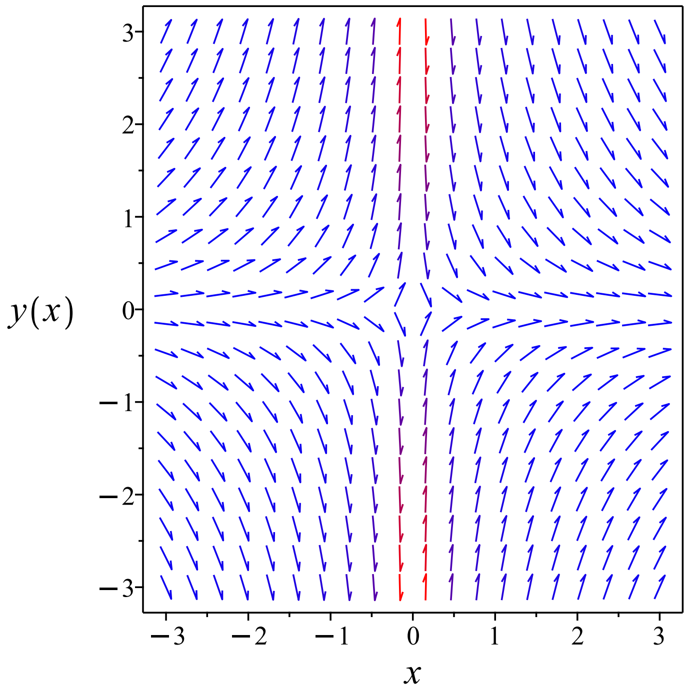


Figure 15: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^2}$$

Verified OK.

1.9.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$x^2(u'(x)\, x + u(x)) + 2u(x)\, x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{3}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{3}{x}\,dx \\
\ln(u) &= -3\ln(x) + c_2 \\
u &= \mathrm{e}^{-3\ln(x)+c_2} \\
&= \frac{c_2}{x^3}
\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= \frac{c_2}{x^2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2}{x^2} \tag{1}$$

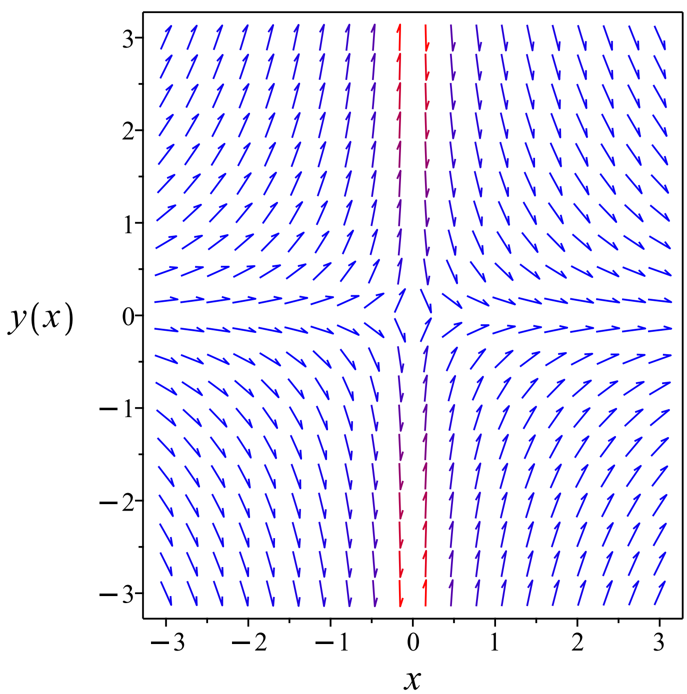


Figure 16: Slope field plot

Verification of solutions

$$y = \frac{c_2}{x^2}$$

Verified OK.

1.9.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 17: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{x^2} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy \end{aligned}$$

Which results in

$$S = x^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2xy \\ S_y &= x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 y = c_1$$

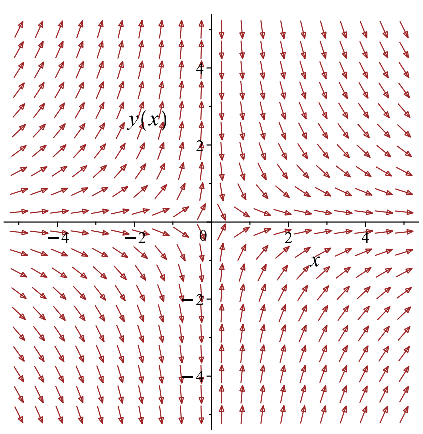
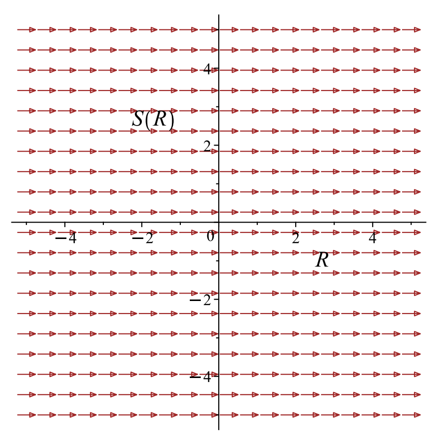
Which simplifies to

$$x^2 y = c_1$$

Which gives

$$y = \frac{c_1}{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{2y}{x}$

</td><td>$R = x$
$S = x^2 y$</td><td>$\frac{dS}{dR} = 0$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^2} \tag{1}$$

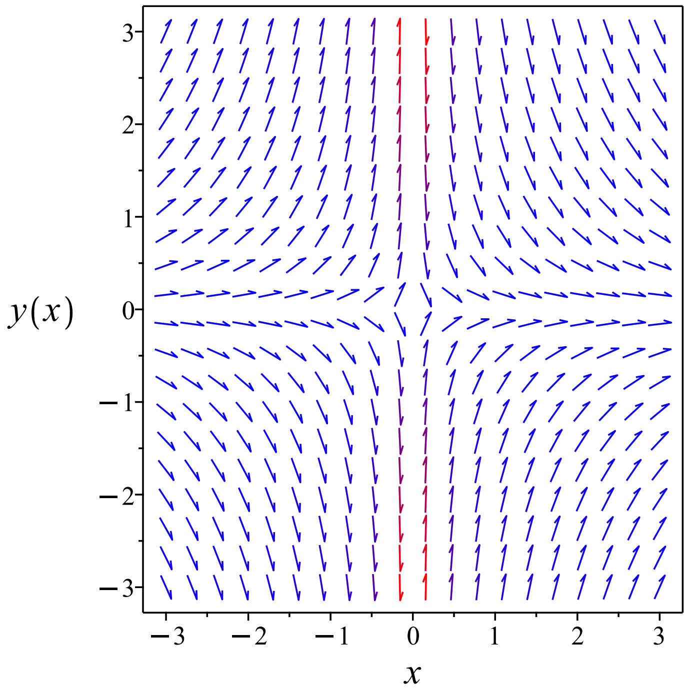


Figure 17: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^2}$$

Verified OK.

1.9.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{2y}\right)\mathrm{d}y = \left(\frac{1}{x}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x}\right)\mathrm{d}x + \left(-\frac{1}{2y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x}$$
$$N(x,y) = -\frac{1}{2y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)$$
$$= 0$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{2y}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x
\end{aligned}$$

$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{2y}$. Therefore equation (4) becomes

$$-\frac{1}{2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{2y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{1}{2y}\right) \mathrm{d}y \\
f(y) &= -\frac{\ln(y)}{2} + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{\ln(y)}{2}$$

The solution becomes

$$y = \frac{e^{-2c_1}}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-2c_1}}{x^2} \tag{1}$$

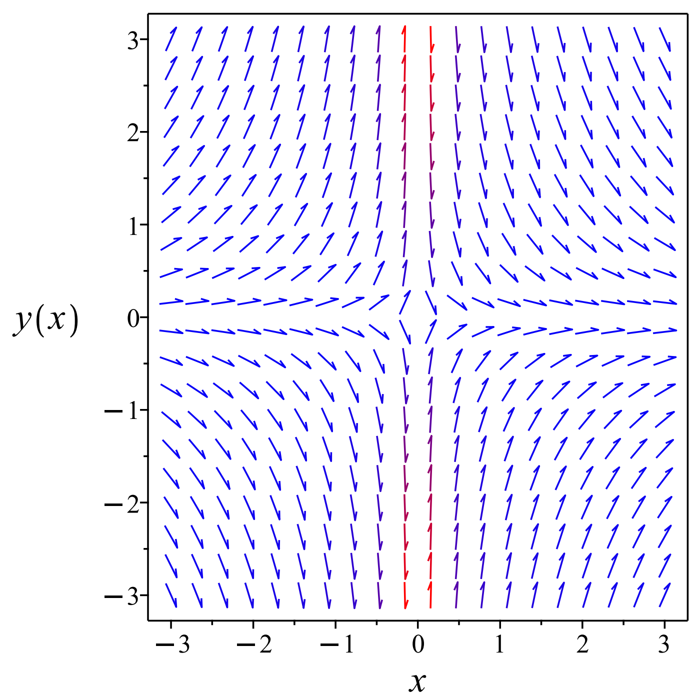


Figure 18: Slope field plot

Verification of solutions

$$y = \frac{e^{-2c_1}}{x^2}$$

Verified OK.

1.9.6 Maple step by step solution

Let's solve

$x^2y' + 2yx = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (x^2y' + 2yx)\, dx = \int 0 dx + c_1$

- Evaluate integral

  $x^2y = c_1$

- Solve for y

  $y = \frac{c_1}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(x^2*diff(y(x),x)+2*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.064 (sec). Leaf size: 16

```
DSolve[x^2*y'[x]+2*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x^2}$$
$$y(x) \to 0$$

1.10 problem 24

Internal problem ID [12583]
Internal file name [`OUTPUT/11235_Thursday_October_19_2023_04_43_35_PM_20815667/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 = 1}$$

1.10.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2+1} dy = x + c_1$$
$$\arctan(y) = x + c_1$$

Solving for y gives these solutions

$$y_1 = \tan(x + c_1)$$

Summary
The solution(s) found are the following

$$y = \tan(x + c_1) \tag{1}$$

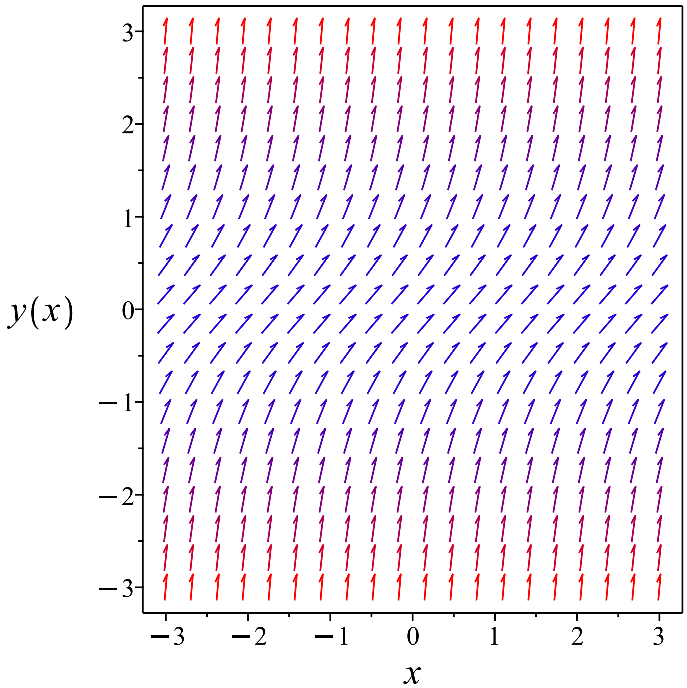


Figure 19: Slope field plot

Verification of solutions

$$y = \tan(x + c_1)$$

Verified OK.

1.10.2 Maple step by step solution

Let's solve

$y' - y^2 = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int 1 dx + c_1$

- Evaluate integral

$$\arctan(y) = x + c_1$$

- Solve for y

$$y = \tan(x + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(y(x),x)-y(x)^2=1,y(x), singsol=all)
```

$$y(x) = \tan(c_1 + x)$$

✓ Solution by Mathematica

Time used: 0.213 (sec). Leaf size: 24

```
DSolve[y'[x]-y[x]^2==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan(x + c_1)$$
$$y(x) \to -i$$
$$y(x) \to i$$

1.11 problem 25

Internal problem ID [12584]
Internal file name [`OUTPUT/11236_Thursday_October_19_2023_04_43_35_PM_46001342/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{2x^2y'' + y'x - y = 0}$$

1.11.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$2x^2(r(r-1))x^{r-2} + xrx^{r-1} - x^r = 0$$

Simplifying gives

$$2r(r-1)\,x^r + r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$2r(r-1) + r - 1 = 0$$

Or

$$2r^2 - r - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = -\frac{1}{2}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_1 x + \frac{c_2}{\sqrt{x}}$$

Summary
The solution(s) found are the following

$$y = c_1 x + \frac{c_2}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1 x + \frac{c_2}{\sqrt{x}}$$

Verified OK.

1.11.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$2x^2 y'' + y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = -\frac{1}{2x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{2x}dx\right)}dx \\
&= \int e^{-\frac{\ln(x)}{2}}\,dx \\
&= \int \frac{1}{\sqrt{x}}dx \\
&= 2\sqrt{x}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{2x^2}}{\frac{1}{x}} \\
&= -\frac{1}{2x}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{y(\tau)}{2x} &= 0
\end{aligned}$$

But in terms of τ

$$-\frac{1}{2x} = -\frac{2}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{2y(\tau)}{\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 2\tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r - 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau} + c_2\tau^2$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1}{2\sqrt{x}} + 4c_2x$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{2\sqrt{x}} + 4c_2 x \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{2\sqrt{x}} + 4c_2 x$$

Verified OK.

1.11.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$2x^2 y'' + y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = -\frac{1}{2x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{2x^2} - \frac{1}{2x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \frac{5v'(x)}{2x} &= 0 \\
v''(x) + \frac{5v'(x)}{2x} &= 0 \qquad (7)
\end{aligned}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \frac{5u(x)}{2x} = 0 \qquad (8)
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{5u}{2x}
\end{aligned}
$$

Where $f(x) = -\frac{5}{2x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{5}{2x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{5}{2x}\, dx \\
\ln(u) &= -\frac{5\ln(x)}{2} + c_1 \\
u &= \mathrm{e}^{-\frac{5\ln(x)}{2} + c_1} \\
&= \frac{c_1}{x^{\frac{5}{2}}}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{2c_1}{3x^{\frac{3}{2}}} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right) x \\
&= \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right) x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right) x \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right) x
$$

Verified OK.

1.11.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0
\end{aligned} \tag{1}
$$

If the term $AB''+BB'+CB$ is zero, then this method works and can be used to solve

$$ABv''+\left(2AB'+B^2\right)v'=0$$

By Using $u=v'$ which reduces the order of the above ode to one. The new ode is

$$ABu'+\left(2AB'+B^2\right)u=0$$

The above ode is first order ode which is solved for u. Now a new ode $v'=u$ is solved for v as first order ode. Then the final solution is obtain from $y=Bv$.

This method works only if the term $AB''+BB'+CB$ is zero. The given ODE shows that

$$\begin{aligned}A&=2x^2\\B&=x\\C&=-1\\F&=0\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}AB''+BB'+CB&=\left(2x^2\right)(0)+(x)(1)+(-1)(x)\\&=0\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$2x^3v''+\left(5x^2\right)v'=0$$

Now by applying $v'=u$ the above becomes

$$2x^3u'(x)+5x^2u(x)=0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}u'&=F(x,u)\\&=f(x)g(u)\\&=-\frac{5u}{2x}\end{aligned}$$

Where $f(x) = -\frac{5}{2x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{5}{2x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{5}{2x}\,dx \\
\ln(u) &= -\frac{5\ln(x)}{2} + c_1 \\
u &= e^{-\frac{5\ln(x)}{2} + c_1} \\
&= \frac{c_1}{x^{\frac{5}{2}}}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= \frac{c_1}{x^{\frac{5}{2}}}
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{c_1}{x^{\frac{5}{2}}}\,\mathrm{d}x \\
&= -\frac{2c_1}{3x^{\frac{3}{2}}} + c_2
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y(x) &= Bv \\
&= (x)\left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right) \\
&= \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)x
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)x \tag{1}$$

Verification of solutions

$$y = \left(-\frac{2c_1}{3x^{\frac{3}{2}}} + c_2\right)x$$

Verified OK.

1.11.5 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
2x^2y'' + y'x - y &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 2x^2\\
B &= x \tag{3}\\
C &= -1
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{5}{16x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 5\\
t &= 16x^2
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{5}{16x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 21: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{5}{16x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{5}{16x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{5}{16}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{4} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{4}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{5}{16x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{4}$	$-\frac{1}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{1}{4}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{4} - \left(-\frac{1}{4}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{4x} + (-)(0) \\ &= -\frac{1}{4x} \\ &= -\frac{1}{4x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{4x}\right)(0) + \left(\left(\frac{1}{4x^2}\right) + \left(-\frac{1}{4x}\right)^2 - \left(\frac{5}{16x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int -\frac{1}{4x} dx} \\ &= \frac{1}{x^{\frac{1}{4}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{2x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{4}} \\
&= z_1 \left(\frac{1}{x^{\frac{1}{4}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{\sqrt{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{2x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2x^{\frac{3}{2}}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{\sqrt{x}}\right) + c_2 \left(\frac{1}{\sqrt{x}}\left(\frac{2x^{\frac{3}{2}}}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{\sqrt{x}} + \frac{2c_2 x}{3} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1}{\sqrt{x}} + \frac{2c_2x}{3}$$

Verified OK.

1.11.6 Maple step by step solution

Let's solve

$2y''x^2 + y'x - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{2x} + \frac{y}{2x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2x} - \frac{y}{2x^2} = 0$

- Multiply by denominators of the ODE

  $2y''x^2 + y'x - y = 0$

- Make a change of variables

  $t = \ln(x)$

- ☐ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)x^2+\frac{d}{dt}y(t)-y(t)=0$$

- Simplify

$$2\frac{d^2}{dt^2}y(t)-\frac{d}{dt}y(t)-y(t)=0$$

- Isolate 2nd derivative

$$\frac{d^2}{dt^2}y(t)=\frac{\frac{d}{dt}y(t)}{2}+\frac{y(t)}{2}$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^2}{dt^2}y(t)-\frac{\frac{d}{dt}y(t)}{2}-\frac{y(t)}{2}=0$$

- Characteristic polynomial of ODE

$$r^2-\frac{1}{2}r-\frac{1}{2}=0$$

- Factor the characteristic polynomial

$$\frac{(2r+1)(r-1)}{2}=0$$

- Roots of the characteristic polynomial

$$r=\left(1,-\frac{1}{2}\right)$$

- 1st solution of the ODE

$$y_1(t)=\mathrm{e}^t$$

- 2nd solution of the ODE

$$y_2(t)=\mathrm{e}^{-\frac{t}{2}}$$

- General solution of the ODE

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

- Substitute in solutions

$$y(t)=c_1\mathrm{e}^t+c_2\mathrm{e}^{-\frac{t}{2}}$$

- Change variables back using $t=\ln(x)$

$$y=c_1x+\frac{c_2}{\sqrt{x}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(2*x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x}} + c_2x$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 18

```
DSolve[2*x^2*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{\sqrt{x}} + c_2x$$

1.12 problem 26

Internal problem ID [12585]
Internal file name [`OUTPUT/11237_Thursday_October_19_2023_04_43_36_PM_55985488/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'x = \sin(x)}$$

1.12.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{\sin(x)}{x}\, \mathrm{d}x \\ &= \mathrm{Si}(x) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \mathrm{Si}(x) + c_1 \tag{1}$$

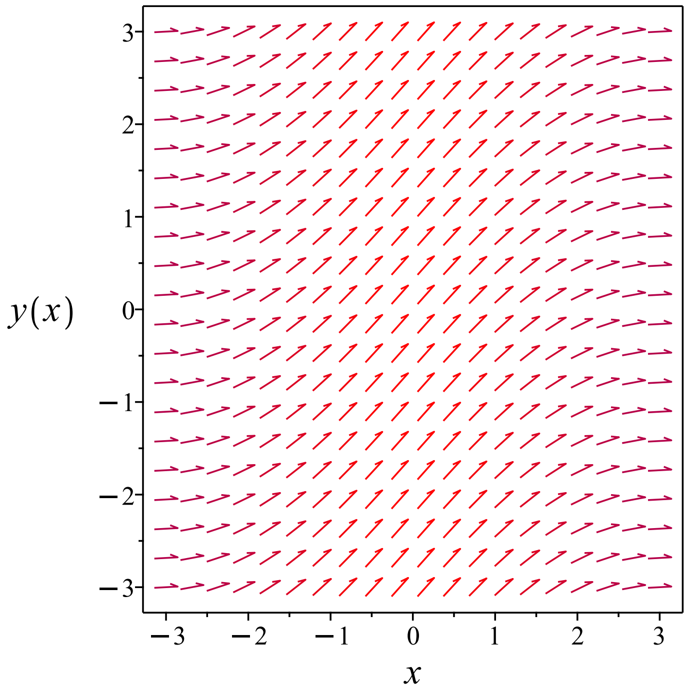


Figure 20: Slope field plot

Verification of solutions

$$y = \operatorname{Si}(x) + c_1$$

Verified OK.

1.12.2 Maple step by step solution

Let's solve

$y'x = \sin(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' = \frac{\sin(x)}{x}$

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{\sin(x)}{x} dx + c_1$

- Evaluate integral

$$y = \mathrm{Si}(x) + c_1$$

- Solve for y

$$y = \mathrm{Si}(x) + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(x*diff(y(x),x)-sin(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{Si}\,(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 10

```
DSolve[x*y'[x]-Sin[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \mathrm{Si}(x) + c_1$$

1.13 problem 27

Internal problem ID [12586]
Internal file name [OUTPUT/11238_Thursday_October_19_2023_04_43_36_PM_6767101/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + 3y = 0}$$

1.13.1 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{1}{3y} dy = \int dx$$
$$-\frac{\ln(y)}{3} = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{y^{\frac{1}{3}}} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{y^{\frac{1}{3}}} = c_2 e^{x}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{-3x}}{c_2^3} \tag{1}$$

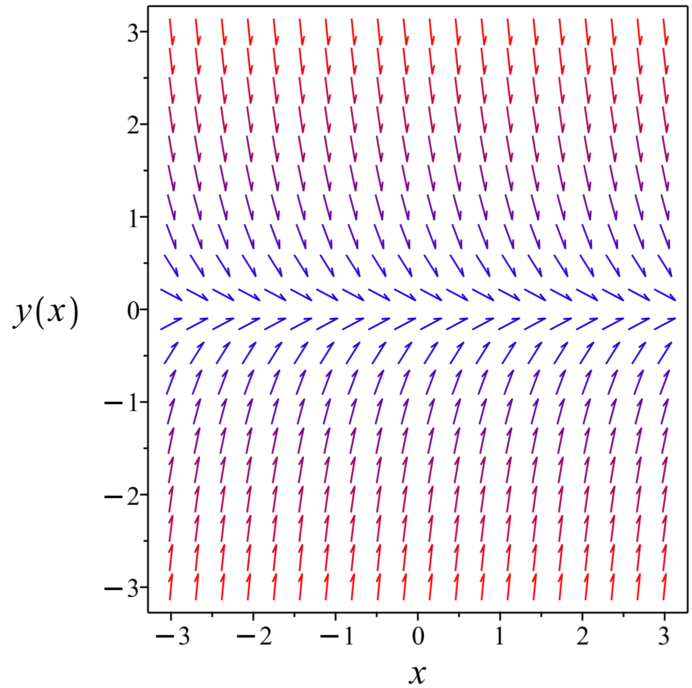


Figure 21: Slope field plot

Verification of solutions

$$y = \frac{e^{-3x}}{c_2^3}$$

Verified OK.

1.13.2 Maple step by step solution

Let's solve

$y' + 3y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -3$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int (-3)\, dx + c_1$

- Evaluate integral

$$\ln(y) = -3x + c_1$$

- Solve for y

$$y = e^{-3x+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 18

```
DSolve[y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-3x}$$
$$y(x) \to 0$$

1.14 problem 28

Internal problem ID [12587]
Internal file name [`OUTPUT/11239_Thursday_October_19_2023_04_43_37_PM_74600233/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 3y' - 10y = 0}$$

1.14.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = -10$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} - 10 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda - 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = -10$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-3^2 - (4)(1)(-10)} \\
&= \frac{3}{2} \pm \frac{7}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{3}{2} + \frac{7}{2} \\
\lambda_2 &= \frac{3}{2} - \frac{7}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 5 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(5)x} + c_2 e^{(-2)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{5x} + c_2 \mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{5x} + c_2 \mathrm{e}^{-2x} \tag{1}$$

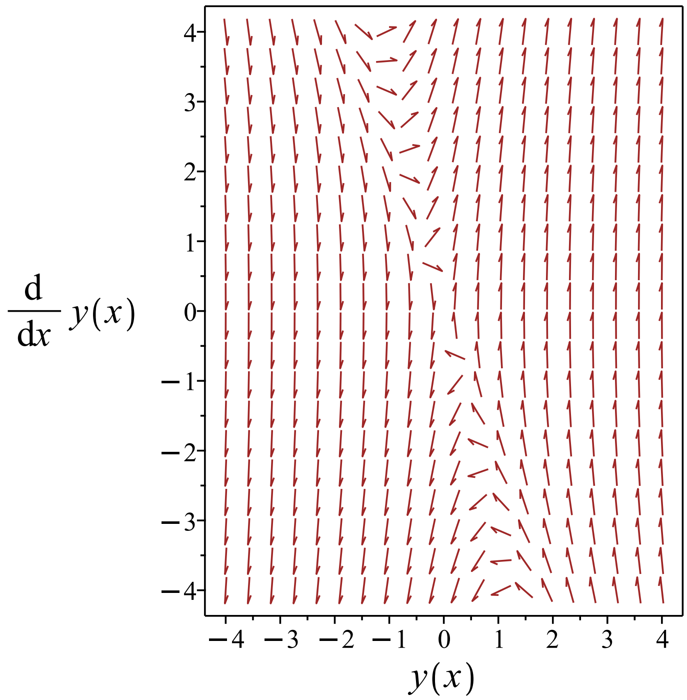


Figure 22: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{5x} + c_2 e^{-2x}$$

Verified OK.

1.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' - 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= -10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{49}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 49 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{49z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 25: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{49}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{7x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx}\\ &= z_1 e^{\frac{3x}{2}}\\ &= z_1\left(\mathrm{e}^{\frac{3x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\mathrm{e}^{7x}}{7}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-2x}\right) + c_2 \left(e^{-2x} \left(\frac{e^{7x}}{7}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 e^{-2x} + \frac{c_2 e^{5x}}{7} \tag{1}
$$

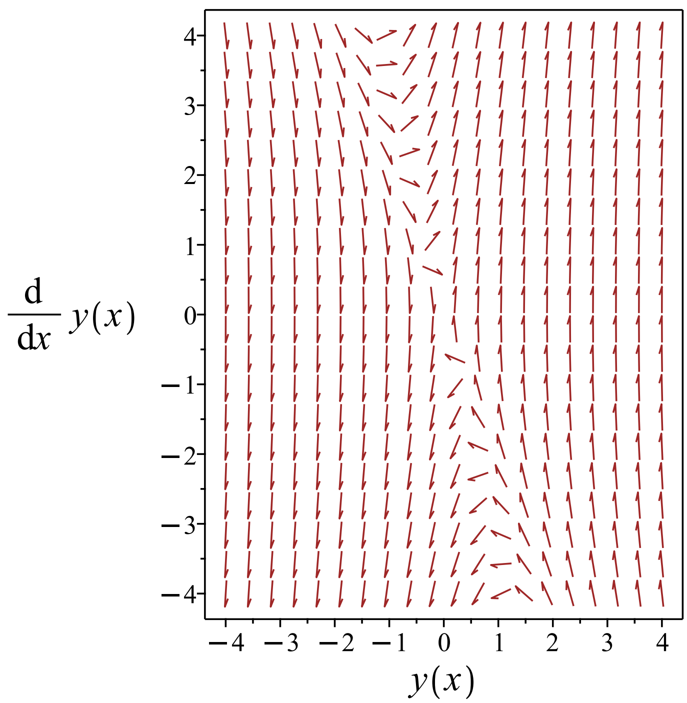


Figure 23: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 e^{-2x} + \frac{c_2 e^{5x}}{7}
$$

Verified OK.

1.14.3 Maple step by step solution

Let's solve

$y'' - 3y' - 10y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 3r - 10 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-5) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 5)$

- 1st solution of the ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{5x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} + c_2 e^{5x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)-10*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 \mathrm{e}^{7x} + c_2\right) \mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 22

```
DSolve[y''[x]-3*y'[x]-10*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(c_2 e^{7x} + c_1\right)$$

1.15 problem 29

Internal problem ID [12588]
Internal file name [`OUTPUT/11240_Thursday_October_19_2023_04_43_38_PM_37522487/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + y = 0}$$

1.15.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)} \\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-x} + c_2 x\, e^{-x} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + x\, e^{-x} c_2 \tag{1}$$

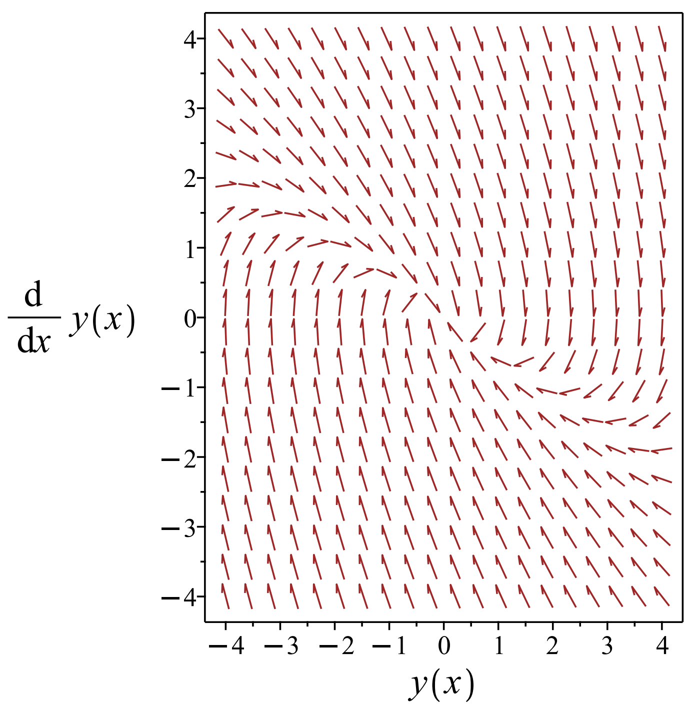


Figure 24: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + x\, e^{-x} c_2$$

Verified OK.

1.15.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 2\, dx} \\ &= \mathrm{e}^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ (\mathrm{e}^{x} y)'' &= 0 \end{aligned}$$

Integrating once gives

$$(\mathrm{e}^{x} y)' = c_1$$

Integrating again gives

$$(\mathrm{e}^{x} y) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{x}}$$

Or

$$y = \mathrm{e}^{-x} c_1 x + c_2 \mathrm{e}^{-x}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-x} c_1 x + c_2 \mathrm{e}^{-x} \tag{1}$$

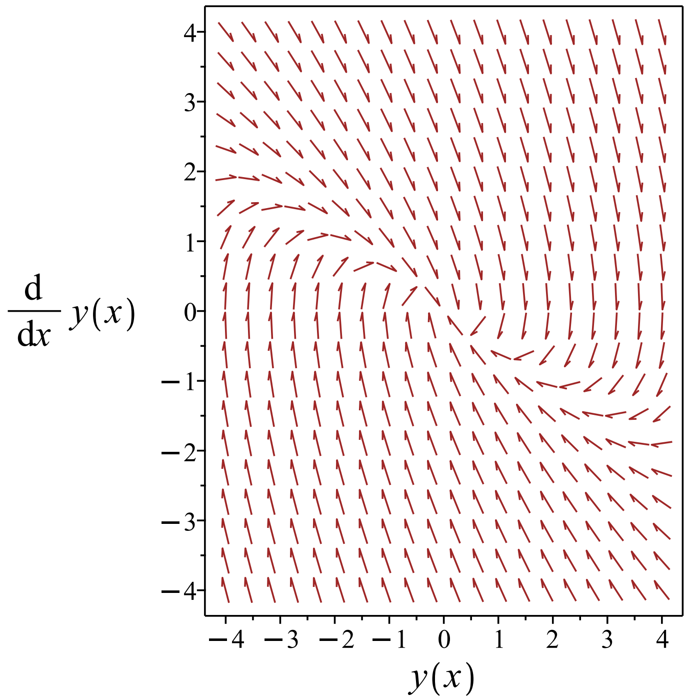


Figure 25: Slope field plot

Verification of solutions

$$y = e^{-x} c_1 x + c_2 e^{-x}$$

Verified OK.

1.15.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 27: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-x}\right) + c_2\left(e^{-x}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-x} + x\, e^{-x} c_2 \tag{1}
$$

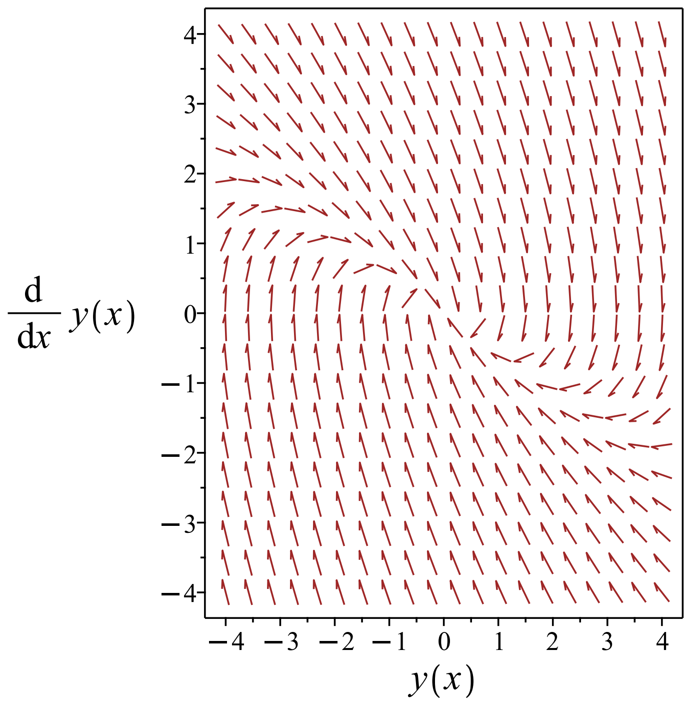


Figure 26: Slope field plot

Verification of solutions

$$
y = c_1 e^{-x} + x\, e^{-x} c_2
$$

Verified OK.

1.15.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the ODE

  $y_1(x) = e^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-x} + x\,e^{-x} c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 18

```
DSolve[y''[x]+2*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(c_2x + c_1)$$

1.16 problem 30

Internal problem ID [12589]
Internal file name [`OUTPUT/11241_Thursday_October_19_2023_04_43_38_PM_48557998/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 30.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 7y'' + 12y' = 0}$$

The characteristic equation is

$$\lambda^3 - 7\lambda^2 + 12\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 4$$
$$\lambda_3 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + \mathrm{e}^{3x} c_2 + \mathrm{e}^{4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = \mathrm{e}^{3x}$$
$$y_3 = \mathrm{e}^{4x}$$

Summary
The solution(s) found are the following

$$y = c_1 + e^{3x} c_2 + e^{4x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 + e^{3x} c_2 + e^{4x} c_3$$

Verified OK.

1.16.1 Maple step by step solution

Let's solve

$y''' - 7y'' + 12y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 7y_3(x) - 12y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 7y_3(x) - 12y_2(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -12 & 7 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -12 & 7 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{4x} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = e^{3x} c_2 \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + e^{4x} c_3 \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{e^{3x} c_2}{9} + \frac{e^{4x} c_3}{16} + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$3)-7*diff(y(x),x$2)+12*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 e^{3x} + c_3 e^{4x}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 30

```
DSolve[y'''[x]-7*y''[x]+12*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}c_1 e^{3x} + \frac{1}{4}c_2 e^{4x} + c_3$$

1.17 problem 31

Internal problem ID [12590]
Internal file name [`OUTPUT/11242_Thursday_October_19_2023_04_43_39_PM_53530479/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2y'x - y = 0}$$

1.17.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{2x} \end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= \frac{1}{2x}\, dx \\
\int \frac{1}{y}\, dy &= \int \frac{1}{2x}\, dx \\
\ln(y) &= \frac{\ln(x)}{2} + c_1 \\
y &= \mathrm{e}^{\frac{\ln(x)}{2} + c_1} \\
&= c_1\sqrt{x}
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1\sqrt{x} \tag{1}
$$

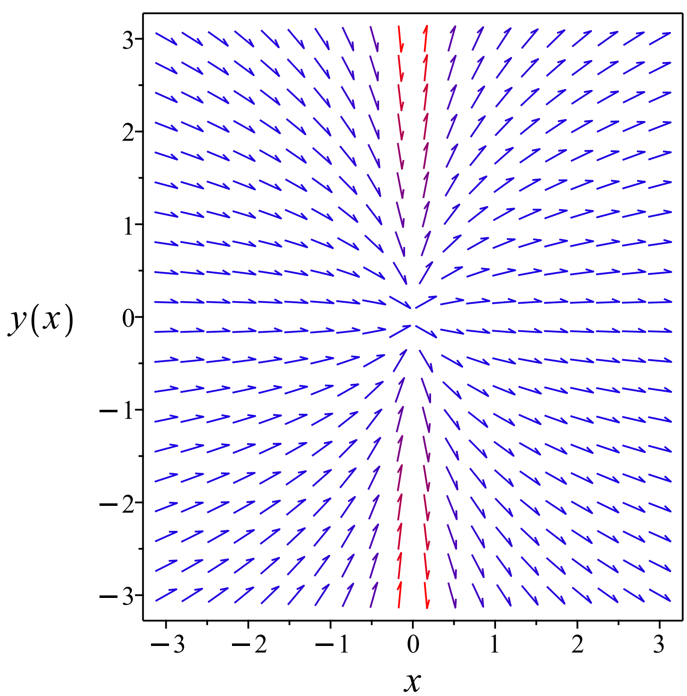


Figure 27: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1\sqrt{x}
$$

Verified OK.

1.17.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{2x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{2x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{2x}dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{x}}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{\sqrt{x}} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x}}$ results in

$$y = c_1\sqrt{x}$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x} \tag{1}$$

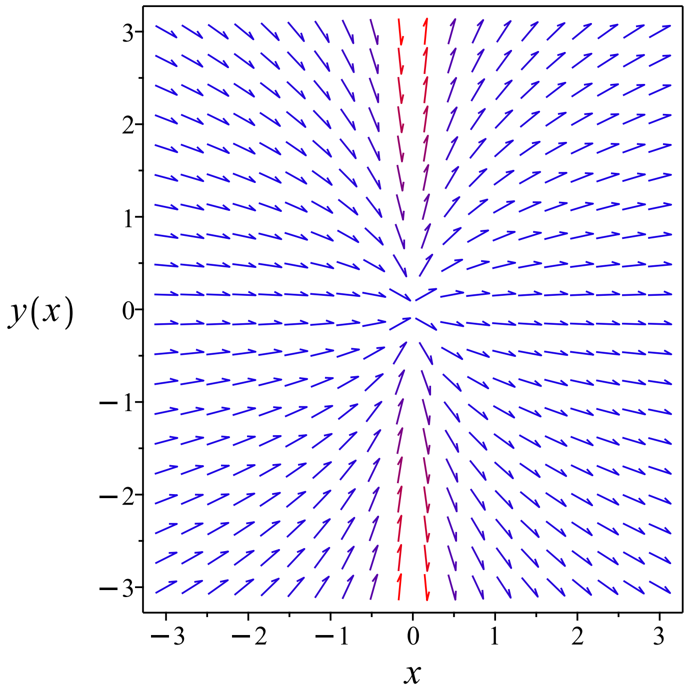


Figure 28: Slope field plot

Verification of solutions

$$y = c_1\sqrt{x}$$

Verified OK.

1.17.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2(u'(x)\,x + u(x))\,x - u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{2x} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{2x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{2x}\,dx \\
\ln(u) &= -\frac{\ln(x)}{2} + c_2 \\
u &= \mathrm{e}^{-\frac{\ln(x)}{2}+c_2} \\
&= \frac{c_2}{\sqrt{x}}
\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= \sqrt{x}\,c_2
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \sqrt{x}\,c_2 \tag{1}$$

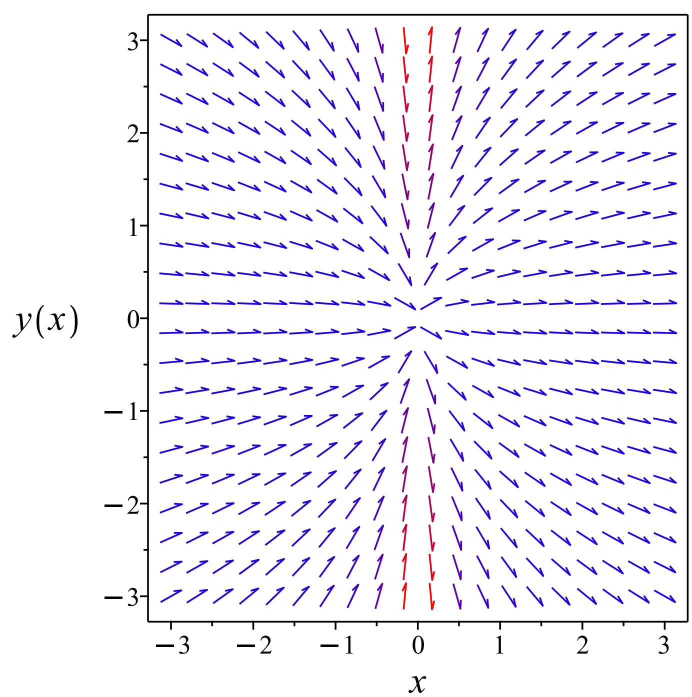


Figure 29: Slope field plot

Verification of solutions

$$y = \sqrt{x}\, c_2$$

Verified OK.

1.17.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 30: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \sqrt{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{x}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sqrt{x}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{2x^{\frac{3}{2}}} \\ S_y &= \frac{1}{\sqrt{x}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\sqrt{x}} = c_1$$

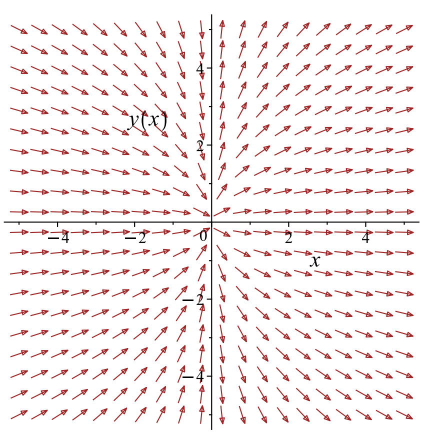
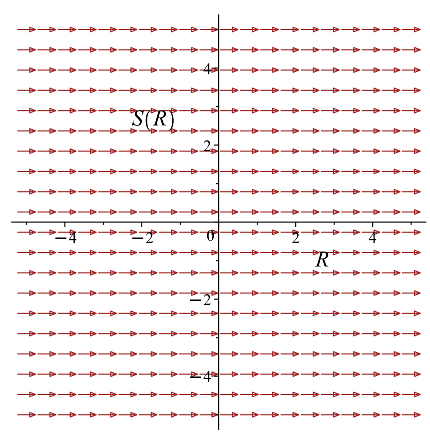
Which simplifies to

$$\frac{y}{\sqrt{x}} = c_1$$

Which gives

$$y = c_1\sqrt{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{2x}$ 	$R = x$ $S = \frac{y}{\sqrt{x}}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1\sqrt{x} \tag{1}$$

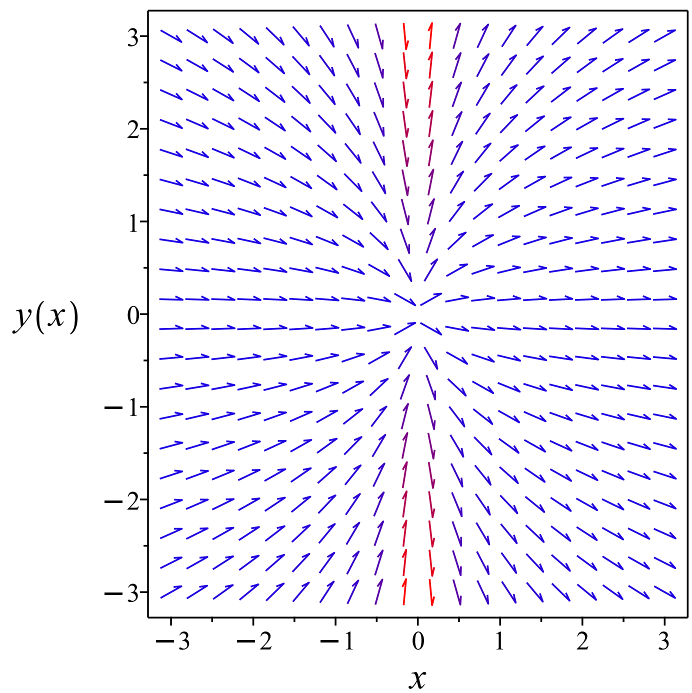


Figure 30: Slope field plot

Verification of solutions

$$y = c_1\sqrt{x}$$

Verified OK.

1.17.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{2}{y}\right) \mathrm{d}y &= \left(\frac{1}{x}\right) \mathrm{d}x\\ \left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{2}{y}\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x}\\ N(x, y) &= \frac{2}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{x} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2}{y}$. Therefore equation (4) becomes

$$
\frac{2}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{2}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{2}{y}\right) \mathrm{d}y \\
f(y) &= 2\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + 2\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + 2\ln(y)$$

The solution becomes

$$y = e^{\frac{\ln(x)}{2}+\frac{c_1}{2}}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{\ln(x)}{2}+\frac{c_1}{2}} \tag{1}$$

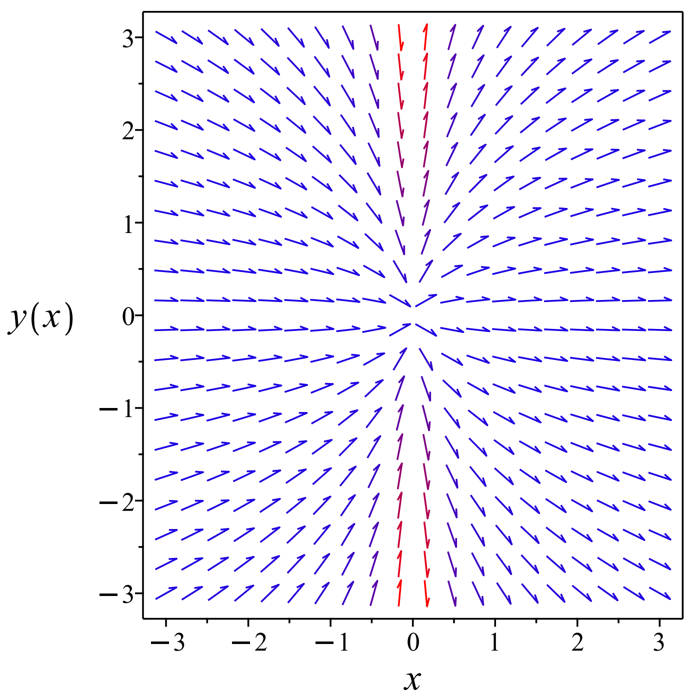


Figure 31: Slope field plot

Verification of solutions

$$y = e^{\frac{\ln(x)}{2}+\frac{c_1}{2}}$$

Verified OK.

1.17.6 Maple step by step solution

Let's solve

$2y'x - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{2x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{2x}dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{\ln(x)}{2} + c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{e^{-2c_1}x}}{e^{-2c_1}}, y = -\frac{\sqrt{e^{-2c_1}x}}{e^{-2c_1}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(2*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 18

```
DSolve[2*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1\sqrt{x}$$
$$y(x) \to 0$$

1.18 problem 32

Internal problem ID [12591]
Internal file name [`OUTPUT/11243_Thursday_October_19_2023_04_43_39_PM_53610816/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x^2y'' - y'x = 0}$$

The ODE is

$$x^2y'' - y'x = 0$$

Or

$$x(y''x - y') = 0$$

For $x \neq 0$ the above simplifies to

$$y''x - y' = 0$$

1.18.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} + 0 = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r + 0 = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r + 0 = 0$$

Or

$$r^2 - 2r = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 0$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^2 + c_1$$

Summary
The solution(s) found are the following

$$y = c_2x^2 + c_1 \tag{1}$$

Verification of solutions

$$y = c_2x^2 + c_1$$

Verified OK. `{x <> 0}`

1.18.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)\, x^2 - p(x)\, x = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= \frac{p}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned} \frac{1}{p}\, dp &= \frac{1}{x}\, dx \\ \int \frac{1}{p}\, dp &= \int \frac{1}{x}\, dx \\ \ln(p) &= \ln(x) + c_1 \\ p &= \mathrm{e}^{\ln(x)+c_1} \\ &= c_1 x \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_1 x$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_1 x \,\mathrm{d}x \\ &= \frac{c_1 x^2}{2} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^2}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2}{2} + c_2$$

Verified OK. `{x <> 0}`

1.18.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - y' x = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 33: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x^2} \, dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2 \left(1 \left(\frac{x^2}{2}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + \frac{c_2 x^2}{2} \tag{1}$$

Verification of solutions

$$y = c_1 + \frac{c_2 x^2}{2}$$

Verified OK. `{x <> 0}`

1.18.4 Maple step by step solution

Let's solve

$y''x^2 - y'x = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} = 0$

- Multiply by denominators of the ODE

  $y''x - y' = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  ◦ Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

$y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x - \frac{\frac{d}{dt}y(t)}{x} = 0$

- Simplify

  $\frac{\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t)}{x} = 0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = 2\frac{d}{dt}y(t)$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 2r = 0$

- Factor the characteristic polynomial

  $r(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (0, 2)$

- 1st solution of the ODE

  $y_1(t) = 1$

- 2nd solution of the ODE

  $y_2(t) = e^{2t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 + c_2 e^{2t}$

- Change variables back using $t = \ln(x)$

  $y = c_2 x^2 + c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1x^2 + c_2$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 17

```
DSolve[x^2*y''[x]-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1x^2}{2} + c_2$$

1.19 problem 33

Internal problem ID [12592]
Internal file name [`OUTPUT/11244_Thursday_October_19_2023_04_43_40_PM_30015343/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2y'' + 6y'x + 4y = 0}$$

1.19.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 6xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 6r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 6r + 4 = 0$$

Or

$$r^2 + 5r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -4$$
$$r_2 = -1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^4} + \frac{c_2}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^4} + \frac{c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^4} + \frac{c_2}{x}$$

Verified OK.

1.19.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + 6y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{6}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{6}{x}dx\right)}dx \\ &= \int e^{-6\ln(x)}\, dx \\ &= \int \frac{1}{x^6}dx \\ &= -\frac{1}{5x^5} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{4}{x^2}}{\frac{1}{x^{12}}} \\ &= 4x^{10} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + 4x^{10}y(\tau) = 0$$

But in terms of τ

$$4x^{10} = \frac{4}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 4y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 4\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 4\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 4 = 0$$

Or

$$25r^2 - 25r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{5}$$
$$r_2 = \frac{4}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{1}{5}} + c_2 \tau^{\frac{4}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{4}{5}} \left(-\frac{1}{x^5}\right)^{\frac{1}{5}}}{5} + \frac{c_2 5^{\frac{1}{5}} \left(-\frac{1}{x^5}\right)^{\frac{4}{5}}}{5}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 5^{\frac{4}{5}} \left(-\frac{1}{x^5}\right)^{\frac{1}{5}}}{5} + \frac{c_2 5^{\frac{1}{5}} \left(-\frac{1}{x^5}\right)^{\frac{4}{5}}}{5} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 5^{\frac{4}{5}} \left(-\frac{1}{x^5}\right)^{\frac{1}{5}}}{5} + \frac{c_2 5^{\frac{1}{5}} \left(-\frac{1}{x^5}\right)^{\frac{4}{5}}}{5}$$

Verified OK.

1.19.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' + 6y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{6}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\, x^3} + \frac{6}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= \frac{5c}{2} \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2} y(\tau) + \frac{5c\left(\frac{d}{d\tau} y(\tau)\right)}{2} + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{-\frac{5c\tau}{4}} \left(c_1 \cosh\left(\frac{3c\tau}{4}\right) + i c_2 \sinh\left(\frac{3c\tau}{4}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{c_1\cosh\left(\frac{3\ln(x)}{2}\right) + ic_2\sinh\left(\frac{3\ln(x)}{2}\right)}{x^{\frac{5}{2}}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1\cosh\left(\frac{3\ln(x)}{2}\right) + ic_2\sinh\left(\frac{3\ln(x)}{2}\right)}{x^{\frac{5}{2}}} \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1\cosh\left(\frac{3\ln(x)}{2}\right) + ic_2\sinh\left(\frac{3\ln(x)}{2}\right)}{x^{\frac{5}{2}}}$$

Verified OK.

1.19.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + 6y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{6}{x} \\
q(x) &= \frac{4}{x^2}
\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{6n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \frac{4v'(x)}{x} = 0$$
$$v''(x) + \frac{4v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{4u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{4u}{x} \end{aligned}$$

Where $f(x) = -\frac{4}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{4}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{4}{x}\,dx \\
\ln(u) &= -4\ln(x) + c_1 \\
u &= \mathrm{e}^{-4\ln(x)+c_1} \\
&= \frac{c_1}{x^4}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{3x^3} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \frac{-\frac{c_1}{3x^3} + c_2}{x} \\
&= \frac{3c_2x^3 - c_1}{3x^4}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{-\frac{c_1}{3x^3} + c_2}{x} \tag{1}
$$

Verification of solutions

$$
y = \frac{-\frac{c_1}{3x^3} + c_2}{x}
$$

Verified OK.

1.19.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x^2 y'' + 6y'x + 4y\right) dx = 0$$
$$x^2 y' + 4yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{4}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' + \frac{4y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{4}{x} dx}$$
$$= x^4$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x^2}\right)$$
$$\frac{d}{dx}\left(x^4 y\right) = \left(x^4\right)\left(\frac{c_1}{x^2}\right)$$
$$d\left(x^4 y\right) = \left(c_1 x^2\right) dx$$

Integrating gives

$$x^4 y = \int c_1 x^2 \, dx$$
$$x^4 y = \frac{c_1 x^3}{3} + c_2$$

Dividing both sides by the integrating factor $\mu = x^4$ results in

$$y = \frac{c_1}{3x} + \frac{c_2}{x^4}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{3x} + \frac{c_2}{x^4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{3x} + \frac{c_2}{x^4}$$

Verified OK.

1.19.6 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2 y'' + 6y'x + 4y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x^2 y'' + 6y'x + 4y\right) dx = 0$$
$$x^2 y' + 4yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{4}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' + \frac{4y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{4}{x} dx}$$
$$= x^4$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(x^4 y) &= (x^4)\left(\frac{c_1}{x^2}\right) \\
\mathrm{d}(x^4 y) &= (c_1 x^2)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^4 y &= \int c_1 x^2 \, \mathrm{d}x \\
x^4 y &= \frac{c_1 x^3}{3} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^4$ results in

$$
y = \frac{c_1}{3x} + \frac{c_2}{x^4}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1}{3x} + \frac{c_2}{x^4} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{3x} + \frac{c_2}{x^4}
$$

Verified OK.

1.19.7 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
x^2 y'' + 6y'x + 4y &= 0 & (1) \\
Ay'' + By' + Cy &= 0 & (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= 6x \\
C &= 4
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 35: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + (-)(0) \\ &= -\frac{1}{x} \\ &= -\frac{1}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6x}{x^2} \, dx} \\
&= z_1 e^{-3\ln(x)} \\
&= z_1 \left(\frac{1}{x^3}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-6\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^3}{3}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^4}\right) + c_2 \left(\frac{1}{x^4} \left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1}{x^4} + \frac{c_2}{3x} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x^4} + \frac{c_2}{3x}
$$

Verified OK.

1.19.8 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x^2 \\
q(x) &= 6x \\
r(x) &= 4 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 6
\end{aligned}
$$

Therefore (1) becomes

$$
2 - (6) + (4) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$x^2y' + 4yx = c_1$$

We now have a first order ode to solve which is

$$x^2y' + 4yx = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{c_1}{x^2} \end{aligned}$$

Hence the ode is

$$y' + \frac{4y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{4}{x} dx} \\ &= x^4 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(x^4y\right) &= \left(x^4\right)\left(\frac{c_1}{x^2}\right) \\ \mathrm{d}\left(x^4y\right) &= \left(c_1x^2\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$x^4y = \int c_1x^2 \, dx$$
$$x^4y = \frac{c_1x^3}{3} + c_2$$

Dividing both sides by the integrating factor $\mu = x^4$ results in

$$y = \frac{c_1}{3x} + \frac{c_2}{x^4}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{3x} + \frac{c_2}{x^4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{3x} + \frac{c_2}{x^4}$$

Verified OK.

1.19.9 Maple step by step solution

Let's solve

$y''x^2 + 6y'x + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{6y'}{x} - \frac{4y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{6y'}{x} + \frac{4y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 + 6y'x + 4y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

- Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^2 + 6\frac{d}{dt}y(t) + 4y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 4y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + 5r + 4 = 0$

- Factor the characteristic polynomial

  $(r+4)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-4, -1)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-4t}$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{-t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{-4t} + c_2 \mathrm{e}^{-t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^4} + \frac{c_2}{x}$

- Simplify

  $y = \frac{c_1}{x^4} + \frac{c_2}{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+6*x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^3 + c_2}{x^4}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]+6*x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^3 + c_1}{x^4}$$

1.20 problem 34

Internal problem ID [12593]
Internal file name [`OUTPUT/11245_Thursday_October_19_2023_04_43_41_PM_14908028/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - 5y'x + 9y = 0}$$

1.20.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 5xrx^{r-1} + 9x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 5r\,x^r + 9x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 5r + 9 = 0$$

Or

$$r^2 - 6r + 9 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 3 \\ r_2 &= 3 \end{aligned}$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1 x^3 + c_2 x^3 \ln(x)$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x^3 + c_2 x^3 \ln(x) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 x^3 + c_2 x^3 \ln(x)$$

Verified OK.

1.20.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - 5y'x + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= \frac{9}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-(\int p(x)dx)}\,dx \\ &= \int \mathrm{e}^{-(\int -\frac{5}{x}dx)}\,dx \\ &= \int e^{5\ln(x)}\,dx \\ &= \int x^5 dx \\ &= \frac{x^6}{6}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{9}{x^2}}{x^{10}} \\ &= \frac{9}{x^{12}}\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{9y(\tau)}{x^{12}} &= 0\end{aligned}$$

But in terms of τ

$$\frac{9}{x^{12}} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{6}\,\sqrt{x^6}\,(c_1 - c_2\ln(2) + c_2\ln(x^6) - c_2\ln(3))}{6}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{6}\,\sqrt{x^6}\,(c_1 - c_2 \ln(2) + c_2 \ln(x^6) - c_2 \ln(3))}{6} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{6}\,\sqrt{x^6}\,(c_1 - c_2 \ln(2) + c_2 \ln(x^6) - c_2 \ln(3))}{6}$$

Verified OK.

1.20.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' - 5y'x + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= \frac{9}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{3\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{3}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{3}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{5}{x}\frac{3\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{3\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 3\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{3\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 x^3
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^3 \tag{1}
$$

Verification of solutions

$$
y = c_1 x^3
$$

Verified OK.

1.20.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 5y'x + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= \frac{9}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{5n}{x^2} + \frac{9}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{u}{x}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= (c_1 \ln(x) + c_2)\, x^3 \\
&= (c_1 \ln(x) + c_2)\, x^3
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 \ln(x) + c_2)\, x^3 \tag{1}$$

Verification of solutions

$$y = (c_1 \ln(x) + c_2)\, x^3$$

Verified OK.

1.20.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 5y'x + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -5x \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 37: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x}{x^2}\,dx} \\
&= z_1 e^{\frac{5\ln(x)}{2}} \\
&= z_1 \left(x^{\frac{5}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^3
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^3\right) + c_2\left(x^3(\ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^3 + c_2 x^3 \ln(x) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^3 + c_2 x^3 \ln(x)
$$

Verified OK.

1.20.6 Maple step by step solution

Let's solve

$y''x^2 - 5y'x + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y'}{x} - \frac{9y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5y'}{x} + \frac{9y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 - 5y'x + 9y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - 5\frac{d}{dt}y(t) + 9y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 6\frac{d}{dt}y(t) + 9y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r-3)^2 = 0$
- Root of the characteristic polynomial

  $r = 3$
- 1st solution of the ODE

  $y_1(t) = e^{3t}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^{3t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 e^{3t} + c_2 t\,e^{3t}$
- Change variables back using $t = \ln(x)$

  $y = c_1 x^3 + c_2 x^3 \ln(x)$
- Simplify

  $y = x^3(c_1 + \ln(x)\,c_2)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x$2)-5*x*diff(y(x),x)+9*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^3(c_1 + c_2 \ln(x))$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]-5*x*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3(3c_2 \log(x) + c_1)$$

1.21 problem 35

Internal problem ID [12594]
Internal file name [`OUTPUT/11246_Thursday_October_19_2023_04_43_42_PM_12799126/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 35.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^2 - 4y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 2\sqrt{y} \tag{1}$$
$$y' = -2\sqrt{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{2\sqrt{y}}dy = \int dx$$
$$\sqrt{y} = x + c_1$$

Summary
The solution(s) found are the following

$$\sqrt{y} = x + c_1 \tag{1}$$

Verification of solutions

$$\sqrt{y} = x + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{2\sqrt{y}} dy = \int dx$$
$$-\sqrt{y} = x + c_2$$

Summary

The solution(s) found are the following

$$-\sqrt{y} = x + c_2 \tag{1}$$

Verification of solutions

$$-\sqrt{y} = x + c_2$$

Verified OK.

1.21.1 Maple step by step solution

Let's solve

$y'^2 - 4y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{y}} = 2$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y}} dx = \int 2dx + c_1$

- Evaluate integral

  $2\sqrt{y} = 2x + c_1$

- Solve for y

  $y = \frac{1}{4}c_1^2 + c_1 x + x^2$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   <- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)^2-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = (x - c_1)^2$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 38

```
DSolve[(y'[x])^2-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}(-2x + c_1)^2$$
$$y(x) \to \frac{1}{4}(2x + c_1)^2$$
$$y(x) \to 0$$

1.22 problem 36

Internal problem ID [12595]
Internal file name [`OUTPUT/11247_Thursday_October_19_2023_04_43_42_PM_58881340/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 36.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_nonlinear_p_but_separable"**

Maple gives the following as the ode type

`[[_homogeneous, `class G`]]`

$$\boxed{y'^2 - 9yx = 0}$$

1.22.1 Solving as first order nonlinear p but separable ode

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = 9x, g = y$. Hence the ode is

$$(y')^2 = 9xy$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$9x > 0$$
$$y > 0$$

Under the above assumption the differential equations become separable and can be written as

$$y' = \sqrt{f}\,\sqrt{g}$$
$$y' = -\sqrt{f}\,\sqrt{g}$$

Therefore

$$\frac{1}{\sqrt{g}}\,dy = \left(\sqrt{f}\right)\,dx$$
$$-\frac{1}{\sqrt{g}}\,dy = \left(\sqrt{f}\right)\,dx$$

Replacing $f(x), g(y)$ by their values gives

$$\frac{1}{\sqrt{y}}\,dy = \left(\sqrt{9}\,\sqrt{x}\right)\,dx$$
$$-\frac{1}{\sqrt{y}}\,dy = \left(\sqrt{9}\,\sqrt{x}\right)\,dx$$

Integrating now gives the solutions.

$$\int \frac{1}{\sqrt{y}}dy = \int \sqrt{9}\,\sqrt{x}dx + c_1$$
$$\int -\frac{1}{\sqrt{y}}dy = \int \sqrt{9}\,\sqrt{x}dx + c_1$$

Integrating gives

$$2\sqrt{y} = \frac{2x^{\frac{3}{2}}\sqrt{9}}{3} + c_1$$
$$-2\sqrt{y} = \frac{2x^{\frac{3}{2}}\sqrt{9}}{3} + c_1$$

Therefore

$$y = x^{\frac{3}{2}}c_1 + x^3 + \frac{c_1^2}{4}$$
$$y = x^{\frac{3}{2}}c_1 + x^3 + \frac{c_1^2}{4}$$

Summary

The solution(s) found are the following

$$y = x^{\frac{3}{2}}c_1 + x^3 + \frac{c_1^2}{4} \tag{1}$$
$$y = x^{\frac{3}{2}}c_1 + x^3 + \frac{c_1^2}{4} \tag{2}$$

Verification of solutions

$$y = x^{\frac{3}{2}} c_1 + x^3 + \frac{c_1^2}{4}$$

Verified OK. `{0 < y, 0 < 9*x}`

$$y = x^{\frac{3}{2}} c_1 + x^3 + \frac{c_1^2}{4}$$

Verified OK. `{0 < y, 0 < 9*x}`

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      1st order, trying the canonical coordinates of the invariance group
      <- 1st order, canonical coordinates successful
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 69

```
dsolve(diff(y(x),x)^2-9*x*y(x)=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= 0 \\
y(x) &= x^3 + 2\sqrt{c_1 x}\, x + c_1 \\
y(x) &= x^3 - 2\sqrt{c_1 x}\, x + c_1 \\
y(x) &= x^3 - 2\sqrt{c_1 x}\, x + c_1 \\
y(x) &= x^3 + 2\sqrt{c_1 x}\, x + c_1
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 46

```
DSolve[(y'[x])^2-9*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \frac{1}{4}\left(-2x^{3/2} + c_1\right)^2 \\
y(x) &\to \frac{1}{4}\left(2x^{3/2} + c_1\right)^2 \\
y(x) &\to 0
\end{aligned}
$$

1.23 problem 37

Internal problem ID [12596]
Internal file name [OUTPUT/11248_Thursday_October_19_2023_04_43_44_PM_72759423/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises page 14
Problem number: 37.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y'^2 = x^6}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = x^3 \tag{1}$$
$$y' = -x^3 \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int x^3 \, \mathrm{d}x \\ &= \frac{x^4}{4} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x^4}{4} + c_1 \tag{1}$$

Verification of solutions

$$y = \frac{x^4}{4} + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\begin{aligned} y &= \int -x^3 \, \mathrm{d}x \\ &= -\frac{x^4}{4} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^4}{4} + c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{x^4}{4} + c_2$$

Verified OK.

1.23.1 Maple step by step solution

Let's solve

$y'^2 = x^6$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y'^2 dx = \int x^6 dx + c_1$

- Cannot compute integral

  $\int y'^2 dx = \frac{x^7}{7} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x)^2=x^6,y(x), singsol=all)
```

$$y(x) = \frac{x^4}{4} + c_1$$
$$y(x) = -\frac{x^4}{4} + c_1$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 29

```
DSolve[(y'[x])^2==x^6,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^4}{4} + c_1$$
$$y(x) \to \frac{x^4}{4} + c_1$$

2 Chapter 1. Introduction. Exercises 1.3, page 27

2.1 problem 1

Internal problem ID [12597]
Internal file name [`OUTPUT/11249_Thursday_October_19_2023_04_43_44_PM_32428753/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - 2yx = 0}$$

2.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= 2xy \end{aligned}$$

Where $f(x) = 2x$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= 2x\,dx \\
\int \frac{1}{y}\,dy &= \int 2x\,dx \\
\ln(y) &= x^2 + c_1 \\
y &= \mathrm{e}^{x^2+c_1} \\
&= c_1\mathrm{e}^{x^2}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1\mathrm{e}^{x^2} \tag{1}
$$

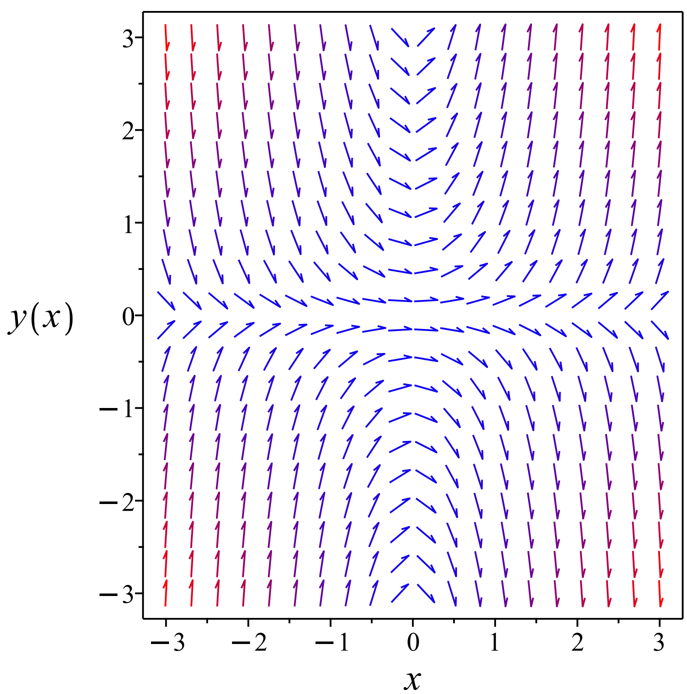


Figure 32: Slope field plot

Verification of solutions

$$
y = c_1\mathrm{e}^{x^2}
$$

Verified OK.

2.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - 2yx = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -2x dx} \\ &= e^{-x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx} \mu y &= 0 \\ \frac{d}{dx}\left(e^{-x^2} y\right) &= 0 \end{aligned}$$

Integrating gives

$$e^{-x^2} y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x^2}$ results in

$$y = c_1 e^{x^2}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x^2} \tag{1}$$

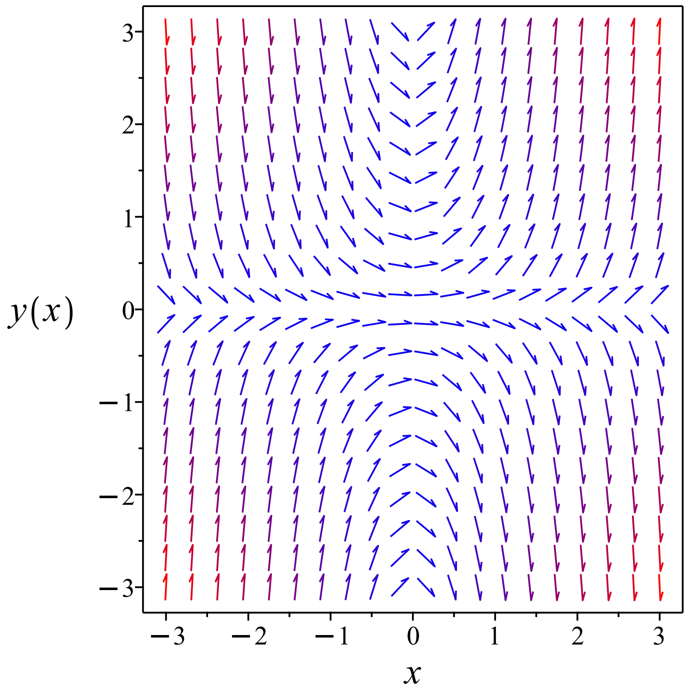


Figure 33: Slope field plot

Verification of solutions

$$y = c_1 e^{x^2}$$

Verified OK.

2.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - 2u(x)\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(2x^2-1)}{x} \end{aligned}$$

Where $f(x) = \frac{2x^2-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{2x^2-1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{2x^2-1}{x}\,dx \\
\ln(u) &= x^2 - \ln(x) + c_2 \\
u &= e^{x^2-\ln(x)+c_2} \\
&= c_2 e^{x^2-\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2 e^{x^2}}{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= c_2 e^{x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2 e^{x^2} \tag{1}
$$

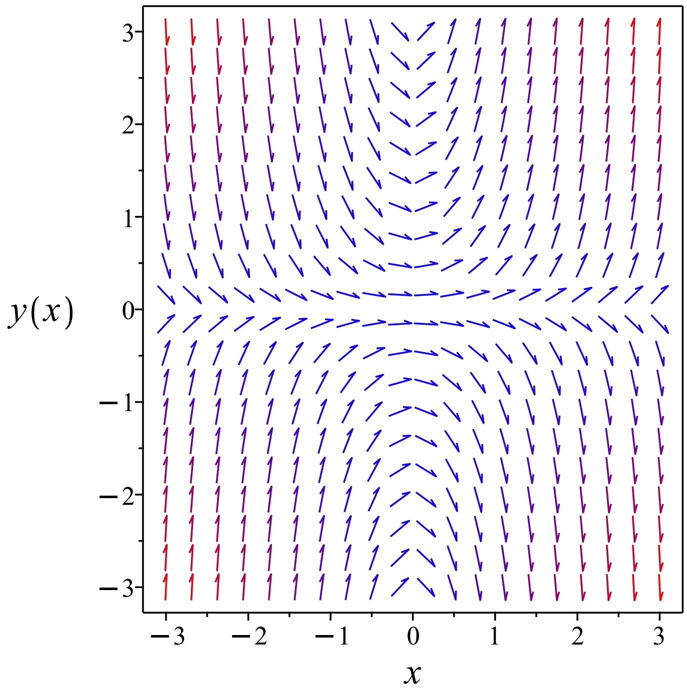


Figure 34: Slope field plot

Verification of solutions

$$y = c_2 e^{x^2}$$

Verified OK.

2.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2xy$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 41: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{x^2}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 2xy$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -2x\,\mathrm{e}^{-x^2} y \\ S_y &= \mathrm{e}^{-x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x^2} y = c_1$$

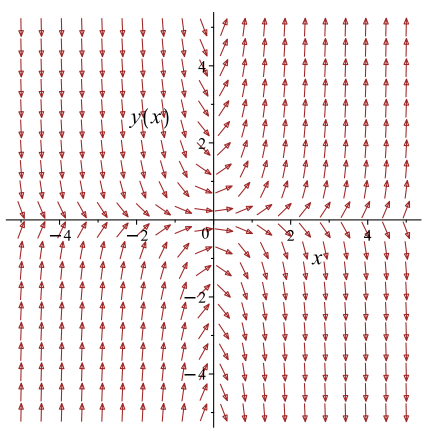
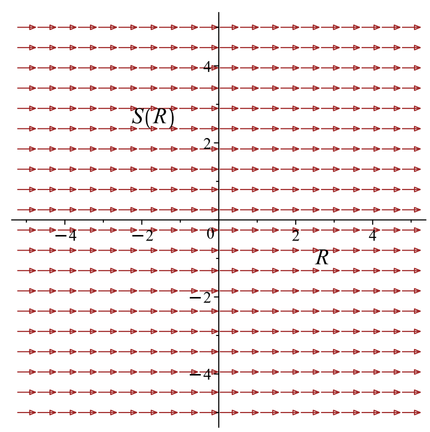
Which simplifies to

$$e^{-x^2} y = c_1$$

Which gives

$$y = c_1 e^{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2xy$ 	$R = x$ $S = e^{-x^2} y$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 e^{x^2} \tag{1}$$

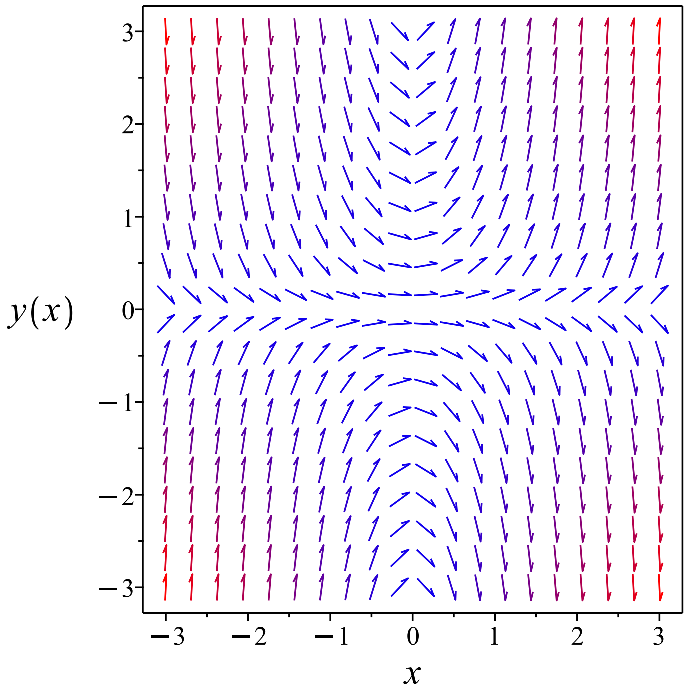


Figure 35: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{x^2}$$

Verified OK.

2.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{2y}\right) \mathrm{d}y &= (x)\,\mathrm{d}x \\
(-x)\,\mathrm{d}x + \left(\frac{1}{2y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x \\
N(x, y) &= \frac{1}{2y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x\\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x \, \mathrm{d}x\\
\phi &= -\frac{x^2}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y}$. Therefore equation (4) becomes

$$
\frac{1}{2y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{2y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2y}\right) \mathrm{d}y$$
$$f(y) = \frac{\ln(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\ln(y)}{2}$$

The solution becomes

$$y = \mathrm{e}^{x^2+2c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x^2+2c_1} \tag{1}$$

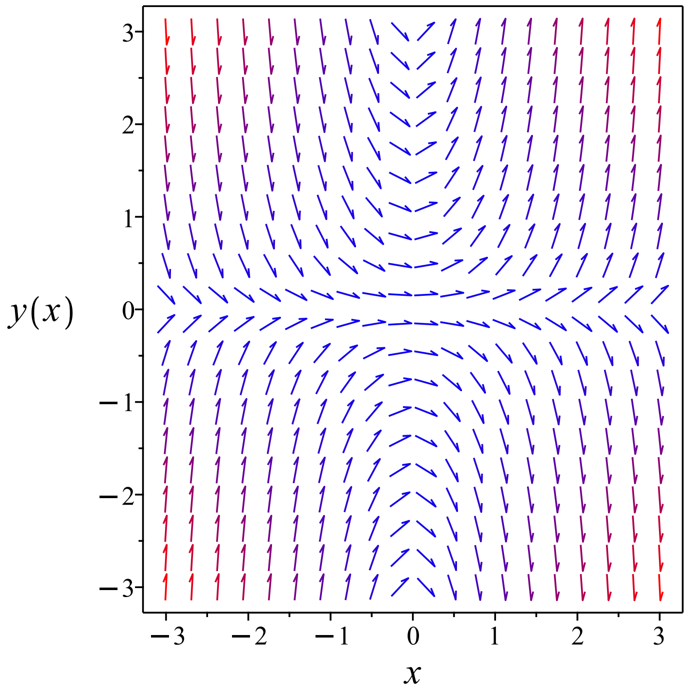


Figure 36: Slope field plot

Verification of solutions

$$y = e^{x^2+2c_1}$$

Verified OK.

2.1.6 Maple step by step solution

Let's solve

$y' - 2yx = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int 2xdx + c_1$

- Evaluate integral

$$\ln(y) = x^2 + c_1$$

- Solve for y

$$y = e^{x^2+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)-2*x*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{x^2} c_1$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 18

```
DSolve[y'[x]-2*x*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x^2}$$
$$y(x) \to 0$$

2.2 problem 2

Internal problem ID [12598]
Internal file name [`OUTPUT/11250_Thursday_October_19_2023_04_43_44_PM_64725706/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = x^2 + 2x - 1}$$

2.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x^2 + 2x - 1 \end{aligned}$$

Hence the ode is

$$y' + y = x^2 + 2x - 1$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x^2+2x-1\right)$$
$$\frac{d}{dx}(e^x y) = (e^x)\left(x^2+2x-1\right)$$
$$d(e^x y) = \left(\left(x^2+2x-1\right)e^x\right)\,dx$$

Integrating gives

$$e^x y = \int \left(x^2+2x-1\right)e^x\,dx$$
$$e^x y = \left(x^2-1\right)e^x + c_1$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(x^2-1\right)e^x + c_1 e^{-x}$$

which simplifies to

$$y = x^2 - 1 + c_1 e^{-x}$$

Summary
The solution(s) found are the following

$$y = x^2 - 1 + c_1 e^{-x} \tag{1}$$

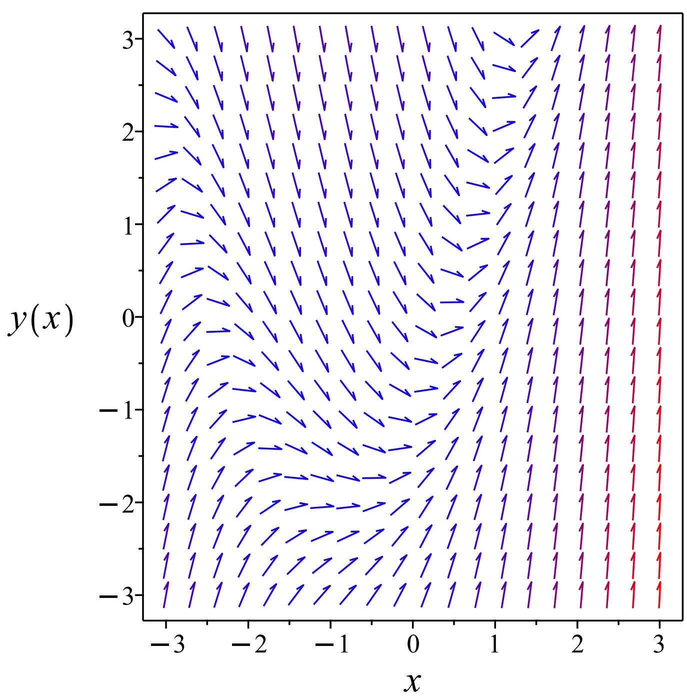


Figure 37: Slope field plot

Verification of solutions

$$y = x^2 - 1 + c_1 e^{-x}$$

Verified OK.

2.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= x^2 + 2x - y - 1 \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 44: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{-x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy \end{aligned}$$

Which results in

$$S = e^x y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x^2 + 2x - y - 1$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^x y \\ S_y &= e^x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (x^2 + 2x - 1) e^x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (R^2 + 2R - 1) e^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \left(R^2 - 1\right) e^R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\, e^x = \left(x^2 - 1\right) e^x + c_1$$

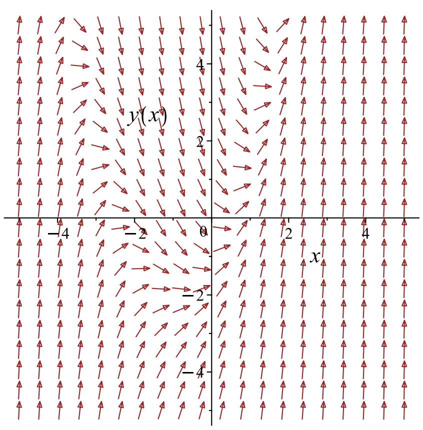
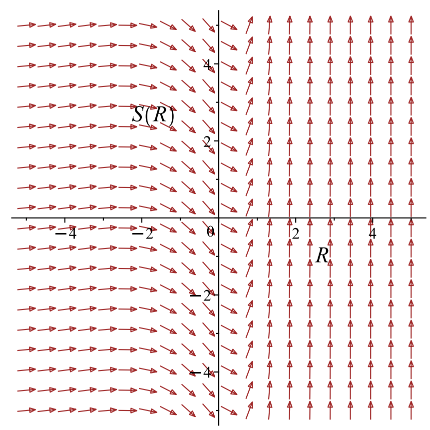
Which simplifies to

$$y\, e^x = \left(x^2 - 1\right) e^x + c_1$$

Which gives

$$y = \left(x^2 e^x - e^x + c_1\right) e^{-x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2 + 2x - y - 1$ 	$R = x$ $S = e^x y$	$\frac{dS}{dR} = \left(R^2 + 2R - 1\right) e^R$ 

Summary

The solution(s) found are the following

$$y = \left(x^2 e^x - e^x + c_1\right) e^{-x} \tag{1}$$

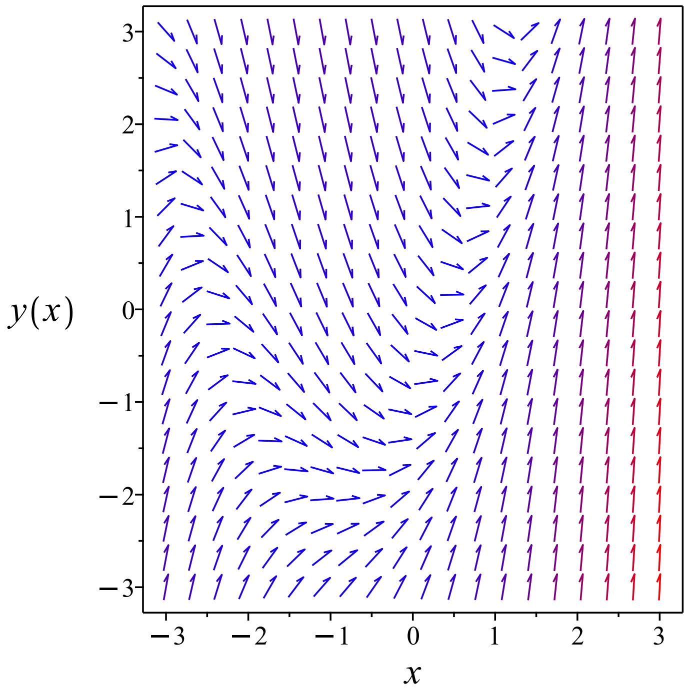


Figure 38: Slope field plot

Verification of solutions

$$y = (x^2 e^x - e^x + c_1) e^{-x}$$

Verified OK.

2.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= \left(x^2 + 2x - y - 1\right) \mathrm{d}x \\ \left(-x^2 - 2x + y + 1\right) \mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -x^2 - 2x + y + 1 \\ N(x, y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 - 2x + y + 1\right) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((1) - (0)) \\ &= 1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{x} \\ &= \mathrm{e}^{x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{x}\left(-x^{2} - 2x + y + 1\right) \\ &= -\mathrm{e}^{x}\left(x^{2} + 2x - y - 1\right) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{x}(1) \\ &= \mathrm{e}^{x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{x}\left(x^{2} + 2x - y - 1\right)\right) + \left(\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\mathrm{e}^{x}\left(x^{2}+2x-y-1\right) \mathrm{d}x$$
$$\phi = -\left(x^{2}-y-1\right) \mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\left(x^{2}-y-1\right) \mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\left(x^{2}-y-1\right) \mathrm{e}^{x}$$

The solution becomes

$$y = \left(x^{2}\mathrm{e}^{x} - \mathrm{e}^{x} + c_1\right) \mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = \left(x^{2}\mathrm{e}^{x} - \mathrm{e}^{x} + c_1\right) \mathrm{e}^{-x} \tag{1}$$

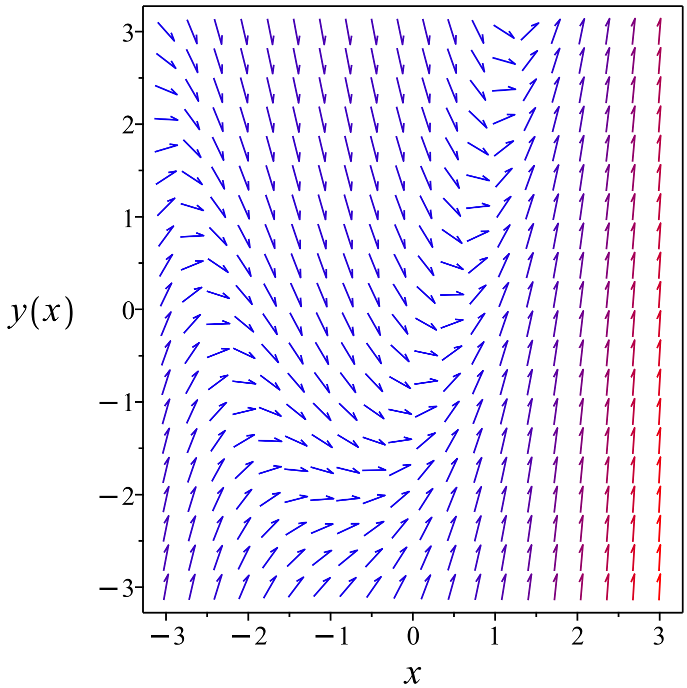


Figure 39: Slope field plot

Verification of solutions

$$y = \left(x^2 e^x - e^x + c_1\right) e^{-x}$$

Verified OK.

2.2.4 Maple step by step solution

Let's solve

$y' + y = x^2 + 2x - 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + x^2 + 2x - 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = x^2 + 2x - 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + y\right) = \mu(x)\left(x^2 + 2x - 1\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\,(y'+y) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\,(x^2+2x-1)\, dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\,(x^2+2x-1)\, dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)(x^2+2x-1)dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int (x^2+2x-1)\mathrm{e}^{x}dx + c_1}{\mathrm{e}^{x}}$
- Evaluate the integrals on the rhs

  $y = \frac{(x^2-1)\mathrm{e}^{x} + c_1}{\mathrm{e}^{x}}$
- Simplify

  $y = x^2 - 1 + c_1\mathrm{e}^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)+y(x)=x^2+2*x-1,y(x), singsol=all)
```

$$y(x) = x^2 - 1 + c_1 \mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.088 (sec). Leaf size: 18

```
DSolve[y'[x]+y[x]==x^2+2*x-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + c_1 e^{-x} - 1$$

2.3 problem 3

Internal problem ID [12599]
Internal file name [`OUTPUT/11251_Thursday_October_19_2023_04_43_45_PM_85570290/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' - 6y = 0}$$

2.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 6 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-6)}\\ &= \frac{1}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{5}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{5}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(3)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{3x} + c_2 e^{-2x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{3x} + c_2 e^{-2x} \tag{1}$$

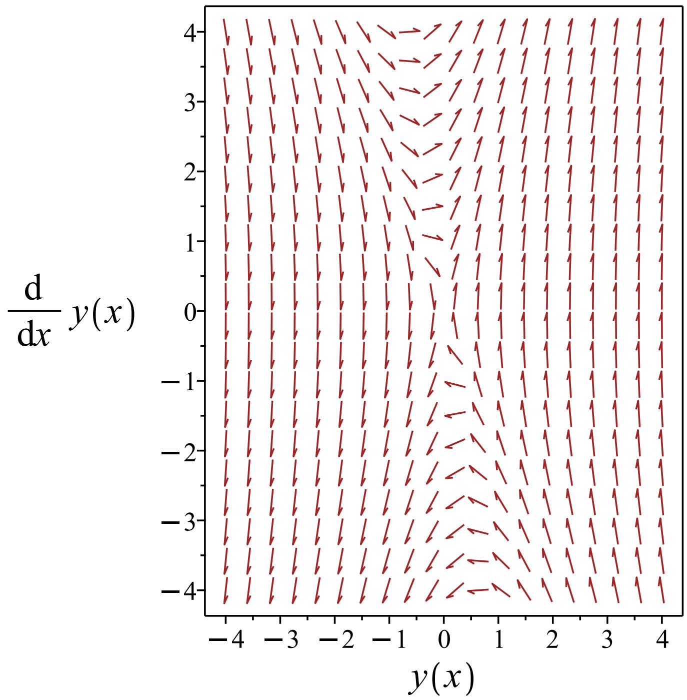


Figure 40: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{-2x}$$

Verified OK.

2.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 47: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{5x}}{5}\right)
\end{aligned}$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{5x}}{5}\right)\right)$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + \frac{\mathrm{e}^{3x} c_2}{5} \tag{1}$$

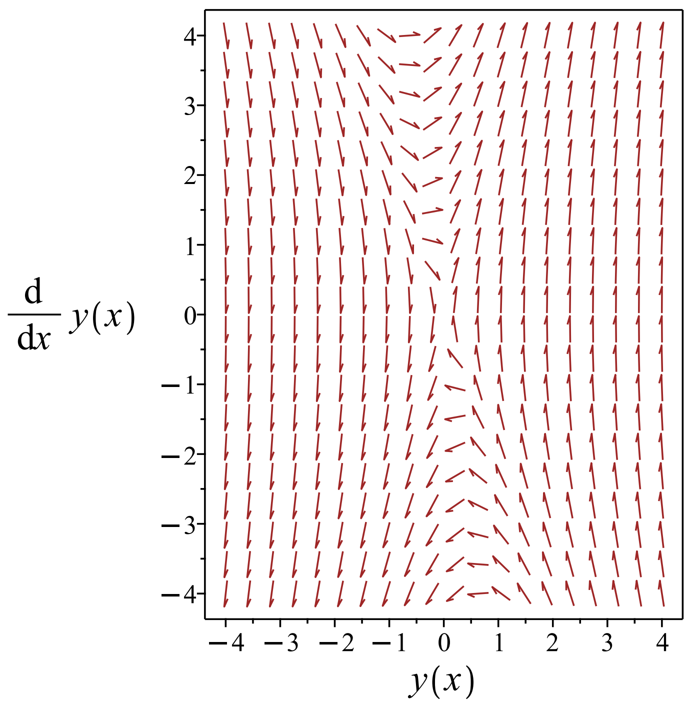


Figure 41: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + \frac{\mathrm{e}^{3x} c_2}{5}$$

Verified OK.

2.3.3 Maple step by step solution

Let's solve

$y'' - y' - 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - r - 6 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r - 3) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 3)$

- 1st solution of the ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} + e^{3x} c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-diff(y(x),x)-6*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 \mathrm{e}^{5x} + c_2\right) \mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 22

```
DSolve[y''[x]-y'[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(c_2 e^{5x} + c_1\right)$$

2.4 problem 4

Internal problem ID [12600]
Internal file name [`OUTPUT/11252_Thursday_October_19_2023_04_43_46_PM_97973919/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - x\sqrt{y} = 0}$$

2.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x\sqrt{y} \end{aligned}$$

Where $f(x) = x$ and $g(y) = \sqrt{y}$. Integrating both sides gives

$$\frac{1}{\sqrt{y}}\, dy = x\, dx$$

$$\int \frac{1}{\sqrt{y}}\, dy = \int x\, dx$$

$$2\sqrt{y} = \frac{x^2}{2} + c_1$$

The solution is

$$2\sqrt{y} - \frac{x^2}{2} - c_1 = 0$$

Summary

The solution(s) found are the following

$$2\sqrt{y} - \frac{x^2}{2} - c_1 = 0 \tag{1}$$

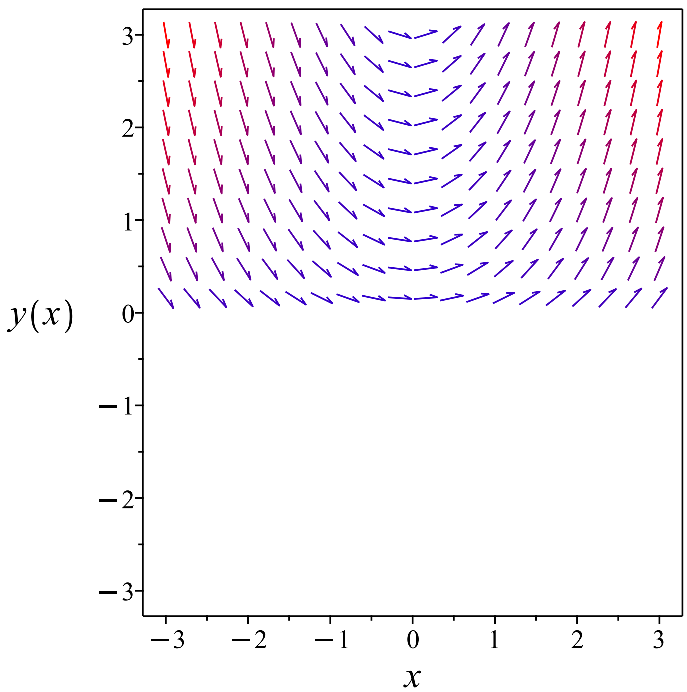


Figure 42: Slope field plot

Verification of solutions

$$2\sqrt{y} - \frac{x^2}{2} - c_1 = 0$$

Verified OK.

2.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x\sqrt{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 49: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{1}{x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x\sqrt{y}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{\sqrt{y}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{\sqrt{R}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = 2\sqrt{R} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{x^2}{2} = 2\sqrt{y} + c_1
$$

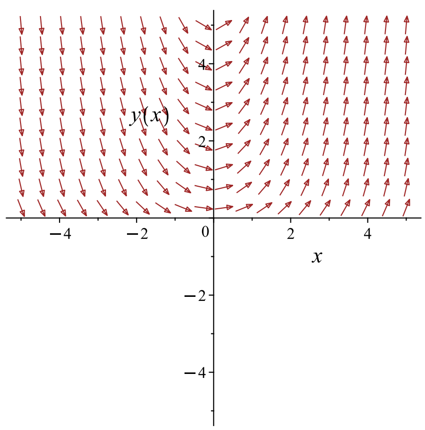
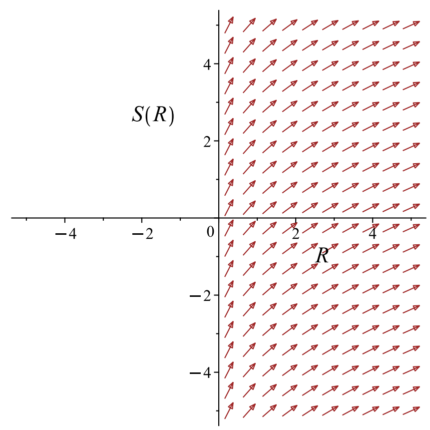
Which simplifies to

$$
\frac{x^2}{2} = 2\sqrt{y} + c_1
$$

Which gives

$$
y = \frac{1}{16}x^4 - \frac{1}{4}c_1 x^2 + \frac{1}{4}c_1^2
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = x\sqrt{y}$

</td><td>$R = y$
$S = \frac{x^2}{2}$</td><td>$\frac{dS}{dR} = \frac{1}{\sqrt{R}}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{1}{16}x^4 - \frac{1}{4}c_1x^2 + \frac{1}{4}c_1^2 \tag{1}$$

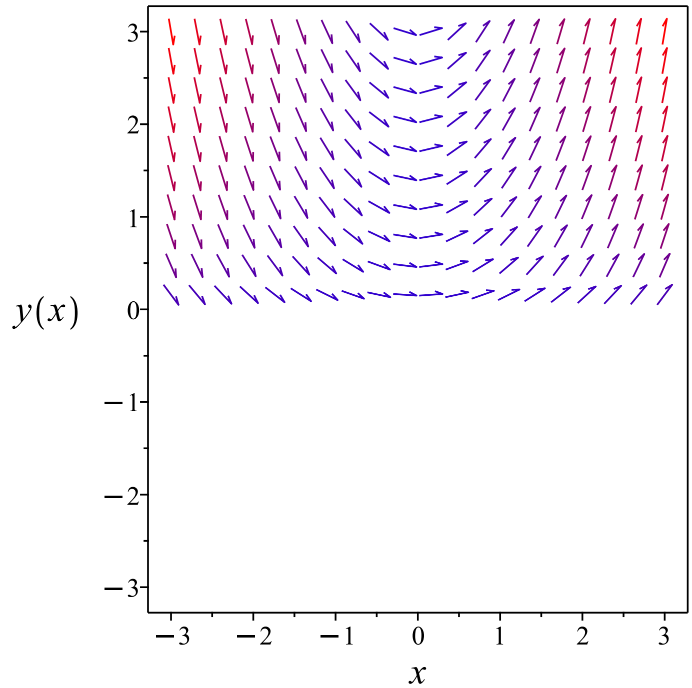


Figure 43: Slope field plot

Verification of solutions

$$y = \frac{1}{16}x^4 - \frac{1}{4}c_1x^2 + \frac{1}{4}c_1^2$$

Verified OK.

2.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{\sqrt{y}}\right) \mathrm{d}y = (x)\,\mathrm{d}x$$
$$(-x)\,\mathrm{d}x + \left(\frac{1}{\sqrt{y}}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = \frac{1}{\sqrt{y}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{y}}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -x\, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{y}}$. Therefore equation (4) becomes

$$
\frac{1}{\sqrt{y}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{\sqrt{y}}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{\sqrt{y}}\right) \mathrm{d}y$$
$$f(y) = 2\sqrt{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + 2\sqrt{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + 2\sqrt{y}$$

The solution becomes

$$y = \frac{1}{16}x^4 + \frac{1}{4}c_1 x^2 + \frac{1}{4}c_1^2$$

Summary

The solution(s) found are the following

$$y = \frac{1}{16}x^4 + \frac{1}{4}c_1 x^2 + \frac{1}{4}c_1^2 \tag{1}$$

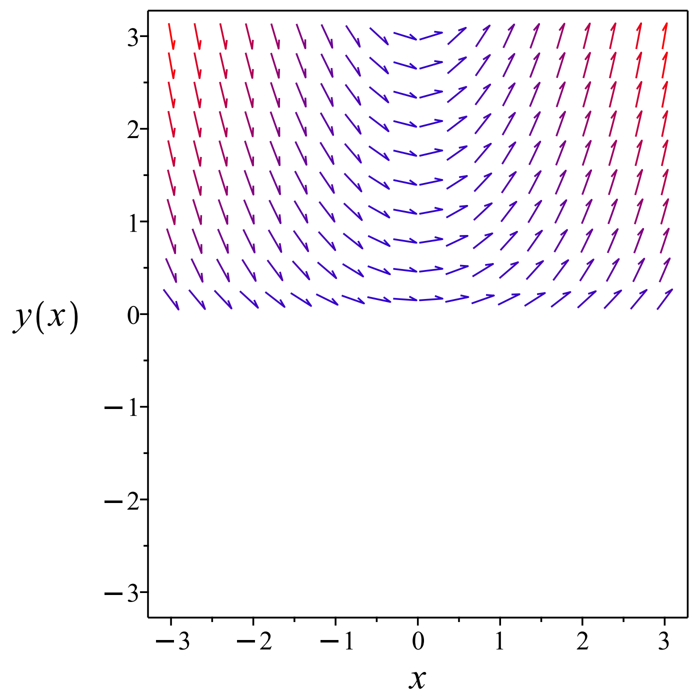


Figure 44: Slope field plot

Verification of solutions

$$y = \frac{1}{16}x^4 + \frac{1}{4}c_1x^2 + \frac{1}{4}c_1^2$$

Verified OK.

2.4.4 Maple step by step solution

Let's solve

$y' - x\sqrt{y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{y}} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y}}dx = \int xdx + c_1$

- Evaluate integral

$$2\sqrt{y} = \frac{x^2}{2} + c_1$$

- Solve for y

$$y = \frac{1}{16}x^4 + \frac{1}{4}c_1x^2 + \frac{1}{4}c_1^2$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)=x*y(x)^(1/2),y(x), singsol=all)
```

$$\sqrt{y(x)} - \frac{x^2}{4} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.2 (sec). Leaf size: 24

```
DSolve[y'[x]==x*y[x]^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{16}(x^2 + 2c_1)^2$$
$$y(x) \to 0$$

2.5 problem 5

Internal problem ID [12601]
Internal file name [`OUTPUT/11253_Thursday_October_19_2023_04_43_47_PM_87594477/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - y = 0}$$

2.5.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = \mathrm{e}^{x} c_1 + c_2 \mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x} c_1 + c_2 \mathrm{e}^{-x} \tag{1}$$

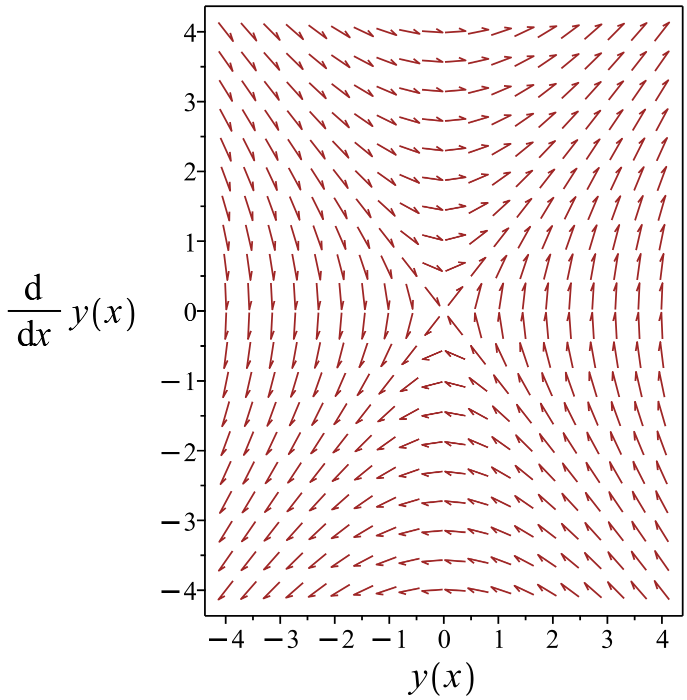


Figure 45: Slope field plot

<u>Verification of solutions</u>

$$y = e^{x} c_1 + c_2 e^{-x}$$

Verified OK.

2.5.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y' y'' - y y' = 0$$

Integrating the above w.r.t x gives

$$\int (y' y'' - y y')\, dx = 0$$

$$\frac{y'^2}{2} - \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2c_1} \tag{1}$$

$$y' = -\sqrt{y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{y^2+2c_1}}dy = \int dx$$
$$\ln\left(y+\sqrt{y^2+2c_1}\right) = x+c_2$$

Raising both side to exponential gives

$$y+\sqrt{y^2+2c_1} = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$y+\sqrt{y^2+2c_1} = c_3\mathrm{e}^{x}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{y^2+2c_1}}dy = \int dx$$
$$-\ln\left(y+\sqrt{y^2+2c_1}\right) = x+c_4$$

Raising both side to exponential gives

$$\frac{1}{y+\sqrt{y^2+2c_1}} = \mathrm{e}^{x+c_4}$$

Which simplifies to

$$\frac{1}{y+\sqrt{y^2+2c_1}} = c_5\mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(\mathrm{e}^{2x}c_3^2-2c_1\right)\mathrm{e}^{-x}}{2c_3} \tag{1}$$
$$y = -\frac{\left(2c_1c_5^2\mathrm{e}^{2x}-1\right)\mathrm{e}^{-x}}{2c_5} \tag{2}$$

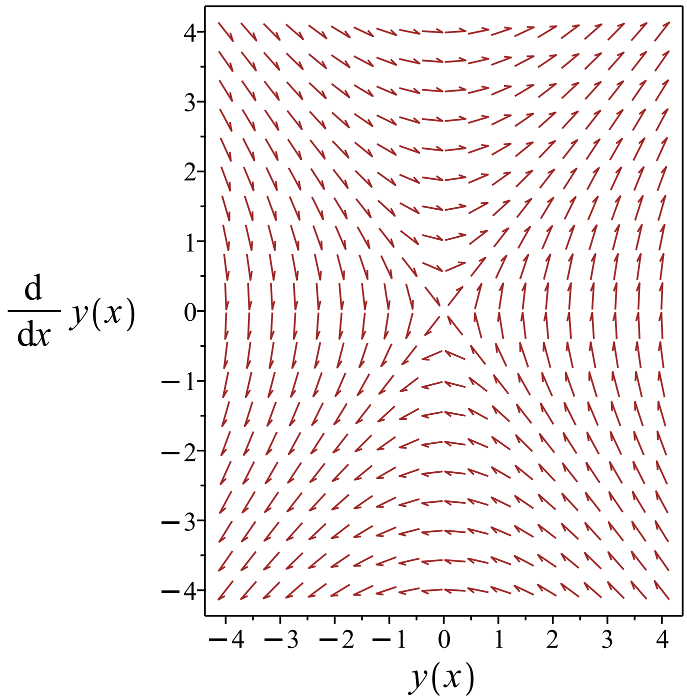


Figure 46: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\left(e^{2x} c_3^2 - 2c_1\right) e^{-x}}{2c_3}$$

Verified OK.

$$y = -\frac{\left(2c_1 c_5^2 e^{2x} - 1\right) e^{-x}}{2c_5}$$

Verified OK.

2.5.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \qquad (5) \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 52: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= e^{-x} \int \frac{1}{e^{-2x}}\, dx \\ &= e^{-x} \left(\frac{e^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2}\right)\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_2 e^{x}}{2} + c_1 e^{-x} \tag{1}$$

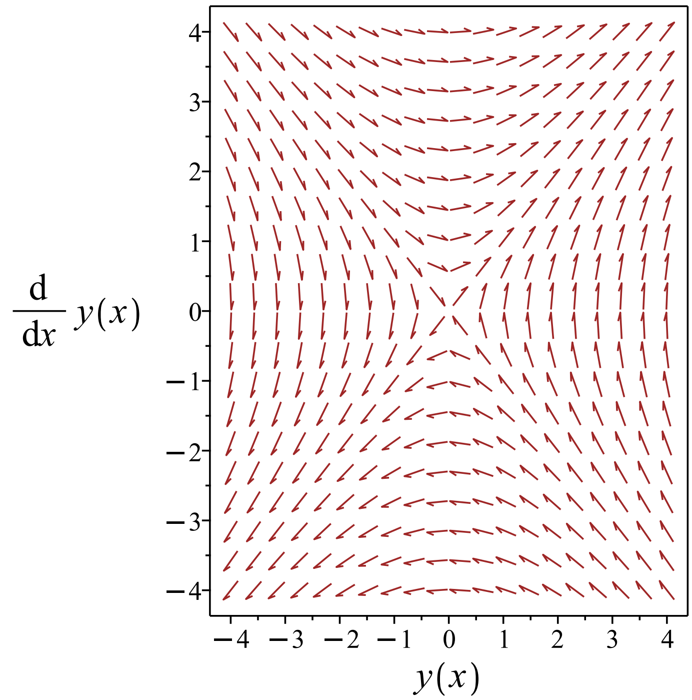


Figure 47: Slope field plot

Verification of solutions

$$y = \frac{c_2 e^x}{2} + c_1 e^{-x}$$

Verified OK.

2.5.4 Maple step by step solution

Let's solve

$y'' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

$r = (-1, 1)$

- 1st solution of the ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-x} + c_2 e^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x} + c_2 e^{x}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 20

```
DSolve[y''[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x} + c_2 e^{-x}$$

2.6 problem 6

Internal problem ID [12602]
Internal file name [OUTPUT/11254_Thursday_October_19_2023_04_43_48_PM_68138707/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - 3y^{\frac{2}{3}} = 0}$$

2.6.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{3y^{\frac{2}{3}}} dy = \int dx$$

$$y^{\frac{1}{3}} = x + c_1$$

Summary
The solution(s) found are the following

$$y^{\frac{1}{3}} = x + c_1 \tag{1}$$

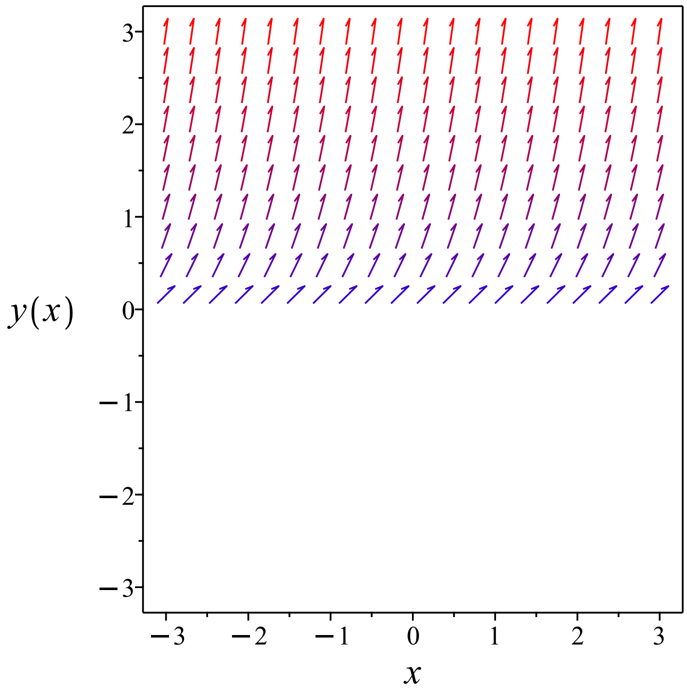


Figure 48: Slope field plot

Verification of solutions

$$y^{\frac{1}{3}} = x + c_1$$

Verified OK.

2.6.2 Maple step by step solution

Let's solve

$y' - 3y^{\frac{2}{3}} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{2}{3}}} = 3$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{2}{3}}} dx = \int 3dx + c_1$

- Evaluate integral

$$3y^{\frac{1}{3}} = 3x + c_1$$

- Solve for y

$$y = x^3 + c_1x^2 + \frac{1}{3}c_1^2x + \frac{1}{27}c_1^3$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)=3*y(x)^(2/3),y(x), singsol=all)
```

$$y(x)^{\frac{1}{3}} - c_1 - x = 0$$

✓ Solution by Mathematica

Time used: 0.166 (sec). Leaf size: 22

```
DSolve[y'[x]==3*y[x]^(2/3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{27}(3x + c_1)^3$$
$$y(x) \rightarrow 0$$

2.7 problem 7

Internal problem ID [12603]
Internal file name [`OUTPUT/11255_Thursday_October_19_2023_04_43_48_PM_53777959/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x \ln (x) y' - (\ln (x) + 1) y = 0}$$

2.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{(\ln (x) + 1) y}{x \ln (x)} \end{aligned}$$

Where $f(x) = \frac{\ln(x)+1}{x\ln(x)}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{\ln(x)+1}{x\ln(x)}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{\ln(x)+1}{x\ln(x)}\,dx \\
\ln(y) &= \ln(x) + \ln(\ln(x)) + c_1 \\
y &= e^{\ln(x)+\ln(\ln(x))+c_1} \\
&= c_1 e^{\ln(x)+\ln(\ln(x))}
\end{aligned}
$$

Which simplifies to

$$y = c_1 x \ln(x)$$

Summary

The solution(s) found are the following

$$y = c_1 x \ln(x) \tag{1}$$

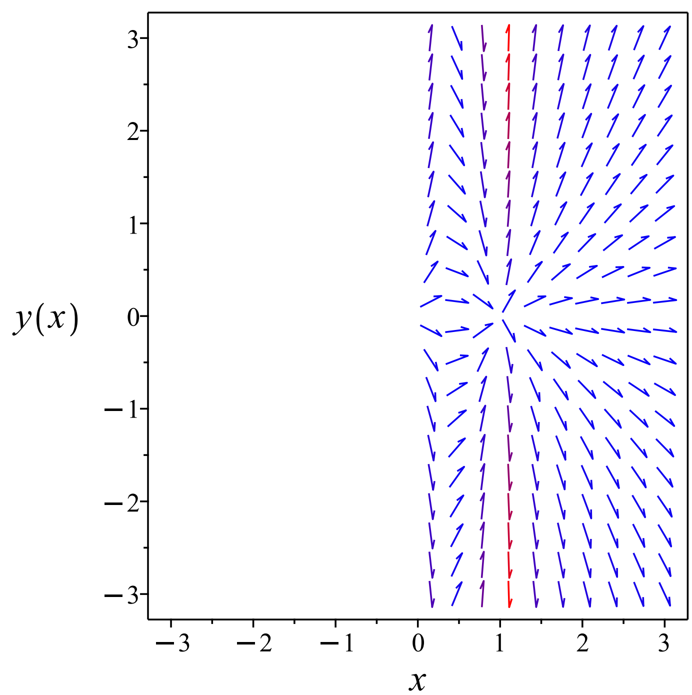


Figure 49: Slope field plot

Verification of solutions

$$y = c_1 x \ln(x)$$

Verified OK.

2.7.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{\ln (x) + 1}{x \ln (x)} \\
q(x) &= 0
\end{aligned}$$

Hence the ode is

$$y' - \frac{(\ln (x) + 1) y}{x \ln (x)} = 0$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{\ln(x)+1}{x \ln(x)} dx} \\
&= \mathrm{e}^{-\ln(x) - \ln(\ln(x))}
\end{aligned}$$

Which simplifies to

$$\mu = \frac{1}{x \ln (x)}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{y}{x \ln (x)}\right) &= 0
\end{aligned}$$

Integrating gives

$$\frac{y}{x \ln (x)} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x \ln(x)}$ results in

$$y = c_1 x \ln (x)$$

Summary
The solution(s) found are the following

$$y = c_1 x \ln (x) \tag{1}$$

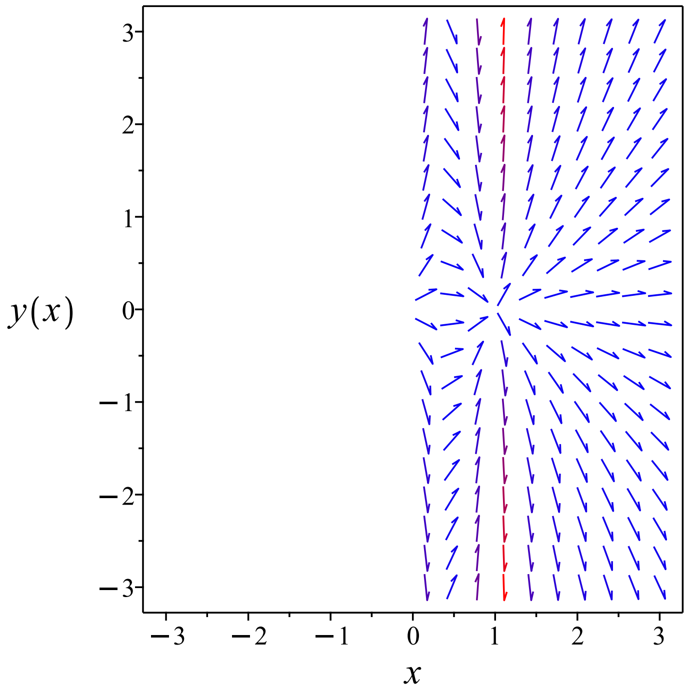


Figure 50: Slope field plot

Verification of solutions

$$y = c_1 x \ln(x)$$

Verified OK.

2.7.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x \ln(x)\,(u'(x)\,x + u(x)) - (\ln(x) + 1)\,u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u}{x \ln(x)} \end{aligned}$$

Where $f(x) = \frac{1}{x \ln(x)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{1}{x \ln(x)}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{1}{x \ln(x)}\,dx \\
\ln(u) &= \ln(\ln(x)) + c_2 \\
u &= \mathrm{e}^{\ln(\ln(x))+c_2} \\
&= \ln(x)\,c_2
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= x \ln(x)\,c_2
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = x \ln(x)\,c_2 \tag{1}$$

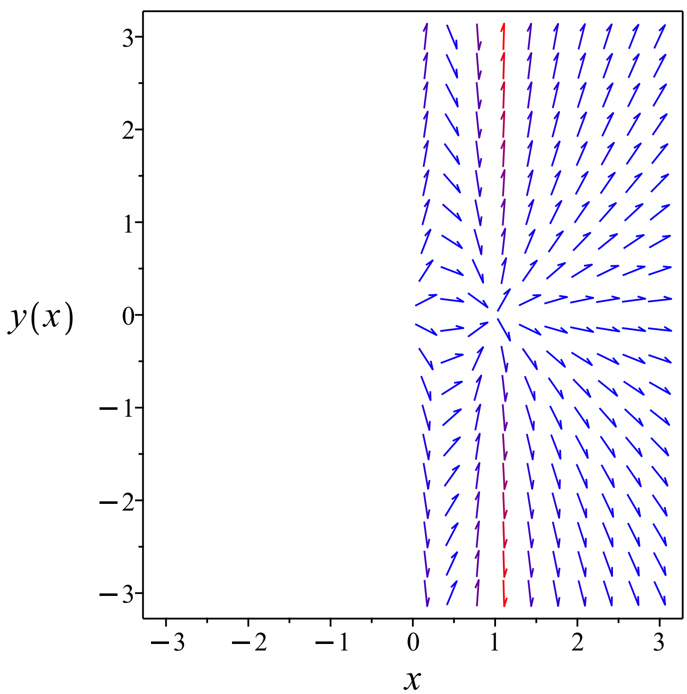


Figure 51: Slope field plot

Verification of solutions

$$y = x \ln(x) c_2$$

Verified OK.

2.7.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{(\ln(x) + 1) y}{x \ln(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 55: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{\ln(x) + \ln(\ln(x))}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\ln(x)+\ln(\ln(x))}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x \ln(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{(\ln(x) + 1) y}{x \ln(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{(\ln(x) + 1) y}{x^2 \ln(x)^2} \\ S_y &= \frac{1}{x \ln(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x \ln(x)} = c_1$$

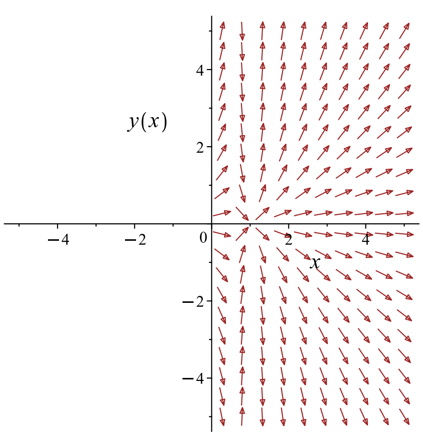
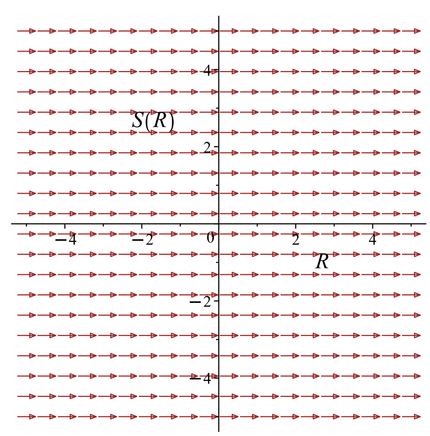
Which simplifies to

$$\frac{y}{x \ln(x)} = c_1$$

Which gives

$$y = c_1 x \ln(x)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{(\ln(x)+1)y}{x\ln(x)}$ 	$R = x$ $S = \frac{y}{x \ln(x)}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 x \ln(x) \tag{1}$$

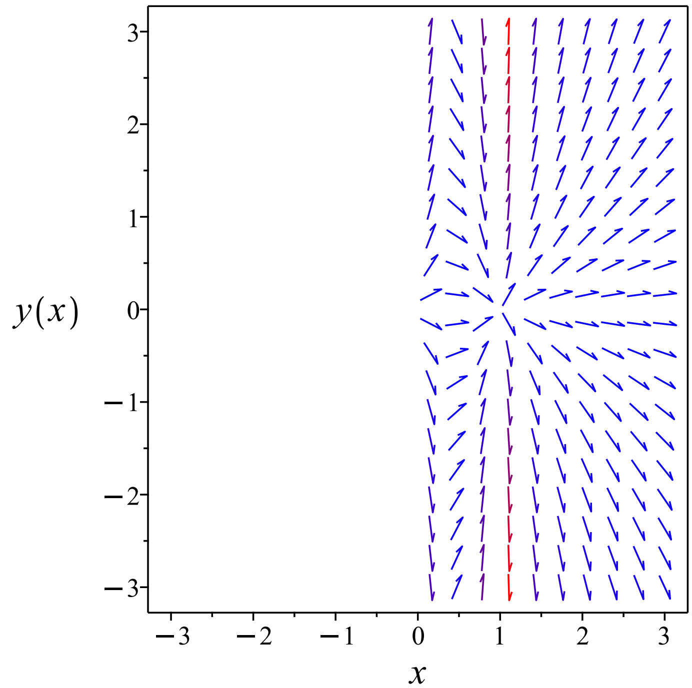


Figure 52: Slope field plot

Verification of solutions

$$y = c_1 x \ln(x)$$

Verified OK.

2.7.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y}\right)\mathrm{d}y = \left(\frac{\ln(x)+1}{x\ln(x)}\right)\mathrm{d}x$$
$$\left(-\frac{\ln(x)+1}{x\ln(x)}\right)\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{\ln(x)+1}{x\ln(x)}$$
$$N(x,y) = \frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\ln(x)+1}{x\ln(x)}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{\ln(x) + 1}{x \ln(x)} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) - \ln(\ln(x)) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(\ln(x)) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(\ln(x)) + \ln(y)$$

The solution becomes

$$y = e^{c_1} \ln(x)\, x$$

Summary

The solution(s) found are the following

$$y = e^{c_1} \ln(x)\, x \tag{1}$$

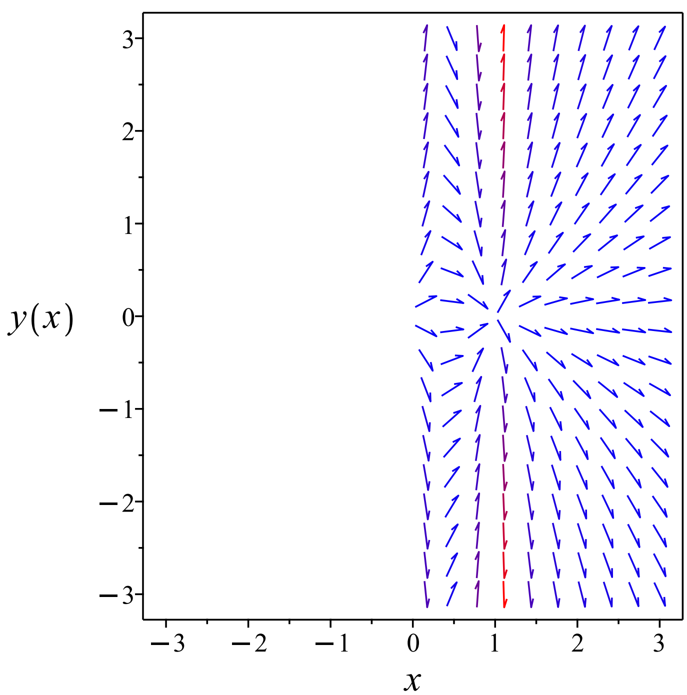


Figure 53: Slope field plot

Verification of solutions

$$y = e^{c_1} \ln(x)\, x$$

Verified OK.

2.7.6 Maple step by step solution

Let's solve

$x \ln(x) y' - (\ln(x) + 1) y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{\ln(x)+1}{x \ln(x)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{\ln(x)+1}{x \ln(x)} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + \ln(\ln(x)) + c_1$

- Solve for y

  $y = e^{c_1} \ln(x) x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve((x*ln(x))*diff(y(x),x)-(1+ln(x))*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \ln(x) x$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 16

```
DSolve[(x*Log[x])*y'[x]-(1+Log[x])*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x \log(x)$$
$$y(x) \to 0$$

2.8 problem 8 a(i)

Internal problem ID [12604]
Internal file name [`OUTPUT/11256_Thursday_October_19_2023_04_43_49_PM_52831655/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 8 a(i).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - y' - 2y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = -5]$$

2.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= -2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - y' - 2y = 0$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.8.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda \,\mathrm{e}^{\lambda x} - 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(-2)} \\
&= \frac{1}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{2} + \frac{3}{2}$$

$$\lambda_2 = \frac{1}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^{2x} + c_2 e^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} - c_2 e^{-x}$$

substituting $y' = -5$ and $x = 0$ in the above gives

$$-5 = 2c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 3$$

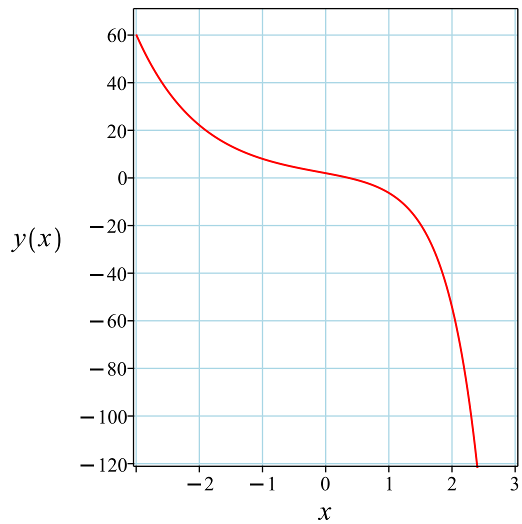
Substituting these values back in above solution results in

$$y = -e^{2x} + 3\,e^{-x}$$

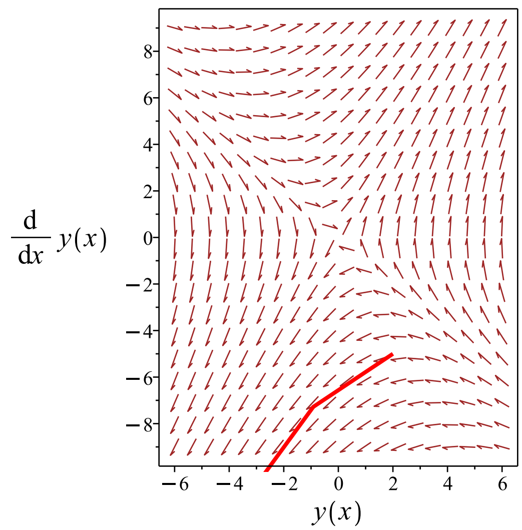
Summary

The solution(s) found are the following

$$y = -e^{2x} + 3\,e^{-x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -e^{2x} + 3\,e^{-x}$$

Verified OK.

2.8.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 58: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$
2 = c_1 + \frac{c_2}{3} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-x} + \frac{2c_2 \mathrm{e}^{2x}}{3}
$$

substituting $y' = -5$ and $x = 0$ in the above gives

$$
-5 = -c_1 + \frac{2c_2}{3} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= -3
\end{aligned}
$$

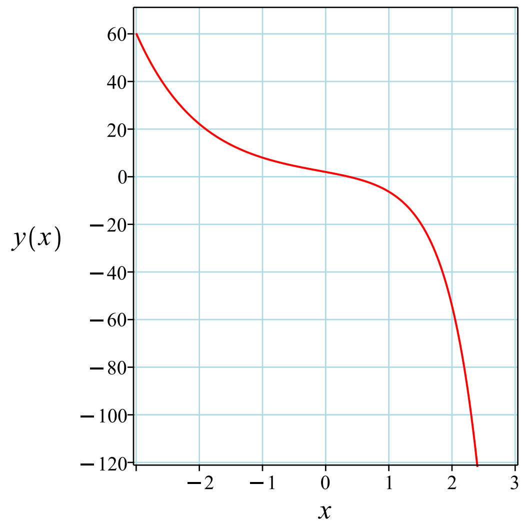
Substituting these values back in above solution results in

$$
y = -\mathrm{e}^{2x} + 3\,\mathrm{e}^{-x}
$$

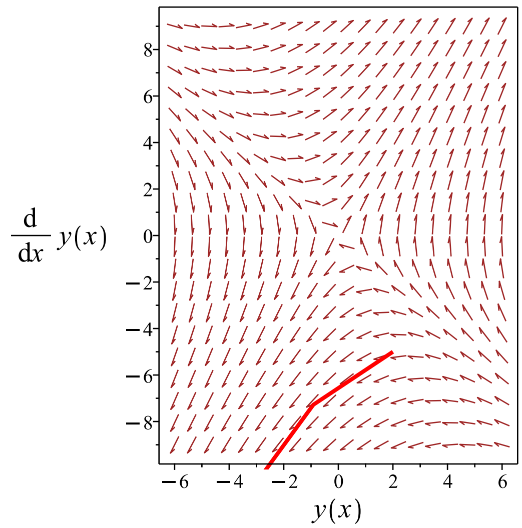
Summary

The solution(s) found are the following

$$
y = -\mathrm{e}^{2x} + 3\,\mathrm{e}^{-x} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -e^{2x} + 3\,e^{-x}$$

Verified OK.

2.8.4 Maple step by step solution

Let's solve

$$\left[y'' - y' - 2y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = -5\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r + 1)\,(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the ODE

$y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x}$

□ Check validity of solution $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x}$

  ◦ Use initial condition $y(0) = 2$

    $2 = c_1 + c_2$

  ◦ Compute derivative of the solution

    $y' = -c_1 \mathrm{e}^{-x} + 2c_2 \mathrm{e}^{2x}$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = -5$

    $-5 = -c_1 + 2c_2$

  ◦ Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = -1\}$

  ◦ Substitute constant values into general solution and simplify

    $y = -\mathrm{e}^{2x} + 3\,\mathrm{e}^{-x}$

- Solution to the IVP

  $y = -\mathrm{e}^{2x} + 3\,\mathrm{e}^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2)-diff(y(x),x)-2*y(x)=0,y(0) = 2, D(y)(0) = -5],y(x), singsol=all)
```

$$y(x) = 3\,\mathrm{e}^{-x} - \mathrm{e}^{2x}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 19

```
DSolve[{y''[x]-y'[x]-2*y[x]==0,{y[0]==2,y'[0]==-5}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{-x}\left(e^{3x} - 3\right)$$

2.9 problem 8 a(ii)

Internal problem ID [12605]
Internal file name [`OUTPUT/11257_Thursday_October_19_2023_04_43_50_PM_93576080/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 8 a(ii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' - 2y = 0}$$

With initial conditions

$$[y(1) = 3, y'(1) = -1]$$

2.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = -1$$
$$q(x) = -2$$
$$F = 0$$

Hence the ode is

$$y'' - y' - 2y = 0$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

2.9.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-2)} \\ &= \frac{1}{2} \pm \frac{3}{2} \end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{2} + \frac{3}{2}$$

$$\lambda_2 = \frac{1}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^{2x} + c_2 e^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = c_1 e^2 + c_2 e^{-1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} - c_2 e^{-x}$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = 2c_1 e^2 - c_2 e^{-1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{2\,e^{-2}}{3}$$
$$c_2 = \frac{7\,e}{3}$$

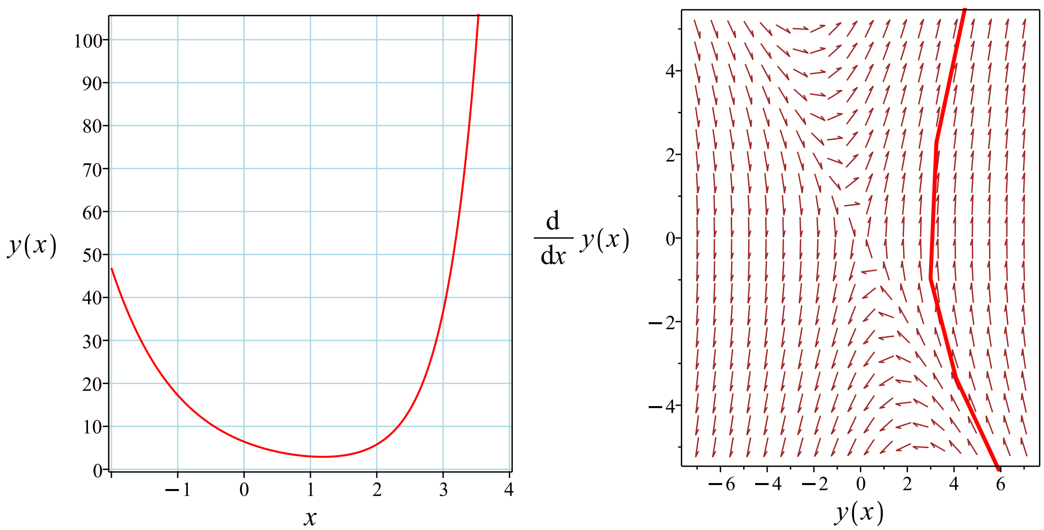
Substituting these values back in above solution results in

$$y = \frac{2\,e^{2x} e^{-2}}{3} + \frac{7\,e^{-x} e}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{2\,e^{2x} e^{-2}}{3} + \frac{7\,e^{-x} e}{3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{2\,e^{2x} e^{-2}}{3} + \frac{7\,e^{-x} e}{3}$$

Verified OK.

2.9.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 60: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = c_1 \mathrm{e}^{-1} + \frac{c_2 \mathrm{e}^{2}}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \mathrm{e}^{-x} + \frac{2 c_2 \mathrm{e}^{2x}}{3}$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = -c_1 e^{-1} + \frac{2c_2 e^2}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{7\,e}{3}$$
$$c_2 = 2\,e^{-2}$$

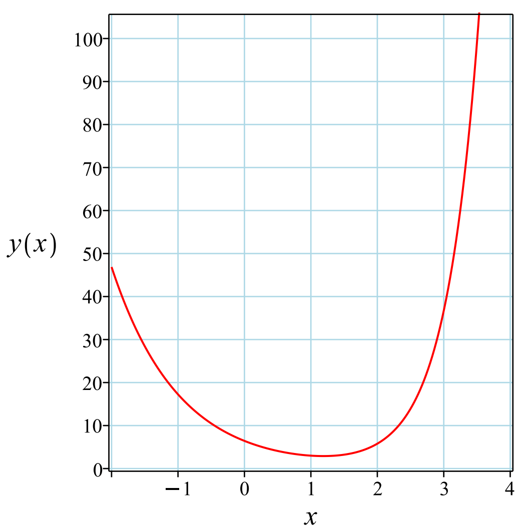
Substituting these values back in above solution results in

$$y = \frac{2\,e^{2x} e^{-2}}{3} + \frac{7\,e^{-x} e}{3}$$

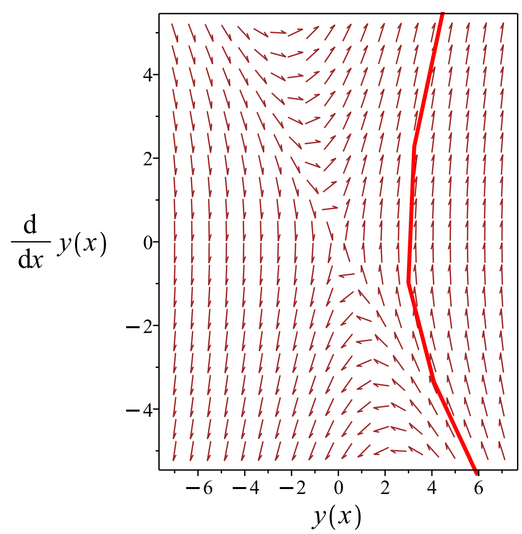
<u>Summary</u>

The solution(s) found are the following

$$y = \frac{2\,e^{2x} e^{-2}}{3} + \frac{7\,e^{-x} e}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{2\,e^{2x} e^{-2}}{3} + \frac{7\,e^{-x} e}{3}$$

Verified OK.

2.9.4 Maple step by step solution

Let's solve

$\left[y'' - y' - 2y = 0, y(1) = 3, y'\Big|_{\{x=1\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-x} + c_2 e^{2x}$

□ Check validity of solution $y = c_1 e^{-x} + c_2 e^{2x}$

  - Use initial condition $y(1) = 3$

    $3 = c_1 e^{-1} + c_2 e^{2}$

  - Compute derivative of the solution

    $y' = -c_1 e^{-x} + 2c_2 e^{2x}$

  - Use the initial condition $y'\Big|_{\{x=1\}} = -1$

    $-1 = -c_1 e^{-1} + 2c_2 e^{2}$

  - Solve for c_1 and c_2

    $\left\{c_1 = \frac{7}{3\,e^{-1}}, c_2 = \frac{2}{3\,e^{2}}\right\}$

- Substitute constant values into general solution and simplify

  $$y = \frac{2\,e^{2x-2}}{3} + \frac{7\,e^{1-x}}{3}$$

- Solution to the IVP

  $$y = \frac{2\,e^{2x-2}}{3} + \frac{7\,e^{1-x}}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$2)-diff(y(x),x)-2*y(x)=0,y(1) = 3, D(y)(1) = -1],y(x), singsol=all)
```

$$y(x) = \frac{7\,e^{1-x}}{3} + \frac{2\,e^{2x-2}}{3}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 28

```
DSolve[{y''[x]-y'[x]-2*y[x]==0,{y[1]==3,y'[1]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{7e^{1-x}}{3} + \frac{2}{3}e^{2x-2}$$

2.10 problem 8 b(i)

Internal problem ID [12606]
Internal file name [`OUTPUT/11258_Thursday_October_19_2023_04_43_52_PM_25991731/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 8 b(i).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' - 2y = 0}$$

With initial conditions

$$[y(0) = 1, y(2) = 0]$$

2.10.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-2)} \\ &= \frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{3}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 e^{2x} + c_2 e^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$0 = c_1 e^4 + c_2 e^{-2} \tag{1A}$$

substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{e^6 - 1}$$
$$c_2 = \frac{e^6}{e^6 - 1}$$

Substituting these values back in above solution results in

$$y = \frac{-e^{2x+6} - e^{-x+6} + e^{12-x} + e^{2x}}{e^{12} - 2\,e^6 + 1}$$

Which simplifies to

$$y = \frac{-e^{2x+6} - e^{-x+6} + e^{12-x} + e^{2x}}{e^{12} - 2\,e^6 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-e^{2x+6} - e^{-x+6} + e^{12-x} + e^{2x}}{e^{12} - 2\,e^6 + 1} \tag{1}$$

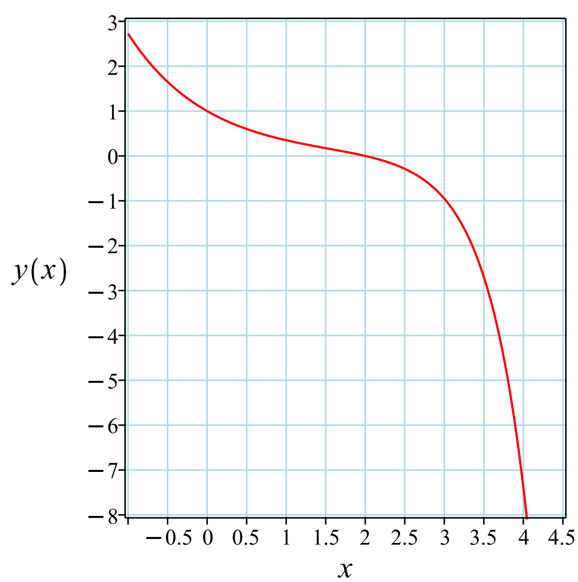


Figure 58: Solution plot

Verification of solutions

$$y = \frac{-\mathrm{e}^{2x+6} - \mathrm{e}^{-x+6} + \mathrm{e}^{12-x} + \mathrm{e}^{2x}}{\mathrm{e}^{12} - 2\,\mathrm{e}^{6} + 1}$$

Verified OK.

2.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 62: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{3x}}{3}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$0 = c_1 e^{-2} + \frac{c_2 e^{4}}{3} \tag{1A}$$

substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + \frac{c_2}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{e^6}{e^6 - 1} \\ c_2 &= -\frac{3}{e^6 - 1} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{-e^{2x+6} - e^{-x+6} + e^{12-x} + e^{2x}}{e^{12} - 2\,e^6 + 1}$$

Which simplifies to

$$y = \frac{-e^{2x+6} - e^{-x+6} + e^{12-x} + e^{2x}}{e^{12} - 2\,e^6 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-e^{2x+6} - e^{-x+6} + e^{12-x} + e^{2x}}{e^{12} - 2\,e^6 + 1} \tag{1}$$

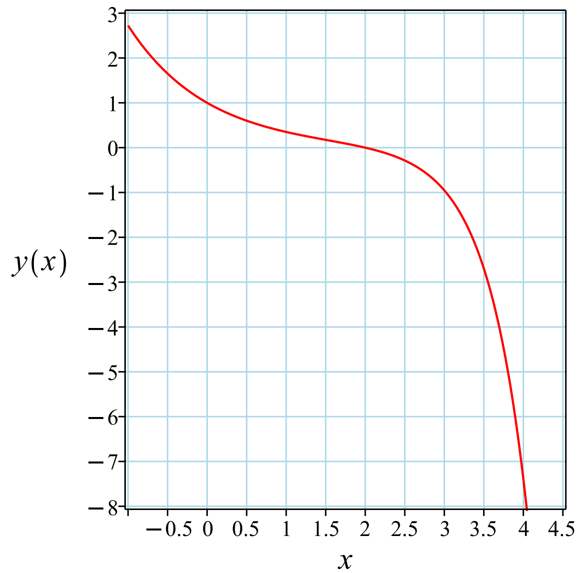


Figure 59: Solution plot

Verification of solutions

$$y = \frac{-e^{2x+6} - e^{-x+6} + e^{12-x} + e^{2x}}{e^{12} - 2e^6 + 1}$$

Verified OK.

2.10.3 Maple step by step solution

Let's solve

$[y'' - y' - 2y = 0, y(0) = 1, y(2) = 0]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the ODE

$$y_2(x) = e^{2x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{-x} + c_2 e^{2x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 24

```
dsolve([diff(y(x),x$2)-diff(y(x),x)-2*y(x)=0,y(0) = 1, y(2) = 0],y(x), singsol=all)
```

$$y(x) = \frac{e^{6-x} - e^{2x}}{e^6 - 1}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 29

```
DSolve[{y''[x]-y'[x]-2*y[x]==0,{y[0]==1,y[2]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-x}(e^6 - e^{3x})}{e^6 - 1}$$

2.11 problem 8 b(ii)

Internal problem ID [12607]
Internal file name [`OUTPUT/11259_Thursday_October_19_2023_04_43_53_PM_93038368/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 8 b(ii).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' - 2y = 0}$$

With initial conditions

$$[y(0) = 0, y'(2) = 1]$$

2.11.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda \,\mathrm{e}^{\lambda x} - 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(-2)}\\ &= \frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{2} + \frac{3}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} - c_2 e^{-x}$$

substituting $y' = 1$ and $x = 2$ in the above gives

$$1 = 2c_1 e^4 - c_2 e^{-2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{e^2}{2\,e^6 + 1}$$
$$c_2 = -\frac{e^2}{2\,e^6 + 1}$$

Substituting these values back in above solution results in

$$y = \frac{-e^{-x+2} - 2\,e^{8-x} + 2\,e^{2x+8} + e^{2x+2}}{4\,e^{12} + 4\,e^6 + 1}$$

Which simplifies to

$$y = \frac{-e^{-x+2} - 2\,e^{8-x} + 2\,e^{2x+8} + e^{2x+2}}{4\,e^{12} + 4\,e^6 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-e^{-x+2} - 2\,e^{8-x} + 2\,e^{2x+8} + e^{2x+2}}{4\,e^{12} + 4\,e^6 + 1} \tag{1}$$

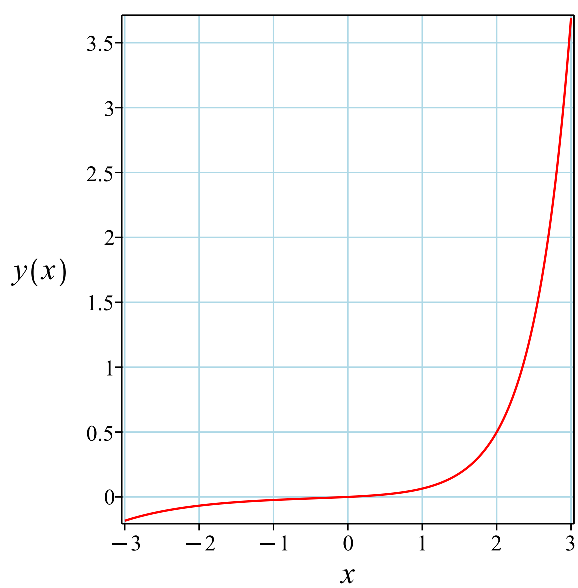


Figure 60: Solution plot

Verification of solutions

$$y = \frac{-\mathrm{e}^{-x+2} - 2\,\mathrm{e}^{8-x} + 2\,\mathrm{e}^{2x+8} + \mathrm{e}^{2x+2}}{4\,\mathrm{e}^{12} + 4\,\mathrm{e}^{6} + 1}$$

Verified OK.

2.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 64: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{3x}}{3}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} + \frac{2c_2 e^{2x}}{3}$$

substituting $y' = 1$ and $x = 2$ in the above gives

$$1 = -c_1 e^{-2} + \frac{2c_2 e^{4}}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{e^2}{2\,e^6 + 1}$$
$$c_2 = \frac{3\,e^2}{2\,e^6 + 1}$$

Substituting these values back in above solution results in

$$y = \frac{-e^{-x+2} - 2\,e^{8-x} + 2\,e^{2x+8} + e^{2x+2}}{4\,e^{12} + 4\,e^{6} + 1}$$

Which simplifies to

$$y = \frac{-e^{-x+2} - 2\,e^{8-x} + 2\,e^{2x+8} + e^{2x+2}}{4\,e^{12} + 4\,e^{6} + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{-e^{-x+2} - 2\,e^{8-x} + 2\,e^{2x+8} + e^{2x+2}}{4\,e^{12} + 4\,e^{6} + 1} \tag{1}$$

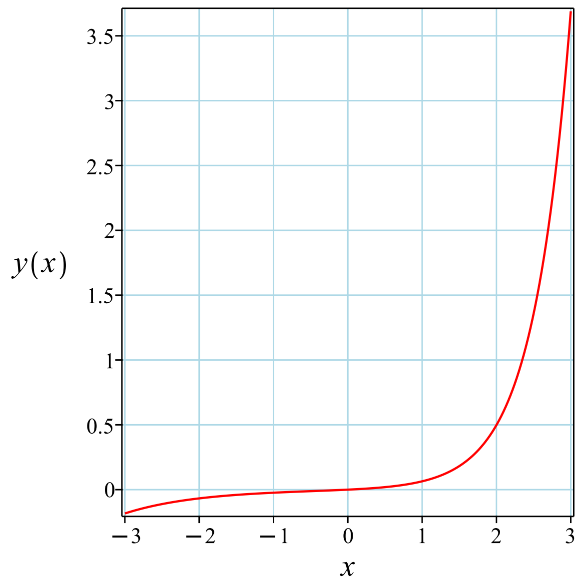


Figure 61: Solution plot

Verification of solutions

$$y = \frac{-e^{-x+2} - 2\,e^{8-x} + 2\,e^{2x+8} + e^{2x+2}}{4\,e^{12} + 4\,e^{6} + 1}$$

Verified OK.

2.11.3 Maple step by step solution

Let's solve

$$\left[y'' - y' - 2y = 0, y(0) = 0, y'\Big|_{\{x=2\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-x} + c_2 e^{2x}$

- [ ] Check validity of solution $y = c_1 e^{-x} + c_2 e^{2x}$
  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -c_1 e^{-x} + 2c_2 e^{2x}$

  - Use the initial condition $y'\Big|_{\{x=2\}} = 1$

    $1 = -c_1 e^{-2} + 2c_2 e^4$

  - Solve for c_1 and c_2

    $\left\{c_1 = -\frac{1}{2\,e^4 + e^{-2}}, c_2 = \frac{1}{2\,e^4 + e^{-2}}\right\}$

  - Substitute constant values into general solution and simplify

    $y = \frac{e^{-x+2}(e^{3x} - 1)}{2\,e^6 + 1}$

- Solution to the IVP

  $y = \frac{e^{-x+2}(e^{3x} - 1)}{2\,e^6 + 1}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 25

```
dsolve([diff(y(x),x$2)-diff(y(x),x)-2*y(x)=0,y(0) = 0, D(y)(2) = 1],y(x), singsol=all)
```

$$y(x) = \frac{e^{2-x}(e^{3x} - 1)}{2\,e^{6} + 1}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 29

```
DSolve[{y''[x]-y'[x]-2*y[x]==0,{y[0]==0,y'[2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{2-x}(e^{3x} - 1)}{1 + 2e^{6}}$$

2.12 problem 9

Internal problem ID [12608]
Internal file name [`OUTPUT/11260_Thursday_October_19_2023_04_43_54_PM_11144062/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 9.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{x^3 y''' - 3x^2 y'' + 6y'x - 6y = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3 y''' - 3x^2 y'' + 6y'x - 6y = 0$$

gives

$$6x\lambda\, x^{\lambda-1} - 3x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} - 6x^\lambda = 0$$

Which simplifies to

$$6\lambda\, x^\lambda - 3\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda - 6x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$6\lambda - 3\lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) - 6 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

This table summarises the result

root	multiplicity	type of root
1	1	real root
2	1	real root
3	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_3 x^3 + c_2 x^2 + c_1 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = x$$
$$y_2 = x^2$$
$$y_3 = x^3$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_3 x^3 + c_2 x^2 + c_1 x \tag{1}$$

Verification of solutions

$$y = c_3x^3 + c_2x^2 + c_1x$$

Verified OK.

2.12.1 Maple step by step solution

Let's solve

$x^3y''' - 3y''x^2 + 6y'x - 6y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{6y}{x^3} + \frac{3(y''x-2y')}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{3y''}{x} + \frac{6y'}{x^2} - \frac{6y}{x^3} = 0$

- Multiply by denominators of the ODE

  $x^3y''' - 3y''x^2 + 6y'x - 6y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  - Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3}y(t)\right)t'(x)^3 + 3t'(x)\,t''(x)\left(\frac{d^2}{dt^2}y(t)\right) + t'''(x)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) - 3\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 + 6\frac{d}{dt}y(t) - 6y(t) = 0$

- Simplify

  $\frac{d^3}{dt^3}y(t) - 6\frac{d^2}{dt^2}y(t) + 11\frac{d}{dt}y(t) - 6y(t) = 0$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = 6y_3(t) - 11y_2(t) + 6y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 6y_3(t) - 11y_2(t) + 6y_1(t)\right]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$

- Rewrite the system as

  $\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{3} = e^{3t} \cdot \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_{1} + c_2 \overrightarrow{y}_{2} + c_3 \overrightarrow{y}_{3}$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 e^{t} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] + c_2 e^{2t} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right] + c_3 e^{3t} \cdot \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y(t) = c_1 e^{t} + \frac{c_2 e^{2t}}{4} + \frac{c_3 e^{3t}}{9}$$

- Change variables back using $t = \ln(x)$

$$y = c_1 x + \frac{1}{4} c_2 x^2 + \frac{1}{9} c_3 x^3$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x^3*diff(y(x),x$3)-3*x^2*diff(y(x),x$2)+6*x*diff(y(x),x)-6*y(x)=0,y(x), singsol=all)
```

$$y(x) = x(c_1x^2 + c_3x + c_2)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 19

```
DSolve[x^3*y'''[x]-3*x^2*y''[x]+6*x*y'[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(x(c_3x + c_2) + c_1)$$

2.13 problem 10 (a)

Internal problem ID [12609]
Internal file name [`OUTPUT/11261_Thursday_October_19_2023_04_43_55_PM_69370016/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 10 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 4y'x + 6y = 0}$$

With initial conditions

$$[y(1) = 0, y(2) = -4]$$

2.13.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned}r_1 &= 2\\ r_2 &= 3\end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2 x^3 + c_1 x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x^3 + c_1 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -4$ and $x = 2$ in the above gives

$$-4 = 8c_2 + 4c_1 \tag{1A}$$

substituting $y = 0$ and $x = 1$ in the above gives

$$0 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = -x^3 + x^2$$

Summary
The solution(s) found are the following

$$y = -x^3 + x^2 \tag{1}$$

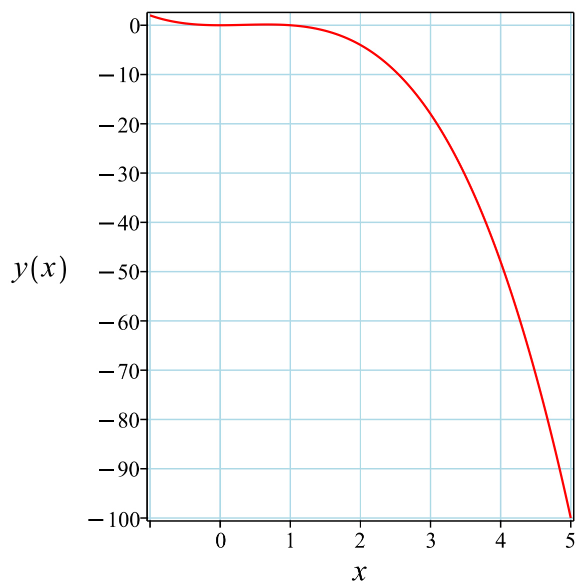


Figure 62: Solution plot

Verification of solutions

$$y = -x^3 + x^2$$

Verified OK.

2.13.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\, dx} \\
&= e^{\int -\frac{4}{x}\, dx} \\
&= \frac{1}{x^2}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\frac{y}{x^2}\right)'' &= 0
\end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x^2}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{x^2}\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\frac{1}{x^2}}$$

Or

$$y = c_1 x^3 + c_2 x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x^3 + c_2 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -4$ and $x = 2$ in the above gives

$$-4 = 8c_1 + 4c_2 \tag{1A}$$

substituting $y = 0$ and $x = 1$ in the above gives

$$0 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = -x^3 + x^2$$

Summary

The solution(s) found are the following

$$y = -x^3 + x^2 \tag{1}$$

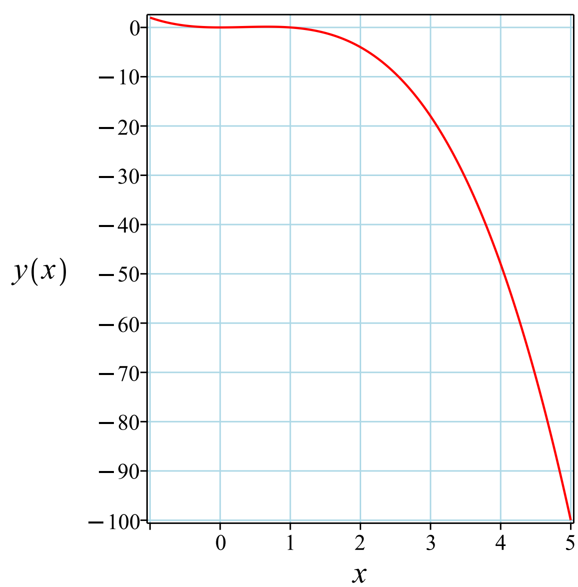


Figure 63: Solution plot

Verification of solutions

$$y = -x^3 + x^2$$

Verified OK.

2.13.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{4}{x}dx\right)}dx \\ &= \int e^{4\ln(x)}\, dx \\ &= \int x^4 dx \\ &= \frac{x^5}{5} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{6}{x^2}}{x^8} \\
&= \frac{6}{x^{10}}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^{10}} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{6}{x^{10}} = \frac{6}{25\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0
$$

Simplifying gives

$$
25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
25r(r-1) + 0 + 6 = 0
$$

Or

$$
25r^2 - 25r + 6 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{2}{5} \\
r_2 &= \frac{3}{5}
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$
y(\tau) = c_1 y_1 + c_2 y_2
$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$
y(\tau) = c_1 \tau^{\frac{2}{5}} + c_2 \tau^{\frac{3}{5}}
$$

The above solution is now transformed back to y using (6) which results in

$$
y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -4$ and $x = 2$ in the above gives

$$
-4 = \frac{4\, 5^{\frac{2}{5}} \left(5^{\frac{1}{5}} c_1 + 2c_2\right)}{5} \tag{1A}
$$

substituting $y = 0$ and $x = 1$ in the above gives

$$
0 = \frac{5^{\frac{2}{5}} \left(5^{\frac{1}{5}} c_1 + c_2\right)}{5} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 5^{\frac{2}{5}} \\
c_2 &= -5^{\frac{3}{5}}
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = -\left(x^5\right)^{\frac{3}{5}} + \left(x^5\right)^{\frac{2}{5}}$$

Summary

The solution(s) found are the following

$$y = -\left(x^5\right)^{\frac{3}{5}} + \left(x^5\right)^{\frac{2}{5}} \tag{1}$$

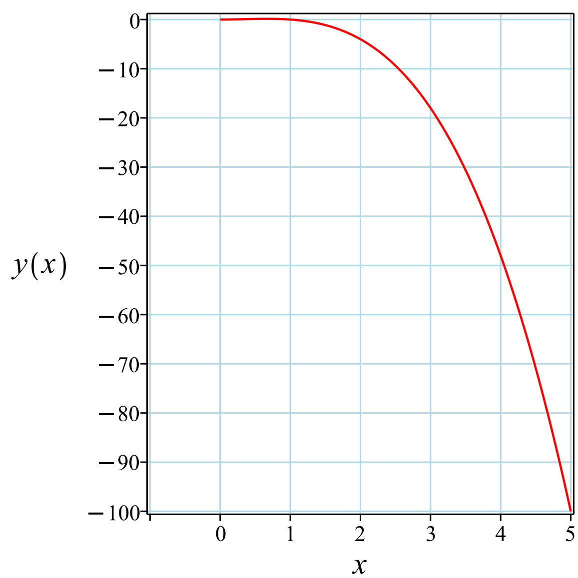


Figure 64: Solution plot

Verification of solutions

$$y = -\left(x^5\right)^{\frac{3}{5}} + \left(x^5\right)^{\frac{2}{5}}$$

Verified OK.

2.13.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{4}{x} \\
q(x) &= \frac{6}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{5c\sqrt{6}}{6}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) = 0
$$

$$
\frac{d^2}{d\tau^2}y(\tau) - \frac{5c\sqrt{6}\left(\frac{d}{d\tau}y(\tau)\right)}{6} + c^2 y(\tau) = 0 \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\,c\tau}{12}} \left(c_1 \cosh\left(\frac{\sqrt{6}\,c\tau}{12}\right) + ic_2 \sinh\left(\frac{\sqrt{6}\,c\tau}{12}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{6}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -4$ and $x = 2$ in the above gives

$$-4 = 2c_2 i + 6c_1 \tag{1A}$$

substituting $y = 0$ and $x = 1$ in the above gives

$$0 = c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}
c_1 &= 0 \\
c_2 &= 2i
\end{aligned}$$

Substituting these values back in above solution results in

$$y = -2x^{\frac{5}{2}} \sinh\left(\frac{\ln(x)}{2}\right)$$

Summary

The solution(s) found are the following

$$y = -2x^{\frac{5}{2}} \sinh\left(\frac{\ln(x)}{2}\right) \tag{1}$$

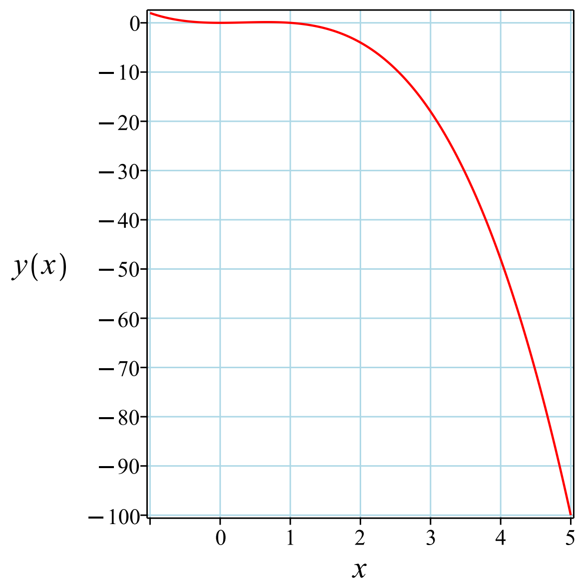


Figure 65: Solution plot

Verification of solutions

$$y = -2x^{\frac{5}{2}} \sinh\left(\frac{\ln(x)}{2}\right)$$

Verified OK.

2.13.5 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{6}{x^2}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\
&= \frac{6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\
&= \frac{6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\
&= 0
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{4}{x}}{2}} \\
&= x^2
\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x^2 \tag{4}$$

Applying this change of variable to the original ode results in

$$x^4 v''(x) = 0$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x + c_2)\, (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = (c_1 x + c_2)\, x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (c_1 x + c_2)\, x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -4$ and $x = 2$ in the above gives

$$-4 = 8c_1 + 4c_2 \tag{1A}$$

substituting $y = 0$ and $x = 1$ in the above gives

$$0 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -1 \\ c_2 &= 1 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -x^2(x - 1)$$

<u>Summary</u>
The solution(s) found are the following

$$y = -x^2(x - 1) \tag{1}$$

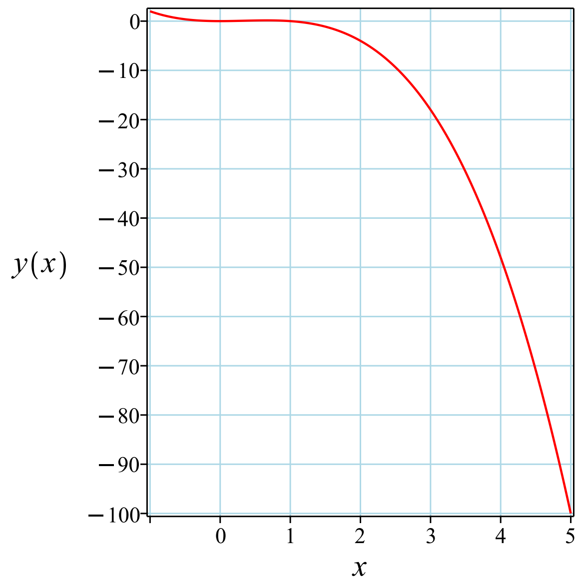


Figure 66: Solution plot

<u>Verification of solutions</u>

$$y = -x^2(x-1)$$

Verified OK.

2.13.6 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\ \ln(u) &= -2\ln(x) + c_1 \\ u &= \mathrm{e}^{-2\ln(x)+c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{x} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(-\frac{c_1}{x} + c_2\right)x^3 \\ &= (c_2 x - c_1)\,x^2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \left(-\frac{c_1}{x} + c_2\right)x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -4$ and $x = 2$ in the above gives

$$-4 = -4c_1 + 8c_2 \tag{1A}$$

substituting $y = 0$ and $x = 1$ in the above gives

$$0 = -c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -1 \\ c_2 &= -1 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -x^2(x-1)$$

Summary
The solution(s) found are the following

$$y = -x^2(x-1) \tag{1}$$

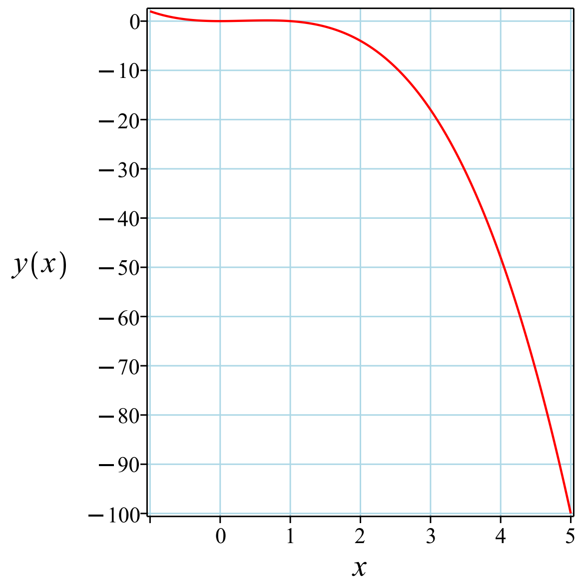


Figure 67: Solution plot

<u>Verification of solutions</u>

$$y = -x^2(x-1)$$

Verified OK.

2.13.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 67: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\
&= z_1 e^{2\ln(x)} \\
&= z_1 \left(x^2\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(x)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2x^3 + c_1x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -4$ and $x = 2$ in the above gives

$$-4 = 8c_2 + 4c_1 \tag{1A}$$

substituting $y = 0$ and $x = 1$ in the above gives

$$0 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= -1
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = -x^3 + x^2$$

Summary

The solution(s) found are the following

$$y = -x^3 + x^2 \tag{1}$$

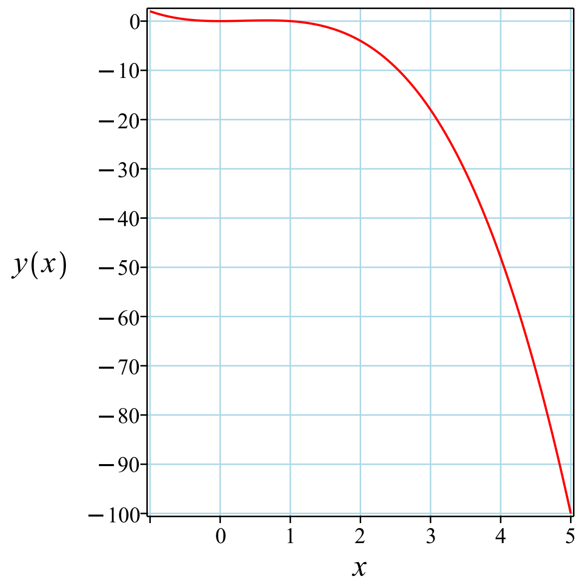


Figure 68: Solution plot

Verification of solutions

$$y = -x^3 + x^2$$

Verified OK.

2.13.8 Maple step by step solution

Let's solve

$[y''x^2 - 4y'x + 6y = 0, y(1) = 0, y(2) = -4]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{x} - \frac{6y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x} + \frac{6y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 - 4y'x + 6y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

- Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^2 - 4\frac{d}{dt}y(t) + 6y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 5\frac{d}{dt}y(t) + 6y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{2t} + c_2 e^{3t}$

- Change variables back using $t = \ln(x)$

$$y = c_2x^3 + c_1x^2$$

- Simplify

$$y = x^2(c_2x + c_1)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve([x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=0,y(1) = 0, y(2) = -4],y(x), singsol=all)
```

$$y(x) = -x^3 + x^2$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 13

```
DSolve[{x^2*y''[x]-4*x*y'[x]+6*y[x]==0,{y[1]==0,y[2]==-4}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to -\left((x-1)x^2\right)$$

2.14 problem 10 (b)

Internal problem ID [12610]
Internal file name [`OUTPUT/11262_Thursday_October_19_2023_04_43_56_PM_83509366/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 10 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
  x)]`]]
```

$$\boxed{x^2 y'' - 4y'x + 6y = 0}$$

With initial conditions

$$[y(2) = 4, y'(1) = 0]$$

2.14.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 3 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2 x^3 + c_1 x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x^3 + c_1 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = 8c_2 + 4c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_2 x^2 + 2c_1 x$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = 3c_2 + 2c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -3$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = 2x^3 - 3x^2$$

Summary

The solution(s) found are the following

$$y = 2x^3 - 3x^2 \tag{1}$$

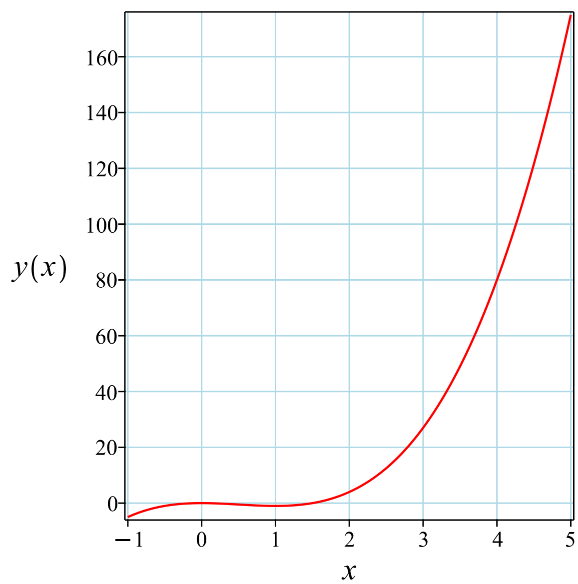


Figure 69: Solution plot

Verification of solutions

$$y = 2x^3 - 3x^2$$

Verified OK.

2.14.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -\frac{4}{x}\, dx} \\ &= \frac{1}{x^2} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\frac{y}{x^2}\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x^2}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{x^2}\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\frac{1}{x^2}}$$

Or

$$y = c_1 x^3 + c_2 x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x^3 + c_2 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = 8c_1 + 4c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1x^2 + 2c_2x$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = 3c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -3$$

Substituting these values back in above solution results in

$$y = 2x^3 - 3x^2$$

Summary

The solution(s) found are the following

$$y = 2x^3 - 3x^2 \tag{1}$$

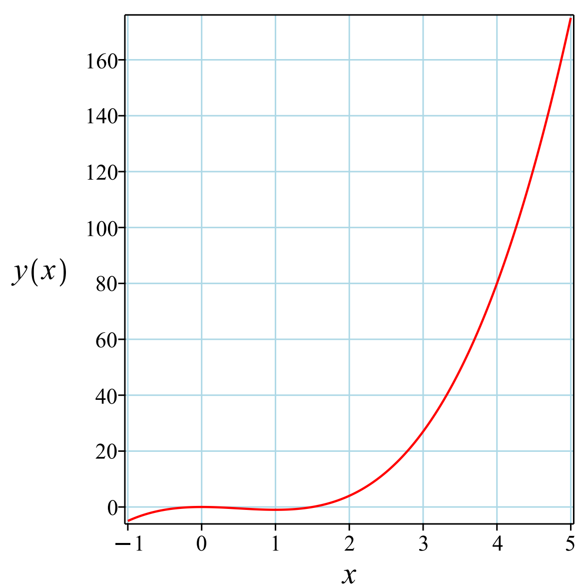


Figure 70: Solution plot

Verification of solutions

$$y = 2x^3 - 3x^2$$

Verified OK.

2.14.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{6}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{4}{x}dx\right)}dx \\ &= \int e^{4\ln(x)}\,dx \\ &= \int x^4 dx \\ &= \frac{x^5}{5} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{6}{x^2}}{x^8} \\ &= \frac{6}{x^{10}} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^{10}} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{6}{x^{10}} = \frac{6}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 6 = 0$$

Or

$$25r^2 - 25r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{2}{5}$$
$$r_2 = \frac{3}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{2}{5}} + c_2 \tau^{\frac{3}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = \frac{4\, 5^{\frac{2}{5}} \left(5^{\frac{1}{5}} c_1 + 2c_2\right)}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2c_1 5^{\frac{3}{5}} x^4}{5 (x^5)^{\frac{3}{5}}} + \frac{3c_2 5^{\frac{2}{5}} x^4}{5 (x^5)^{\frac{2}{5}}}$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = \frac{\left(2\, 5^{\frac{1}{5}} c_1 + 3c_2\right) 5^{\frac{2}{5}}}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -3\,5^{\frac{2}{5}}$$
$$c_2 = 2\,5^{\frac{3}{5}}$$

Substituting these values back in above solution results in

$$y = 2\left(x^5\right)^{\frac{3}{5}} - 3\left(x^5\right)^{\frac{2}{5}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = 2\left(x^5\right)^{\frac{3}{5}} - 3\left(x^5\right)^{\frac{2}{5}} \tag{1}$$

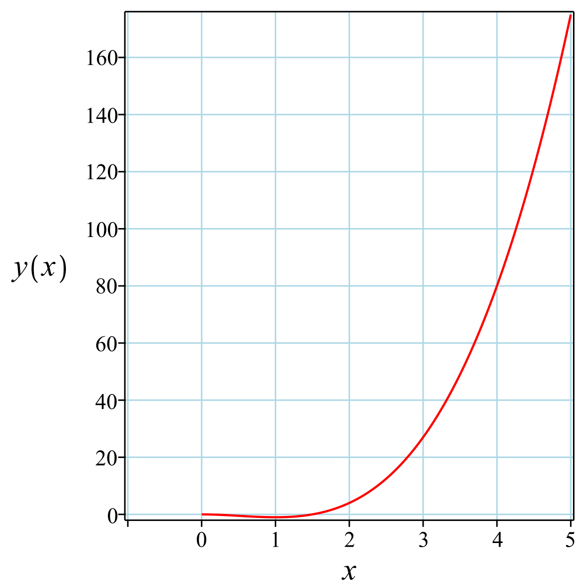


Figure 71: Solution plot

<u>Verification of solutions</u>

$$y = 2\left(x^5\right)^{\frac{3}{5}} - 3\left(x^5\right)^{\frac{2}{5}}$$

Verified OK.

2.14.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{5c\sqrt{6}}{6} \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) - \frac{5c\sqrt{6}\left(\frac{d}{d\tau} y(\tau)\right)}{6} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\,c\tau}{12}} \left(c_1 \cosh\left(\frac{\sqrt{6}\,c\tau}{12}\right) + i c_2 \sinh\left(\frac{\sqrt{6}\,c\tau}{12}\right) \right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{6}\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{6}\sqrt{\frac{1}{x^2}}\,x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right) \right)
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right) \right) \qquad (1)
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$
4 = 2c_2 i + 6c_1 \qquad (1A)
$$

Taking derivative of the solution gives

$$
y' = \frac{5x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)}{2} + x^{\frac{5}{2}} \left(\frac{c_1 \sinh\left(\frac{\ln(x)}{2}\right)}{2x} + \frac{i c_2 \cosh\left(\frac{\ln(x)}{2}\right)}{2x} \right)
$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = \frac{5c_1}{2} + \frac{c_2 i}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = -5i$$

Substituting these values back in above solution results in

$$y = 5x^{\frac{5}{2}} \sinh\left(\frac{\ln(x)}{2}\right) - x^{\frac{5}{2}} \cosh\left(\frac{\ln(x)}{2}\right)$$

Which simplifies to

$$y = -\left(-5\sinh\left(\frac{\ln(x)}{2}\right) + \cosh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{5}{2}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\left(-5\sinh\left(\frac{\ln(x)}{2}\right) + \cosh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{5}{2}} \tag{1}$$

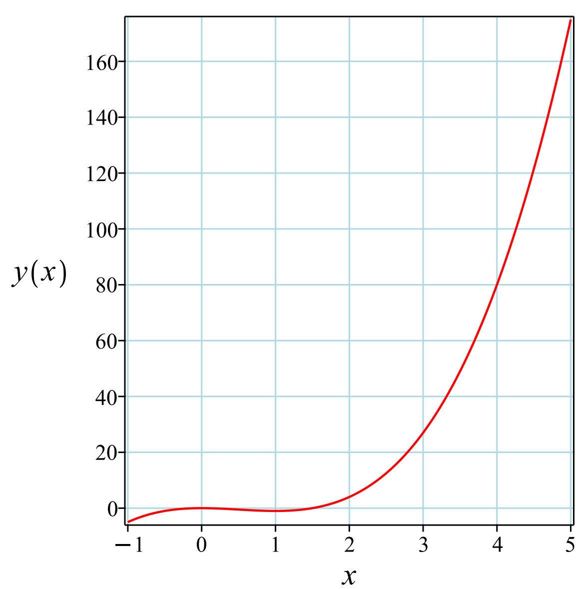


Figure 72: Solution plot

Verification of solutions

$$y = -\left(-5\sinh\left(\frac{\ln(x)}{2}\right) + \cosh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{5}{2}}$$

Verified OK.

2.14.5 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\ &= \frac{6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{-\frac{4}{x}}{2}} \\ &= x^2 \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x^2 \tag{4}$$

Applying this change of variable to the original ode results in

$$x^4 v''(x) = 0$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x + c_2)\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = (c_1 x + c_2)\, x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (c_1 x + c_2)\, x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = 8c_1 + 4c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 x^2 + 2(c_1 x + c_2)\, x$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = 3c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -3$$

Substituting these values back in above solution results in

$$y = x^2(2x - 3)$$

Summary

The solution(s) found are the following

$$y = x^2(2x - 3) \tag{1}$$

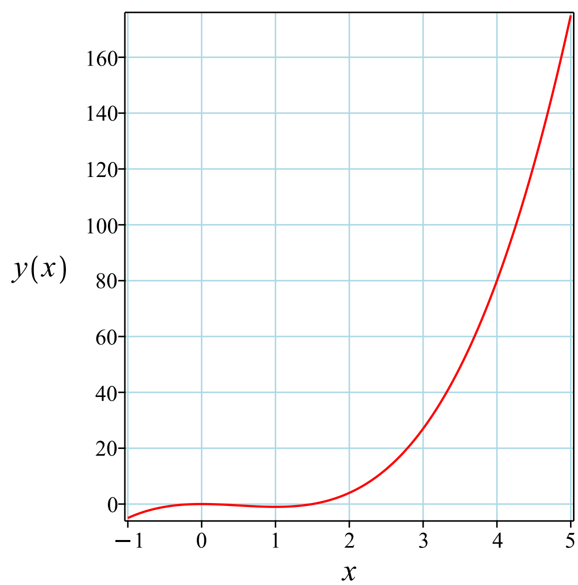


Figure 73: Solution plot

Verification of solutions

$$y = x^2(2x - 3)$$

Verified OK.

2.14.6 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{2}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{2}{x}\, dx \\ \ln(u) &= -2\ln(x) + c_1 \\ u &= \mathrm{e}^{-2\ln(x)+c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1}{x} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(-\frac{c_1}{x} + c_2\right) x^3 \\ &= (c_2 x - c_1)\, x^2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \left(-\frac{c_1}{x} + c_2\right) x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = -4c_1 + 8c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 x + 3\left(-\frac{c_1}{x} + c_2\right) x^2$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = -2c_1 + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = x^2(2x - 3)$$

Summary

The solution(s) found are the following

$$y = x^2(2x - 3) \tag{1}$$

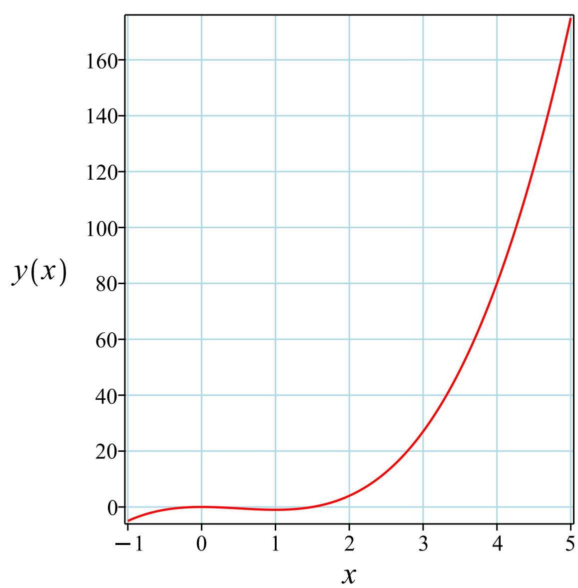


Figure 74: Solution plot

Verification of solutions

$$y = x^2(2x - 3)$$

Verified OK.

2.14.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 69: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\
&= z_1 e^{2\ln(x)} \\
&= z_1\left(x^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(x)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_2 x^3 + c_1 x^2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$
4 = 8c_2 + 4c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = 3c_2x^2 + 2c_1x$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = 3c_2 + 2c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -3$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = 2x^3 - 3x^2$$

Summary

The solution(s) found are the following

$$y = 2x^3 - 3x^2 \tag{1}$$

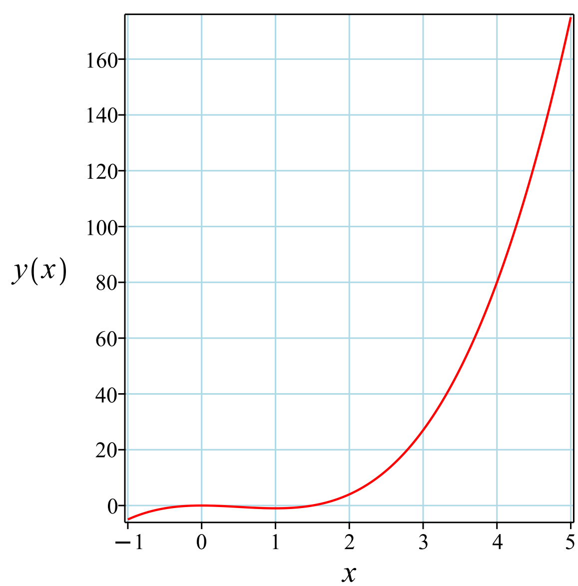


Figure 75: Solution plot

Verification of solutions

$$y = 2x^3 - 3x^2$$

Verified OK.

2.14.8 Maple step by step solution

Let's solve

$$\left[y''x^2 - 4y'x + 6y = 0, y(2) = 4, y'\Big|_{\{x=1\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{4y'}{x} - \frac{6y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{4y'}{x} + \frac{6y}{x^2} = 0$

- Multiply by denominators of the ODE

$y''x^2 - 4y'x + 6y = 0$

- Make a change of variables

$t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - 4\frac{d}{dt}y(t) + 6y(t) = 0$

- Simplify

$\frac{d^2}{dt^2}y(t) - 5\frac{d}{dt}y(t) + 6y(t) = 0$

- Characteristic polynomial of ODE

$r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{2t} + c_2 e^{3t}$

- Change variables back using $t = \ln(x)$

  $y = c_2 x^3 + c_1 x^2$

- Simplify

  $y = x^2(c_2 x + c_1)$

☐ Check validity of solution $y = x^2(c_2 x + c_1)$

  - Use initial condition $y(2) = 4$

    $4 = 8c_2 + 4c_1$

  - Compute derivative of the solution

    $y' = 2x(c_2 x + c_1) + c_2 x^2$

  - Use the initial condition $y'\Big|_{\{x=1\}} = 0$

    $0 = 3c_2 + 2c_1$

  - Solve for c_1 and c_2

    $\{c_1 = -3, c_2 = 2\}$

  - Substitute constant values into general solution and simplify

    $y = 2x^3 - 3x^2$

- Solution to the IVP

$$y = 2x^3 - 3x^2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve([x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=0,y(2) = 4, D(y)(1) = 0],y(x), singsol=al
```

$$y(x) = 2x^3 - 3x^2$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 14

```
DSolve[{x^2*y''[x]-4*x*y'[x]+6*y[x]==0,{y'[1]==0,y[2]==4}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to x^2(2x - 3)$$

2.15 problem 10 (c)

Internal problem ID [12611]
Internal file name [OUTPUT/11263_Thursday_October_19_2023_04_43_58_PM_73155439/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 10 (c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 4y'x + 6y = 0}$$

With initial conditions

$$[y(1) = 1, y'(2) = -12]$$

2.15.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^3 + c_1x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2x^3 + c_1x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_2x^2 + 2c_1x$$

substituting $y' = -12$ and $x = 2$ in the above gives

$$-12 = 12c_2 + 4c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$y = -2x^3 + 3x^2$$

Summary

The solution(s) found are the following

$$y = -2x^3 + 3x^2 \tag{1}$$

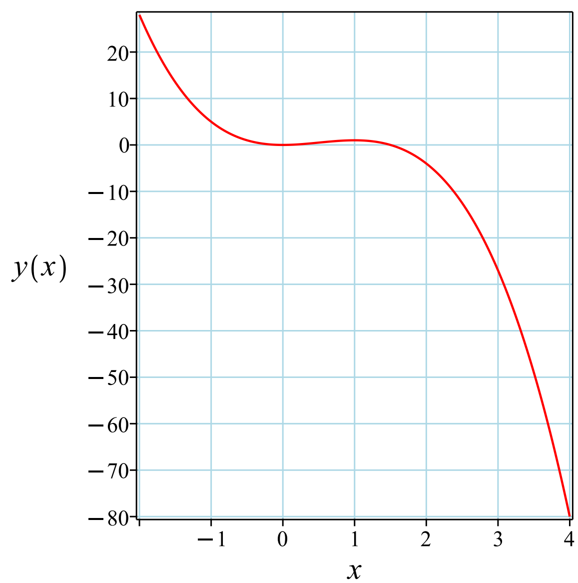


Figure 76: Solution plot

Verification of solutions

$$y = -2x^3 + 3x^2$$

Verified OK.

2.15.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -\frac{4}{x}\,dx} \\
&= \frac{1}{x^2}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\frac{y}{x^2}\right)'' &= 0
\end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x^2}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{x^2}\right) = c_1x + c_2$$

Hence the solution is

$$y = \frac{c_1x + c_2}{\frac{1}{x^2}}$$

Or

$$y = c_1x^3 + c_2x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1x^3 + c_2x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1x^2 + 2c_2x$$

substituting $y' = -12$ and $x = 2$ in the above gives

$$-12 = 12c_1 + 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = -2x^3 + 3x^2$$

Summary

The solution(s) found are the following

$$y = -2x^3 + 3x^2 \tag{1}$$

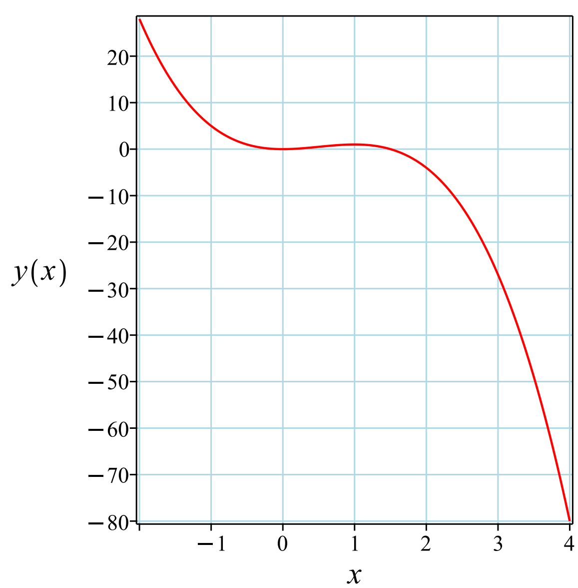


Figure 77: Solution plot

Verification of solutions

$$y = -2x^3 + 3x^2$$

Verified OK.

2.15.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x) dx\right)} dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{4}{x} dx\right)} dx \\ &= \int e^{4\ln(x)}\, dx \\ &= \int x^4 dx \\ &= \frac{x^5}{5} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{6}{x^2}}{x^8} \\ &= \frac{6}{x^{10}} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^{10}} = 0$$

But in terms of τ

$$\frac{6}{x^{10}} = \frac{6}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 6 = 0$$

Or

$$25r^2 - 25r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{2}{5} \\
r_2 &= \frac{3}{5}
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$
y(\tau) = c_1 y_1 + c_2 y_2
$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$
y(\tau) = c_1 \tau^{\frac{2}{5}} + c_2 \tau^{\frac{3}{5}}
$$

The above solution is now transformed back to y using (6) which results in

$$
y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$
1 = \frac{5^{\frac{2}{5}} \left(5^{\frac{1}{5}} c_1 + c_2\right)}{5} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{2 c_1 5^{\frac{3}{5}} x^4}{5 (x^5)^{\frac{3}{5}}} + \frac{3 c_2 5^{\frac{2}{5}} x^4}{5 (x^5)^{\frac{2}{5}}}
$$

substituting $y' = -12$ and $x = 2$ in the above gives

$$
-12 = \frac{4\left(5^{\frac{1}{5}} c_1 + 3 c_2\right) 5^{\frac{2}{5}}}{5} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3\,5^{\frac{2}{5}}$$
$$c_2 = -2\,5^{\frac{3}{5}}$$

Substituting these values back in above solution results in

$$y = -2(x^5)^{\frac{3}{5}} + 3(x^5)^{\frac{2}{5}}$$

Summary

The solution(s) found are the following

$$y = -2(x^5)^{\frac{3}{5}} + 3(x^5)^{\frac{2}{5}} \tag{1}$$

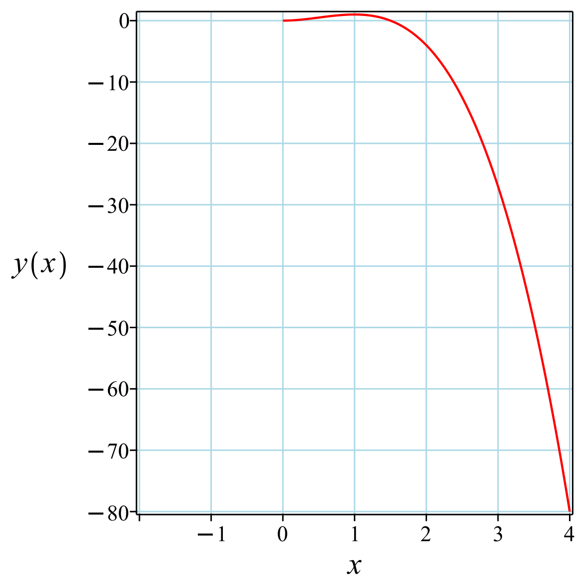


Figure 78: Solution plot

Verification of solutions

$$y = -2(x^5)^{\frac{3}{5}} + 3(x^5)^{\frac{2}{5}}$$

Verified OK.

2.15.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{5c\sqrt{6}}{6} \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) - \frac{5c\sqrt{6}\left(\frac{d}{d\tau} y(\tau)\right)}{6} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\, c\tau}{12}} \left(c_1 \cosh\left(\frac{\sqrt{6}\, c\tau}{12}\right) + i c_2 \sinh\left(\frac{\sqrt{6}\, c\tau}{12}\right) \right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dx \\
&= \frac{\int \sqrt{6}\, \sqrt{\frac{1}{x^2}} dx}{c} \\
&= \frac{\sqrt{6}\, \sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right) \right)
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right) \right) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{5x^{\frac{3}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right) \right)}{2} + x^{\frac{5}{2}} \left(\frac{c_1 \sinh\left(\frac{\ln(x)}{2}\right)}{2x} + \frac{i c_2 \cosh\left(\frac{\ln(x)}{2}\right)}{2x} \right)
$$

substituting $y' = -12$ and $x = 2$ in the above gives

$$-12 = 4c_2 i + 8c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 5i$$

Substituting these values back in above solution results in

$$y = -5x^{\frac{5}{2}} \sinh\left(\frac{\ln(x)}{2}\right) + x^{\frac{5}{2}} \cosh\left(\frac{\ln(x)}{2}\right)$$

Which simplifies to

$$y = \left(-5\sinh\left(\frac{\ln(x)}{2}\right) + \cosh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{5}{2}}$$

Summary
The solution(s) found are the following

$$y = \left(-5\sinh\left(\frac{\ln(x)}{2}\right) + \cosh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{5}{2}} \tag{1}$$

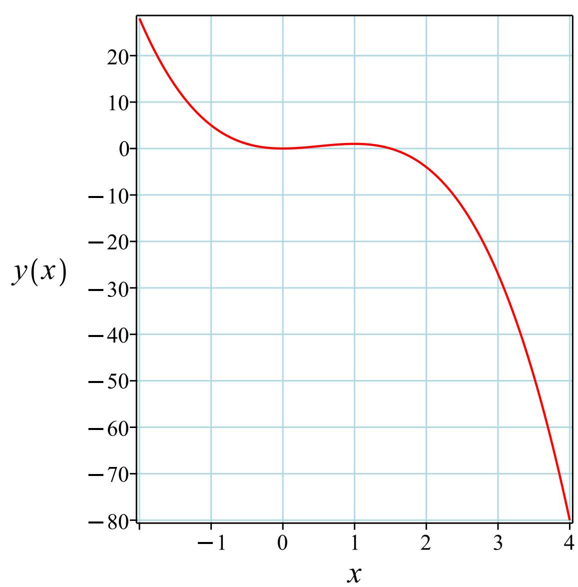


Figure 79: Solution plot

Verification of solutions

$$y = \left(-5 \sinh\left(\frac{\ln(x)}{2}\right) + \cosh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{5}{2}}$$

Verified OK.

2.15.5 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\ &= \frac{6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{-\frac{4}{x}}{2}} \\ &= x^2 \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x^2 \tag{4}$$

Applying this change of variable to the original ode results in

$$x^4 v''(x) = 0$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x + c_2)\, (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = (c_1 x + c_2)\, x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (c_1 x + c_2)\, x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 x^2 + 2(c_1 x + c_2)\, x$$

substituting $y' = -12$ and $x = 2$ in the above gives

$$-12 = 12c_1 + 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = -x^2(2x - 3)$$

Summary

The solution(s) found are the following

$$y = -x^2(2x - 3) \tag{1}$$

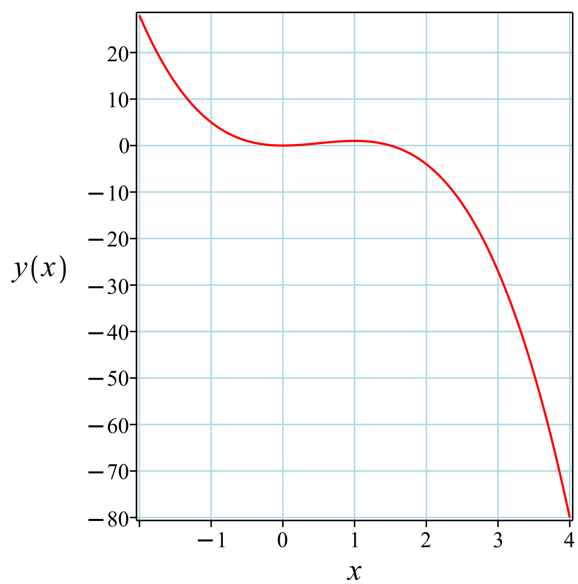


Figure 80: Solution plot

Verification of solutions

$$y = -x^2(2x - 3)$$

Verified OK.

2.15.6 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{2u}{x}
\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\
\ln(u) &= -2\ln(x) + c_1 \\
u &= e^{-2\ln(x)+c_1} \\
&= \frac{c_1}{x^2}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{x} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{x} + c_2\right)x^3 \\
&= (c_2 x - c_1)\,x^2
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \left(-\frac{c_1}{x} + c_2\right)x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = -c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 x + 3\left(-\frac{c_1}{x} + c_2\right) x^2$$

substituting $y' = -12$ and $x = 2$ in the above gives

$$-12 = -4c_1 + 12c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -3$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$y = -x^2(2x - 3)$$

Summary
The solution(s) found are the following

$$y = -x^2(2x - 3) \tag{1}$$

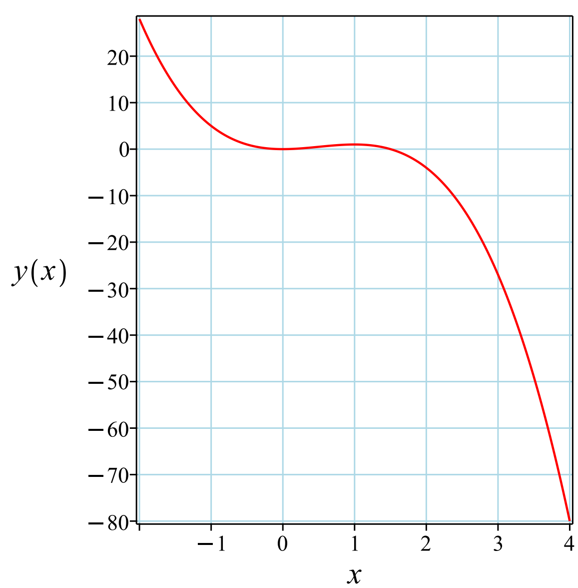


Figure 81: Solution plot

Verification of solutions

$$y = -x^2(2x - 3)$$

Verified OK.

2.15.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 71: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\
&= z_1 e^{2\ln(x)} \\
&= z_1 \left(x^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(x)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_2 x^3 + c_1 x^2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$
1 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = 3c_2x^2 + 2c_1x$$

substituting $y' = -12$ and $x = 2$ in the above gives

$$-12 = 12c_2 + 4c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$y = -2x^3 + 3x^2$$

Summary

The solution(s) found are the following

$$y = -2x^3 + 3x^2 \tag{1}$$

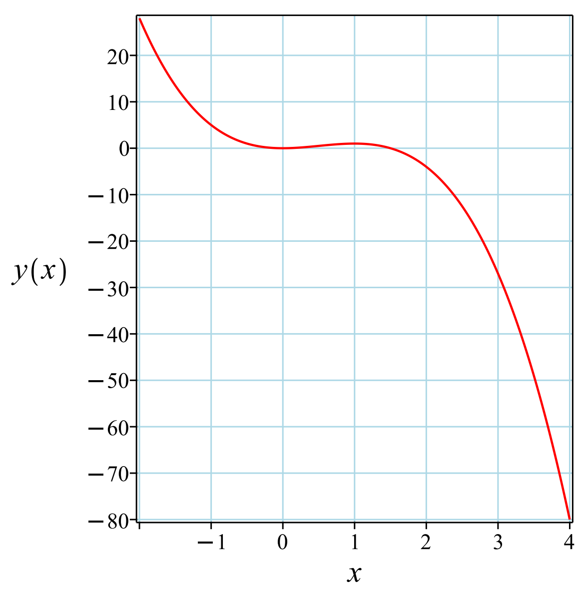


Figure 82: Solution plot

Verification of solutions

$$y = -2x^3 + 3x^2$$

Verified OK.

2.15.8 Maple step by step solution

Let's solve

$$\left[y''x^2 - 4y'x + 6y = 0, y(1) = 1, y'\Big|_{\{x=2\}} = -12\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{x} - \frac{6y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x} + \frac{6y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 - 4y'x + 6y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - 4\frac{d}{dt}y(t) + 6y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 5\frac{d}{dt}y(t) + 6y(t) = 0$

- Characteristic polynomial of ODE

$r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{2t} + c_2 e^{3t}$

- Change variables back using $t = \ln(x)$

  $y = c_2 x^3 + c_1 x^2$

- Simplify

  $y = x^2(c_2 x + c_1)$

☐ Check validity of solution $y = x^2(c_2 x + c_1)$

  - Use initial condition $y(1) = 1$

    $1 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = 2x(c_2 x + c_1) + c_2 x^2$

  - Use the initial condition $y'\Big|_{\{x=2\}} = -12$

    $-12 = 12c_2 + 4c_1$

  - Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = -2\}$

  - Substitute constant values into general solution and simplify

    $y = -2x^3 + 3x^2$

- Solution to the IVP

$$y = -2x^3 + 3x^2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=0,y(1) = 1, D(y)(2) = -12],y(x), singsol=
```

$$y(x) = -2x^3 + 3x^2$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 14

```
DSolve[{x^2*y''[x]-4*x*y'[x]+6*y[x]==0,{y[1]==1,y'[2]==-12}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to (3 - 2x)x^2$$

2.16 problem 10 (d)

Internal problem ID [12612]
Internal file name [`OUTPUT/11264_Thursday_October_19_2023_04_43_59_PM_1173136/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 10 (d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 4y'x + 6y = 0}$$

With initial conditions

$$[y'(1) = 3, y'(2) = 0]$$

2.16.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 3 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2 x^3 + c_1 x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x^3 + c_1 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = 3c_2 x^2 + 2c_1 x$$

substituting $y' = 0$ and $x = 2$ in the above gives

$$0 = 12c_2 + 4c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_2x^2 + 2c_1x$$

substituting $y' = 3$ and $x = 1$ in the above gives

$$3 = 3c_2 + 2c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = -x^3 + 3x^2$$

Summary
The solution(s) found are the following

$$y = -x^3 + 3x^2 \tag{1}$$

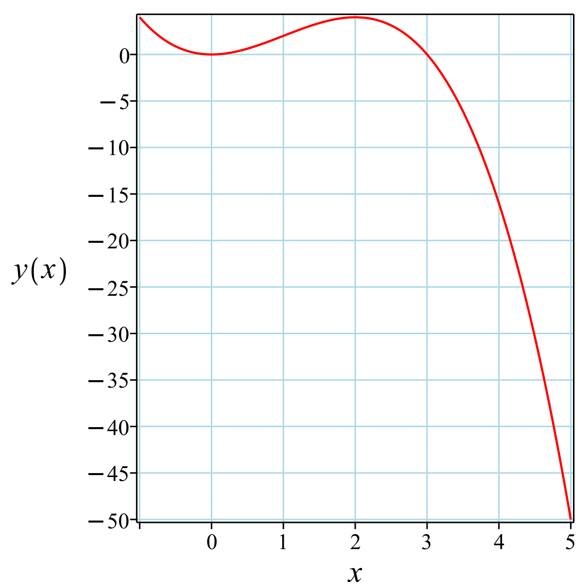


Figure 83: Solution plot

Verification of solutions

$$y = -x^3 + 3x^2$$

Verified OK.

2.16.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^{2}+p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -\frac{4}{x}\,dx} \\
&= \frac{1}{x^{2}}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\frac{y}{x^{2}}\right)'' &= 0
\end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x^{2}}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{x^{2}}\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\frac{1}{x^{2}}}$$

Or

$$y = c_1 x^{3} + c_2 x^{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x^{3} + c_2 x^{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = 3c_1x^2 + 2c_2x$$

substituting $y' = 0$ and $x = 2$ in the above gives

$$0 = 12c_1 + 4c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1x^2 + 2c_2x$$

substituting $y' = 3$ and $x = 1$ in the above gives

$$3 = 3c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = -x^3 + 3x^2$$

Summary

The solution(s) found are the following

$$y = -x^3 + 3x^2 \tag{1}$$

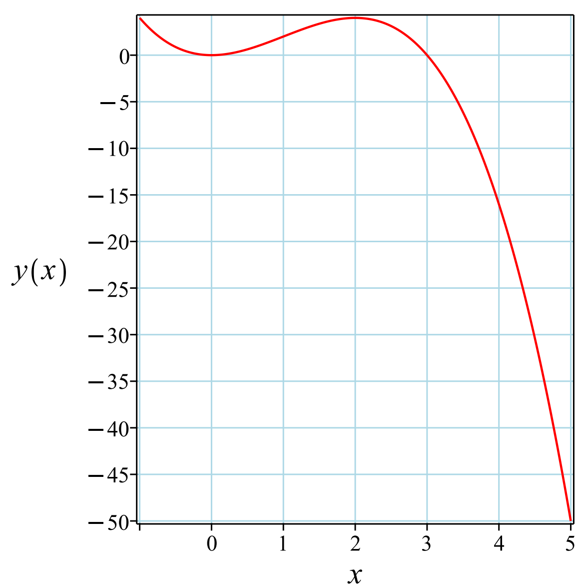


Figure 84: Solution plot

Verification of solutions

$$y = -x^3 + 3x^2$$

Verified OK.

2.16.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{6}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{4}{x}dx\right)} dx \\
&= \int e^{4\ln(x)}\, dx \\
&= \int x^4 dx \\
&= \frac{x^5}{5}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{6}{x^2}}{x^8} \\
&= \frac{6}{x^{10}}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^{10}} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{6}{x^{10}} = \frac{6}{25\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0
$$

Simplifying gives

$$
25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
25r(r-1) + 0 + 6 = 0
$$

Or

$$
25r^2 - 25r + 6 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{2}{5}$$
$$r_2 = \frac{3}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{2}{5}} + c_2 \tau^{\frac{3}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = \frac{2 c_1 5^{\frac{3}{5}} x^4}{5 (x^5)^{\frac{3}{5}}} + \frac{3 c_2 5^{\frac{2}{5}} x^4}{5 (x^5)^{\frac{2}{5}}}$$

substituting $y' = 0$ and $x = 2$ in the above gives

$$0 = \frac{4 \left(5^{\frac{1}{5}} c_1 + 3 c_2\right) 5^{\frac{2}{5}}}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2 c_1 5^{\frac{3}{5}} x^4}{5 (x^5)^{\frac{3}{5}}} + \frac{3 c_2 5^{\frac{2}{5}} x^4}{5 (x^5)^{\frac{2}{5}}}$$

substituting $y' = 3$ and $x = 1$ in the above gives

$$3 = \frac{\left(2\,5^{\frac{1}{5}} c_1 + 3c_2\right) 5^{\frac{2}{5}}}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3\,5^{\frac{2}{5}}$$
$$c_2 = -5^{\frac{3}{5}}$$

Substituting these values back in above solution results in

$$y = -\left(x^5\right)^{\frac{3}{5}} + 3\left(x^5\right)^{\frac{2}{5}}$$

Summary
The solution(s) found are the following

$$y = -\left(x^5\right)^{\frac{3}{5}} + 3\left(x^5\right)^{\frac{2}{5}} \tag{1}$$

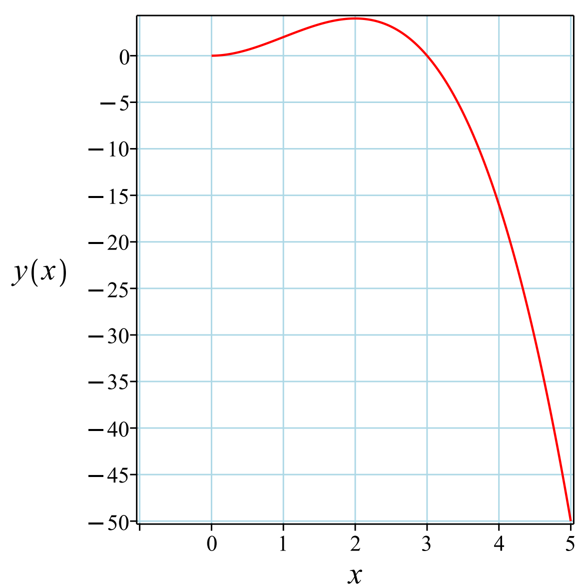


Figure 85: Solution plot

Verification of solutions

$$y = -\left(x^5\right)^{\frac{3}{5}} + 3\left(x^5\right)^{\frac{2}{5}}$$

Verified OK.

2.16.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\, x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{5c\sqrt{6}}{6} \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) - \frac{5c\sqrt{6}\left(\frac{d}{d\tau} y(\tau)\right)}{6} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\,c\tau}{12}} \left(c_1 \cosh\left(\frac{\sqrt{6}\,c\tau}{12}\right) + i c_2 \sinh\left(\frac{\sqrt{6}\,c\tau}{12}\right) \right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dx \\
&= \frac{\int \sqrt{6}\sqrt{\frac{1}{x^2}} dx}{c} \\
&= \frac{\sqrt{6}\sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right) \right)
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right) \right) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$
y' = \frac{5x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)}{2} + x^{\frac{5}{2}} \left(\frac{c_1 \sinh\left(\frac{\ln(x)}{2}\right)}{2x} + \frac{i c_2 \cosh\left(\frac{\ln(x)}{2}\right)}{2x} \right)
$$

substituting $y' = 0$ and $x = 2$ in the above gives

$$
0 = 4c_2 i + 8c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = \frac{5x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)}{2} + x^{\frac{5}{2}}\left(\frac{c_1 \sinh\left(\frac{\ln(x)}{2}\right)}{2x} + \frac{ic_2 \cosh\left(\frac{\ln(x)}{2}\right)}{2x}\right)$$

substituting $y' = 3$ and $x = 1$ in the above gives

$$3 = \frac{5c_1}{2} + \frac{c_2 i}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 4i$$

Substituting these values back in above solution results in

$$y = 2x^{\frac{5}{2}} \cosh\left(\frac{\ln(x)}{2}\right) - 4x^{\frac{5}{2}} \sinh\left(\frac{\ln(x)}{2}\right)$$

Which simplifies to

$$y = 2\left(-2\sinh\left(\frac{\ln(x)}{2}\right) + \cosh\left(\frac{\ln(x)}{2}\right)\right)x^{\frac{5}{2}}$$

Summary

The solution(s) found are the following

$$y = 2\left(-2\sinh\left(\frac{\ln(x)}{2}\right) + \cosh\left(\frac{\ln(x)}{2}\right)\right)x^{\frac{5}{2}} \tag{1}$$

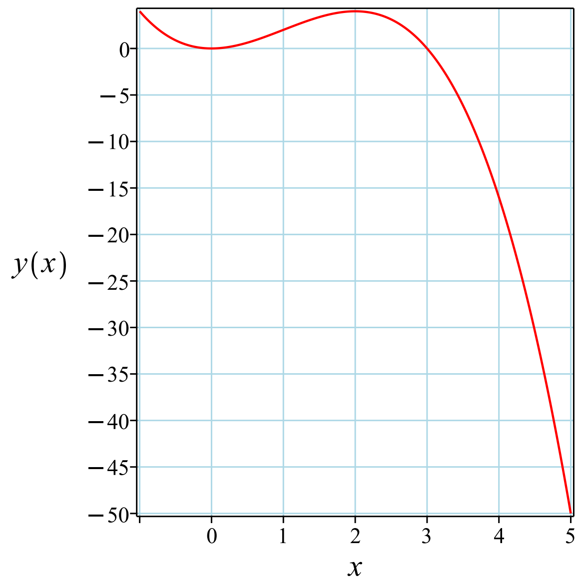


Figure 86: Solution plot

Verification of solutions

$$y = 2\left(-2\sinh\left(\frac{\ln(x)}{2}\right) + \cosh\left(\frac{\ln(x)}{2}\right)\right)x^{\frac{5}{2}}$$

Verified OK.

2.16.5 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{6}{x^2}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\
&= \frac{6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\
&= \frac{6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{4}{x}}{2}} \\
&= x^2
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, x^2 \tag{4}
$$

Applying this change of variable to the original ode results in

$$
x^4 v''(x) = 0
$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$
v(x) = c_1 x + c_2
$$

Now that $v(x)$ is known, then

$$
\begin{aligned}
y &= v(x)\, z(x) \\
&= (c_1 x + c_2)\, (z(x))
\end{aligned} \tag{7}
$$

But from (5)

$$
z(x) = x^2
$$

Hence (7) becomes

$$y = (c_1 x + c_2) x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (c_1 x + c_2) x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = c_1 x^2 + 2(c_1 x + c_2) x$$

substituting $y' = 0$ and $x = 2$ in the above gives

$$0 = 12c_1 + 4c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 x^2 + 2(c_1 x + c_2) x$$

substituting $y' = 3$ and $x = 1$ in the above gives

$$3 = 3c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -1 \\ c_2 &= 3 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -x^2(x - 3)$$

Summary

The solution(s) found are the following

$$y = -x^2(x - 3) \tag{1}$$

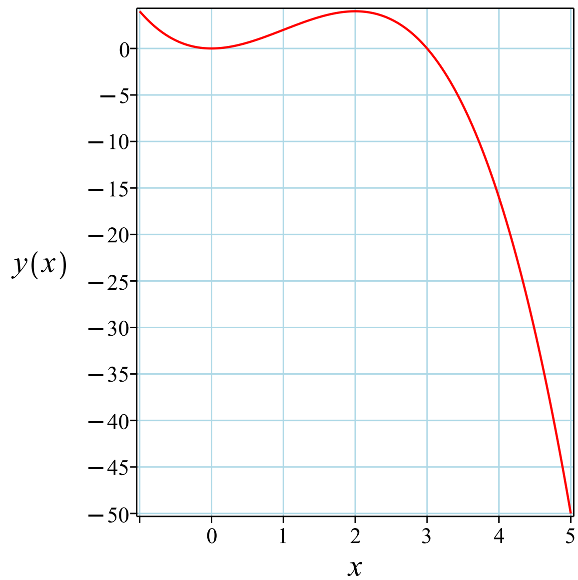


Figure 87: Solution plot

Verification of solutions

$$y = -x^2(x-3)$$

Verified OK.

2.16.6 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= -\frac{4}{x}\\ q(x) &= \frac{6}{x^2}\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \frac{2v'(x)}{x} = 0$$
$$v''(x) + \frac{2v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\ \ln(u) &= -2\ln(x) + c_1 \\ u &= \mathrm{e}^{-2\ln(x)+c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{x} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{x} + c_2\right) x^3 \\
&= (c_2 x - c_1)\, x^2
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \left(-\frac{c_1}{x} + c_2\right) x^3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$
y' = c_1 x + 3\left(-\frac{c_1}{x} + c_2\right) x^2
$$

substituting $y' = 0$ and $x = 2$ in the above gives

$$
0 = -4c_1 + 12c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 x + 3\left(-\frac{c_1}{x} + c_2\right) x^2
$$

substituting $y' = 3$ and $x = 1$ in the above gives

$$
3 = -2c_1 + 3c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -3 \\
c_2 &= -1
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = -x^2(x - 3)$$

Summary
The solution(s) found are the following

$$y = -x^2(x - 3) \tag{1}$$

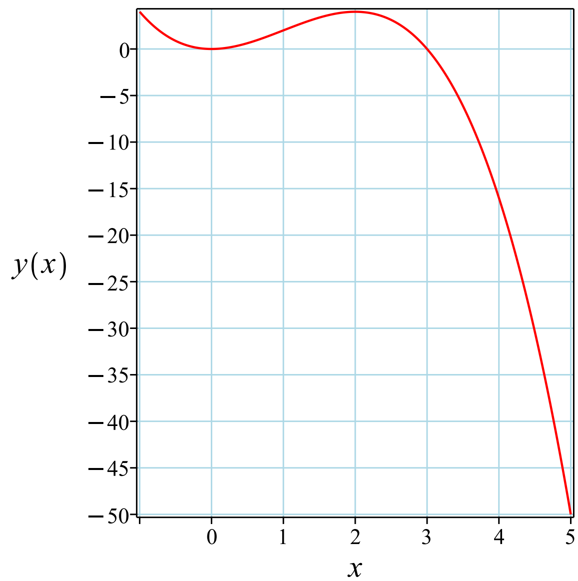


Figure 88: Solution plot

Verification of solutions

$$y = -x^2(x - 3)$$

Verified OK.

2.16.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 73: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x}{x^2} \, dx} \\
&= z_1 e^{2\ln(x)} \\
&= z_1 \left(x^2\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^2\right) + c_2 \left(x^2 (x)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x^3 + c_1 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. Taking derivative of the solution gives

$$y' = 3c_2 x^2 + 2c_1 x$$

substituting $y' = 0$ and $x = 2$ in the above gives

$$0 = 12c_2 + 4c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_2 x^2 + 2c_1 x$$

substituting $y' = 3$ and $x = 1$ in the above gives

$$3 = 3c_2 + 2c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = -x^3 + 3x^2$$

Summary
The solution(s) found are the following

$$y = -x^3 + 3x^2 \tag{1}$$

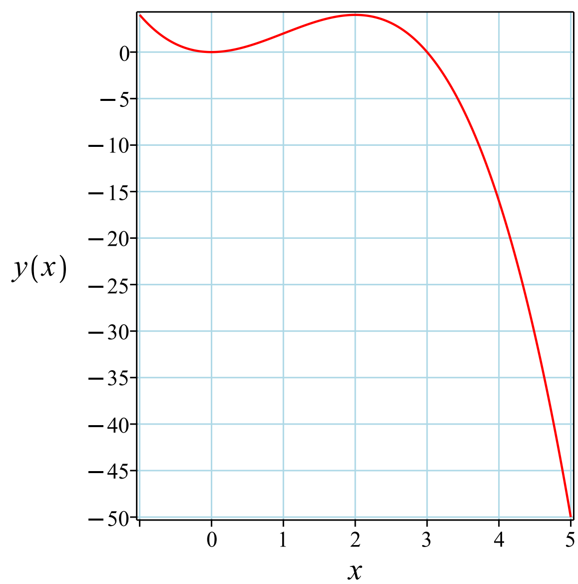


Figure 89: Solution plot

Verification of solutions

$$y = -x^3 + 3x^2$$

Verified OK.

2.16.8 Maple step by step solution

Let's solve

$$\left[y''x^2-4y'x+6y=0, y'\Big|_{\{x=1\}}=3, y'\Big|_{\{x=2\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{4y'}{x}-\frac{6y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{4y'}{x}+\frac{6y}{x^2}=0$

- Multiply by denominators of the ODE

  $y''x^2-4y'x+6y=0$

- Make a change of variables

  $t=\ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y'=\left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y'=\frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y''=\left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2+t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y''=\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)x^2-4\frac{d}{dt}y(t)+6y(t)=0$

- Simplify

  $\frac{d^2}{dt^2}y(t)-5\frac{d}{dt}y(t)+6y(t)=0$

- Characteristic polynomial of ODE

$r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{2t} + c_2 e^{3t}$

- Change variables back using $t = \ln(x)$

  $y = c_2 x^3 + c_1 x^2$

- Simplify

  $y = x^2(c_2 x + c_1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=0,D(y)(1) = 3, D(y)(2) = 0],y(x), singsol
```

$$y(x) = -x^3 + 3x^2$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 13

```
DSolve[{x^2*y''[x]-4*x*y'[x]+6*y[x]==0,{y'[1]==3,y'[2]==0}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to -\left((x-3)x^2\right)$$

2.17 problem 10 (e)

Internal problem ID [12613]
Internal file name [`OUTPUT/11265_Thursday_October_19_2023_04_44_01_PM_78049625/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 10 (e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 4y'x + 6y = 0}$$

With initial conditions

$$[y(0) = 0, y(2) = 4]$$

2.17.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2 x^3 + c_1 x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x^3 + c_1 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = 8c_2 + 4c_1 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = 0 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1 - 2c_2$$

Substituting these values back in above solution results in

$$y = c_2x^3 - 2c_2x^2 + x^2$$

Which simplifies to

$$y = x^2(1 + (x - 2) c_2)$$

Summary
The solution(s) found are the following

$$y = x^2(1 + (x - 2) c_2) \tag{1}$$

Verification of solutions

$$y = x^2(1 + (x - 2) c_2)$$

Verified OK.

2.17.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x) y' + \frac{(p(x)^2 + p'(x)) y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -\frac{4}{x}\,dx} \\ &= \frac{1}{x^2} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\frac{y}{x^2}\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x^2}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{x^2}\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\frac{1}{x^2}}$$

Or

$$y = c_1 x^3 + c_2 x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x^3 + c_2 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = 8c_1 + 4c_2 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = 0 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_2 = 1 - 2c_1$$

Substituting these values back in above solution results in

$$y = c_1 x^3 - 2c_1 x^2 + x^2$$

Which simplifies to

$$y = x^2 (1 + (x - 2) c_1)$$

Summary
The solution(s) found are the following

$$y = x^2 (1 + (x - 2) c_1) \tag{1}$$

Verification of solutions

$$y = x^2 (1 + (x - 2) c_1)$$

Verified OK.

2.17.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{4}{x}dx\right)}dx \\ &= \int e^{4\ln(x)}\,dx \\ &= \int x^4dx \\ &= \frac{x^5}{5} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{6}{x^2}}{x^8} \\
&= \frac{6}{x^{10}}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^{10}} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{6}{x^{10}} = \frac{6}{25\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0
$$

Simplifying gives

$$
25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
25r(r-1) + 0 + 6 = 0
$$

Or

$$
25r^2 - 25r + 6 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{2}{5}$$
$$r_2 = \frac{3}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{2}{5}} + c_2 \tau^{\frac{3}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = \frac{4\, 5^{\frac{2}{5}} \left(5^{\frac{1}{5}} c_1 + 2c_2\right)}{5} \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = 0 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{2\, 5^{\frac{4}{5}} c_2}{5} + 5^{\frac{2}{5}}$$

Substituting these values back in above solution results in

$$y = \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} - \frac{2\, 5^{\frac{2}{5}} (x^5)^{\frac{2}{5}} c_2}{5} + (x^5)^{\frac{2}{5}}$$

Which simplifies to

$$y = \frac{(x^5)^{\frac{2}{5}} \left(5 + \left((x^5)^{\frac{1}{5}} - 2\right) c_2 5^{\frac{2}{5}}\right)}{5}$$

Summary
The solution(s) found are the following

$$y = \frac{(x^5)^{\frac{2}{5}} \left(5 + \left((x^5)^{\frac{1}{5}} - 2\right) c_2 5^{\frac{2}{5}}\right)}{5} \tag{1}$$

Verification of solutions

$$y = \frac{(x^5)^{\frac{2}{5}} \left(5 + \left((x^5)^{\frac{1}{5}} - 2\right) c_2 5^{\frac{2}{5}}\right)}{5}$$

Verified OK.

2.17.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{6}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{5c\sqrt{6}}{6}
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{5c\sqrt{6}\,\left(\frac{d}{d\tau}y(\tau)\right)}{6} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\,c\tau}{12}}\left(c_1\cosh\left(\frac{\sqrt{6}\,c\tau}{12}\right) + ic_2\sinh\left(\frac{\sqrt{6}\,c\tau}{12}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{6}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right) \right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right) \right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = 2c_2 i + 6c_1 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = 0 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{2}{3} - \frac{c_2 i}{3}$$

Substituting these values back in above solution results in

$$y = -\frac{i x^{\frac{5}{2}} \cosh\left(\frac{\ln(x)}{2}\right) c_2}{3} + i x^{\frac{5}{2}} \sinh\left(\frac{\ln(x)}{2}\right) c_2 + \frac{2 x^{\frac{5}{2}} \cosh\left(\frac{\ln(x)}{2}\right)}{3}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left((c_2 i - 2) \cosh\left(\frac{\ln(x)}{2}\right) - 3 i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{5}{2}}}{3} \tag{1}$$

Verification of solutions

$$y = -\frac{\left((c_2 i - 2) \cosh\left(\frac{\ln(x)}{2}\right) - 3 i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{5}{2}}}{3}$$

Verified OK.

2.17.5 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\ &= \frac{6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{-\frac{4}{x}}{2}} \\ &= x^2 \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x^2 \tag{4}$$

Applying this change of variable to the original ode results in

$$x^4 v''(x) = 0$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$v(x) = c_1x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1x + c_2)\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = (c_1x + c_2)\, x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (c_1x + c_2)\, x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = 8c_1 + 4c_2 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = 0 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_2 = 1 - 2c_1$$

Substituting these values back in above solution results in

$$y = x^2(c_1x - 2c_1 + 1)$$

Which simplifies to

$$y = x^2(1 + (x - 2)\, c_1)$$

Summary
The solution(s) found are the following

$$y = x^2(1 + (x - 2)\, c_1) \tag{1}$$

Verification of solutions

$$y = x^2(1 + (x - 2)\, c_1)$$

Verified OK.

2.17.6 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\ \ln(u) &= -2\ln(x) + c_1 \\ u &= \mathrm{e}^{-2\ln(x)+c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{x} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(-\frac{c_1}{x} + c_2\right)x^3 \\ &= (c_2 x - c_1)\,x^2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \left(-\frac{c_1}{x} + c_2\right)x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = -4c_1 + 8c_2 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = 0 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1 + 2c_2$$

Substituting these values back in above solution results in

$$y = x^2(c_2 x - 2c_2 + 1)$$

Which simplifies to

$$y = x^2(1 + (x - 2)\, c_2)$$

Summary
The solution(s) found are the following

$$y = x^2(1 + (x - 2)\, c_2) \tag{1}$$

Verification of solutions

$$y = x^2(1 + (x - 2)\, c_2)$$

Verified OK.

2.17.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 4y'x + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 75: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4x}{x^2} \, dx} \\ &= z_1 e^{2\ln(x)} \\ &= z_1 \left(x^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{4 \ln(x)}}{(y_1)^2} \, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (x^2) + c_2 (x^2(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x^3 + c_1 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = 8c_2 + 4c_1 \tag{1A}$$

substituting $y = 0$ and $x = 0$ in the above gives

$$0 = 0 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1 - 2c_2$$

Substituting these values back in above solution results in

$$y = c_2 x^3 - 2c_2 x^2 + x^2$$

Which simplifies to

$$y = x^2(1 + (x - 2) c_2)$$

Summary

The solution(s) found are the following

$$y = x^2(1 + (x - 2) c_2) \tag{1}$$

Verification of solutions

$$y = x^2(1 + (x - 2) c_2)$$

Verified OK.

2.17.8 Maple step by step solution

Let's solve

$[y''x^2 - 4y'x + 6y = 0, y(0) = 0, y(2) = 4]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{x} - \frac{6y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x} + \frac{6y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 - 4y'x + 6y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

$y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^2 - 4\frac{d}{dt}y(t) + 6y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 5\frac{d}{dt}y(t) + 6y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{2t}$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{3t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{3t}$

- Change variables back using $t = \ln(x)$

  $y = c_2 x^3 + c_1 x^2$

- Simplify

  $y = x^2 (c_2 x + c_1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve([x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=0,y(0) = 0, y(2) = 4],y(x), singsol=all)
```

$$y(x) = x^2(1 + c_1(x - 2))$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 23

```
DSolve[{x^2*y''[x]-4*x*y'[x]+6*y[x]==0,{y[0]==0,y[2]==4}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{2}x^2(x - c_1x + 2c_1)$$

2.18 problem 10 (f)

Internal problem ID [12614]
Internal file name [`OUTPUT/11266_Thursday_October_19_2023_04_44_02_PM_1413369/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 1. Introduction. Exercises 1.3, page 27
Problem number: 10 (f).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

Unable to solve or complete the solution.

$$\boxed{x^2y'' - 4y'x + 6y = 0}$$

With initial conditions

$$[y(0) = 2, y'(2) = -1]$$

2.18.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 3 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2 x^3 + c_1 x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x^3 + c_1 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = 0 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_2 x^2 + 2c_1 x$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = 12c_2 + 4c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

2.18.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -\frac{4}{x}\,dx} \\
&= \frac{1}{x^2}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\frac{y}{x^2}\right)'' &= 0
\end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x^2}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{x^2}\right) = c_1x + c_2$$

Hence the solution is

$$y = \frac{c_1x + c_2}{\frac{1}{x^2}}$$

Or

$$y = c_1x^3 + c_2x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x^3 + c_2 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = 0 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1 x^2 + 2c_2 x$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = 12c_1 + 4c_2 \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

2.18.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int -\frac{4}{x}dx\right)} dx \\
&= \int e^{4\ln(x)}\, dx \\
&= \int x^4 dx \\
&= \frac{x^5}{5}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{6}{x^2}}{x^8} \\
&= \frac{6}{x^{10}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + \frac{6y(\tau)}{x^{10}} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{6}{x^{10}} = \frac{6}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 6 = 0$$

Or

$$25r^2 - 25r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{2}{5}$$
$$r_2 = \frac{3}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{2}{5}} + c_2\tau^{\frac{3}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{3}{5}}(x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}}(x^5)^{\frac{3}{5}}}{5}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y=\frac{c_1 5^{\frac{3}{5}}\left(x^5\right)^{\frac{2}{5}}}{5}+\frac{c_2 5^{\frac{2}{5}}\left(x^5\right)^{\frac{3}{5}}}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2=0 \tag{1A}$$

Taking derivative of the solution gives

$$y'=\frac{2c_1 5^{\frac{3}{5}}x^4}{5\left(x^5\right)^{\frac{3}{5}}}+\frac{3c_2 5^{\frac{2}{5}}x^4}{5\left(x^5\right)^{\frac{2}{5}}}$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1=\frac{4\left(5^{\frac{1}{5}}c_1+3c_2\right)5^{\frac{2}{5}}}{5} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

2.18.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y''-4y'x+6y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=-\frac{4}{x}$$
$$q(x)=\frac{6}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{5c\sqrt{6}}{6}
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{5c\sqrt{6}\left(\frac{d}{d\tau}y(\tau)\right)}{6} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\,c\tau}{12}}\left(c_1\cosh\left(\frac{\sqrt{6}\,c\tau}{12}\right) + ic_2\sinh\left(\frac{\sqrt{6}\,c\tau}{12}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{6}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}\,x\ln\left(x\right)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{5}{2}}\left(c_1\cosh\left(\frac{\ln\left(x\right)}{2}\right) + ic_2\sinh\left(\frac{\ln\left(x\right)}{2}\right)\right)
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = x^{\frac{5}{2}}\left(c_1\cosh\left(\frac{\ln\left(x\right)}{2}\right) + ic_2\sinh\left(\frac{\ln\left(x\right)}{2}\right)\right) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$
2 = 0 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{5x^{\frac{3}{2}}\left(c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)\right)}{2} + x^{\frac{5}{2}}\left(\frac{c_1\sinh\left(\frac{\ln(x)}{2}\right)}{2x} + \frac{ic_2\cosh\left(\frac{\ln(x)}{2}\right)}{2x}\right)
$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$
-1 = 4c_2 i + 8c_1 \tag{2A}
$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

2.18.5 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\ &= \frac{6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{-\frac{4}{x}}{2}} \\ &= x^2 \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x^2 \tag{4}$$

Applying this change of variable to the original ode results in

$$x^4 v''(x) = 0$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x + c_2)\, (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = (c_1 x + c_2)\, x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (c_1 x + c_2)\, x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = 0 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 x^2 + 2(c_1 x + c_2)\, x$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = 12c_1 + 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

<u>Verification of solutions</u> N/A

2.18.6 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\ \ln(u) &= -2\ln(x) + c_1 \\ u &= \mathrm{e}^{-2\ln(x)+c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{x} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(-\frac{c_1}{x} + c_2\right)x^3 \\ &= (c_2 x - c_1)\,x^2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \left(-\frac{c_1}{x} + c_2\right)x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = 0 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 x + 3\left(-\frac{c_1}{x} + c_2\right) x^2$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = -4c_1 + 12c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

2.18.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 4y'x + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 77: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\
&= z_1 e^{2\ln(x)} \\
&= z_1\left(x^2\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(x)\right)
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2x^3 + c_1x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = 0 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_2x^2 + 2c_1x$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = 12c_2 + 4c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

2.18.8 Maple step by step solution

Let's solve

$\left[y''x^2 - 4y'x + 6y = 0, y(0) = 2, y'\Big|_{\{x=2\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{x} - \frac{6y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x} + \frac{6y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 - 4y'x + 6y = 0$

- Make a change of variables

$t = \ln(x)$

☐ Substitute the change of variables back into the ODE

- Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^2 - 4\frac{d}{dt}y(t) + 6y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 5\frac{d}{dt}y(t) + 6y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{2t} + c_2 e^{3t}$

- Change variables back using $t = \ln(x)$

  $y = c_2 x^3 + c_1 x^2$

- Simplify

  $y = x^2(c_2 x + c_1)$

□ Check validity of solution $y = x^2(c_2 x + c_1)$

  - Use initial condition $y(0) = 2$

    $2 = 0$

  - Compute derivative of the solution

    $y' = 2x(c_2 x + c_1) + c_2 x^2$

  - Use the initial condition $y'\Big|_{\{x=2\}} = -1$

    $-1 = 12c_2 + 4c_1$

  - Solve for c_1 and c_2
  - The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✗ Solution by Maple

```
dsolve([x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=0,y(0) = 2, D(y)(2) = -1],y(x), singsol=a
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{x^2*y''[x]-4*x*y'[x]+6*y[x]==0,{y[0]==2,y'[2]==-1}},y[x],x,IncludeSingularSolutions
```

{}

3 Chapter 2. The Initial Value Problem. Exercises 2.1, page 40

3.1 problem 1 (A)

Internal problem ID [12615]
Internal file name [`OUTPUT/11267_Friday_November_03_2023_06_29_29_AM_64086885/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 1 (A).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = 1 - x}$$

3.1.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 1 - x \, \mathrm{d}x \\ &= x - \frac{1}{2}x^2 + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x - \frac{1}{2}x^2 + c_1 \tag{1}$$

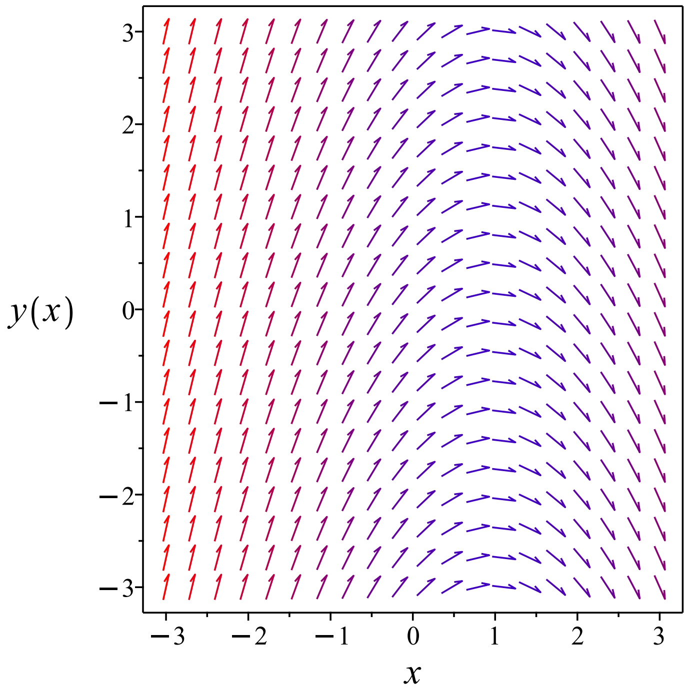


Figure 90: Slope field plot

Verification of solutions

$$y = x - \frac{1}{2}x^2 + c_1$$

Verified OK.

3.1.2 Maple step by step solution

Let's solve

$y' = 1 - x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int (1 - x)\, dx + c_1$

- Evaluate integral

  $y = x - \frac{1}{2}x^2 + c_1$

- Solve for y

$$y = x - \frac{1}{2}x^2 + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=1-x,y(x), singsol=all)
```

$$y(x) = -\frac{1}{2}x^2 + x + c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 16

```
DSolve[y'[x]==1-x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2}{2} + x + c_1$$

3.2 problem 1 (B)

Internal problem ID [12616]
Internal file name [`OUTPUT/11268_Friday_November_03_2023_06_29_32_AM_784968/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 1 (B).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = x - 1}$$

3.2.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int x - 1 \, \mathrm{d}x \\ &= \frac{1}{2}x^2 - x + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{2}x^2 - x + c_1 \tag{1}$$

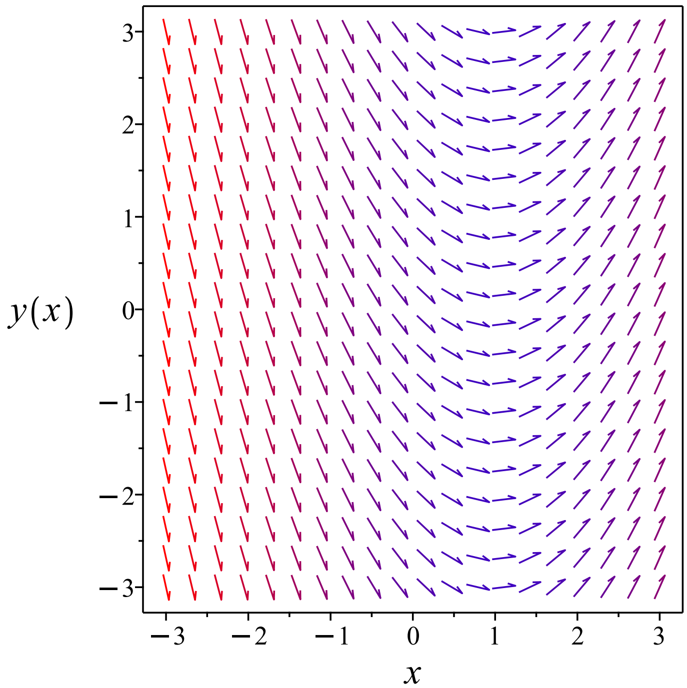


Figure 91: Slope field plot

Verification of solutions

$$y = \frac{1}{2}x^2 - x + c_1$$

Verified OK.

3.2.2 Maple step by step solution

Let's solve

$y' = x - 1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int (x-1)\, dx + c_1$

- Evaluate integral

  $y = \frac{1}{2}x^2 - x + c_1$

- Solve for y

$$y = \frac{1}{2}x^2 - x + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)=x-1,y(x), singsol=all)
```

$$y(x) = \frac{1}{2}x^2 - x + c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 18

```
DSolve[y'[x]==x-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} - x + c_1$$

3.3 problem 2 (C)

Internal problem ID [12617]
Internal file name [`OUTPUT/11269_Friday_November_03_2023_06_29_32_AM_5344487/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 2 (C).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y = 1}$$

3.3.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{1-y} dy = \int dx$$
$$-\ln(1-y) = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{1-y} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{1-y} = c_2 e^{x}$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{-x}}{c_2} + 1 \tag{1}$$

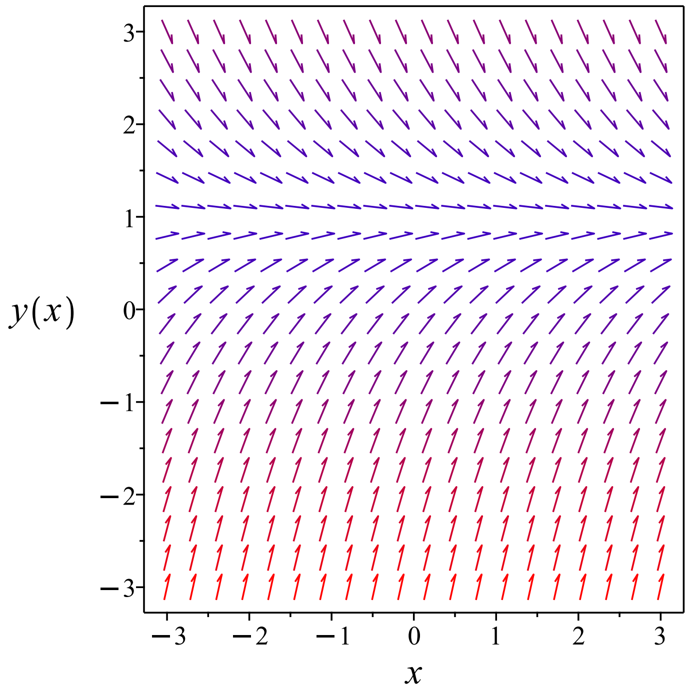


Figure 92: Slope field plot

Verification of solutions

$$y = -\frac{e^{-x}}{c_2} + 1$$

Verified OK.

3.3.2 Maple step by step solution

Let's solve

$y' + y = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1-y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{1-y} dx = \int 1 dx + c_1$

- Evaluate integral

$$-\ln(1-y) = x + c_1$$

- Solve for y

$$y = -e^{-x-c_1} + 1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=1-y(x),y(x), singsol=all)
```

$$y(x) = c_1 e^{-x} + 1$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 20

```
DSolve[y'[x]==1-y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow 1 + c_1 e^{-x}$$
$$y(x) \rightarrow 1$$

3.4 problem 2 (D)

Internal problem ID [12618]
Internal file name [`OUTPUT/11270_Friday_November_03_2023_06_29_33_AM_11055370/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 2 (D).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y = 1}$$

3.4.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y+1} dy = \int dx$$
$$\ln(y+1) = x + c_1$$

Raising both side to exponential gives

$$y + 1 = e^{x+c_1}$$

Which simplifies to

$$y + 1 = c_2 e^x$$

Summary
The solution(s) found are the following

$$y = c_2 e^x - 1 \tag{1}$$

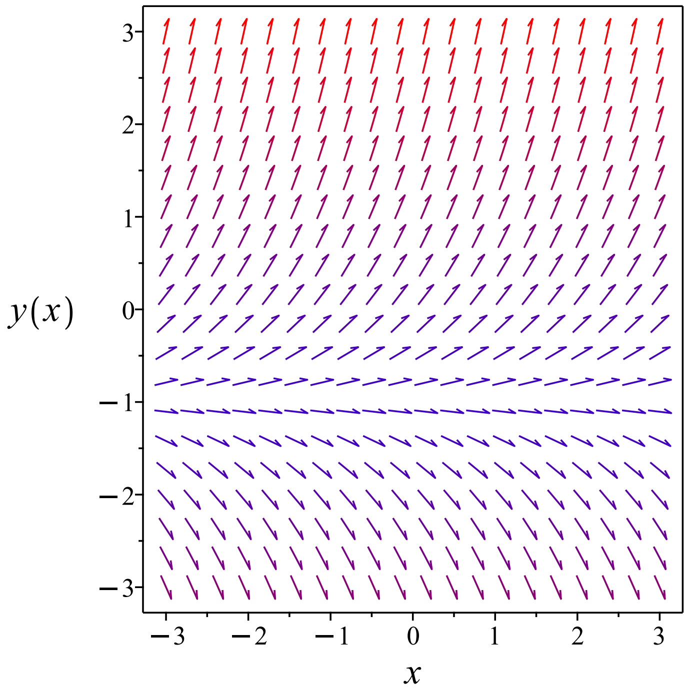


Figure 93: Slope field plot

Verification of solutions

$$y = c_2 e^x - 1$$

Verified OK.

3.4.2 Maple step by step solution

Let's solve

$y' - y = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y+1} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y+1} dx = \int 1 dx + c_1$

- Evaluate integral

$$\ln(y+1) = x + c_1$$

- Solve for y

$$y = e^{x+c_1} - 1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)=1+y(x),y(x), singsol=all)
```

$$y(x) = -1 + c_1 e^x$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 18

```
DSolve[y'[x]==1+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -1 + c_1 e^x$$
$$y(x) \to -1$$

3.5 problem 3 (E)

Internal problem ID [12619]
Internal file name [`OUTPUT/11271_Friday_November_03_2023_06_29_33_AM_19494184/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 3 (E).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 = -4}$$

3.5.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 - 4} dy = \int dx$$
$$\frac{\ln(y-2)}{4} - \frac{\ln(y+2)}{4} = x + c_1$$

The above can be written as

$$\left(\frac{1}{4}\right)(\ln(y-2) - \ln(y+2)) = x + c_1$$
$$\ln(y-2) - \ln(y+2) = (4)(x + c_1)$$
$$= 4c_1 + 4x$$

Raising both side to exponential gives

$$e^{\ln(y-2) - \ln(y+2)} = 4c_1 e^{4x}$$

Which simplifies to

$$\frac{y-2}{y+2} = c_2 e^{4x}$$

Summary

The solution(s) found are the following

$$y = -\frac{2(c_2 e^{4x}+1)}{-1+c_2 e^{4x}} \tag{1}$$

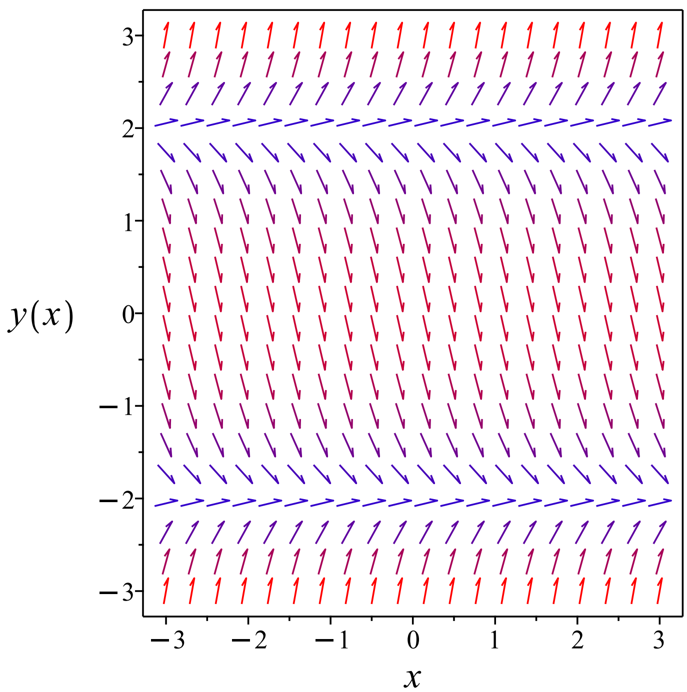


Figure 94: Slope field plot

Verification of solutions

$$y = -\frac{2(c_2 e^{4x}+1)}{-1+c_2 e^{4x}}$$

Verified OK.

3.5.2 Maple step by step solution

Let's solve

$y' - y^2 = -4$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-4} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2-4}dx = \int 1dx + c_1$

- Evaluate integral

  $\frac{\ln(y-2)}{4} - \frac{\ln(y+2)}{4} = x + c_1$

- Solve for y

  $y = -\frac{2(e^{4c_1+4x}+1)}{-1+e^{4c_1+4x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)=y(x)^2-4,y(x), singsol=all)
```

$$y(x) = \frac{-2c_1 e^{4x} - 2}{-1 + c_1 e^{4x}}$$

✓ Solution by Mathematica

Time used: 1.066 (sec). Leaf size: 40

```
DSolve[y'[x]==y[x]^2-4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2 - 2e^{4(x+c_1)}}{1 + e^{4(x+c_1)}}$$
$$y(x) \to -2$$
$$y(x) \to 2$$

3.6 problem 3 (F)

Internal problem ID [12620]
Internal file name [`OUTPUT/11272_Friday_November_03_2023_06_29_34_AM_54095803/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 3 (F).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y^2 = 4}$$

3.6.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y^2+4} dy = \int dx$$
$$-\frac{\ln(y-2)}{4} + \frac{\ln(y+2)}{4} = x + c_1$$

The above can be written as

$$\left(-\frac{1}{4}\right)(\ln(y-2) - \ln(y+2)) = x + c_1$$
$$\ln(y-2) - \ln(y+2) = (-4)(x + c_1)$$
$$= -4x - 4c_1$$

Raising both side to exponential gives

$$e^{\ln(y-2)-\ln(y+2)} = -4c_1 e^{-4x}$$

Which simplifies to

$$\frac{y-2}{y+2} = c_2 \mathrm{e}^{-4x}$$

Summary

The solution(s) found are the following

$$y = -\frac{2(c_2 \mathrm{e}^{-4x}+1)}{-1+c_2 \mathrm{e}^{-4x}} \tag{1}$$

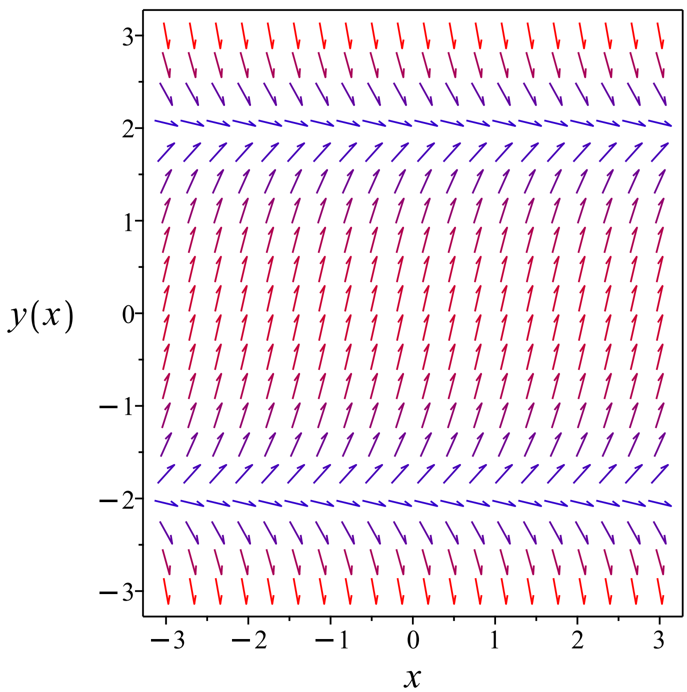


Figure 95: Slope field plot

Verification of solutions

$$y = -\frac{2(c_2 \mathrm{e}^{-4x}+1)}{-1+c_2 \mathrm{e}^{-4x}}$$

Verified OK.

3.6.2 Maple step by step solution

Let's solve

$y' + y^2 = 4$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{4-y^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{4-y^2} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\frac{\ln(y-2)}{4} + \frac{\ln(y+2)}{4} = x + c_1$

- Solve for y

  $y = \frac{2(\mathrm{e}^{4c_1+4x}+1)}{-1+\mathrm{e}^{4c_1+4x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)=4-y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{2c_1 e^{4x} + 2}{-1 + c_1 e^{4x}}$$

✓ Solution by Mathematica

Time used: 0.278 (sec). Leaf size: 45

```
DSolve[y'[x]==4-y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2(e^{4x} - e^{4c_1})}{e^{4x} + e^{4c_1}}$$
$$y(x) \to -2$$
$$y(x) \to 2$$

3.7 problem 4 (G)

Internal problem ID [12621]
Internal file name [`OUTPUT/11273_Friday_November_03_2023_06_29_34_AM_7363715/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 4 (G).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - yx = 0}$$

3.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= xy \end{aligned}$$

Where $f(x) = x$ and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\,dy = x\,dx$$
$$\int \frac{1}{y}\,dy = \int x\,dx$$
$$\ln(y) = \frac{x^2}{2} + c_1$$
$$y = \mathrm{e}^{\frac{x^2}{2}+c_1}$$
$$= c_1 \mathrm{e}^{\frac{x^2}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{x^2}{2}} \tag{1}$$

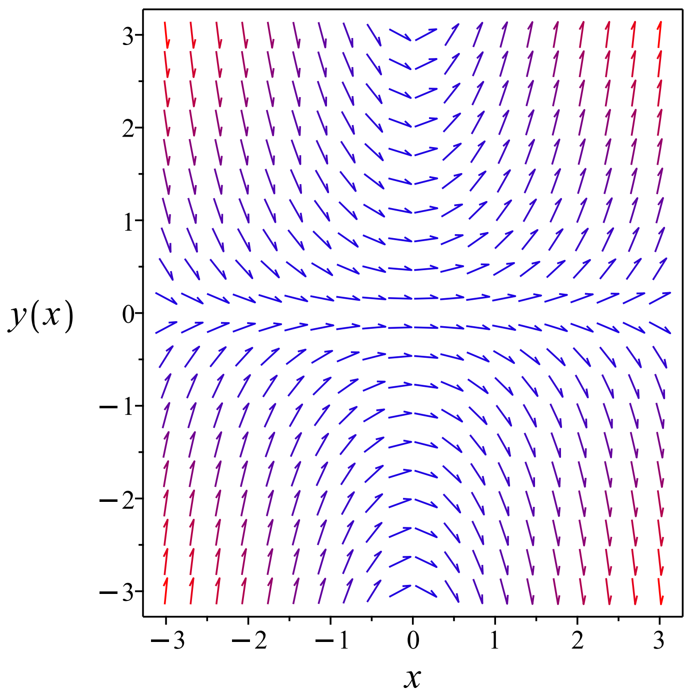


Figure 96: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{x^2}{2}}$$

Verified OK.

3.7.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -x \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - yx = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -x dx} \\ &= e^{-\frac{x^2}{2}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(e^{-\frac{x^2}{2}} y\right) &= 0 \end{aligned}$$

Integrating gives

$$e^{-\frac{x^2}{2}} y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x^2}{2}}$ results in

$$y = c_1 e^{\frac{x^2}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{\frac{x^2}{2}} \tag{1}$$

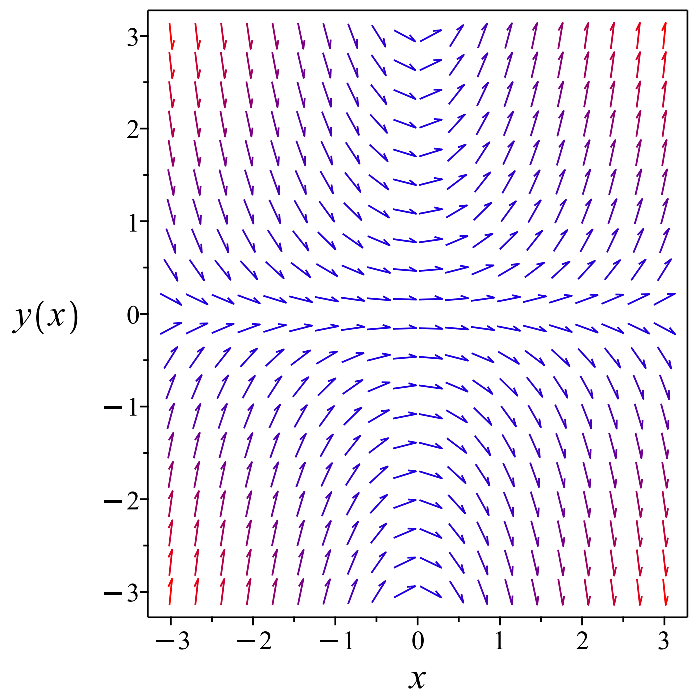


Figure 97: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x^2}{2}}$$

Verified OK.

3.7.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - u(x)\, x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(x^2 - 1)}{x} \end{aligned}$$

Where $f(x) = \frac{x^2-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{x^2-1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{x^2-1}{x}\,dx \\
\ln(u) &= \frac{x^2}{2} - \ln(x) + c_2 \\
u &= \mathrm{e}^{\frac{x^2}{2}-\ln(x)+c_2} \\
&= c_2 \mathrm{e}^{\frac{x^2}{2}-\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2 \mathrm{e}^{\frac{x^2}{2}}}{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= c_2 \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_2 \mathrm{e}^{\frac{x^2}{2}} \tag{1}
$$

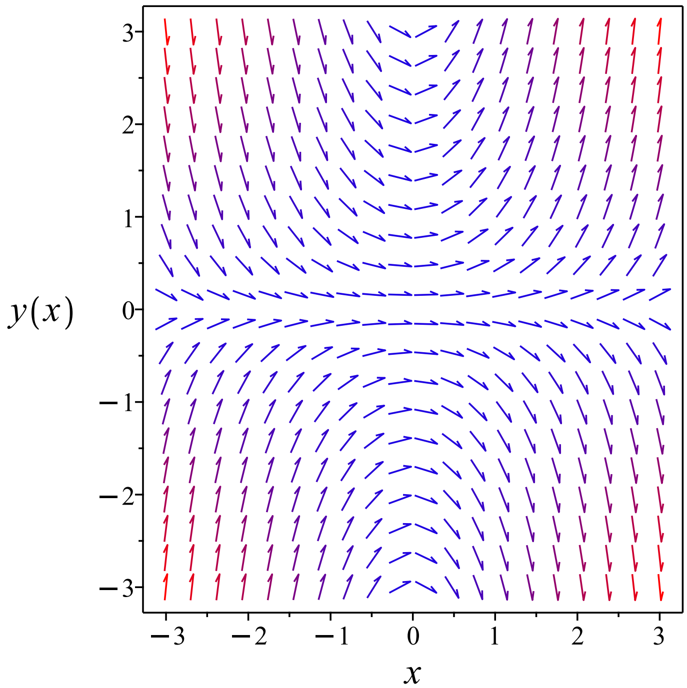


Figure 98: Slope field plot

Verification of solutions

$$y = c_2 e^{\frac{x^2}{2}}$$

Verified OK.

3.7.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = xy$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 85: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\frac{x^2}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{x^2}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{x^2}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = xy$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -x\,e^{-\frac{x^2}{2}} y \\ S_y &= e^{-\frac{x^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-\frac{x^2}{2}} y = c_1$$

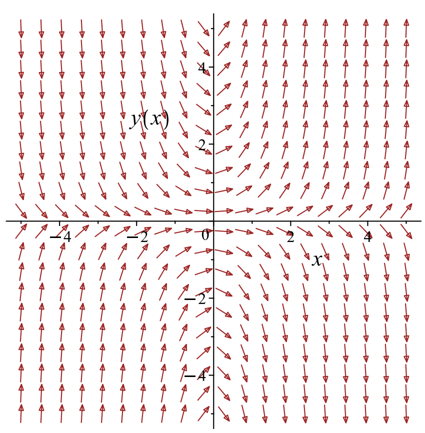
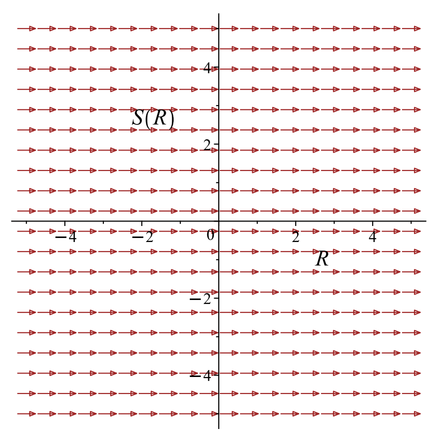
Which simplifies to

$$e^{-\frac{x^2}{2}} y = c_1$$

Which gives

$$y = c_1 e^{\frac{x^2}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = xy$ 	$R = x$ $S = e^{-\frac{x^2}{2}} y$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 e^{\frac{x^2}{2}} \tag{1}$$

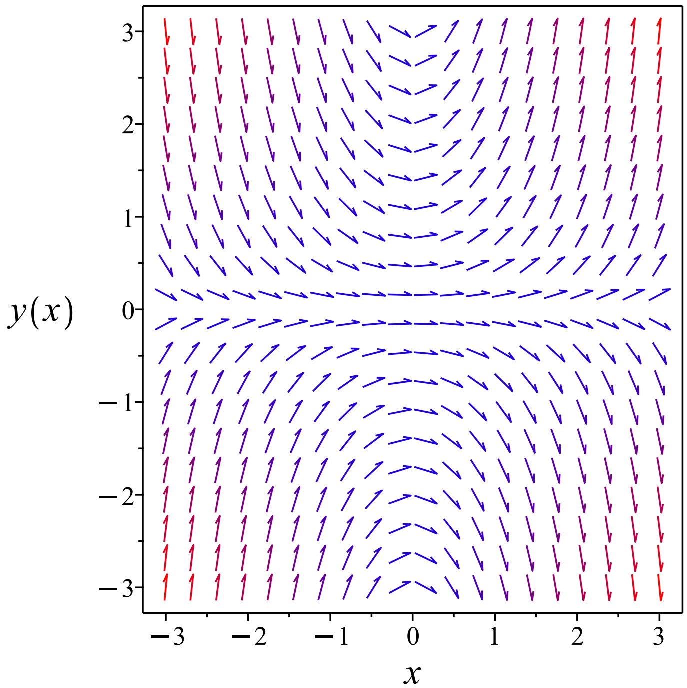


Figure 99: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x^2}{2}}$$

Verified OK.

3.7.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y}\right)\mathrm{d}y = (x)\,\mathrm{d}x$$
$$(-x)\,\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = \frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \ln(y)$$

The solution becomes

$$y = e^{\frac{x^2}{2} + c_1}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{x^2}{2} + c_1} \tag{1}$$

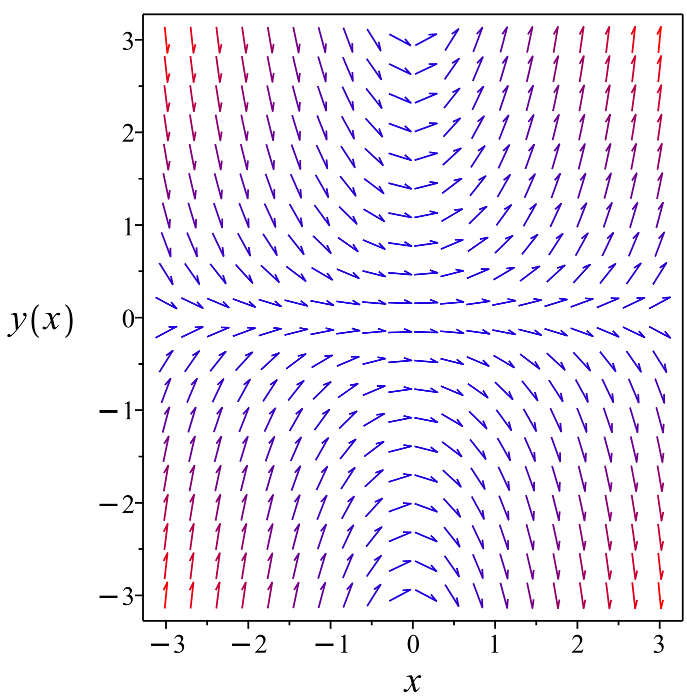


Figure 100: Slope field plot

Verification of solutions

$$y = e^{\frac{x^2}{2}+c_1}$$

Verified OK.

3.7.6 Maple step by step solution

Let's solve

$y' - yx = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int x dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{x^2}{2} + c_1$

- Solve for y

  $y = e^{\frac{x^2}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=x*y(x),y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{x^2}{2}}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 22

```
DSolve[y'[x]==x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{x^2}{2}}$$
$$y(x) \to 0$$

3.8 problem 4 (H)

Internal problem ID [12622]
Internal file name [`OUTPUT/11274_Friday_November_03_2023_06_29_35_AM_27176372/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 4 (H).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' + yx = 0}$$

3.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -xy \end{aligned}$$

Where $f(x) = -x$ and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\,dy = -x\,dx$$
$$\int \frac{1}{y}\,dy = \int -x\,dx$$
$$\ln(y) = -\frac{x^2}{2} + c_1$$
$$y = \mathrm{e}^{-\frac{x^2}{2}+c_1}$$
$$= c_1 \mathrm{e}^{-\frac{x^2}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{x^2}{2}} \tag{1}$$

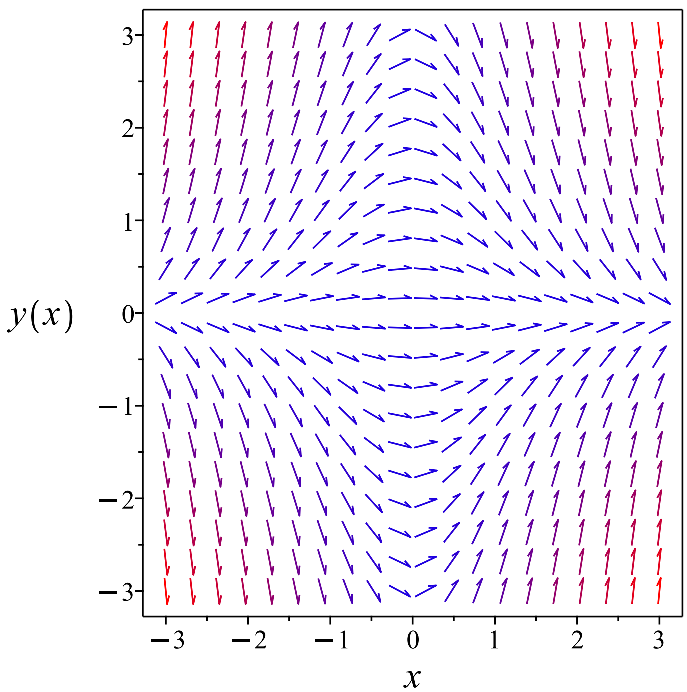


Figure 101: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{x^2}{2}}$$

Verified OK.

3.8.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = x$$
$$q(x) = 0$$

Hence the ode is

$$y' + yx = 0$$

The integrating factor μ is

$$\mu = e^{\int x dx}$$
$$= e^{\frac{x^2}{2}}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(e^{\frac{x^2}{2}} y\right) = 0$$

Integrating gives

$$e^{\frac{x^2}{2}} y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x^2}{2}}$ results in

$$y = c_1 e^{-\frac{x^2}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-\frac{x^2}{2}} \tag{1}$$

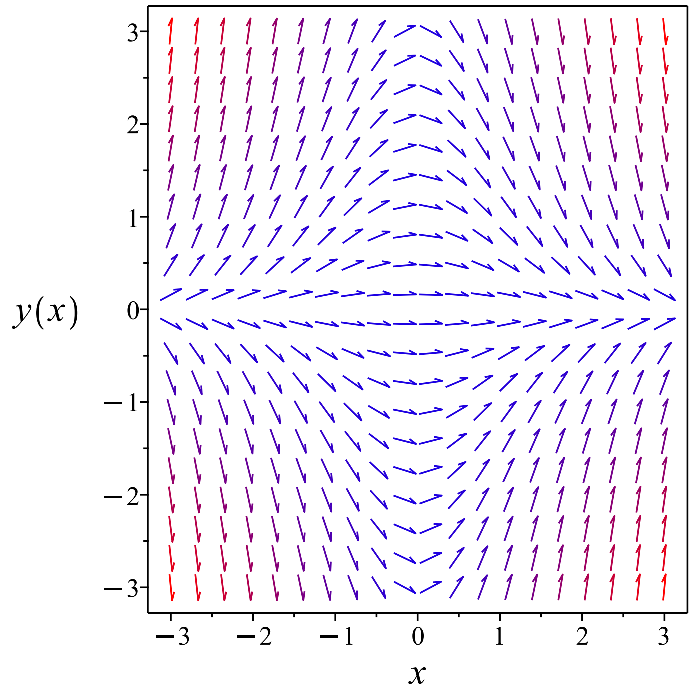


Figure 102: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x^2}{2}}$$

Verified OK.

3.8.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) + u(x)\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{u(x^2+1)}{x}\end{aligned}$$

Where $f(x) = -\frac{x^2+1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{x^2+1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{x^2+1}{x}\,dx \\
\ln(u) &= -\frac{x^2}{2} - \ln(x) + c_2 \\
u &= \mathrm{e}^{-\frac{x^2}{2}-\ln(x)+c_2} \\
&= c_2\mathrm{e}^{-\frac{x^2}{2}-\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2\mathrm{e}^{-\frac{x^2}{2}}}{x}$$

Therefore the solution y is

$$\begin{aligned}
y &= xu \\
&= c_2\mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2\mathrm{e}^{-\frac{x^2}{2}} \tag{1}$$

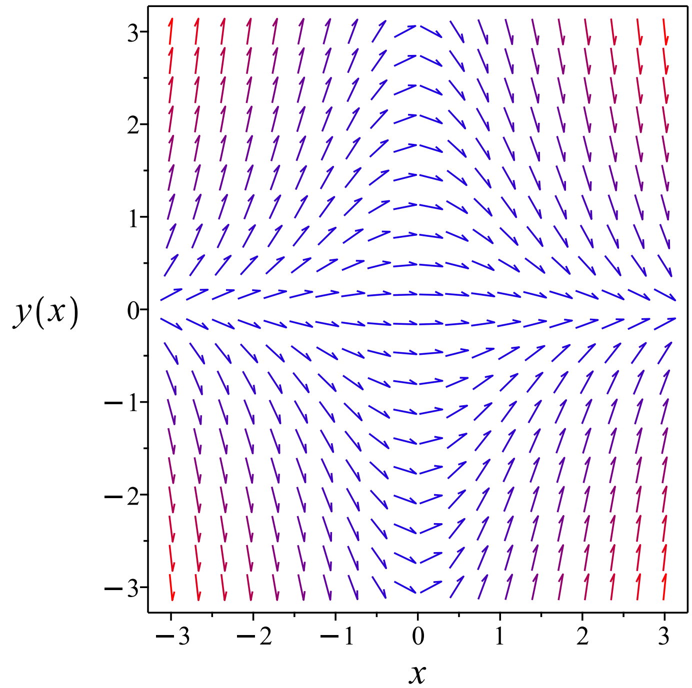


Figure 103: Slope field plot

Verification of solutions

$$y = c_2 e^{-\frac{x^2}{2}}$$

Verified OK.

3.8.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -xy \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 88: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = 0$$
$$\eta(x, y) = e^{-\frac{x^2}{2}} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{x^2}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{x^2}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -xy$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= x\, e^{\frac{x^2}{2}} y \\ S_y &= e^{\frac{x^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\frac{x^2}{2}} y = c_1$$

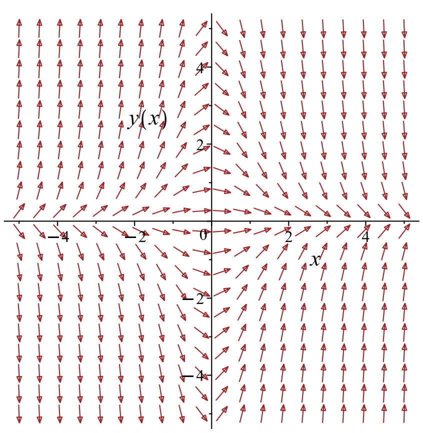
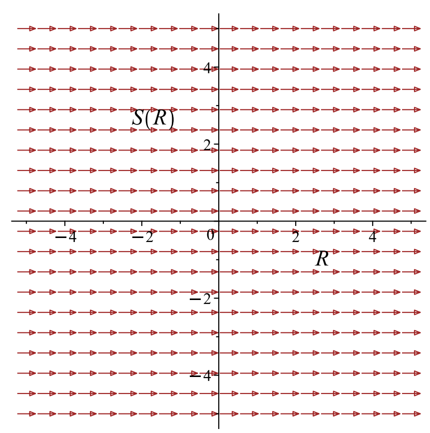
Which simplifies to

$$e^{\frac{x^2}{2}} y = c_1$$

Which gives

$$y = c_1 e^{-\frac{x^2}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -xy$ 	$R = x$ $S = e^{\frac{x^2}{2}} y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 e^{-\frac{x^2}{2}} \tag{1}$$

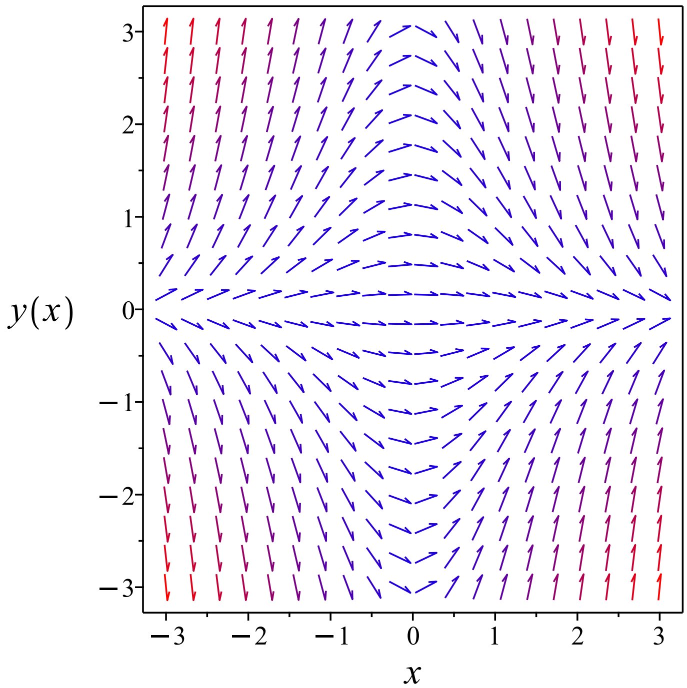


Figure 104: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x^2}{2}}$$

Verified OK.

3.8.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right) \mathrm{d}y = (x)\,\mathrm{d}x$$
$$(-x)\,\mathrm{d}x + \left(-\frac{1}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$
-\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= -\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \ln(y)$$

The solution becomes

$$y = e^{-\frac{x^2}{2} - c_1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{-\frac{x^2}{2} - c_1} \tag{1}$$

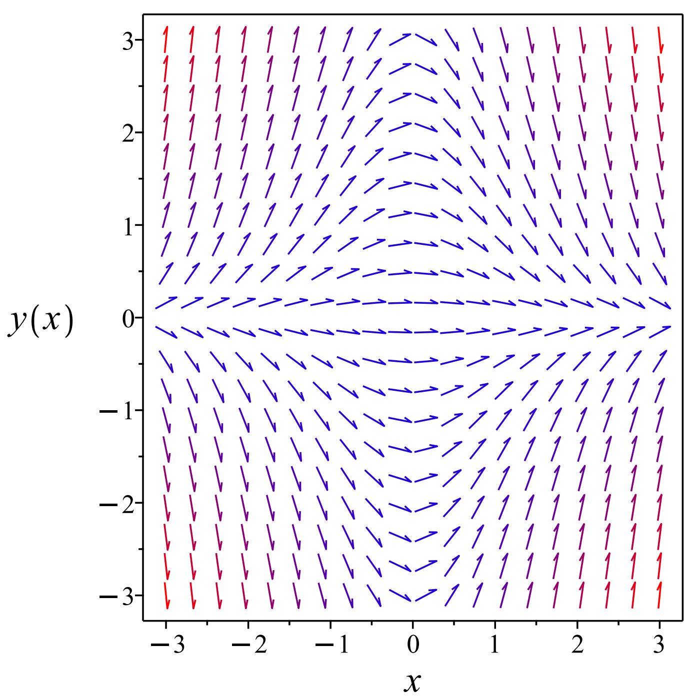


Figure 105: Slope field plot

Verification of solutions

$$y = e^{-\frac{x^2}{2}-c_1}$$

Verified OK.

3.8.6 Maple step by step solution

Let's solve

$y' + yx = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -xdx + c_1$

- Evaluate integral

  $\ln(y) = -\frac{x^2}{2} + c_1$

- Solve for y

  $y = e^{-\frac{x^2}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=-x*y(x),y(x), singsol=all)
```

$$y(x) = e^{-\frac{x^2}{2}} c_1$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 22

```
DSolve[y'[x]==-x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\frac{x^2}{2}}$$
$$y(x) \to 0$$

3.9 problem 5 (I)

Internal problem ID [12623]
Internal file name [`OUTPUT/11275_Friday_November_03_2023_06_29_36_AM_57195971/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 5 (I).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + y^2 = x^2}$$

3.9.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x^2 - y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 - y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2$, $f_1(x) = 0$ and $f_2(x) = -1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= x^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) + x^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sqrt{x}\left(c_1 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + c_2 \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)\right)$$

The above shows that

$$u'(x) = \left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) c_2 + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1\right) x^{\frac{3}{2}}$$

Using the above in (1) gives the solution

$$y = \frac{\left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) c_2 + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1\right) x}{c_1 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + c_2 \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_3\right) x}{c_3 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_3\right) x}{c_3 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)} \tag{1}$$

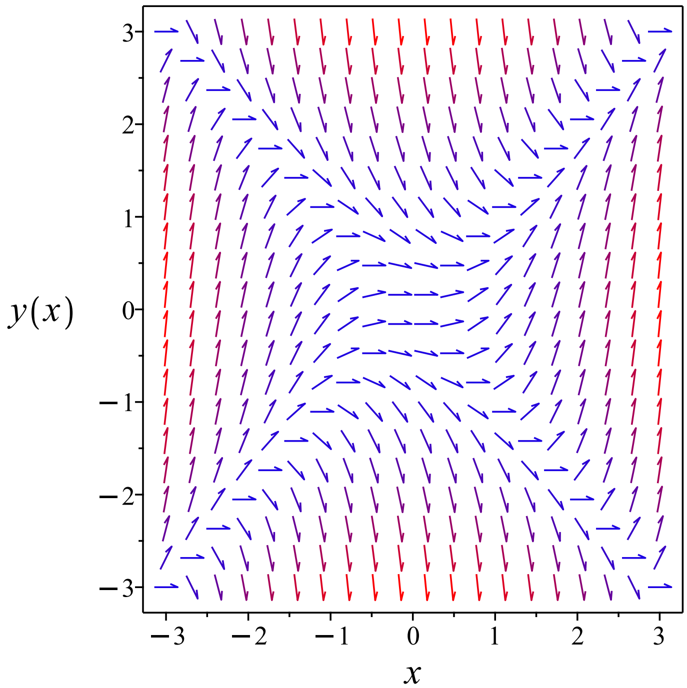


Figure 106: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_3\right) x}{c_3 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
dsolve(diff(y(x),x)=x^2-y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{x\left(\text{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1 - \text{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right)\right)}{\text{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_1 + \text{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

✓ Solution by Mathematica

Time used: 0.184 (sec). Leaf size: 197

```
DSolve[y'[x]==x^2-y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{-ix^2\left(2\,\text{BesselJ}\left(-\frac{3}{4}, \frac{ix^2}{2}\right) + c_1\left(\text{BesselJ}\left(-\frac{5}{4}, \frac{ix^2}{2}\right) - \text{BesselJ}\left(\frac{3}{4}, \frac{ix^2}{2}\right)\right)\right) - c_1\,\text{BesselJ}\left(-\frac{1}{4}, \frac{ix^2}{2}\right)}{2x\left(\text{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right) + c_1\,\text{BesselJ}\left(-\frac{1}{4}, \frac{ix^2}{2}\right)\right)}$$

$$y(x) \to \frac{ix^2\,\text{BesselJ}\left(-\frac{5}{4}, \frac{ix^2}{2}\right) - ix^2\,\text{BesselJ}\left(\frac{3}{4}, \frac{ix^2}{2}\right) + \text{BesselJ}\left(-\frac{1}{4}, \frac{ix^2}{2}\right)}{2x\,\text{BesselJ}\left(-\frac{1}{4}, \frac{ix^2}{2}\right)}$$

3.10 problem 5 (J)

Internal problem ID [12624]
Internal file name [`OUTPUT/11276_Friday_November_03_2023_06_29_36_AM_9378555/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 5 (J).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - y^2 = -x^2}$$

3.10.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -x^2 + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -x^2 + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -x^2$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= -x^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - x^2u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \sqrt{x}\left(c_1 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + c_2 \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)\right)$$

The above shows that

$$u'(x) = \left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right)c_2 + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right)c_1\right)x^{\frac{3}{2}}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right)c_2 + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right)c_1\right)x}{c_1 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + c_2 \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-\operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right)c_3 + \operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right)\right)x}{c_3 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-\operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right)c_3 + \operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right)\right)x}{c_3 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)} \tag{1}$$

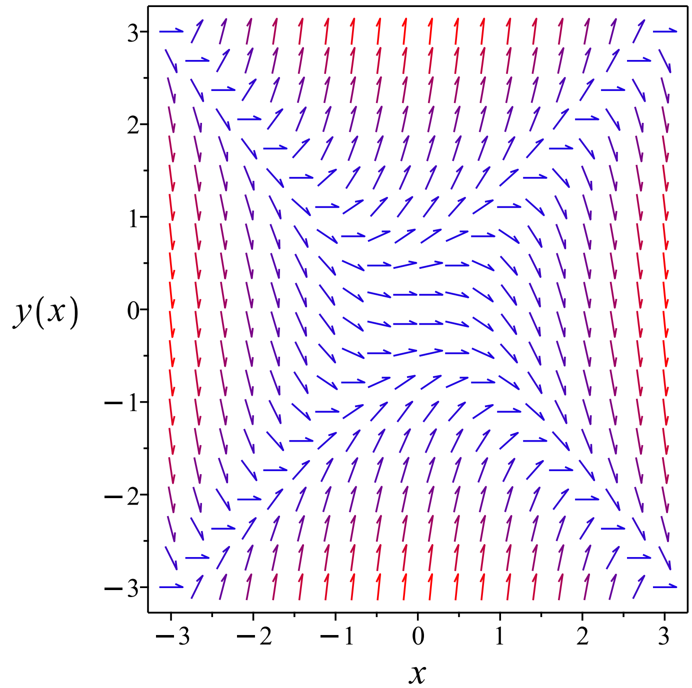


Figure 107: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\left(-\operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_3 + \operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right)\right) x}{c_3 \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(diff(y(x),x)=y(x)^2-x^2,y(x), singsol=all)
```

$$y(x) = -\frac{x\left(\operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1 - \operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right)\right)}{\operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_1 + \operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

✓ Solution by Mathematica

Time used: 0.178 (sec). Leaf size: 196

```
DSolve[y'[x]==y[x]^2-x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{ix^2\left(2\operatorname{BesselJ}\left(-\frac{3}{4}, \frac{ix^2}{2}\right) + c_1\left(\operatorname{BesselJ}\left(-\frac{5}{4}, \frac{ix^2}{2}\right) - \operatorname{BesselJ}\left(\frac{3}{4}, \frac{ix^2}{2}\right)\right)\right) + c_1\operatorname{BesselJ}\left(-\frac{1}{4}, \frac{ix^2}{2}\right)}{2x\left(\operatorname{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right) + c_1\operatorname{BesselJ}\left(-\frac{1}{4}, \frac{ix^2}{2}\right)\right)}$$

$$y(x) \to -\frac{ix^2\operatorname{BesselJ}\left(-\frac{5}{4}, \frac{ix^2}{2}\right) - ix^2\operatorname{BesselJ}\left(\frac{3}{4}, \frac{ix^2}{2}\right) + \operatorname{BesselJ}\left(-\frac{1}{4}, \frac{ix^2}{2}\right)}{2x\operatorname{BesselJ}\left(-\frac{1}{4}, \frac{ix^2}{2}\right)}$$

3.11 problem 6

Internal problem ID [12625]
Internal file name [`OUTPUT/11277_Friday_November_03_2023_06_29_37_AM_53530479/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = x}$$

3.11.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = x$$

Hence the ode is

$$y' - y = x$$

The integrating factor μ is

$$\mu = e^{\int (-1)dx}$$
$$= e^{-x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(x)$$
$$\frac{d}{dx}(e^{-x}y) = (e^{-x})(x)$$
$$d(e^{-x}y) = (x\,e^{-x})\,dx$$

Integrating gives

$$e^{-x}y = \int x\,e^{-x}\,dx$$
$$e^{-x}y = -(x+1)\,e^{-x} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = -e^{x}(x+1)\,e^{-x} + e^{x}c_1$$

which simplifies to

$$y = e^{x}c_1 - x - 1$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{x}c_1 - x - 1 \tag{1}$$

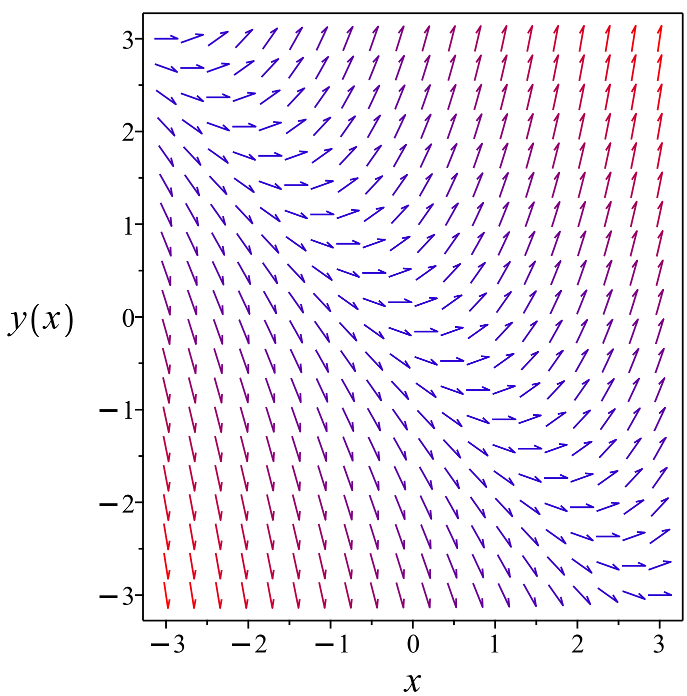


Figure 108: Slope field plot

Verification of solutions

$$y = e^x c_1 - x - 1$$

Verified OK.

3.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= y + x \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 91: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= e^x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^x} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = y + x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x\, e^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\, e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -(R+1)\,e^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x} y = -(x+1)\,e^{-x} + c_1$$

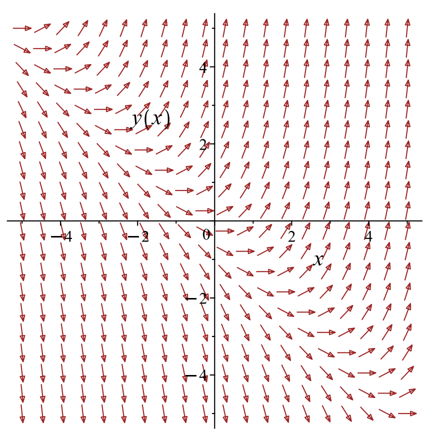
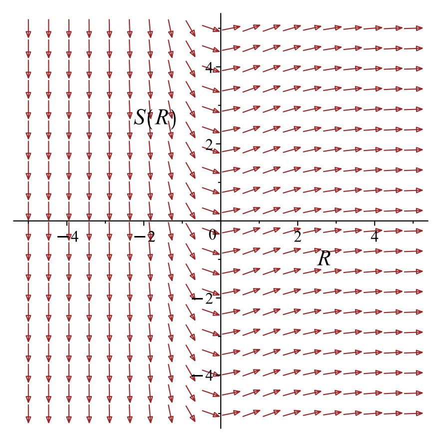
Which simplifies to

$$(x+y+1)\,e^{-x} - c_1 = 0$$

Which gives

$$y = -\left(x\,e^{-x} + e^{-x} - c_1\right) e^{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y + x$ 	$R = x$ $S = e^{-x} y$	$\frac{dS}{dR} = R\,e^{-R}$ 

Summary

The solution(s) found are the following

$$y = -\left(x\,e^{-x} + e^{-x} - c_1\right) e^{x} \tag{1}$$

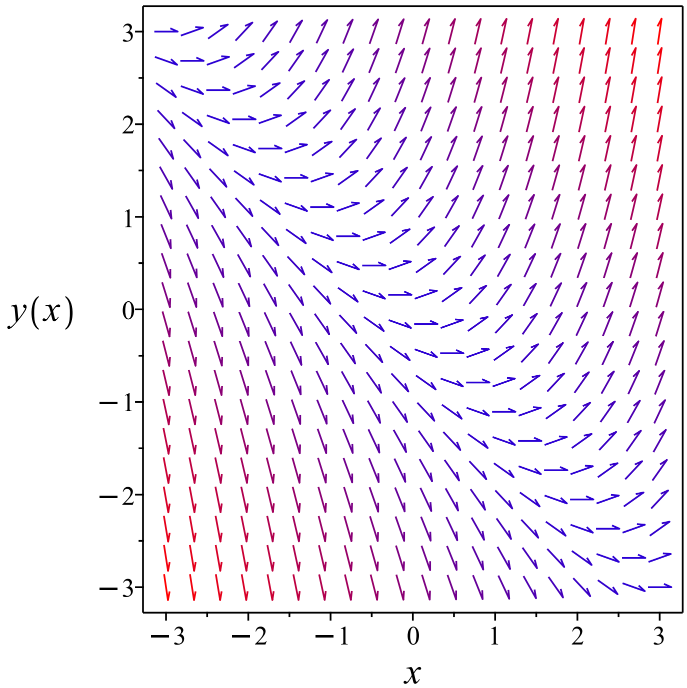


Figure 109: Slope field plot

Verification of solutions

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Verified OK.

3.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (y + x)\,\mathrm{d}x\\ (-y - x)\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -y - x$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y - x)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-1) - (0)) \\
&= -1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-x} \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-x}(-y - x) \\
&= -\mathrm{e}^{-x}(y + x)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-x}(1) \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-x}(y + x)\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\mathrm{e}^{-x}(y+x) \,\mathrm{d}x$$

$$\phi = (x+y+1)\,\mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (x+y+1)\,\mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (x+y+1)\,\mathrm{e}^{-x}$$

The solution becomes

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x} \tag{1}$$

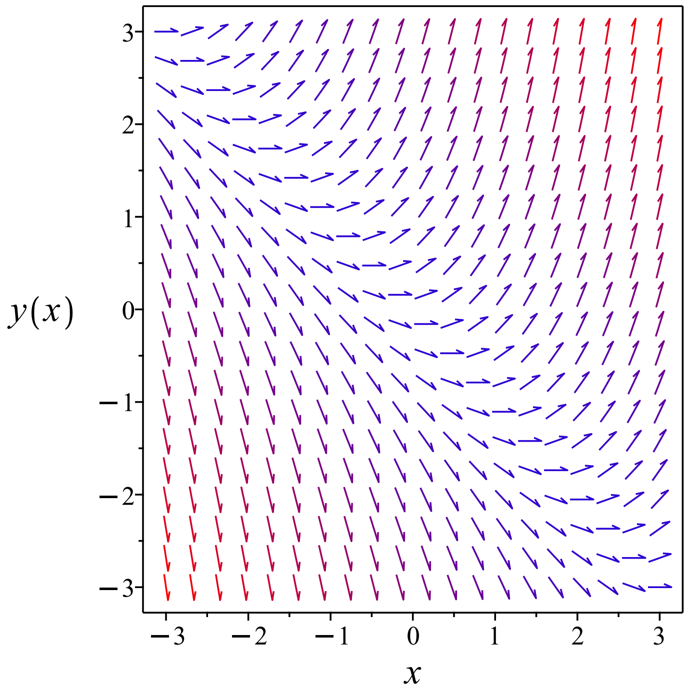


Figure 110: Slope field plot

Verification of solutions

$$y = -\left(x\,e^{-x} + e^{-x} - c_1\right) e^{x}$$

Verified OK.

3.11.4 Maple step by step solution

Let's solve

$y' - y = x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - y) = \mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'-y) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,x dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,x dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)x dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int x\,\mathrm{e}^{-x} dx + c_1}{\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y = \frac{-(x+1)\mathrm{e}^{-x} + c_1}{\mathrm{e}^{-x}}$
- Simplify

  $y = \mathrm{e}^{x} c_1 - x - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)=x+y(x),y(x), singsol=all)
```

$$y(x) = -x - 1 + c_1 e^x$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 16

```
DSolve[y'[x]==x+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x + c_1 e^x - 1$$

3.12 problem 7

Internal problem ID [12626]
Internal file name [`OUTPUT/11278_Friday_November_03_2023_06_29_38_AM_98852067/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - yx = 0}$$

3.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= xy \end{aligned}$$

Where $f(x) = x$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= x\,dx \\
\int \frac{1}{y}\,dy &= \int x\,dx \\
\ln(y) &= \frac{x^2}{2} + c_1 \\
y &= \mathrm{e}^{\frac{x^2}{2}+c_1} \\
&= c_1 \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\frac{x^2}{2}} \tag{1}
$$

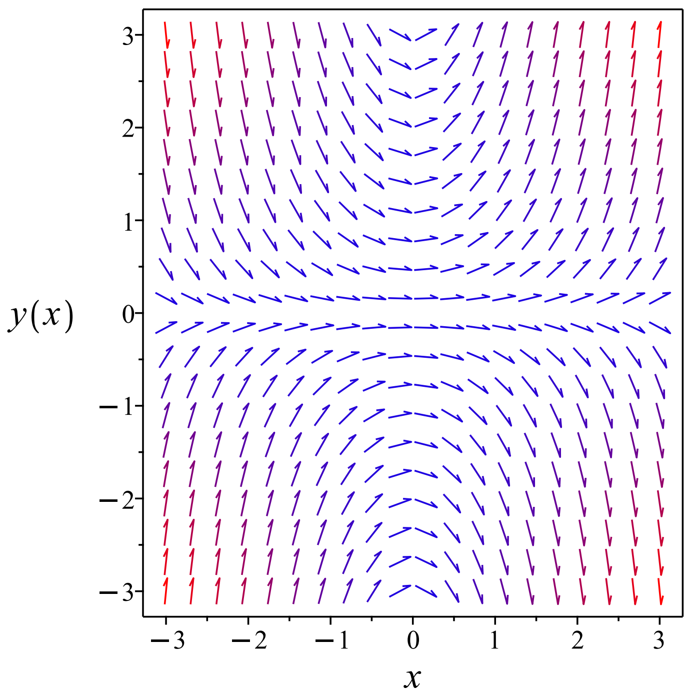


Figure 111: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{\frac{x^2}{2}}
$$

Verified OK.

3.12.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -x \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - yx = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -x dx} \\ &= e^{-\frac{x^2}{2}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(e^{-\frac{x^2}{2}}y\right) &= 0 \end{aligned}$$

Integrating gives

$$e^{-\frac{x^2}{2}}y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x^2}{2}}$ results in

$$y = c_1 e^{\frac{x^2}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{\frac{x^2}{2}} \tag{1}$$

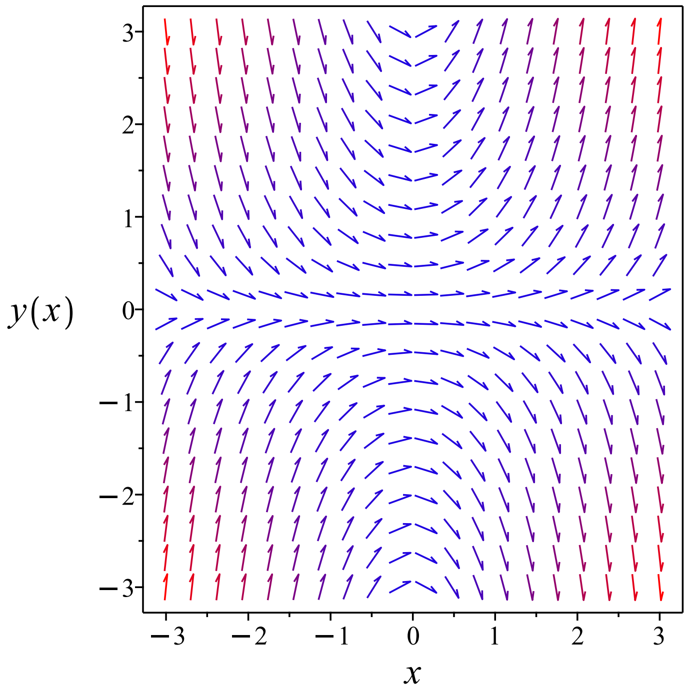


Figure 112: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x^2}{2}}$$

Verified OK.

3.12.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - u(x)\, x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(x^2 - 1)}{x} \end{aligned}$$

Where $f(x) = \frac{x^2-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{x^2-1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{x^2-1}{x}\, dx \\
\ln (u) &= \frac{x^2}{2} - \ln (x) + c_2 \\
u &= \mathrm{e}^{\frac{x^2}{2} - \ln(x) + c_2} \\
&= c_2 \mathrm{e}^{\frac{x^2}{2} - \ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2 \mathrm{e}^{\frac{x^2}{2}}}{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= c_2 \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2 \mathrm{e}^{\frac{x^2}{2}} \tag{1}
$$

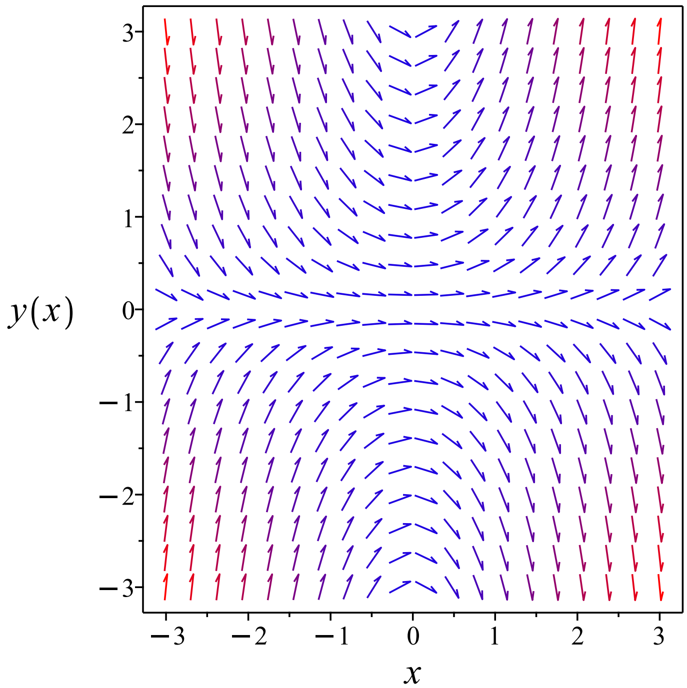


Figure 113: Slope field plot

Verification of solutions

$$y = c_2 e^{\frac{x^2}{2}}$$

Verified OK.

3.12.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = xy$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 94: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{\frac{x^2}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{x^2}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{x^2}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = xy$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -x\,e^{-\frac{x^2}{2}} y \\ S_y &= e^{-\frac{x^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-\frac{x^2}{2}} y = c_1$$

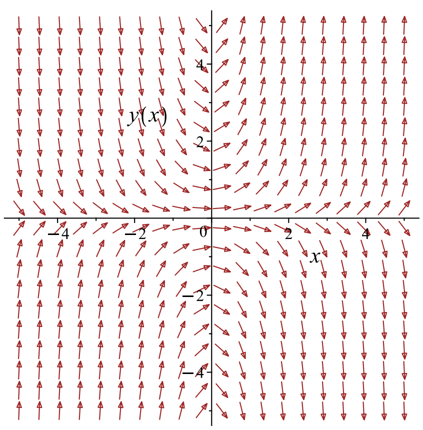
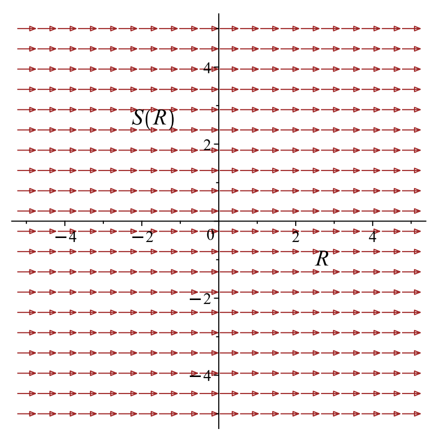
Which simplifies to

$$e^{-\frac{x^2}{2}} y = c_1$$

Which gives

$$y = c_1 e^{\frac{x^2}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = xy$ 	$R = x$ $S = e^{-\frac{x^2}{2}} y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{x^2}{2}} \tag{1}$$

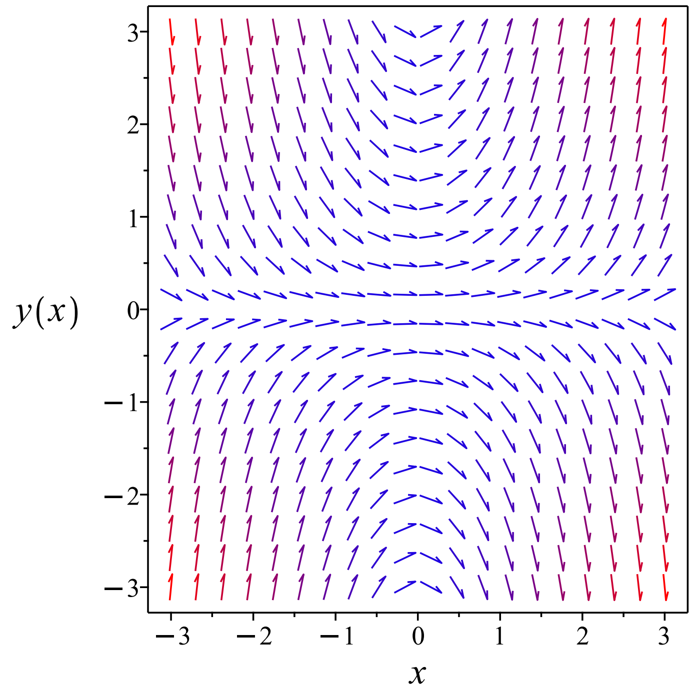


Figure 114: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x^2}{2}}$$

Verified OK.

3.12.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y &= (x)\,\mathrm{d}x \\ (-x)\,\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x \\ N(x, y) &= \frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \ln(y)$$

The solution becomes

$$y = e^{\frac{x^2}{2}+c_1}$$

Summary
The solution(s) found are the following

$$y = e^{\frac{x^2}{2}+c_1} \tag{1}$$

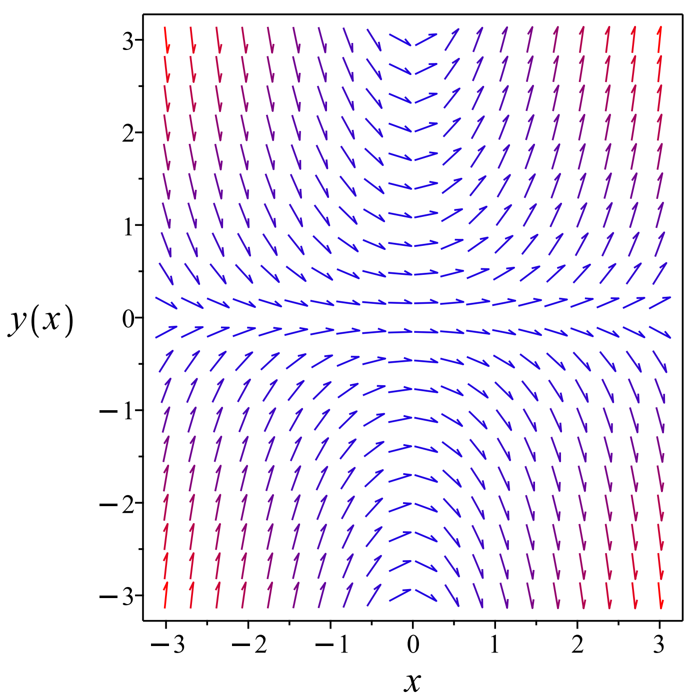


Figure 115: Slope field plot

Verification of solutions

$$y = e^{\frac{x^2}{2}+c_1}$$

Verified OK.

3.12.6 Maple step by step solution

Let's solve

$y' - yx = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int x dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{x^2}{2} + c_1$

- Solve for y

  $y = e^{\frac{x^2}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=x*y(x),y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{x^2}{2}}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 22

```
DSolve[y'[x]==x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{x^2}{2}}$$
$$y(x) \to 0$$

3.13 problem 8

Internal problem ID [12627]
Internal file name [`OUTPUT/11279_Friday_November_03_2023_06_29_38_AM_68378690/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differential-Type", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{x}{y} = 0}$$

3.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x}{y} \end{aligned}$$

Where $f(x) = x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\, dy = x\, dx$$

$$\int \frac{1}{\frac{1}{y}}\, dy = \int x\, dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c_1$$

Which results in

$$y = \sqrt{x^2 + 2c_1}$$

$$y = -\sqrt{x^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x^2 + 2c_1} \tag{1}$$

$$y = -\sqrt{x^2 + 2c_1} \tag{2}$$

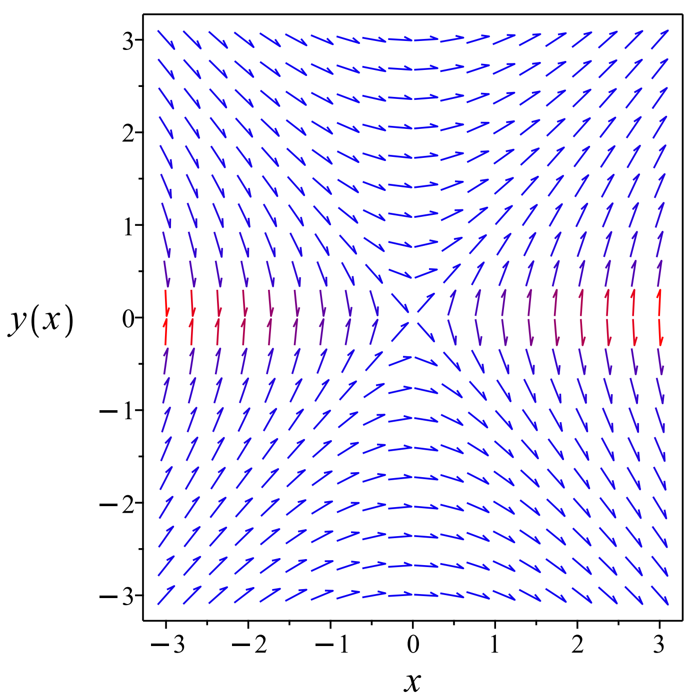


Figure 116: Slope field plot

Verification of solutions

$$y = \sqrt{x^2 + 2c_1}$$

Verified OK.

$$y = -\sqrt{x^2 + 2c_1}$$

Verified OK.

3.13.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{1}{u(x)} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2-1}{xu} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2-1}{u}}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{\frac{u^2-1}{u}}\,du &= \int -\frac{1}{x}\,dx \\ \frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} &= -\ln(x) + c_2 \end{aligned}$$

The above can be written as

$$\begin{aligned} \left(\frac{1}{2}\right)(\ln(u-1) + \ln(u+1)) &= -\ln(x) + 2c_2 \\ \ln(u-1) + \ln(u+1) &= (2)(-\ln(x) + 2c_2) \\ &= -2\ln(x) + 4c_2 \end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u-1)+\ln(u+1)} = e^{-2\ln(x)+2c_2}$$

Which simplifies to

$$
\begin{aligned}
u^2 - 1 &= \frac{2c_2}{x^2} \\
&= \frac{c_3}{x^2}
\end{aligned}
$$

The solution is

$$
u(x)^2 - 1 = \frac{c_3}{x^2}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\frac{y^2}{x^2} - 1 &= \frac{c_3}{x^2} \\
\frac{y^2}{x^2} - 1 &= \frac{c_3}{x^2}
\end{aligned}
$$

Which simplifies to

$$
-(-y + x)(y + x) = c_3
$$

<u>Summary</u>

The solution(s) found are the following

$$
-(-y + x)(y + x) = c_3 \tag{1}
$$

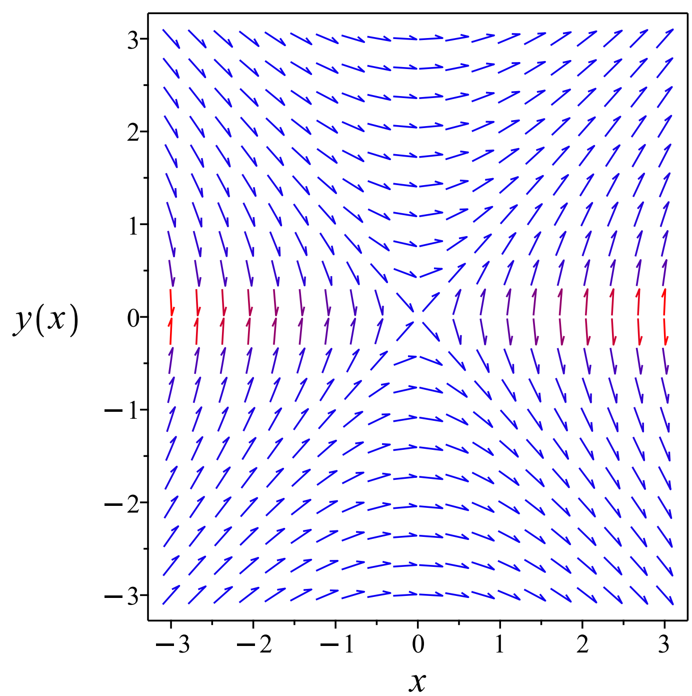


Figure 117: Slope field plot

Verification of solutions

$$-(-y+x)(y+x) = c_3$$

Verified OK.

3.13.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x}{y} \tag{1}$$

Which becomes

$$(y)\,dy = (x)\,dx \tag{2}$$

But the RHS is complete differential because

$$(x)\,dx = d\left(\frac{x^2}{2}\right)$$

Hence (2) becomes

$$(y)\,dy = d\left(\frac{x^2}{2}\right)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{x^2+2c_1} + c_1$$
$$y = -\sqrt{x^2+2c_1} + c_1$$

Summary
The solution(s) found are the following

$$y = \sqrt{x^2+2c_1} + c_1 \tag{1}$$
$$y = -\sqrt{x^2+2c_1} + c_1 \tag{2}$$

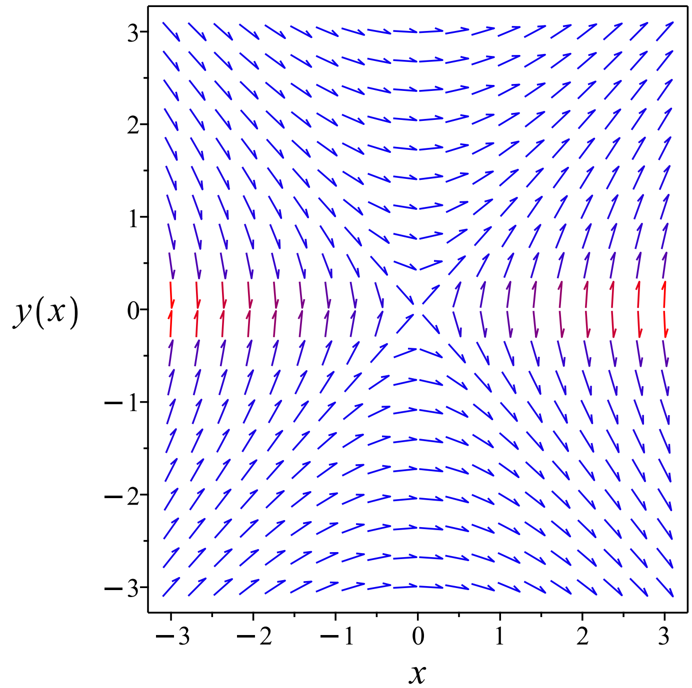


Figure 118: Slope field plot

Verification of solutions

$$y = \sqrt{x^2 + 2c_1} + c_1$$

Verified OK.

$$y = -\sqrt{x^2 + 2c_1} + c_1$$

Verified OK.

3.13.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 97: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{x} \\
\eta(x,y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

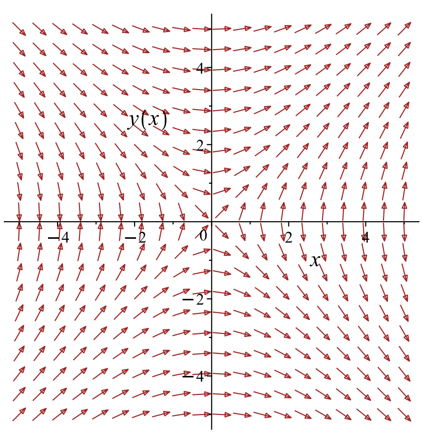
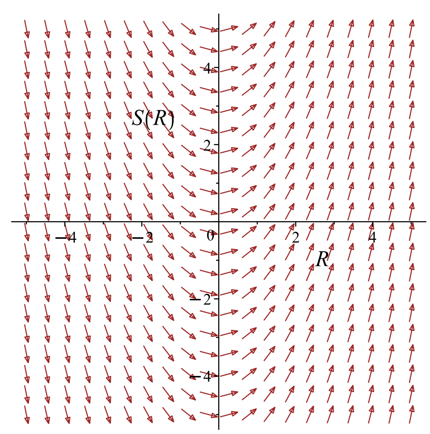
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x}{y}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1 \tag{1}$$

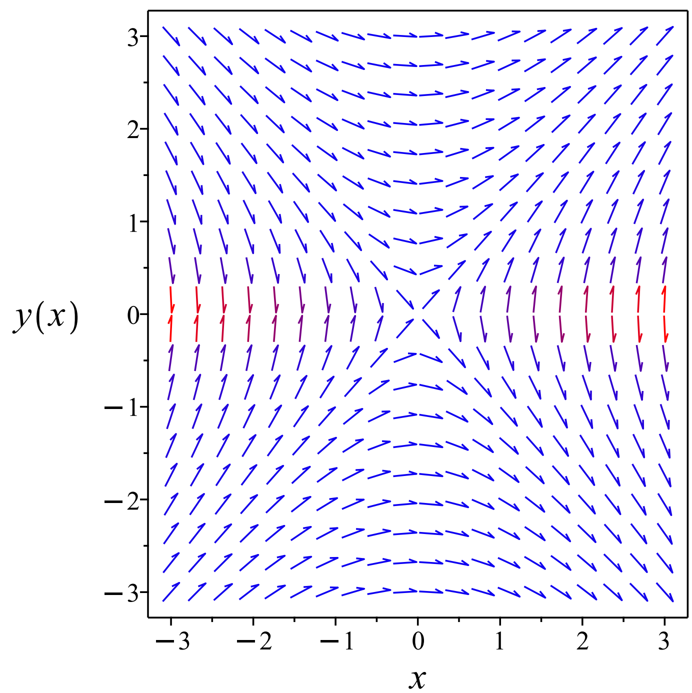


Figure 119: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Verified OK.

3.13.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1A}$$

Therefore

$$(y)\, dy = (x)\, dx$$
$$(-x)\, dx + (y)\, dy = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(y)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y$. Therefore equation (4) becomes

$$y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y) \, \mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} + \frac{y^2}{2} = c_1 \tag{1}$$

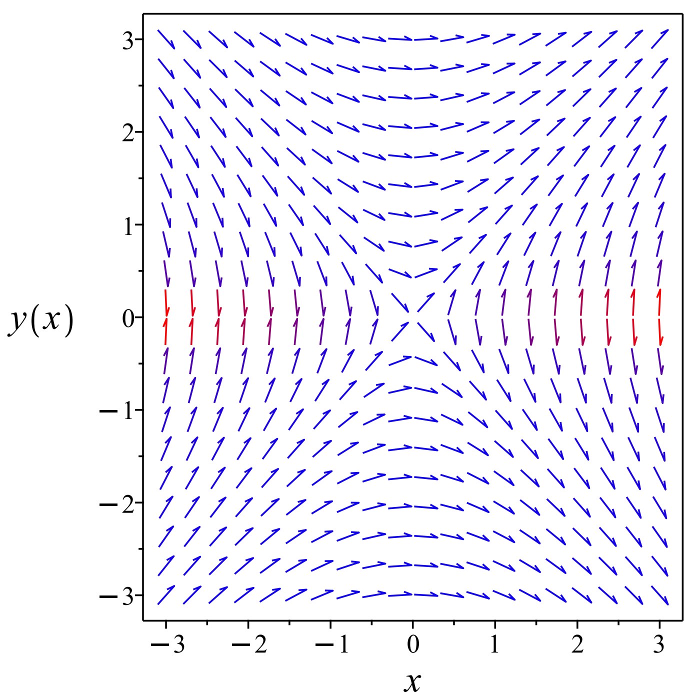


Figure 120: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} + \frac{y^2}{2} = c_1$$

Verified OK.

3.13.6 Maple step by step solution

Let's solve

$y' - \frac{x}{y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = x$

- Integrate both sides with respect to x

  $\int yy'dx = \int xdx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = \sqrt{x^2 + 2c_1}, y = -\sqrt{x^2 + 2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x)=x/y(x),y(x), singsol=all)
```

$$y(x) = \sqrt{x^2 + c_1}$$
$$y(x) = -\sqrt{x^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.127 (sec). Leaf size: 35

```
DSolve[y'[x]==x/y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x^2 + 2c_1}$$
$$y(x) \to \sqrt{x^2 + 2c_1}$$

3.14 problem 9

Internal problem ID [12628]
Internal file name [`OUTPUT/11280_Friday_November_03_2023_06_29_39_AM_84437884/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y}{x} = 0}$$

3.14.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\,dy = \frac{1}{x}\,dx$$
$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$
$$\ln(y) = \ln(x) + c_1$$
$$y = e^{\ln(x)+c_1}$$
$$= c_1 x$$

Summary

The solution(s) found are the following

$$y = c_1 x \tag{1}$$

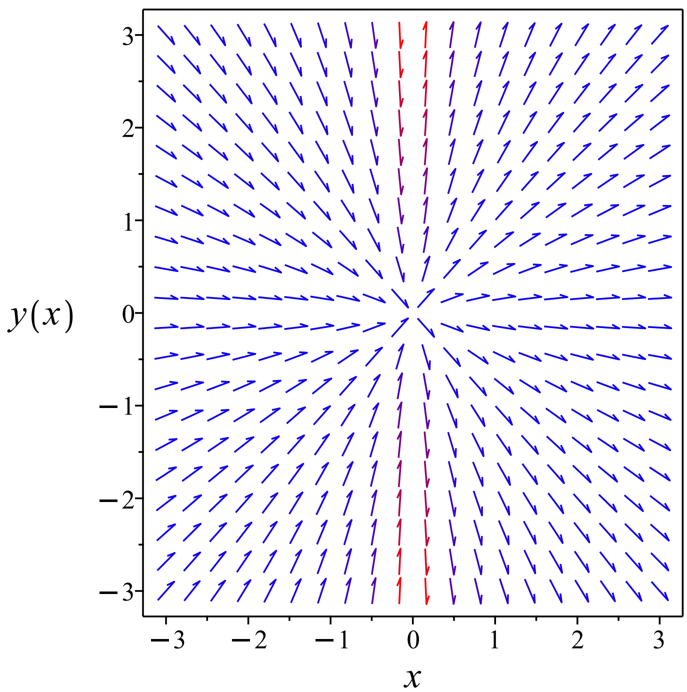


Figure 121: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

3.14.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x$$

Summary
The solution(s) found are the following

$$y = c_1 x \tag{1}$$

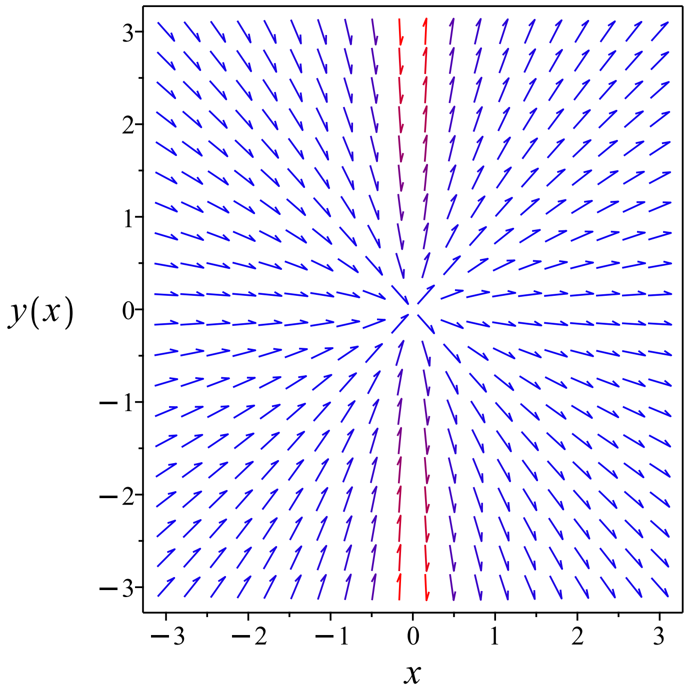


Figure 122: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

3.14.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x = 0$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0\,\mathrm{d}x \\ &= c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= c_2 x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x \tag{1}$$

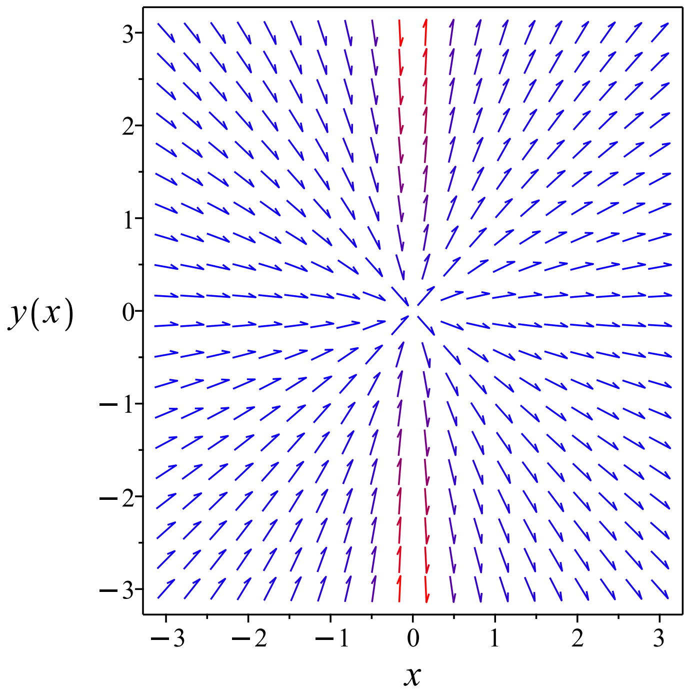


Figure 123: Slope field plot

Verification of solutions

$$y = c_2 x$$

Verified OK.

3.14.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 100: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = c_1$$

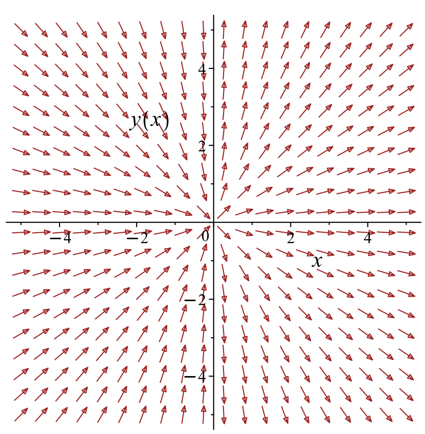
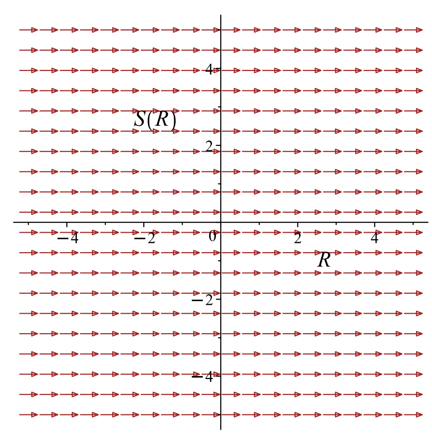
Which simplifies to

$$\frac{y}{x} = c_1$$

Which gives

$$y = c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 x \tag{1}$$

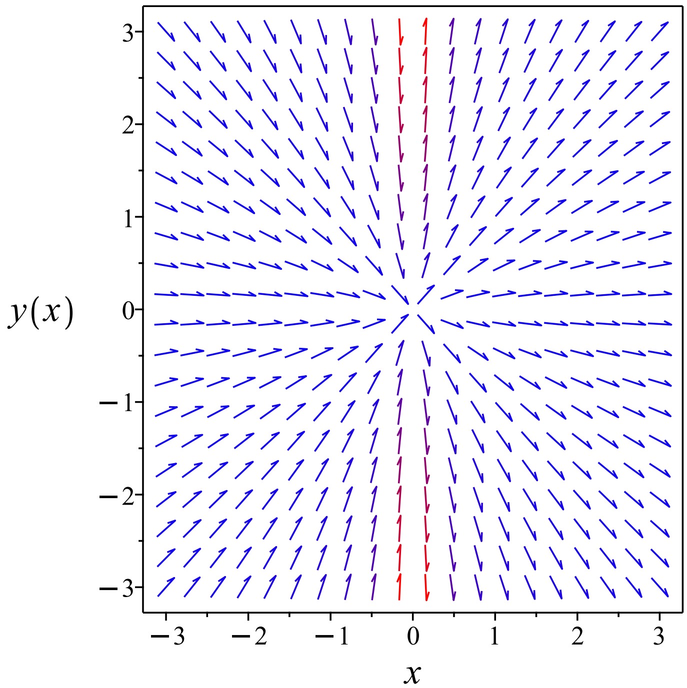


Figure 124: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 x$$

Verified OK.

3.14.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y}\right) \mathrm{d}y &= \left(\frac{1}{x}\right) \mathrm{d}x \\
\left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{1}{y}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= \frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x \\
\phi &= -\ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y}\right)\mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y)$$

The solution becomes

$$y = e^{c_1} x$$

Summary

The solution(s) found are the following

$$y = e^{c_1} x \tag{1}$$

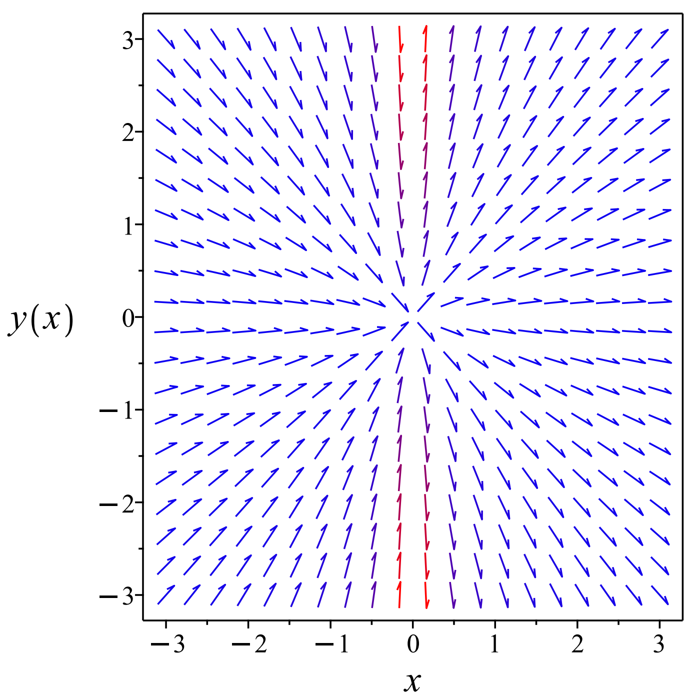


Figure 125: Slope field plot

Verification of solutions

$$y = e^{c_1} x$$

Verified OK.

3.14.6 Maple step by step solution

Let's solve

$y' - \frac{y}{x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = e^{c_1} x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve(diff(y(x),x)=y(x)/x,y(x), singsol=all)
```

$$y(x) = c_1 x$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 14

```
DSolve[y'[x]==y[x]/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x$$
$$y(x) \to 0$$

3.15 problem 10

Internal problem ID [12629]
Internal file name [`OUTPUT/11281_Friday_November_03_2023_06_29_40_AM_41313843/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 = 1}$$

3.15.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2+1} dy = x + c_1$$
$$\arctan(y) = x + c_1$$

Solving for y gives these solutions

$$y_1 = \tan(x + c_1)$$

Summary
The solution(s) found are the following

$$y = \tan(x + c_1) \tag{1}$$

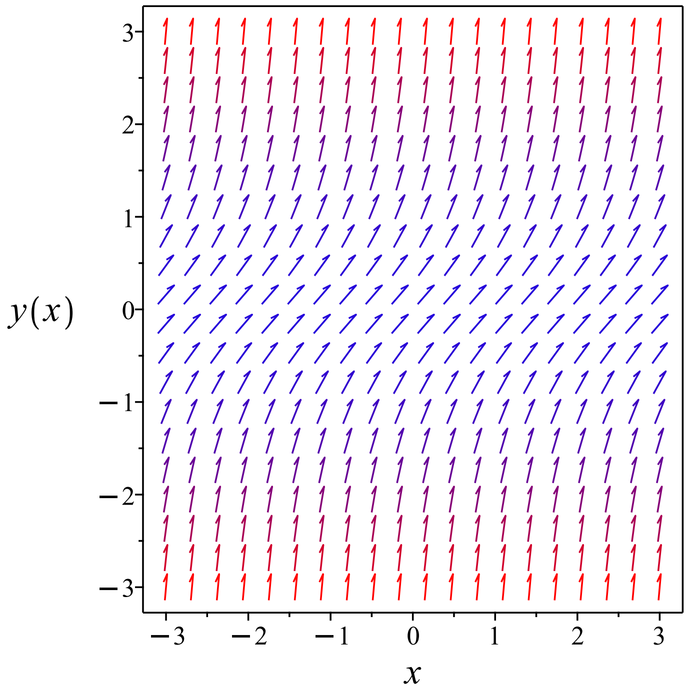


Figure 126: Slope field plot

Verification of solutions

$$y = \tan(x + c_1)$$

Verified OK.

3.15.2 Maple step by step solution

Let's solve

$y' - y^2 = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int 1 dx + c_1$

- Evaluate integral

$$\arctan(y) = x + c_1$$

- Solve for y

$$y = \tan(x + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(y(x),x)=1+y(x)^2,y(x), singsol=all)
```

$$y(x) = \tan(c_1 + x)$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 24

```
DSolve[y'[x]==1+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan(x + c_1)$$
$$y(x) \to -i$$
$$y(x) \to i$$

3.16 problem 11

Internal problem ID [12630]
Internal file name [OUTPUT/11282_Friday_November_03_2023_06_29_40_AM_69834863/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y^2 + 3y = 0}$$

3.16.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2 - 3y} dy = \int dx$$
$$\frac{\ln(y-3)}{3} - \frac{\ln(y)}{3} = x + c_1$$

The above can be written as

$$\left(\frac{1}{3}\right)(\ln(y-3) - \ln(y)) = x + c_1$$
$$\ln(y-3) - \ln(y) = (3)(x + c_1)$$
$$= 3c_1 + 3x$$

Raising both side to exponential gives

$$e^{\ln(y-3)-\ln(y)} = 3c_1 e^{3x}$$

Which simplifies to

$$\frac{y-3}{y} = e^{3x} c_2$$

Summary

The solution(s) found are the following

$$y = -\frac{3}{-1 + e^{3x} c_2} \tag{1}$$

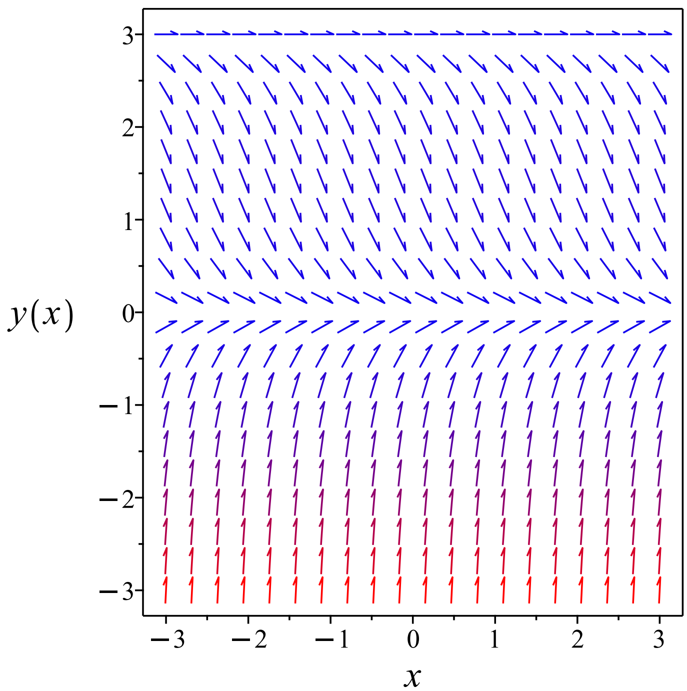


Figure 127: Slope field plot

Verification of solutions

$$y = -\frac{3}{-1 + e^{3x} c_2}$$

Verified OK.

3.16.2 Maple step by step solution

Let's solve

$y' - y^2 + 3y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2-3y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2-3y}dx = \int 1dx + c_1$

- Evaluate integral

  $\frac{\ln(y-3)}{3} - \frac{\ln(y)}{3} = x + c_1$

- Solve for y

  $y = -\frac{3}{e^{3c_1+3x}-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)=y(x)^2-3*y(x),y(x), singsol=all)
```

$$y(x) = \frac{3}{1 + 3c_1 e^{3x}}$$

✓ Solution by Mathematica

Time used: 0.352 (sec). Leaf size: 29

```
DSolve[y'[x]==y[x]^2-3*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3}{1 + e^{3(x+c_1)}}$$
$$y(x) \to 0$$
$$y(x) \to 3$$

3.17 problem 12

Internal problem ID [12631]
Internal file name [`OUTPUT/11283_Friday_November_03_2023_06_29_41_AM_22483055/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"abelFirstKind"**

Maple gives the following as the ode type

`[_Abel]`

Unable to solve or complete the solution.

$$\boxed{y' - y^3 = x^3}$$

3.17.1 Solving as abelFirstKind ode

This is Abel first kind ODE, it has the form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Comparing the above to given ODE which is

$$y' = x^3 + y^3 \tag{1}$$

Therefore

$$\begin{aligned} f_0(x) &= x^3 \\ f_1(x) &= 0 \\ f_2(x) &= 0 \\ f_3(x) &= 1 \end{aligned}$$

Since $f_2(x) = 0$ then we check the Abel invariant to see if it depends on x or not. The Abel invariant is given by

$$-\frac{f_1^3}{f_0^2 f_3}$$

Which when evaluating gives

$$\frac{1}{x^9}$$

Since the Abel invariant depends on x then unable to solve this ode at this time.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=x^3+y(x)^3,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==x^3+y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

3.18 problem 13

Internal problem ID [12632]
Internal file name [`OUTPUT/11284_Friday_November_03_2023_06_29_41_AM_62814595/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - |y| = 0}$$

3.18.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{|y|} dy = x + c_1$$

$$\begin{cases} -\ln(y) & y < 0 \\ \text{undefined} & y = 0 \\ \ln(y) & 0 < y \end{cases} = x + c_1$$

Solving for y gives these solutions

$$\begin{aligned} y_1 &= \mathrm{e}^{-x-c_1} \\ &= \frac{\mathrm{e}^{-x}}{c_1} \\ y_2 &= \mathrm{e}^{x+c_1} \\ &= \mathrm{e}^{x} c_1 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-x}}{c_1} \tag{1}$$

$$y = e^{x} c_1 \tag{2}$$

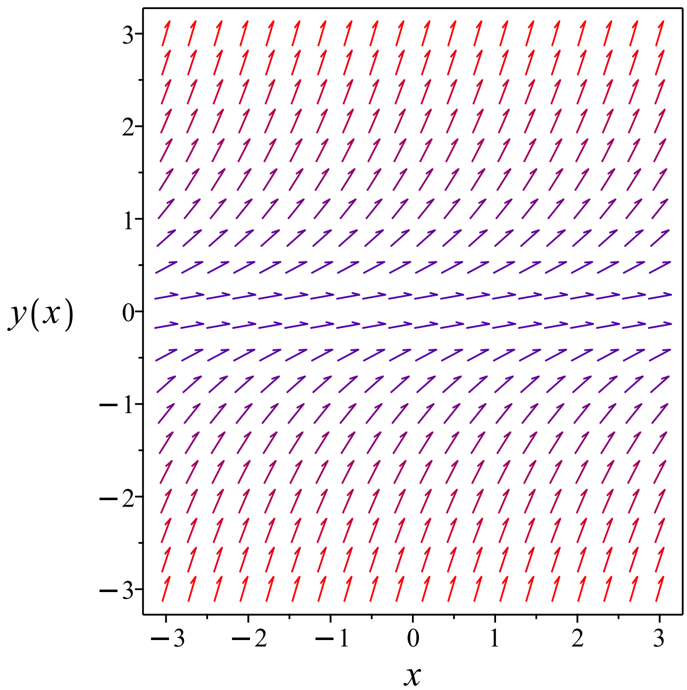


Figure 128: Slope field plot

Verification of solutions

$$y = \frac{e^{-x}}{c_1}$$

Verified OK.

$$y = e^{x} c_1$$

Verified OK.

3.18.2 Maple step by step solution

Let's solve

$y' - |y| = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{|y|} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{|y|}dx = \int 1dx + c_1$

- Evaluate integral

  $\left\{\begin{array}{cc} -\ln\left(y\right) & y<0 \\ \mathit{undefined} & y=0 \\ \ln\left(y\right) & 0<y \end{array}\right. = x + c_1$

- Solve for y

  $\{y = \mathrm{e}^{-x-c_1}, y = \mathrm{e}^{x+c_1}\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)=abs(y(x)),y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= \frac{e^{-x}}{c_1} \\
y(x) &= c_1 e^{x}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.229 (sec). Leaf size: 29

```
DSolve[y'[x]==Abs[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \text{InverseFunction}\left[\int_1^{\#1} \frac{1}{|K[1]|} dK[1]\&\right][x + c_1] \\
y(x) &\to 0
\end{aligned}
$$

3.19 problem 14

Internal problem ID [12633]
Internal file name [`OUTPUT/11285_Friday_November_03_2023_06_29_43_AM_42828604/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - e^{-y+x} = 0}$$

3.19.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= e^{-y}e^{x} \end{aligned}$$

Where $f(x) = e^{x}$ and $g(y) = e^{-y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{e^{-y}}\, dy &= e^{x}\, dx \\ \int \frac{1}{e^{-y}}\, dy &= \int e^{x}\, dx \\ e^{y} &= e^{x} + c_1 \end{aligned}$$

Which results in

$$y = \ln\left(e^{x} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \ln\left(e^{x} + c_1\right) \tag{1}$$

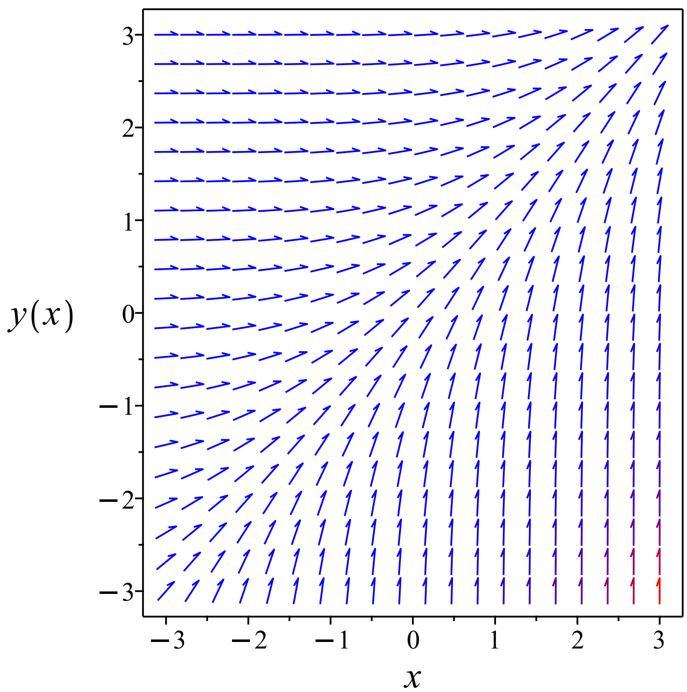


Figure 129: Slope field plot

Verification of solutions

$$y = \ln\left(e^{x} + c_1\right)$$

Verified OK.

3.19.2 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = \mathrm{e}^{-y+x} \tag{1}$$

And using the substitution $u = \mathrm{e}^{y}$ then

$$u' = y'\mathrm{e}^{y}$$

The above shows that

$$\begin{aligned} y' &= u'(x)\,\mathrm{e}^{-y} \\ &= \frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$\frac{u'(x)}{u} = \frac{\mathrm{e}^{x}}{u}$$

The above simplifies to

$$u'(x) = \mathrm{e}^{x} \tag{2}$$

Now ode (2) is solved for $u(x)$ Integrating both sides gives

$$\begin{aligned} u(x) &= \int \mathrm{e}^{x}\,\mathrm{d}x \\ &= \mathrm{e}^{x} + c_1 \end{aligned}$$

Substituting the solution found for $u(x)$ in $u = \mathrm{e}^{y}$ gives

$$\begin{aligned} y &= \ln\left(u(x)\right) \\ &= \ln\left(\mathrm{e}^{x} + c_1\right) \\ &= \ln\left(\mathrm{e}^{x} + c_1\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \ln\left(\mathrm{e}^{x} + c_1\right) \tag{1}$$

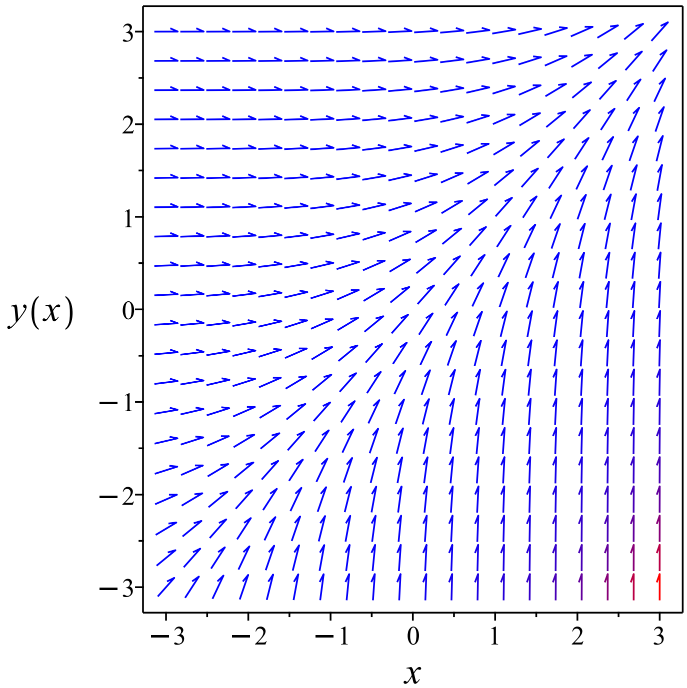


Figure 130: Slope field plot

Verification of solutions

$$y = \ln\left(e^{x} + c_1\right)$$

Verified OK.

3.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = e^{-y+x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 106: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \mathrm{e}^{-x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{e^{-x}} dx \end{aligned}$$

Which results in

$$S = e^{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = e^{-y+x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= e^{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x} = e^{y} + c_1$$

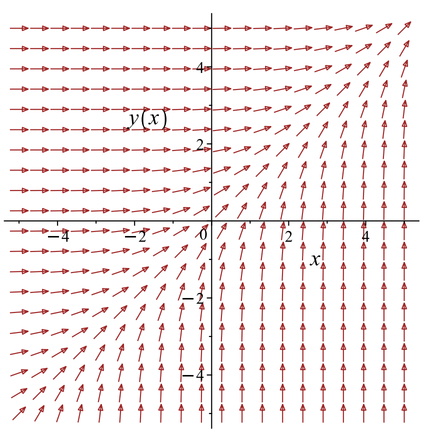
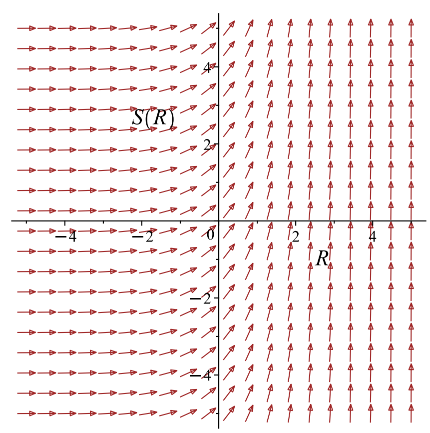
Which simplifies to

$$e^{x} = e^{y} + c_1$$

Which gives

$$y = \ln\left(e^{x} - c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = e^{-y+x}$ 	$R = y$ $S = e^{x}$	$\frac{dS}{dR} = e^{R}$ 

Summary
The solution(s) found are the following

$$y = \ln\left(e^{x} - c_1\right) \tag{1}$$

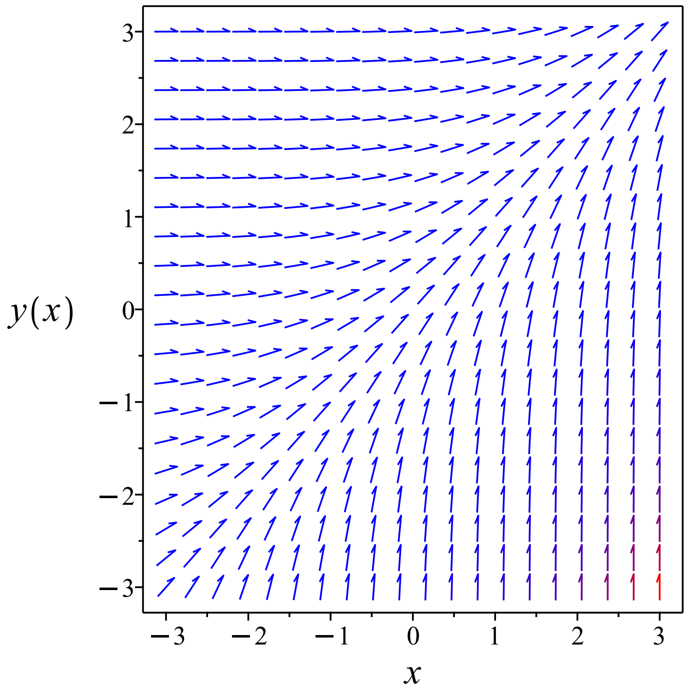


Figure 131: Slope field plot

Verification of solutions

$$y = \ln\left(e^{x} - c_1\right)$$

Verified OK.

3.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^y)\,\mathrm{d}y &= (\mathrm{e}^x)\,\mathrm{d}x\\ (-\mathrm{e}^x)\,\mathrm{d}x + (\mathrm{e}^y)\,\mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\mathrm{e}^x\\ N(x, y) &= \mathrm{e}^y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\mathrm{e}^x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\mathrm{e}^{x} \mathrm{d}x$$
$$\phi = -\mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{y}$. Therefore equation (4) becomes

$$\mathrm{e}^{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \mathrm{e}^{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int (\mathrm{e}^{y}) \mathrm{d}y$$
$$f(y) = \mathrm{e}^{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\mathrm{e}^{x} + \mathrm{e}^{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -e^x + e^y$$

The solution becomes

$$y = \ln(e^x + c_1)$$

Summary

The solution(s) found are the following

$$y = \ln(e^x + c_1) \tag{1}$$

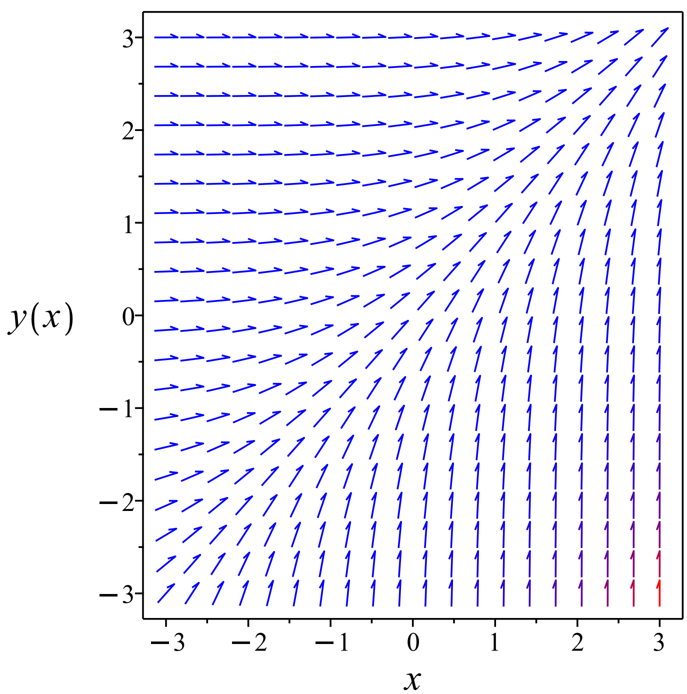


Figure 132: Slope field plot

Verification of solutions

$$y = \ln(e^x + c_1)$$

Verified OK.

3.19.5 Maple step by step solution

Let's solve

$y' - e^{-y+x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' e^{y} = e^{x}$

- Integrate both sides with respect to x

  $\int y' e^{y} dx = \int e^{x} dx + c_1$

- Evaluate integral

  $e^{y} = e^{x} + c_1$

- Solve for y

  $y = \ln\left(e^{x} + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(diff(y(x),x)=exp(x-y(x)),y(x), singsol=all)
```

$$y(x) = \ln\left(e^{x} + c_1\right)$$

✓ Solution by Mathematica

Time used: 1.319 (sec). Leaf size: 12

```
DSolve[y'[x]==Exp[x-y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log\left(e^x + c_1\right)$$

3.20 problem 15

Internal problem ID [12634]
Internal file name [`OUTPUT/11286_Friday_November_03_2023_06_29_43_AM_51568707/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \ln(y + x) = 0}$$

3.20.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \ln(y + x) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \ln(y+x)(b_3 - a_2) - \ln(y+x)^2 a_3 - \frac{x a_2 + y a_3 + a_1}{y+x} - \frac{x b_2 + y b_3 + b_1}{y+x} = 0 \quad \text{(5E)}$$

Putting the above in normal form gives

$$-\frac{\ln(y+x)^2 a_3 x + \ln(y+x)^2 a_3 y + \ln(y+x) x a_2 - \ln(y+x) x b_3 + \ln(y+x) y a_2 - \ln(y+x) y b_3 + x a_2}{y+x}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-\ln(y+x)^2 a_3 x - \ln(y+x)^2 a_3 y - \ln(y+x) x a_2 + \ln(y+x) x b_3 \\ &- \ln(y+x) y a_2 + \ln(y+x) y b_3 - x a_2 - y a_3 + b_2 y - y b_3 - a_1 - b_1 = 0 \end{aligned} \quad \text{(6E)}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \ln(y+x)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \ln(y+x) = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-v_3^2 a_3 v_1 - v_3^2 a_3 v_2 - v_3 v_1 a_2 - v_3 v_2 a_2 + v_3 v_1 b_3 \\ &+ v_3 v_2 b_3 - v_1 a_2 - v_2 a_3 + b_2 v_2 - v_2 b_3 - a_1 - b_1 = 0 \end{aligned} \quad \text{(7E)}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &-v_3^2 a_3 v_1 + (b_3 - a_2) v_1 v_3 - v_1 a_2 - v_3^2 a_3 v_2 \\ &+ (b_3 - a_2) v_2 v_3 + (-a_3 + b_2 - b_3) v_2 - a_1 - b_1 = 0 \end{aligned} \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-a_2 &= 0 \\
-a_3 &= 0 \\
-a_1 - b_1 &= 0 \\
b_3 - a_2 &= 0 \\
-a_3 + b_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -b_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -1 \\
\eta &= 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 1 - (\ln (y + x))\,(-1) \\
&= 1 + \ln (y + x) \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{1 + \ln (y + x)} dy \end{aligned}$$

Which results in

$$S = -\mathrm{e}^{-1} \operatorname{expIntegral}_1 (-\ln (y + x) - 1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \ln (y + x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{1 + \ln (y + x)} \\ S_y &= \frac{1}{1 + \ln (y + x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

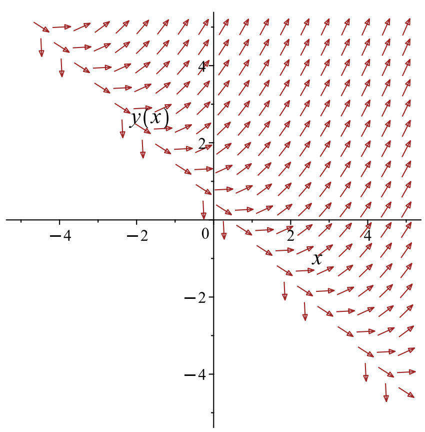
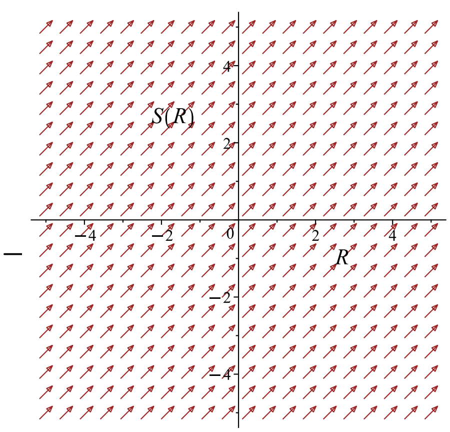
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-e^{-1} \operatorname{expIntegral}_1\left(-\ln\left(y+x\right)-1\right) = x + c_1$$

Which simplifies to

$$-e^{-1} \operatorname{expIntegral}_1\left(-\ln\left(y+x\right)-1\right) = x + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \ln\left(y+x\right)$ 	$R = x$ $S = -e^{-1} \operatorname{expIntegral}_1 (-$	$\frac{dS}{dR} = 1$ 

Summary
The solution(s) found are the following

$$-e^{-1} \operatorname{expIntegral}_1\left(-\ln\left(y+x\right)-1\right) = x + c_1 \tag{1}$$

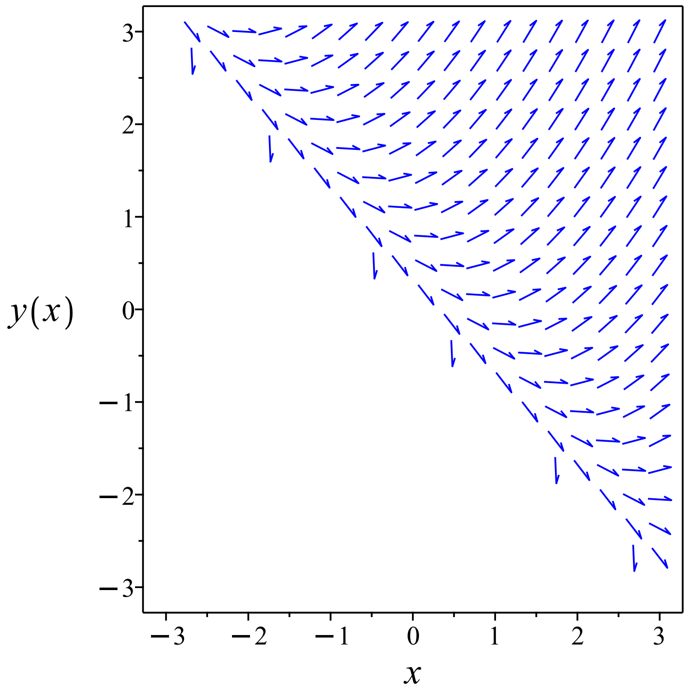


Figure 133: Slope field plot

Verification of solutions

$$-\mathrm{e}^{-1}\,\mathrm{expIntegral}_1\left(-\ln\left(y+x\right)-1\right)=x+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 29

```
dsolve(diff(y(x),x)=ln(x+y(x)),y(x), singsol=all)
```

$$y(x) = e^{\text{RootOf}\left(c_1 e - x\, e - \text{expIntegral}_1(-\_Z - 1)\right)} - x$$

✓ Solution by Mathematica

Time used: 0.207 (sec). Leaf size: 22

```
DSolve[y'[x]==Log[x+y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{\text{ExpIntegralEi}(\log(x+y(x))+1)}{e} - x = c_1, y(x)\right]$$

3.21 problem 16

Internal problem ID [12635]
Internal file name [`OUTPUT/11287_Friday_November_03_2023_06_29_44_AM_69163266/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{2x - y}{3y + x} = 0}$$

3.21.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{2x - u(x)\,x}{3u\,(x)\,x + x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u^2 + 2u - 2}{x\,(3u + 1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{3u^2+2u-2}{3u+1}$. Integrating both sides gives

$$\frac{1}{\frac{3u^2+2u-2}{3u+1}}\, du = -\frac{1}{x}\, dx$$
$$\int \frac{1}{\frac{3u^2+2u-2}{3u+1}}\, du = \int -\frac{1}{x}\, dx$$
$$\frac{\ln\left(3u^2+2u-2\right)}{2} = -\ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$\sqrt{3u^2+2u-2} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{3u^2+2u-2} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{3u\left(x\right)^2 + 2u\left(x\right) - 2} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{3u\left(x\right)^2 + 2u\left(x\right) - 2} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{3y^2}{x^2} + \frac{2y}{x} - 2} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$
$$\sqrt{\frac{3y^2+2yx-2x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{3y^2+2yx-2x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x} \tag{1}$$

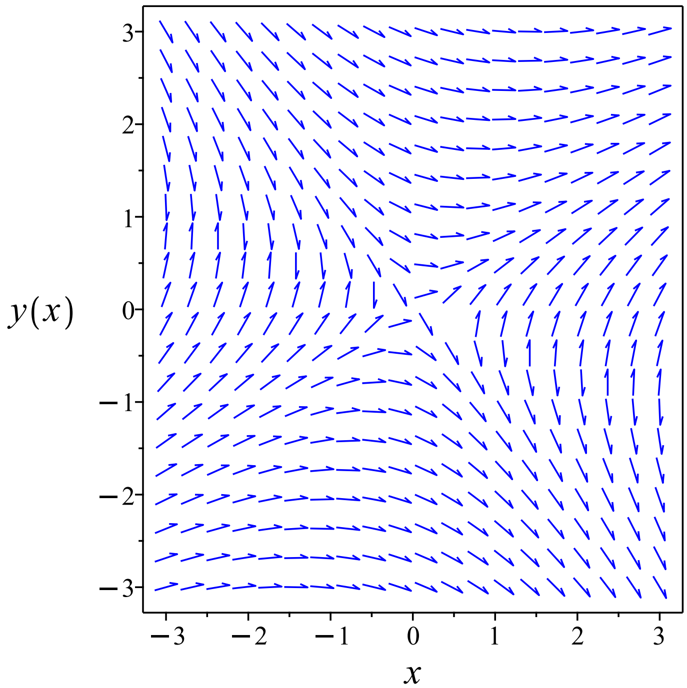


Figure 134: Slope field plot

Verification of solutions

$$\sqrt{\frac{3y^2+2yx-2x^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

3.21.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{2x-y}{3y+x} \tag{1}$$

Which becomes

$$(3y)\,dy = (-x)\,dy + (2x-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (2x-y)\,dx = d\left(x^2 - xy\right)$$

Hence (2) becomes

$$(3y)\,dy = d\left(x^2 - xy\right)$$

Integrating both sides gives gives these solutions

$$y = -\frac{x}{3} + \frac{\sqrt{7x^2 + 6c_1}}{3} + c_1$$
$$y = -\frac{x}{3} - \frac{\sqrt{7x^2 + 6c_1}}{3} + c_1$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{3} + \frac{\sqrt{7x^2 + 6c_1}}{3} + c_1 \tag{1}$$
$$y = -\frac{x}{3} - \frac{\sqrt{7x^2 + 6c_1}}{3} + c_1 \tag{2}$$

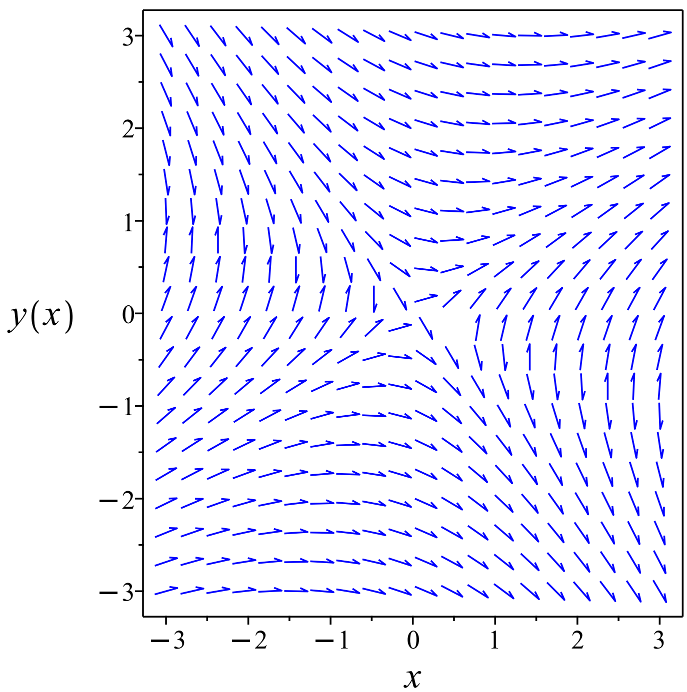


Figure 135: Slope field plot

Verification of solutions

$$y = -\frac{x}{3} + \frac{\sqrt{7x^2 + 6c_1}}{3} + c_1$$

Verified OK.

$$y = -\frac{x}{3} - \frac{\sqrt{7x^2 + 6c_1}}{3} + c_1$$

Verified OK.

3.21.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{-2x + y}{3y + x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(-2x + y)(b_3 - a_2)}{3y + x} - \frac{(-2x + y)^2 a_3}{(3y + x)^2} \\ & - \left(\frac{2}{3y + x} + \frac{-2x + y}{(3y + x)^2}\right)(x a_2 + y a_3 + a_1) \\ & - \left(-\frac{1}{3y + x} + \frac{-6x + 3y}{(3y + x)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2a_2 + 4x^2a_3 - 8x^2b_2 - 2x^2b_3 + 12xya_2 - 4xya_3 - 6xyb_2 - 12xyb_3 - 3y^2a_2 + 8y^2a_3 - 9y^2b_2 + 3y^2b_3 -}{(3y+x)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$-2x^2a_2 - 4x^2a_3 + 8x^2b_2 + 2x^2b_3 - 12xya_2 + 4xya_3 + 6xyb_2 + 12xyb_3 + 3y^2a_2 - 8y^2a_3 + 9y^2b_2 - 3y^2b_3 + 7xb_1 - 7ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2v_1^2 - 12a_2v_1v_2 + 3a_2v_2^2 - 4a_3v_1^2 + 4a_3v_1v_2 - 8a_3v_2^2 + 8b_2v_1^2 + 6b_2v_1v_2 + 9b_2v_2^2 + 2b_3v_1^2 + 12b_3v_1v_2 - 3b_3v_2^2 - 7a_1v_2 + 7b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-2a_2 - 4a_3 + 8b_2 + 2b_3)\,v_1^2 + (-12a_2 + 4a_3 + 6b_2 + 12b_3)\,v_1v_2 + 7b_1v_1 + (3a_2 - 8a_3 + 9b_2 - 3b_3)\,v_2^2 - 7a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-7a_1 &= 0\\
7b_1 &= 0\\
-12a_2 + 4a_3 + 6b_2 + 12b_3 &= 0\\
-2a_2 - 4a_3 + 8b_2 + 2b_3 &= 0\\
3a_2 - 8a_3 + 9b_2 - 3b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_2 + b_3 \\ a_3 &= \frac{3b_2}{2} \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{-2x+y}{3y+x}\right)(x) \\ &= \frac{-2x^2+2xy+3y^2}{3y+x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2x^2+2xy+3y^2}{3y+x}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(-2x^2 + 2xy + 3y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-2x + y}{3y + x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2x - y}{2x^2 - 2xy - 3y^2} \\
S_y &= \frac{-3y - x}{2x^2 - 2xy - 3y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

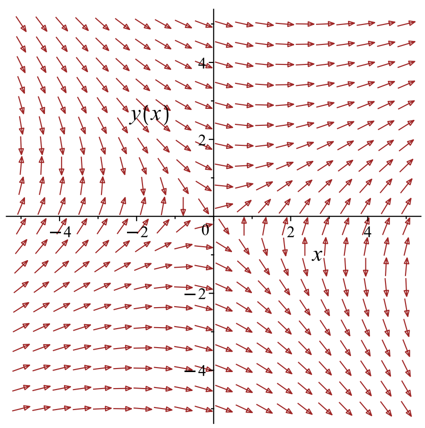
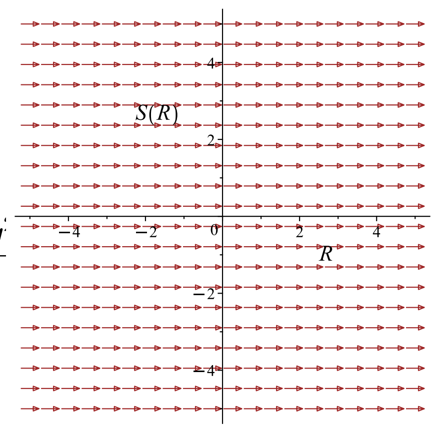
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(3y^2 + 2yx - 2x^2)}{2} = c_1$$

Which simplifies to

$$\frac{\ln\left(3y^2+2yx-2x^2\right)}{2}=c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=-\frac{-2x+y}{3y+x}$ 	$R = x$ $S = \frac{\ln\left(-2x^2+2xy+3y\right.}{2}$	$\frac{dS}{dR}=0$ 

Summary

The solution(s) found are the following

$$\frac{\ln\left(3y^2+2yx-2x^2\right)}{2}=c_1 \tag{1}$$

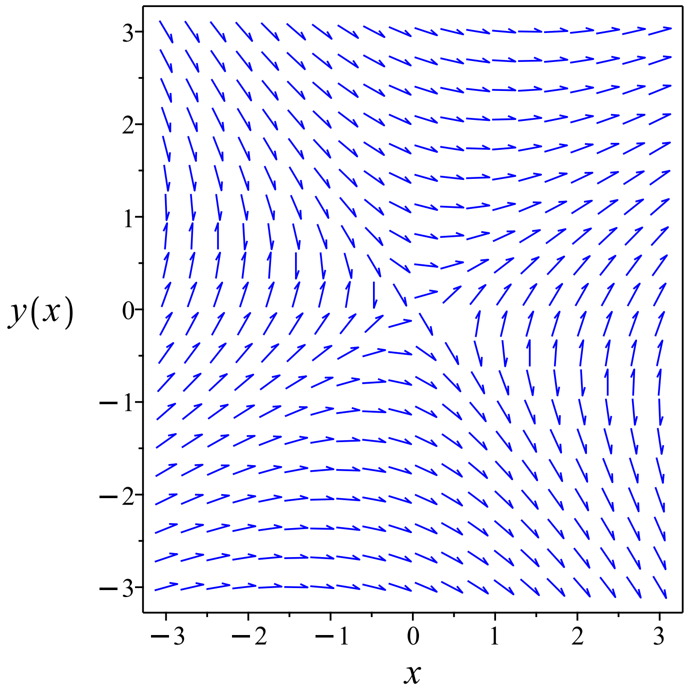


Figure 136: Slope field plot

Verification of solutions

$$\frac{\ln(3y^2 + 2yx - 2x^2)}{2} = c_1$$

Verified OK.

3.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3y + x) \, \mathrm{d}y &= (2x - y) \, \mathrm{d}x \\ (-2x + y) \, \mathrm{d}x + (3y + x) \, \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -2x + y$$
$$N(x, y) = 3y + x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2x + y) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3y + x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -2x + y \, \mathrm{d}x$$

$$\phi = -x(-y + x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y + x$. Therefore equation (4) becomes

$$3y + x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 3y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (3y) \, \mathrm{d}y$$

$$f(y) = \frac{3y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x(-y + x) + \frac{3y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x(-y + x) + \frac{3y^2}{2}$$

Summary

The solution(s) found are the following

$$-x(-y + x) + \frac{3y^2}{2} = c_1 \tag{1}$$

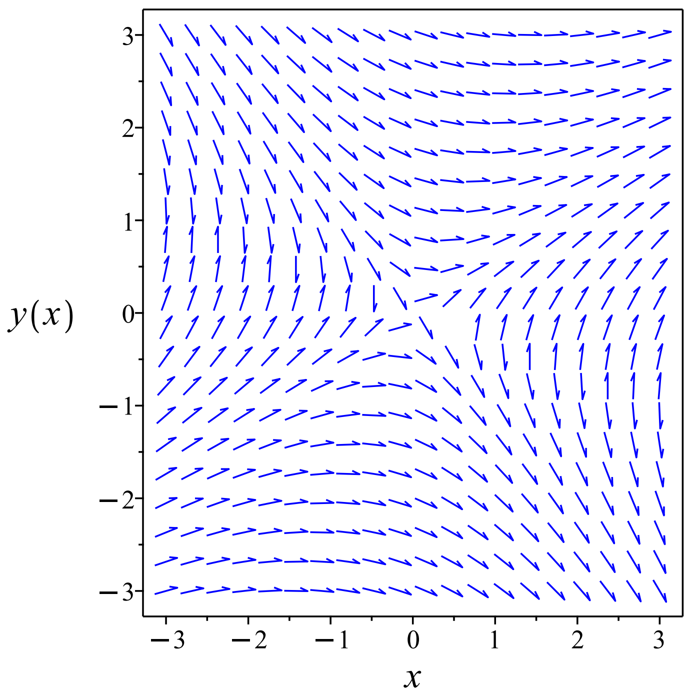


Figure 137: Slope field plot

Verification of solutions

$$-x(-y + x) + \frac{3y^2}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.079 (sec). Leaf size: 53

```
dsolve(diff(y(x),x)=(2*x-y(x))/(x+3*y(x)),y(x), singsol=all)
```

$$y(x) = \frac{-c_1 x - \sqrt{7c_1^2 x^2 + 3}}{3c_1}$$
$$y(x) = \frac{-c_1 x + \sqrt{7c_1^2 x^2 + 3}}{3c_1}$$

✓ Solution by Mathematica

Time used: 0.812 (sec). Leaf size: 114

```
DSolve[y'[x]==(2*x-y[x])/(x+3*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}\left(-x - \sqrt{7x^2 + 3e^{2c_1}}\right)$$
$$y(x) \to \frac{1}{3}\left(-x + \sqrt{7x^2 + 3e^{2c_1}}\right)$$
$$y(x) \to \frac{1}{3}\left(-\sqrt{7}\sqrt{x^2} - x\right)$$
$$y(x) \to \frac{1}{3}\left(\sqrt{7}\sqrt{x^2} - x\right)$$

3.22 problem 17

Internal problem ID [12636]
Internal file name [`OUTPUT/11288_Friday_November_03_2023_06_29_45_AM_12419405/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.1, page 40
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$y' - \frac{1}{\sqrt{15 - x^2 - y^2}} = 0$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x)=1/sqrt(15-x^2-y(x)^2),y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]==1/Sqrt[15-x^2-y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

Not solved

4 Chapter 2. The Initial Value Problem. Exercises 2.2, page 53

4.1 problem 1

Internal problem ID [12637]
Internal file name [`OUTPUT/11289_Friday_November_03_2023_06_29_46_AM_16542071/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{3y}{(-5+x)(x+3)} = \mathrm{e}^{-x}}$$

4.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{(-5+x)(x+3)}$$
$$q(x) = \mathrm{e}^{-x}$$

Hence the ode is

$$y' - \frac{3y}{(-5+x)(x+3)} = \mathrm{e}^{-x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{(-5+x)(x+3)}dx}$$
$$= e^{-\frac{3\ln(-5+x)}{8}+\frac{3\ln(x+3)}{8}}$$

Which simplifies to

$$\mu = \frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(e^{-x}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(x+3)^{\frac{3}{8}}y}{(-5+x)^{\frac{3}{8}}}\right) = \left(\frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}\right)\left(e^{-x}\right)$$
$$\mathrm{d}\left(\frac{(x+3)^{\frac{3}{8}}y}{(-5+x)^{\frac{3}{8}}}\right) = \left(\frac{e^{-x}(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}\right)\mathrm{d}x$$

Integrating gives

$$\frac{(x+3)^{\frac{3}{8}}y}{(-5+x)^{\frac{3}{8}}} = \int \frac{e^{-x}(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}\,\mathrm{d}x$$
$$\frac{(x+3)^{\frac{3}{8}}y}{(-5+x)^{\frac{3}{8}}} = \int \frac{e^{-x}(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}dx + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}$ results in

$$y = \frac{(-5+x)^{\frac{3}{8}}\left(\int \frac{e^{-x}(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}dx\right)}{(x+3)^{\frac{3}{8}}} + \frac{c_1(-5+x)^{\frac{3}{8}}}{(x+3)^{\frac{3}{8}}}$$

which simplifies to

$$y = \frac{(-5+x)^{\frac{3}{8}}\left(\int \frac{e^{-x}(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}dx + c_1\right)}{(x+3)^{\frac{3}{8}}}$$

Summary

The solution(s) found are the following

$$y = \frac{(-5+x)^{\frac{3}{8}}\left(\int \frac{e^{-x}(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}dx + c_1\right)}{(x+3)^{\frac{3}{8}}} \tag{1}$$

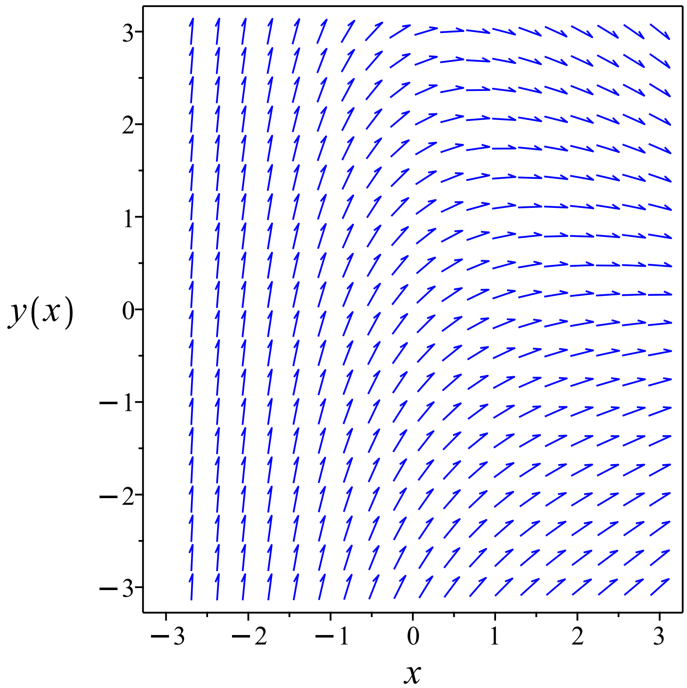


Figure 138: Slope field plot

Verification of solutions

$$y = \frac{(-5+x)^{\frac{3}{8}} \left(\int \frac{e^{-x}(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}} dx + c_1\right)}{(x+3)^{\frac{3}{8}}}$$

Verified OK.

4.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 e^{-x} - 2x e^{-x} - 15 e^{-x} + 3y}{(-5+x)(x+3)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 109: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{\frac{3\ln(-5+x)}{8} - \frac{3\ln(x+3)}{8}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{3\ln(-5+x)}{8} - \frac{3\ln(x+3)}{8}}} dy \end{aligned}$$

Which results in

$$S = e^{\ln\left(\frac{1}{(-5+x)^{\frac{3}{8}}}\right) + \ln\left((x+3)^{\frac{3}{8}}\right)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2 e^{-x} - 2x e^{-x} - 15 e^{-x} + 3y}{(-5+x)(x+3)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3y}{(-5+x)^{\frac{11}{8}} (x+3)^{\frac{5}{8}}} \\ S_y &= \frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{-x}(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{-R}(R+3)^{\frac{3}{8}}}{(-5+R)^{\frac{3}{8}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \int \frac{e^{-R}(R+3)^{\frac{3}{8}}}{(-5+R)^{\frac{3}{8}}} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(x+3)^{\frac{3}{8}} y}{(-5+x)^{\frac{3}{8}}} = \int \frac{e^{-x}(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}} dx + c_1$$

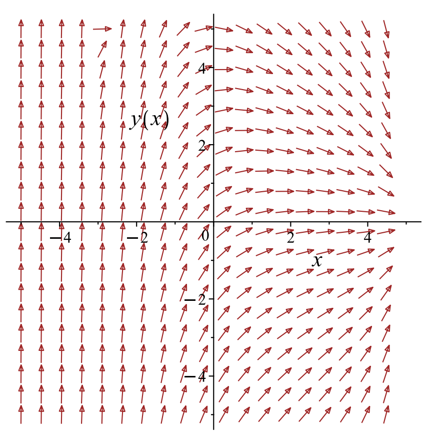
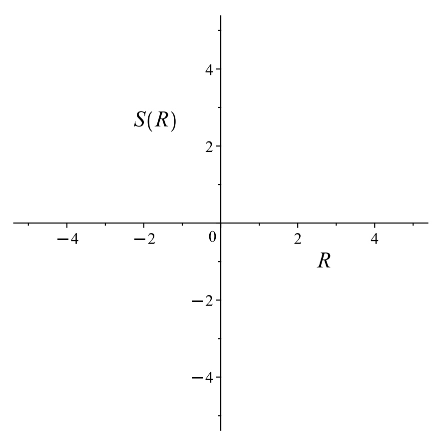
Which simplifies to

$$\frac{(x+3)^{\frac{3}{8}} y}{(-5+x)^{\frac{3}{8}}} = \int \frac{e^{-x}(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}} dx + c_1$$

Which gives

$$y = \frac{(-5+x)^{\frac{3}{8}} \left(\int \frac{e^{-x}(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}} dx + c_1 \right)}{(x+3)^{\frac{3}{8}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{x^2 e^{-x} - 2x e^{-x} - 15 e^{-x} + 3y}{(-5+x)(x+3)}$

</td><td>$R = x$
$S = \frac{(x+3)^{\frac{3}{8}} y}{(-5+x)^{\frac{3}{8}}}$</td><td>$\frac{dS}{dR} = \frac{e^{-R}(R+3)^{\frac{3}{8}}}{(-5+R)^{\frac{3}{8}}}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{(-5+x)^{\frac{3}{8}} \left(\int \frac{e^{-x}(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}dx + c_1\right)}{(x+3)^{\frac{3}{8}}} \tag{1}$$

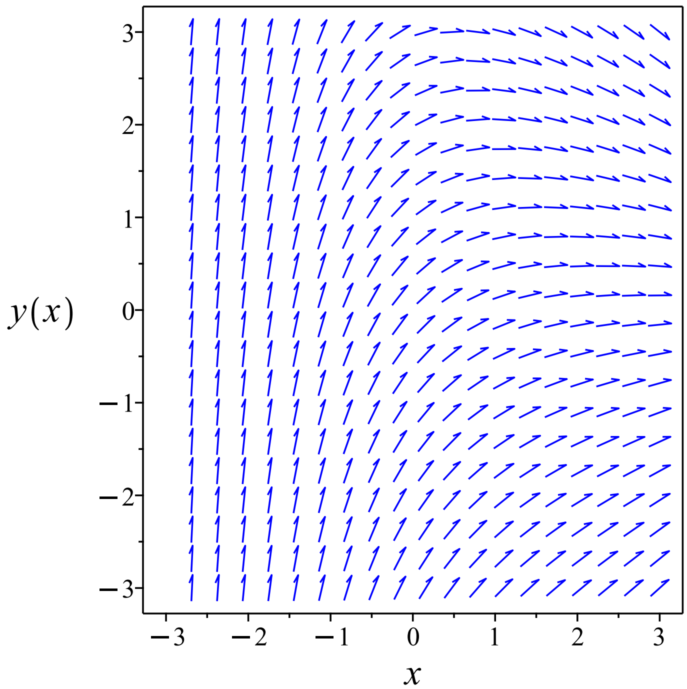


Figure 139: Slope field plot

Verification of solutions

$$y = \frac{(-5+x)^{\frac{3}{8}} \left(\int \frac{e^{-x}(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}dx + c_1\right)}{(x+3)^{\frac{3}{8}}}$$

Verified OK.

4.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(\frac{3y}{(-5+x)(x+3)}+\mathrm{e}^{-x}\right)\mathrm{d}x$$

$$\left(-\frac{3y}{(-5+x)(x+3)}-\mathrm{e}^{-x}\right)\mathrm{d}x+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{3y}{(-5+x)(x+3)} - e^{-x}\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{3y}{(-5+x)(x+3)} - e^{-x}\right)\\ &= -\frac{3}{(-5+x)(x+3)}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(-\frac{3}{(-5+x)(x+3)}\right) - (0)\right)\\ &= -\frac{3}{(-5+x)(x+3)}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{3}{(-5+x)(x+3)}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{3\ln(-5+x)}{8} + \frac{3\ln(x+3)}{8}}\\ &= \frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}\left(-\frac{3y}{(-5+x)(x+3)} - \mathrm{e}^{-x}\right) \\ &= -\frac{(x^2-2x-15)\,\mathrm{e}^{-x}+3y}{(-5+x)^{\frac{11}{8}}(x+3)^{\frac{5}{8}}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}(1) \\ &= \frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\frac{(x^2-2x-15)\,\mathrm{e}^{-x}+3y}{(-5+x)^{\frac{11}{8}}(x+3)^{\frac{5}{8}}}\right) + \left(\frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{(x^2-2x-15)\,\mathrm{e}^{-x}+3y}{(-5+x)^{\frac{11}{8}}(x+3)^{\frac{5}{8}}}\,\mathrm{d}x\end{aligned}$$

$$\phi = \int^{x} -\frac{(\_a^2-2\_a-15)\,\mathrm{e}^{-\_a}+3y}{(-5+\_a)^{\frac{11}{8}}(\_a+3)^{\frac{5}{8}}}d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}$. Therefore equation (4) becomes

$$\frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}} = \frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^x -\frac{(\_a^2 - 2\_a - 15)\,\mathrm{e}^{-\_a} + 3y}{(-5+\_a)^{\frac{11}{8}}\,(\_a+3)^{\frac{5}{8}}} d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^x -\frac{(\_a^2 - 2\_a - 15)\,\mathrm{e}^{-\_a} + 3y}{(-5+\_a)^{\frac{11}{8}}\,(\_a+3)^{\frac{5}{8}}} d\_a$$

Summary

The solution(s) found are the following

$$\int^x -\frac{(\_a^2 - 2\_a - 15)\,\mathrm{e}^{-\_a} + 3y}{(-5+\_a)^{\frac{11}{8}}\,(\_a+3)^{\frac{5}{8}}} d\_a = c_1 \tag{1}$$

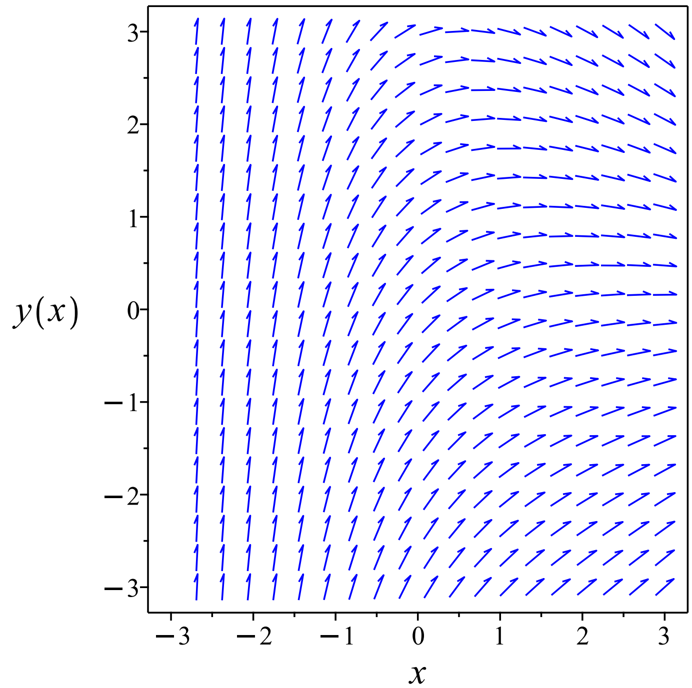


Figure 140: Slope field plot

Verification of solutions

$$\int^{x} -\frac{(\_a^2 - 2\_a - 15)\,\mathrm{e}^{-\_a} + 3y}{(-5+\_a)^{\frac{11}{8}}\,(\_a+3)^{\frac{5}{8}}} d\_a = c_1$$

Verified OK.

4.1.4 Maple step by step solution

Let's solve

$y' - \frac{3y}{(-5+x)(x+3)} = \mathrm{e}^{-x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{3y}{(-5+x)(x+3)} + \mathrm{e}^{-x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{3y}{(-5+x)(x+3)} = \mathrm{e}^{-x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $$\mu(x)\left(y' - \frac{3y}{(-5+x)(x+3)}\right) = \mu(x)\,\mathrm{e}^{-x}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $$\mu(x)\left(y' - \frac{3y}{(-5+x)(x+3)}\right) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

  $$\mu'(x) = -\frac{3\mu(x)}{(-5+x)(x+3)}$$

- Solve to find the integrating factor

  $$\mu(x) = \frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}$$

- Integrate both sides with respect to x

  $$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,\mathrm{e}^{-x} dx + c_1$$

- Evaluate the integral on the lhs

  $$\mu(x)\,y = \int \mu(x)\,\mathrm{e}^{-x} dx + c_1$$

- Solve for y

  $$y = \frac{\int \mu(x)\mathrm{e}^{-x} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}}$

  $$y = \frac{(-5+x)^{\frac{3}{8}}\left(\int \frac{\mathrm{e}^{-x}(x+3)^{\frac{3}{8}}}{(-5+x)^{\frac{3}{8}}} dx + c_1\right)}{(x+3)^{\frac{3}{8}}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve(diff(y(x),x)=3*y(x)/( (x-5)*(x+3))+exp(-x),y(x), singsol=all)
```

$$y(x) = \frac{\left(\int \frac{e^{-x}(x+3)^{\frac{3}{8}}}{(x-5)^{\frac{3}{8}}}dx + c_1\right)(x-5)^{\frac{3}{8}}}{(x+3)^{\frac{3}{8}}}$$

✓ Solution by Mathematica

Time used: 15.323 (sec). Leaf size: 57

```
DSolve[y'[x]==3*y[x]/( (x-5)*(x+3))+Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(5-x)^{3/8}\left(\int_1^x \frac{e^{-K[1]}(K[1]+3)^{3/8}}{(5-K[1])^{3/8}}dK[1] + c_1\right)}{(x+3)^{3/8}}$$

4.2 problem 2

Internal problem ID [12638]
Internal file name [`OUTPUT/11290_Friday_November_03_2023_06_29_47_AM_65750474/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y' - \frac{xy}{x^2 + y^2} = 0}$$

4.2.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - \frac{x^2 u(x)}{x^2 + u(x)^2\, x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^3}{x\,(u^2+1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^3}{u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^3}{u^2+1}}\, du = -\frac{1}{x}\, dx$$
$$\int \frac{1}{\frac{u^3}{u^2+1}}\, du = \int -\frac{1}{x}\, dx$$
$$-\frac{1}{2u^2} + \ln(u) = -\ln(x) + c_2$$

The solution is

$$-\frac{1}{2u(x)^2} + \ln(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x^2}{2y^2} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$
$$-\frac{x^2}{2y^2} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2y^2} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0 \tag{1}$$

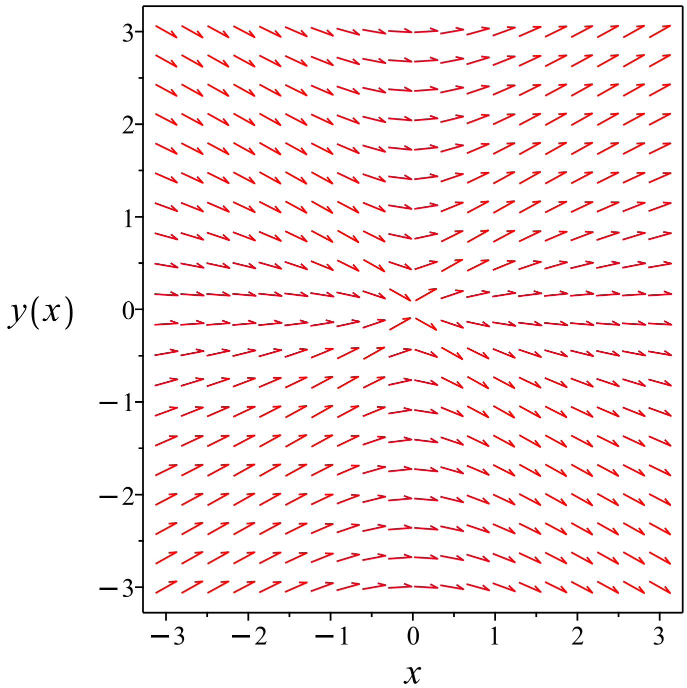


Figure 141: Slope field plot

Verification of solutions

$$-\frac{x^2}{2y^2} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Verified OK.

4.2.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{xy}{x^2 + y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{xy(b_3 - a_2)}{x^2 + y^2} - \frac{x^2 y^2 a_3}{(x^2 + y^2)^2} - \left(\frac{y}{x^2 + y^2} - \frac{2x^2 y}{(x^2 + y^2)^2}\right)(x a_2 + y a_3 + a_1) - \left(\frac{x}{x^2 + y^2} - \frac{2x\, y^2}{(x^2 + y^2)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-3x^2 y^2 b_2 + 2x\, y^3 a_2 - 2x\, y^3 b_3 + y^4 a_3 - y^4 b_2 + x^3 b_1 - x^2 y a_1 - x\, y^2 b_1 + y^3 a_1}{(x^2 + y^2)^2} = 0$$

Setting the numerator to zero gives

$$3x^2 y^2 b_2 - 2x\, y^3 a_2 + 2x\, y^3 b_3 - y^4 a_3 + y^4 b_2 - x^3 b_1 + x^2 y a_1 + x\, y^2 b_1 - y^3 a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2 v_1 v_2^3 - a_3 v_2^4 + 3b_2 v_1^2 v_2^2 + b_2 v_2^4 + 2b_3 v_1 v_2^3 + a_1 v_1^2 v_2 - a_1 v_2^3 - b_1 v_1^3 + b_1 v_1 v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_1 v_1^3 + 3b_2 v_1^2 v_2^2 + a_1 v_1^2 v_2 + (-2a_2 + 2b_3)\, v_1 v_2^3 + b_1 v_1 v_2^2 + (-a_3 + b_2)\, v_2^4 - a_1 v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ -a_1 &= 0 \\ -b_1 &= 0 \\ 3b_2 &= 0 \\ -2a_2 + 2b_3 &= 0 \\ -a_3 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\,\xi \\ &= y - \left(\frac{xy}{x^2 + y^2}\right)(x) \\ &= \frac{y^3}{x^2 + y^2} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^3}{x^2+y^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^2}{2y^2} + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{xy}{x^2+y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x}{y^2} \\ S_y &= \frac{x^2+y^2}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln(y)\, y^2 - x^2}{2y^2} = c_1$$

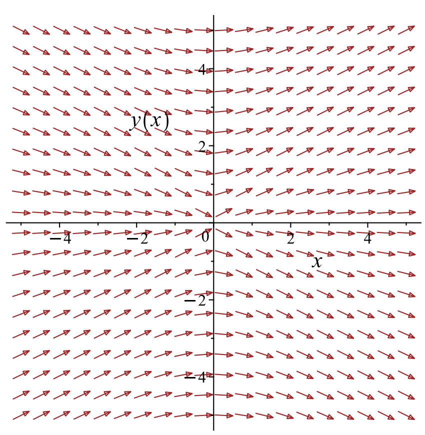
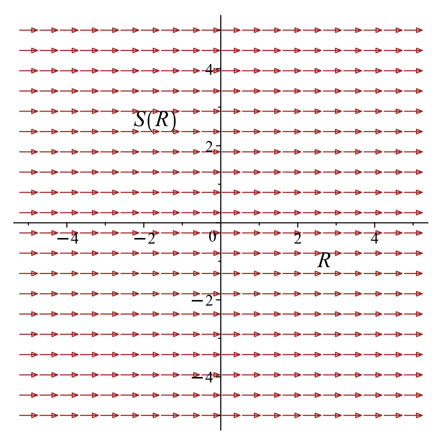
Which simplifies to

$$\frac{2\ln(y)\, y^2 - x^2}{2y^2} = c_1$$

Which gives

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{-2c_1} x^2\right)}{2} + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$ 	$R = x$ $S = \frac{2\ln(y)\, y^2 - x^2}{2y^2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{-2c_1} x^2\right)}{2}+c_1} \tag{1}$$

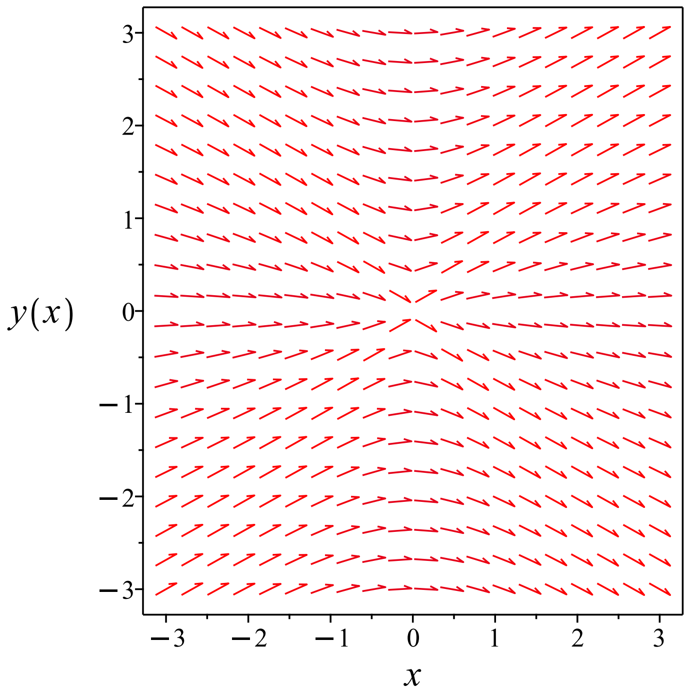


Figure 142: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{-2c_1} x^2\right)}{2}+c_1}$$

Verified OK.

4.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 + y^2)\,\mathrm{d}y &= (xy)\,\mathrm{d}x \\ (-xy)\,\mathrm{d}x + (x^2 + y^2)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -xy \\ N(x, y) &= x^2 + y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-xy) \\ &= -x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2+y^2)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^2+y^2}((-x)-(2x))\\ &= -\frac{3x}{x^2+y^2}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\frac{1}{xy}((2x)-(-x))\\ &= -\frac{3}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B\,\mathrm{d}y}\\ &= e^{\int -\frac{3}{y}\,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(y)}\\ &= \frac{1}{y^3}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{y^3}(-xy)\\ &= -\frac{x}{y^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{y^3}\left(x^2+y^2\right) \\ &= \frac{x^2+y^2}{y^3}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\frac{x}{y^2}\right) + \left(\frac{x^2+y^2}{y^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{y^2}\,\mathrm{d}x \\ \phi &= -\frac{x^2}{2y^2} + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x^2}{y^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2+y^2}{y^3}$. Therefore equation (4) becomes

$$\frac{x^2+y^2}{y^3} = \frac{x^2}{y^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2y^2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2y^2} + \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{-2c_1}x^2\right)}{2}+c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{-2c_1}x^2\right)}{2}+c_1} \tag{1}$$

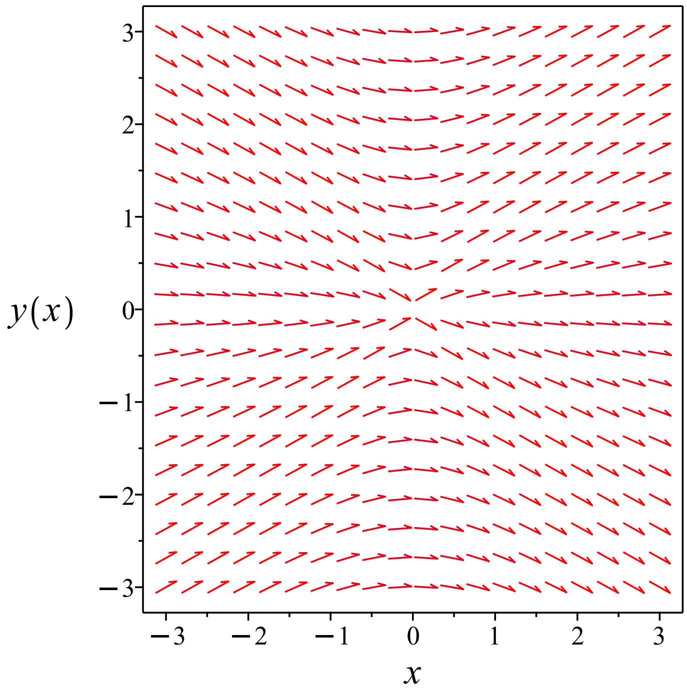


Figure 143: Slope field plot

Verification of solutions

$$y = e^{\frac{\operatorname{LambertW}\left(e^{-2c_1}x^2\right)}{2}+c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)=x*y(x)/( x^2+y(x)^2),y(x), singsol=all)
```

$$y(x) = \sqrt{\frac{1}{\text{LambertW}\left(c_1 x^2\right)}}\, x$$

✓ Solution by Mathematica

Time used: 11.187 (sec). Leaf size: 49

```
DSolve[y'[x]==x*y[x]/( x^2+y[x]^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{\sqrt{W\left(e^{-2c_1} x^2\right)}}$$
$$y(x) \to \frac{x}{\sqrt{W\left(e^{-2c_1} x^2\right)}}$$
$$y(x) \to 0$$

4.3 problem 3

Internal problem ID [12639]
Internal file name [`OUTPUT/11291_Friday_November_03_2023_06_29_53_AM_9194465/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{1}{yx} = 0}$$

4.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{1}{xy} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = \frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int \frac{1}{x}\,dx$$

$$\frac{y^2}{2} = \ln(x) + c_1$$

Which results in

$$y = \sqrt{2\ln(x) + 2c_1}$$

$$y = -\sqrt{2\ln(x) + 2c_1}$$

Summary
The solution(s) found are the following

$$y = \sqrt{2\ln(x) + 2c_1} \tag{1}$$

$$y = -\sqrt{2\ln(x) + 2c_1} \tag{2}$$

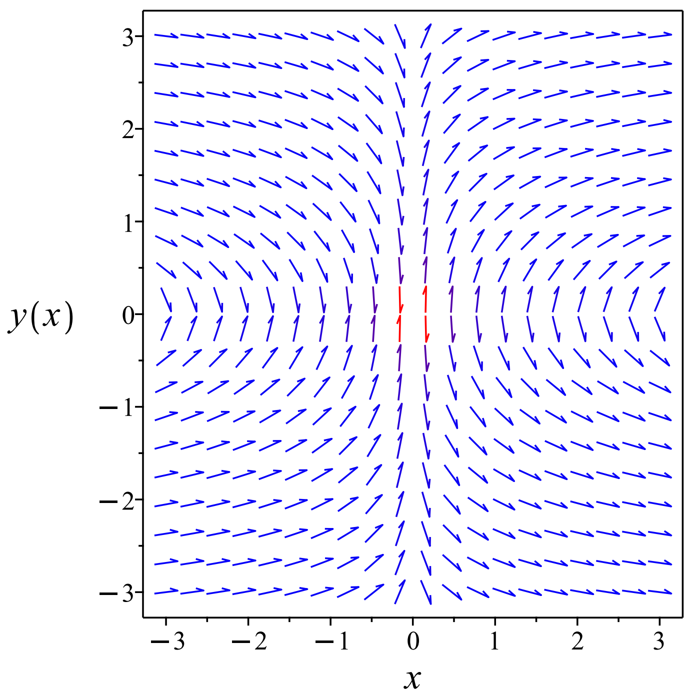


Figure 144: Slope field plot

Verification of solutions

$$y = \sqrt{2\ln(x) + 2c_1}$$

Verified OK.

$$y = -\sqrt{2\ln(x) + 2c_1}$$

Verified OK.

4.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{1}{xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 112: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= x\\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx\\ &= \int \frac{1}{x} dx\end{aligned}$$

Which results in

$$S = \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{1}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

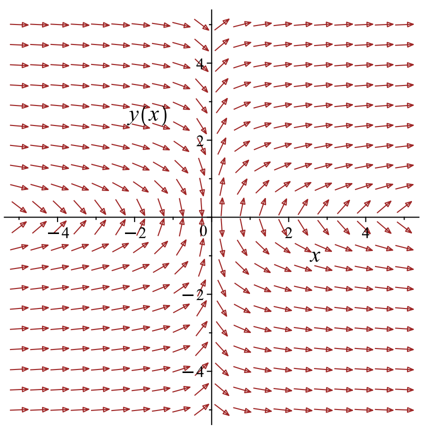
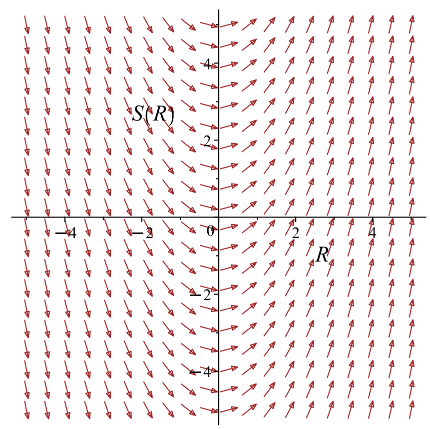
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\ln(x) = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1}{xy}$ 	$R = y$ $S = \ln(x)$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$\ln(x) = \frac{y^2}{2} + c_1 \tag{1}$$

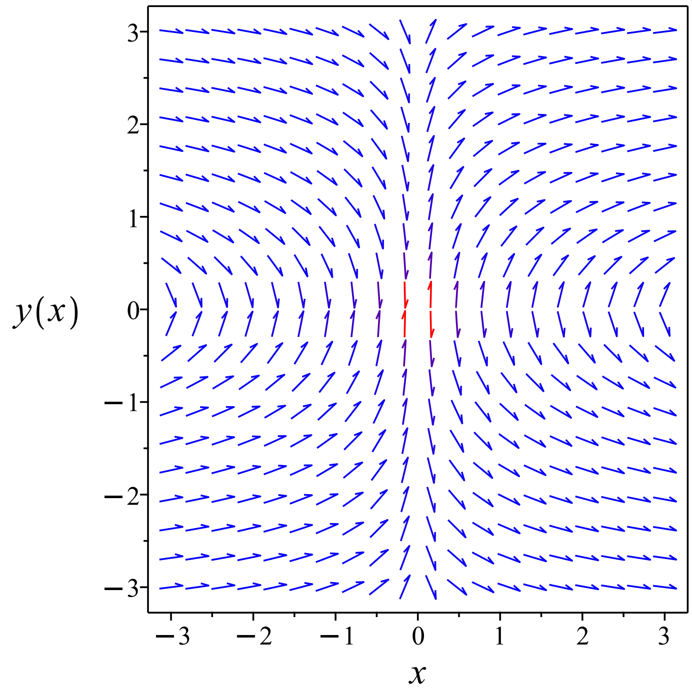


Figure 145: Slope field plot

Verification of solutions

$$\ln(x) = \frac{y^2}{2} + c_1$$

Verified OK.

4.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(y) \, \mathrm{d}y = \left(\frac{1}{x}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x}\right) \mathrm{d}x + (y) \, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y$. Therefore equation (4) becomes

$$
y = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int (y) \, \mathrm{d}y \\
f(y) &= \frac{y^2}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\ln(x) + \frac{y^2}{2} = c_1 \tag{1}$$

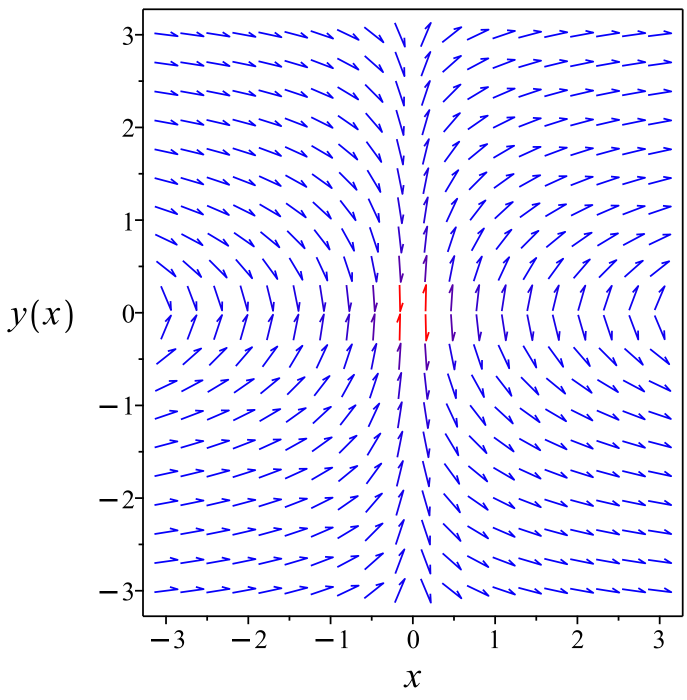


Figure 146: Slope field plot

Verification of solutions

$$-\ln(x) + \frac{y^2}{2} = c_1$$

Verified OK.

4.3.4 Maple step by step solution

Let's solve

$y' - \frac{1}{yx} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int yy'dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \ln(x) + c_1$

- Solve for y

  $\left\{y = \sqrt{2\ln(x) + 2c_1}, y = -\sqrt{2\ln(x) + 2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve(diff(y(x),x)=1/( x*y(x)),y(x), singsol=all)
```

$$y(x) = \sqrt{2\ln(x) + c_1}$$
$$y(x) = -\sqrt{2\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.1 (sec). Leaf size: 40

```
DSolve[y'[x]==1/(x*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2}\sqrt{\log(x) + c_1}$$
$$y(x) \to \sqrt{2}\sqrt{\log(x) + c_1}$$

4.4 problem 4

Internal problem ID [12640]
Internal file name [`OUTPUT/11292_Friday_November_03_2023_06_29_54_AM_80971990/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \ln(y-1) = 0}$$

4.4.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\ln(y-1)} dy = \int dx$$
$$-\operatorname{expIntegral}_1(-\ln(y-1)) = x + c_1$$

Raising both side to exponential gives

$$e^{-\operatorname{expIntegral}_1(-\ln(y-1))} = e^{x+c_1}$$

Which simplifies to

$$e^{-\operatorname{expIntegral}_1(-\ln(y-1))} = c_2 e^x$$

Summary
The solution(s) found are the following

$$y = e^{\operatorname{RootOf}(\operatorname{expIntegral}_1(-\_Z)+\ln(c_2)+x)} + 1 \tag{1}$$

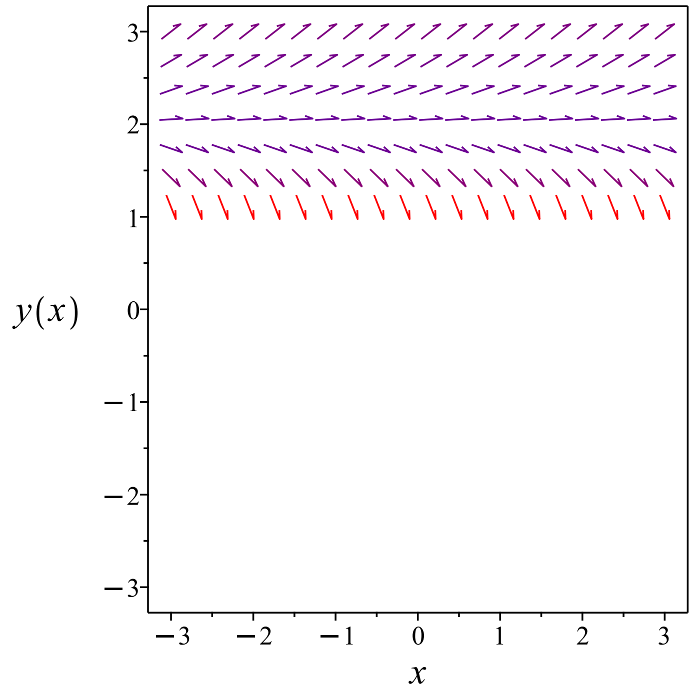


Figure 147: Slope field plot

<u>Verification of solutions</u>

$$y = \mathrm{e}^{\mathrm{RootOf}(\mathrm{expIntegral}_1(-\_Z)+\ln(c_2)+x)} + 1$$

Verified OK.

4.4.2 Maple step by step solution

Let's solve

$y' - \ln(y-1) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\ln(y-1)} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\ln(y-1)}dx = \int 1dx + c_1$

- Evaluate integral

$$-\mathrm{Ei}_1(-\ln(y-1)) = x + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)=ln(y(x)-1),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\mathrm{RootOf}(x+\mathrm{expIntegral}_1(-\_Z)+c_1)} + 1$$

✓ Solution by Mathematica

Time used: 0.29 (sec). Leaf size: 21

```
DSolve[y'[x]==Log[y[x]-1],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \mathrm{InverseFunction}[\mathrm{LogIntegral}(\#1-1)\&][x+c_1]$$
$$y(x) \to 2$$

4.5 problem 5

Internal problem ID [12641]
Internal file name [`OUTPUT/11293_Friday_November_03_2023_06_29_54_AM_30618932/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \sqrt{(y+2)(y-1)} = 0}$$

4.5.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\sqrt{(y+2)(y-1)}} dy = \int dx$$
$$\ln\left(y + \frac{1}{2} + \sqrt{y^2 + y - 2}\right) = x + c_1$$

Raising both side to exponential gives

$$y + \frac{1}{2} + \sqrt{y^2 + y - 2} = e^{x + c_1}$$

Which simplifies to

$$y + \frac{1}{2} + \sqrt{y^2 + y - 2} = c_2 e^x$$

Summary
The solution(s) found are the following

$$y = \frac{(4\,e^{2x} c_2^2 - 4c_2 e^x + 9)\,e^{-x}}{8c_2} \tag{1}$$

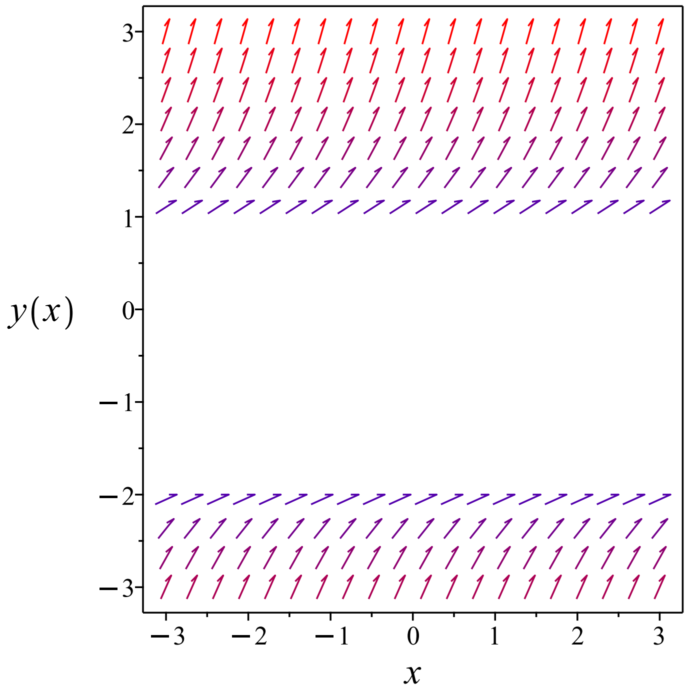


Figure 148: Slope field plot

Verification of solutions

$$y = \frac{(4\,e^{2x}c_2^2 - 4c_2e^x + 9)\,e^{-x}}{8c_2}$$

Verified OK.

4.5.2 Maple step by step solution

Let's solve

$y' - \sqrt{(y+2)\,(y-1)} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{(y+2)(y-1)}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{(y+2)(y-1)}}dx = \int 1dx + c_1$

- Evaluate integral

$$\ln\left(y+\frac{1}{2}+\sqrt{-2+y^2+y}\right)=x+c_1$$

- Solve for y

$$y=\frac{4\left(e^{x+c_1}\right)^2-4\,e^{x+c_1}+9}{8\,e^{x+c_1}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x)=sqrt( (y(x)+2)*(y(x)-1)),y(x), singsol=all)
```

$$x+\ln(2)-\ln\left(1+2y(x)+2\sqrt{(y(x)+2)(-1+y(x))}\right)+c_1=0$$

✓ Solution by Mathematica

Time used: 0.497 (sec). Leaf size: 41

```
DSolve[y'[x]==Sqrt[ (y[x]+2)*(y[x]-1)],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{1}{4}\left(-e^{-x-c_1}-9e^{x+c_1}-2\right)$$
$$y(x)\to -2$$
$$y(x)\to 1$$

4.6 problem 6

Internal problem ID [12642]
Internal file name [`OUTPUT/11294_Friday_November_03_2023_06_29_55_AM_26256630/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{y}{y - x} = 0}$$

4.6.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)\,x}{u\,(x)\,x - x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(u-2)}{x\,(u-1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u-2)}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u-2)}{u-1}} du = -\frac{1}{x} dx$$
$$\int \frac{1}{\frac{u(u-2)}{u-1}} du = \int -\frac{1}{x} dx$$
$$\frac{\ln (u(u-2))}{2} = -\ln (x) + c_2$$

Raising both side to exponential gives

$$\sqrt{u (u-2)} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u (u-2)} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u (x) (u (x)-2)} = \frac{c_3 e^{c_2}}{x}$$

The solution is

$$\sqrt{u (x) (u (x)-2)} = \frac{c_3 e^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y \left(\frac{y}{x}-2\right)}{x}} = \frac{c_3 e^{c_2}}{x}$$
$$\sqrt{\frac{y (-2x+y)}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

<u>Summary</u>

The solution(s) found are the following

$$\sqrt{\frac{y (-2x+y)}{x^2}} = \frac{c_3 e^{c_2}}{x} \tag{1}$$

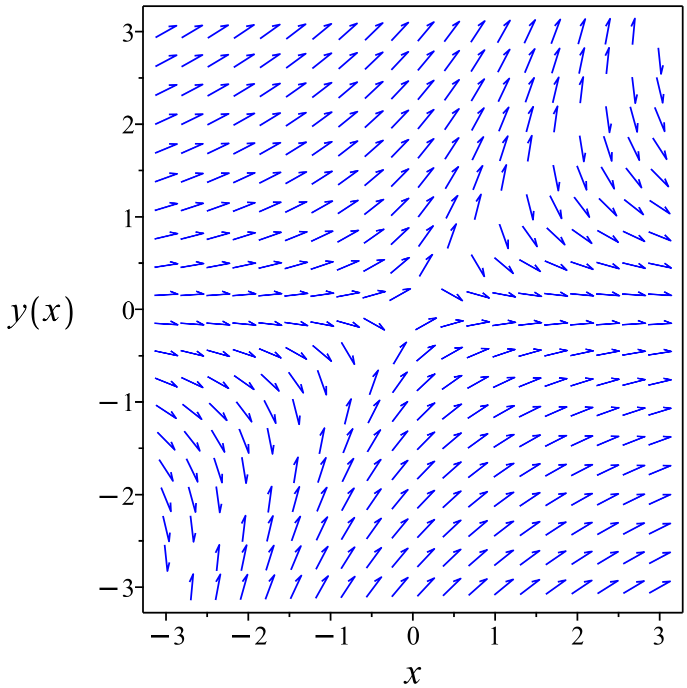


Figure 149: Slope field plot

Verification of solutions

$$\sqrt{\frac{y(-2x+y)}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

4.6.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{y}{y-x} \tag{1}$$

Which becomes

$$(-y)\,dy = (-x)\,dy + (-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (-y)\,dx = d(-xy)$$

Hence (2) becomes

$$(-y)\,dy = d(-xy)$$

Integrating both sides gives gives these solutions

$$y = x + \sqrt{x^2 - 2c_1} + c_1$$
$$y = x - \sqrt{x^2 - 2c_1} + c_1$$

Summary

The solution(s) found are the following

$$y = x + \sqrt{x^2 - 2c_1} + c_1 \tag{1}$$
$$y = x - \sqrt{x^2 - 2c_1} + c_1 \tag{2}$$

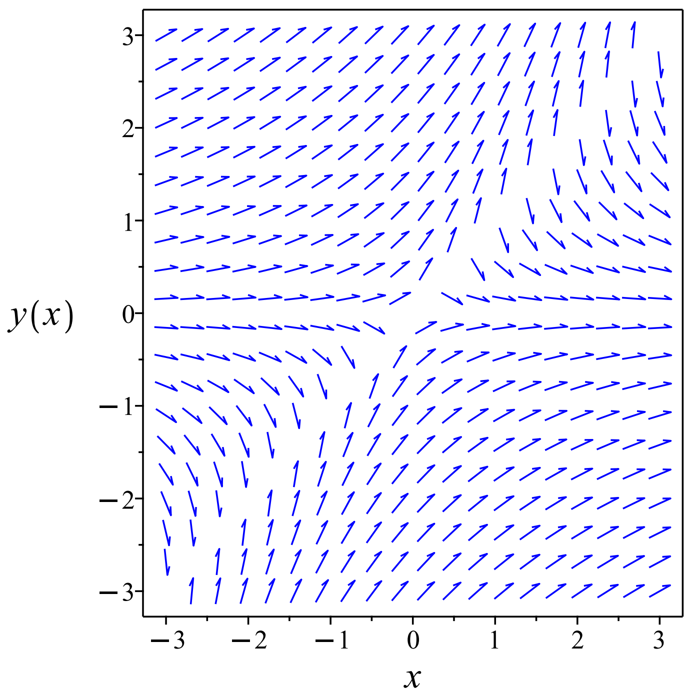


Figure 150: Slope field plot

Verification of solutions

$$y = x + \sqrt{x^2 - 2c_1} + c_1$$

Verified OK.

$$y = x - \sqrt{x^2 - 2c_1} + c_1$$

Verified OK.

4.6.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y}{y-x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 + \frac{y(b_3 - a_2)}{y-x} - \frac{y^2 a_3}{(y-x)^2} - \frac{y(x a_2 + y a_3 + a_1)}{(y-x)^2} \\
- \left(\frac{1}{y-x} - \frac{y}{(y-x)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{2x^2 b_2 - 2xy b_2 - y^2 a_2 - 2y^2 a_3 + y^2 b_2 + y^2 b_3 + x b_1 - y a_1}{(-y+x)^2} = 0
$$

Setting the numerator to zero gives

$$
2x^2 b_2 - 2xy b_2 - y^2 a_2 - 2y^2 a_3 + y^2 b_2 + y^2 b_3 + x b_1 - y a_1 = 0 \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_2^2 - 2a_3v_2^2 + 2b_2v_1^2 - 2b_2v_1v_2 + b_2v_2^2 + b_3v_2^2 - a_1v_2 + b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$2b_2v_1^2 - 2b_2v_1v_2 + b_1v_1 + (-a_2 - 2a_3 + b_2 + b_3)\, v_2^2 - a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -a_1 &= 0 \\ -2b_2 &= 0 \\ 2b_2 &= 0 \\ -a_2 - 2a_3 + b_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -2a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y}{y - x}\right)(x) \\
&= \frac{2xy - y^2}{-y + x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2xy - y^2}{-y + x}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln\left(y(-2x + y)\right)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{y}{y - x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{2x - y} \\
S_y &= \frac{-y + x}{y(2x - y)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

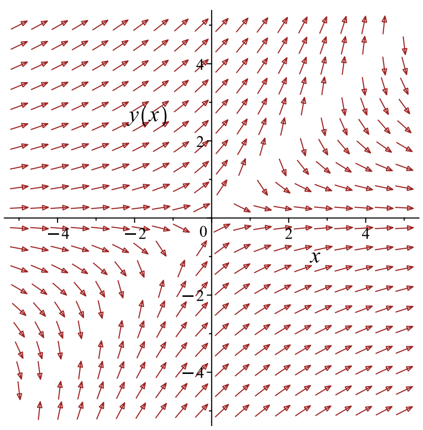
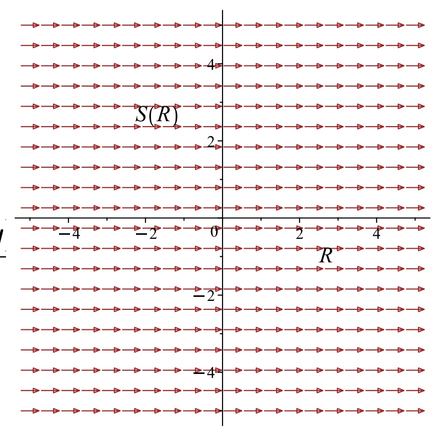
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(y)}{2} + \frac{\ln(-2x + y)}{2} = c_1
$$

Which simplifies to

$$
\frac{\ln(y)}{2} + \frac{\ln(-2x + y)}{2} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{y-x}$ 	$R = x$ $S = \frac{\ln(y)}{2} + \frac{\ln(-2x+y)}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{2} + \frac{\ln(-2x+y)}{2} = c_1 \tag{1}$$

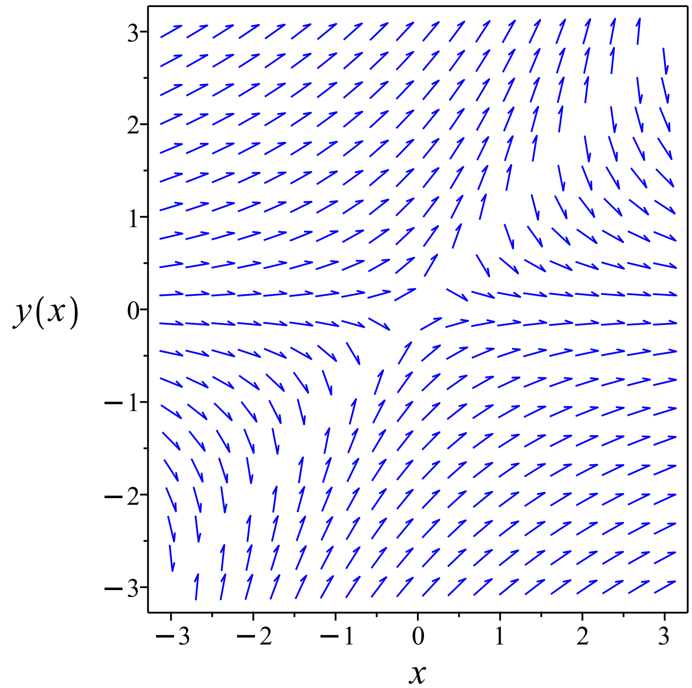


Figure 151: Slope field plot

Verification of solutions

$$\frac{\ln(y)}{2}+\frac{\ln(-2x+y)}{2}=c_1$$

Verified OK.

4.6.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y - x)\, \mathrm{d}y &= (y)\, \mathrm{d}x \\ (-y)\, \mathrm{d}x + (y - x)\, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -y \\ N(x, y) &= y - x \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y) \\ &= -1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y - x) \\ &= -1 \end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M && (1)\\
\frac{\partial \phi}{\partial y} &= N && (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -y\,\mathrm{d}x\\
\phi &= -xy + f(y) && (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=y-x$. Therefore equation (4) becomes

$$
y - x = -x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int (y)\,\mathrm{d}y\\
f(y) &= \frac{y^2}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -xy + \frac{1}{2}y^2 + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -xy + \frac{1}{2}y^2$$

Summary

The solution(s) found are the following

$$-yx + \frac{y^2}{2} = c_1 \tag{1}$$

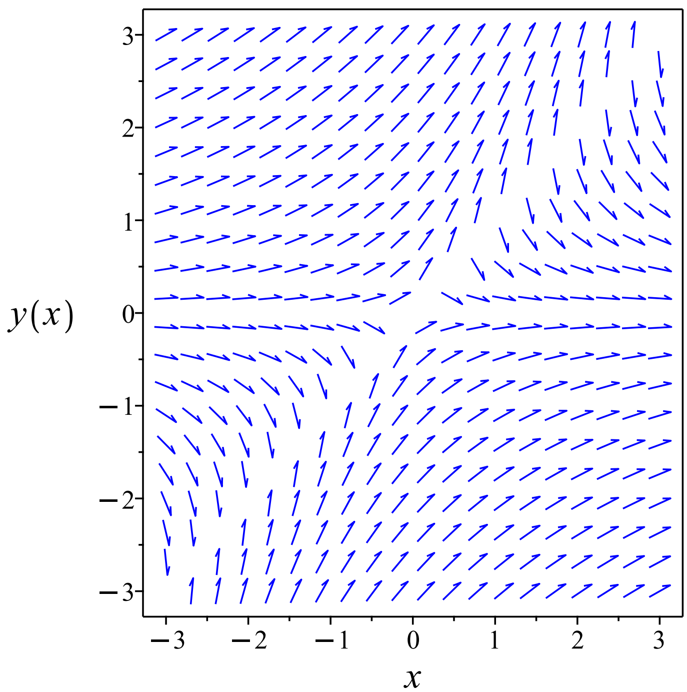


Figure 152: Slope field plot

Verification of solutions

$$-yx + \frac{y^2}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x)=y(x)/(y(x)-x),y(x), singsol=all)
```

$$y(x) = x - \sqrt{x^2 - 2c_1}$$
$$y(x) = x + \sqrt{x^2 - 2c_1}$$

✓ Solution by Mathematica

Time used: 0.836 (sec). Leaf size: 80

```
DSolve[y'[x]==y[x]/(y[x]-x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - \sqrt{x^2 - e^{2c_1}}$$
$$y(x) \to x + \sqrt{x^2 - e^{2c_1}}$$
$$y(x) \to 0$$
$$y(x) \to x - \sqrt{x^2}$$
$$y(x) \to \sqrt{x^2} + x$$

4.7 problem 7

Internal problem ID [12643]
Internal file name [`OUTPUT/11295_Friday_November_03_2023_06_29_56_AM_6075644/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{x}{y^2} = 0}$$

4.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x}{y^2} \end{aligned}$$

Where $f(x) = x$ and $g(y) = \frac{1}{y^2}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y^2}}\,dy = x\,dx$$

$$\int \frac{1}{\frac{1}{y^2}}\,dy = \int x\,dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} + c_1$$

Which results in

$$y = \frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{2}$$

$$y = -\frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\,(12x^2 + 24c_1)^{\frac{1}{3}}}{4}$$

$$y = -\frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\,(12x^2 + 24c_1)^{\frac{1}{3}}}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{2} \tag{1}$$

$$y = -\frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\,(12x^2 + 24c_1)^{\frac{1}{3}}}{4} \tag{2}$$

$$y = -\frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\,(12x^2 + 24c_1)^{\frac{1}{3}}}{4} \tag{3}$$

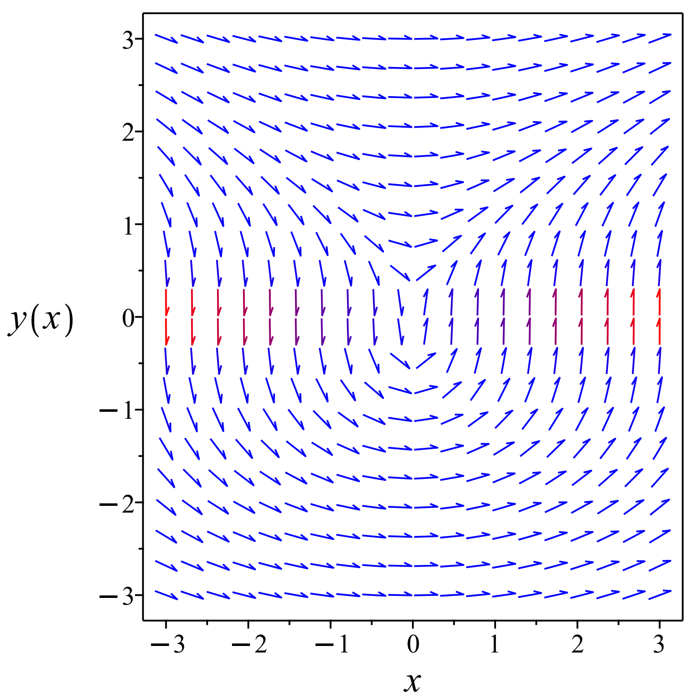


Figure 153: Slope field plot

Verification of solutions

$$y = \frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{2}$$

Verified OK.

$$y = -\frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\,(12x^2 + 24c_1)^{\frac{1}{3}}}{4}$$

Verified OK.

$$y = -\frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\,(12x^2 + 24c_1)^{\frac{1}{3}}}{4}$$

Verified OK.

4.7.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x}{y^2} \tag{1}$$

Which becomes

$$(y^2)\,dy = (x)\,dx \tag{2}$$

But the RHS is complete differential because

$$(x)\,dx = d\left(\frac{x^2}{2}\right)$$

Hence (2) becomes

$$(y^2)\,dy = d\left(\frac{x^2}{2}\right)$$

Integrating both sides gives gives these solutions

$$\begin{aligned}
y &= \frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{2} + c_1 \\
y &= -\frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\,(12x^2 + 24c_1)^{\frac{1}{3}}}{4} + c_1 \\
y &= -\frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\,(12x^2 + 24c_1)^{\frac{1}{3}}}{4} + c_1
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(12x^{2}+24c_{1}\right)^{\frac{1}{3}}}{2}+c_{1} \tag{1}$$

$$y = -\frac{\left(12x^{2}+24c_{1}\right)^{\frac{1}{3}}}{4}+\frac{i\sqrt{3}\,\left(12x^{2}+24c_{1}\right)^{\frac{1}{3}}}{4}+c_{1} \tag{2}$$

$$y = -\frac{\left(12x^{2}+24c_{1}\right)^{\frac{1}{3}}}{4}-\frac{i\sqrt{3}\,\left(12x^{2}+24c_{1}\right)^{\frac{1}{3}}}{4}+c_{1} \tag{3}$$

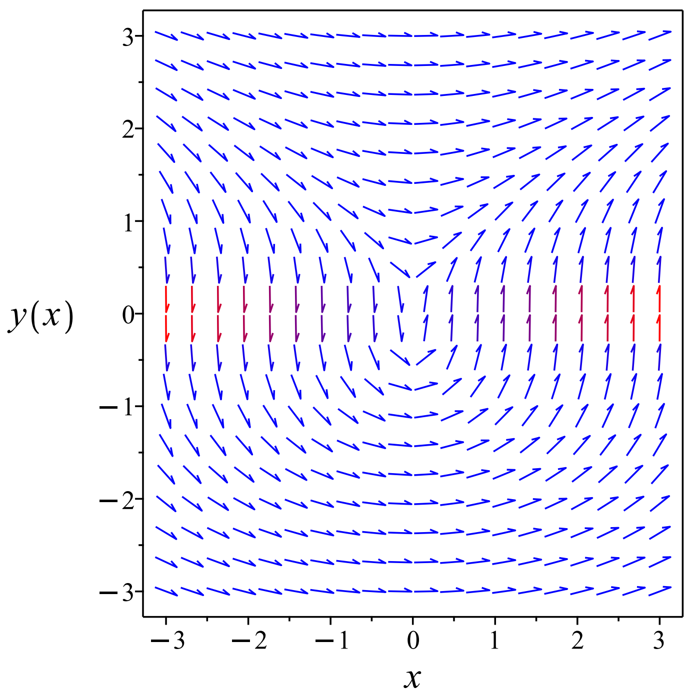


Figure 154: Slope field plot

Verification of solutions

$$y = \frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{2} + c_1$$

Verified OK.

$$y = -\frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\,(12x^2 + 24c_1)^{\frac{1}{3}}}{4} + c_1$$

Verified OK.

$$y = -\frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\,(12x^2 + 24c_1)^{\frac{1}{3}}}{4} + c_1$$

Verified OK.

4.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x}{y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 117: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= \frac{1}{x} \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x}{y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^3}{3} + c_1 \tag{4}$$

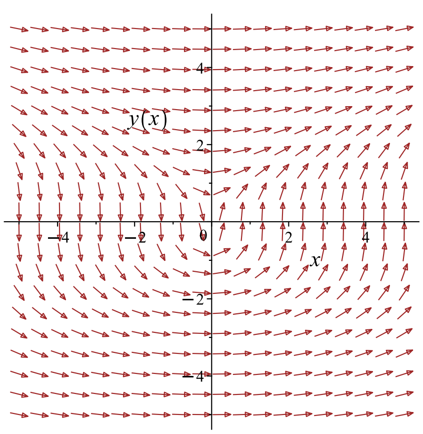
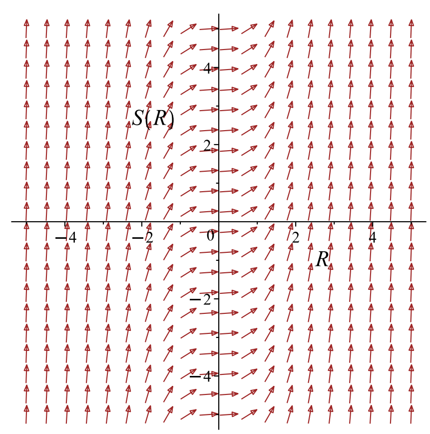
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \frac{y^3}{3} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = \frac{y^3}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x}{y^2}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = R^2$ 

Summary
The solution(s) found are the following

$$\frac{x^2}{2} = \frac{y^3}{3} + c_1 \tag{1}$$

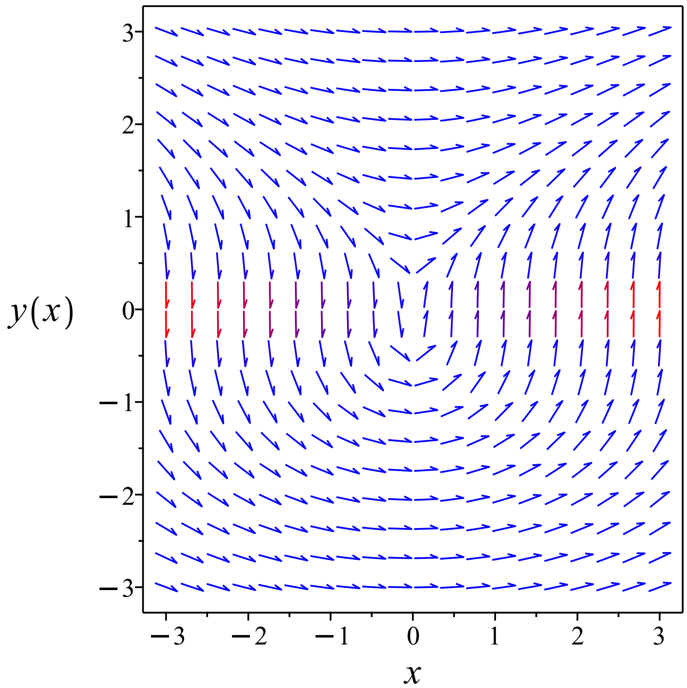


Figure 155: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = \frac{y^3}{3} + c_1$$

Verified OK.

4.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y^2)\,\mathrm{d}y &= (x)\,\mathrm{d}x \\ (-x)\,\mathrm{d}x + (y^2)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x \\ N(x, y) &= y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^2) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y^2$. Therefore equation (4) becomes

$$y^2 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y^2) \, \mathrm{d}y$$
$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^3}{3}$$

Summary

The solution(s) found are the following

$$\frac{y^3}{3} - \frac{x^2}{2} = c_1 \tag{1}$$

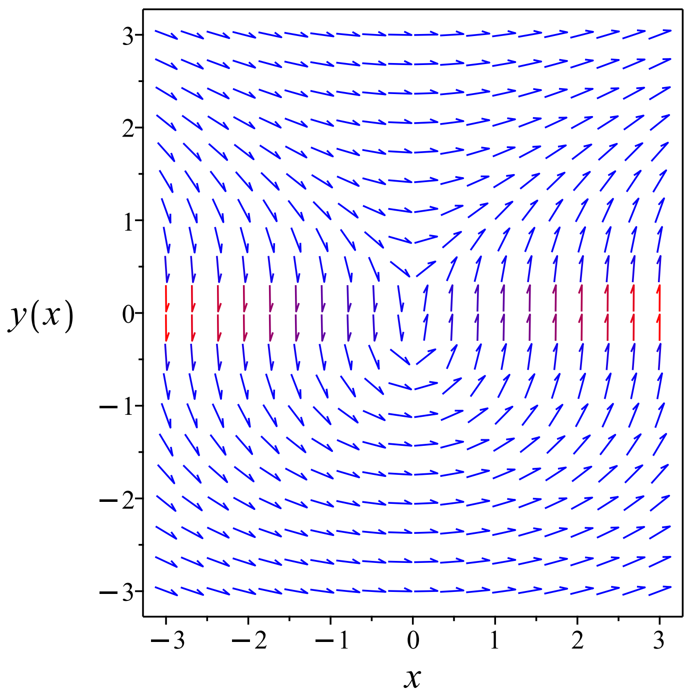


Figure 156: Slope field plot

Verification of solutions

$$\frac{y^3}{3} - \frac{x^2}{2} = c_1$$

Verified OK.

4.7.5 Maple step by step solution

Let's solve

$y' - \frac{x}{y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'y^2 = x$

- Integrate both sides with respect to x

  $\int y'y^2 dx = \int x dx + c_1$

- Evaluate integral

  $\frac{y^3}{3} = \frac{x^2}{2} + c_1$

- Solve for y

  $y = \frac{(12x^2 + 24c_1)^{\frac{1}{3}}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 65

```
dsolve(diff(y(x),x)=x/y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{(12x^2 + 8c_1)^{\frac{1}{3}}}{2}$$
$$y(x) = -\frac{(12x^2 + 8c_1)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{4}$$
$$y(x) = \frac{(12x^2 + 8c_1)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.283 (sec). Leaf size: 79

```
DSolve[y'[x]==x/y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt[3]{-\frac{3}{2}}\sqrt[3]{x^2 + 2c_1}$$
$$y(x) \to \sqrt[3]{\frac{3}{2}}\sqrt[3]{x^2 + 2c_1}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{\frac{3}{2}}\sqrt[3]{x^2 + 2c_1}$$

4.8 problem 8

Internal problem ID [12644]
Internal file name [`OUTPUT/11296_Friday_November_03_2023_06_29_57_AM_4938376/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{\sqrt{y}}{x} = 0}$$

4.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\sqrt{y}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \sqrt{y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\sqrt{y}}\,dy &= \frac{1}{x}\,dx \\ \int \frac{1}{\sqrt{y}}\,dy &= \int \frac{1}{x}\,dx \\ 2\sqrt{y} &= \ln(x) + c_1 \end{aligned}$$

The solution is

$$2\sqrt{y} - \ln(x) - c_1 = 0$$

Summary
The solution(s) found are the following

$$2\sqrt{y} - \ln(x) - c_1 = 0 \tag{1}$$

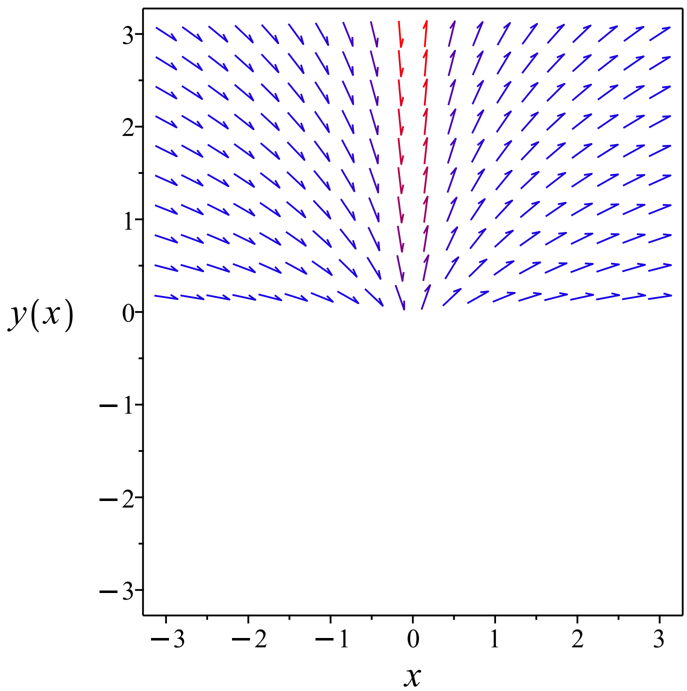


Figure 157: Slope field plot

Verification of solutions

$$2\sqrt{y} - \ln(x) - c_1 = 0$$

Verified OK.

4.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\sqrt{y}}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 120: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x \\
\eta(x, y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x} dx
\end{aligned}
$$

Which results in

$$
S = \ln(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{\sqrt{y}}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{y}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{R}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\sqrt{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = 2\sqrt{y} + c_1$$

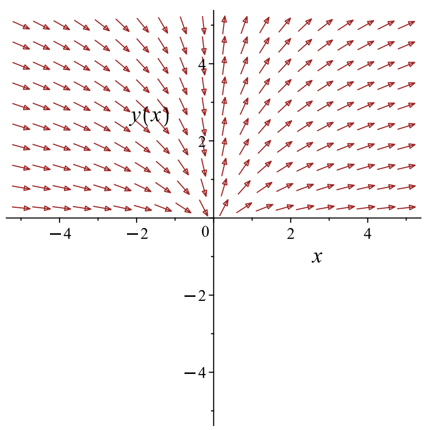
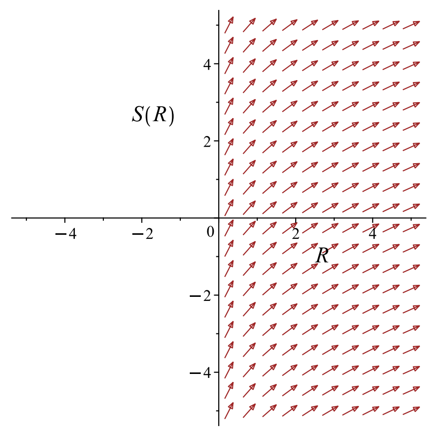
Which simplifies to

$$\ln(x) = 2\sqrt{y} + c_1$$

Which gives

$$y = \frac{\ln(x)^2}{4} - \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{y}}{x}$ 	$R = y$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{1}{\sqrt{R}}$ 

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^2}{4} - \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4} \tag{1}$$

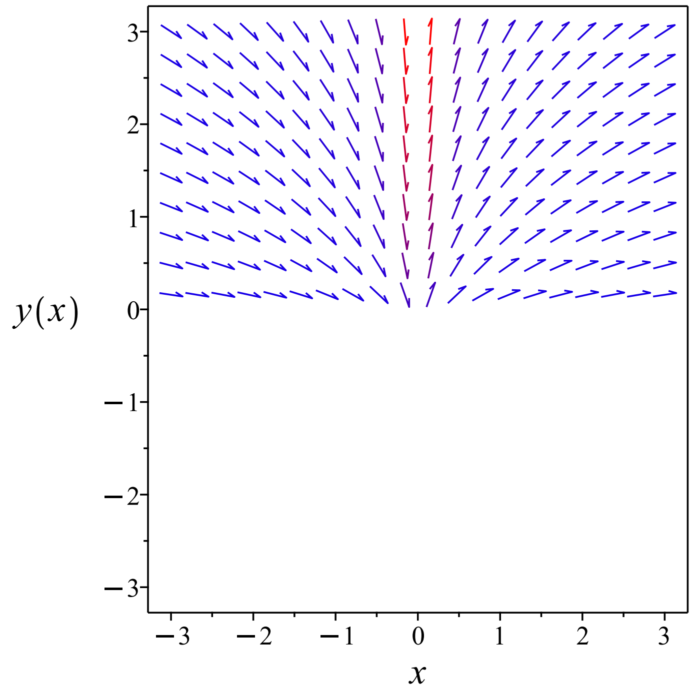


Figure 158: Slope field plot

Verification of solutions

$$y = \frac{\ln(x)^2}{4} - \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$$

Verified OK.

4.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{\sqrt{y}}\right)\mathrm{d}y = \left(\frac{1}{x}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{\sqrt{y}}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x}$$
$$N(x,y) = \frac{1}{\sqrt{y}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{y}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{y}}$. Therefore equation (4) becomes

$$
\frac{1}{\sqrt{y}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{\sqrt{y}}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{\sqrt{y}}\right)\mathrm{d}y \\
f(y) &= 2\sqrt{y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + 2\sqrt{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + 2\sqrt{y}$$

The solution becomes

$$y = \frac{\ln(x)^2}{4} + \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^2}{4} + \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4} \tag{1}$$

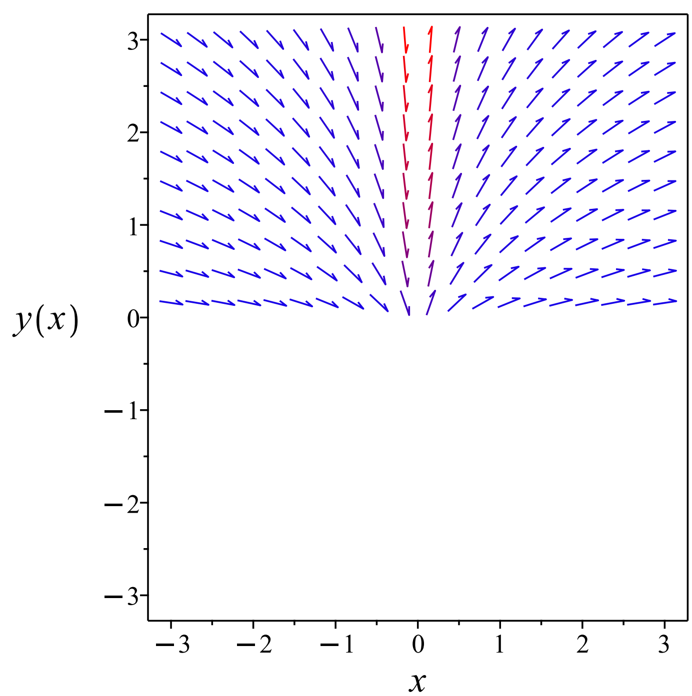


Figure 159: Slope field plot

Verification of solutions

$$y = \frac{\ln (x)^2}{4} + \frac{c_1 \ln (x)}{2} + \frac{c_1^2}{4}$$

Verified OK.

4.8.4 Maple step by step solution

Let's solve

$y' - \frac{\sqrt{y}}{x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{y}} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y}} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $2\sqrt{y} = \ln (x) + c_1$

- Solve for y

  $y = \frac{\ln(x)^2}{4} + \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)=sqrt(y(x))/x,y(x), singsol=all)
```

$$\sqrt{y(x)} - \frac{\ln(x)}{2} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.202 (sec). Leaf size: 21

```
DSolve[y'[x]==Sqrt[y[x]]/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}(\log(x) + c_1)^2$$
$$y(x) \to 0$$

4.9 problem 9

Internal problem ID [12645]
Internal file name [`OUTPUT/11297_Friday_November_03_2023_06_29_58_AM_57080782/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{xy}{1-y} = 0}$$

4.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{xy}{y-1} \end{aligned}$$

Where $f(x) = -x$ and $g(y) = \frac{y}{y-1}$. Integrating both sides gives

$$\frac{1}{\frac{y}{y-1}}\,dy = -x\,dx$$

$$\int \frac{1}{\frac{y}{y-1}}\,dy = \int -x\,dx$$

$$y - \ln(y) = -\frac{x^2}{2} + c_1$$

Which results in

$$y = e^{-\operatorname{LambertW}\left(-e^{\frac{x^2}{2}-c_1}\right)+\frac{x^2}{2}-c_1}$$

Summary

The solution(s) found are the following

$$y = e^{-\operatorname{LambertW}\left(-e^{\frac{x^2}{2}-c_1}\right)+\frac{x^2}{2}-c_1} \tag{1}$$

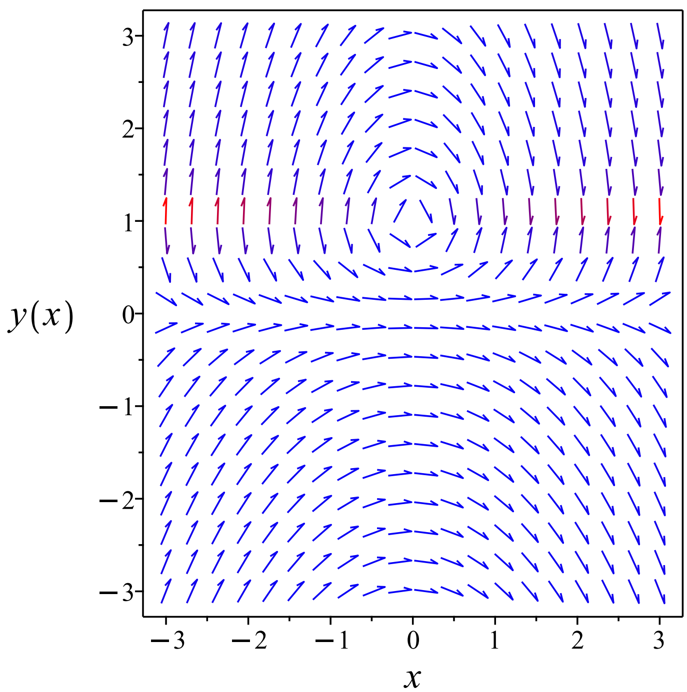


Figure 160: Slope field plot

Verification of solutions

$$y = e^{-\operatorname{LambertW}\left(-e^{\frac{x^2}{2}-c_1}\right)+\frac{x^2}{2}-c_1}$$

Verified OK.

4.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{xy}{y-1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 123: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = -\frac{1}{x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{xy}{y-1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{y-1}{y} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R-1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = R - \ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{x^2}{2} = y - \ln(y) + c_1
$$

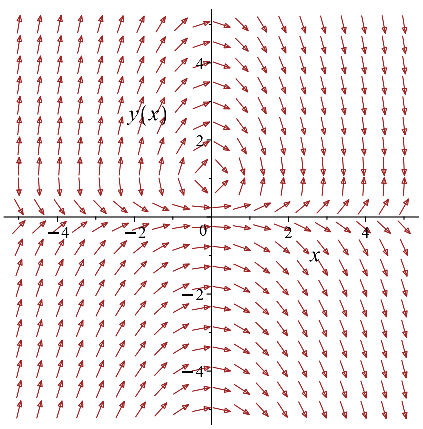
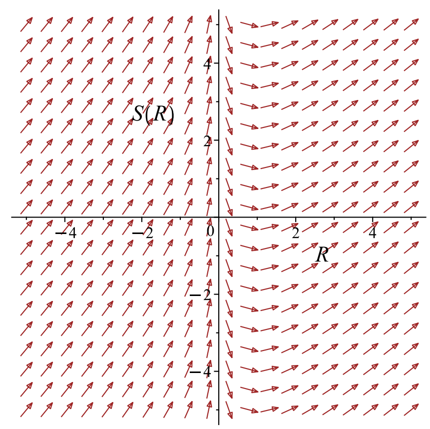
Which simplifies to

$$
-\frac{x^2}{2} = y - \ln(y) + c_1
$$

Which gives

$$
y = \mathrm{e}^{-\operatorname{LambertW}\left(-\mathrm{e}^{\frac{x^2}{2}+c_1}\right)+\frac{x^2}{2}+c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{xy}{y-1}$ 	$R = y$ $S = -\frac{x^2}{2}$	$\frac{dS}{dR} = \frac{R-1}{R}$ 

Summary

The solution(s) found are the following

$$y = e^{-\operatorname{LambertW}\left(-e^{\frac{x^2}{2}+c_1}\right)+\frac{x^2}{2}+c_1} \tag{1}$$

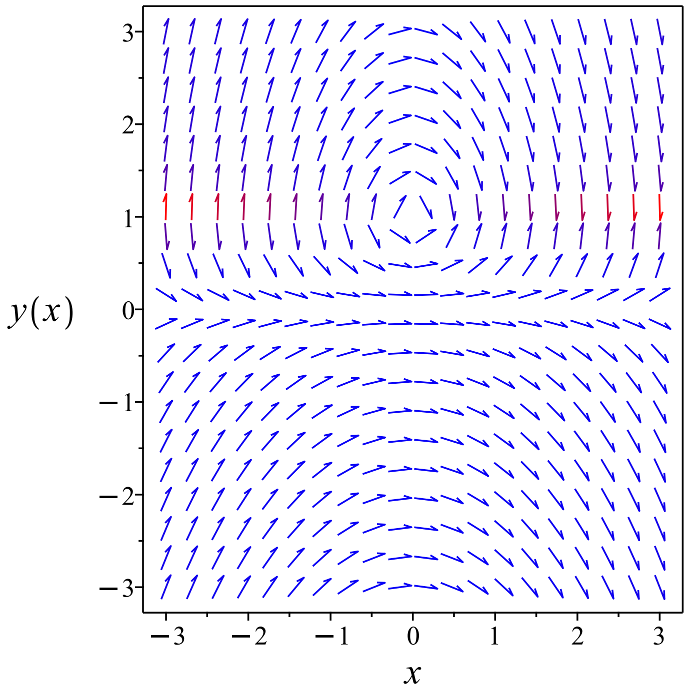


Figure 161: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{-\mathrm{LambertW}\left(-\mathrm{e}^{\frac{x^2}{2}+c_1}\right)+\frac{x^2}{2}+c_1}
$$

Verified OK.

4.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{y-1}{y}\right) \mathrm{d}y &= (x)\,\mathrm{d}x \\
(-x)\,\mathrm{d}x + \left(-\frac{y-1}{y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x \\
N(x, y) &= -\frac{y-1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y-1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y-1}{y}$. Therefore equation (4) becomes

$$-\frac{y-1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{y-1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1-y}{y}\right)\mathrm{d}y\\ f(y) &= -y + \ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - y + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - y + \ln(y)$$

The solution becomes

$$y = e^{-\text{LambertW}\left(-e^{\frac{x^2}{2}+c_1}\right)+\frac{x^2}{2}+c_1}$$

Summary

The solution(s) found are the following

$$y = e^{-\text{LambertW}\left(-e^{\frac{x^2}{2}+c_1}\right)+\frac{x^2}{2}+c_1} \tag{1}$$

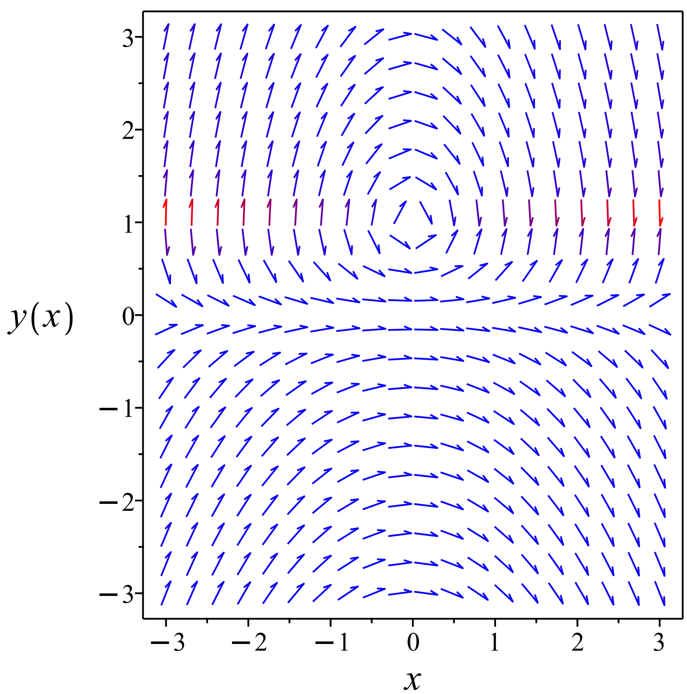


Figure 162: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\mathrm{e}^{\frac{x^2}{2}+c_1}\right)+\frac{x^2}{2}+c_1}$$

Verified OK.

4.9.4 Maple step by step solution

Let's solve

$y' - \frac{xy}{1-y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(1-y)}{y} = x$

- Integrate both sides with respect to x

  $\int \frac{y'(1-y)}{y}dx = \int xdx + c_1$

- Evaluate integral

  $-y + \ln(y) = \frac{x^2}{2} + c_1$

- Solve for y

  $y = \mathrm{e}^{-LambertW\left(-\mathrm{e}^{\frac{x^2}{2}+c_1}\right)+\frac{x^2}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)=x*y(x)/(1-y(x)),y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}\left(-\mathrm{e}^{\frac{x^2}{2}+c_1}\right)$$

✓ Solution by Mathematica

Time used: 3.96 (sec). Leaf size: 29

```
DSolve[y'[x]==x*y[x]/(1-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -W\left(-e^{\frac{x^2}{2}-c_1}\right)$$
$$y(x) \to 0$$

4.10 problem 10

Internal problem ID [12646]
Internal file name [OUTPUT/11298_Friday_November_03_2023_06_29_58_AM_89943328/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{y' - (yx)^{\frac{1}{3}} = 0}$$

4.10.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= (xy)^{\frac{1}{3}} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + (xy)^{\frac{1}{3}} (b_3 - a_2) - (xy)^{\frac{2}{3}} a_3 - \frac{y(xa_2 + ya_3 + a_1)}{3 (xy)^{\frac{2}{3}}} - \frac{x(xb_2 + yb_3 + b_1)}{3 (xy)^{\frac{2}{3}}} = 0 \quad (5E)$$

Putting the above in normal form gives

$$-\frac{3(xy)^{\frac{4}{3}} a_3 + 4xya_2 - 2xyb_3 - 3b_2(xy)^{\frac{2}{3}} + x^2b_2 + y^2a_3 + xb_1 + ya_1}{3 (xy)^{\frac{2}{3}}} = 0$$

Setting the numerator to zero gives

$$-3(xy)^{\frac{4}{3}} a_3 + 3b_2(xy)^{\frac{2}{3}} - x^2b_2 - 4xya_2 + 2xyb_3 - y^2a_3 - xb_1 - ya_1 = 0 \quad (6E)$$

Since the PDE has radicals, simplifying gives

$$-3xy(xy)^{\frac{1}{3}} a_3 - x^2b_2 - 4xya_2 + 2xyb_3 + 3b_2(xy)^{\frac{2}{3}} - y^2a_3 - xb_1 - ya_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, (xy)^{\frac{1}{3}}, (xy)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, (xy)^{\frac{1}{3}} = v_3, (xy)^{\frac{2}{3}} = v_4\right\}$$

The above PDE (6E) now becomes

$$-3v_1v_2v_3a_3 - 4v_1v_2a_2 - v_2^2a_3 - v_1^2b_2 + 2v_1v_2b_3 - v_2a_1 - v_1b_1 + 3b_2v_4 = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$-v_1^2b_2 - 3v_1v_2v_3a_3 + (-4a_2 + 2b_3) v_1v_2 - v_1b_1 - v_2^2a_3 - v_2a_1 + 3b_2v_4 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-a_1 &= 0 \\
-3a_3 &= 0 \\
-a_3 &= 0 \\
-b_1 &= 0 \\
-b_2 &= 0 \\
3b_2 &= 0 \\
-4a_2 + 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= 2y
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{2y}{x} \\
&= \frac{2y}{x}
\end{aligned}$$

This is easily solved to give

$$y = c_1 x^2$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x^2}$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \ln(x) \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (xy)^{\frac{1}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{2y}{x^3} \\ R_y &= \frac{1}{x^2} \\ S_x &= \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x^2}{x (xy)^{\frac{1}{3}} - 2y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^{\frac{1}{3}} - 2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(8R^2 - 1)}{4} + \frac{\ln\left(2R^{\frac{2}{3}} + 4R^{\frac{4}{3}} + 1\right)}{4} - \frac{\ln\left(2R^{\frac{2}{3}} - 1\right)}{2} + c_1 \tag{4}$$

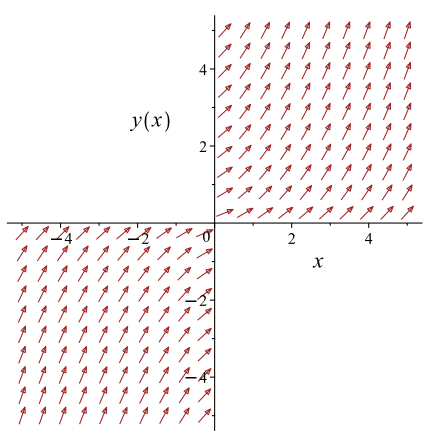
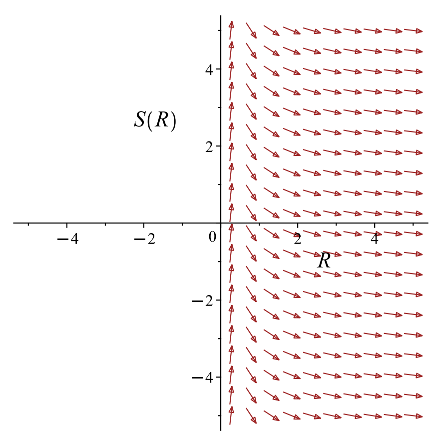
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = -\frac{\ln\left(\frac{8y^2}{x^4} - 1\right)}{4} + \frac{\ln\left(2\left(\frac{y}{x^2}\right)^{\frac{2}{3}} + 4\left(\frac{y}{x^2}\right)^{\frac{4}{3}} + 1\right)}{4} - \frac{\ln\left(2\left(\frac{y}{x^2}\right)^{\frac{2}{3}} - 1\right)}{2} + c_1$$

Which simplifies to

$$\ln(x) = -\frac{\ln\left(\frac{8y^2}{x^4} - 1\right)}{4} + \frac{\ln\left(2\left(\frac{y}{x^2}\right)^{\frac{2}{3}} + 4\left(\frac{y}{x^2}\right)^{\frac{4}{3}} + 1\right)}{4} - \frac{\ln\left(2\left(\frac{y}{x^2}\right)^{\frac{2}{3}} - 1\right)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = (xy)^{\frac{1}{3}}$

</td><td>$R = \frac{y}{x^2}$
$S = \ln(x)$</td><td>$\frac{dS}{dR} = \frac{1}{R^{\frac{1}{3}} - 2R}$

</td></tr>
</table>

Summary
The solution(s) found are the following

$$\ln(x) = -\frac{\ln\left(\frac{8y^2}{x^4}-1\right)}{4}+\frac{\ln\left(2\left(\frac{y}{x^2}\right)^{\frac{2}{3}}+4\left(\frac{y}{x^2}\right)^{\frac{4}{3}}+1\right)}{4}-\frac{\ln\left(2\left(\frac{y}{x^2}\right)^{\frac{2}{3}}-1\right)}{2}+c_1 \tag{1}$$

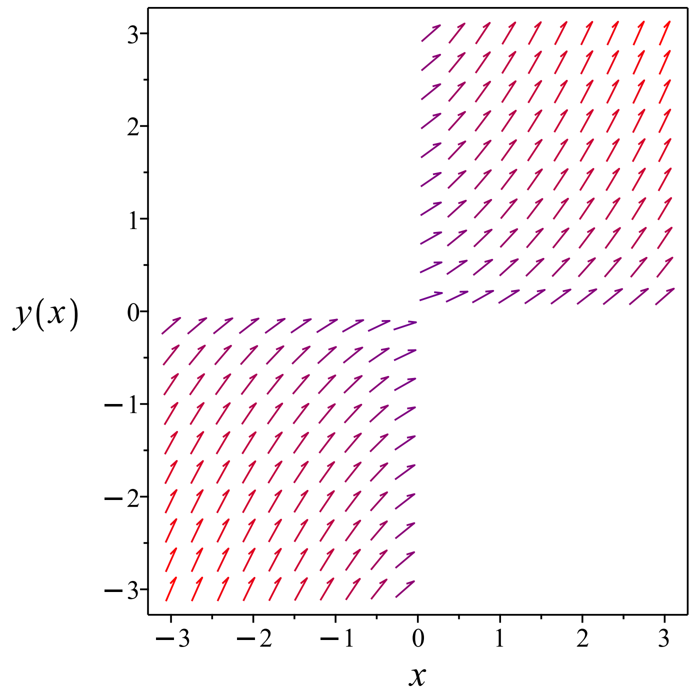


Figure 163: Slope field plot

Verification of solutions

$$\ln(x) = -\frac{\ln\left(\frac{8y^2}{x^4}-1\right)}{4}+\frac{\ln\left(2\left(\frac{y}{x^2}\right)^{\frac{2}{3}}+4\left(\frac{y}{x^2}\right)^{\frac{4}{3}}+1\right)}{4}-\frac{\ln\left(2\left(\frac{y}{x^2}\right)^{\frac{2}{3}}-1\right)}{2}+c_1$$

Verified OK.

4.10.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left((xy)^{\frac{1}{3}}\right)\mathrm{d}x \\ \left(-(xy)^{\frac{1}{3}}\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -(xy)^{\frac{1}{3}} \\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-(xy)^{\frac{1}{3}}\right) \\ &= -\frac{x}{3\,(xy)^{\frac{2}{3}}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\frac{x}{3\,(xy)^{\frac{2}{3}}}\right) - (0)\right) \\ &= -\frac{x}{3\,(xy)^{\frac{2}{3}}}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{(xy)^{\frac{1}{3}}}\left((0) - \left(-\frac{x}{3\,(xy)^{\frac{2}{3}}}\right)\right) \\ &= -\frac{1}{3y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}y} \\ &= e^{\int -\frac{1}{3y} \,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{\ln(y)}{3}} \\ &= \frac{1}{y^{\frac{1}{3}}}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^{\frac{1}{3}}}\left(-(xy)^{\frac{1}{3}}\right) \\
&= -\frac{(xy)^{\frac{1}{3}}}{y^{\frac{1}{3}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^{\frac{1}{3}}}(1) \\
&= \frac{1}{y^{\frac{1}{3}}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{(xy)^{\frac{1}{3}}}{y^{\frac{1}{3}}}\right) + \left(\frac{1}{y^{\frac{1}{3}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{(xy)^{\frac{1}{3}}}{y^{\frac{1}{3}}} \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{3x(xy)^{\frac{1}{3}}}{4y^{\frac{1}{3}}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= -\frac{x^2}{4 (xy)^{\frac{2}{3}} y^{\frac{1}{3}}} + \frac{x(xy)^{\frac{1}{3}}}{4y^{\frac{4}{3}}} + f'(y) \\ &= 0 + f'(y) \end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$\frac{1}{y^{\frac{1}{3}}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^{\frac{1}{3}}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{1}{y^{\frac{1}{3}}}\right) \mathrm{d}y$$

$$f(y) = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{3x(xy)^{\frac{1}{3}}}{4y^{\frac{1}{3}}} + \frac{3y^{\frac{2}{3}}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{3x(xy)^{\frac{1}{3}}}{4y^{\frac{1}{3}}} + \frac{3y^{\frac{2}{3}}}{2}$$

Summary

The solution(s) found are the following

$$-\frac{3x(yx)^{\frac{1}{3}}}{4y^{\frac{1}{3}}}+\frac{3y^{\frac{2}{3}}}{2}=c_1 \tag{1}$$

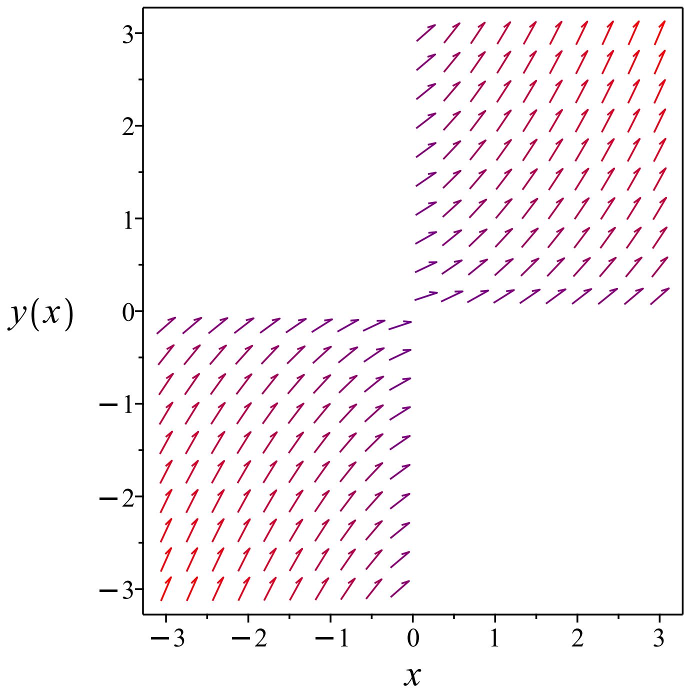


Figure 164: Slope field plot

Verification of solutions

$$-\frac{3x(yx)^{\frac{1}{3}}}{4y^{\frac{1}{3}}}+\frac{3y^{\frac{2}{3}}}{2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 87

```
dsolve(diff(y(x),x)=(x*y(x))^(1/3),y(x), singsol=all)
```

$$-\frac{\left(\left(-4x^5c_1+32y(x)^2\,c_1x+2x\right)\left(y(x)\,x\right)^{\frac{2}{3}}+\left(x^3+4(y(x)\,x)^{\frac{1}{3}}\,y(x)\right)\left(x^4c_1-8c_1y(x)^2+1\right)\right)x}{\left(x^4-8y\left(x\right)^2\right)\left(-2\left(y\left(x\right)x\right)^{\frac{2}{3}}+x^2\right)^2}$$
$$=0$$

✓ Solution by Mathematica

Time used: 4.979 (sec). Leaf size: 35

```
DSolve[y'[x]==(x*y[x])^(1/3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{\left(3x^{4/3}+4c_1\right)^{3/2}}{6\sqrt{6}}$$
$$y(x)\to 0$$

4.11 problem 11

Internal problem ID [12647]
Internal file name [OUTPUT/11299_Friday_November_03_2023_06_30_00_AM_53092840/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \sqrt{\frac{y-4}{x}} = 0}$$

4.11.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \sqrt{\frac{-4+y}{x}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\sqrt{\frac{-4+y}{x}}\,(b_3-a_2)-\frac{(-4+y)\,a_3}{x}+\frac{(-4+y)\,(xa_2+ya_3+a_1)}{2\sqrt{\frac{-4+y}{x}}\,x^2}-\frac{xb_2+yb_3+b_1}{2\sqrt{\frac{-4+y}{x}}\,x}=0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2b_2\sqrt{\frac{-4+y}{x}}\,x^2-2a_3\sqrt{\frac{-4+y}{x}}\,xy+8a_3\sqrt{\frac{-4+y}{x}}\,x-x^2b_2-xya_2+xyb_3+y^2a_3+4xa_2-xb_1-8xb_3+ya_1-}{2\sqrt{\frac{-4+y}{x}}\,x^2}$$
$$=0$$

Setting the numerator to zero gives

$$2b_2\sqrt{\frac{-4+y}{x}}\,x^2-2a_3\sqrt{\frac{-4+y}{x}}\,xy+8a_3\sqrt{\frac{-4+y}{x}}\,x-x^2b_2-xya_2 \\ +xyb_3+y^2a_3+4xa_2-xb_1-8xb_3+ya_1-4ya_3-4a_1=0 \tag{6E}$$

Simplifying the above gives

$$-2(-4+y)\,xa_2+2(-4+y)\,xb_3+2b_2\sqrt{\frac{-4+y}{x}}\,x^2-2a_3\sqrt{\frac{-4+y}{x}}\,xy \\ +8a_3\sqrt{\frac{-4+y}{x}}\,x-x^2b_2+xya_2-xyb_3+y^2a_3-4xa_2-xb_1+ya_1-4ya_3-4a_1=0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$2b_2\sqrt{\frac{-4+y}{x}}\,x^2-2a_3\sqrt{\frac{-4+y}{x}}\,xy+8a_3\sqrt{\frac{-4+y}{x}}\,x-x^2b_2-xya_2 \\ +xyb_3+y^2a_3+4xa_2-xb_1-8xb_3+ya_1-4ya_3-4a_1=0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{\frac{-4+y}{x}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{\frac{-4+y}{x}} = v_3\right\}$$

The above PDE (6E) now becomes

$$-2a_3v_3v_1v_2 + 2b_2v_3v_1^2 - v_1v_2a_2 + 8a_3v_3v_1 + v_2^2a_3 - v_1^2b_2 + v_1v_2b_3 + v_2a_1 + 4v_1a_2 - 4v_2a_3 - v_1b_1 - 8v_1b_3 - 4a_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$2b_2v_3v_1^2 - v_1^2b_2 - 2a_3v_3v_1v_2 + (b_3 - a_2)\,v_1v_2 + 8a_3v_3v_1 + (4a_2 - b_1 - 8b_3)\,v_1 + v_2^2a_3 + (a_1 - 4a_3)\,v_2 - 4a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_3 &= 0\\
-4a_1 &= 0\\
-2a_3 &= 0\\
8a_3 &= 0\\
-b_2 &= 0\\
2b_2 &= 0\\
a_1 - 4a_3 &= 0\\
b_3 - a_2 &= 0\\
4a_2 - b_1 - 8b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= -4b_3 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -4 + y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -4 + y - \left(\sqrt{\frac{-4 + y}{x}}\right)(x) \\
&= -x\sqrt{-\frac{4}{x} + \frac{y}{x}} + y - 4 \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-x\sqrt{-\frac{4}{x}+\frac{y}{x}}+y-4} dy
\end{aligned}
$$

Which results in

$$
S = \ln\left(-4+y-x\right) - 2\,\operatorname{arctanh}\left(\sqrt{-\frac{4}{x}+\frac{y}{x}}\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \sqrt{\frac{-4+y}{x}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{\sqrt{x}+\sqrt{-4+y}}{\sqrt{x}\,\left(4-y+x\right)} \\
S_y &= -\frac{\sqrt{x}+\sqrt{-4+y}}{\sqrt{-4+y}\,\left(4-y+x\right)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\left(-\sqrt{x}-\sqrt{-4+y}\right)\left(\sqrt{\frac{-4+y}{x}}\,\sqrt{x}-\sqrt{-4+y}\right)}{\sqrt{x}\,\left(4-y+x\right)\sqrt{-4+y}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

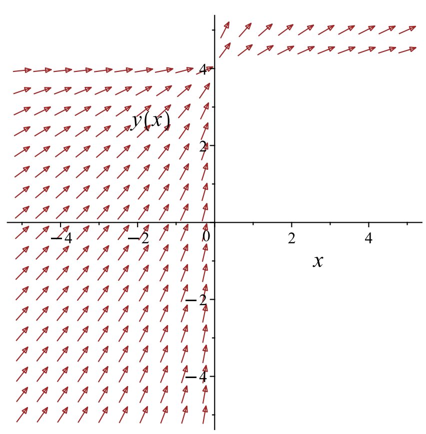
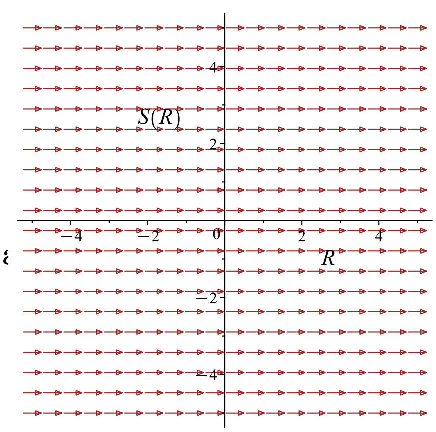
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(-4+y-x\right) - 2\,\mathrm{arctanh}\left(\frac{\sqrt{y-4}}{\sqrt{x}}\right) = c_1$$

Which simplifies to

$$\ln\left(-4+y-x\right) - 2\,\mathrm{arctanh}\left(\frac{\sqrt{y-4}}{\sqrt{x}}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sqrt{\frac{-4+y}{x}}$ 	$R = x$ $S = \ln\left(-4+y-x\right) - 2$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\ln\left(-4+y-x\right) - 2\,\mathrm{arctanh}\left(\frac{\sqrt{y-4}}{\sqrt{x}}\right) = c_1 \tag{1}$$

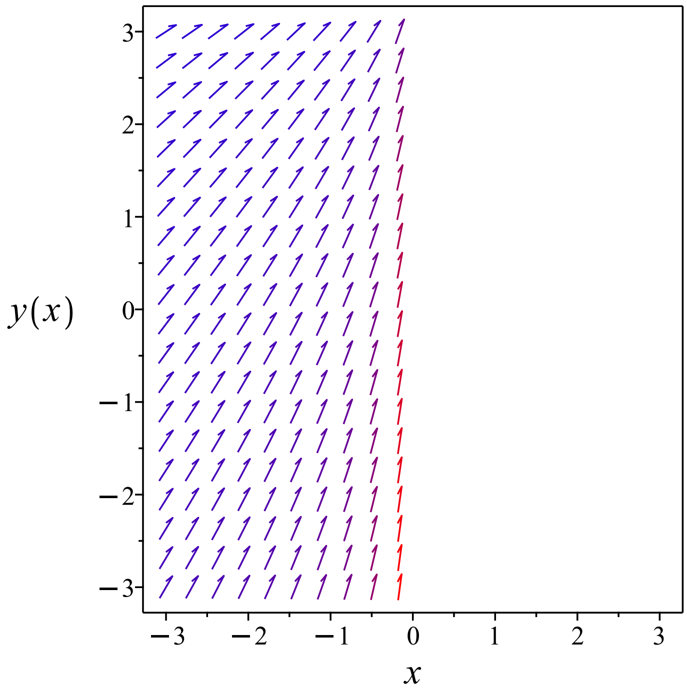


Figure 165: Slope field plot

Verification of solutions

$$\ln\left(-4+y-x\right)-2\,\text{arctanh}\left(\frac{\sqrt{y-4}}{\sqrt{x}}\right)=c_1$$

Verified OK.

4.11.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\mathrm{d}y = \left(\sqrt{\frac{-4+y}{x}}\right) \mathrm{d}x
$$

$$
\left(-\sqrt{\frac{-4+y}{x}}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\sqrt{\frac{-4+y}{x}} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\sqrt{\frac{-4+y}{x}}\right) \\
&= -\frac{1}{2\sqrt{\frac{-4+y}{x}}\, x}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{1}{2\sqrt{\frac{-4+y}{x}}\, x}\right) - (0)\right) \\
&= -\frac{1}{2\sqrt{\frac{-4+y}{x}}\, x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{\sqrt{\frac{-4+y}{x}}}\left((0) - \left(-\frac{1}{2\sqrt{\frac{-4+y}{x}}\, x}\right)\right) \\
&= -\frac{1}{-8+2y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{1}{-8+2y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{\ln(-4+y)}{2}} \\
&= \frac{1}{\sqrt{-4+y}}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{\sqrt{-4+y}}\left(-\sqrt{\frac{-4+y}{x}}\right) \\ &= -\frac{\sqrt{\frac{-4+y}{x}}}{\sqrt{-4+y}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{\sqrt{-4+y}}(1) \\ &= \frac{1}{\sqrt{-4+y}}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(-\frac{\sqrt{\frac{-4+y}{x}}}{\sqrt{-4+y}}\right) + \left(\frac{1}{\sqrt{-4+y}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{\sqrt{\frac{-4+y}{x}}}{\sqrt{-4+y}}\,\mathrm{d}x$$

$$\phi = -\frac{2x\sqrt{\frac{-4+y}{x}}}{\sqrt{-4+y}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= -\frac{1}{\sqrt{\frac{-4+y}{x}}\,\sqrt{-4+y}} + \frac{x\sqrt{\frac{-4+y}{x}}}{(-4+y)^{\frac{3}{2}}} + f'(y) \qquad (4)\\
&= 0 + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{-4+y}}$. Therefore equation (4) becomes

$$
\frac{1}{\sqrt{-4+y}} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{\sqrt{-4+y}}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{\sqrt{-4+y}}\right) \mathrm{d}y\\
f(y) &= 2\sqrt{-4+y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{2x\sqrt{\frac{-4+y}{x}}}{\sqrt{-4+y}} + 2\sqrt{-4+y} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{2x\sqrt{\frac{-4+y}{x}}}{\sqrt{-4+y}} + 2\sqrt{-4+y}
$$

Summary

The solution(s) found are the following

$$
-\frac{2x\sqrt{\frac{y-4}{x}}}{\sqrt{y-4}}+2\sqrt{y-4}=c_1 \tag{1}
$$

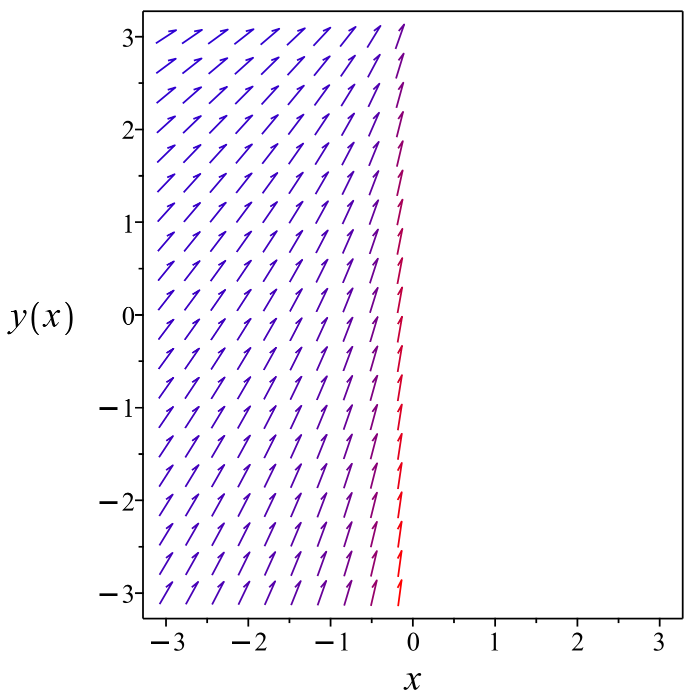


Figure 166: Slope field plot

Verification of solutions

$$
-\frac{2x\sqrt{\frac{y-4}{x}}}{\sqrt{y-4}}+2\sqrt{y-4}=c_1
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 38

```
dsolve(diff(y(x),x)=sqrt( (y(x)-4)/x ),y(x), singsol=all)
```

$$-\ln\left(\frac{-y(x)+4+x}{x}\right)+2\,\mathrm{arctanh}\left(\sqrt{\frac{y(x)-4}{x}}\right)-\ln(x)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.294 (sec). Leaf size: 29

```
DSolve[y'[x]==Sqrt[ (y[x]-4)/x ],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to x+c_1\sqrt{x}+4+\frac{c_1{}^2}{4}$$
$$y(x)\to 4$$

4.12 problem 12

Internal problem ID [12648]
Internal file name [`OUTPUT/11300_Friday_November_03_2023_06_30_02_AM_71545787/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y' + \frac{y}{x} - y^{\frac{1}{4}} = 0}$$

4.12.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-y^{\frac{1}{4}}x + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 126: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{y^{\frac{1}{4}}}{x^{\frac{3}{4}}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^{\frac{1}{4}}}{x^{\frac{3}{4}}}} dy \end{aligned}$$

Which results in

$$S = \frac{4y^{\frac{3}{4}} x^{\frac{3}{4}}}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-y^{\frac{1}{4}} x + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^{\frac{3}{4}}}{x^{\frac{1}{4}}} \\ S_y &= \frac{x^{\frac{3}{4}}}{y^{\frac{1}{4}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^{\frac{3}{4}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^{\frac{3}{4}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{4R^{\frac{7}{4}}}{7} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{4y^{\frac{3}{4}}x^{\frac{3}{4}}}{3} = \frac{4x^{\frac{7}{4}}}{7} + c_1$$

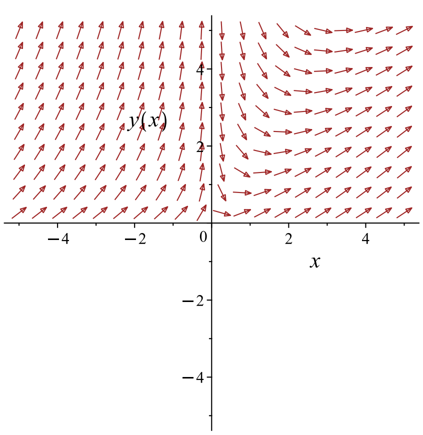
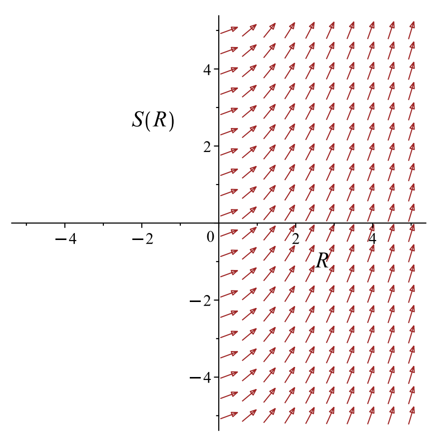
Which simplifies to

$$\frac{4y^{\frac{3}{4}}x^{\frac{3}{4}}}{3} = \frac{4x^{\frac{7}{4}}}{7} + c_1$$

Which gives

$$y = \frac{3\,3^{\frac{1}{3}}28^{\frac{2}{3}}\left(\frac{4x^{\frac{7}{4}}+7c_1}{x^{\frac{3}{4}}}\right)^{\frac{4}{3}}}{784}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-y^{\frac{1}{4}}x+y}{x}$ 	$R = x$ $S = \frac{4y^{\frac{3}{4}}x^{\frac{3}{4}}}{3}$	$\frac{dS}{dR} = R^{\frac{3}{4}}$ 

Summary
The solution(s) found are the following

$$y = \frac{3\,3^{\frac{1}{3}} 28^{\frac{2}{3}} \left(\frac{4x^{\frac{7}{4}}+7c_1}{x^{\frac{3}{4}}}\right)^{\frac{4}{3}}}{784} \tag{1}$$

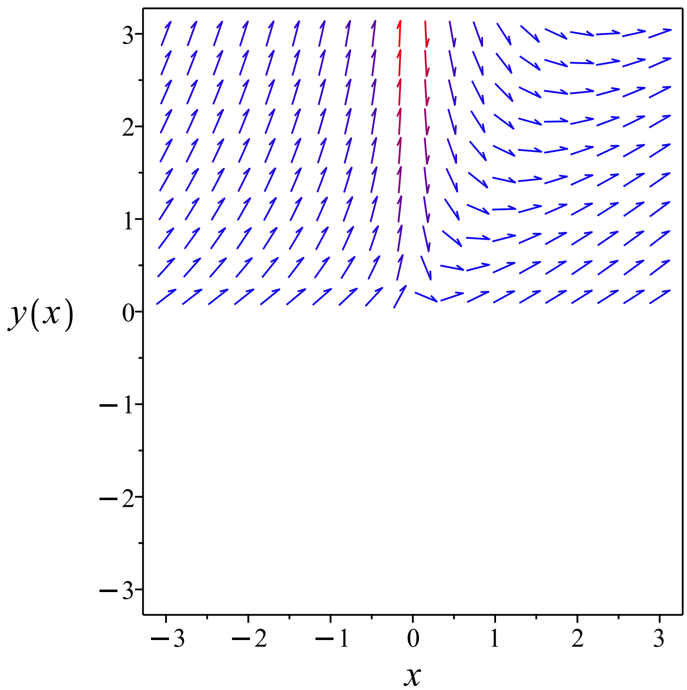


Figure 167: Slope field plot

Verification of solutions

$$y = \frac{3\,3^{\frac{1}{3}} 28^{\frac{2}{3}} \left(\frac{4x^{\frac{7}{4}}+7c_1}{x^{\frac{3}{4}}}\right)^{\frac{4}{3}}}{784}$$

Verified OK.

4.12.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{-y^{\frac{1}{4}}x + y}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + y^{\frac{1}{4}} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= 1 \\ n &= \frac{1}{4} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^{\frac{1}{4}}$ gives

$$y'\frac{1}{y^{\frac{1}{4}}} = -\frac{y^{\frac{3}{4}}}{x} + 1 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^{\frac{3}{4}} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = \frac{3}{4y^{\frac{1}{4}}}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{4w'(x)}{3} &= -\frac{w(x)}{x} + 1 \\
w' &= -\frac{3w}{4x} + \frac{3}{4} \qquad (7)
\end{aligned}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{3}{4x} \\
q(x) &= \frac{3}{4}
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{3w(x)}{4x} = \frac{3}{4}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{3}{4x}dx} \\
&= x^{\frac{3}{4}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{3}{4}\right) \\
\frac{d}{dx}\left(x^{\frac{3}{4}}w\right) &= \left(x^{\frac{3}{4}}\right)\left(\frac{3}{4}\right) \\
d\left(x^{\frac{3}{4}}w\right) &= \left(\frac{3x^{\frac{3}{4}}}{4}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^{\frac{3}{4}}w &= \int \frac{3x^{\frac{3}{4}}}{4} dx \\
x^{\frac{3}{4}}w &= \frac{3x^{\frac{7}{4}}}{7} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^{\frac{3}{4}}$ results in

$$w(x) = \frac{3x}{7} + \frac{c_1}{x^{\frac{3}{4}}}$$

Replacing w in the above by $y^{\frac{3}{4}}$ using equation (5) gives the final solution.

$$y^{\frac{3}{4}} = \frac{3x}{7} + \frac{c_1}{x^{\frac{3}{4}}}$$

Summary

The solution(s) found are the following

$$y^{\frac{3}{4}} = \frac{3x}{7} + \frac{c_1}{x^{\frac{3}{4}}} \tag{1}$$

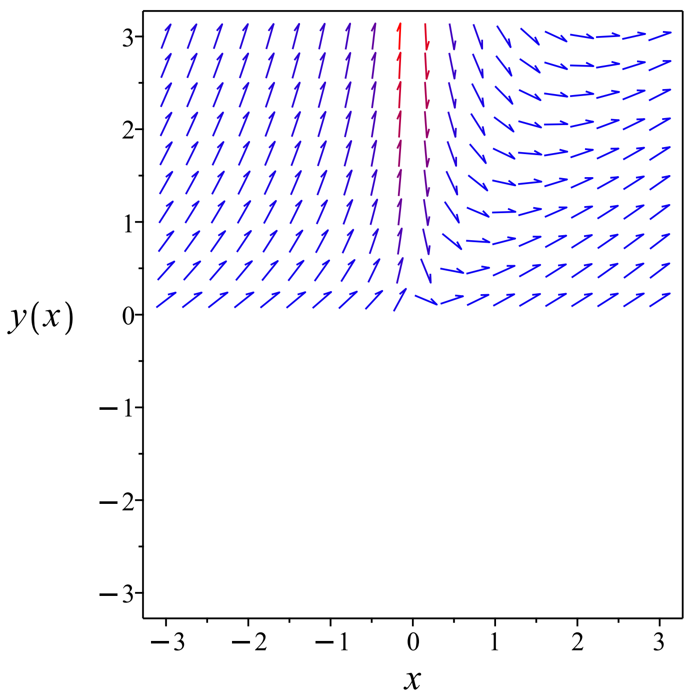


Figure 168: Slope field plot

Verification of solutions

$$y^{\frac{3}{4}} = \frac{3x}{7} + \frac{c_1}{x^{\frac{3}{4}}}$$

Verified OK.

4.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(y^{\frac{1}{4}}x-y\right)\mathrm{d}x\\ \left(-y^{\frac{1}{4}}x+y\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y^{\frac{1}{4}}x+y\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y^{\frac{1}{4}}x + y\right) \\ &= -\frac{x}{4y^{\frac{3}{4}}} + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}\left(\left(-\frac{x}{4y^{\frac{3}{4}}} + 1\right) - (1)\right) \\ &= -\frac{1}{4y^{\frac{3}{4}}}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{-y^{\frac{1}{4}}x + y}\left((1) - \left(-\frac{x}{4y^{\frac{3}{4}}} + 1\right)\right) \\ &= -\frac{x}{y^{\frac{3}{4}}\left(4y^{\frac{1}{4}}x - 4y\right)}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - \left(-\frac{x}{4y^{\frac{3}{4}}} + 1\right)}{x\left(-y^{\frac{1}{4}}x + y\right) - y\,(x)} \\
&= -\frac{1}{4xy}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{1}{4t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{1}{4t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{\ln(t)}{4}} \\
&= \frac{1}{t^{\frac{1}{4}}}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{(xy)^{\frac{1}{4}}}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{(xy)^{\frac{1}{4}}}\left(-y^{\frac{1}{4}}x + y\right) \\
&= \frac{-y^{\frac{1}{4}}x + y}{(xy)^{\frac{1}{4}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(xy)^{\frac{1}{4}}}(x) \\
&= \frac{x}{(xy)^{\frac{1}{4}}}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-y^{\frac{1}{4}}x + y}{(xy)^{\frac{1}{4}}}\right) + \left(\frac{x}{(xy)^{\frac{1}{4}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-y^{\frac{1}{4}}x + y}{(xy)^{\frac{1}{4}}}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{4(xy)^{\frac{3}{4}}\left(7y^{\frac{3}{4}} - 3x\right)}{21y^{\frac{3}{4}}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{\left(7y^{\frac{3}{4}} - 3x\right)x}{7\,(xy)^{\frac{1}{4}}\,y^{\frac{3}{4}}} + \frac{(xy)^{\frac{3}{4}}}{y} - \frac{(xy)^{\frac{3}{4}}\left(7y^{\frac{3}{4}} - 3x\right)}{7y^{\frac{7}{4}}} + f'(y) \tag{4}
$$

$$
= \frac{x}{(xy)^{\frac{1}{4}}} + f'(y)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{(xy)^{\frac{1}{4}}}$. Therefore equation (4) becomes

$$\frac{x}{(xy)^{\frac{1}{4}}} = \frac{x}{(xy)^{\frac{1}{4}}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{4(xy)^{\frac{3}{4}} \left(7y^{\frac{3}{4}} - 3x\right)}{21y^{\frac{3}{4}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{4(xy)^{\frac{3}{4}} \left(7y^{\frac{3}{4}} - 3x\right)}{21y^{\frac{3}{4}}}$$

Summary

The solution(s) found are the following

$$\frac{4(yx)^{\frac{3}{4}} \left(7y^{\frac{3}{4}} - 3x\right)}{21y^{\frac{3}{4}}} = c_1 \tag{1}$$

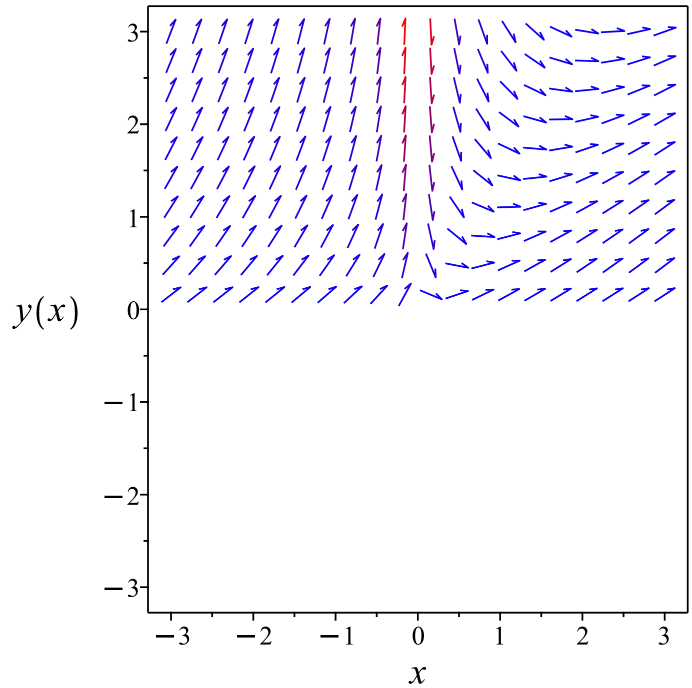


Figure 169: Slope field plot

Verification of solutions

$$\frac{4(yx)^{\frac{3}{4}}\left(7y^{\frac{3}{4}}-3x\right)}{21y^{\frac{3}{4}}}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)=-y(x)/x+y(x)^(1/4),y(x), singsol=all)
```

$$y(x)^{\frac{3}{4}} - \frac{3x}{7} - \frac{c_1}{x^{\frac{3}{4}}} = 0$$

✓ Solution by Mathematica

Time used: 9.843 (sec). Leaf size: 31

```
DSolve[y'[x]==-y[x]/x+y[x]^(1/4),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\left(3x + \frac{7c_1}{x^{3/4}}\right)^{4/3}}{7\sqrt[3]{7}}$$

4.13 problem 13

Internal problem ID [12649]
Internal file name [`OUTPUT/11301_Friday_November_03_2023_06_30_04_AM_273934/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 4y = -5}$$

With initial conditions

$$[y(1) = 4]$$

4.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -4$$
$$q(x) = -5$$

Hence the ode is

$$y' - 4y = -5$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

4.13.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{4y-5} dy = \int dx$$
$$\frac{\ln(4y-5)}{4} = x + c_1$$

Raising both side to exponential gives

$$(4y-5)^{\frac{1}{4}} = e^{x+c_1}$$

Which simplifies to

$$(4y-5)^{\frac{1}{4}} = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = \frac{c_2^4 e^4}{4} + \frac{5}{4}$$

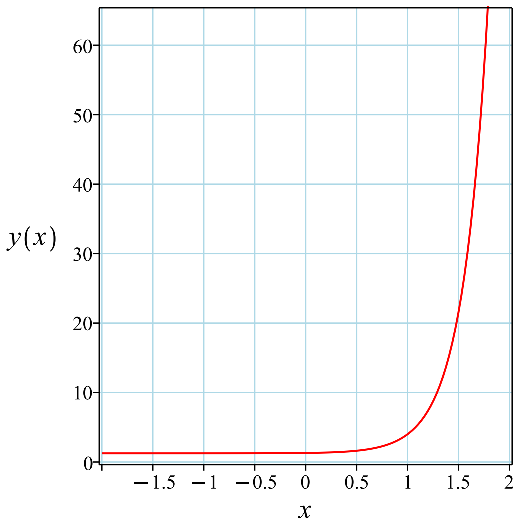
$$c_2 = 11^{\frac{1}{4}} e^{-1}$$

Substituting c_2 found above in the general solution gives

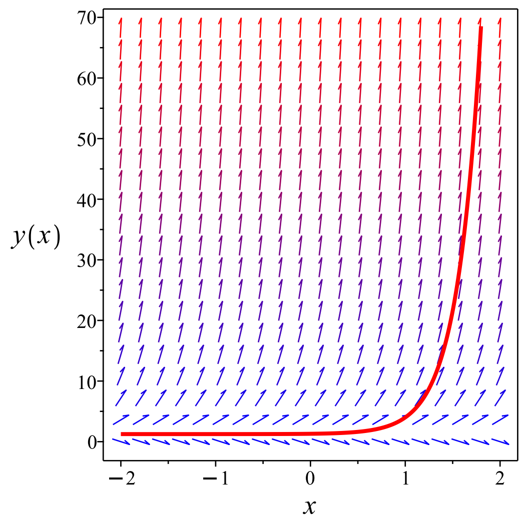
$$y = \frac{11\, e^{-4} e^{4x}}{4} + \frac{5\, e^{-4} e^4}{4}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{11\, e^{-4} e^{4x}}{4} + \frac{5\, e^{-4} e^4}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{11\,\mathrm{e}^{-4}\mathrm{e}^{4x}}{4} + \frac{5\,\mathrm{e}^{-4}\mathrm{e}^{4}}{4}$$

Verified OK.

4.13.3 Maple step by step solution

Let's solve

$[y' - 4y = -5, y(1) = 4]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{4y-5} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{4y-5}dx = \int 1dx + c_1$

- Evaluate integral

  $\frac{\ln(4y-5)}{4} = x + c_1$

- Solve for y

$$y = \frac{e^{4c_1+4x}}{4} + \frac{5}{4}$$

- Use initial condition $y(1) = 4$

$$4 = \frac{e^{4c_1+4}}{4} + \frac{5}{4}$$

- Solve for c_1

$$c_1 = -1 + \frac{\ln(11)}{4}$$

- Substitute $c_1 = -1 + \frac{\ln(11)}{4}$ into general solution and simplify

$$y = \frac{11\,e^{4x-4}}{4} + \frac{5}{4}$$

- Solution to the IVP

$$y = \frac{11\,e^{4x-4}}{4} + \frac{5}{4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)=4*y(x)-5,y(1) = 4],y(x), singsol=all)
```

$$y(x) = \frac{5}{4} + \frac{11\,e^{-4+4x}}{4}$$

✓ Solution by Mathematica
Time used: 0.043 (sec). Leaf size: 20

```
DSolve[{y'[x]==4*y[x]-5,{y[1]==4}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{11}{4}e^{4x-4} + \frac{5}{4}$$

4.14 problem 14

Internal problem ID [12650]
Internal file name [`OUTPUT/11302_Friday_November_03_2023_06_30_05_AM_62961360/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + 3y = 1}$$

With initial conditions

$$[y(-2) = 1]$$

4.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 3$$
$$q(x) = 1$$

Hence the ode is

$$y' + 3y = 1$$

The domain of $p(x) = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -2$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -2$ is also inside this domain. Hence solution exists and is unique.

4.14.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-3y+1} dy = \int dx$$
$$-\frac{\ln(-3y+1)}{3} = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{(-3y+1)^{\frac{1}{3}}} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{(-3y+1)^{\frac{1}{3}}} = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = -2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_2^3 - e^6}{3c_2^3}$$

$$c_2 = -\frac{4^{\frac{1}{3}} e^2}{2}$$

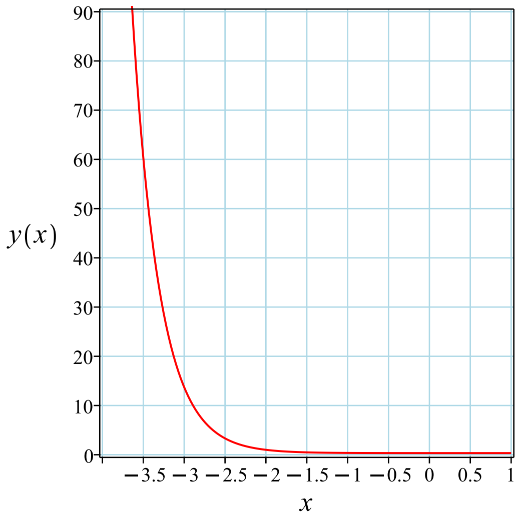
Substituting c_2 found above in the general solution gives

$$y = \frac{2\, e^{-6} e^{-3x}}{3} + \frac{e^{-6} e^6}{3}$$

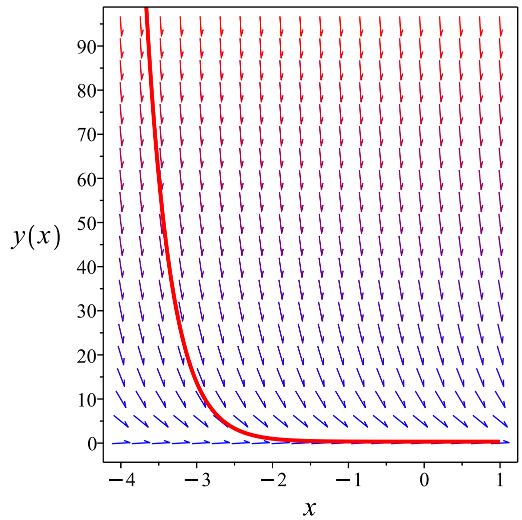
Summary

The solution(s) found are the following

$$y = \frac{2\, e^{-6} e^{-3x}}{3} + \frac{e^{-6} e^6}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2\,e^{-6}e^{-3x}}{3} + \frac{e^{-6}e^{6}}{3}$$

Verified OK.

4.14.3 Maple step by step solution

Let's solve

$[y' + 3y = 1, y(-2) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-3y+1} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-3y+1}dx = \int 1dx + c_1$

- Evaluate integral

  $-\frac{\ln(-3y+1)}{3} = x + c_1$

- Solve for y

$y = -\frac{e^{-3x-3c_1}}{3} + \frac{1}{3}$

- Use initial condition $y(-2) = 1$

  $1 = -\frac{e^{6-3c_1}}{3} + \frac{1}{3}$

- Solve for c_1

  $c_1 = 2 - \frac{\ln(2)}{3} - \frac{I\pi}{3}$

- Substitute $c_1 = 2 - \frac{\ln(2)}{3} - \frac{I\pi}{3}$ into general solution and simplify

  $y = \frac{2\,e^{-3x-6}}{3} + \frac{1}{3}$

- Solution to the IVP

  $y = \frac{2\,e^{-3x-6}}{3} + \frac{1}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)+3*y(x)=1,y(-2) = 1],y(x), singsol=all)
```

$$y(x) = \frac{1}{3} + \frac{2\,e^{-6-3x}}{3}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 20

```
DSolve[{y'[x]+3*y[x]==1,{y[-2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{3}e^{-3(x+2)} + \frac{1}{3}$$

4.15 problem 15

Internal problem ID [12651]
Internal file name [`OUTPUT/11303_Friday_November_03_2023_06_30_05_AM_72975530/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - ay = b}$$

With initial conditions

$$[y(c) = d]$$

4.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -a$$
$$q(x) = b$$

Hence the ode is

$$y' - ay = b$$

The domain of $p(x) = -a$ is

$$\{-\infty < x < \infty\}$$

But the point $x_0 = c$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

4.15.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{ay+b} dy = \int dx$$
$$\frac{\ln(ay+b)}{a} = x + c_1$$

Raising both side to exponential gives

$$e^{\frac{\ln(ay+b)}{a}} = e^{x+c_1}$$

Which simplifies to

$$(ay+b)^{\frac{1}{a}} = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = c$ and $y = d$ in the above solution gives an equation to solve for the constant of integration.

$$d = \frac{(c_2 e^c)^a - b}{a}$$

$$c_2 = e^{\frac{\ln(ad+b)}{a} - c}$$

Substituting c_2 found above in the general solution gives

$$y = \frac{\left((ad+b)^{\frac{1}{a}} e^{-c+x}\right)^a - b}{a}$$

Summary

The solution(s) found are the following

$$y = \frac{(ad+b)\, e^{-(c-x)a} - b}{a} \tag{1}$$

Verification of solutions

$$y = \frac{(ad+b)\, e^{-(c-x)a} - b}{a}$$

Verified OK. `{positive}`

4.15.3 Maple step by step solution

Let's solve

$[y' - ay = b, y(c) = d]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{ay+b} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{ay+b}dx = \int 1dx + c_1$

- Evaluate integral

  $\frac{\ln(ay+b)}{a} = x + c_1$

- Solve for y

  $y = \frac{e^{c_1 a + ax} - b}{a}$

- Use initial condition $y(c) = d$

  $d = \frac{e^{c_1 a + ac} - b}{a}$

- Solve for c_1

  $c_1 = \frac{-ac + \ln(ad+b)}{a}$

- Substitute $c_1 = \frac{-ac + \ln(ad+b)}{a}$ into general solution and simplify

  $y = \frac{(ad+b)e^{-(c-x)a} - b}{a}$

- Solution to the IVP

  $y = \frac{(ad+b)e^{-(c-x)a} - b}{a}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve([diff(y(x),x)=a*y(x)+b,y(c) = d],y(x), singsol=all)
```

$$y(x) = \frac{(ad+b)\,e^{-a(c-x)} - b}{a}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 39

```
DSolve[{y'[x]==a*y[x]+b,{y[c]==d}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-ac}(b(e^{ax} - e^{ac}) + ade^{ax})}{a}$$

4.16 problem 16

Internal problem ID [12652]
Internal file name [`OUTPUT/11304_Friday_November_03_2023_06_30_06_AM_4856061/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = x^2 + e^x - \sin(x)}$$

With initial conditions

$$[y(2) = -1]$$

4.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= x^2 + e^x - \sin(x) \end{aligned}$$

Hence the ode is

$$y' = x^2 + e^x - \sin(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = x^2 + e^x - \sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

4.16.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int x^2 + e^x - \sin(x)\, dx \\ &= \frac{x^3}{3} + \cos(x) + e^x + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{8}{3} + \cos(2) + e^2 + c_1$$

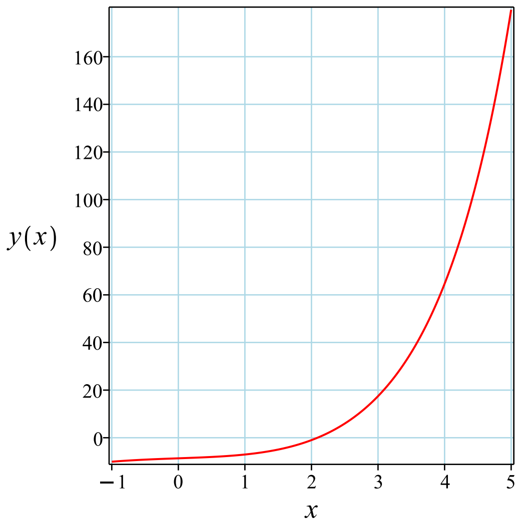
$$c_1 = -\frac{11}{3} - \cos(2) - e^2$$

Substituting c_1 found above in the general solution gives

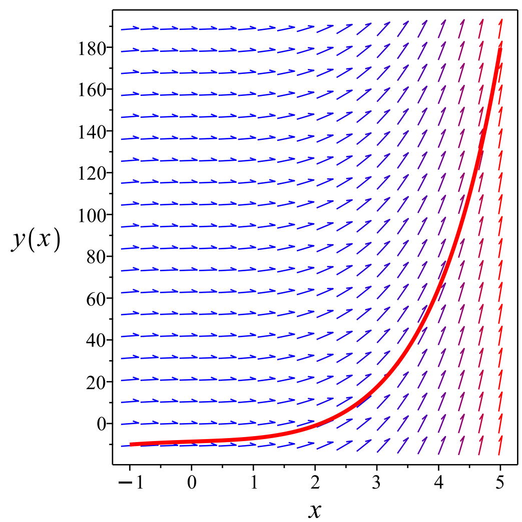
$$y = \frac{x^3}{3} + \cos(x) + e^x - \frac{11}{3} - \cos(2) - e^2$$

Summary
The solution(s) found are the following

$$y = \frac{x^3}{3} + \cos(x) + e^x - \frac{11}{3} - \cos(2) - e^2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^3}{3} + \cos(x) + e^x - \frac{11}{3} - \cos(2) - e^2$$

Verified OK.

4.16.3 Maple step by step solution

Let's solve

$[y' = x^2 + e^x - \sin(x), y(2) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int (x^2 + e^x - \sin(x))\, dx + c_1$

- Evaluate integral

  $y = \frac{x^3}{3} + \cos(x) + e^x + c_1$

- Solve for y

  $y = \frac{x^3}{3} + \cos(x) + e^x + c_1$

- Use initial condition $y(2) = -1$

  $-1 = \frac{8}{3} + \cos(2) + e^2 + c_1$

- Solve for c_1

  $c_1 = -\frac{11}{3} - \cos(2) - e^2$
- Substitute $c_1 = -\frac{11}{3} - \cos(2) - e^2$ into general solution and simplify

  $y = \frac{x^3}{3} + \cos(x) + e^x - \frac{11}{3} - \cos(2) - e^2$
- Solution to the IVP

  $y = \frac{x^3}{3} + \cos(x) + e^x - \frac{11}{3} - \cos(2) - e^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 23

```
dsolve([diff(y(x),x)=x^2+exp(x)-sin(x),y(2) = -1],y(x), singsol=all)
```

$$y(x) = \frac{x^3}{3} + \cos(x) + e^x - \frac{11}{3} - \cos(2) - e^2$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 30

```
DSolve[{y'[x]==x^2+Exp[x]-Sin[x],{y[2]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{3} + e^x + \cos(x) - e^2 - \frac{11}{3} - \cos(2)$$

4.17 problem 17

Internal problem ID [12653]
Internal file name [`OUTPUT/11305_Friday_November_03_2023_06_30_07_AM_38579209/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - yx = \frac{1}{x^2+1}}$$

With initial conditions

$$[y(-5) = 0]$$

4.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -x$$
$$q(x) = \frac{1}{x^2+1}$$

Hence the ode is

$$y' - yx = \frac{1}{x^2 + 1}$$

The domain of $p(x) = -x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -5$ is inside this domain. The domain of $q(x) = \frac{1}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -5$ is also inside this domain. Hence solution exists and is unique.

4.17.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -x dx} \\ &= \mathrm{e}^{-\frac{x^2}{2}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{x^2 + 1}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{x^2}{2}} y\right) &= \left(\mathrm{e}^{-\frac{x^2}{2}}\right)\left(\frac{1}{x^2 + 1}\right) \\ \mathrm{d}\left(\mathrm{e}^{-\frac{x^2}{2}} y\right) &= \left(\frac{\mathrm{e}^{-\frac{x^2}{2}}}{x^2 + 1}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{-\frac{x^2}{2}} y &= \int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{x^2 + 1} \mathrm{d}x \\ \mathrm{e}^{-\frac{x^2}{2}} y &= \int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{x^2 + 1} dx + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{x^2}{2}}$ results in

$$y = \mathrm{e}^{\frac{x^2}{2}}\left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{x^2 + 1} dx\right) + c_1 \mathrm{e}^{\frac{x^2}{2}}$$

which simplifies to

$$y = e^{\frac{x^2}{2}} \left(\int \frac{e^{-\frac{x^2}{2}}}{x^2+1} dx + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = -5$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = e^{\frac{25}{2}} \left(\int^{-5} \frac{e^{-\frac{\_a^2}{2}}}{\_a^2+1} d\_a + c_1\right)$$

$$c_1 = -\left(\int^{-5} \frac{e^{-\frac{\_a^2}{2}}}{\_a^2+1} d\_a\right)$$

Substituting c_1 found above in the general solution gives

$$y = e^{\frac{x^2}{2}} \left(\int \frac{e^{-\frac{x^2}{2}}}{x^2+1} dx - \left(\int^{-5} \frac{e^{-\frac{\_a^2}{2}}}{\_a^2+1} d\_a\right)\right)$$

Summary

The solution(s) found are the following

$$y = e^{\frac{x^2}{2}} \left(\int \frac{e^{-\frac{x^2}{2}}}{x^2+1} dx - \left(\int^{-5} \frac{e^{-\frac{\_a^2}{2}}}{\_a^2+1} d\_a\right)\right) \tag{1}$$

Verification of solutions

$$y = e^{\frac{x^2}{2}} \left(\int \frac{e^{-\frac{x^2}{2}}}{x^2+1} dx - \left(\int^{-5} \frac{e^{-\frac{\_a^2}{2}}}{\_a^2+1} d\_a\right)\right)$$

Verified OK.

4.17.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y\,x^3 + xy + 1}{x^2+1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 132: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\frac{x^2}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{x^2}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{x^2}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y\,x^3 + xy + 1}{x^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -x\,e^{-\frac{x^2}{2}} y \\ S_y &= e^{-\frac{x^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{-\frac{x^2}{2}}}{x^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{-\frac{R^2}{2}}}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{e^{-\frac{R^2}{2}}}{R^2+1} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-\frac{x^2}{2}} y = \int \frac{e^{-\frac{x^2}{2}}}{x^2+1} dx + c_1$$

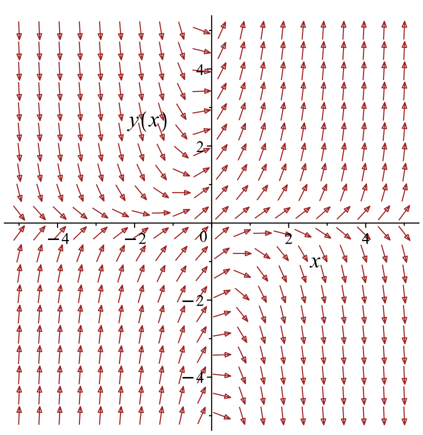
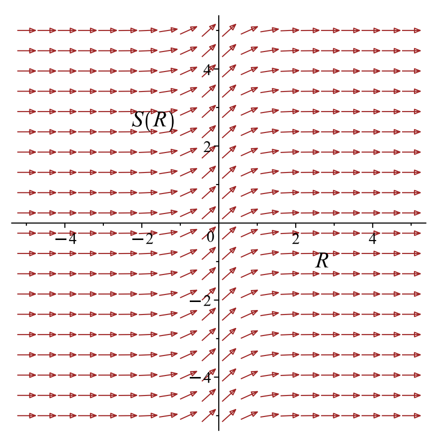
Which simplifies to

$$e^{-\frac{x^2}{2}} y = \int \frac{e^{-\frac{x^2}{2}}}{x^2+1} dx + c_1$$

Which gives

$$y = \left(\int \frac{e^{-\frac{x^2}{2}}}{x^2+1} dx + c_1\right) e^{\frac{x^2}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y x^3+xy+1}{x^2+1}$ 	$R = x$ $S = e^{-\frac{x^2}{2}} y$	$\frac{dS}{dR} = \frac{e^{-\frac{R^2}{2}}}{R^2+1}$ 

Initial conditions are used to solve for c_1. Substituting $x = -5$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = e^{\frac{25}{2}} \left(\int^{-5} \frac{e^{-\frac{\_a^2}{2}}}{\_a^2 + 1} d\_a + c_1 \right)$$

$$c_1 = -\left(\int^{-5} \frac{e^{-\frac{\_a^2}{2}}}{\_a^2 + 1} d\_a \right)$$

Substituting c_1 found above in the general solution gives

$$y = e^{\frac{x^2}{2}} \left(\int \frac{e^{-\frac{x^2}{2}}}{x^2 + 1} dx - \left(\int^{-5} \frac{e^{-\frac{\_a^2}{2}}}{\_a^2 + 1} d\_a \right) \right)$$

Summary

The solution(s) found are the following

$$y = e^{\frac{x^2}{2}} \left(\int \frac{e^{-\frac{x^2}{2}}}{x^2 + 1} dx - \left(\int^{-5} \frac{e^{-\frac{\_a^2}{2}}}{\_a^2 + 1} d\_a \right) \right) \tag{1}$$

Verification of solutions

$$y = e^{\frac{x^2}{2}} \left(\int \frac{e^{-\frac{x^2}{2}}}{x^2 + 1} dx - \left(\int^{-5} \frac{e^{-\frac{\_a^2}{2}}}{\_a^2 + 1} d\_a \right) \right)$$

Verified OK.

4.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(xy + \frac{1}{x^2 + 1}\right)\mathrm{d}x$$
$$\left(-xy - \frac{1}{x^2 + 1}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -xy - \frac{1}{x^2 + 1}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-xy - \frac{1}{x^2 + 1}\right)$$
$$= -x$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-x) - (0)) \\
&= -x
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -x \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{x^2}{2}} \\
&= \mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-\frac{x^2}{2}}\left(-xy - \frac{1}{x^2+1}\right) \\
&= -\frac{\mathrm{e}^{-\frac{x^2}{2}}\left(y\,x^3 + xy + 1\right)}{x^2+1}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\frac{x^2}{2}}(1) \\
&= \mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(-\frac{\mathrm{e}^{-\frac{x^2}{2}}\left(y\,x^3 + xy + 1\right)}{x^2 + 1}\right) + \left(\mathrm{e}^{-\frac{x^2}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int -\frac{\mathrm{e}^{-\frac{x^2}{2}}\left(y\,x^3 + xy + 1\right)}{x^2 + 1}\,\mathrm{d}x$$

$$\phi = \int_{-5}^{x} -\frac{\mathrm{e}^{-\frac{\_a^2}{2}}\left(\_a^3 y + \_ay + 1\right)}{\_a^2 + 1} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= \int_{-5}^{x} -\frac{\mathrm{e}^{-\frac{\_a^2}{2}}\left(\_a^3 + \_a\right)}{\_a^2 + 1} d\_a + f'(y) \\ &= -\left(\int_{-5}^{x} \_a\,\mathrm{e}^{-\frac{\_a^2}{2}} d\_a\right) + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \mathrm{e}^{-\frac{x^2}{2}}$. Therefore equation (4) becomes

$$\mathrm{e}^{-\frac{x^2}{2}} = -\left(\int_{-5}^{x} \_a\,\mathrm{e}^{-\frac{\_a^2}{2}} d\_a\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \int_{-5}^{x} \_a\,\mathrm{e}^{-\frac{\_a^2}{2}} d\_a + \mathrm{e}^{-\frac{x^2}{2}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\int_{-5}^{x} \_a\,\mathrm{e}^{-\frac{\_a^2}{2}} d\_a + \mathrm{e}^{-\frac{x^2}{2}}\right)\mathrm{d}y$$

$$f(y) = \left(\int_{-5}^{x} \_a\,\mathrm{e}^{-\frac{\_a^2}{2}} d\_a + \mathrm{e}^{-\frac{x^2}{2}}\right)y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int_{-5}^{x} -\frac{\mathrm{e}^{-\frac{\_a^2}{2}}(\_a^3 y + \_a y + 1)}{\_a^2 + 1} d\_a + \left(\int_{-5}^{x} \_a\,\mathrm{e}^{-\frac{\_a^2}{2}} d\_a + \mathrm{e}^{-\frac{x^2}{2}}\right)y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int_{-5}^{x} -\frac{\mathrm{e}^{-\frac{\_a^2}{2}}(\_a^3 y + \_a y + 1)}{\_a^2 + 1} d\_a + \left(\int_{-5}^{x} \_a\,\mathrm{e}^{-\frac{\_a^2}{2}} d\_a + \mathrm{e}^{-\frac{x^2}{2}}\right)y$$

The solution becomes

$$y = \frac{\int_{-5}^{x} \frac{\mathrm{e}^{-\frac{\_a^2}{2}}}{\_a^2+1} d\_a + c_1}{\mathrm{e}^{-\frac{x^2}{2}} - \left(\int_{-5}^{x} \frac{\mathrm{e}^{-\frac{\_a^2}{2}}\_a}{\_a^2+1} d\_a\right) - \left(\int_{-5}^{x} \frac{\mathrm{e}^{-\frac{\_a^2}{2}}\_a^3}{\_a^2+1} d\_a\right) + \int_{-5}^{x} \_a\,\mathrm{e}^{-\frac{\_a^2}{2}} d\_a}$$

Initial conditions are used to solve for c_1. Substituting $x = -5$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 \mathrm{e}^{\frac{25}{2}}$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\int_{-5}^{x} \frac{\mathrm{e}^{-\frac{\_a^2}{2}}}{\_a^2+1} d\_a}{\mathrm{e}^{-\frac{x^2}{2}} - \left(\int_{-5}^{x} \frac{\mathrm{e}^{-\frac{\_a^2}{2}}\_a}{\_a^2+1} d\_a\right) - \left(\int_{-5}^{x} \frac{\mathrm{e}^{-\frac{\_a^2}{2}}\_a^3}{\_a^2+1} d\_a\right) + \int_{-5}^{x} \_a\,\mathrm{e}^{-\frac{\_a^2}{2}} d\_a}$$

Summary
The solution(s) found are the following

$$y = \frac{\int_{-5}^{x} \frac{e^{-\frac{\_a^2}{2}}}{\_a^2+1} d\_a}{e^{-\frac{x^2}{2}} - \left(\int_{-5}^{x} \frac{e^{-\frac{\_a^2}{2}} \_a}{\_a^2+1} d\_a\right) - \left(\int_{-5}^{x} \frac{e^{-\frac{\_a^2}{2}} \_a^3}{\_a^2+1} d\_a\right) + \int_{-5}^{x} \_a\, e^{-\frac{\_a^2}{2}} d\_a} \tag{1}$$

Verification of solutions

$$y = \frac{\int_{-5}^{x} \frac{e^{-\frac{\_a^2}{2}}}{\_a^2+1} d\_a}{e^{-\frac{x^2}{2}} - \left(\int_{-5}^{x} \frac{e^{-\frac{\_a^2}{2}} \_a}{\_a^2+1} d\_a\right) - \left(\int_{-5}^{x} \frac{e^{-\frac{\_a^2}{2}} \_a^3}{\_a^2+1} d\_a\right) + \int_{-5}^{x} \_a\, e^{-\frac{\_a^2}{2}} d\_a}$$

Verified OK.

4.17.5 Maple step by step solution

Let's solve

$\left[y' - yx = \frac{1}{x^2+1}, y(-5) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = yx + \frac{1}{x^2+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - yx = \frac{1}{x^2+1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - yx\right) = \frac{\mu(x)}{x^2+1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - yx\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)\, x$

- Solve to find the integrating factor

  $\mu(x) = e^{-\frac{x^2}{2}}$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{x^2+1} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \frac{\mu(x)}{x^2+1} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x)}{x^2+1} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \mathrm{e}^{-\frac{x^2}{2}}$

$$y = \frac{\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{x^2+1} dx + c_1}{\mathrm{e}^{-\frac{x^2}{2}}}$$

- Simplify

$$y = \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{x^2+1} dx + c_1\right) \mathrm{e}^{\frac{x^2}{2}}$$

- Use initial condition $y(-5) = 0$

$$0 = \left(\int^{-5} \frac{\mathrm{e}^{-\frac{\_a^2}{2}}}{\_a^2+1} d\_a + c_1\right) \mathrm{e}^{\frac{25}{2}}$$

- Solve for c_1

$$c_1 = -\left(\int^{-5} \frac{\mathrm{e}^{-\frac{\_a^2}{2}}}{\_a^2+1} d\_a\right)$$

- Substitute $c_1 = -\left(\int^{-5} \frac{\mathrm{e}^{-\frac{\_a^2}{2}}}{\_a^2+1} d\_a\right)$ into general solution and simplify

$$y = \mathrm{e}^{\frac{x^2}{2}} \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{x^2+1} dx - \left(\int^{-5} \frac{\mathrm{e}^{-\frac{\_a^2}{2}}}{\_a^2+1} d\_a\right)\right)$$

- Solution to the IVP

$$y = \mathrm{e}^{\frac{x^2}{2}} \left(\int \frac{\mathrm{e}^{-\frac{x^2}{2}}}{x^2+1} dx - \left(\int^{-5} \frac{\mathrm{e}^{-\frac{\_a^2}{2}}}{\_a^2+1} d\_a\right)\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 31

```
dsolve([diff(y(x),x)=x*y(x)+1/(1+x^2),y(-5) = 0],y(x), singsol=all)
```

$$y(x) = \left(\int_{-5}^{x} \frac{\mathrm{e}^{-\frac{\_z1^2}{2}}}{\_z1^2 + 1} d\_z1\right) \mathrm{e}^{\frac{x^2}{2}}$$

✓ Solution by Mathematica

Time used: 0.478 (sec). Leaf size: 41

```
DSolve[{y'[x]==x*y[x]+1/(1+x^2),{y[-5]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{x^2}{2}} \int_{-5}^{x} \frac{e^{-\frac{1}{2}K[1]^2}}{K[1]^2 + 1} dK[1]$$

4.18 problem 18

Internal problem ID [12654]
Internal file name [`OUTPUT/11306_Friday_November_03_2023_06_30_08_AM_48642611/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y}{x} = \cos(x)}$$

With initial conditions

$$[y(-1) = 0]$$

4.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \cos(x)$$

Hence the ode is

$$y' - \frac{y}{x} = \cos(x)$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = \cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

4.18.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\cos(x)) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)(\cos(x)) \\ d\left(\frac{y}{x}\right) &= \left(\frac{\cos(x)}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{\cos(x)}{x} dx \\ \frac{y}{x} &= \operatorname{Ci}(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x \operatorname{Ci}(x) + c_1 x$$

which simplifies to

$$y = x(\operatorname{Ci}(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -i\pi - \text{Ci}(1) - c_1$$

$$c_1 = -i\pi - \text{Ci}(1)$$

Substituting c_1 found above in the general solution gives

$$y = -i\pi x - \text{Ci}(1)\, x + x\, \text{Ci}(x)$$

Summary

The solution(s) found are the following

$$y = -i\pi x - \text{Ci}(1)\, x + x\, \text{Ci}(x) \tag{1}$$

Verification of solutions

$$y = -i\pi x - \text{Ci}(1)\, x + x\, \text{Ci}(x)$$

Verified OK.

4.18.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x = \cos(x)$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{\cos(x)}{x}\, \mathrm{d}x \\ &= \text{Ci}(x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x(\text{Ci}(x) + c_2) \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -i\pi - \text{Ci}(1) - c_2$$

$$c_2 = -i\pi - \text{Ci}(1)$$

Substituting c_2 found above in the general solution gives

$$y = -i\pi x - \operatorname{Ci}(1)\, x + x \operatorname{Ci}(x)$$

Summary

The solution(s) found are the following

$$y = -i\pi x - \operatorname{Ci}(1)\, x + x \operatorname{Ci}(x) \tag{1}$$

Verification of solutions

$$y = -i\pi x - \operatorname{Ci}(1)\, x + x \operatorname{Ci}(x)$$

Verified OK.

4.18.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + \cos(x)\, x}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 135: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y + \cos(x)\, x}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\cos(x)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\cos(R)}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \operatorname{Ci}(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \operatorname{Ci}(x) + c_1$$

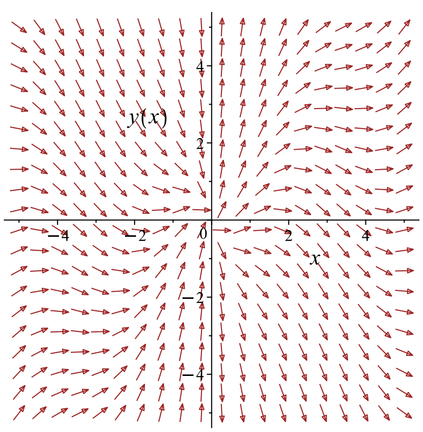
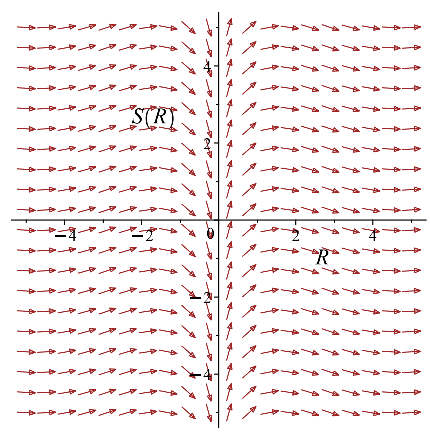
Which simplifies to

$$\frac{y}{x} = \operatorname{Ci}(x) + c_1$$

Which gives

$$y = x(\operatorname{Ci}(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\cos(x)x}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \frac{\cos(R)}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -i\pi - \operatorname{Ci}(1) - c_1$$

$$c_1 = -i\pi - \operatorname{Ci}(1)$$

Substituting c_1 found above in the general solution gives

$$y = -i\pi x - \operatorname{Ci}(1)\, x + x\, \operatorname{Ci}(x)$$

Summary
The solution(s) found are the following

$$y = -i\pi x - \operatorname{Ci}(1)\, x + x\, \operatorname{Ci}(x) \tag{1}$$

Verification of solutions

$$y = -i\pi x - \operatorname{Ci}(1)\, x + x\, \operatorname{Ci}(x)$$

Verified OK.

4.18.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{y}{x} + \cos(x)\right)\mathrm{d}x$$

$$\left(-\frac{y}{x} - \cos(x)\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{y}{x} - \cos(x) \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{x} - \cos(x)\right) \\ &= -\frac{1}{x} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\frac{1}{x}\right) - (0)\right) \\ &= -\frac{1}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(x)} \\ &= \frac{1}{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x}\left(-\frac{y}{x} - \cos(x)\right) \\ &= \frac{-y - \cos(x)\, x}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x}(1) \\ &= \frac{1}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-y - \cos(x)\, x}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \overline{M} \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int \frac{-y - \cos(x)\, x}{x^2} \, dx$$
$$\phi = -\operatorname{Ci}(x) + \frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\operatorname{Ci}(x) + \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\operatorname{Ci}(x) + \frac{y}{x}$$

The solution becomes

$$y = x(\operatorname{Ci}(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -i\pi - \operatorname{Ci}(1) - c_1$$

$$c_1 = -i\pi - \operatorname{Ci}(1)$$

Substituting c_1 found above in the general solution gives

$$y = -i\pi x - \operatorname{Ci}(1)\, x + x \operatorname{Ci}(x)$$

Summary

The solution(s) found are the following

$$y = -i\pi x - \operatorname{Ci}(1)\, x + x \operatorname{Ci}(x) \tag{1}$$

Verification of solutions

$$y = -i\pi x - \operatorname{Ci}(1)\, x + x \operatorname{Ci}(x)$$

Verified OK.

4.18.6 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{x} = \cos(x), y(-1) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + \cos(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = \cos(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)\cos(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

$\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x) \cos(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x) \cos(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) \cos(x) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \frac{\cos(x)}{x} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(\mathrm{Ci}(x) + c_1)$

- Use initial condition $y(-1) = 0$

  $0 = -\mathrm{I}\pi - \mathrm{Ci}(1) - c_1$

- Solve for c_1

  $c_1 = -\mathrm{I}\pi - \mathrm{Ci}(1)$

- Substitute $c_1 = -\mathrm{I}\pi - \mathrm{Ci}(1)$ into general solution and simplify

  $y = (\mathrm{Ci}(x) - \mathrm{I}\pi - \mathrm{Ci}(1))\, x$

- Solution to the IVP

  $y = (\mathrm{Ci}(x) - \mathrm{I}\pi - \mathrm{Ci}(1))\, x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)=y(x)/x+cos(x),y(-1) = 0],y(x), singsol=all)
```

$$y(x) = (\mathrm{Ci}\,(x) - \mathrm{Ci}\,(1) - i\pi)\, x$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 14

```
DSolve[{y'[x]==y[x]/x+Cos[x],{y[-1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(\text{CosIntegral}(x) - \text{CosIntegral}(-1))$$

4.19 problem 19

Internal problem ID [12655]
Internal file name [`OUTPUT/11307_Friday_November_03_2023_06_30_09_AM_28264325/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y}{x} = \tan(x)}$$

With initial conditions

$$[y(\pi) = 0]$$

4.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \tan(x)$$

Hence the ode is

$$y' - \frac{y}{x} = \tan(x)$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = \pi$ is inside this domain. The domain of $q(x) = \tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z117 \vee \frac{1}{2}\pi + \pi\_Z117 < x\right\}$$

And the point $x_0 = \pi$ is also inside this domain. Hence solution exists and is unique.

4.19.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\tan(x)) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)(\tan(x)) \\ d\left(\frac{y}{x}\right) &= \left(\frac{\tan(x)}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{\tan(x)}{x} dx \\ \frac{y}{x} &= -i\ln(x) - i\left(\int -\frac{2}{x(e^{2ix}+1)}dx\right) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(-i\ln(x) - i\left(\int -\frac{2}{x(e^{2ix}+1)}dx\right)\right) + c_1 x$$

which simplifies to

$$y = x\left(2i\left(\int \frac{1}{x\left(e^{2ix}+1\right)}dx\right) - i\ln\left(x\right) + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \pi\left(2i\left(\int^{\pi} \frac{1}{\_a\left(e^{2i\_a}+1\right)}d\_a\right) - i\ln\left(\pi\right) + c_1\right)$$

$$c_1 = -2i\left(\int^{\pi} \frac{1}{\_a\left(e^{2i\_a}+1\right)}d\_a\right) + i\ln\left(\pi\right)$$

Substituting c_1 found above in the general solution gives

$$y = x\left(2i\left(\int \frac{1}{x\left(e^{2ix}+1\right)}dx\right) - i\ln\left(x\right) - 2i\left(\int^{\pi} \frac{1}{\_a\left(e^{2i\_a}+1\right)}d\_a\right) + i\ln\left(\pi\right)\right)$$

But this does not satisfy the initial conditions. Hence no solution can be found.

Verification of solutions N/A

4.19.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x = \tan\left(x\right)$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{\tan\left(x\right)}{x}\,\mathrm{d}x \\ &= -i\ln\left(x\right) - i\left(\int -\frac{2}{x\left(e^{2ix}+1\right)}dx\right) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x\left(-i\ln\left(x\right) - i\left(\int -\frac{2}{x\left(e^{2ix}+1\right)}dx\right) + c_2\right) \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \pi\left(-i\ln\left(\pi\right) + 2i\left(\int^{\pi} \frac{1}{\_a\left(e^{2i\_a}+1\right)}d\_a\right) + c_2\right)$$

$$c_2 = -2i\left(\int^{\pi} \frac{1}{\_a\left(e^{2i\_a}+1\right)}d\_a\right)+i\ln\left(\pi\right)$$

Substituting c_2 found above in the general solution gives

$$y = x\left(-i\ln\left(x\right)-i\left(\int -\frac{2}{x\left(e^{2ix}+1\right)}dx\right)+i\ln\left(\pi\right)-2i\left(\int^{\pi} \frac{1}{\_a\left(e^{2i\_a}+1\right)}d\_a\right)\right)$$

But this does not satisfy the initial conditions. Hence no solution can be found.

Verification of solutions N/A

4.19.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{x\tan\left(x\right)+y}{x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 138: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x \tan(x) + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\tan(x)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\tan(R)}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{\tan(R)}{R} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \int \frac{\tan(x)}{x} dx + c_1$$

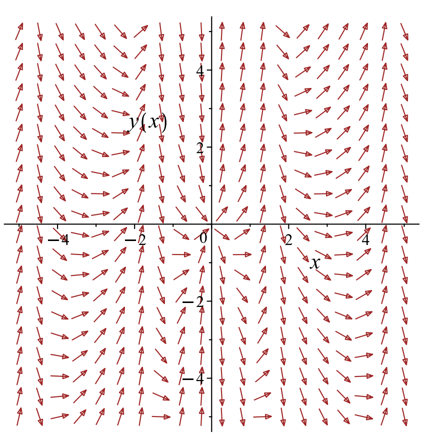
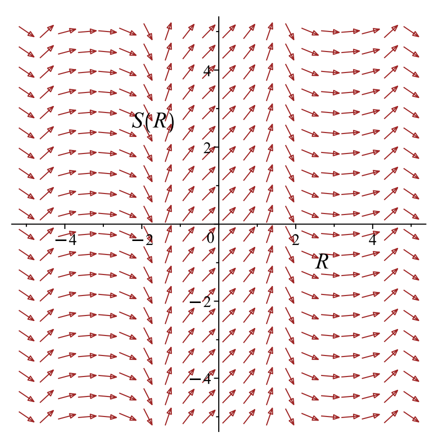
Which simplifies to

$$\frac{y}{x} = \int \frac{\tan(x)}{x} dx + c_1$$

Which gives

$$y = \left(\int \frac{\tan(x)}{x} dx + c_1\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x\tan(x)+y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \frac{\tan(R)}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \left(\int^{\pi} \frac{\tan(\_a)}{\_a} d\_a + c_1\right) \pi$$

$$c_1 = -\left(\int^{\pi} \frac{\tan(\_a)}{\_a} d\_a\right)$$

Substituting c_1 found above in the general solution gives

$$y = \left(\int \frac{\tan(x)}{x} dx - \left(\int^{\pi} \frac{\tan(\_a)}{\_a} d\_a\right)\right) x$$

But this does not satisfy the initial conditions. Hence no solution can be found.

Verification of solutions N/A

4.19.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{y}{x} + \tan(x)\right)\mathrm{d}x$$
$$\left(-\frac{y}{x} - \tan(x)\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{y}{x} - \tan(x)$$
$$N(x,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{x} - \tan(x)\right)\\ &= -\frac{1}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(-\frac{1}{x}\right) - (0)\right)\\ &= -\frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{1}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(x)}\\ &= \frac{1}{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x}\left(-\frac{y}{x}-\tan(x)\right)\\ &= \frac{-x\tan(x)-y}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x}(1)\\ &= \frac{1}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{-x\tan(x)-y}{x^2}\right)+\left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \frac{-x \tan(x) - y}{x^2} \, \mathrm{d}x \\
\phi &= \int_{\pi}^{x} \frac{-\_a \tan(\_a) - y}{\_a^2} d\_a + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \int_{\pi}^{x} -\frac{1}{\_a^2} d\_a + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$
\frac{1}{x} = -\left(\int_{\pi}^{x} \frac{1}{\_a^2} d\_a\right) + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{\left(\int_{\pi}^{x} \frac{1}{\_a^2} d\_a\right) x + 1}{x}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{\left(\int_{\pi}^{x} \frac{1}{\_a^2} d\_a\right) x + 1}{x}\right) \mathrm{d}y \\
f(y) &= \frac{\left(\left(\int_{\pi}^{x} \frac{1}{\_a^2} d\_a\right) x + 1\right) y}{x} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \int_{\pi}^{x} \frac{-\_a \tan(\_a) - y}{\_a^2} d\_a + \frac{\left(\left(\int_{\pi}^{x} \frac{1}{\_a^2} d\_a\right) x + 1\right) y}{x} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int_{\pi}^{x} \frac{-\_a \tan(\_a) - y}{\_a^2} d\_a + \frac{\left(\left(\int_{\pi}^{x} \frac{1}{\_a^2} d\_a\right) x + 1\right) y}{x}$$

The solution becomes

$$y = c_1 x + \left(\int_{\pi}^{x} \frac{\tan(\_a)}{\_a} d\_a\right) x$$

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \pi c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \left(\int_{\pi}^{x} \frac{\tan(\_a)}{\_a} d\_a\right) x$$

Summary
The solution(s) found are the following

$$y = \left(\int_{\pi}^{x} \frac{\tan(\_a)}{\_a} d\_a\right) x \tag{1}$$

Verification of solutions

$$y = \left(\int_{\pi}^{x} \frac{\tan(\_a)}{\_a} d\_a\right) x$$

Verified OK. `{positive}`

4.19.6 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{x} = \tan(x), y(\pi) = 0\right]$

- Highest derivative means the order of the ODE is 1

y'

- Isolate the derivative

  $y' = \frac{y}{x} + \tan(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = \tan(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)\tan(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\tan(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\tan(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\tan(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = \left(\int \frac{\tan(x)}{x} dx + c_1\right) x$

- Evaluate the integrals on the rhs

  $y = \left(-\mathrm{I}\ln(x) - \mathrm{I}\left(\int -\frac{2}{x\left((\mathrm{e}^{\mathrm{I}x})^2+1\right)} dx\right) + c_1\right) x$

- Simplify

  $y = \left(-\mathrm{I}\ln(x) + 2\,\mathrm{I}\left(\int \frac{1}{x(\mathrm{e}^{2\,\mathrm{I}x}+1)} dx\right) + c_1\right) x$

- Use initial condition $y(\pi) = 0$

  $0 = \left(-\mathrm{I}\ln(\pi) + 2\,\mathrm{I}\left(\int^{\pi} \frac{1}{\_a(\mathrm{e}^{2\,\mathrm{I}\_a}+1)} d\_a\right) + c_1\right) \pi$

- Solve for c_1

  $c_1 = \mathrm{I}\ln(\pi) - 2\,\mathrm{I}\left(\int^{\pi} \frac{1}{\_a\left(e^{2\,\mathrm{I}\_a}+1\right)}d\_a\right)$

- Substitute $c_1 = \mathrm{I}\ln(\pi) - 2\,\mathrm{I}\left(\int^{\pi} \frac{1}{\_a\left(e^{2\,\mathrm{I}\_a}+1\right)}d\_a\right)$ into general solution and simplify

  $y = \mathrm{I}\left(\ln(\pi) - \ln(x) - 2\left(\int^{\pi} \frac{1}{\_a\left(e^{2\,\mathrm{I}\_a}+1\right)}d\_a\right) + 2\left(\int \frac{1}{x\left(e^{2\,\mathrm{I}x}+1\right)}dx\right)\right)x$

- Solution to the IVP

  $y = \mathrm{I}\left(\ln(\pi) - \ln(x) - 2\left(\int^{\pi} \frac{1}{\_a\left(e^{2\,\mathrm{I}\_a}+1\right)}d\_a\right) + 2\left(\int \frac{1}{x\left(e^{2\,\mathrm{I}x}+1\right)}dx\right)\right)x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve([diff(y(x),x)=y(x)/x+tan(x),y(Pi) = 0],y(x), singsol=all)
```

$$y(x) = \left(\int_{\pi}^{x} \frac{\tan(\_z1)}{\_z1}d\_z1\right)x$$

✓ Solution by Mathematica

Time used: 1.98 (sec). Leaf size: 22

```
DSolve[{y'[x]==y[x]/x+Tan[x],{y[Pi]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\int_{\pi}^{x} \frac{\tan(K[1])}{K[1]}dK[1]$$

4.20 problem 20

Internal problem ID [12656]
Internal file name [`OUTPUT/11308_Friday_November_03_2023_06_30_10_AM_86605867/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y}{-x^2 + 4} = \sqrt{x}}$$

With initial conditions

$$[y(3) = 4]$$

4.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x^2 - 4}$$
$$q(x) = \sqrt{x}$$

Hence the ode is

$$y' + \frac{y}{x^2 - 4} = \sqrt{x}$$

The domain of $p(x) = \frac{1}{x^2-4}$ is

$$\{-\infty \le x < -2, -2 < x < 2, 2 < x \le \infty\}$$

And the point $x_0 = 3$ is inside this domain. The domain of $q(x) = \sqrt{x}$ is

$$\{0 \le x\}$$

And the point $x_0 = 3$ is also inside this domain. Hence solution exists and is unique.

4.20.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x^2-4}dx} \\ &= e^{\frac{\ln(x-2)}{4} - \frac{\ln(x+2)}{4}}\end{aligned}$$

Which simplifies to

$$\mu = \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\sqrt{x}\right) \\ \frac{d}{dx}\left(\frac{(x-2)^{\frac{1}{4}} y}{(x+2)^{\frac{1}{4}}}\right) &= \left(\frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}\right)\left(\sqrt{x}\right) \\ d\left(\frac{(x-2)^{\frac{1}{4}} y}{(x+2)^{\frac{1}{4}}}\right) &= \left(\frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}\right) dx\end{aligned}$$

Integrating gives

$$\frac{(x-2)^{\frac{1}{4}} y}{(x+2)^{\frac{1}{4}}} = \int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} dx$$

$$\frac{(x-2)^{\frac{1}{4}} y}{(x+2)^{\frac{1}{4}}} = \frac{2(x+2)^{\frac{3}{4}}\sqrt{x}\,(x-2)^{\frac{1}{4}}}{3} + \frac{\left(\int \frac{\frac{4}{3}-\frac{4x}{3}}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}dx\right)\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}{\sqrt{x}\,(x-2)^{\frac{3}{4}}(x+2)^{\frac{1}{4}}} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$ results in

$$y = \frac{(x+2)^{\frac{1}{4}} \left(\frac{2(x+2)^{\frac{3}{4}} \sqrt{x}\, (x-2)^{\frac{1}{4}}}{3} + \frac{\left(\int \frac{\frac{4}{3}-\frac{4x}{3}}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}} dx\right) \left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}{\sqrt{x}\, (x-2)^{\frac{3}{4}} (x+2)^{\frac{1}{4}}}\right)}{(x-2)^{\frac{1}{4}}} + \frac{c_1(x+2)^{\frac{1}{4}}}{(x-2)^{\frac{1}{4}}}$$

which simplifies to

$$y = \frac{3c_1(x+2)^{\frac{1}{4}} (x-2)^{\frac{3}{4}} \sqrt{x} + 2x^3 - 4\left(\int \frac{x-1}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}} dx\right) \left(x^2(x-2)^3 (x+2)\right)^{\frac{1}{4}} - 8x}{\sqrt{x}\, (3x-6)}$$

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = \frac{\left(3c_1 5^{\frac{1}{4}} \sqrt{3} + 30 - 4\left(\int^3 \frac{\_a-1}{\left(\_a^2(\_a-2)^3(\_a+2)\right)^{\frac{1}{4}}} d\_a\right) 45^{\frac{1}{4}}\right) \sqrt{3}}{9}$$

$$c_1 = -\frac{2\sqrt{3}\, 5^{\frac{3}{4}}}{3} + \frac{4\, 5^{\frac{3}{4}}}{5} + \frac{4\left(\int^3 \frac{\_a-1}{\left(\_a^2(\_a-2)^3(\_a+2)\right)^{\frac{1}{4}}} d\_a\right)}{3}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{3\left(-\frac{2\sqrt{3}\, 5^{\frac{3}{4}}}{3} + \frac{4\, 5^{\frac{3}{4}}}{5} + \frac{4\left(\int^3 \frac{\_a-1}{\left(\_a^2(\_a-2)^3(\_a+2)\right)^{\frac{1}{4}}} d\_a\right)}{3}\right) (x+2)^{\frac{1}{4}} (x-2)^{\frac{3}{4}} \sqrt{x} + 2x^3 - 4\left(\int \frac{x-1}{\left(x^2(x-2)^3(x+\right.}\right.}{\sqrt{x}\, (3x-6)}$$

But this does not satisfy the initial conditions. Hence no solution can be found.

Verification of solutions N/A

4.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^{\frac{5}{2}} - 4\sqrt{x} - y}{x^2 - 4}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 141: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \mathrm{e}^{-\frac{\ln(x-2)}{4}+\frac{\ln(x+2)}{4}}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi}=\frac{dy}{\eta}=dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S &= \int\frac{1}{\eta}dy\\ &= \int\frac{1}{\mathrm{e}^{-\frac{\ln(x-2)}{4}+\frac{\ln(x+2)}{4}}}dy\end{aligned}$$

Which results in

$$S=\mathrm{e}^{\ln\left((x-2)^{\frac{1}{4}}\right)+\ln\left(\frac{1}{(x+2)^{\frac{1}{4}}}\right)}y$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{x^{\frac{5}{2}}-4\sqrt{x}-y}{x^2-4}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{(x-2)^{\frac{3}{4}} (x+2)^{\frac{5}{4}}} \\
S_y &= \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{\sqrt{R}\,(R-2)^{\frac{1}{4}}}{(R+2)^{\frac{1}{4}}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \int \frac{\sqrt{R}\,(R-2)^{\frac{1}{4}}}{(R+2)^{\frac{1}{4}}} dR + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{(x-2)^{\frac{1}{4}} y}{(x+2)^{\frac{1}{4}}} = \int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} dx + c_1
$$

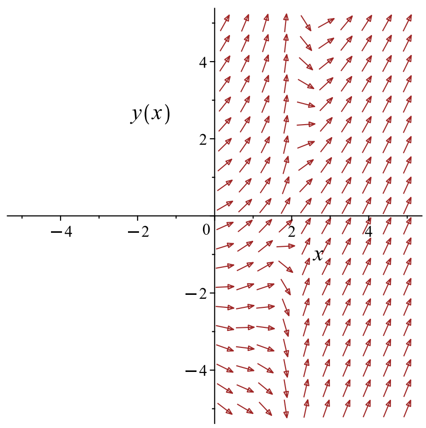
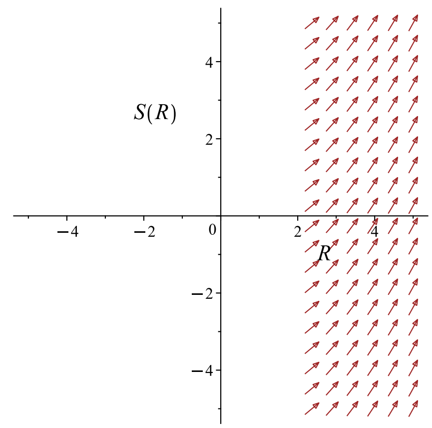
Which simplifies to

$$
\frac{(x-2)^{\frac{1}{4}} y}{(x+2)^{\frac{1}{4}}} = \int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} dx + c_1
$$

Which gives

$$
y = \frac{(x+2)^{\frac{1}{4}} \left(\int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} dx + c_1 \right)}{(x-2)^{\frac{1}{4}}}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{x^{\frac{5}{2}}-4\sqrt{x}-y}{x^{2}-4}$

</td><td>$R = x$
$S = \frac{(x-2)^{\frac{1}{4}} y}{(x+2)^{\frac{1}{4}}}$</td><td>$\frac{dS}{dR} = \frac{\sqrt{R}\,(R-2)^{\frac{1}{4}}}{(R+2)^{\frac{1}{4}}}$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = 5^{\frac{1}{4}} \left(\int^{3} \frac{\sqrt{\_a}\,(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}} d\_a + c_1\right)$$

$$c_1 = -\left(\int^{3} \frac{\sqrt{\_a}\,(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}} d\_a\right) + \frac{4\,5^{\frac{3}{4}}}{5}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{(x+2)^{\frac{1}{4}} \left(\int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} dx - \left(\int^{3} \frac{\sqrt{\_a}\,(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}} d\_a\right) + \frac{4\,5^{\frac{3}{4}}}{5}\right)}{(x-2)^{\frac{1}{4}}}$$

Summary

The solution(s) found are the following

$$y = \frac{(x+2)^{\frac{1}{4}} \left(\int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} dx - \left(\int^{3} \frac{\sqrt{\_a}\,(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}} d\_a\right) + \frac{4\,5^{\frac{3}{4}}}{5}\right)}{(x-2)^{\frac{1}{4}}} \quad (1)$$

Verification of solutions

$$y = \frac{(x+2)^{\frac{1}{4}} \left(\int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}dx - \left(\int^{3} \frac{\sqrt{\_a}\,(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}}d\_a\right) + \frac{4\,5^{\frac{3}{4}}}{5}\right)}{(x-2)^{\frac{1}{4}}}$$

Verified OK.

4.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
dy &= \left(\frac{y}{-x^2+4} + \sqrt{x}\right) dx
\end{aligned}
$$

$$
\left(-\frac{y}{-x^2+4} - \sqrt{x}\right) dx + dy = 0 \tag{2A}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{y}{-x^2+4} - \sqrt{x} \\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{-x^2+4} - \sqrt{x}\right) \\
&= \frac{1}{x^2-4}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{1}{-x^2+4}\right) - (0)\right) \\
&= \frac{1}{x^2-4}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, dx} \\
&= e^{\int \frac{1}{x^2-4} \, dx}
\end{aligned}
$$

The result of integrating gives

$$\mu = e^{\frac{\ln(x-2)}{4} - \frac{\ln(x+2)}{4}}$$
$$= \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\overline{M} = \mu M$$
$$= \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} \left(-\frac{y}{-x^2+4} - \sqrt{x}\right)$$
$$= \frac{\left(\frac{y}{x^2-4} - \sqrt{x}\right)(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$$

And

$$\overline{N} = \mu N$$
$$= \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}(1)$$
$$= \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{dy}{dx} = 0$$
$$\left(\frac{\left(\frac{y}{x^2-4} - \sqrt{x}\right)(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}\right) + \left(\frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}\right)\frac{dy}{dx} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{\left(\frac{y}{x^2-4} - \sqrt{x}\right)(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} \,\mathrm{d}x$$

$$\phi = \int_3^x \frac{\left(\frac{y}{\_a^2-4} - \sqrt{\_a}\right)(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \int_3^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}} d\_a + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$. Therefore equation (4) becomes

$$\frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} = \int_3^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}} d\_a + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{-\left(\int_3^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}} d\_a\right)(x+2)^{\frac{1}{4}} + (x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{-\left(\int_3^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}} d\_a\right)(x+2)^{\frac{1}{4}} + (x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}\right) \mathrm{d}y$$

$$f(y) = \frac{\left(-\left(\int_3^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}} d\_a\right)(x+2)^{\frac{1}{4}} + (x-2)^{\frac{1}{4}}\right) y}{(x+2)^{\frac{1}{4}}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int_3^x \frac{\left(\frac{y}{\_a^2-4} - \sqrt{\_a}\right)(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}} d\_a + \frac{\left(-\left(\int_3^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}} d\_a\right)(x+2)^{\frac{1}{4}} + (x-2)^{\frac{1}{4}}\right) y}{(x+2)^{\frac{1}{4}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int_3^x \frac{\left(\frac{y}{\_a^2-4} - \sqrt{\_a}\right)(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}} d\_a + \frac{\left(-\left(\int_3^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}} d\_a\right)(x+2)^{\frac{1}{4}} + (x-2)^{\frac{1}{4}}\right) y}{(x+2)^{\frac{1}{4}}}$$

The solution becomes

$$y = \frac{(x+2)^{\frac{1}{4}} \left(c_1 + \int_3^x \frac{(\_a-2)^{\frac{1}{4}}\sqrt{\_a}}{(\_a+2)^{\frac{1}{4}}} d\_a\right)}{(x-2)^{\frac{1}{4}}}$$

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = 5^{\frac{1}{4}} c_1$$

$$c_1 = \frac{4\, 5^{\frac{3}{4}}}{5}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{4(x+2)^{\frac{1}{4}}\, 5^{\frac{3}{4}} + 5(x+2)^{\frac{1}{4}} \left(\int_3^x \frac{(\_a-2)^{\frac{1}{4}}\sqrt{\_a}}{(\_a+2)^{\frac{1}{4}}} d\_a\right)}{5\,(x-2)^{\frac{1}{4}}}$$

Summary
The solution(s) found are the following

$$y = \frac{4(x+2)^{\frac{1}{4}} 5^{\frac{3}{4}} + 5(x+2)^{\frac{1}{4}} \left(\int_3^x \frac{(\_a-2)^{\frac{1}{4}} \sqrt{\_a}}{(\_a+2)^{\frac{1}{4}}} d\_a\right)}{5\,(x-2)^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{4(x+2)^{\frac{1}{4}} 5^{\frac{3}{4}} + 5(x+2)^{\frac{1}{4}} \left(\int_3^x \frac{(\_a-2)^{\frac{1}{4}} \sqrt{\_a}}{(\_a+2)^{\frac{1}{4}}} d\_a\right)}{5\,(x-2)^{\frac{1}{4}}}$$

Verified OK.

4.20.5 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{-x^2+4} = \sqrt{x}, y(3) = 4\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x^2-4} + \sqrt{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x^2-4} = \sqrt{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x^2-4}\right) = \mu(x)\sqrt{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{x^2-4}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x^2-4}$

- Solve to find the integrating factor

  $\mu(x) = \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\,\sqrt{x} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \mu(x)\,\sqrt{x} dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)\sqrt{x} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$

$$y = \frac{(x+2)^{\frac{1}{4}} \left(\int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} dx + c_1\right)}{(x-2)^{\frac{1}{4}}}$$

- Evaluate the integrals on the rhs

$$y = \frac{(x+2)^{\frac{1}{4}} \left(\frac{2(x+2)^{\frac{3}{4}}\sqrt{x}\,(x-2)^{\frac{1}{4}}}{3} + \frac{\left(\int \frac{\frac{4}{3}-\frac{4x}{3}}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}} dx\right)\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}{\sqrt{x}\,(x-2)^{\frac{3}{4}}(x+2)^{\frac{1}{4}}} + c_1\right)}{(x-2)^{\frac{1}{4}}}$$

- Simplify

$$y = \frac{3c_1(x+2)^{\frac{1}{4}}(x-2)^{\frac{3}{4}}\sqrt{x} + 2x^3 - 4\left(\int \frac{x-1}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}} dx\right)\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}} - 8x}{\sqrt{x}\,(3x-6)}$$

- Use initial condition $y(3) = 4$

$$4 = \frac{\left(3c_1 5^{\frac{1}{4}}\sqrt{3} + 30 - 4\left(\int^3 \frac{\_\_a - 1}{\left(\_\_a^2(\_\_a-2)^3(\_\_a+2)\right)^{\frac{1}{4}}} d\_\_a\right) 45^{\frac{1}{4}}\right)\sqrt{3}}{9}$$

- Solve for c_1

$$c_1 = -\frac{2\sqrt{3}\,5^{\frac{3}{4}}}{3} + \frac{4\,5^{\frac{3}{4}}}{5} + \frac{4\left(\int^3 \frac{\_\_a-1}{\left(\_\_a^2(\_\_a-2)^3(\_\_a+2)\right)^{\frac{1}{4}}} d\_\_a\right)}{3}$$

- Substitute $c_1 = -\frac{2\sqrt{3}\,5^{\frac{3}{4}}}{3} + \frac{4\,5^{\frac{3}{4}}}{5} + \frac{4\left(\int^3 \frac{\_\_a-1}{\left(\_\_a^2(\_\_a-2)^3(\_\_a+2)\right)^{\frac{1}{4}}} d\_\_a\right)}{3}$ into general solution and simplif

$$y = -\frac{20\left(\left(\int \frac{x-1}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}} dx\right)\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}} - \sqrt{x}\left(-\frac{\sqrt{3}\,5^{\frac{3}{4}}}{2} + \int^3 \frac{\_\_a-1}{\left(\_\_a^2(\_\_a-2)^3(\_\_a+2)\right)^{\frac{1}{4}}} d\_\_a + \frac{3\,5^{\frac{3}{4}}}{5}\right)(x+2)^{\frac{1}{4}}(x-}{\sqrt{x}\,(15x-30)}$$

- Solution to the IVP

$$y = -\frac{20\left(\left(\int \frac{x-1}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}dx\right)\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}} - \sqrt{x}\left(-\frac{\sqrt{3}\,5^{\frac{3}{4}}}{2} + \int^3 \frac{\_a-1}{\left(\_a^2(\_a-2)^3(\_a+2)\right)^{\frac{1}{4}}}d\_a + \frac{3\,5^{\frac{3}{4}}}{5}\right)(x+2)^{\frac{1}{4}}(x-}{\sqrt{x}\,(15x-30)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 41

```
dsolve([diff(y(x),x)=y(x)/(4-x^2)+sqrt(x),y(3) = 4],y(x), singsol=all)
```

$$y(x) = \frac{\left(4\,5^{\frac{3}{4}} + 5\left(\int_3^x \frac{\sqrt{\_z1}\,(\_z1-2)^{\frac{1}{4}}}{(2+\_z1)^{\frac{1}{4}}}d\_z1\right)\right)(x+2)^{\frac{1}{4}}}{5\,(x-2)^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 2.843 (sec). Leaf size: 202

```
DSolve[{y'[x]==y[x]/(4-x^2)+Sqrt[x],{y[3]==4}},y[x],x,IncludeSingularSolutions -> True]
```

$y(x)$

$$\rightarrow \frac{\left(\frac{1}{45}+\frac{i}{45}\right)\sqrt[4]{x+2}\left((10-10i)x^{3/2}\operatorname{AppellF1}\left(\frac{3}{2},\frac{3}{4},\frac{1}{4},\frac{5}{2},\frac{x}{2},-\frac{x}{2}\right) - (30-30i)\sqrt{x}\operatorname{AppellF1}\left(\frac{1}{2},\frac{3}{4},\frac{1}{4},\frac{3}{2},\frac{x}{2},-\right.\right.}{}$$

4.21 problem 21

Internal problem ID [12657]
Internal file name [`OUTPUT/11309_Friday_November_03_2023_06_30_12_AM_76026492/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' - \frac{y}{-x^2 + 4} = \sqrt{x}}$$

With initial conditions

$$[y(1) = -3]$$

4.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x^2 - 4} \\ q(x) &= \sqrt{x} \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x^2 - 4} = \sqrt{x}$$

The domain of $p(x) = \frac{1}{x^2-4}$ is

$$\{-\infty \leq x < -2, -2 < x < 2, 2 < x \leq \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \sqrt{x}$ is

$$\{0 \leq x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

4.21.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x^2-4}dx} \\ &= e^{\frac{\ln(x-2)}{4} - \frac{\ln(x+2)}{4}}\end{aligned}$$

Which simplifies to

$$\mu = \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\sqrt{x}\right) \\ \frac{d}{dx}\left(\frac{(x-2)^{\frac{1}{4}} y}{(x+2)^{\frac{1}{4}}}\right) &= \left(\frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}\right)\left(\sqrt{x}\right) \\ d\left(\frac{(x-2)^{\frac{1}{4}} y}{(x+2)^{\frac{1}{4}}}\right) &= \left(\frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}\right) dx\end{aligned}$$

Integrating gives

$$\frac{(x-2)^{\frac{1}{4}} y}{(x+2)^{\frac{1}{4}}} = \int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}\, dx$$

$$\frac{(x-2)^{\frac{1}{4}} y}{(x+2)^{\frac{1}{4}}} = \frac{2(x+2)^{\frac{3}{4}}\sqrt{x}\,(x-2)^{\frac{1}{4}}}{3} + \frac{\left(\int \frac{\frac{4}{3}-\frac{4x}{3}}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}dx\right)\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}{\sqrt{x}\,(x-2)^{\frac{3}{4}}(x+2)^{\frac{1}{4}}} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$ results in

$$y = \frac{(x+2)^{\frac{1}{4}}\left(\frac{2(x+2)^{\frac{3}{4}}\sqrt{x}\,(x-2)^{\frac{1}{4}}}{3} + \frac{\left(\int \frac{\frac{4}{3}-\frac{4x}{3}}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}dx\right)\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}{\sqrt{x}\,(x-2)^{\frac{3}{4}}(x+2)^{\frac{1}{4}}}\right)}{(x-2)^{\frac{1}{4}}} + \frac{c_1(x+2)^{\frac{1}{4}}}{(x-2)^{\frac{1}{4}}}$$

which simplifies to

$$y = \frac{3c_1(x+2)^{\frac{1}{4}}(x-2)^{\frac{3}{4}}\sqrt{x} + 2x^3 - 4\left(\int \frac{x-1}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}dx\right)\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}} - 8x}{\sqrt{x}\,(3x-6)}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = -c_1 3^{\frac{1}{4}}(-1)^{\frac{3}{4}} + 2 + \frac{4\left(\int^1 \frac{\_a-1}{\left(\_a^2(\_a-2)^3(\_a+2)\right)^{\frac{1}{4}}}d\_a\right)(-3)^{\frac{1}{4}}}{3}$$

$$c_1 = \frac{i\left(5i\sqrt{2}\,3^{\frac{3}{4}} - 5\,3^{\frac{3}{4}}\sqrt{2} - 8\left(\int^1 \frac{\_a-1}{\left(\_a^2(\_a-2)^3(\_a+2)\right)^{\frac{1}{4}}}d\_a\right)\right)}{6}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\frac{i\left(5i\sqrt{2}\,3^{\frac{3}{4}} - 5\,3^{\frac{3}{4}}\sqrt{2} - 8\left(\int^1 \frac{\_a-1}{\left(\_a^2(\_a-2)^3(\_a+2)\right)^{\frac{1}{4}}}d\_a\right)\right)(x+2)^{\frac{1}{4}}(x-2)^{\frac{3}{4}}\sqrt{x}}{2} + 2x^3 - 4\left(\int \frac{x-1}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}dx\right)\left(x^2(\right.}{\sqrt{x}\,(3x-6)}$$

But this does not satisfy the initial conditions. Hence no solution can be found.

Verification of solutions N/A

4.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x^{\frac{5}{2}} - 4\sqrt{x} - y}{x^2 - 4} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 144: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \mathrm{e}^{-\frac{\ln(x-2)}{4}+\frac{\ln(x+2)}{4}}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\mathrm{e}^{-\frac{\ln(x-2)}{4}+\frac{\ln(x+2)}{4}}} dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{\ln\left((x-2)^{\frac{1}{4}}\right)+\ln\left(\frac{1}{(x+2)^{\frac{1}{4}}}\right)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^{\frac{5}{2}} - 4\sqrt{x} - y}{x^2 - 4}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{(x-2)^{\frac{3}{4}} (x+2)^{\frac{5}{4}}} \\
S_y &= \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{\sqrt{R}\,(R-2)^{\frac{1}{4}}}{(R+2)^{\frac{1}{4}}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \int \frac{\sqrt{R}\,(R-2)^{\frac{1}{4}}}{(R+2)^{\frac{1}{4}}} dR + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{(x-2)^{\frac{1}{4}}\, y}{(x+2)^{\frac{1}{4}}} = \int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} dx + c_1
$$

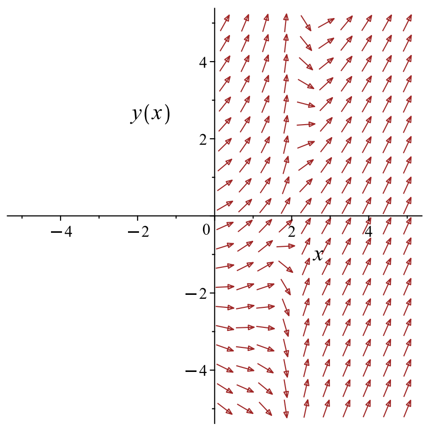
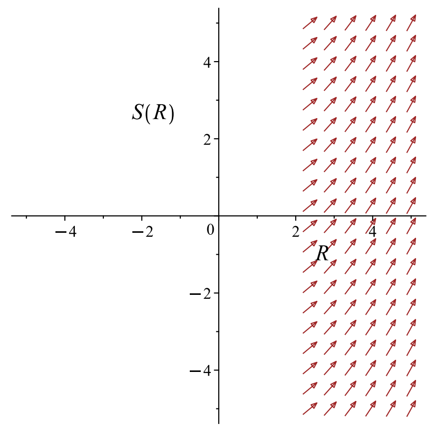
Which simplifies to

$$
\frac{(x-2)^{\frac{1}{4}}\, y}{(x+2)^{\frac{1}{4}}} = \int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} dx + c_1
$$

Which gives

$$
y = \frac{(x+2)^{\frac{1}{4}} \left(\int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} dx + c_1\right)}{(x-2)^{\frac{1}{4}}}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^{\frac{5}{2}}-4\sqrt{x}-y}{x^2-4}$ 	$R = x$ $S = \frac{(x-2)^{\frac{1}{4}} y}{(x+2)^{\frac{1}{4}}}$	$\frac{dS}{dR} = \frac{\sqrt{R}\,(R-2)^{\frac{1}{4}}}{(R+2)^{\frac{1}{4}}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = -3^{\frac{1}{4}} \left(\int^{1} \frac{\sqrt{\_a}\,(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}} d\_a + c_1 \right) (-1)^{\frac{3}{4}}$$

$$c_1 = -\frac{i\sqrt{2}\,3^{\frac{3}{4}}}{2} - \frac{3^{\frac{3}{4}}\sqrt{2}}{2} - \left(\int^{1} \frac{\sqrt{\_a}\,(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}} d\_a \right)$$

Substituting c_1 found above in the general solution gives

$$y = \frac{(x+2)^{\frac{1}{4}} \left(\int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} dx - \frac{i\sqrt{2}\,3^{\frac{3}{4}}}{2} - \frac{3^{\frac{3}{4}}\sqrt{2}}{2} - \left(\int^{1} \frac{\sqrt{\_a}\,(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}} d\_a \right) \right)}{(x-2)^{\frac{1}{4}}}$$

Summary

The solution(s) found are the following

$$y = \frac{(x+2)^{\frac{1}{4}} \left(\int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} dx - \frac{i\sqrt{2}\,3^{\frac{3}{4}}}{2} - \frac{3^{\frac{3}{4}}\sqrt{2}}{2} - \left(\int^{1} \frac{\sqrt{\_a}\,(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}} d\_a \right) \right)}{(x-2)^{\frac{1}{4}}} \quad (1)$$

Verification of solutions

$$y = \frac{(x+2)^{\frac{1}{4}} \left(\int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}dx - \frac{i\sqrt{2}\,3^{\frac{3}{4}}}{2} - \frac{3^{\frac{3}{4}}\sqrt{2}}{2} - \left(\int^{1} \frac{\sqrt{\_a}\,(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}}d\_a\right)\right)}{(x-2)^{\frac{1}{4}}}$$

Verified OK. `{positive}`

4.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$dy = \left(\frac{y}{-x^2+4} + \sqrt{x}\right) dx$$

$$\left(-\frac{y}{-x^2+4} - \sqrt{x}\right) dx + dy = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{y}{-x^2+4} - \sqrt{x} \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{-x^2+4} - \sqrt{x}\right) \\ &= \frac{1}{x^2-4} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\frac{1}{-x^2+4}\right) - (0)\right) \\ &= \frac{1}{x^2-4} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,dx} \\ &= e^{\int \frac{1}{x^2-4}\,dx} \end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\frac{\ln(x-2)}{4}-\frac{\ln(x+2)}{4}} \\
&= \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}\left(-\frac{y}{-x^2+4}-\sqrt{x}\right) \\
&= \frac{\left(\frac{y}{x^2-4}-\sqrt{x}\right)(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}(1) \\
&= \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{\left(\frac{y}{x^2-4}-\sqrt{x}\right)(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}\right)+\left(\frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{\left(\frac{y}{x^2-4} - \sqrt{x}\right)(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} \,\mathrm{d}x$$

$$\phi = \int_1^x \frac{\left(\frac{y}{\_a^2-4} - \sqrt{\_a}\right)(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \int_1^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}} d\_a + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$. Therefore equation (4) becomes

$$\frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}} = \int_1^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}} d\_a + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{\left(\int_1^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}} d\_a\right)(x+2)^{\frac{1}{4}} - (x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$$

$$= \frac{-\left(\int_1^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}} d\_a\right)(x+2)^{\frac{1}{4}} + (x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$$

Integrating the above w.r.t y results in

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{-\left(\int_1^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}} d\_a\right)(x+2)^{\frac{1}{4}} + (x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}\right) \mathrm{d}y$$

$$f(y) = \frac{\left(-\left(\int_1^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}}d\_a\right)(x+2)^{\frac{1}{4}} + (x-2)^{\frac{1}{4}}\right)y}{(x+2)^{\frac{1}{4}}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int_1^x \frac{\left(\frac{y}{\_a^2-4} - \sqrt{\_a}\right)(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}}d\_a + \frac{\left(-\left(\int_1^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}}d\_a\right)(x+2)^{\frac{1}{4}} + (x-2)^{\frac{1}{4}}\right)y}{(x+2)^{\frac{1}{4}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int_1^x \frac{\left(\frac{y}{\_a^2-4} - \sqrt{\_a}\right)(\_a-2)^{\frac{1}{4}}}{(\_a+2)^{\frac{1}{4}}}d\_a + \frac{\left(-\left(\int_1^x \frac{(\_a-2)^{\frac{1}{4}}}{(\_a^2-4)(\_a+2)^{\frac{1}{4}}}d\_a\right)(x+2)^{\frac{1}{4}} + (x-2)^{\frac{1}{4}}\right)y}{(x+2)^{\frac{1}{4}}}$$

The solution becomes

$$y = \frac{(x+2)^{\frac{1}{4}}\left(c_1 + \int_1^x \frac{(\_a-2)^{\frac{1}{4}}\sqrt{\_a}}{(\_a+2)^{\frac{1}{4}}}d\_a\right)}{(x-2)^{\frac{1}{4}}}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = \left(\frac{1}{2} - \frac{i}{2}\right)c_1 3^{\frac{1}{4}}\sqrt{2}$$

$$c_1 = \left(-\frac{1}{2} - \frac{i}{2}\right)3^{\frac{3}{4}}\sqrt{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-i\sqrt{2}\, 3^{\frac{3}{4}} (x+2)^{\frac{1}{4}} - \sqrt{2}\, 3^{\frac{3}{4}} (x+2)^{\frac{1}{4}} + 2(x+2)^{\frac{1}{4}} \left(\int_1^x \frac{(\_a-2)^{\frac{1}{4}} \sqrt{\_a}}{(\_a+2)^{\frac{1}{4}}} d\_a\right)}{2\,(x-2)^{\frac{1}{4}}}$$

Summary

The solution(s) found are the following

$$y = \frac{-i\sqrt{2}\, 3^{\frac{3}{4}} (x+2)^{\frac{1}{4}} - \sqrt{2}\, 3^{\frac{3}{4}} (x+2)^{\frac{1}{4}} + 2(x+2)^{\frac{1}{4}} \left(\int_1^x \frac{(\_a-2)^{\frac{1}{4}} \sqrt{\_a}}{(\_a+2)^{\frac{1}{4}}} d\_a\right)}{2\,(x-2)^{\frac{1}{4}}} \tag{1}$$

Verification of solutions

$$y = \frac{-i\sqrt{2}\, 3^{\frac{3}{4}} (x+2)^{\frac{1}{4}} - \sqrt{2}\, 3^{\frac{3}{4}} (x+2)^{\frac{1}{4}} + 2(x+2)^{\frac{1}{4}} \left(\int_1^x \frac{(\_a-2)^{\frac{1}{4}} \sqrt{\_a}}{(\_a+2)^{\frac{1}{4}}} d\_a\right)}{2\,(x-2)^{\frac{1}{4}}}$$

Verified OK. `{positive}`

4.21.5 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{-x^2+4} = \sqrt{x}, y(1) = -3\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x^2-4} + \sqrt{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x^2-4} = \sqrt{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x^2-4}\right) = \mu(x)\sqrt{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x^2-4}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x^2-4}$

- Solve to find the integrating factor

$$\mu(x) = \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\sqrt{x}dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \mu(x)\sqrt{x}dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)\sqrt{x}dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}$

$$y = \frac{(x+2)^{\frac{1}{4}}\left(\int \frac{\sqrt{x}\,(x-2)^{\frac{1}{4}}}{(x+2)^{\frac{1}{4}}}dx + c_1\right)}{(x-2)^{\frac{1}{4}}}$$

- Evaluate the integrals on the rhs

$$y = \frac{(x+2)^{\frac{1}{4}}\left(\frac{2(x+2)^{\frac{3}{4}}\sqrt{x}\,(x-2)^{\frac{1}{4}}}{3} + \frac{\left(\int \frac{\frac{4}{3}-\frac{4x}{3}}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}dx\right)\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}{\sqrt{x}\,(x-2)^{\frac{3}{4}}(x+2)^{\frac{1}{4}}} + c_1\right)}{(x-2)^{\frac{1}{4}}}$$

- Simplify

$$y = \frac{3c_1(x+2)^{\frac{1}{4}}(x-2)^{\frac{3}{4}}\sqrt{x} + 2x^3 - 4\left(\int \frac{x-1}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}dx\right)\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}} - 8x}{\sqrt{x}\,(3x-6)}$$

- Use initial condition $y(1) = -3$

$$-3 = -c_1 3^{\frac{1}{4}}(-1)^{\frac{3}{4}} + 2 + \frac{4\left(\int^1 \frac{\_a-1}{\left(\_a^2(\_a-2)^3(\_a+2)\right)^{\frac{1}{4}}}d\_a\right)(-3)^{\frac{1}{4}}}{3}$$

- Solve for c_1

$$c_1 = \frac{\mathrm{I}}{6}\left(5\,\mathrm{I}\sqrt{2}\,3^{\frac{3}{4}} - 5\,3^{\frac{3}{4}}\sqrt{2} - 8\left(\int^1 \frac{\_a-1}{\left(\_a^2(\_a-2)^3(\_a+2)\right)^{\frac{1}{4}}}d\_a\right)\right)$$

- Substitute $c_1 = \frac{\mathrm{I}}{6}\left(5\,\mathrm{I}\sqrt{2}\,3^{\frac{3}{4}} - 5\,3^{\frac{3}{4}}\sqrt{2} - 8\left(\int^1 \frac{\_a-1}{\left(\_a^2(\_a-2)^3(\_a+2)\right)^{\frac{1}{4}}}d\_a\right)\right)$ into general solution

$$y = -\frac{8\left(\left(\int \frac{x-1}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}dx\right)\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}+\sqrt{x}\left(\left(\frac{5}{8}+\frac{5\,\mathrm{I}}{8}\right)3^{\frac{3}{4}}\sqrt{2}+\mathrm{I}\left(\int^1 \frac{\_a-1}{\left(\_a^2(\_a-2)^3(\_a+2)\right)^{\frac{1}{4}}}d\_a\right)\right)(x+2)^{\frac{1}{4}}}{\sqrt{x}\,(6x-12)}$$

- Solution to the IVP

$$y = -\frac{8\left(\left(\int \frac{x-1}{\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}}dx\right)\left(x^2(x-2)^3(x+2)\right)^{\frac{1}{4}}+\sqrt{x}\left(\left(\frac{5}{8}+\frac{5\,\mathrm{I}}{8}\right)3^{\frac{3}{4}}\sqrt{2}+\mathrm{I}\left(\int^1 \frac{\_a-1}{\left(\_a^2(\_a-2)^3(\_a+2)\right)^{\frac{1}{4}}}d\_a\right)\right)(x+2)^{\frac{1}{4}}}{\sqrt{x}\,(6x-12)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 44

```
dsolve([diff(y(x),x)=y(x)/(4-x^2)+sqrt(x),y(1) = -3],y(x), singsol=all)
```

$$y(x) = -\frac{(x+2)^{\frac{1}{4}}\left(-2\left(\int_1^x \frac{\sqrt{\_z1}\,(\_z1-2)^{\frac{1}{4}}}{(2+\_z1)^{\frac{1}{4}}}d\_z1\right)+(1+i)\sqrt{2}\,3^{\frac{3}{4}}\right)}{2\,(x-2)^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 158

```
DSolve[{y'[x]==y[x]/(4-x^2)+Sqrt[x],{y[1]==-3}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[4]{x+2}\left(4x^{3/2}\,\text{AppellF1}\left(\frac{3}{2},\frac{3}{4},\frac{1}{4},\frac{5}{2},\frac{x}{2},-\frac{x}{2}\right)-12\sqrt{x}\,\text{AppellF1}\left(\frac{1}{2},\frac{3}{4},\frac{1}{4},\frac{3}{2},\frac{x}{2},-\frac{x}{2}\right)-4\,\text{AppellF1}\left(\frac{3}{2},\frac{3}{4},\frac{1}{4},\frac{5}{2},\right.\right.}{9\sqrt[4]{2-x}}$$

4.22 problem 22

Internal problem ID [12658]
Internal file name [`OUTPUT/11310_Friday_November_03_2023_06_30_15_AM_34322720/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - y\cot(x) = \csc(x)}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 1\right]$$

4.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\cot(x)$$
$$q(x) = \csc(x)$$

Hence the ode is

$$y' - y\cot(x) = \csc(x)$$

The domain of $p(x) = -\cot(x)$ is

$$\{x < \pi\_Z118 \lor \pi\_Z118 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The domain of $q(x) = \csc(x)$ is

$$\{x < \pi\_Z118 \lor \pi\_Z118 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

4.22.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\cot(x)dx} \\ &= \frac{1}{\sin(x)}\end{aligned}$$

Which simplifies to

$$\mu = \csc(x)$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\csc(x)) \\ \frac{d}{dx}(\csc(x)\,y) &= (\csc(x))(\csc(x)) \\ d(\csc(x)\,y) &= \csc(x)^2\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\csc(x)\,y &= \int \csc(x)^2\,dx \\ \csc(x)\,y &= -\cot(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \csc(x)$ results in

$$y = -\sin(x)\cot(x) + \sin(x)\,c_1$$

which simplifies to

$$y = \sin(x) c_1 - \cos(x)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

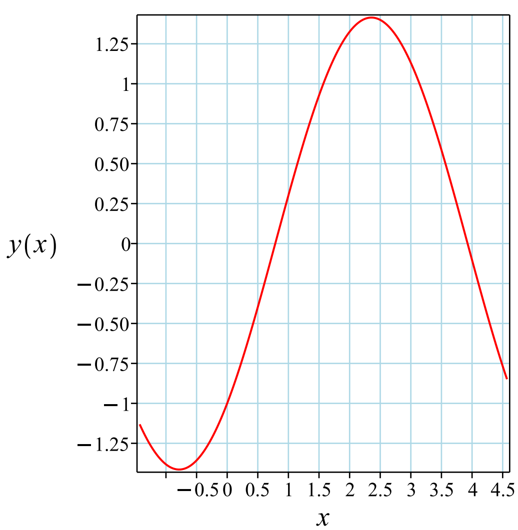
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

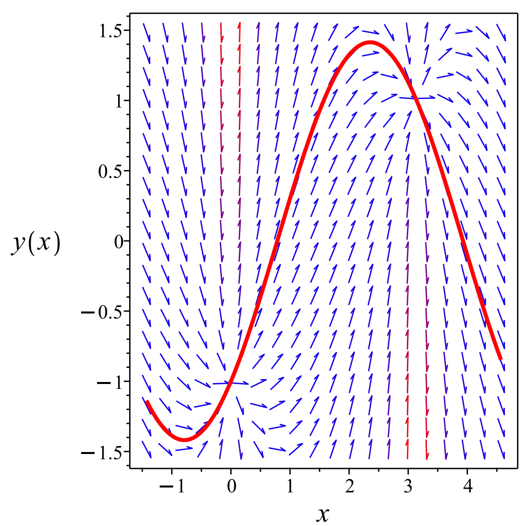
$$y = -\cos(x) + \sin(x)$$

Summary
The solution(s) found are the following

$$y = -\cos(x) + \sin(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\cos(x) + \sin(x)$$

Verified OK.

4.22.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= y\cot(x) + \csc(x) \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 147: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \sin(x)\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\sin(x)} dy\end{aligned}$$

Which results in

$$S = \frac{y}{\sin(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = y\cot(x) + \csc(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= -\csc(x)\cot(x)\,y\\ S_y &= \csc(x)\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \csc(x)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \csc(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\cot(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\csc(x)\, y = -\cot(x) + c_1$$

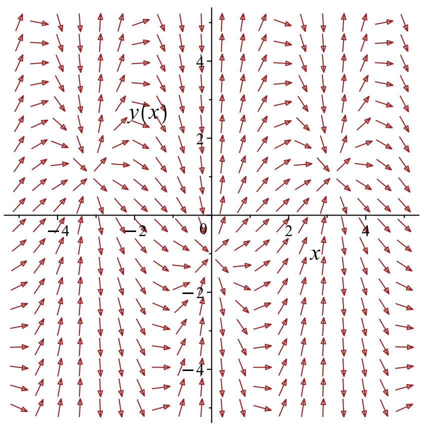
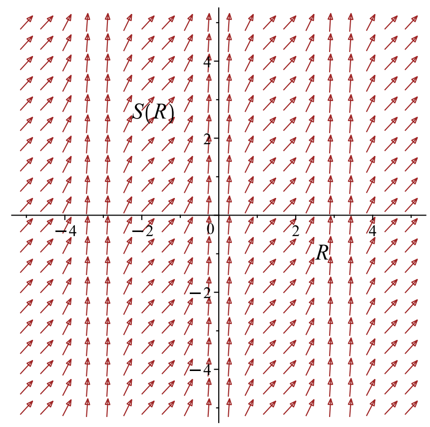
Which simplifies to

$$\csc(x)\, y = -\cot(x) + c_1$$

Which gives

$$y = -\frac{\cot(x) - c_1}{\csc(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y \cot(x) + \csc(x)$ 	$R = x$ $S = \csc(x) y$	$\frac{dS}{dR} = \csc(R)^2$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

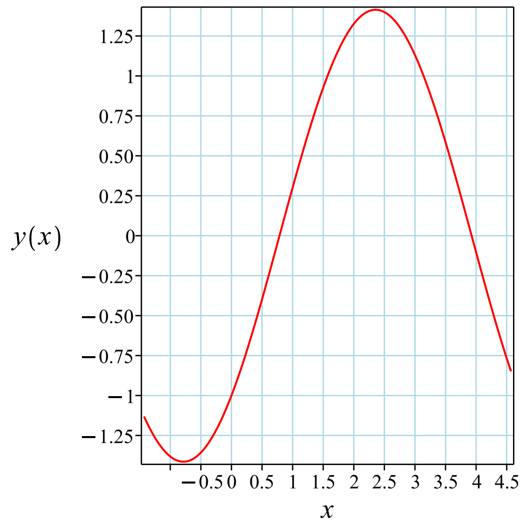
Substituting c_1 found above in the general solution gives

$$y = -\sin(x)\cot(x) + \sin(x)$$

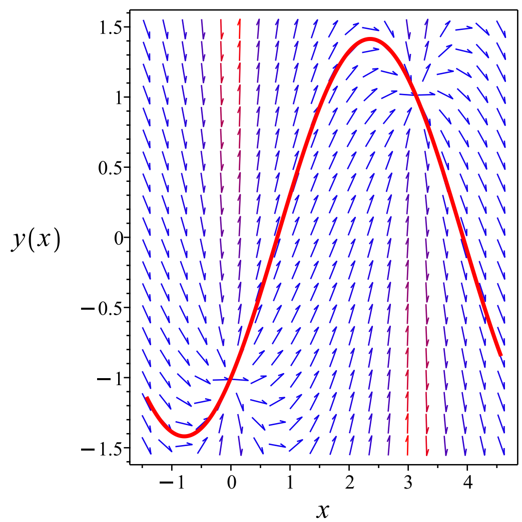
Summary

The solution(s) found are the following

$$y = -\sin(x)\cot(x) + \sin(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\sin(x)\cot(x) + \sin(x)$$

Verified OK.

4.22.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (y\cot(x) + \csc(x))\,\mathrm{d}x\\ (-y\cot(x) - \csc(x))\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -y\cot(x) - \csc(x)\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y\cot(x) - \csc(x))\\ &= -\cot(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((-\cot(x)) - (0))\\ &= -\cot(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\cot(x)\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(\sin(x))}\\ &= \csc(x)\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \csc(x)\,(-y\cot(x) - \csc(x))\\ &= \csc(x)^2\,(-\cos(x)\,y - 1)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \csc(x)\,(1)\\ &= \csc(x)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\csc(x)^2\,(-\cos(x)\,y - 1)\right) + (\csc(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x\\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\csc(x)^2\,(-\cos(x)\,y - 1)\,\mathrm{d}x\end{aligned}$$

$$\phi = \cot(x) + \csc(x)\,y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \csc(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \csc(x)$. Therefore equation (4) becomes

$$\csc(x) = \csc(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \cot(x) + \csc(x)\, y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cot(x) + \csc(x)\, y$$

The solution becomes

$$y = -\frac{\cot(x) - c_1}{\csc(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

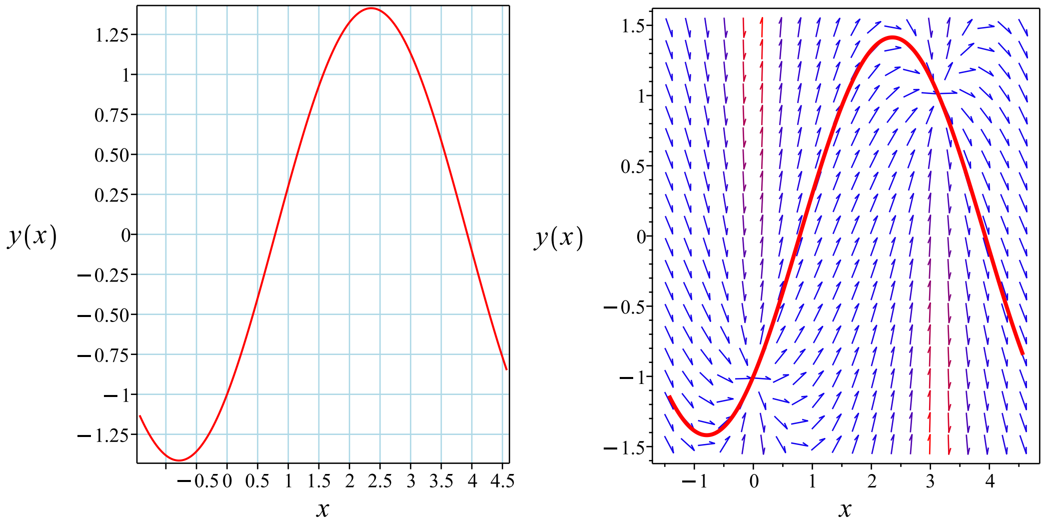
Substituting c_1 found above in the general solution gives

$$y = -\sin(x)\cot(x) + \sin(x)$$

Summary

The solution(s) found are the following

$$y = -\sin(x)\cot(x) + \sin(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\sin(x)\cot(x) + \sin(x)$$

Verified OK.

4.22.5 Maple step by step solution

Let's solve

$\left[y' - y\cot(x) = \csc(x), y\left(\frac{\pi}{2}\right) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y\cot(x) + \csc(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y\cot(x) = \csc(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y'-y\cot(x)\right)=\mu(x)\csc(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$\mu(x)\left(y'-y\cot(x)\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

$\mu'(x)=-\mu(x)\cot(x)$

- Solve to find the integrating factor

$\mu(x)=\frac{1}{\sin(x)}$

- Integrate both sides with respect to x

$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\csc(x)\,dx+c_1$

- Evaluate the integral on the lhs

$\mu(x)\,y=\int\mu(x)\csc(x)\,dx+c_1$

- Solve for y

$y=\frac{\int\mu(x)\csc(x)dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\frac{1}{\sin(x)}$

$y=\sin(x)\left(\int\frac{\csc(x)}{\sin(x)}dx+c_1\right)$

- Evaluate the integrals on the rhs

$y=\sin(x)\left(-\cot(x)+c_1\right)$

- Simplify

$y=\sin(x)\,c_1-\cos(x)$

- Use initial condition $y\left(\frac{\pi}{2}\right)=1$

$1=c_1$

- Solve for c_1

$c_1=1$

- Substitute $c_1=1$ into general solution and simplify

$y=-\cos(x)+\sin(x)$

- Solution to the IVP

$y=-\cos(x)+\sin(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 11

```
dsolve([diff(y(x),x)=cot(x)*y(x)+csc(x),y(1/2*Pi) = 1],y(x), singsol=all)
```

$$y(x) = -\cos(x) + \sin(x)$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 12

```
DSolve[{y'[x]==Cot[x]*y[x]+Csc[x],{y[Pi/2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x) - \cos(x)$$

4.23 problem 23

Internal problem ID [12659]
Internal file name [`OUTPUT/11311_Friday_November_03_2023_06_30_16_AM_72689874/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' + x\sqrt{1 - y^2} = 0}$$

With initial conditions

$$[y(0) = 1]$$

4.23.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -x\sqrt{-y^2 + 1} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-x\sqrt{-y^2+1}\right)\\ &= \frac{xy}{\sqrt{-y^2+1}}\end{aligned}$$

$\frac{\partial f}{\partial y}$ is not defined at $y = 1$ therefore existence and uniqueness theorem do not apply. Solution exist but not guaranteed to be unique.

4.23.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= -x\sqrt{-y^2+1}\end{aligned}$$

Where $f(x) = -x$ and $g(y) = \sqrt{-y^2+1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\sqrt{-y^2+1}}\,dy &= -x\,dx\\ \int \frac{1}{\sqrt{-y^2+1}}\,dy &= \int -x\,dx\\ \arcsin(y) &= -\frac{x^2}{2} + c_1\end{aligned}$$

Which results in

$$y = \sin\left(-\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \sin(c_1)$$

$$c_1 = \frac{\pi}{2}$$

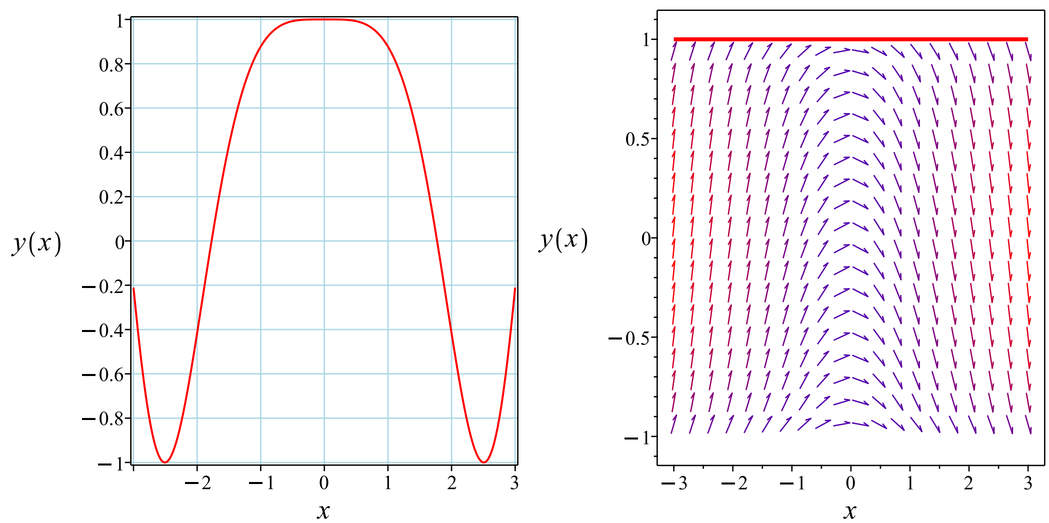
Substituting c_1 found above in the general solution gives

$$y = \cos\left(\frac{x^2}{2}\right)$$

Summary

The solution(s) found are the following

$$y = \cos\left(\frac{x^2}{2}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \cos\left(\frac{x^2}{2}\right)$$

Verified OK.

4.23.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -x\sqrt{-y^2+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 150: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{1}{x} \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{1}{x}} dx
\end{aligned}
$$

Which results in

$$
S = -\frac{x^2}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -x\sqrt{-y^2+1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{-y^2 + 1}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2 + 1}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arcsin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{2} = \arcsin(y) + c_1$$

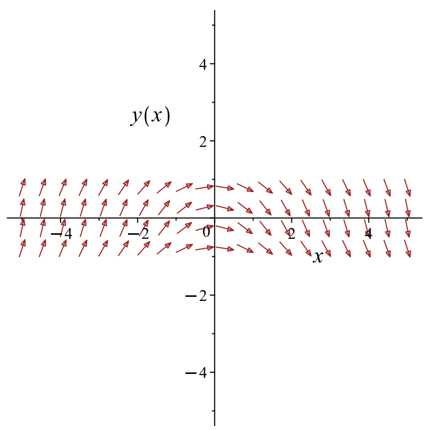
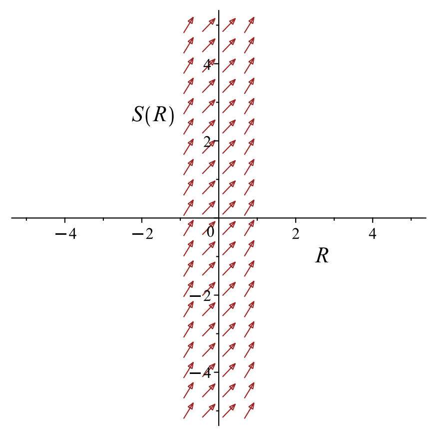
Which simplifies to

$$-\frac{x^2}{2} = \arcsin(y) + c_1$$

Which gives

$$y = -\sin\left(\frac{x^2}{2} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -x\sqrt{-y^2+1}$ 	$R = y$ $S = -\frac{x^2}{2}$	$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\sin(c_1)$$

$$c_1 = -\frac{\pi}{2}$$

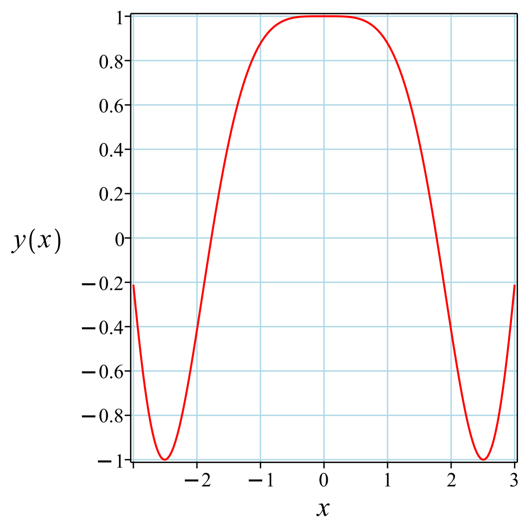
Substituting c_1 found above in the general solution gives

$$y = \cos\left(\frac{x^2}{2}\right)$$

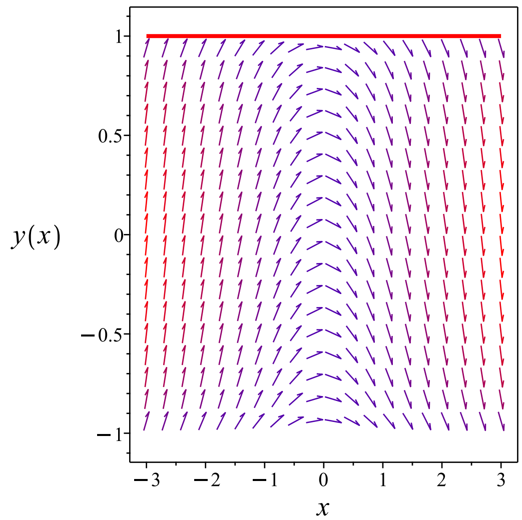
Summary

The solution(s) found are the following

$$y = \cos\left(\frac{x^2}{2}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cos\left(\frac{x^2}{2}\right)$$

Verified OK.

4.23.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{\sqrt{-y^2+1}}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(-\frac{1}{\sqrt{-y^2+1}}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x\\ N(x, y) &= -\frac{1}{\sqrt{-y^2+1}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{\sqrt{-y^2+1}}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x=\int M\, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x=\int -x\, \mathrm{d}x$$
$$\phi=-\frac{x^2}{2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=-\frac{1}{\sqrt{-y^2+1}}$. Therefore equation (4) becomes

$$-\frac{1}{\sqrt{-y^2+1}}=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=-\frac{1}{\sqrt{-y^2+1}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y=\int \left(-\frac{1}{\sqrt{-y^2+1}}\right) \mathrm{d}y$$
$$f(y)=-\arcsin(y)+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \arcsin(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \arcsin(y)$$

The solution becomes

$$y = -\sin\left(\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\sin(c_1)$$

$$c_1 = -\frac{\pi}{2}$$

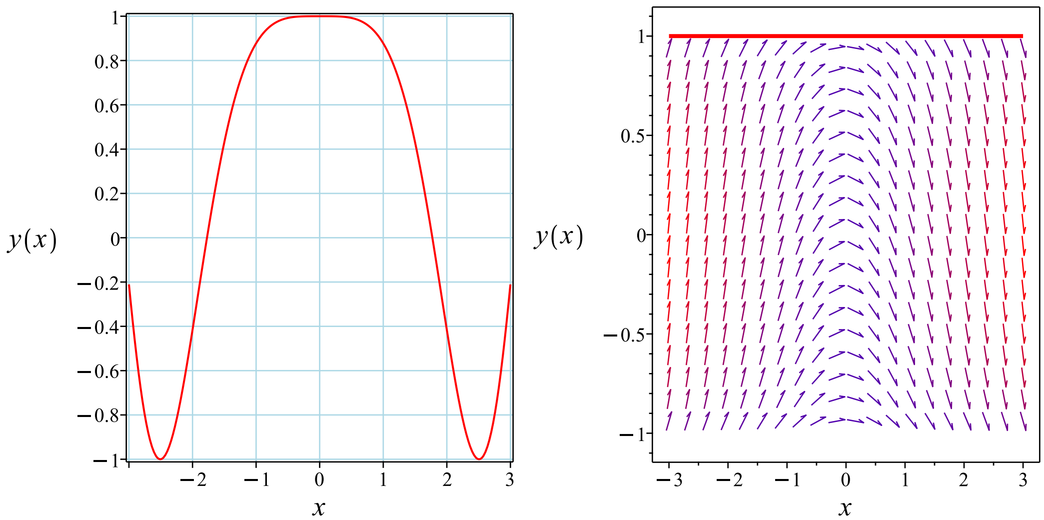
Substituting c_1 found above in the general solution gives

$$y = \cos\left(\frac{x^2}{2}\right)$$

Summary

The solution(s) found are the following

$$y = \cos\left(\frac{x^2}{2}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \cos\left(\frac{x^2}{2}\right)$$

Verified OK.

4.23.5 Maple step by step solution

Let's solve

$\left[y' + x\sqrt{1-y^2} = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}} = -x$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y^2}}dx = \int -xdx + c_1$

- Evaluate integral

  $\arcsin(y) = -\frac{x^2}{2} + c_1$

- Solve for y

$$y = \sin\left(-\frac{x^2}{2} + c_1\right)$$

- Use initial condition $y(0) = 1$

$$1 = \sin(c_1)$$

- Solve for c_1

$$c_1 = \frac{\pi}{2}$$

- Substitute $c_1 = \frac{\pi}{2}$ into general solution and simplify

$$y = \cos\left(\frac{x^2}{2}\right)$$

- Solution to the IVP

$$y = \cos\left(\frac{x^2}{2}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=-x*sqrt(1-y(x)^2),y(0) = 1],y(x), singsol=all)
```

$$y(x) = 1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 6

```
DSolve[{y'[x]==-x*Sqrt[1-y[x]^2],{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1$$

4.24 problem 24

Internal problem ID [12660]
Internal file name [`OUTPUT/11312_Friday_November_03_2023_06_30_17_AM_68810174/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.2, page 53
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{y' - \frac{\sqrt{x^2+4y}}{2} = -\frac{x}{2}}$$

With initial conditions

$$[y(6) = -9]$$

4.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2} \end{aligned}$$

The x domain of $f(x,y)$ when $y=-9$ is

$$\{6 \leq x \leq \infty, -\infty \leq x \leq -6\}$$

And the point $x_0 = 6$ is inside this domain. The y domain of $f(x,y)$ when $x=6$ is

$$\{-9 \leq y\}$$

And the point $y_0 = -9$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{2}+\frac{\sqrt{x^2+4y}}{2}\right)\\ &= \frac{1}{\sqrt{x^2+4y}}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -9$ is

$$\{-\infty \leq x < -6, 6 < x \leq \infty\}$$

But the point $x_0 = 6$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

4.24.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{x}{2}+\frac{\sqrt{x^2+4y}}{2}\\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}b_2 + \left(-\frac{x}{2}+\frac{\sqrt{x^2+4y}}{2}\right)(b_3 - a_2) - \left(-\frac{x}{2}+\frac{\sqrt{x^2+4y}}{2}\right)^2 a_3\\ -\left(-\frac{1}{2}+\frac{x}{2\sqrt{x^2+4y}}\right)(xa_2 + ya_3 + a_1) - \frac{xb_2 + yb_3 + b_1}{\sqrt{x^2+4y}} = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$
-\frac{\left(x^2+4y\right)^{\frac{3}{2}} a_3+\sqrt{x^2+4y}\,x^2 a_3-2x^3 a_3-4\sqrt{x^2+4y}\,x a_2+2\sqrt{x^2+4y}\,x b_3-2\sqrt{x^2+4y}\,y a_3+4x^2 a_2-}{4\sqrt{x^2+4y}}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
&-\left(x^2+4y\right)^{\frac{3}{2}} a_3-\sqrt{x^2+4y}\,x^2 a_3+2x^3 a_3+4\sqrt{x^2+4y}\,x a_2 \\
&-2\sqrt{x^2+4y}\,x b_3+2\sqrt{x^2+4y}\,y a_3-4x^2 a_2+2x^2 b_3+6xy a_3 \\
&+2\sqrt{x^2+4y}\,a_1+4b_2\sqrt{x^2+4y}-2x a_1-4x b_2-8y a_2+4y b_3-4b_1=0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-\left(x^2+4y\right)^{\frac{3}{2}} a_3+2\left(x^2+4y\right) x a_3-\sqrt{x^2+4y}\,x^2 a_3-2\left(x^2+4y\right) a_2 \\
&+2\left(x^2+4y\right) b_3+4\sqrt{x^2+4y}\,x a_2-2\sqrt{x^2+4y}\,x b_3+2\sqrt{x^2+4y}\,y a_3-2x^2 a_2 \\
&-2xy a_3+2\sqrt{x^2+4y}\,a_1+4b_2\sqrt{x^2+4y}-2x a_1-4x b_2-4y b_3-4b_1=0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&2x^3 a_3-2\sqrt{x^2+4y}\,x^2 a_3-4x^2 a_2+2x^2 b_3+4\sqrt{x^2+4y}\,x a_2 \\
&-2\sqrt{x^2+4y}\,x b_3+6xy a_3-2\sqrt{x^2+4y}\,y a_3-2x a_1-4x b_2 \\
&+2\sqrt{x^2+4y}\,a_1+4b_2\sqrt{x^2+4y}-8y a_2+4y b_3-4b_1=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^2+4y}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{x^2+4y}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2v_1^3 a_3-2v_3 v_1^2 a_3-4v_1^2 a_2+4v_3 v_1 a_2+6v_1 v_2 a_3-2v_3 v_2 a_3+2v_1^2 b_3 \\
&-2v_3 v_1 b_3-2v_1 a_1+2v_3 a_1-8v_2 a_2-4v_1 b_2+4b_2 v_3+4v_2 b_3-4b_1=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$2v_1^3a_3 - 2v_3v_1^2a_3 + (-4a_2 + 2b_3)\, v_1^2 + 6v_1v_2a_3 + (4a_2 - 2b_3)\, v_1v_3 \\ + (-2a_1 - 4b_2)\, v_1 - 2v_3v_2a_3 + (-8a_2 + 4b_3)\, v_2 + (2a_1 + 4b_2)\, v_3 - 4b_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_3 &= 0 \\ 2a_3 &= 0 \\ 6a_3 &= 0 \\ -4b_1 &= 0 \\ -2a_1 - 4b_2 &= 0 \\ 2a_1 + 4b_2 &= 0 \\ -8a_2 + 4b_3 &= 0 \\ -4a_2 + 2b_3 &= 0 \\ 4a_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -2b_2 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= 2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -2 \\ \eta &= x \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= x - \left(-\frac{x}{2} + \frac{\sqrt{x^2 + 4y}}{2}\right)(-2) \\
&= \sqrt{x^2 + 4y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\sqrt{x^2 + 4y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\sqrt{x^2 + 4y}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x}{2} + \frac{\sqrt{x^2 + 4y}}{2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x}{2\sqrt{x^2+4y}} \\
S_y &= \frac{1}{\sqrt{x^2+4y}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R}{2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\sqrt{x^2+4y}}{2} = \frac{x}{2} + c_1
$$

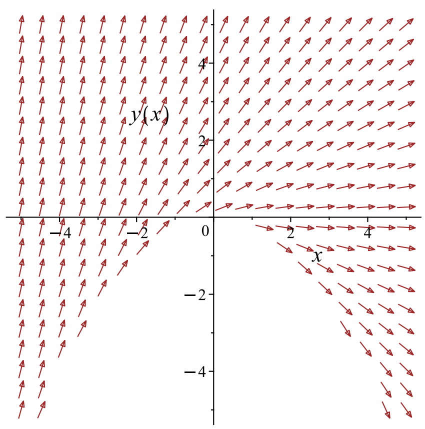
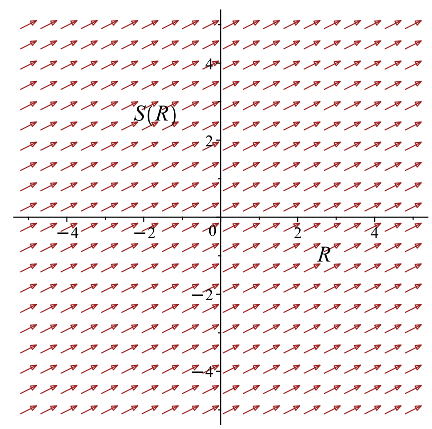
Which simplifies to

$$
\frac{\sqrt{x^2+4y}}{2} = \frac{x}{2} + c_1
$$

Which gives

$$
y = c_1^2 + c_1 x
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}$ 	$R = x$ $S = \frac{\sqrt{x^2 + 4y}}{2}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 6$ and $y = -9$ in the above solution gives an equation to solve for the constant of integration.

$$-9 = c_1^2 + 6c_1$$

$$c_1 = -3$$

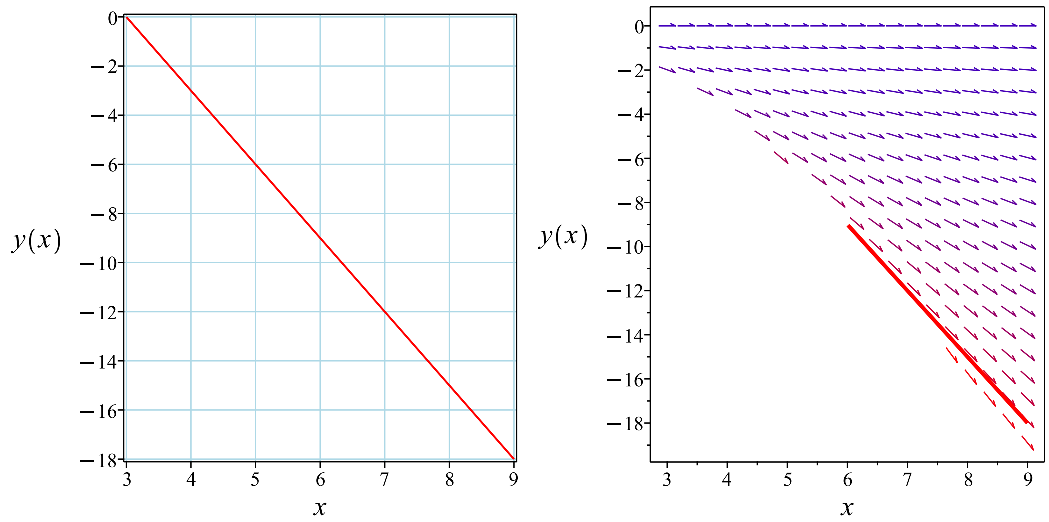
Substituting c_1 found above in the general solution gives

$$y = 9 - 3x$$

Summary

The solution(s) found are the following

$$y = 9 - 3x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 9 - 3x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
differential order: 1; found: 2 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 1.328 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)=(-x+sqrt(x^2+4*y(x)))/2,y(6) = -9],y(x), singsol=all)
```

$$y(x) = 9 - 3x$$
$$y(x) = -\frac{x^2}{4}$$

✓ Solution by Mathematica

Time used: 0.987 (sec). Leaf size: 10

```
DSolve[{y'[x]==(-x+Sqrt[x^2+4*y[x]])/2,{y[6]==-9}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 9 - 3x$$

5 Chapter 2. The Initial Value Problem. Exercises 2.3.1, page 57

5.1 problem 1

Internal problem ID [12661]
Internal file name [`OUTPUT/11313_Friday_November_03_2023_06_30_19_AM_14899875/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.1, page 57
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = 3x + 1}$$

With initial conditions

$$[y(1) = 2]$$

5.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 3x + 1 \end{aligned}$$

Hence the ode is

$$y' = 3x + 1$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = 3x + 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

5.1.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 3x + 1 \, \mathrm{d}x \\ &= \frac{3}{2}x^2 + x + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{5}{2} + c_1$$

$$c_1 = -\frac{1}{2}$$

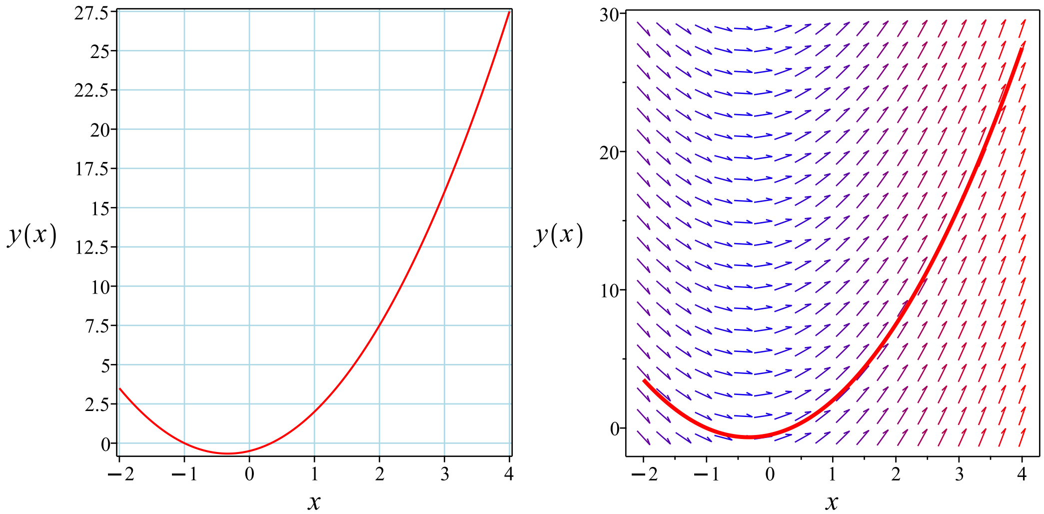
Substituting c_1 found above in the general solution gives

$$y = \frac{3}{2}x^2 + x - \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{3}{2}x^2 + x - \frac{1}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{3}{2}x^2 + x - \frac{1}{2}$$

Verified OK.

5.1.3 Maple step by step solution

Let's solve

$[y' = 3x + 1, y(1) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int (3x + 1)\, dx + c_1$

- Evaluate integral

  $y = \frac{3}{2}x^2 + x + c_1$

- Solve for y

  $y = \frac{3}{2}x^2 + x + c_1$

- Use initial condition $y(1) = 2$

  $2 = \frac{5}{2} + c_1$

- Solve for c_1

  $c_1 = -\frac{1}{2}$
- Substitute $c_1 = -\frac{1}{2}$ into general solution and simplify

  $y = \frac{3}{2}x^2 + x - \frac{1}{2}$
- Solution to the IVP

  $y = \frac{3}{2}x^2 + x - \frac{1}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve([diff(y(x),x)=3*x+1,y(1) = 2],y(x), singsol=all)
```

$$y(x) = \frac{3}{2}x^2 + x - \frac{1}{2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 17

```
DSolve[{y'[x]==3*x+1,{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3x^2}{2} + x - \frac{1}{2}$$

5.2 problem 2

Internal problem ID [12662]
Internal file name [`OUTPUT/11314_Friday_November_03_2023_06_30_19_AM_54321075/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.1, page 57
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = x + \frac{1}{x}}$$

With initial conditions

$$[y(1) = 2]$$

5.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \frac{x^2 + 1}{x}$$

Hence the ode is

$$y' = \frac{x^2 + 1}{x}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{x^2+1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

5.2.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{x^2+1}{x}\, dx \\ &= \frac{x^2}{2} + \ln(x) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{1}{2} + c_1$$

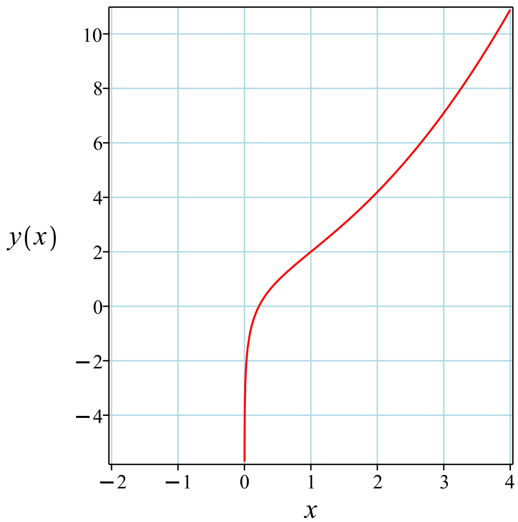
$$c_1 = \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

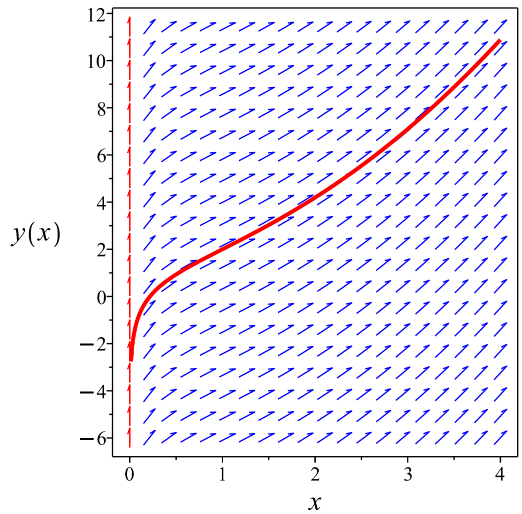
$$y = \frac{x^2}{2} + \ln(x) + \frac{3}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2} + \ln(x) + \frac{3}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^2}{2} + \ln(x) + \frac{3}{2}$$

Verified OK.

5.2.3 Maple step by step solution

Let's solve

$\left[y' = x + \frac{1}{x}, y(1) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \left(x + \frac{1}{x}\right) dx + c_1$

- Evaluate integral

  $y = \frac{x^2}{2} + \ln(x) + c_1$

- Solve for y

  $y = \frac{x^2}{2} + \ln(x) + c_1$

- Use initial condition $y(1) = 2$

  $2 = \frac{1}{2} + c_1$

- Solve for c_1

  $c_1 = \frac{3}{2}$
- Substitute $c_1 = \frac{3}{2}$ into general solution and simplify

  $y = \frac{x^2}{2} + \ln(x) + \frac{3}{2}$
- Solution to the IVP

  $y = \frac{x^2}{2} + \ln(x) + \frac{3}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)=x+1/x,y(1) = 2],y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} + \ln(x) + \frac{3}{2}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 18

```
DSolve[{y'[x]==x+1/x,{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(x^2 + 2\log(x) + 3)$$

5.3 problem 3

Internal problem ID [12663]
Internal file name [`OUTPUT/11315_Friday_November_03_2023_06_30_20_AM_49692135/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.1, page 57
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = 2\sin(x)}$$

With initial conditions

$$[y(\pi) = 1]$$

5.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = 2\sin(x)$$

Hence the ode is

$$y' = 2\sin(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is inside this domain. The domain of $q(x) = 2\sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is also inside this domain. Hence solution exists and is unique.

5.3.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 2\sin(x)\, \mathrm{d}x \\ &= -2\cos(x) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + 2$$

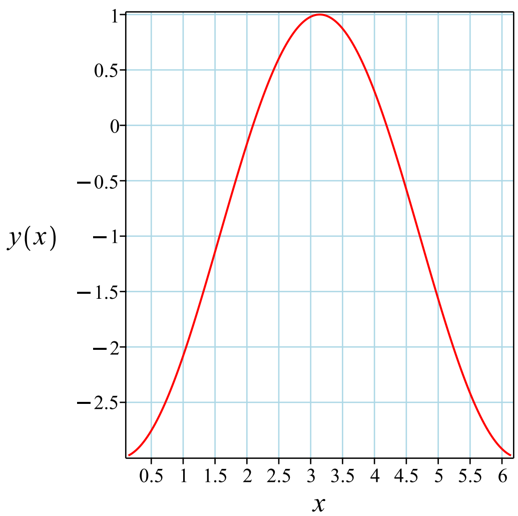
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

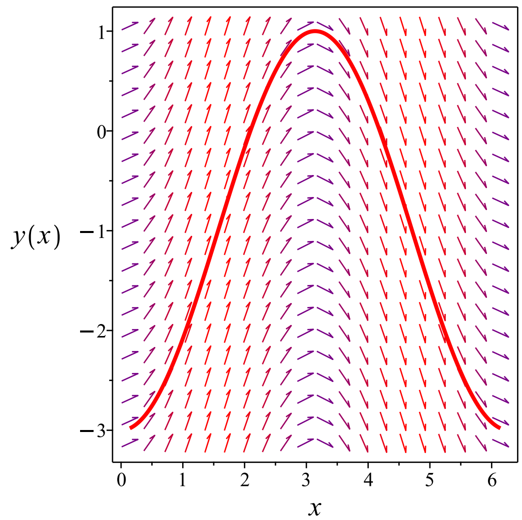
$$y = -2\cos(x) - 1$$

Summary
The solution(s) found are the following

$$y = -2\cos(x) - 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -2\cos(x) - 1$$

Verified OK.

5.3.3 Maple step by step solution

Let's solve

$[y' = 2\sin(x), y(\pi) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int 2\sin(x)\, dx + c_1$

- Evaluate integral

  $y = -2\cos(x) + c_1$

- Solve for y

  $y = -2\cos(x) + c_1$

- Use initial condition $y(\pi) = 1$

  $1 = c_1 + 2$

- Solve for c_1

  $c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

  $y = -2\cos(x) - 1$

- Solution to the IVP

  $y = -2\cos(x) - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)=2*sin(x),y(Pi) = 1],y(x), singsol=all)
```

$$y(x) = -2\cos(x) - 1$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 11

```
DSolve[{y'[x]==2*Sin[x],{y[Pi]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2\cos(x) - 1$$

5.4 problem 4

Internal problem ID [12664]
Internal file name [`OUTPUT/11316_Friday_November_03_2023_06_30_20_AM_22437125/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.1, page 57
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = x \sin(x)}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 1\right]$$

5.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= x \sin(x) \end{aligned}$$

Hence the ode is

$$y' = x \sin(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The domain of $q(x) = x \sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

5.4.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int x \sin(x) \, \mathrm{d}x \\ &= -\cos(x) x + \sin(x) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 1 + c_1$$

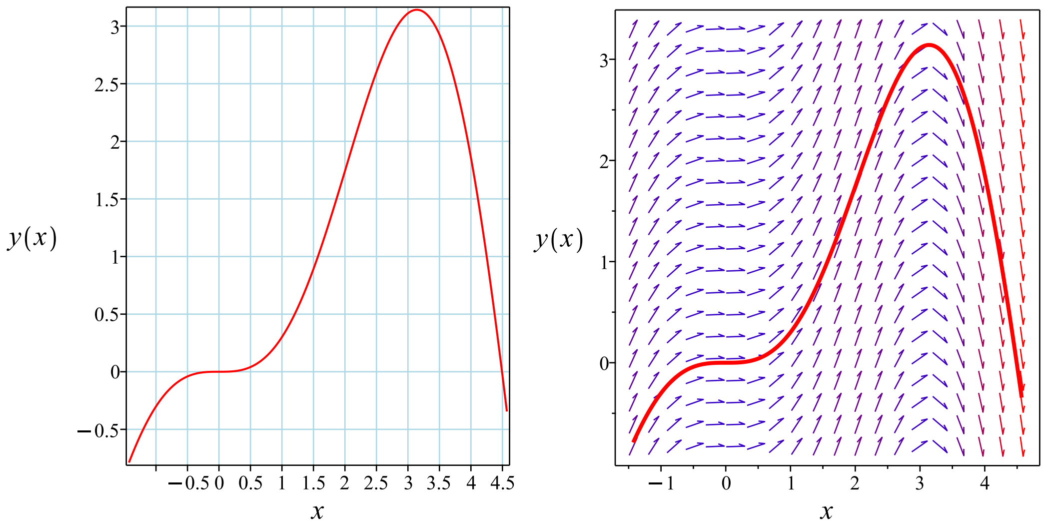
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = -\cos(x) x + \sin(x)$$

Summary
The solution(s) found are the following

$$y = -\cos(x) x + \sin(x) \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\cos(x)\, x + \sin(x)$$

Verified OK.

5.4.3 Maple step by step solution

Let's solve

$\left[y' = x \sin(x), y\left(\frac{\pi}{2}\right) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int x \sin(x)\, dx + c_1$

- Evaluate integral

  $y = -\cos(x)\, x + \sin(x) + c_1$

- Solve for y

  $y = -\cos(x)\, x + \sin(x) + c_1$

- Use initial condition $y\left(\frac{\pi}{2}\right) = 1$

  $1 = 1 + c_1$

- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = -\cos(x) x + \sin(x)$
- Solution to the IVP

  $y = -\cos(x) x + \sin(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve([diff(y(x),x)=x*sin(x),y(1/2*Pi) = 1],y(x), singsol=all)
```

$$y(x) = \sin(x) - \cos(x) x$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 13

```
DSolve[{y'[x]==x*Sin[x],{y[Pi/2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x) - x\cos(x)$$

5.5 problem 5

Internal problem ID [12665]
Internal file name [`OUTPUT/11317_Friday_November_03_2023_06_30_21_AM_40510007/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.1, page 57
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = \frac{1}{x-1}}$$

With initial conditions

$$[y(2) = 1]$$

5.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \frac{1}{x-1}$$

Hence the ode is

$$y' = \frac{1}{x-1}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = \frac{1}{x-1}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

5.5.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{x-1} \, \mathrm{d}x \\ &= \ln(x-1) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

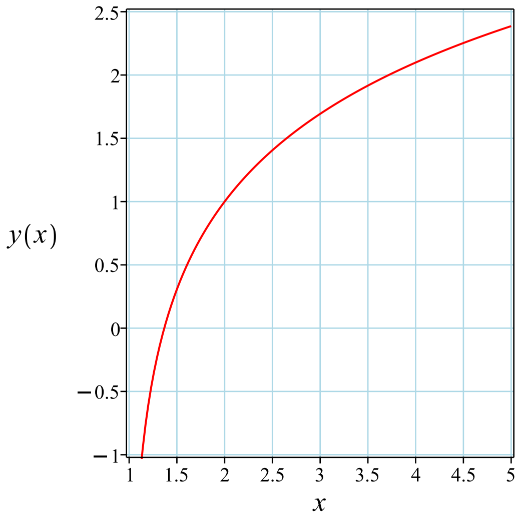
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

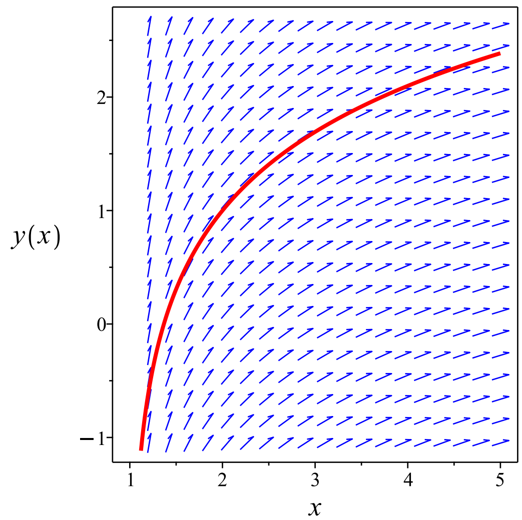
$$y = \ln(x-1) + 1$$

Summary
The solution(s) found are the following

$$y = \ln(x-1) + 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \ln(x-1) + 1$$

Verified OK.

5.5.3 Maple step by step solution

Let's solve

$\left[y' = \frac{1}{x-1}, y(2) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{1}{x-1} dx + c_1$

- Evaluate integral

  $y = \ln(x-1) + c_1$

- Solve for y

  $y = \ln(x-1) + c_1$

- Use initial condition $y(2) = 1$

  $1 = c_1$

- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = \ln(x - 1) + 1$
- Solution to the IVP

  $y = \ln(x - 1) + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)=1/(x-1),y(2) = 1],y(x), singsol=all)
```

$$y(x) = \ln(-1 + x) + 1$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 11

```
DSolve[{y'[x]==1/(x-1),{y[2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(x - 1) + 1$$

5.6 problem 6

Internal problem ID [12666]
Internal file name [`OUTPUT/11318_Friday_November_03_2023_06_30_21_AM_38775294/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.1, page 57
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = \frac{1}{x-1}}$$

With initial conditions

$$[y(0) = 1]$$

5.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \frac{1}{x-1}$$

Hence the ode is

$$y' = \frac{1}{x-1}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1}{x-1}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

5.6.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{x-1}\, \mathrm{d}x \\ &= \ln(x-1) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = i\pi + c_1$$

$$c_1 = -i\pi + 1$$

Substituting c_1 found above in the general solution gives

$$y = \ln(x-1) + 1 - i\pi$$

Summary
The solution(s) found are the following

$$y = \ln(x-1) + 1 - i\pi \tag{1}$$

Verification of solutions

$$y = \ln(x-1) + 1 - i\pi$$

Verified OK.

5.6.3 Maple step by step solution

Let's solve

$\left[y' = \frac{1}{x-1}, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{1}{x-1} dx + c_1$

- Evaluate integral

  $y = \ln (x-1) + c_1$

- Solve for y

  $y = \ln (x-1) + c_1$

- Use initial condition $y(0) = 1$

  $1 = \mathrm{I}\pi + c_1$

- Solve for c_1

  $c_1 = 1 - \mathrm{I}\pi$

- Substitute $c_1 = 1 - \mathrm{I}\pi$ into general solution and simplify

  $y = \ln (x-1) + 1 - \mathrm{I}\pi$

- Solution to the IVP

  $y = \ln (x-1) + 1 - \mathrm{I}\pi$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)=1/(x-1),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \ln(-1 + x) + 1 - i\pi$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 16

```
DSolve[{y'[x]==1/(x-1),{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(x - 1) - i\pi + 1$$

5.7 problem 7

Internal problem ID [12667]
Internal file name [`OUTPUT/11319_Friday_November_03_2023_06_30_22_AM_46393715/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.1, page 57
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = \frac{1}{x^2 - 1}}$$

With initial conditions

$$[y(2) = 1]$$

5.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \frac{1}{x^2 - 1}$$

Hence the ode is

$$y' = \frac{1}{x^2 - 1}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = \frac{1}{x^2-1}$ is

$$\{-\infty \leq x < -1, -1 < x < 1, 1 < x \leq \infty\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

5.7.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{x^2-1} \,\mathrm{d}x \\ &= -\operatorname{arctanh}(x) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\operatorname{arccoth}(2) + \frac{i\pi}{2} + c_1$$

$$c_1 = \operatorname{arccoth}(2) - \frac{i\pi}{2} + 1$$

Substituting c_1 found above in the general solution gives

$$y = -\operatorname{arctanh}(x) + \operatorname{arccoth}(2) - \frac{i\pi}{2} + 1$$

Summary
The solution(s) found are the following

$$y = -\operatorname{arctanh}(x) + \operatorname{arccoth}(2) - \frac{i\pi}{2} + 1 \tag{1}$$

Verification of solutions

$$y = -\operatorname{arctanh}(x) + \operatorname{arccoth}(2) - \frac{i\pi}{2} + 1$$

Verified OK.

5.7.3 Maple step by step solution

Let's solve

$\left[y' = \frac{1}{x^2-1}, y(2) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{1}{x^2-1} dx + c_1$

- Evaluate integral

  $y = -\operatorname{arctanh}(x) + c_1$

- Solve for y

  $y = -\operatorname{arctanh}(x) + c_1$

- Use initial condition $y(2) = 1$

  $1 = -\operatorname{arctanh}\left(\frac{1}{2}\right) + \frac{\mathrm{I}\pi}{2} + c_1$

- Solve for c_1

  $c_1 = \operatorname{arctanh}\left(\frac{1}{2}\right) - \frac{\mathrm{I}\pi}{2} + 1$

- Substitute $c_1 = \operatorname{arctanh}\left(\frac{1}{2}\right) - \frac{\mathrm{I}\pi}{2} + 1$ into general solution and simplify

  $y = -\operatorname{arctanh}(x) + \operatorname{arctanh}\left(\frac{1}{2}\right) - \frac{\mathrm{I}\pi}{2} + 1$

- Solution to the IVP

  $y = -\operatorname{arctanh}(x) + \operatorname{arctanh}\left(\frac{1}{2}\right) - \frac{\mathrm{I}\pi}{2} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 16

```
dsolve([diff(y(x),x)=1/(x^2-1),y(2) = 1],y(x), singsol=all)
```

$$y(x) = -\operatorname{arctanh}(x) + 1 + \operatorname{arctanh}\left(\frac{1}{2}\right) - \frac{i\pi}{2}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 28

```
DSolve[{y'[x]==1/(x^2-1),{y[2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(\log(3-3x) - \log(x+1) - i\pi + 2)$$

5.8 problem 8

Internal problem ID [12668]
Internal file name [`OUTPUT/11320_Friday_November_03_2023_06_30_22_AM_50345438/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.1, page 57
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = \frac{1}{x^2 - 1}}$$

With initial conditions

$$[y(0) = 1]$$

5.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \frac{1}{x^2 - 1}$$

Hence the ode is

$$y' = \frac{1}{x^2 - 1}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1}{x^2-1}$ is

$$\{-\infty \leq x < -1, -1 < x < 1, 1 < x \leq \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

5.8.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{x^2-1}\, \mathrm{d}x \\ &= -\operatorname{arctanh}(x) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

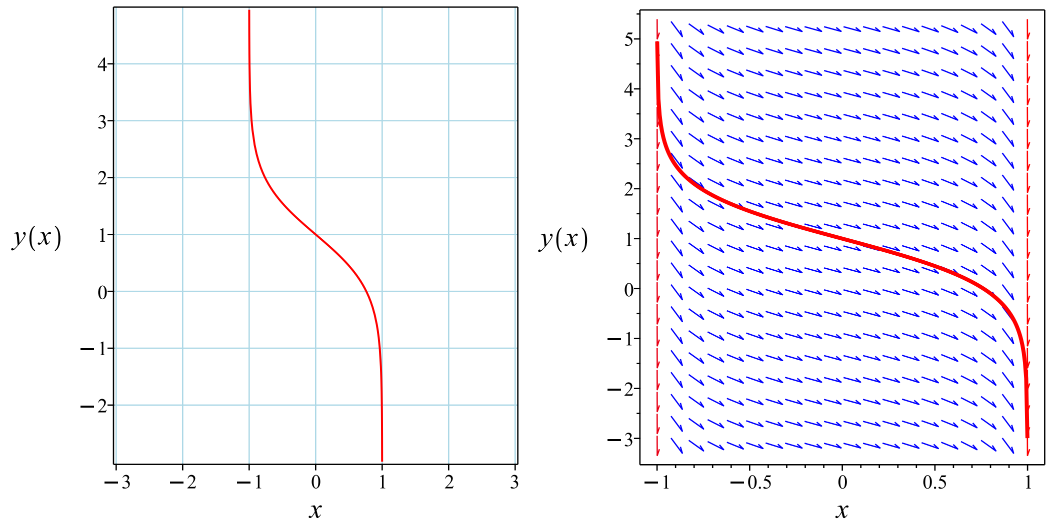
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = -\operatorname{arctanh}(x) + 1$$

Summary
The solution(s) found are the following

$$y = -\operatorname{arctanh}(x) + 1 \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\operatorname{arctanh}(x) + 1$$

Verified OK.

5.8.3 Maple step by step solution

Let's solve

$\left[y' = \frac{1}{x^2-1}, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{1}{x^2-1} dx + c_1$

- Evaluate integral

  $y = -\operatorname{arctanh}(x) + c_1$

- Solve for y

  $y = -\operatorname{arctanh}(x) + c_1$

- Use initial condition $y(0) = 1$

  $1 = c_1$

- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = -\operatorname{arctanh}(x) + 1$
- Solution to the IVP

  $y = -\operatorname{arctanh}(x) + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)=1/(x^2-1),y(0) = 1],y(x), singsol=all)
```

$$y(x) = -\operatorname{arctanh}(x) + 1$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 23

```
DSolve[{y'[x]==1/(x^2-1),{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(\log(1-x) - \log(x+1) + 2)$$

5.9 problem 9

Internal problem ID [12669]
Internal file name [`OUTPUT/11321_Friday_November_03_2023_06_30_23_AM_4631435/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.1, page 57
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = \tan(x)}$$

With initial conditions

$$[y(0) = 0]$$

5.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \tan(x)$$

Hence the ode is

$$y' = \tan(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z117 \vee \frac{1}{2}\pi + \pi\_Z117 < x\right\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

5.9.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \tan(x)\, \mathrm{d}x \\ &= -\ln(\cos(x)) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

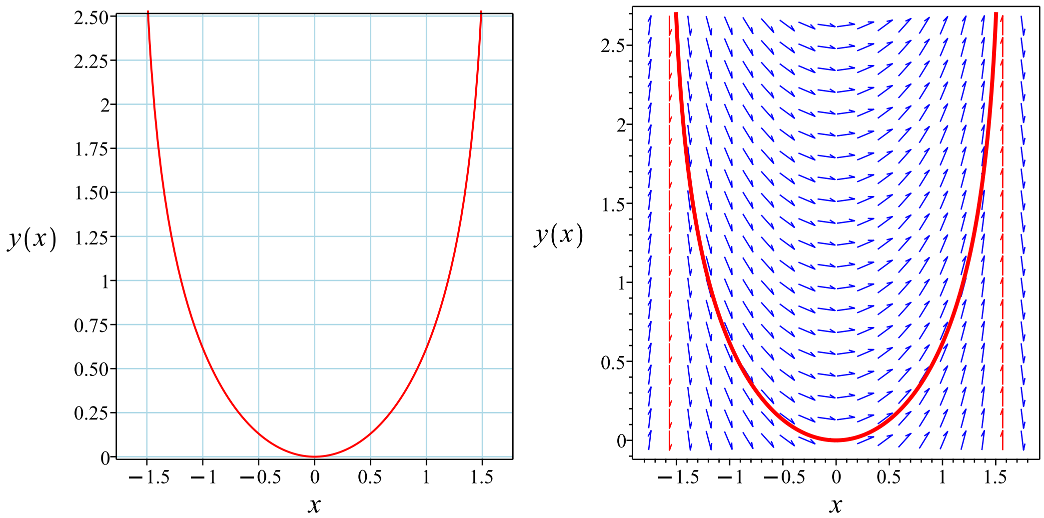
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = -\ln(\cos(x))$$

Summary
The solution(s) found are the following

$$y = -\ln(\cos(x)) \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\ln(\cos(x))$$

Verified OK.

5.9.3 Maple step by step solution

Let's solve

$[y' = \tan(x), y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \tan(x)\, dx + c_1$

- Evaluate integral

  $y = -\ln(\cos(x)) + c_1$

- Solve for y

  $y = -\ln(\cos(x)) + c_1$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = -\ln(\cos(x))$

- Solution to the IVP

  $y = -\ln(\cos(x))$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([diff(y(x),x)=tan(x),y(0) = 0],y(x), singsol=all)
```

$$y(x) = -\ln(\cos(x))$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 10

```
DSolve[{y'[x]==Tan[x],{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\log(\cos(x))$$

5.10 problem 10

Internal problem ID [12670]
Internal file name [`OUTPUT/11322_Friday_November_03_2023_06_30_23_AM_43163742/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.1, page 57
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = \tan(x)}$$

With initial conditions

$$[y(\pi) = 0]$$

5.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \tan(x) \end{aligned}$$

Hence the ode is

$$y' = \tan(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is inside this domain. The domain of $q(x) = \tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z117 \vee \frac{1}{2}\pi + \pi\_Z117 < x\right\}$$

And the point $x_0 = \pi$ is also inside this domain. Hence solution exists and is unique.

5.10.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \tan(x)\, \mathrm{d}x \\ &= -\ln(\cos(x)) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -i\pi + c_1$$

$$c_1 = i\pi$$

Substituting c_1 found above in the general solution gives

$$y = -\ln(\cos(x)) + i\pi$$

Summary

The solution(s) found are the following

$$y = -\ln(\cos(x)) + i\pi \tag{1}$$

Verification of solutions

$$y = -\ln(\cos(x)) + i\pi$$

Verified OK.

5.10.3 Maple step by step solution

Let's solve

$[y' = \tan(x), y(\pi) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \tan(x) dx + c_1$

- Evaluate integral

  $y = -\ln(\cos(x)) + c_1$

- Solve for y

  $y = -\ln(\cos(x)) + c_1$

- Use initial condition $y(\pi) = 0$

  $0 = -\mathrm{I}\pi + c_1$

- Solve for c_1

  $c_1 = \mathrm{I}\pi$

- Substitute $c_1 = \mathrm{I}\pi$ into general solution and simplify

  $y = -\ln(\cos(x)) + \mathrm{I}\pi$

- Solution to the IVP

  $y = -\ln(\cos(x)) + \mathrm{I}\pi$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)=tan(x),y(Pi) = 0],y(x), singsol=all)
```

$$y(x) = -\ln(\cos(x)) + i\pi$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 16

```
DSolve[{y'[x]==Tan[x],{y[Pi]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\log(\cos(x)) + i\pi$$

6 Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63

6.1 problem 1

Internal problem ID [12671]
Internal file name [`OUTPUT/11323_Friday_November_03_2023_06_30_24_AM_55103717/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 3y = 0}$$

With initial conditions

$$[y(0) = -1]$$

6.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -3$$
$$q(x) = 0$$

Hence the ode is

$$y' - 3y = 0$$

The domain of $p(x) = -3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Hence solution exists and is unique.

6.1.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{3y}dy = \int dx$$
$$\frac{\ln(y)}{3} = x + c_1$$

Raising both side to exponential gives

$$y^{\frac{1}{3}} = e^{x+c_1}$$

Which simplifies to

$$y^{\frac{1}{3}} = c_2 e^{x}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_2^3$$

$$c_2 = -1$$

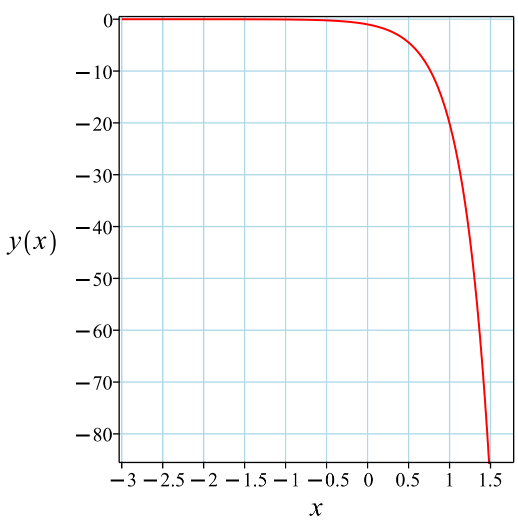
Substituting c_2 found above in the general solution gives

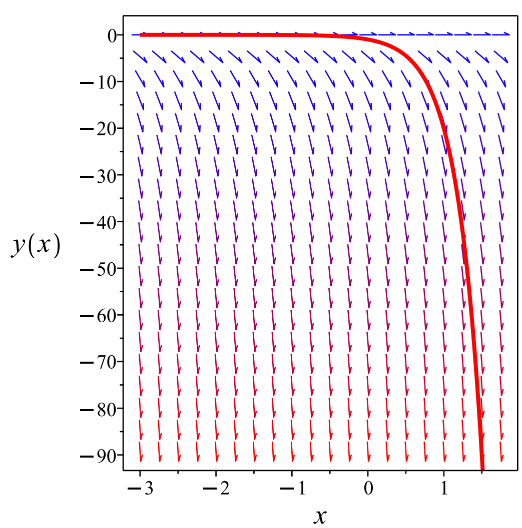
$$y = -e^{3x}$$

Summary

The solution(s) found are the following

$$y = -e^{3x} \tag{1}$$





(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -e^{3x}$$

Verified OK.

6.1.3 Maple step by step solution

Let's solve

$[y' - 3y = 0, y(0) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 3$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 3 dx + c_1$

- Evaluate integral

  $\ln(y) = 3x + c_1$

- Solve for y

  $y = e^{3x+c_1}$

- Use initial condition $y(0) = -1$

  $-1 = e^{c_1}$

- Solve for c_1

  $c_1 = I\pi$

- Substitute $c_1 = I\pi$ into general solution and simplify

  $y = -e^{3x}$

- Solution to the IVP

  $y = -e^{3x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)=3*y(x),y(0) = -1],y(x), singsol=all)
```

$$y(x) = -e^{3x}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 12

```
DSolve[{y'[x]==3*y[x],{y[0]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{3x}$$

6.2 problem 2

Internal problem ID [12672]
Internal file name [`OUTPUT/11324_Friday_November_03_2023_06_30_25_AM_44070852/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' + y = 1}$$

With initial conditions

$$[y(0) = 1]$$

6.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = 1$$

Hence the ode is

$$y' + y = 1$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.2.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{1-y} dy = \int dx$$
$$-\ln(1-y) = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{1-y} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{1-y} = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

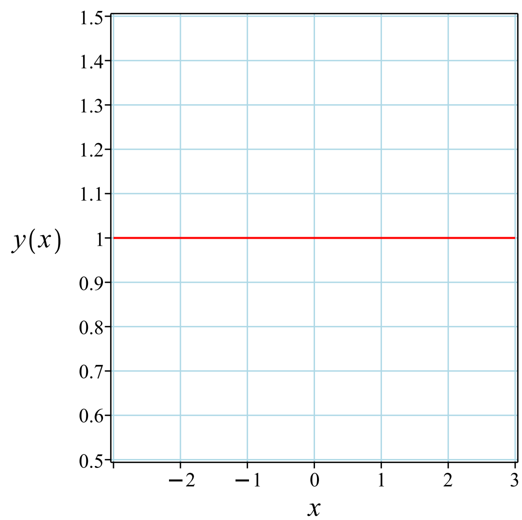
$$1 = \frac{-1+c_2}{c_2}$$

Unable to solve for constant of integration. Since $\lim_{c_1 \to \infty}$ gives $y = -\frac{e^{-x}}{c_2} + 1 = y = 1$ and this result satisfies the given initial condition.

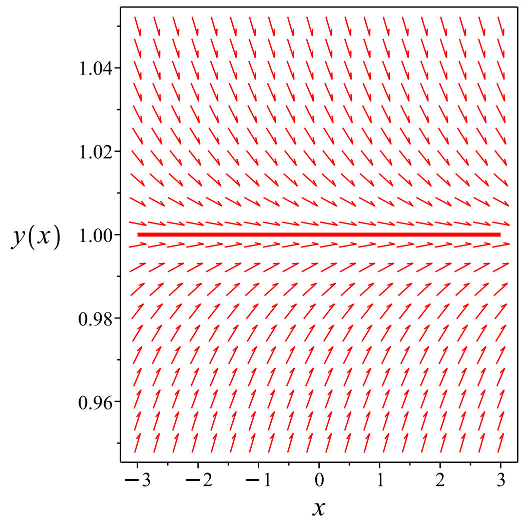
Summary

The solution(s) found are the following

$$y = 1$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1$$

Verified OK.

6.2.3 Maple step by step solution

Let's solve

$[y' + y = 1, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1-y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{1-y} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\ln(1-y) = x + c_1$

- Solve for y

  $y = -e^{-x-c_1} + 1$

- Use initial condition $y(0) = 1$

  $1 = -e^{-c_1} + 1$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=-y(x)+1,y(0) = 1],y(x), singsol=all)
```

$$y(x) = 1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 6

```
DSolve[{y'[x]==-y[x]+1,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1$$

6.3 problem 3

Internal problem ID [12673]
Internal file name [`OUTPUT/11325_Friday_November_03_2023_06_30_25_AM_58471505/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + y = 1}$$

With initial conditions

$$[y(0) = 2]$$

6.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = 1$$

Hence the ode is

$$y' + y = 1$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.3.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{1-y} dy = \int dx$$
$$-\ln(1-y) = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{1-y} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{1-y} = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{-1 + c_2}{c_2}$$

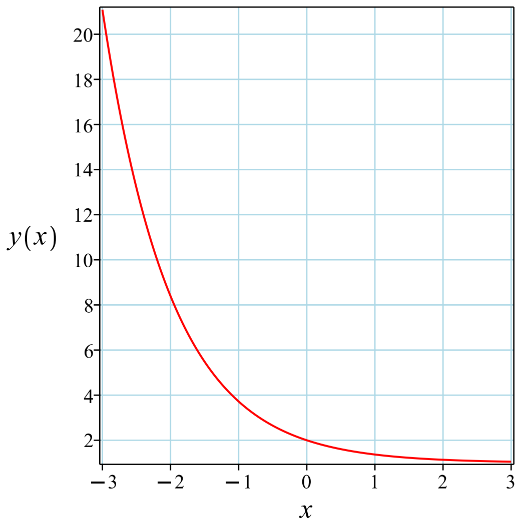
$$c_2 = -1$$

Substituting c_2 found above in the general solution gives

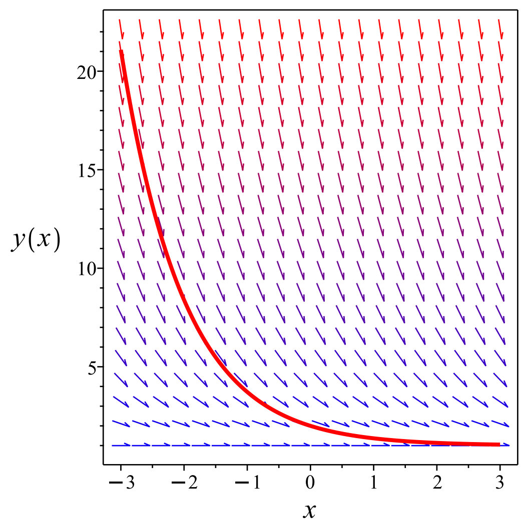
$$y = e^{-x} + 1$$

Summary
The solution(s) found are the following

$$y = e^{-x} + 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{-x} + 1$$

Verified OK.

6.3.3 Maple step by step solution

Let's solve

$[y' + y = 1, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1-y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{1-y} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\ln(1-y) = x + c_1$

- Solve for y

  $y = -e^{-x-c_1} + 1$

- Use initial condition $y(0) = 2$

  $2 = -e^{-c_1} + 1$

- Solve for c_1

  $c_1 = -I\pi$

- Substitute $c_1 = -I\pi$ into general solution and simplify

  $y = e^{-x} + 1$

- Solution to the IVP

  $y = e^{-x} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)=-y(x)+1,y(0) = 2],y(x), singsol=all)
```

$$y(x) = e^{-x} + 1$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 12

```
DSolve[{y'[x]==-y[x]+1,{y[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} + 1$$

6.4 problem 4

Internal problem ID [12674]
Internal file name [`OUTPUT/11326_Friday_November_03_2023_06_30_26_AM_35982736/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - x\,\mathrm{e}^{-x^2+y} = 0}$$

With initial conditions

$$[y(0) = 0]$$

6.4.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= x\,\mathrm{e}^{-x^2+y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x\,\mathrm{e}^{-x^2+y}\right)\\ &= x\,\mathrm{e}^{-x^2+y}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

6.4.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= x\,\mathrm{e}^{-x^2}\mathrm{e}^{y}\end{aligned}$$

Where $f(x) = x\,\mathrm{e}^{-x^2}$ and $g(y) = \mathrm{e}^{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\mathrm{e}^{y}}\,dy &= x\,\mathrm{e}^{-x^2}\,dx\\ \int \frac{1}{\mathrm{e}^{y}}\,dy &= \int x\,\mathrm{e}^{-x^2}\,dx\\ -\mathrm{e}^{-y} &= -\frac{\mathrm{e}^{-x^2}}{2} + c_1\end{aligned}$$

Which results in

$$y = \ln\left(-\frac{2}{-1 + 2c_1\mathrm{e}^{x^2}}\right) + x^2$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln(2) + \ln\left(-\frac{1}{2c_1 - 1}\right)$$

$$c_1 = -\frac{1}{2}$$

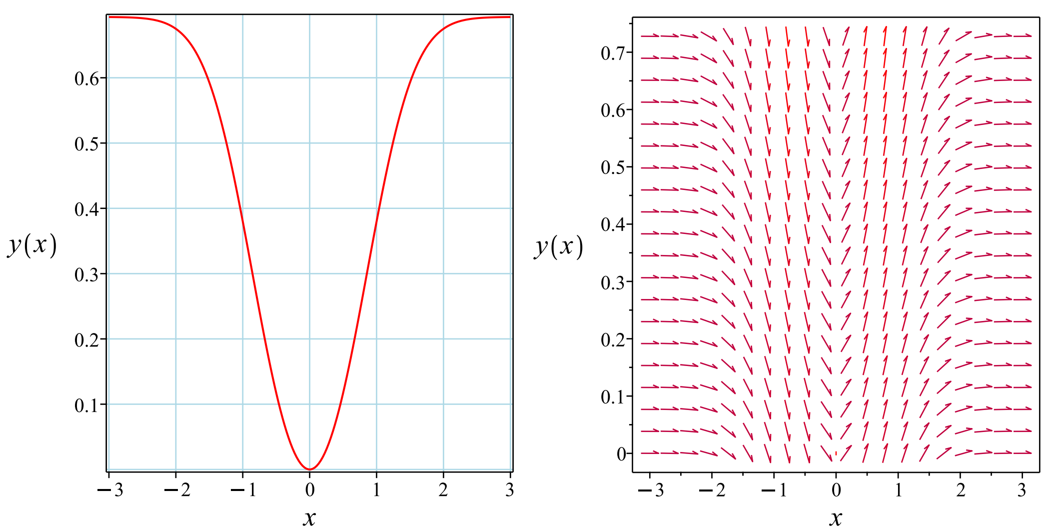
Substituting c_1 found above in the general solution gives

$$y = \ln(2) + \ln\left(\frac{1}{1 + e^{x^2}}\right) + x^2$$

Summary

The solution(s) found are the following

$$y = \ln(2) + \ln\left(\frac{1}{1 + e^{x^2}}\right) + x^2 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \ln(2) + \ln\left(\frac{1}{1 + e^{x^2}}\right) + x^2$$

Verified OK.

6.4.3 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = x\,\mathrm{e}^{-x^2+y} \tag{1}$$

And using the substitution $u = \mathrm{e}^{-y}$ then

$$u' = -y'\mathrm{e}^{-y}$$

The above shows that

$$\begin{aligned} y' &= -u'(x)\,\mathrm{e}^{y} \\ &= -\frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{u} = \frac{x\,\mathrm{e}^{-x^2}}{u}$$

The above simplifies to

$$u'(x) = -x\,\mathrm{e}^{-x^2} \tag{2}$$

Now ode (2) is solved for $u(x)$ Integrating both sides gives

$$\begin{aligned} u(x) &= \int -x\,\mathrm{e}^{-x^2}\,\mathrm{d}x \\ &= \frac{\mathrm{e}^{-x^2}}{2} + c_1 \end{aligned}$$

Substituting the solution found for $u(x)$ in $u = \mathrm{e}^{-y}$ gives

$$\begin{aligned} y &= -\ln\left(u(x)\right) \\ &= -\ln\left(\frac{\mathrm{e}^{-x^2}}{2} + c_1\right) \\ &= \ln\left(2\right) - \ln\left(\mathrm{e}^{-x^2} + 2c_1\right) \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln\left(2\right) - \ln\left(2c_1 + 1\right)$$

$$c_1 = \frac{1}{2}$$

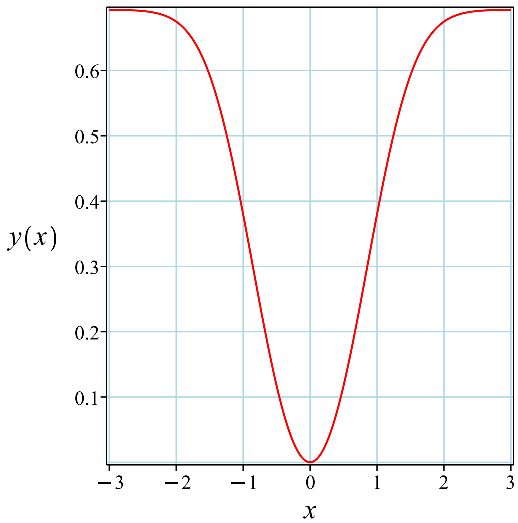
Substituting c_1 found above in the general solution gives

$$y = \ln(2) - \ln\left(e^{-x^2} + 1\right)$$

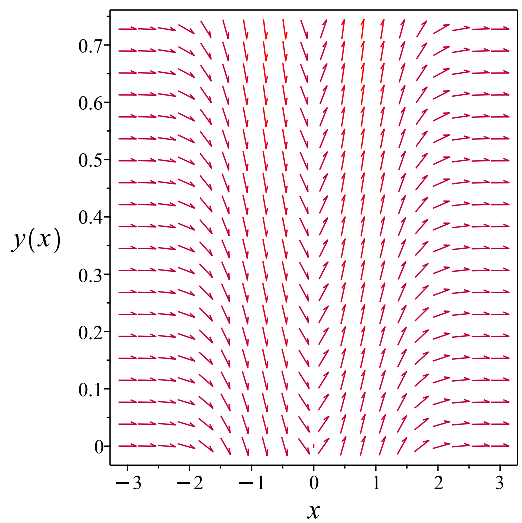
Summary

The solution(s) found are the following

$$y = \ln(2) - \ln\left(e^{-x^2} + 1\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \ln(2) - \ln\left(e^{-x^2} + 1\right)$$

Verified OK.

6.4.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x\,e^{-x^2+y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 166: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{e^{x^2}}{x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta=0$ then in this special case

$$R=y$$

S is found from

$$\begin{aligned}S&=\int\frac{1}{\xi}dx\\&=\int\frac{1}{\frac{e^{x^2}}{x}}dx\end{aligned}$$

Which results in

$$S=-\frac{e^{-x^2}}{2}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=x\,e^{-x^2+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=0\\R_y&=1\\S_x&=x\,e^{-x^2}\\S_y&=0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=e^{-y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -e^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{e^{-x^2}}{2} = -e^{-y} + c_1$$

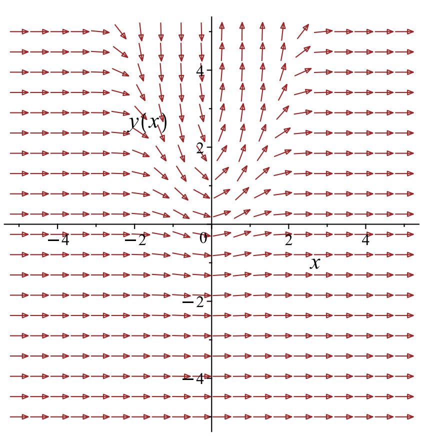
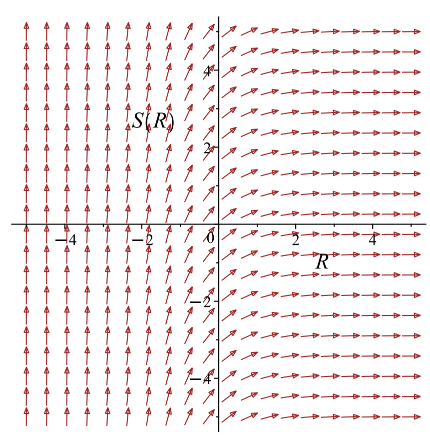
Which simplifies to

$$-\frac{e^{-x^2}}{2} = -e^{-y} + c_1$$

Which gives

$$y = -\ln\left(c_1 e^{x^2} + \frac{1}{2}\right) + x^2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x\,e^{-x^2+y}$ 	$R = y$ $S = -\frac{e^{-x^2}}{2}$	$\frac{dS}{dR} = e^{-R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln(2) - \ln(2c_1 + 1)$$

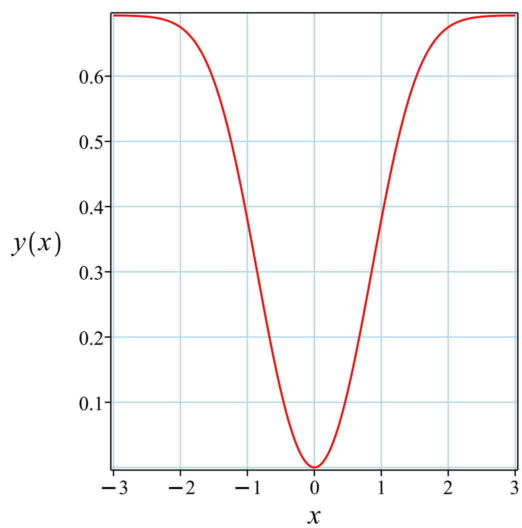
$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

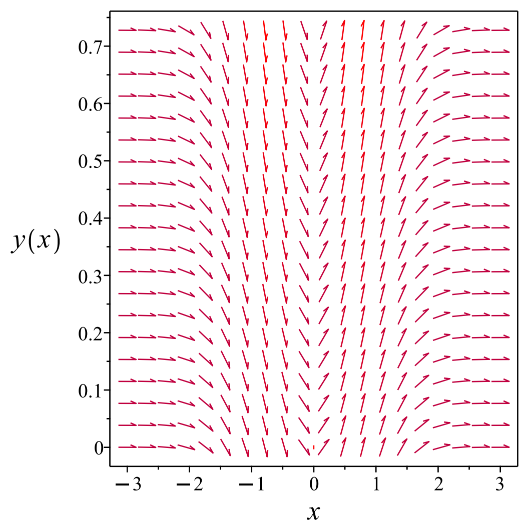
$$y = \ln(2) - \ln\left(1 + e^{x^2}\right) + x^2$$

Summary
The solution(s) found are the following

$$y = \ln(2) - \ln\left(1 + e^{x^2}\right) + x^2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \ln(2) - \ln\left(1 + e^{x^2}\right) + x^2$$

Verified OK.

6.4.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\mathrm{e}^{-y}\right)\mathrm{d}y &= \left(x\,\mathrm{e}^{-x^2}\right)\mathrm{d}x\\ \left(-x\,\mathrm{e}^{-x^2}\right)\mathrm{d}x + \left(\mathrm{e}^{-y}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\,\mathrm{e}^{-x^2}\\ N(x,y) &= \mathrm{e}^{-y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x\,\mathrm{e}^{-x^2}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^{-y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -x\,\mathrm{e}^{-x^2}\,\mathrm{d}x$$

$$\phi = \frac{\mathrm{e}^{-x^2}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-y}$. Therefore equation (4) becomes

$$\mathrm{e}^{-y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \mathrm{e}^{-y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (\mathrm{e}^{-y})\,\mathrm{d}y$$

$$f(y) = -\mathrm{e}^{-y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\mathrm{e}^{-x^2}}{2} - \mathrm{e}^{-y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\mathrm{e}^{-x^2}}{2} - \mathrm{e}^{-y}$$

The solution becomes

$$y = -\ln\left(\frac{1}{2} - c_1 \mathrm{e}^{x^2}\right) + x^2$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln(2) - \ln(-2c_1 + 1)$$

$$c_1 = -\frac{1}{2}$$

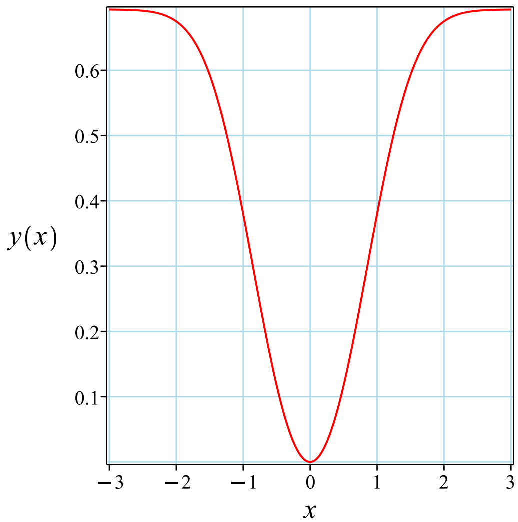
Substituting c_1 found above in the general solution gives

$$y = \ln(2) - \ln\left(1 + \mathrm{e}^{x^2}\right) + x^2$$

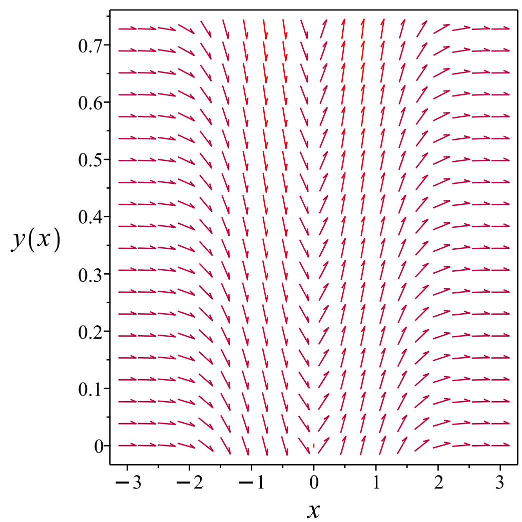
Summary

The solution(s) found are the following

$$y = \ln(2) - \ln\left(1 + \mathrm{e}^{x^2}\right) + x^2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \ln(2) - \ln\left(1 + e^{x^2}\right) + x^2$$

Verified OK.

6.4.6 Maple step by step solution

Let's solve

$\left[y' - x\,e^{-x^2+y} = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{e^y} = \frac{x}{e^{x^2}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{e^y}dx = \int \frac{x}{e^{x^2}}dx + c_1$

- Evaluate integral

  $-\frac{1}{e^y} = -\frac{1}{2\,e^{x^2}} + c_1$

- Solve for y

$$y = \ln\left(-\frac{2}{-1+2c_1 e^{x^2}}\right) + x^2$$

- Use initial condition $y(0) = 0$

$$0 = \ln\left(-\frac{2}{2c_1 - 1}\right)$$

- Solve for c_1

$$c_1 = -\frac{1}{2}$$

- Substitute $c_1 = -\frac{1}{2}$ into general solution and simplify

$$y = \ln(2) + \ln\left(\frac{1}{1+e^{x^2}}\right) + x^2$$

- Solution to the IVP

$$y = \ln(2) + \ln\left(\frac{1}{1+e^{x^2}}\right) + x^2$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 19

```
dsolve([diff(y(x),x)=x*exp(y(x)-x^2),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \ln(2) - \ln\left(1 + e^{x^2}\right) + x^2$$

✓ Solution by Mathematica

Time used: 2.198 (sec). Leaf size: 21

```
DSolve[{y'[x]==x*Exp[y[x]-x^2],{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\log\left(\frac{1}{2}\left(e^{-x^2}+1\right)\right)$$

6.5 problem 5

Internal problem ID [12675]
Internal file name [`OUTPUT/11327_Friday_November_03_2023_06_30_27_AM_65173093/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y}{x} = 0}$$

With initial conditions

$$[y(-1) = 2]$$

6.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{y}{x} = 0$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -1$ is inside this domain. Hence solution exists and is unique.

6.5.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y} \, dy &= \frac{1}{x} \, dx \\ \int \frac{1}{y} \, dy &= \int \frac{1}{x} \, dx \\ \ln(y) &= \ln(x) + c_1 \\ y &= e^{\ln(x)+c_1} \\ &= c_1 x \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -c_1$$

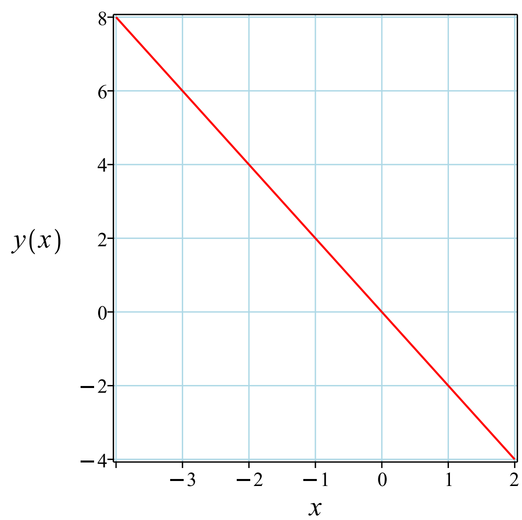
$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

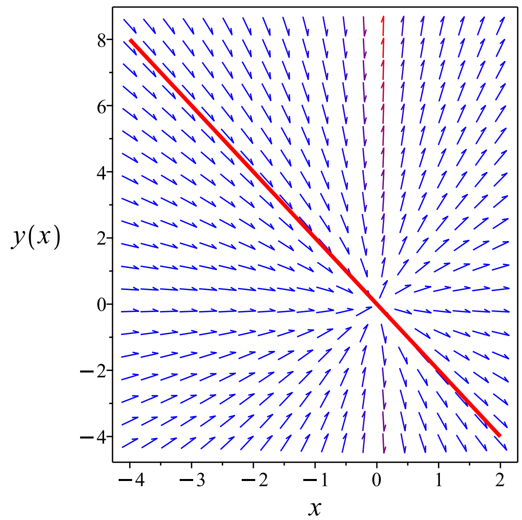
$$y = -2x$$

Summary
The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

6.5.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = 0$$

Integrating gives

$$\frac{y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -c_1$$

$$c_1 = -2$$

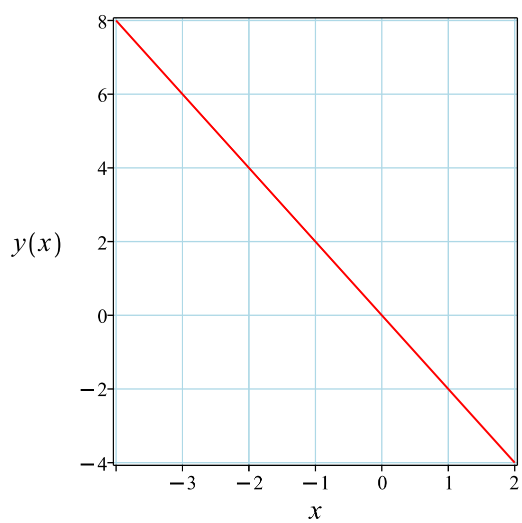
Substituting c_1 found above in the general solution gives

$$y = -2x$$

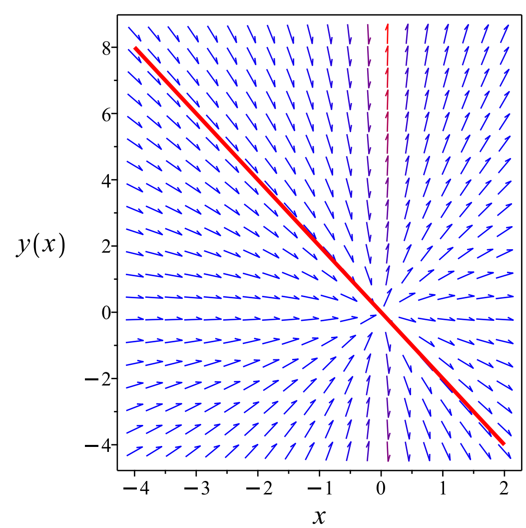
Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

6.5.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x = 0$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0\,\mathrm{d}x \\ &= c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= c_2 x \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -c_2$$

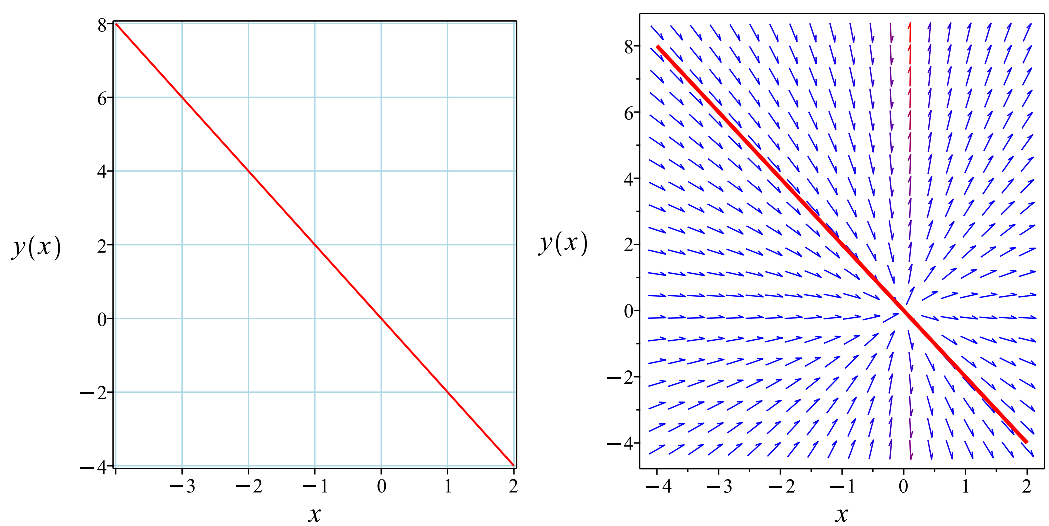
$$c_2 = -2$$

Substituting c_2 found above in the general solution gives

$$y = -2x$$

Summary
The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

6.5.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 169: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = c_1$$

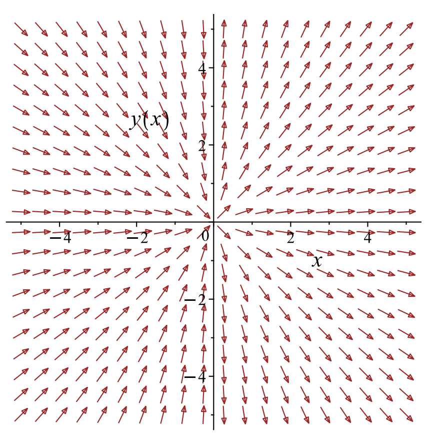
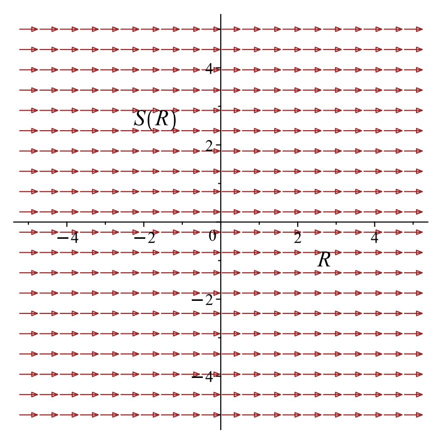
Which simplifies to

$$\frac{y}{x} = c_1$$

Which gives

$$y = c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{y}{x}$
</td><td>$R = x$
$S = \frac{y}{x}$</td><td>$\frac{dS}{dR} = 0$
</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -c_1$$

$$c_1 = -2$$

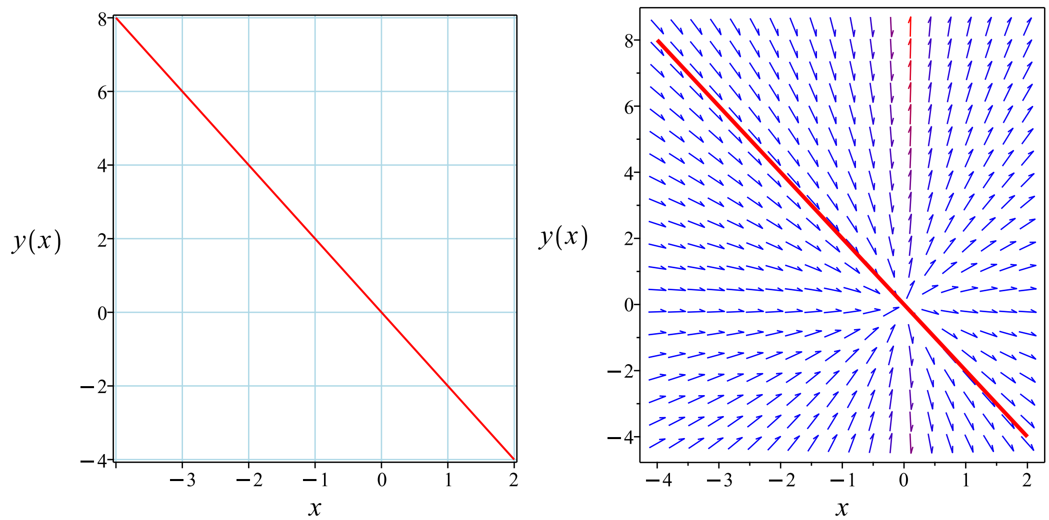
Substituting c_1 found above in the general solution gives

$$y = -2x$$

Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

6.5.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{1}{y}\right) \mathrm{d} y &= \left(\frac{1}{x}\right) \mathrm{d} x \\
\left(-\frac{1}{x}\right) \mathrm{d} x+\left(\frac{1}{y}\right) \mathrm{d} y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= \frac{1}{y}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{x} \, \mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y)$$

The solution becomes

$$y = e^{c_1} x$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -e^{c_1}$$

$$c_1 = \ln(2) + i\pi$$

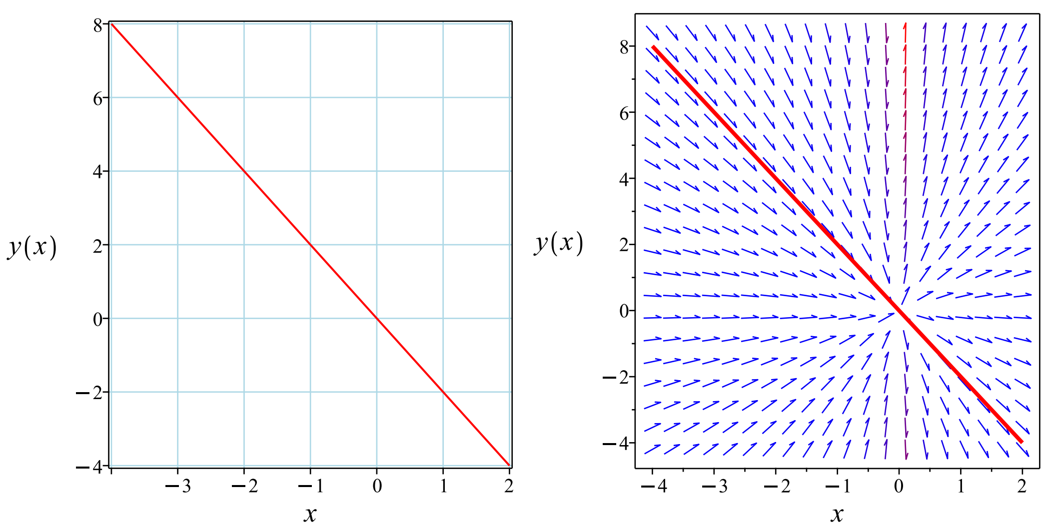
Substituting c_1 found above in the general solution gives

$$y = -2x$$

Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

6.5.7 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{x} = 0, y(-1) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = \mathrm{e}^{c_1} x$

- Use initial condition $y(-1) = 2$

  $2 = -\mathrm{e}^{c_1}$

- Solve for c_1

  $c_1 = \ln(2) + \mathrm{I}\pi$

- Substitute $c_1 = \ln(2) + \mathrm{I}\pi$ into general solution and simplify

  $y = -2x$

- Solution to the IVP

  $y = -2x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 7

```
dsolve([diff(y(x),x)=y(x)/x,y(-1) = 2],y(x), singsol=all)
```

$$y(x) = -2x$$

✓ Solution by Mathematica
Time used: 0.036 (sec). Leaf size: 8

```
DSolve[{y'[x]==y[x]/x,{y[-1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x$$

6.6 problem 6

Internal problem ID [12676]
Internal file name [`OUTPUT/11328_Friday_November_03_2023_06_30_28_AM_68330762/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{2x}{y} = 0}$$

With initial conditions

$$[y(0) = 2]$$

6.6.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{2x}{y} \end{aligned}$$

The x domain of $f(x,y)$ when $y=2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x=0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2x}{y}\right)\\ &= -\frac{2x}{y^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

6.6.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{2x}{y}\end{aligned}$$

Where $f(x) = 2x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = 2x\,dx$$

$$\int \frac{1}{\frac{1}{y}}\,dy = \int 2x\,dx$$

$$\frac{y^2}{2} = x^2 + c_1$$

Which results in

$$y = \sqrt{2x^2 + 2c_1}$$
$$y = -\sqrt{2x^2 + 2c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\sqrt{c_1}\,\sqrt{2}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \sqrt{c_1}\,\sqrt{2}$$

$$c_1 = 2$$

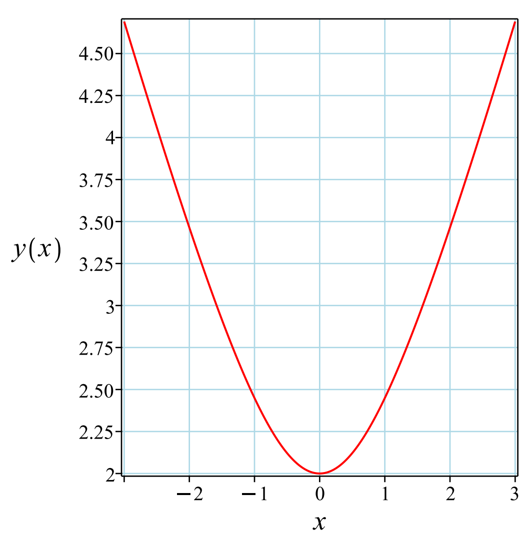
Substituting c_1 found above in the general solution gives

$$y = \sqrt{2x^2 + 4}$$

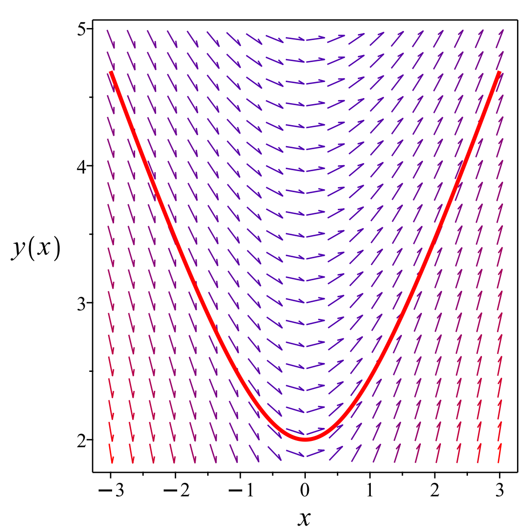
<u>Summary</u>

The solution(s) found are the following

$$y = \sqrt{2x^2 + 4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{2x^2 + 4}$$

Verified OK.

6.6.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{2}{u\,(x)} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2 - 2}{xu} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-2}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2-2}{u}}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{\frac{u^2-2}{u}}\,du &= \int -\frac{1}{x}\,dx \\ \frac{\ln\left(u^2 - 2\right)}{2} &= -\ln\left(x\right) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{u^2 - 2} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u^2 - 2} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u\,(x)^2 - 2} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{u\,(x)^2 - 2} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}-2} = \frac{c_3 e^{c_2}}{x}$$
$$\sqrt{\frac{y^2-2x^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Substituting initial conditions and solving for c_2 gives $c_2 = \frac{\ln\left(\frac{4}{c_3^2}\right)}{2}$. Hence the solution becomes Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration. Unable to solve for constant of integration.

Warning: Failed to find c_3 using initial conditions. Solution could be wrong or there is no solution that satisfies the given initial conditions.

Verification of solutions N/A

6.6.4 Solving as differentialType ode

Writing the ode as

$$y' = \frac{2x}{y} \tag{1}$$

Which becomes

$$(y)\, dy = (2x)\, dx \tag{2}$$

But the RHS is complete differential because

$$(2x)\, dx = d(x^2)$$

Hence (2) becomes

$$(y)\, dy = d(x^2)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{2x^2+2c_1} + c_1$$
$$y = -\sqrt{2x^2+2c_1} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\sqrt{c_1}\,\sqrt{2} + c_1$$

$$c_1 = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{10}}{2}\right) + 2$$

Substituting c_1 found above in the general solution gives

$$y = -\sqrt{2x^2 + 6 + 2\sqrt{5}} + 3 + \sqrt{5}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \sqrt{c_1}\,\sqrt{2} + c_1$$

$$c_1 = -\sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{10}}{2}\right) + 2$$

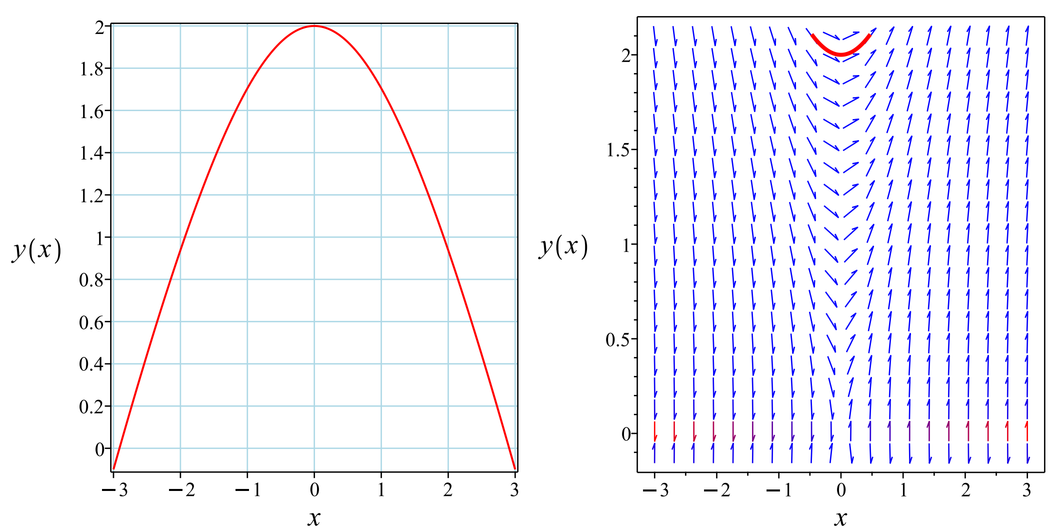
Substituting c_1 found above in the general solution gives

$$y = \sqrt{2x^2 + 6 - 2\sqrt{5}} + 3 - \sqrt{5}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2x^2 + 6 - 2\sqrt{5}} + 3 - \sqrt{5} \tag{1}$$
$$y = -\sqrt{2x^2 + 6 + 2\sqrt{5}} + 3 + \sqrt{5} \tag{2}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \sqrt{2x^2 + 6 - 2\sqrt{5}} + 3 - \sqrt{5}$$

Verified OK.

$$y = -\sqrt{2x^2 + 6 + 2\sqrt{5}} + 3 + \sqrt{5}$$

Verified OK.

6.6.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2x}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 172: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{2x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{2x}} dx \end{aligned}$$

Which results in

$$S = x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 2x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

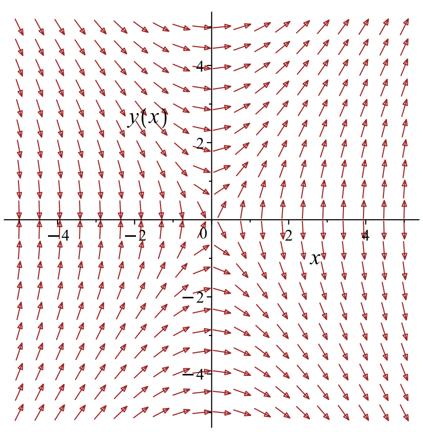
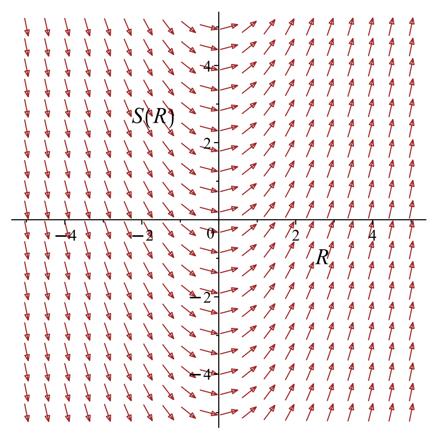
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 = \frac{y^2}{2} + c_1$$

Which simplifies to

$$x^2 = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x}{y}$ 	$R = y$ $S = x^2$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 + 2$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$x^2 = \frac{y^2}{2} - 2$$

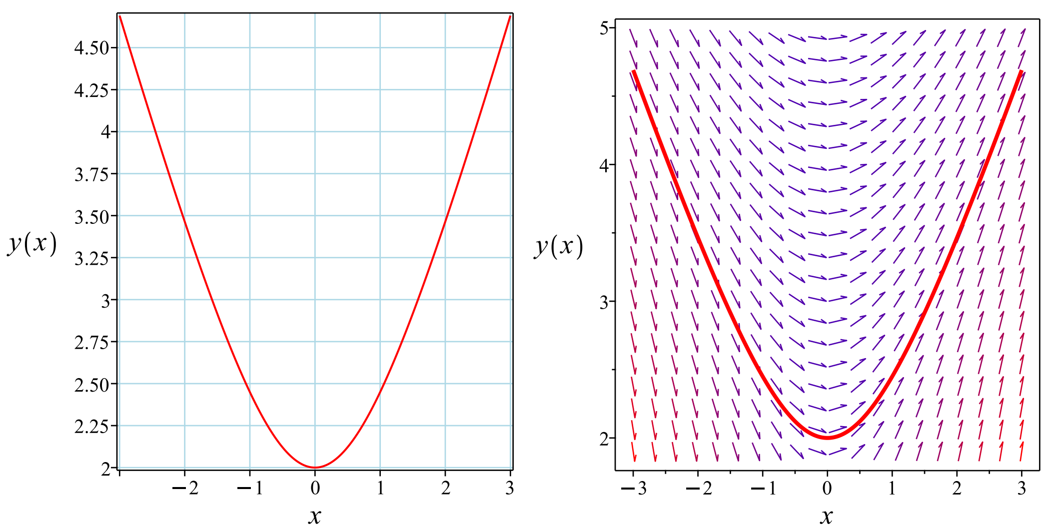
Solving for y from the above gives

$$y = \sqrt{2x^2 + 4}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2x^2 + 4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sqrt{2x^2 + 4}$$

Verified OK.

6.6.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{2}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{y}{2}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\\ N(x,y) &= \frac{y}{2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -x\,\mathrm{d}x$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{2}$. Therefore equation (4) becomes

$$\frac{y}{2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y}{2}\right)\mathrm{d}y$$
$$f(y) = \frac{y^2}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^2}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^2}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

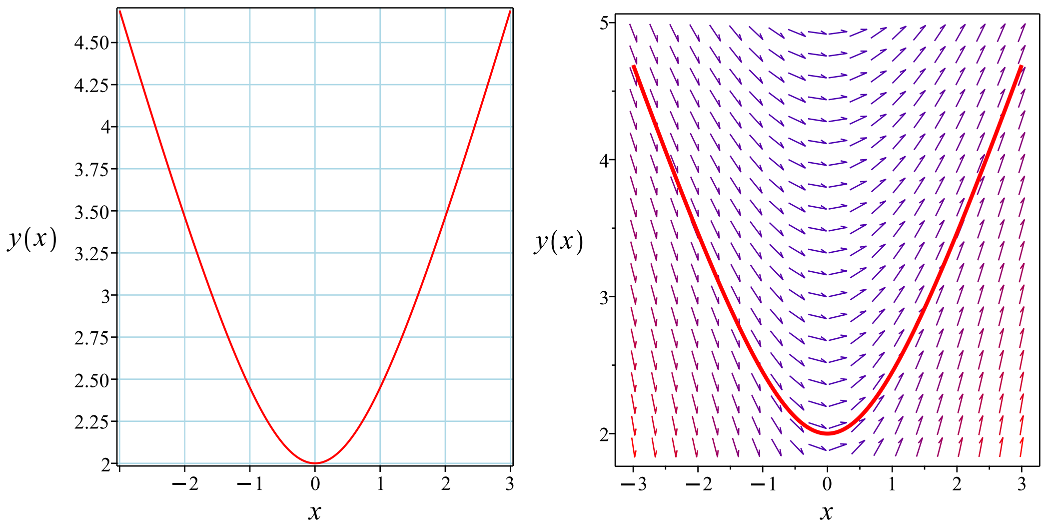
$$-\frac{x^2}{2} + \frac{y^2}{4} = 1$$

Solving for y from the above gives

$$y = \sqrt{2x^2 + 4}$$

Summary
The solution(s) found are the following

$$y = \sqrt{2x^2 + 4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sqrt{2x^2 + 4}$$

Verified OK.

6.6.7 Maple step by step solution

Let's solve

$\left[y' - \frac{2x}{y} = 0, y(0) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = 2x$

- Integrate both sides with respect to x

  $\int yy'dx = \int 2xdx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = x^2 + c_1$
- Solve for y

  $\{y = \sqrt{2x^2 + 2c_1}, y = -\sqrt{2x^2 + 2c_1}\}$
- Use initial condition $y(0) = 2$

  $2 = \sqrt{c_1}\,\sqrt{2}$
- Solve for c_1

  $c_1 = 2$
- Substitute $c_1 = 2$ into general solution and simplify

  $y = \sqrt{2x^2 + 4}$
- Use initial condition $y(0) = 2$

  $2 = -\sqrt{c_1}\,\sqrt{2}$
- Solution does not satisfy initial condition
- Solution to the IVP

  $y = \sqrt{2x^2 + 4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)=2*x/y(x),y(0) = 2],y(x), singsol=all)
```

$$y(x) = \sqrt{2x^2 + 4}$$

✓ Solution by Mathematica

Time used: 0.121 (sec). Leaf size: 20

```
DSolve[{y'[x]==2*x/y[x],{y[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{2}\sqrt{x^2+2}$$

6.7 problem 7

Internal problem ID [12677]
Internal file name [`OUTPUT/11329_Friday_November_03_2023_06_30_29_AM_99627193/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + 2y - y^2 = 0}$$

With initial conditions

$$[y(0) = 1]$$

6.7.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^2 - 2y \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2 - 2y) \\ &= 2y - 2 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

6.7.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{y^2 - 2y} dy &= \int dx \\ \frac{\ln (y-2)}{2} - \frac{\ln (y)}{2} &= x + c_1\end{aligned}$$

The above can be written as

$$\begin{aligned}\left(\frac{1}{2}\right) (\ln (y-2) - \ln (y)) &= x + c_1 \\ \ln (y-2) - \ln (y) &= (2) (x + c_1) \\ &= 2x + 2c_1\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(y-2)-\ln(y)} = 2c_1 e^{2x}$$

Which simplifies to

$$\frac{y-2}{y} = c_2 e^{2x}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{2}{-1 + c_2}$$

$$c_2 = -1$$

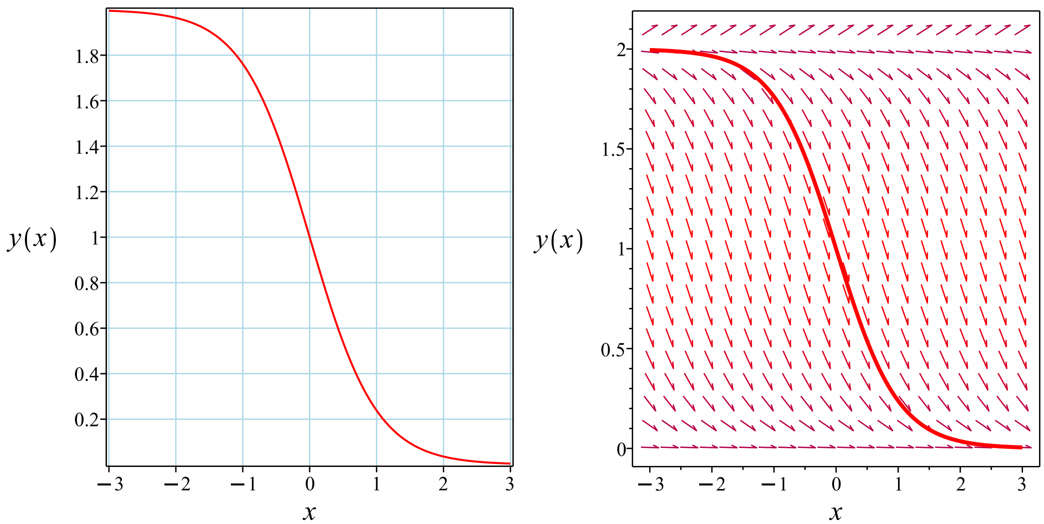
Substituting c_2 found above in the general solution gives

$$y = \frac{2}{e^{2x} + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{2}{e^{2x} + 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{2}{e^{2x} + 1}$$

Verified OK.

6.7.3 Maple step by step solution

Let's solve

$[y' + 2y - y^2 = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-2y+y^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-2y+y^2} dx = \int 1 dx + c_1$

- Evaluate integral

  $\frac{\ln(y-2)}{2} - \frac{\ln(y)}{2} = x + c_1$

- Solve for y

  $y = -\frac{2}{e^{2x+2c_1}-1}$

- Use initial condition $y(0) = 1$

  $1 = -\frac{2}{e^{2c_1}-1}$

- Solve for c_1

  $c_1 = \frac{I}{2}\pi$

- Substitute $c_1 = \frac{I}{2}\pi$ into general solution and simplify

  $y = \frac{2}{e^{2x}+1}$

- Solution to the IVP

  $y = \frac{2}{e^{2x}+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)=-2*y(x)+y(x)^2,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{2}{e^{2x} + 1}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 16

```
DSolve[{y'[x]==-2*y[x]+y[x]^2,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{e^{2x} + 1}$$

6.8 problem 8

Internal problem ID [12678]
Internal file name [`OUTPUT/11330_Friday_November_03_2023_06_30_30_AM_17269816/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - yx = x}$$

With initial conditions

$$[y(1) = 2]$$

6.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -x$$
$$q(x) = x$$

Hence the ode is

$$y' - yx = x$$

The domain of $p(x) = -x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

6.8.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= x(y + 1) \end{aligned}$$

Where $f(x) = x$ and $g(y) = y + 1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y+1}\,dy &= x\,dx \\ \int \frac{1}{y+1}\,dy &= \int x\,dx \\ \ln(y+1) &= \frac{x^2}{2} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$y + 1 = e^{\frac{x^2}{2} + c_1}$$

Which simplifies to

$$y + 1 = c_2 e^{\frac{x^2}{2}}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_2 e^{\frac{1}{2} + c_1} - 1$$

$$c_1 = -\frac{1}{2} + \ln\left(\frac{3}{c_2}\right)$$

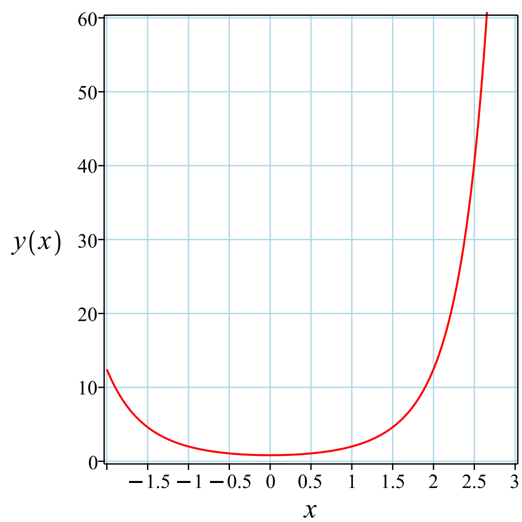
Substituting c_1 found above in the general solution gives

$$y = 3\,\mathrm{e}^{\frac{(x-1)(x+1)}{2}} - 1$$

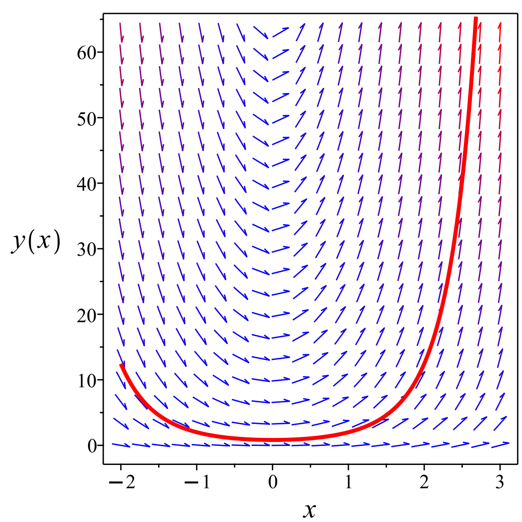
Summary

The solution(s) found are the following

$$y = 3\,\mathrm{e}^{\frac{(x-1)(x+1)}{2}} - 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{\frac{(x-1)(x+1)}{2}} - 1$$

Verified OK.

6.8.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = \mathrm{e}^{\int -x dx}$$
$$= \mathrm{e}^{-\frac{x^2}{2}}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(x) \\
\frac{d}{dx}\left(e^{-\frac{x^2}{2}} y\right) &= \left(e^{-\frac{x^2}{2}}\right)(x) \\
d\left(e^{-\frac{x^2}{2}} y\right) &= \left(x\, e^{-\frac{x^2}{2}}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-\frac{x^2}{2}} y &= \int x\, e^{-\frac{x^2}{2}}\, dx \\
e^{-\frac{x^2}{2}} y &= -e^{-\frac{x^2}{2}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x^2}{2}}$ results in

$$
y = -e^{\frac{x^2}{2}} e^{-\frac{x^2}{2}} + c_1 e^{\frac{x^2}{2}}
$$

which simplifies to

$$
y = -1 + c_1 e^{\frac{x^2}{2}}
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$
2 = -1 + c_1 e^{\frac{1}{2}}
$$

$$
c_1 = 3\, e^{-\frac{1}{2}}
$$

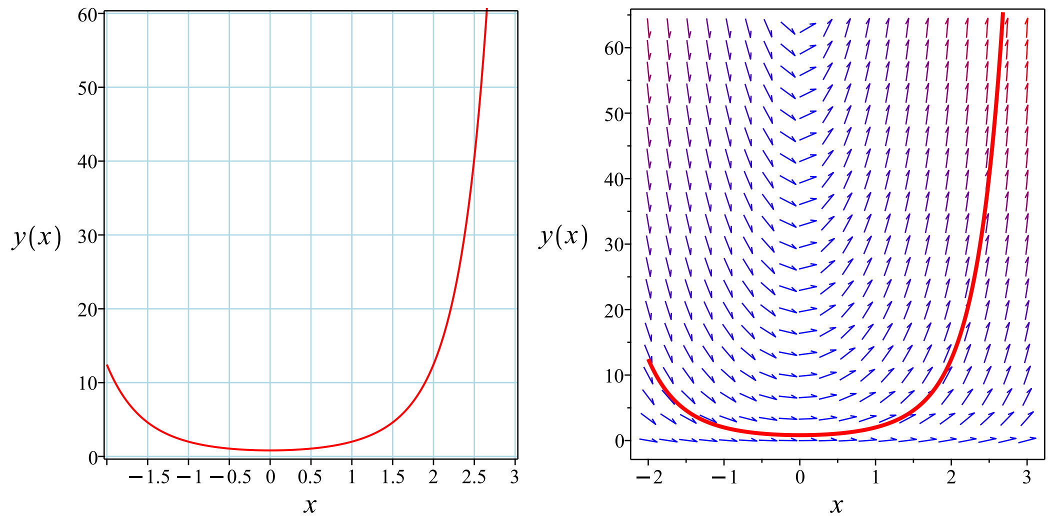
Substituting c_1 found above in the general solution gives

$$
y = 3\, e^{\frac{(x-1)(x+1)}{2}} - 1
$$

Summary

The solution(s) found are the following

$$
y = 3\, e^{\frac{(x-1)(x+1)}{2}} - 1 \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{\frac{(x-1)(x+1)}{2}} - 1$$

Verified OK.

6.8.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = xy + x$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 176: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\frac{x^2}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{x^2}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{x^2}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = xy + x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -x e^{-\frac{x^2}{2}} y \\ S_y &= e^{-\frac{x^2}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x e^{-\frac{x^2}{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R e^{-\frac{R^2}{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -e^{-\frac{R^2}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-\frac{x^2}{2}} y = -e^{-\frac{x^2}{2}} + c_1$$

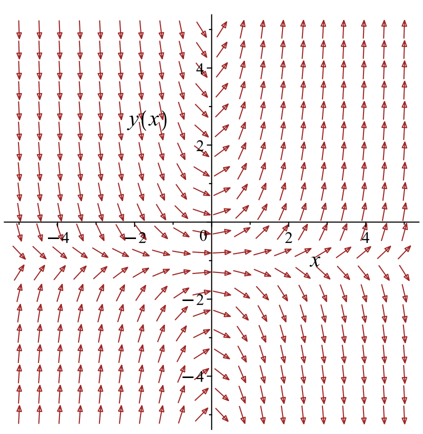
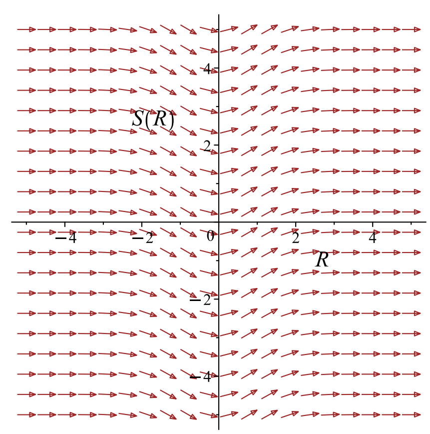
Which simplifies to

$$e^{-\frac{x^2}{2}} y = -e^{-\frac{x^2}{2}} + c_1$$

Which gives

$$y = -\left(e^{-\frac{x^2}{2}} - c_1\right) e^{\frac{x^2}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = xy + x$ 	$R = x$ $S = e^{-\frac{x^2}{2}} y$	$\frac{dS}{dR} = R e^{-\frac{R^2}{2}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -1 + c_1 e^{\frac{1}{2}}$$

$$c_1 = 3\,\mathrm{e}^{-\frac{1}{2}}$$

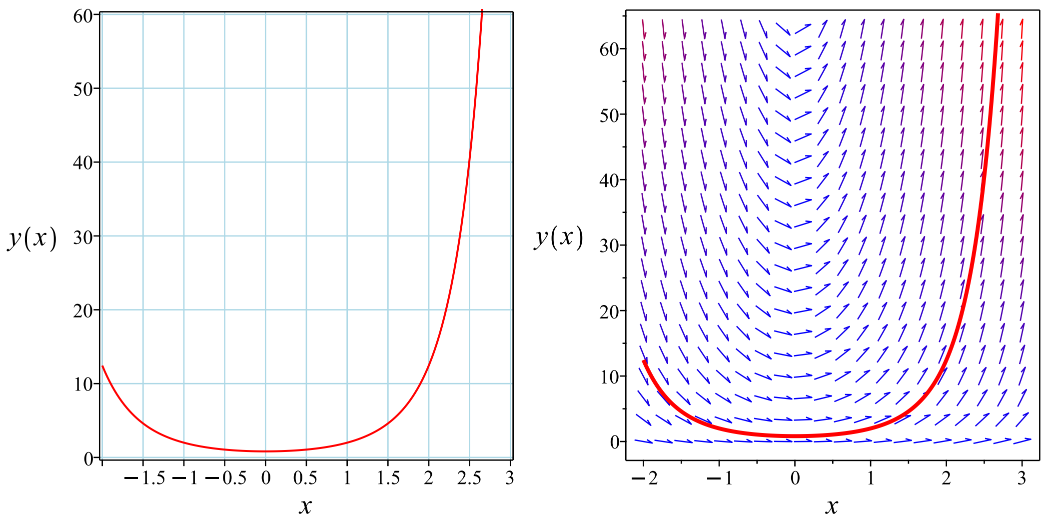
Substituting c_1 found above in the general solution gives

$$y = 3\,\mathrm{e}^{\frac{(x-1)(x+1)}{2}} - 1$$

Summary

The solution(s) found are the following

$$y = 3\,\mathrm{e}^{\frac{(x-1)(x+1)}{2}} - 1 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{\frac{(x-1)(x+1)}{2}} - 1$$

Verified OK.

6.8.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y+1}\right)\mathrm{d}y &= (x)\,\mathrm{d}x \\ (-x)\,\mathrm{d}x + \left(\frac{1}{y+1}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x \\ N(x,y) &= \frac{1}{y+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -x \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y+1}$. Therefore equation (4) becomes

$$
\frac{1}{y+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{1}{y+1}\right) \mathrm{d}y \\
f(y) &= \ln(y+1) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \ln(y+1)$$

The solution becomes

$$y = \mathrm{e}^{\frac{x^2}{2}+c_1} - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \mathrm{e}^{\frac{1}{2}+c_1} - 1$$

$$c_1 = -\frac{1}{2} + \ln(3)$$

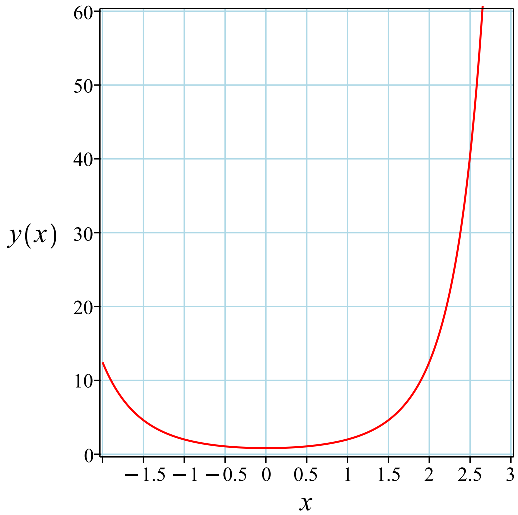
Substituting c_1 found above in the general solution gives

$$y = 3\,\mathrm{e}^{\frac{(x-1)(x+1)}{2}} - 1$$

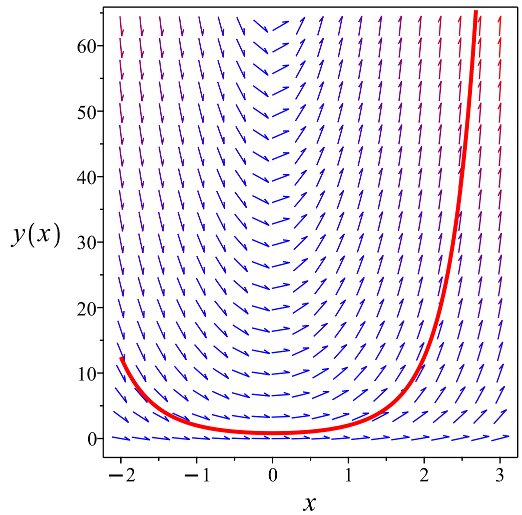
Summary

The solution(s) found are the following

$$y = 3\,\mathrm{e}^{\frac{(x-1)(x+1)}{2}} - 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 3\,\mathrm{e}^{\frac{(x-1)(x+1)}{2}} - 1$$

Verified OK.

6.8.6 Maple step by step solution

Let's solve

$[y' - yx = x, y(1) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y+1} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y+1} dx = \int x dx + c_1$

- Evaluate integral

  $\ln(y+1) = \frac{x^2}{2} + c_1$

- Solve for y

  $y = \mathrm{e}^{\frac{x^2}{2}+c_1} - 1$

- Use initial condition $y(1) = 2$

  $2 = \mathrm{e}^{\frac{1}{2}+c_1} - 1$

- Solve for c_1

  $c_1 = -\frac{1}{2} + \ln(3)$

- Substitute $c_1 = -\frac{1}{2} + \ln(3)$ into general solution and simplify

  $y = 3\,\mathrm{e}^{\frac{(x-1)(x+1)}{2}} - 1$

- Solution to the IVP

  $y = 3\,\mathrm{e}^{\frac{(x-1)(x+1)}{2}} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)=x*y(x)+x,y(1) = 2],y(x), singsol=all)
```

$$y(x) = -1 + 3\,\mathrm{e}^{\frac{(-1+x)(1+x)}{2}}$$

✓ Solution by Mathematica

Time used: 0.099 (sec). Leaf size: 20

```
DSolve[{y'[x]==x*y[x]+x,{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3e^{\frac{1}{2}(x^2-1)} - 1$$

6.9 problem 9

Internal problem ID [12679]
Internal file name [`OUTPUT/11331_Friday_November_03_2023_06_30_31_AM_99325169/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x\,e^{y}+y'=0}$$

With initial conditions

$$[y(0)=0]$$

6.9.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -x\,e^{y} \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(-x\,\mathrm{e}^{y}) \\ &= -x\,\mathrm{e}^{y}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

6.9.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= -x\,\mathrm{e}^{y}\end{aligned}$$

Where $f(x) = -x$ and $g(y) = \mathrm{e}^{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\mathrm{e}^{y}}\,dy &= -x\,dx \\ \int \frac{1}{\mathrm{e}^{y}}\,dy &= \int -x\,dx \\ -\mathrm{e}^{-y} &= -\frac{x^2}{2} + c_1\end{aligned}$$

Which results in

$$y = \ln\left(-\frac{2}{-x^2 + 2c_1}\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln\left(-\frac{1}{c_1}\right)$$

$$c_1 = -1$$

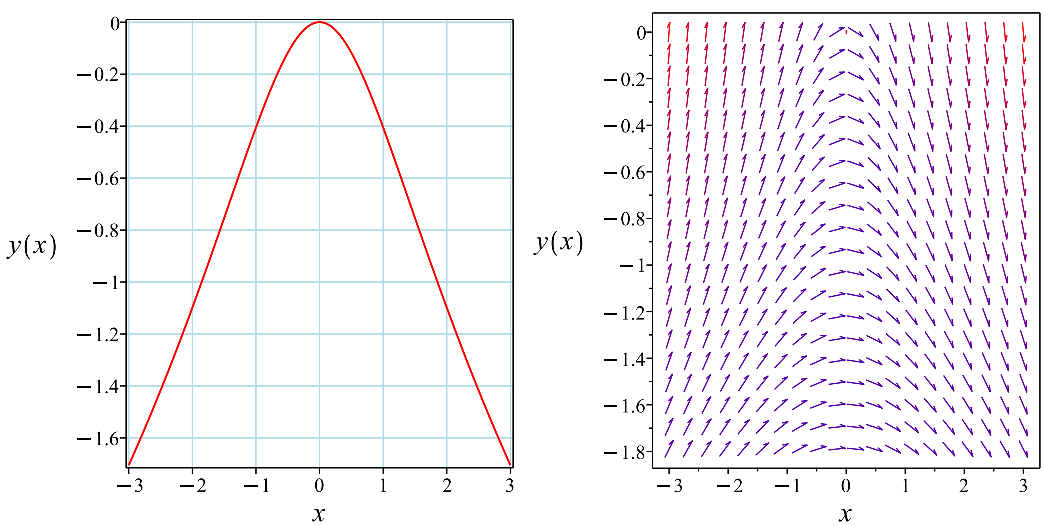
Substituting c_1 found above in the general solution gives

$$y = \ln(2) + \ln\left(\frac{1}{x^2+2}\right)$$

Summary

The solution(s) found are the following

$$y = \ln(2) + \ln\left(\frac{1}{x^2+2}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \ln(2) + \ln\left(\frac{1}{x^2+2}\right)$$

Verified OK.

6.9.3 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = -x\,\mathrm{e}^{y} \tag{1}$$

And using the substitution $u = \mathrm{e}^{-y}$ then

$$u' = -y'\mathrm{e}^{-y}$$

The above shows that

$$\begin{aligned} y' &= -u'(x)\,\mathrm{e}^{y} \\ &= -\frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{u} = -\frac{x}{u}$$

The above simplifies to

$$u'(x) = x \tag{2}$$

Now ode (2) is solved for $u(x)$ Integrating both sides gives

$$\begin{aligned} u(x) &= \int x\, \mathrm{d}x \\ &= \frac{x^2}{2} + c_1 \end{aligned}$$

Substituting the solution found for $u(x)$ in $u = \mathrm{e}^{-y}$ gives

$$\begin{aligned} y &= -\ln(u(x)) \\ &= -\ln\left(\frac{x^2}{2} + c_1\right) \\ &= \ln(2) - \ln\left(x^2 + 2c_1\right) \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\ln(c_1)$$

$$c_1 = 1$$

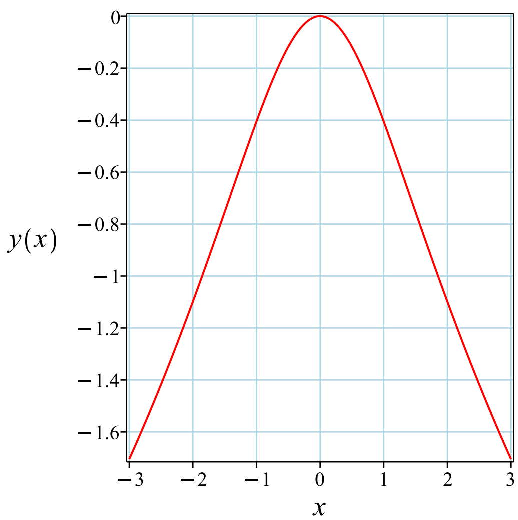
Substituting c_1 found above in the general solution gives

$$y = \ln(2) - \ln(x^2 + 2)$$

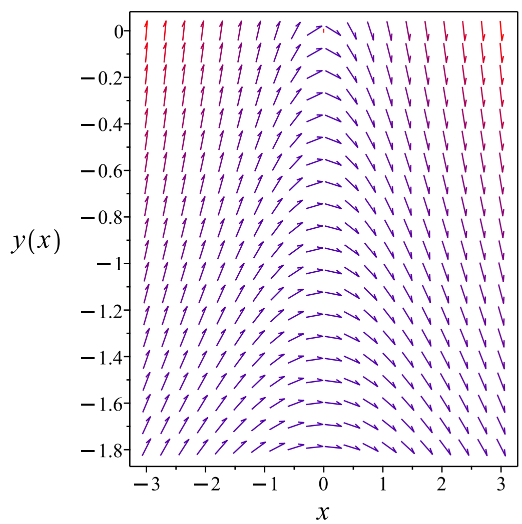
Summary

The solution(s) found are the following

$$y = \ln(2) - \ln(x^2 + 2) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \ln(2) - \ln(x^2 + 2)$$

Verified OK.

6.9.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -x\,e^{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 179: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{1}{x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -x \, \mathrm{e}^{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{-y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{2} = -\mathrm{e}^{-y} + c_1$$

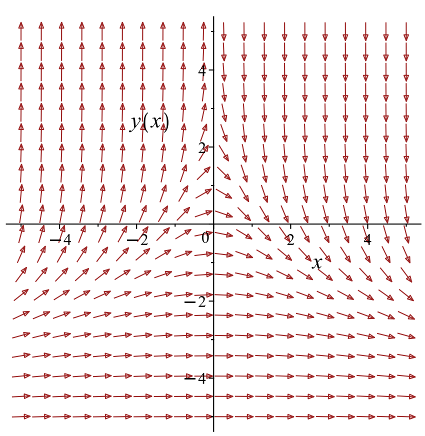
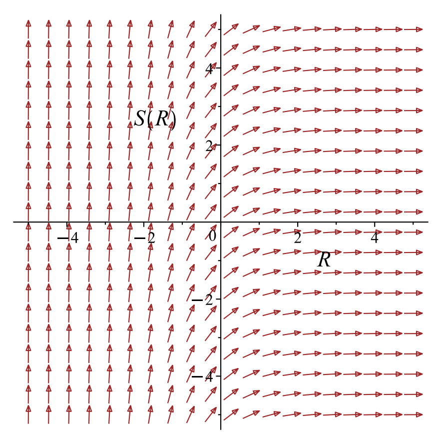
Which simplifies to

$$-\frac{x^2}{2} = -\mathrm{e}^{-y} + c_1$$

Which gives

$$y = -\ln\left(\frac{x^2}{2} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -x\,\mathrm{e}^{y}$ 	$R = y$ $S = -\frac{x^2}{2}$	$\frac{dS}{dR} = \mathrm{e}^{-R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\ln(c_1)$$

$$c_1 = 1$$

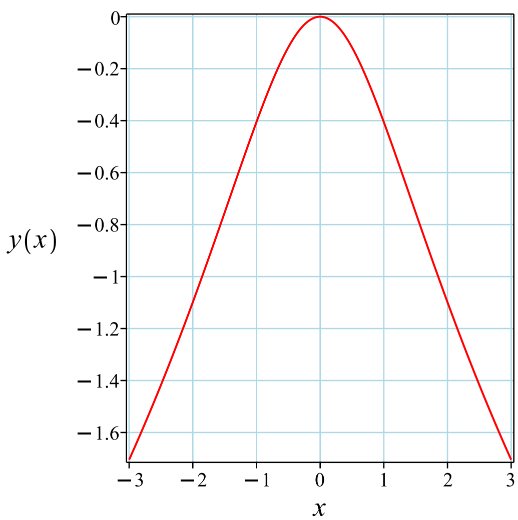
Substituting c_1 found above in the general solution gives

$$y = \ln(2) - \ln(x^2 + 2)$$

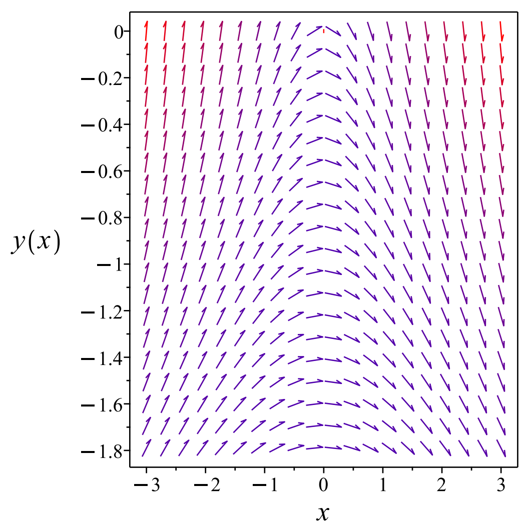
Summary

The solution(s) found are the following

$$y = \ln(2) - \ln(x^2 + 2) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \ln(2) - \ln(x^2 + 2)$$

Verified OK.

6.9.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\mathrm{e}^{-y}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+\left(-\mathrm{e}^{-y}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\\ N(x,y)&=-\mathrm{e}^{-y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-x)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\mathrm{e}^{-y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -x \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\mathrm{e}^{-y}$. Therefore equation (4) becomes

$$
-\mathrm{e}^{-y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\mathrm{e}^{-y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\mathrm{e}^{-y}\right) \mathrm{d}y \\
f(y) &= \mathrm{e}^{-y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + e^{-y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + e^{-y}$$

The solution becomes

$$y = -\ln\left(\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\ln(c_1)$$

$$c_1 = 1$$

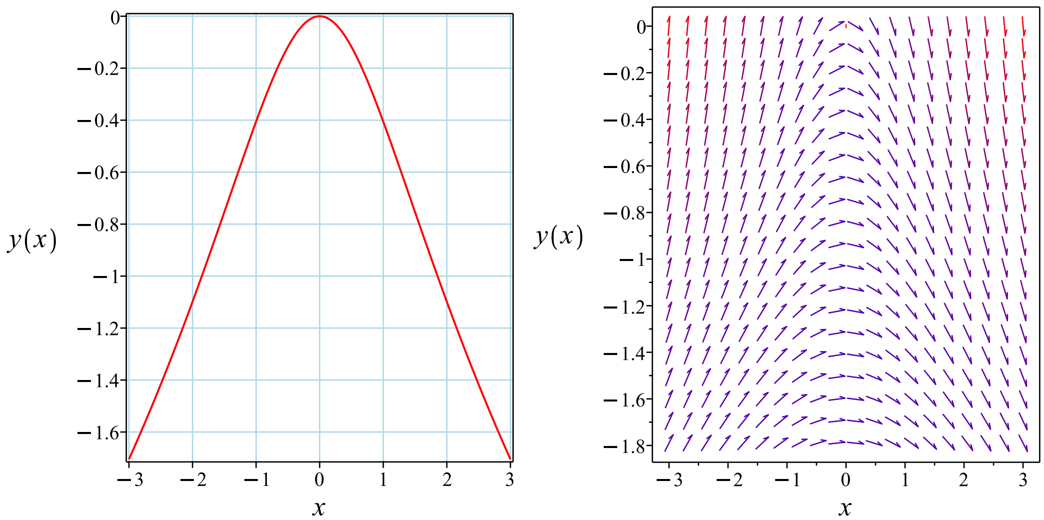
Substituting c_1 found above in the general solution gives

$$y = \ln(2) - \ln(x^2 + 2)$$

Summary

The solution(s) found are the following

$$y = \ln(2) - \ln(x^2 + 2) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \ln(2) - \ln(x^2 + 2)$$

Verified OK.

6.9.6 Maple step by step solution

Let's solve

$[x\,\mathrm{e}^{y} + y' = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\mathrm{e}^{y}} = -x$

- Integrate both sides with respect to x

  $\int \frac{y'}{\mathrm{e}^{y}} dx = \int -x dx + c_1$

- Evaluate integral

  $-\frac{1}{\mathrm{e}^{y}} = -\frac{x^2}{2} + c_1$

- Solve for y

  $y = \ln\left(-\frac{2}{-x^2 + 2c_1}\right)$

- Use initial condition $y(0) = 0$

  $0 = \ln\left(-\frac{1}{c_1}\right)$

- Solve for c_1

  $c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

  $y = \ln(2) + \ln\left(\frac{1}{x^2+2}\right)$

- Solution to the IVP

  $y = \ln(2) + \ln\left(\frac{1}{x^2+2}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 15

```
dsolve([x*exp(y(x))+diff(y(x),x)=0,y(0) = 0],y(x), singsol=all)
```

$$y(x) = \ln(2) - \ln(x^2 + 2)$$

✓ Solution by Mathematica

Time used: 0.476 (sec). Leaf size: 16

```
DSolve[{x*Exp[y[x]]+y'[x]==0,{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(2) - \log(x^2 + 2)$$

6.10 problem 10

Internal problem ID [12680]
Internal file name [`OUTPUT/11332_Friday_November_03_2023_06_30_32_AM_16605047/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y - x^2 y' = 0}$$

With initial conditions

$$[y(1) = 1]$$

6.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x^2}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{y}{x^2} = 0$$

The domain of $p(x) = -\frac{1}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. Hence solution exists and is unique.

6.10.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y}{x^2} \end{aligned}$$

Where $f(x) = \frac{1}{x^2}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= \frac{1}{x^2}\,dx \\ \int \frac{1}{y}\,dy &= \int \frac{1}{x^2}\,dx \\ \ln(y) &= -\frac{1}{x} + c_1 \\ y &= \mathrm{e}^{-\frac{1}{x}+c_1} \\ &= c_1 \mathrm{e}^{-\frac{1}{x}} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 \mathrm{e}^{-1}$$

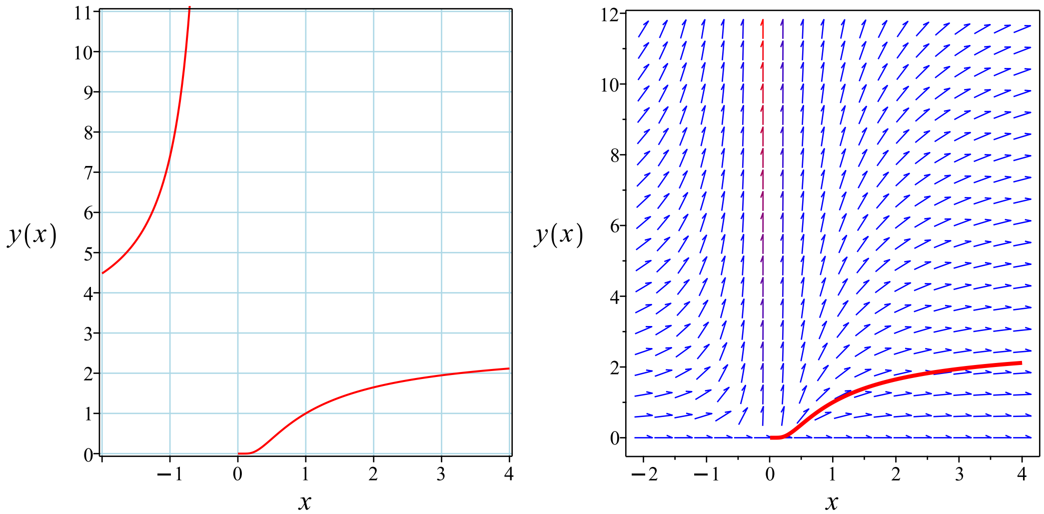
$$c_1 = \mathrm{e}$$

Substituting c_1 found above in the general solution gives

$$y = \mathrm{e}^{\frac{x-1}{x}}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{\frac{x-1}{x}} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{\frac{x-1}{x}}$$

Verified OK.

6.10.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x^2} dx}$$
$$= e^{\frac{1}{x}}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(e^{\frac{1}{x}} y\right) = 0$$

Integrating gives

$$e^{\frac{1}{x}} y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{1}{x}}$ results in

$$y = c_1 e^{-\frac{1}{x}}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 \mathrm{e}^{-1}$$

$$c_1 = \mathrm{e}$$

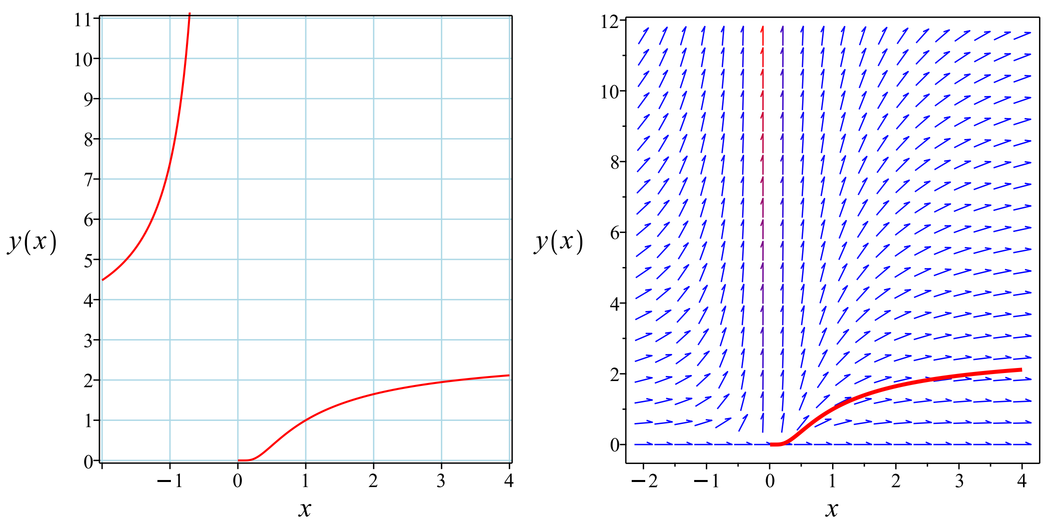
Substituting c_1 found above in the general solution gives

$$y = \mathrm{e}^{\frac{x-1}{x}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{x-1}{x}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{x-1}{x}}$$

Verified OK.

6.10.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x - x^2(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(x-1)}{x^2} \end{aligned}$$

Where $f(x) = -\frac{x-1}{x^2}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{x-1}{x^2}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{x-1}{x^2}\,dx \\ \ln(u) &= -\frac{1}{x} - \ln(x) + c_2 \\ u &= \mathrm{e}^{-\frac{1}{x}-\ln(x)+c_2} \\ &= c_2\mathrm{e}^{-\frac{1}{x}-\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2\mathrm{e}^{-\frac{1}{x}}}{x}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= \mathrm{e}^{-\frac{1}{x}}c_2 \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_2\mathrm{e}^{-1}$$

$$c_2 = \mathrm{e}$$

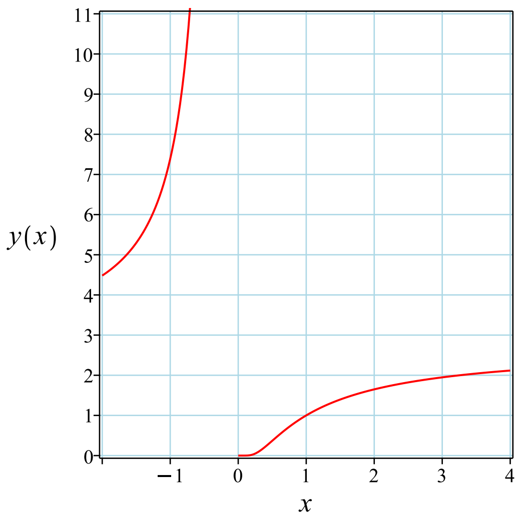
Substituting c_2 found above in the general solution gives

$$y = \mathrm{e}^{\frac{x-1}{x}}$$

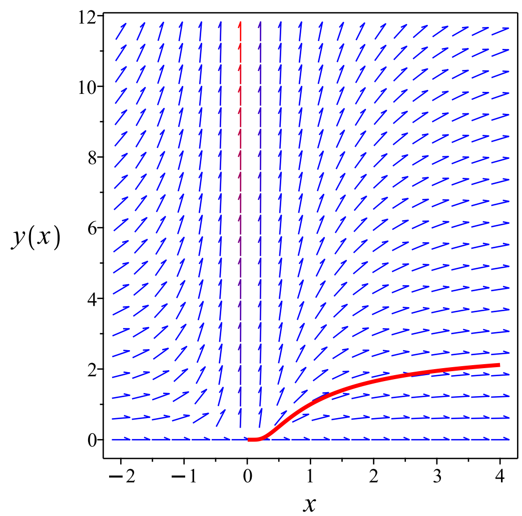
Summary

The solution(s) found are the following

$$y = e^{\frac{x-1}{x}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{\frac{x-1}{x}}$$

Verified OK.

6.10.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 182: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{-\frac{1}{x}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{1}{x}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{1}{x}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{e^{\frac{1}{x}} y}{x^2} \\ S_y &= e^{\frac{1}{x}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\frac{1}{x}} y = c_1$$

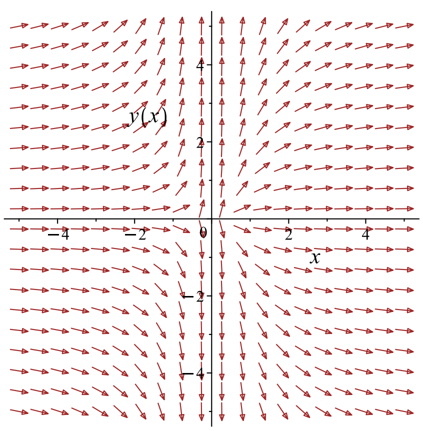
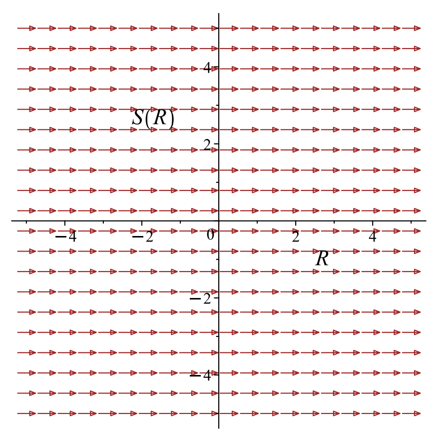
Which simplifies to

$$e^{\frac{1}{x}} y = c_1$$

Which gives

$$y = c_1 e^{-\frac{1}{x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x^2}$ 	$R = x$ $S = e^{\frac{1}{x}} y$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 e^{-1}$$

$$c_1 = \mathrm{e}$$

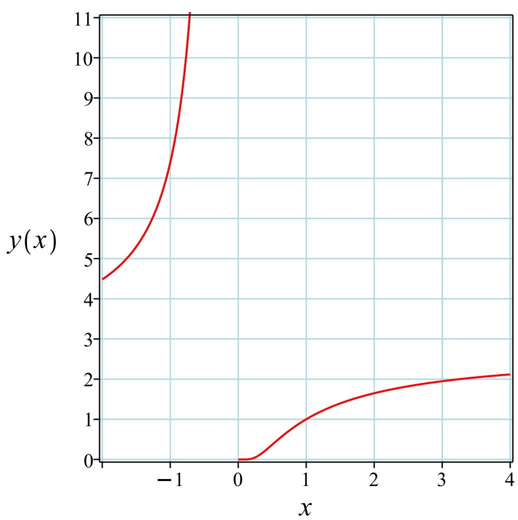
Substituting c_1 found above in the general solution gives

$$y = \mathrm{e}^{\frac{x-1}{x}}$$

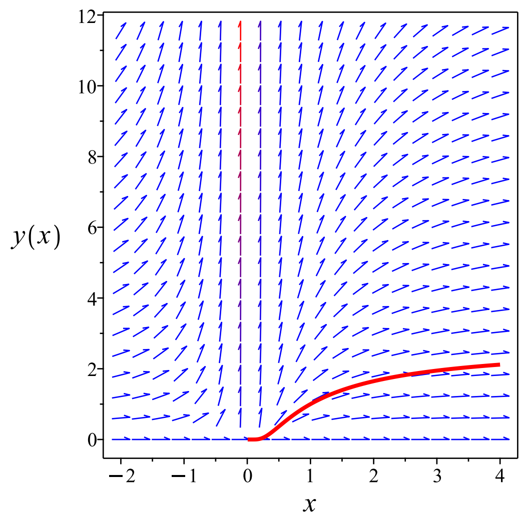
Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{x-1}{x}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{x-1}{x}}$$

Verified OK.

6.10.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y&=\left(\frac{1}{x^2}\right)\mathrm{d}x\\ \left(-\frac{1}{x^2}\right)\mathrm{d}x+\left(\frac{1}{y}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{x^2}\\ N(x,y)&=\frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{x^2}\right)\\ &=0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\frac{1}{x^2} \,\mathrm{d}x$$
$$\phi = \frac{1}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{x} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{x} + \ln(y)$$

The solution becomes

$$y = e^{\frac{c_1 x - 1}{x}}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = e^{-1+c_1}$$

$$c_1 = 1$$

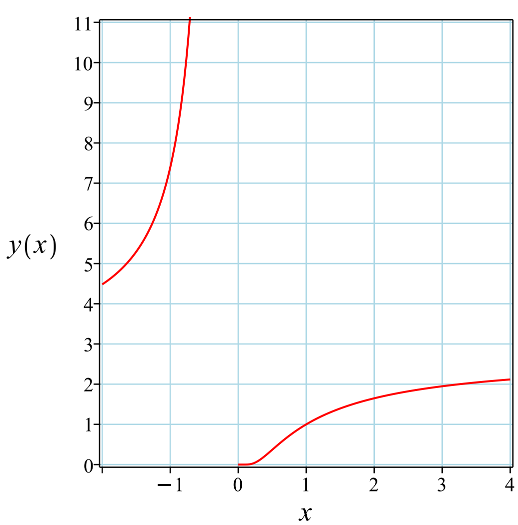
Substituting c_1 found above in the general solution gives

$$y = e^{\frac{x-1}{x}}$$

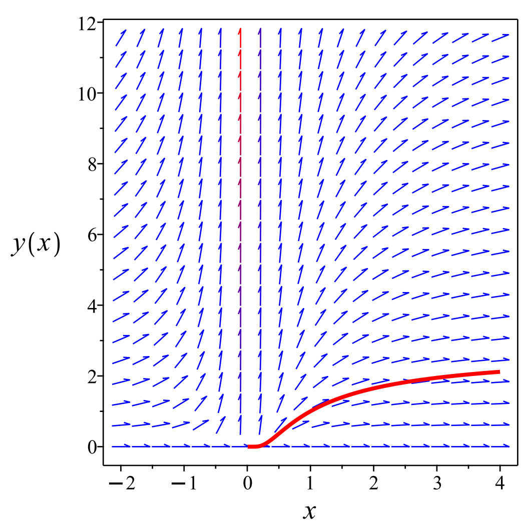
Summary

The solution(s) found are the following

$$y = e^{\frac{x-1}{x}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{\frac{x-1}{x}}$$

Verified OK.

6.10.7 Maple step by step solution

Let's solve

$[y - x^2 y' = 0, y(1) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x^2} dx + c_1$

- Evaluate integral

  $\ln(y) = -\frac{1}{x} + c_1$

- Solve for y

  $y = e^{\frac{c_1 x - 1}{x}}$

- Use initial condition $y(1) = 1$

  $1 = e^{-1+c_1}$

- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

  $y = e^{\frac{x-1}{x}}$

- Solution to the IVP

  $y = e^{\frac{x-1}{x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 13

```
dsolve([y(x)-x^2*diff(y(x),x)=0,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\frac{-1+x}{x}}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 14

```
DSolve[{y[x]-x^2*y'[x]==0,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{1-\frac{1}{x}}$$

6.11 problem 11

Internal problem ID [12681]
Internal file name [`OUTPUT/11333_Friday_November_03_2023_06_30_33_AM_84621573/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{2yy' = 1}$$

6.11.1 Solving as quadrature ode

Integrating both sides gives

$$\int 2y dy = x + c_1$$
$$y^2 = x + c_1$$

Solving for y gives these solutions

$$y_1 = \sqrt{x + c_1}$$
$$y_2 = -\sqrt{x + c_1}$$

Summary
The solution(s) found are the following

$$y = \sqrt{x + c_1} \tag{1}$$
$$y = -\sqrt{x + c_1} \tag{2}$$

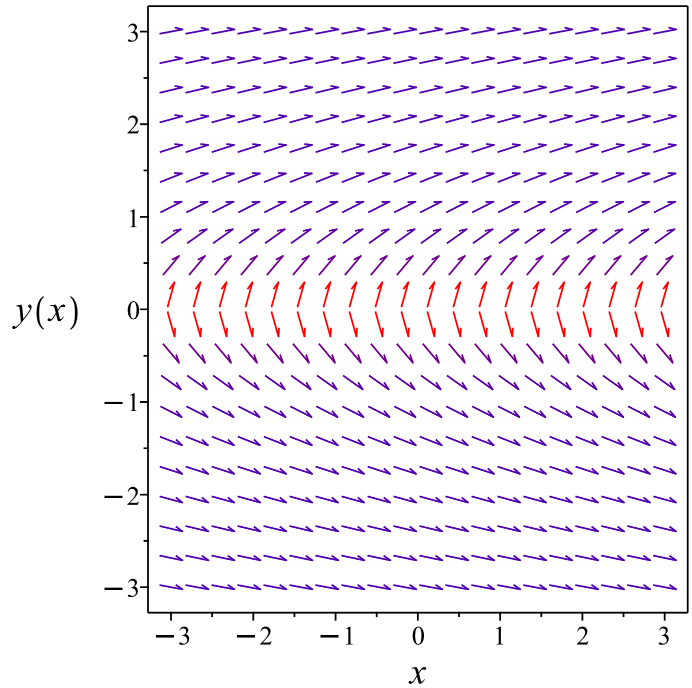


Figure 217: Slope field plot

Verification of solutions

$$y = \sqrt{x + c_1}$$

Verified OK.

$$y = -\sqrt{x + c_1}$$

Verified OK.

6.11.2 Maple step by step solution

Let's solve

$2yy' = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int 2yy'dx = \int 1dx + c_1$

- Evaluate integral

$$y^2 = x + c_1$$

- Solve for y

$$\{y = \sqrt{x + c_1}, y = -\sqrt{x + c_1}\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(2*y(x)*diff(y(x),x)=1,y(x), singsol=all)
```

$$y(x) = \sqrt{c_1 + x}$$
$$y(x) = -\sqrt{c_1 + x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 31

```
DSolve[2*y[x]*y'[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x + 2c_1}$$
$$y(x) \to \sqrt{x + 2c_1}$$

6.12 problem 12

Internal problem ID [12682]
Internal file name [`OUTPUT/11334_Friday_November_03_2023_06_30_33_AM_90158799/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{2xyy' + y^2 = -1}$$

6.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y^2+1}{2xy} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(y) = \frac{y^2+1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2+1}{y}}\, dy = -\frac{1}{2x}\, dx$$

$$\int \frac{1}{\frac{y^2+1}{y}}\, dy = \int -\frac{1}{2x}\, dx$$

$$\frac{\ln (y^2+1)}{2} = -\frac{\ln (x)}{2} + c_1$$

Raising both side to exponential gives

$$\sqrt{y^2+1} = e^{-\frac{\ln(x)}{2}+c_1}$$

Which simplifies to

$$\sqrt{y^2+1} = \frac{c_2}{\sqrt{x}}$$

Which simplifies to

$$\sqrt{1+y^2} = \frac{c_2 e^{c_1}}{\sqrt{x}}$$

The solution is

$$\sqrt{1+y^2} = \frac{c_2 e^{c_1}}{\sqrt{x}}$$

Summary
The solution(s) found are the following

$$\sqrt{1+y^2} = \frac{c_2 e^{c_1}}{\sqrt{x}} \tag{1}$$

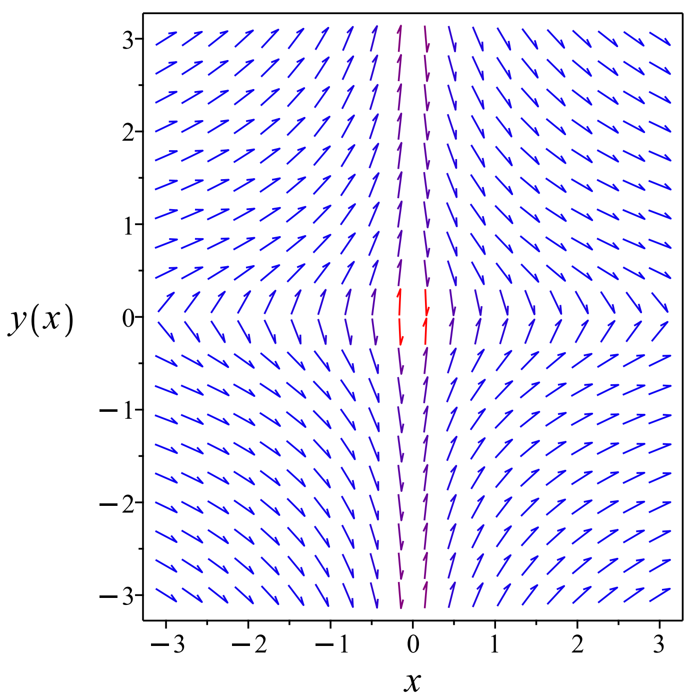


Figure 218: Slope field plot

Verification of solutions

$$\sqrt{1+y^2} = \frac{c_2 \mathrm{e}^{c_1}}{\sqrt{x}}$$

Verified OK.

6.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y^2+1}{2xy}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 186: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= -2x \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-2x} dx \end{aligned}$$

Which results in

$$S = -\frac{\ln(x)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^2 + 1}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{1}{2x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln (R^2 + 1)}{2} + c_1 \tag{4}$$

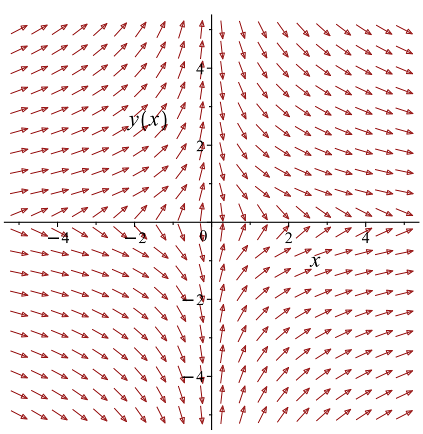
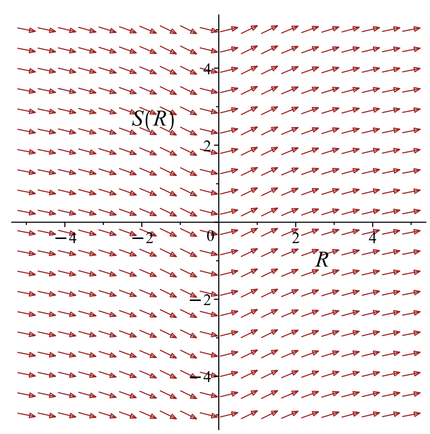
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln (x)}{2} = \frac{\ln (1 + y^2)}{2} + c_1$$

Which simplifies to

$$-\frac{\ln (x)}{2} = \frac{\ln (1 + y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2+1}{2xy}$ 	$R = y$ $S = -\frac{\ln (x)}{2}$	$\frac{dS}{dR} = \frac{R}{R^2+1}$ 

Summary

The solution(s) found are the following

$$-\frac{\ln (x)}{2} = \frac{\ln (1 + y^2)}{2} + c_1 \tag{1}$$

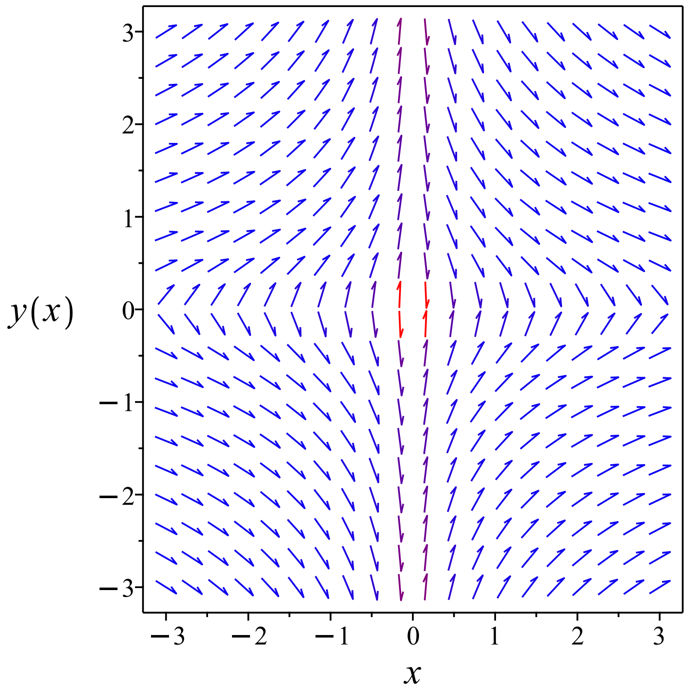


Figure 219: Slope field plot

Verification of solutions

$$-\frac{\ln(x)}{2} = \frac{\ln(1+y^2)}{2} + c_1$$

Verified OK.

6.12.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y^2+1}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{2x}y - \frac{1}{2x}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{2x} \\ f_1(x) &= -\frac{1}{2x} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{2x} - \frac{1}{2x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -\frac{w(x)}{2x} - \frac{1}{2x} \\ w' &= -\frac{w}{x} - \frac{1}{x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = -\frac{1}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{1}{x}\right) \\ \frac{d}{dx}(xw) &= (x)\left(-\frac{1}{x}\right) \\ d(xw) &= -1\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xw &= \int -1\, dx \\ xw &= -x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = -1 + \frac{c_1}{x}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -1 + \frac{c_1}{x}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{x(-x + c_1)}}{x} \\ y(x) &= -\frac{\sqrt{x(-x + c_1)}}{x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{x(-x + c_1)}}{x} \tag{1}$$

$$y = -\frac{\sqrt{x(-x + c_1)}}{x} \tag{2}$$

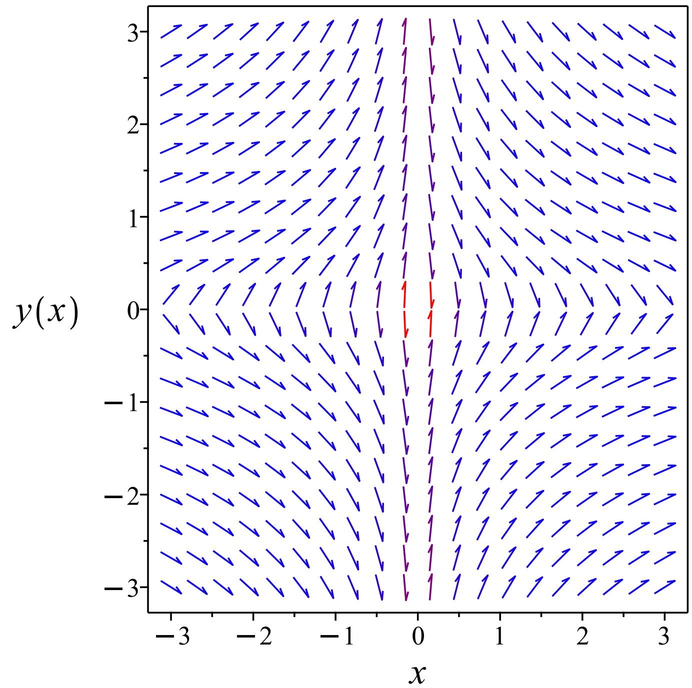


Figure 220: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x\left(-x+c_1\right)}}{x}$$

Verified OK.

$$y = -\frac{\sqrt{x\left(-x+c_1\right)}}{x}$$

Verified OK.

6.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(-\frac{2 y}{y^2+1}\right) \mathrm{d} y &= \left(\frac{1}{x}\right) \mathrm{d} x \\
\left(-\frac{1}{x}\right) \mathrm{d} x+\left(-\frac{2 y}{y^2+1}\right) \mathrm{d} y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= -\frac{2 y}{y^2+1}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{2y}{y^2+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{y^2+1}$. Therefore equation (4) becomes

$$
-\frac{2y}{y^2+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{2y}{y^2+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{2y}{y^2+1}\right) \mathrm{d}y \\
f(y) &= -\ln\left(y^2+1\right) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(y^2 + 1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(y^2 + 1)$$

Summary

The solution(s) found are the following

$$-\ln(x) - \ln(1 + y^2) = c_1 \tag{1}$$

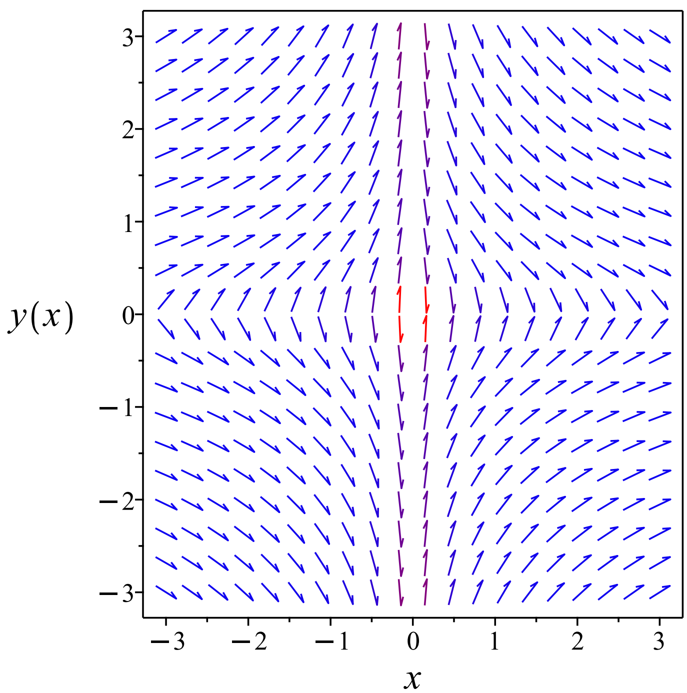


Figure 221: Slope field plot

Verification of solutions

$$-\ln(x) - \ln(1 + y^2) = c_1$$

Verified OK.

6.12.5 Maple step by step solution

Let's solve

$2xyy' + y^2 = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (2xyy' + y^2) dx = \int (-1) dx + c_1$

- Evaluate integral

  $y^2x = -x + c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{x(-x+c_1)}}{x}, y = -\frac{\sqrt{x(-x+c_1)}}{x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(2*x*y(x)*diff(y(x),x)+y(x)^2=-1,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{x\,(c_1 - x)}}{x}$$
$$y(x) = -\frac{\sqrt{x\,(c_1 - x)}}{x}$$

✓ Solution by Mathematica

Time used: 0.471 (sec). Leaf size: 98

```
DSolve[2*x*y[x]*y'[x]+y[x]^2==-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-x+e^{2c_1}}}{\sqrt{x}}$$
$$y(x) \to \frac{\sqrt{-x+e^{2c_1}}}{\sqrt{x}}$$
$$y(x) \to -i$$
$$y(x) \to i$$
$$y(x) \to \frac{\sqrt{-x}}{\sqrt{x}}$$
$$y(x) \to \frac{\sqrt{x}}{\sqrt{-x}}$$

6.13 problem 13

Internal problem ID [12683]
Internal file name [`OUTPUT/11335_Friday_November_03_2023_06_30_34_AM_54833312/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{1 - yx}{x^2} = 0}$$

6.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x} = \frac{1}{x^2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{x}dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(xy) &= (x)\left(\frac{1}{x^2}\right) \\
\mathrm{d}(xy) &= \frac{1}{x}\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
xy &= \int \frac{1}{x}\,\mathrm{d}x \\
xy &= \ln(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
y = \frac{\ln(x)}{x} + \frac{c_1}{x}
$$

which simplifies to

$$
y = \frac{\ln(x) + c_1}{x}
$$

Summary
The solution(s) found are the following

$$
y = \frac{\ln(x) + c_1}{x} \tag{1}
$$

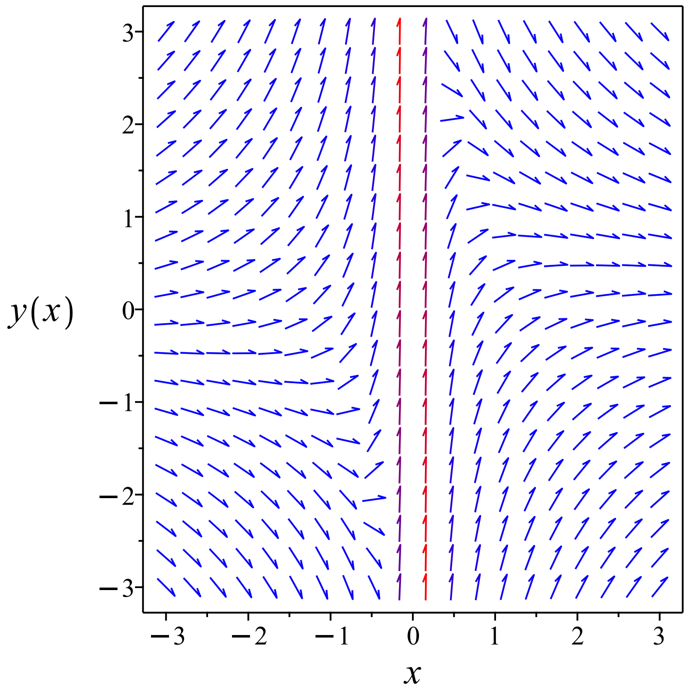


Figure 222: Slope field plot

Verification of solutions

$$y = \frac{\ln(x) + c_1}{x}$$

Verified OK.

6.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{xy - 1}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 189: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = xy$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{xy - 1}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = \ln(x) + c_1$$

Which simplifies to

$$yx = \ln(x) + c_1$$

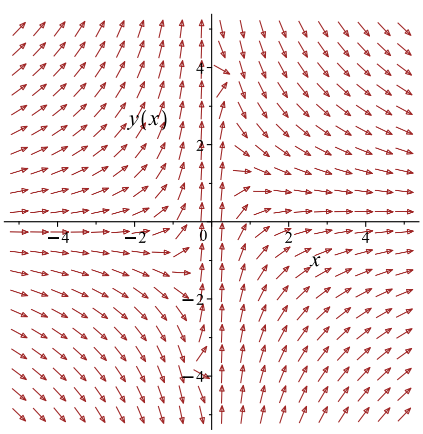
Which gives

$$y = \frac{\ln(x) + c_1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{xy-1}{x^2}$



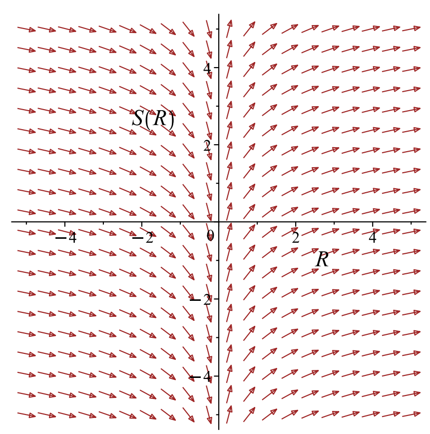
</td><td>

$R = x$

$S = xy$

</td><td>

$\frac{dS}{dR} = \frac{1}{R}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{\ln(x) + c_1}{x} \tag{1}$$

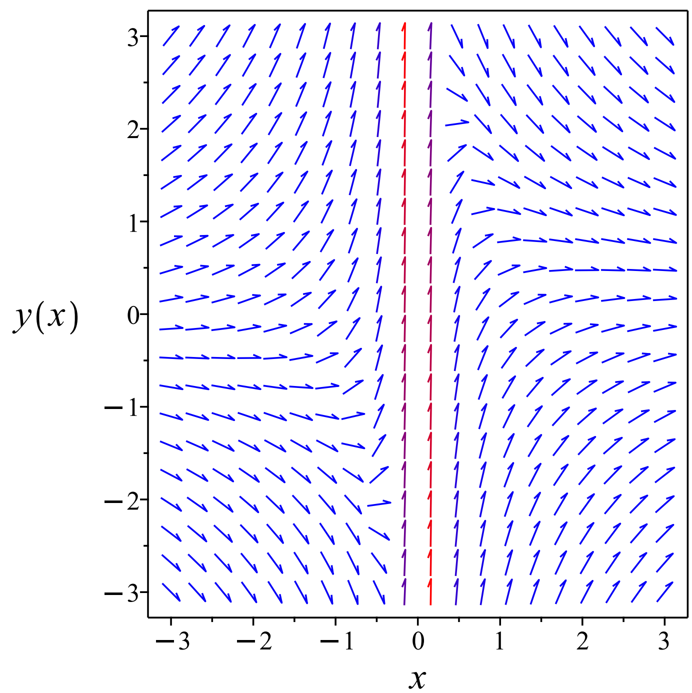


Figure 223: Slope field plot

Verification of solutions

$$y = \frac{\ln(x) + c_1}{x}$$

Verified OK.

6.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{-xy + 1}{x^2}\right) \mathrm{d}x$$
$$\left(-\frac{-xy + 1}{x^2}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{-xy + 1}{x^2}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{-xy + 1}{x^2}\right)$$
$$= \frac{1}{x}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(\frac{1}{x}\right) - (0)\right) \\ &= \frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)} \\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x\left(-\frac{-xy+1}{x^2}\right) \\ &= \frac{xy-1}{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x(1) \\ &= x\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{xy-1}{x}\right) + (x)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{xy-1}{x} \mathrm{d}x$$
$$\phi = xy - \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = xy - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy - \ln(x)$$

The solution becomes

$$y = \frac{\ln(x) + c_1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x) + c_1}{x} \tag{1}$$

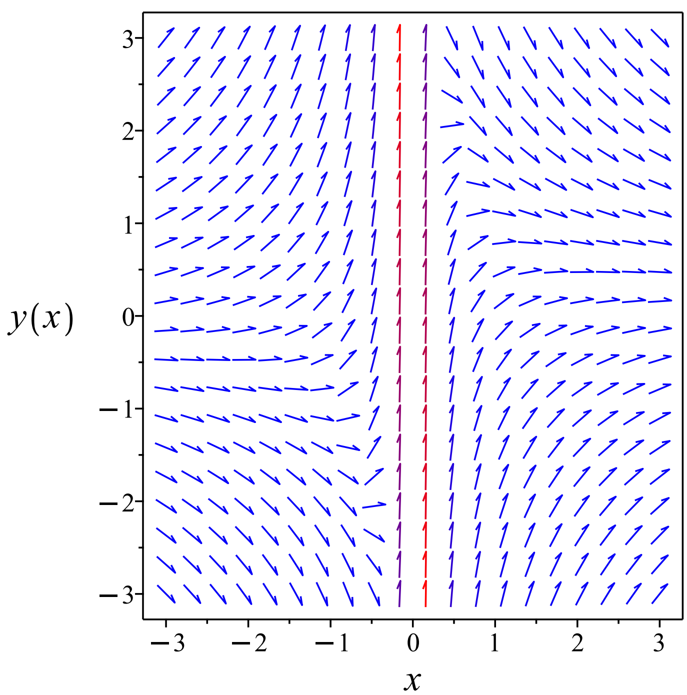


Figure 224: Slope field plot

Verification of solutions

$$y = \frac{\ln(x) + c_1}{x}$$

Verified OK.

6.13.4 Maple step by step solution

Let's solve

$y' - \frac{1-yx}{x^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} + \frac{1}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = \frac{1}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' + \frac{y}{x}\right) = \frac{\mu(x)}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' + \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{x^2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int \frac{1}{x} dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{\ln(x) + c_1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)=(1-x*y(x))/x^2,y(x), singsol=all)
```

$$y(x) = \frac{\ln(x) + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 14

```
DSolve[y'[x]==(1-x*y[x])/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\log(x) + c_1}{x}$$

6.14 problem 14

Internal problem ID [12684]
Internal file name [`OUTPUT/11336_Friday_November_03_2023_06_30_35_AM_26110690/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y' + \frac{y(2x+y)}{x\,(x+2y)} = 0}$$

6.14.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) + \frac{u(x)\,(2x+u(x)\,x)}{x+2u\,(x)\,x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u(u+1)}{x\,(2u+1)} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = \frac{u(u+1)}{2u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u+1)}{2u+1}} du = -\frac{3}{x} dx$$
$$\int \frac{1}{\frac{u(u+1)}{2u+1}} du = \int -\frac{3}{x} dx$$
$$\ln(u(u+1)) = -3\ln(x) + c_2$$

Raising both side to exponential gives

$$u(u+1) = e^{-3\ln(x)+c_2}$$

Which simplifies to

$$u(u+1) = \frac{c_3}{x^3}$$

Which simplifies to

$$u(x)(u(x)+1) = \frac{c_3 e^{c_2}}{x^3}$$

The solution is

$$u(x)(u(x)+1) = \frac{c_3 e^{c_2}}{x^3}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y\left(\frac{y}{x}+1\right)}{x} = \frac{c_3 e^{c_2}}{x^3}$$
$$\frac{y(y+x)}{x^2} = \frac{c_3 e^{c_2}}{x^3}$$

Which simplifies to

$$y(y+x) = \frac{c_3 e^{c_2}}{x}$$

Summary
The solution(s) found are the following

$$y(y+x) = \frac{c_3 e^{c_2}}{x} \tag{1}$$

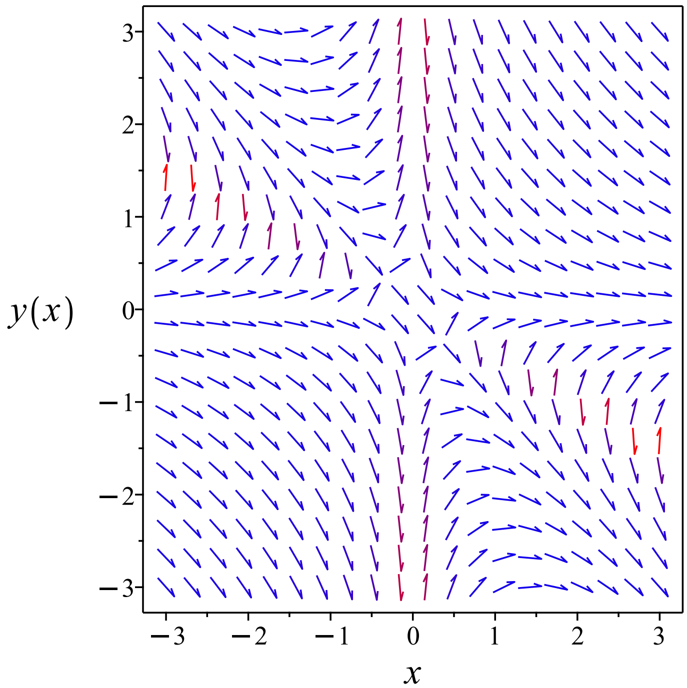


Figure 225: Slope field plot

Verification of solutions

$$y(y+x) = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

6.14.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(2x+y)}{x(x+2y)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{y(2x+y)\,(b_3-a_2)}{x\,(x+2y)} - \frac{y^2(2x+y)^2\,a_3}{x^2\,(x+2y)^2} \\ &- \left(-\frac{2y}{x\,(x+2y)} + \frac{y(2x+y)}{x^2\,(x+2y)} + \frac{y(2x+y)}{x\,(x+2y)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{2x+y}{x\,(x+2y)} - \frac{y}{x\,(x+2y)} + \frac{2y(2x+y)}{x\,(x+2y)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^4b_2 + 6x^3yb_2 + 3x^2y^2a_2 - 6x^2y^2a_3 + 6x^2y^2b_2 - 3x^2y^2b_3 - 6x\,y^3a_3 - 3y^4a_3 + 2x^3b_1 - 2x^2ya_1 + 2x^2yb_1 -}{x^2\,(x+2y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &3x^4b_2 + 6x^3yb_2 + 3x^2y^2a_2 - 6x^2y^2a_3 + 6x^2y^2b_2 - 3x^2y^2b_3 - 6x\,y^3a_3 \\ &- 3y^4a_3 + 2x^3b_1 - 2x^2ya_1 + 2x^2yb_1 - 2x\,y^2a_1 + 2x\,y^2b_1 - 2y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &3a_2v_1^2v_2^2 - 6a_3v_1^2v_2^2 - 6a_3v_1v_2^3 - 3a_3v_2^4 + 3b_2v_1^4 + 6b_2v_1^3v_2 + 6b_2v_1^2v_2^2 \\ &- 3b_3v_1^2v_2^2 - 2a_1v_1^2v_2 - 2a_1v_1v_2^2 - 2a_1v_2^3 + 2b_1v_1^3 + 2b_1v_1^2v_2 + 2b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} & 3b_2v_1^4 + 6b_2v_1^3v_2 + 2b_1v_1^3 + (3a_2 - 6a_3 + 6b_2 - 3b_3)\, v_1^2v_2^2 \\ & + (-2a_1 + 2b_1)\, v_1^2v_2 - 6a_3v_1v_2^3 + (-2a_1 + 2b_1)\, v_1v_2^2 - 3a_3v_2^4 - 2a_1v_2^3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_1 &= 0 \\ -6a_3 &= 0 \\ -3a_3 &= 0 \\ 2b_1 &= 0 \\ 3b_2 &= 0 \\ 6b_2 &= 0 \\ -2a_1 + 2b_1 &= 0 \\ 3a_2 - 6a_3 + 6b_2 - 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{y(2x+y)}{x\,(x+2y)}\right)(x) \\
&= \frac{3xy + 3y^2}{x + 2y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3xy+3y^2}{x+2y}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln\left(y(y+x)\right)}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{y(2x+y)}{x\,(x+2y)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{3x+3y} \\
S_y &= \frac{x+2y}{3y\,(y+x)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{3x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{3R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln (R)}{3} + c_1 \tag{4}
$$

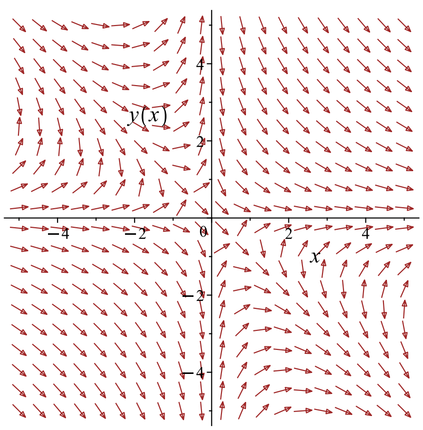
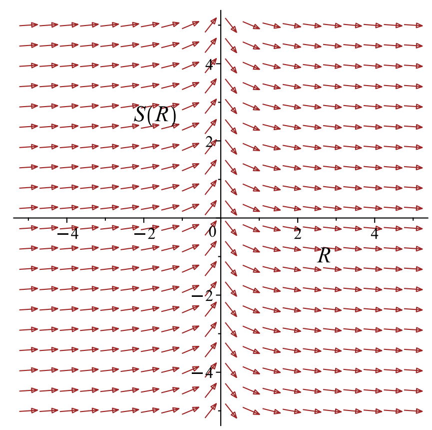
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln (y)}{3} + \frac{\ln (y+x)}{3} = -\frac{\ln (x)}{3} + c_1
$$

Which simplifies to

$$
\frac{\ln (y)}{3} + \frac{\ln (y+x)}{3} = -\frac{\ln (x)}{3} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{y(2x+y)}{x(x+2y)}$
</td><td>$R = x$
$S = \frac{\ln(y)}{3} + \frac{\ln(y+x)}{3}$</td><td>$\frac{dS}{dR} = -\frac{1}{3R}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{3} + \frac{\ln(y+x)}{3} = -\frac{\ln(x)}{3} + c_1 \tag{1}$$

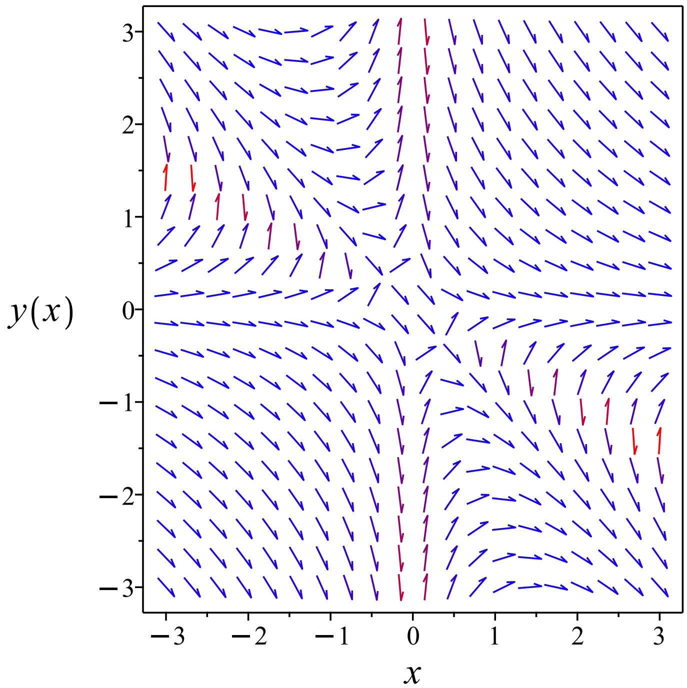


Figure 226: Slope field plot

Verification of solutions

$$\frac{\ln(y)}{3}+\frac{\ln(y+x)}{3}=-\frac{\ln(x)}{3}+c_1$$

Verified OK.

6.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x(x+2y)) \, \mathrm{d}y &= (-y(2x+y)) \, \mathrm{d}x \\ (y(2x+y)) \, \mathrm{d}x + (x(x+2y)) \, \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y(2x+y) \\ N(x, y) &= x(x+2y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(2x+y)) \\ &= 2y + 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x(x+2y)) \\ &= 2y + 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y(2x + y) \, \mathrm{d}x$$
$$\phi = yx(y + x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = (y + x)\,x + xy + f'(y) \tag{4}$$
$$= x(x + 2y) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x(x + 2y)$. Therefore equation (4) becomes

$$x(x + 2y) = x(x + 2y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = yx(y + x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = yx(y + x)$$

Summary

The solution(s) found are the following

$$y(y + x)\,x = c_1 \tag{1}$$

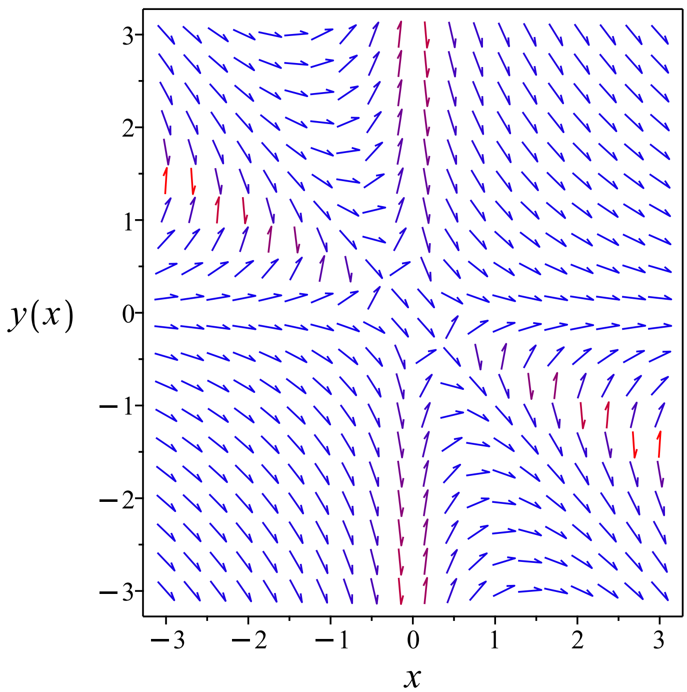


Figure 227: Slope field plot

Verification of solutions

$$y(y + x)\,x = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 71

```
dsolve(diff(y(x),x)=-y(x)*(2*x+y(x))/(x*(2*y(x)+x)),y(x), singsol=all)
```

$$y(x) = \frac{-c_1^2x^2 + \sqrt{c_1x\left(c_1^3x^3 + 4\right)}}{2c_1^2x}$$

$$y(x) = \frac{-c_1^2x^2 - \sqrt{c_1x\left(c_1^3x^3 + 4\right)}}{2c_1^2x}$$

✓ Solution by Mathematica

Time used: 1.084 (sec). Leaf size: 118

```
DSolve[y'[x]==-y[x]*(2*x+y[x])/(x*(2*y[x]+x)),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-x - \frac{\sqrt{x^3 + 4e^{c_1}}}{\sqrt{x}}\right)$$

$$y(x) \to \frac{1}{2}\left(-x + \frac{\sqrt{x^3 + 4e^{c_1}}}{\sqrt{x}}\right)$$

$$y(x) \to -\frac{x^{3/2} + \sqrt{x^3}}{2\sqrt{x}}$$

$$y(x) \to \frac{\sqrt{x^3}}{2\sqrt{x}} - \frac{x}{2}$$

6.15 problem 15

Internal problem ID [12685]
Internal file name [`OUTPUT/11337_Friday_November_03_2023_06_30_36_AM_33531861/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.2, page 63
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y' - \frac{y^2}{1 - yx} = 0}$$

6.15.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y^2}{xy - 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y^2(b_3 - a_2)}{xy - 1} - \frac{y^4 a_3}{(xy - 1)^2} - \frac{y^3(x a_2 + y a_3 + a_1)}{(xy - 1)^2} - \left(-\frac{2y}{xy - 1} + \frac{y^2 x}{(xy - 1)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2y^2b_2 - 2y^4a_3 + x\,y^2b_1 - y^3a_1 - 4xyb_2 - y^2a_2 - y^2b_3 - 2yb_1 + b_2}{(xy - 1)^2} = 0$$

Setting the numerator to zero gives

$$2x^2y^2b_2 - 2y^4a_3 + x\,y^2b_1 - y^3a_1 - 4xyb_2 - y^2a_2 - y^2b_3 - 2yb_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_3v_2^4 + 2b_2v_1^2v_2^2 - a_1v_2^3 + b_1v_1v_2^2 - a_2v_2^2 - 4b_2v_1v_2 - b_3v_2^2 - 2b_1v_2 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$2b_2v_1^2v_2^2 + b_1v_1v_2^2 - 4b_2v_1v_2 - 2a_3v_2^4 - a_1v_2^3 + (-a_2 - b_3)\,v_2^2 - 2b_1v_2 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_1 &= 0 \\
b_2 &= 0 \\
-a_1 &= 0 \\
-2a_3 &= 0 \\
-2b_1 &= 0 \\
-4b_2 &= 0 \\
2b_2 &= 0 \\
-a_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{y^2}{xy - 1}\right)(-x) \\
&= -\frac{y}{xy - 1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{y}{xy-1}} dy \end{aligned}$$

Which results in

$$S = -xy + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^2}{xy - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -y \\ S_y &= -x + \frac{1}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-yx + \ln(y) = c_1$$

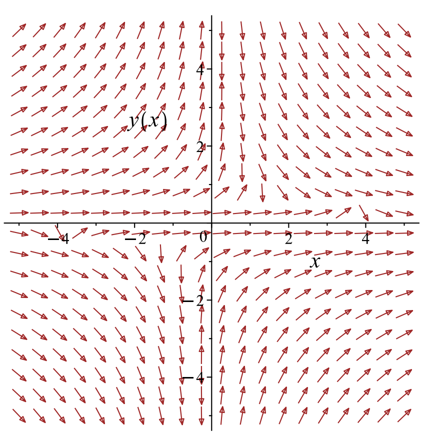
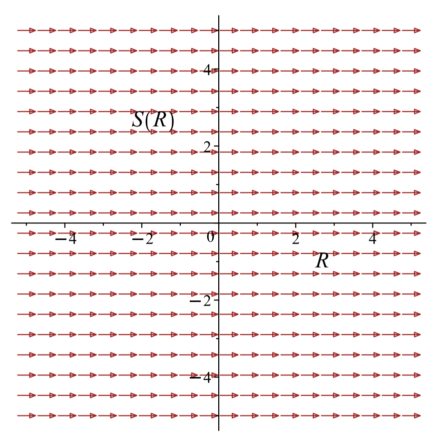
Which simplifies to

$$-yx + \ln(y) = c_1$$

Which gives

$$y = e^{-\operatorname{LambertW}(-e^{c_1} x) + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{y^2}{xy-1}$

</td><td>$R = x$
$S = -xy + \ln(y)$</td><td>$\frac{dS}{dR} = 0$

</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = e^{-\operatorname{LambertW}(-e^{c_1} x)+c_1} \tag{1}$$

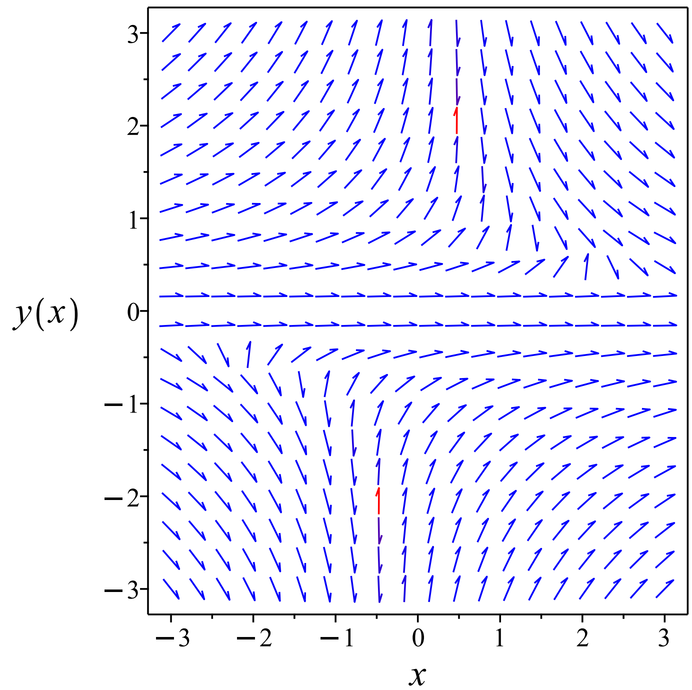


Figure 228: Slope field plot

Verification of solutions

$$y = e^{-\operatorname{LambertW}(-e^{c_1} x)+c_1}$$

Verified OK.

6.15.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{y^2}{-xy + 1}\right)\mathrm{d}x$$

$$\left(-\frac{y^2}{-xy + 1}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{y^2}{-xy + 1}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y^2}{-xy + 1}\right)\\ &= \frac{y(xy - 2)}{(xy - 1)^2}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\
&= 1\left(\left(-\frac{2y}{-xy+1} - \frac{y^2 x}{(-xy+1)^2}\right) - (0)\right)\\
&= \frac{y(xy-2)}{(xy-1)^2}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\
&= \frac{xy-1}{y^2}\left((0) - \left(-\frac{2y}{-xy+1} - \frac{y^2 x}{(-xy+1)^2}\right)\right)\\
&= \frac{-xy+2}{y\,(xy-1)}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}\\
&= \frac{(0) - \left(-\frac{2y}{-xy+1} - \frac{y^2 x}{(-xy+1)^2}\right)}{x\left(-\frac{y^2}{-xy+1}\right) - y\,(1)}\\
&= \frac{-xy+2}{xy-1}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = \frac{-t+2}{t-1}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}\mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int \left(\frac{-t+2}{t-1}\right) \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-t+\ln(t-1)} \\ &= (t-1)\,\mathrm{e}^{-t}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = (xy-1)\,\mathrm{e}^{-xy}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= (xy-1)\,\mathrm{e}^{-xy}\left(-\frac{y^2}{-xy+1}\right) \\ &= y^2\mathrm{e}^{-xy}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= (xy-1)\,\mathrm{e}^{-xy}(1) \\ &= (xy-1)\,\mathrm{e}^{-xy}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(y^2\mathrm{e}^{-xy}\right) + \left((xy-1)\,\mathrm{e}^{-xy}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y^2 \mathrm{e}^{-xy} \, \mathrm{d}x$$

$$\phi = -y \, \mathrm{e}^{-xy} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= -\mathrm{e}^{-xy} + yx \, \mathrm{e}^{-xy} + f'(y) \qquad (4) \\ &= (xy - 1) \, \mathrm{e}^{-xy} + f'(y) \end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (xy - 1) \, \mathrm{e}^{-xy}$. Therefore equation (4) becomes

$$(xy - 1) \, \mathrm{e}^{-xy} = (xy - 1) \, \mathrm{e}^{-xy} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -y \, \mathrm{e}^{-xy} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -y \, \mathrm{e}^{-xy}$$

The solution becomes

$$y = -\frac{\mathrm{LambertW}(c_1 x)}{x}$$

Summary
The solution(s) found are the following

$$y = -\frac{\text{LambertW}(c_1 x)}{x} \tag{1}$$

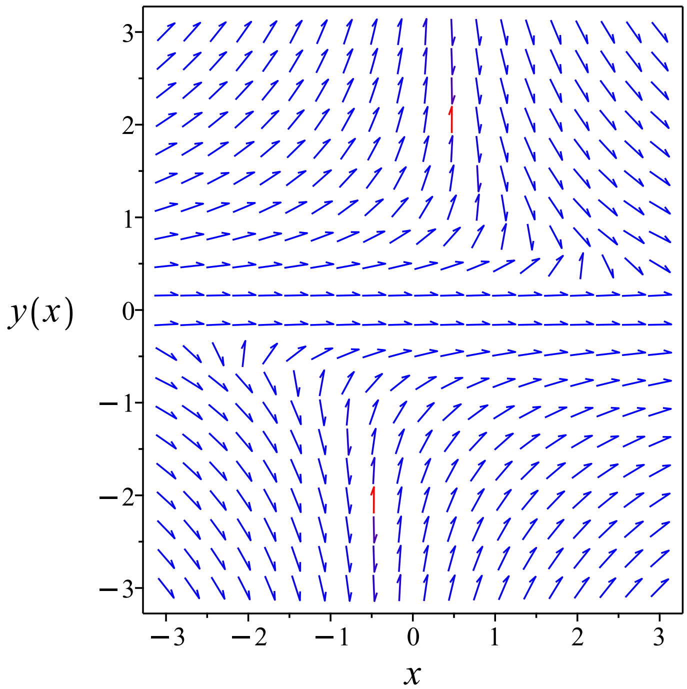


Figure 229: Slope field plot

Verification of solutions

$$y = -\frac{\text{LambertW}(c_1 x)}{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)=y(x)^2/(1-x*y(x)),y(x), singsol=all)
```

$$y(x) = -\frac{\text{LambertW}\left(-x\,\mathrm{e}^{-c_1}\right)}{x}$$

✓ Solution by Mathematica

Time used: 3.256 (sec). Leaf size: 25

```
DSolve[y'[x]==y[x]^2/(1-x*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{W(-e^{-c_1}x)}{x}$$
$$y(x) \to 0$$

7 Chapter 2. The Initial Value Problem. Exercises 2.3.3, page 71

7.1 problem 1

Internal problem ID [12686]
Internal file name [`OUTPUT/11338_Friday_November_03_2023_06_30_37_AM_48167268/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.3, page 71
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 4y = 1}$$

With initial conditions

$$[y(0) = 1]$$

7.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -4$$
$$q(x) = 1$$

Hence the ode is

$$y' - 4y = 1$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.1.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{4y+1}dy = \int dx$$
$$\frac{\ln(4y+1)}{4} = x + c_1$$

Raising both side to exponential gives

$$(4y+1)^{\frac{1}{4}} = e^{x+c_1}$$

Which simplifies to

$$(4y+1)^{\frac{1}{4}} = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_2^4}{4} - \frac{1}{4}$$

$$c_2 = 5^{\frac{1}{4}}$$

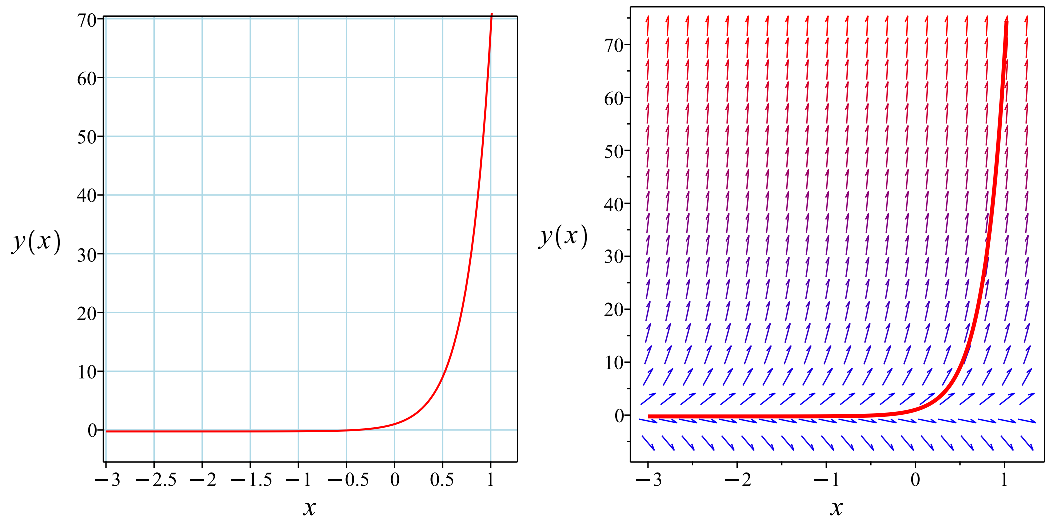
Substituting c_2 found above in the general solution gives

$$y = \frac{5\,e^{4x}}{4} - \frac{1}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{5\,e^{4x}}{4} - \frac{1}{4} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{5\,\mathrm{e}^{4x}}{4} - \frac{1}{4}$$

Verified OK.

7.1.3 Maple step by step solution

Let's solve

$[y' - 4y = 1, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{4y+1} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{4y+1}dx = \int 1dx + c_1$

- Evaluate integral

  $\frac{\ln(4y+1)}{4} = x + c_1$

- Solve for y

$$y = -\frac{1}{4} + \frac{e^{4c_1+4x}}{4}$$

- Use initial condition $y(0) = 1$

$$1 = -\frac{1}{4} + \frac{e^{4c_1}}{4}$$

- Solve for c_1

$$c_1 = \frac{\ln(5)}{4}$$

- Substitute $c_1 = \frac{\ln(5)}{4}$ into general solution and simplify

$$y = \frac{5\,e^{4x}}{4} - \frac{1}{4}$$

- Solution to the IVP

$$y = \frac{5\,e^{4x}}{4} - \frac{1}{4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([diff(y(x),x)=4*y(x)+1,y(0) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{1}{4} + \frac{5\,e^{4x}}{4}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 18

```
DSolve[{y'[x]==4*y[x]+1,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(5e^{4x} - 1\right)$$

7.2 problem 2

Internal problem ID [12687]
Internal file name [`OUTPUT/11339_Friday_November_03_2023_06_30_38_AM_52579092/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.3, page 71
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - yx = 2}$$

With initial conditions

$$[y(0) = 1]$$

7.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -x$$
$$q(x) = 2$$

Hence the ode is

$$y' - yx = 2$$

The domain of $p(x) = -x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.2.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -x dx} \\ &= \mathrm{e}^{-\frac{x^2}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(2) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{x^2}{2}} y\right) &= \left(\mathrm{e}^{-\frac{x^2}{2}}\right)(2) \\ \mathrm{d}\left(\mathrm{e}^{-\frac{x^2}{2}} y\right) &= \left(2\,\mathrm{e}^{-\frac{x^2}{2}}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-\frac{x^2}{2}} y &= \int 2\,\mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x \\ \mathrm{e}^{-\frac{x^2}{2}} y &= \sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{x^2}{2}}$ results in

$$y = \mathrm{e}^{\frac{x^2}{2}}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + c_1 \mathrm{e}^{\frac{x^2}{2}}$$

which simplifies to

$$y = \mathrm{e}^{\frac{x^2}{2}}\left(\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

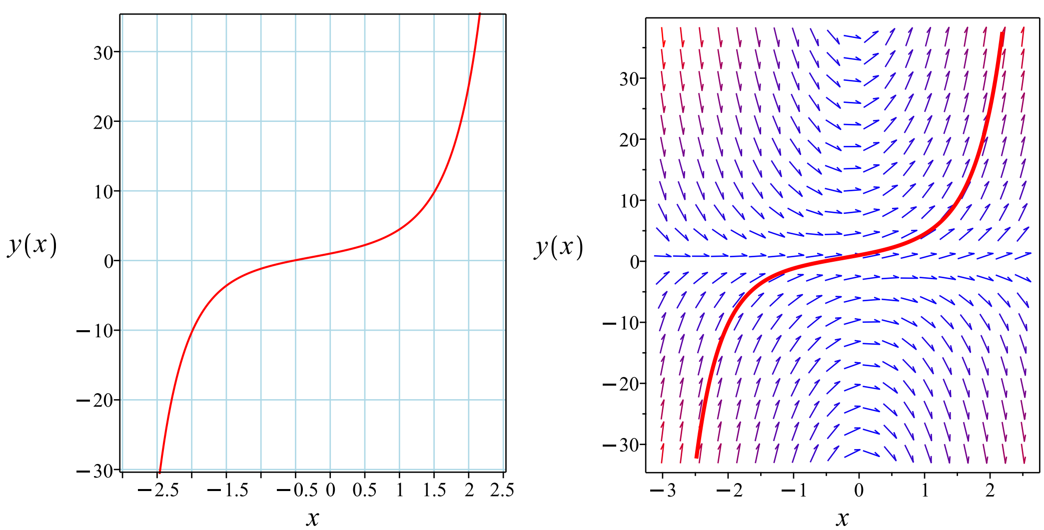
Substituting c_1 found above in the general solution gives

$$y = e^{\frac{x^2}{2}}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + e^{\frac{x^2}{2}}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{x^2}{2}}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + e^{\frac{x^2}{2}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{\frac{x^2}{2}}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + e^{\frac{x^2}{2}}$$

Verified OK.

7.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = xy + 2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 193: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{\frac{x^2}{2}}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{-\frac{x^2}{2}} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = xy + 2
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -x\,\mathrm{e}^{-\frac{x^2}{2}} y \\
S_y &= \mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2\,\mathrm{e}^{-\frac{x^2}{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2\,\mathrm{e}^{-\frac{R^2}{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,R}{2}\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-\frac{x^2}{2}} y = \sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + c_1$$

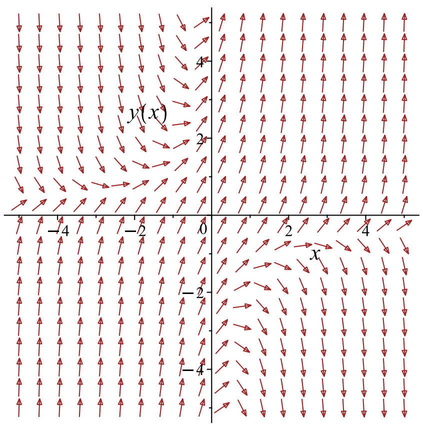
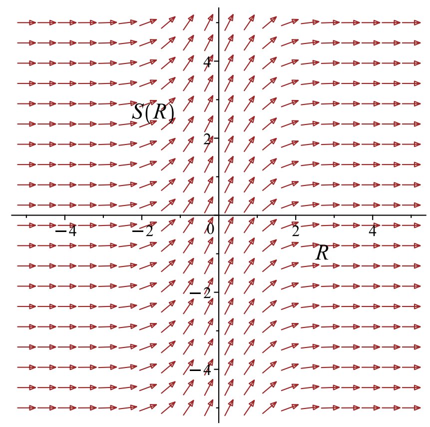
Which simplifies to

$$\mathrm{e}^{-\frac{x^2}{2}} y = \sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + c_1$$

Which gives

$$y = \mathrm{e}^{\frac{x^2}{2}} \left(\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = xy + 2$ 	$R = x$ $S = e^{-\frac{x^2}{2}} y$	$\frac{dS}{dR} = 2\,e^{-\frac{R^2}{2}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

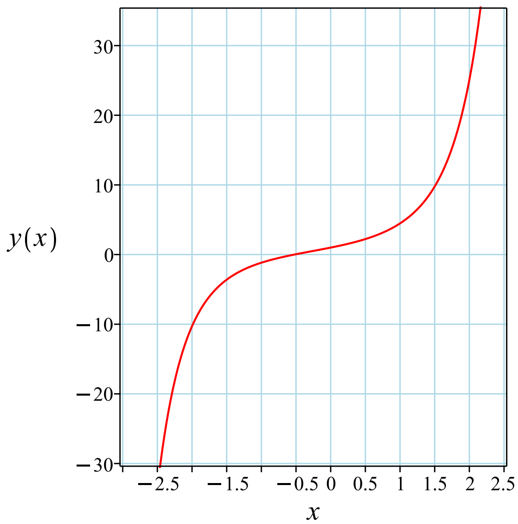
Substituting c_1 found above in the general solution gives

$$y = e^{\frac{x^2}{2}} \sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + e^{\frac{x^2}{2}}$$

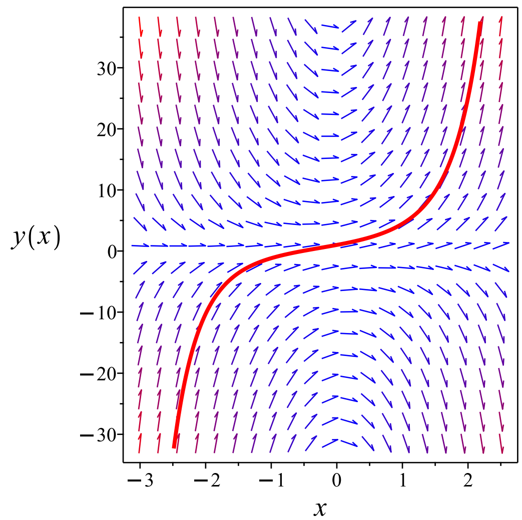
Summary

The solution(s) found are the following

$$y = e^{\frac{x^2}{2}} \sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + e^{\frac{x^2}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \mathrm{e}^{\frac{x^2}{2}} \sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + \mathrm{e}^{\frac{x^2}{2}}$$

Verified OK.

7.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (xy + 2)\,\mathrm{d}x \\ (-xy - 2)\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -xy - 2 \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-xy - 2) \\ &= -x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-x) - (0)) \\
&= -x
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -x \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{x^2}{2}} \\
&= \mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-\frac{x^2}{2}}(-xy - 2) \\
&= -\mathrm{e}^{-\frac{x^2}{2}}(xy + 2)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\frac{x^2}{2}}(1) \\
&= \mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\mathrm{e}^{-\frac{x^2}{2}}(xy + 2)\right) + \left(\mathrm{e}^{-\frac{x^2}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\mathrm{e}^{-\frac{x^2}{2}} (xy + 2) \, \mathrm{d}x$$
$$\phi = \mathrm{e}^{-\frac{x^2}{2}} y - \sqrt{\pi} \sqrt{2} \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{x^2}{2}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-\frac{x^2}{2}}$. Therefore equation (4) becomes

$$\mathrm{e}^{-\frac{x^2}{2}} = \mathrm{e}^{-\frac{x^2}{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-\frac{x^2}{2}} y - \sqrt{\pi} \sqrt{2} \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{-\frac{x^2}{2}} y - \sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right)$$

The solution becomes

$$y = e^{\frac{x^2}{2}} \left(\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

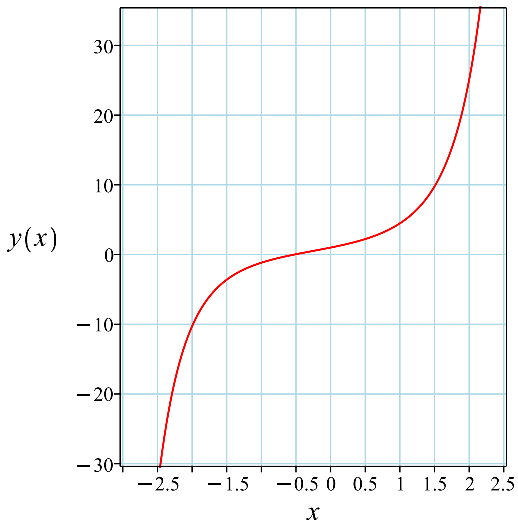
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

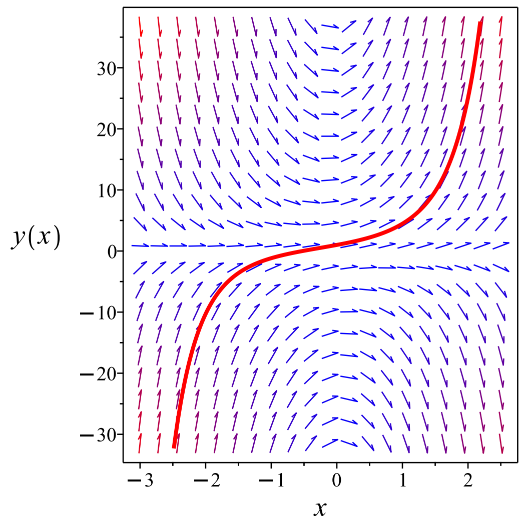
$$y = e^{\frac{x^2}{2}} \sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + e^{\frac{x^2}{2}}$$

Summary
The solution(s) found are the following

$$y = e^{\frac{x^2}{2}} \sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + e^{\frac{x^2}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{x^2}{2}} \sqrt{\pi}\, \sqrt{2}\, \mathrm{erf}\left(\frac{\sqrt{2}\, x}{2}\right) + \mathrm{e}^{\frac{x^2}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve([diff(y(x),x)=x*y(x)+2,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \left(\sqrt{\pi}\, \sqrt{2}\, \mathrm{erf}\left(\frac{x\sqrt{2}}{2}\right) + 1\right) \mathrm{e}^{\frac{x^2}{2}}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 33

```
DSolve[{y'[x]==x*y[x]+2,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{x^2}{2}}\left(\sqrt{2\pi}\text{erf}\left(\frac{x}{\sqrt{2}}\right)+1\right)$$

7.3 problem 3

Internal problem ID [12688]
Internal file name [`OUTPUT/11340_Friday_November_03_2023_06_30_38_AM_52738796/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.3, page 71
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y}{x} = 0}$$

With initial conditions

$$[y(-1) = 2]$$

7.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{y}{x} = 0$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -1$ is inside this domain. Hence solution exists and is unique.

7.3.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\, dy &= \frac{1}{x}\, dx \\ \int \frac{1}{y}\, dy &= \int \frac{1}{x}\, dx \\ \ln(y) &= \ln(x) + c_1 \\ y &= e^{\ln(x)+c_1} \\ &= c_1 x \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -c_1$$

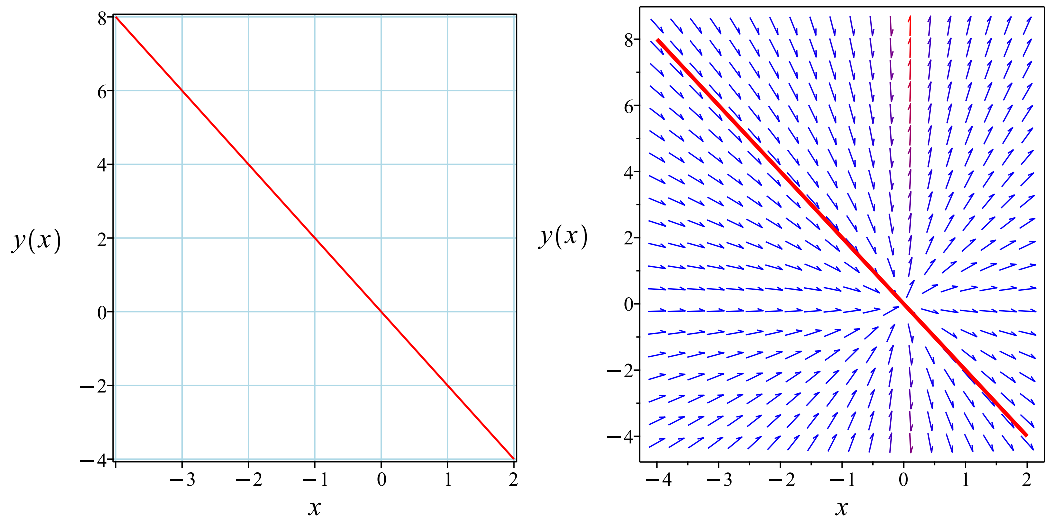
$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$y = -2x$$

Summary
The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -2x$$

Verified OK.

7.3.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = 0$$

Integrating gives

$$\frac{y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -c_1$$

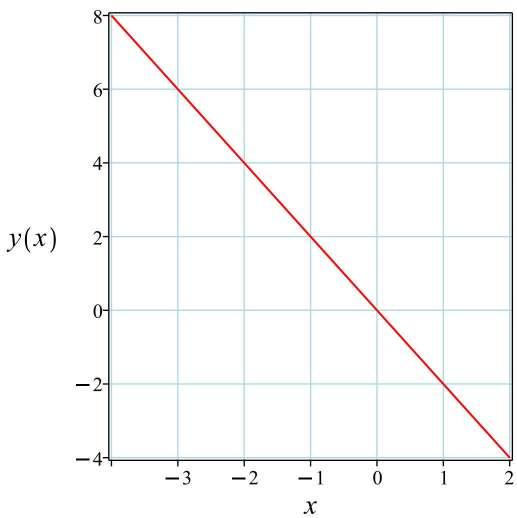
$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

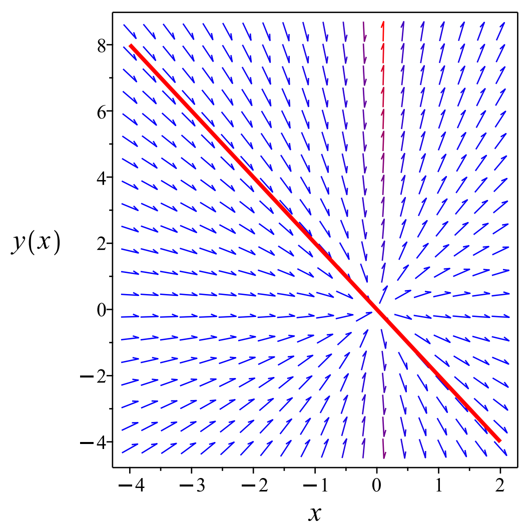
$$y = -2x$$

Summary
The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

7.3.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x = 0$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0 \,\mathrm{d}x \\ &= c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= c_2 x \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -c_2$$

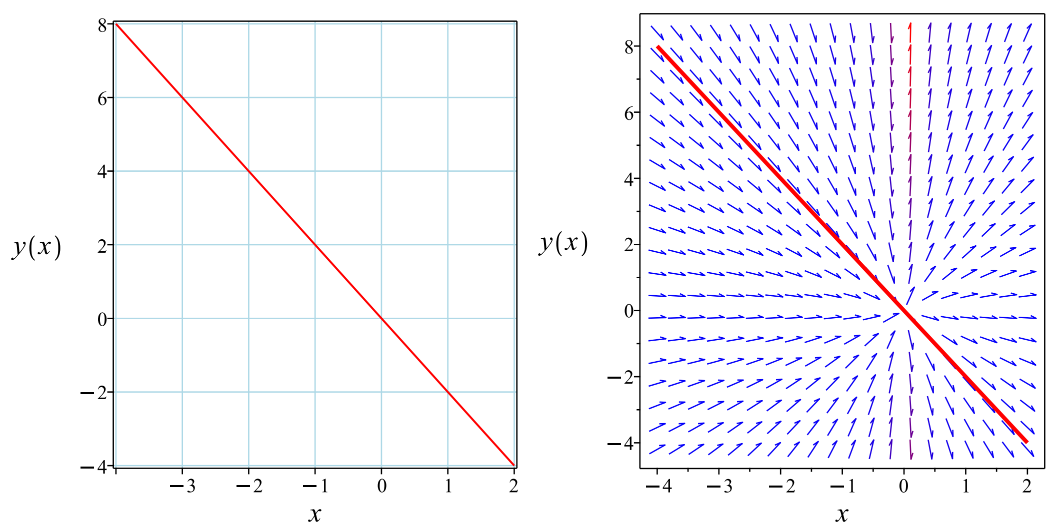
$$c_2 = -2$$

Substituting c_2 found above in the general solution gives

$$y = -2x$$

Summary
The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

7.3.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 195: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = c_1$$

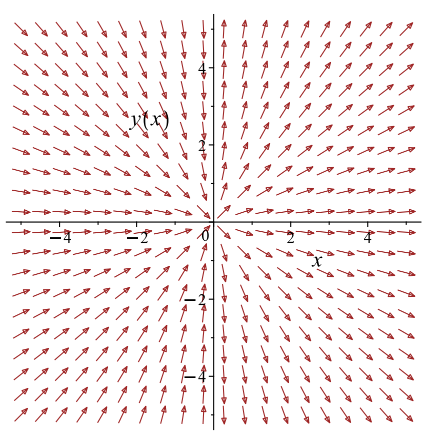
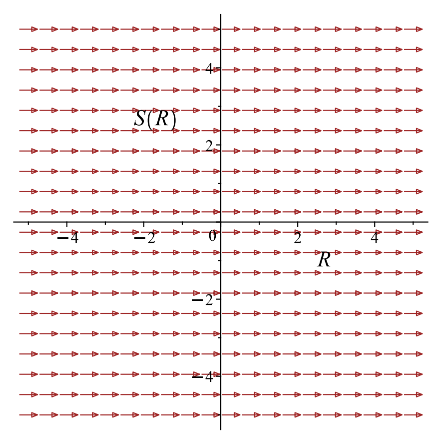
Which simplifies to

$$\frac{y}{x} = c_1$$

Which gives

$$y = c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -c_1$$

$$c_1 = -2$$

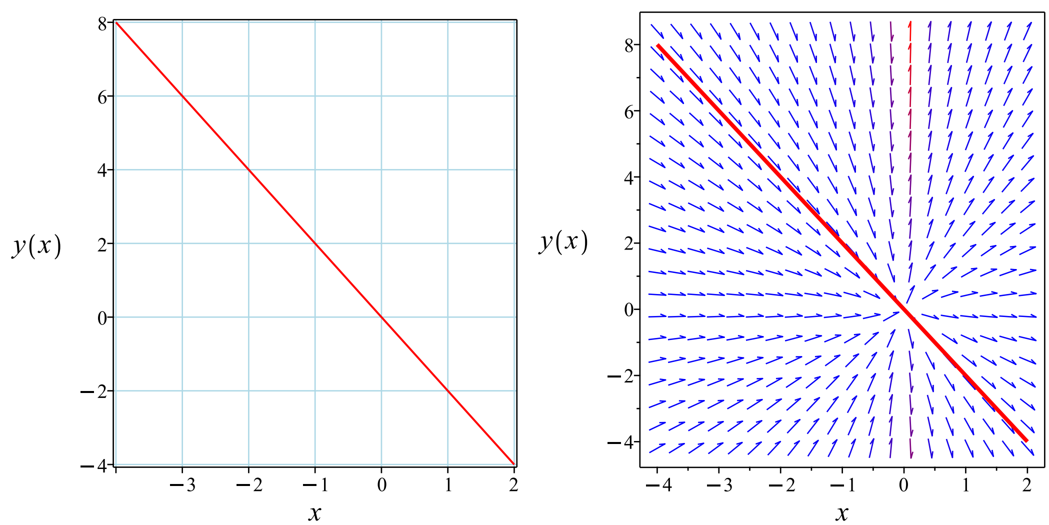
Substituting c_1 found above in the general solution gives

$$y = -2x$$

Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

7.3.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{1}{x}\\ N(x,y) &= \frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{x} \, \mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y)$$

The solution becomes

$$y = e^{c_1} x$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -e^{c_1}$$

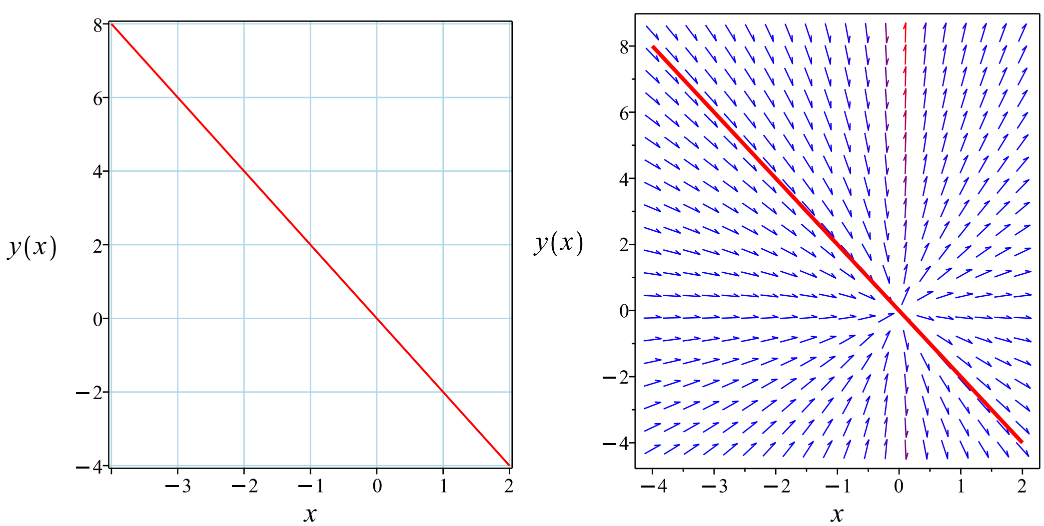
$$c_1 = \ln(2) + i\pi$$

Substituting c_1 found above in the general solution gives

$$y = -2x$$

<u>Summary</u>
The solution(s) found are the following

$$y = -2x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -2x$$

Verified OK.

7.3.7 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{x} = 0, y(-1) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = e^{c_1} x$

- Use initial condition $y(-1) = 2$

  $2 = -e^{c_1}$

- Solve for c_1

  $c_1 = \ln(2) + \mathrm{I}\pi$

- Substitute $c_1 = \ln(2) + \mathrm{I}\pi$ into general solution and simplify

  $y = -2x$

- Solution to the IVP

  $y = -2x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve([diff(y(x),x)=y(x)/x,y(-1) = 2],y(x), singsol=all)
```

$$y(x) = -2x$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 8

```
DSolve[{y'[x]==y[x]/x,{y[-1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x$$

7.4 problem 4

Internal problem ID [12689]
Internal file name [`OUTPUT/11341_Friday_November_03_2023_06_30_39_AM_22100225/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.3, page 71
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' - \frac{y}{x-1} = x^2}$$

With initial conditions

$$[y(0) = 1]$$

7.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x-1}$$
$$q(x) = x^2$$

Hence the ode is

$$y' - \frac{y}{x-1} = x^2$$

The domain of $p(x) = -\frac{1}{x-1}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.4.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x-1}dx} \\ &= \frac{1}{x-1}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(x^2\right) \\ \frac{d}{dx}\left(\frac{y}{x-1}\right) &= \left(\frac{1}{x-1}\right)\left(x^2\right) \\ d\left(\frac{y}{x-1}\right) &= \left(\frac{x^2}{x-1}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x-1} &= \int \frac{x^2}{x-1}\, dx \\ \frac{y}{x-1} &= \frac{x^2}{2} + x + \ln(x-1) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x-1}$ results in

$$y = (x-1)\left(\frac{x^2}{2} + x + \ln(x-1)\right) + c_1(x-1)$$

which simplifies to

$$y = (x-1)\left(\frac{x^2}{2}+x+\ln(x-1)+c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x=0$ and $y=1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -i\pi - c_1$$

$$c_1 = -i\pi - 1$$

Substituting c_1 found above in the general solution gives

$$y = -i\pi x + \frac{x^3}{2} + \ln(x-1)\,x + i\pi + \frac{x^2}{2} - \ln(x-1) - 2x + 1$$

Summary

The solution(s) found are the following

$$y = -i\pi x + \frac{x^3}{2} + \ln(x-1)\,x + i\pi + \frac{x^2}{2} - \ln(x-1) - 2x + 1 \tag{1}$$

Verification of solutions

$$y = -i\pi x + \frac{x^3}{2} + \ln(x-1)\,x + i\pi + \frac{x^2}{2} - \ln(x-1) - 2x + 1$$

Verified OK.

7.4.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^3 - x^2 + y}{x-1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 198: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= x - 1\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x-1} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x-1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^3 - x^2 + y}{x-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{(x-1)^2} \\ S_y &= \frac{1}{x-1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x^2}{x-1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^2}{R-1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + R + \ln(R-1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x-1} = \frac{x^2}{2} + x + \ln(x-1) + c_1$$

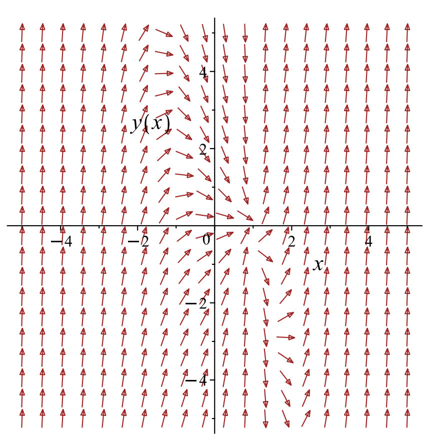
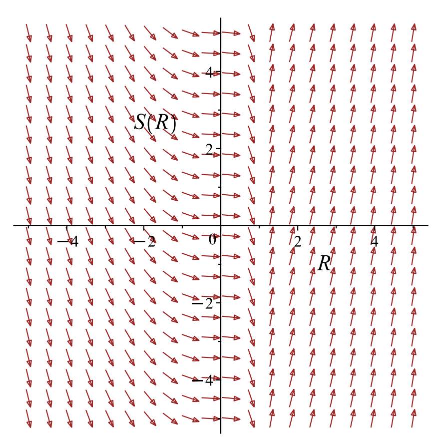
Which simplifies to

$$\frac{y}{x-1} = \frac{x^2}{2} + x + \ln(x-1) + c_1$$

Which gives

$$y = \frac{(x-1)\left(x^2 + 2\ln(x-1) + 2c_1 + 2x\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3-x^2+y}{x-1}$ 	$R = x$ $S = \frac{y}{x-1}$	$\frac{dS}{dR} = \frac{R^2}{R-1}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -i\pi - c_1$$

$$c_1 = -i\pi - 1$$

Substituting c_1 found above in the general solution gives

$$y = -i\pi x + \frac{x^3}{2} + \ln(x-1)\,x + i\pi + \frac{x^2}{2} - \ln(x-1) - 2x + 1$$

Summary

The solution(s) found are the following

$$y = -i\pi x + \frac{x^3}{2} + \ln(x-1)\,x + i\pi + \frac{x^2}{2} - \ln(x-1) - 2x + 1 \tag{1}$$

Verification of solutions

$$y = -i\pi x + \frac{x^3}{2} + \ln(x-1)\,x + i\pi + \frac{x^2}{2} - \ln(x-1) - 2x + 1$$

Verified OK.

7.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{y}{x-1} + x^2\right)\mathrm{d}x$$

$$\left(-\frac{y}{x-1} - x^2\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -\frac{y}{x-1} - x^2 \\
N(x, y) &= 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{x-1} - x^2\right) \\
&= -\frac{1}{x-1}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{1}{x-1}\right) - (0)\right) \\
&= -\frac{1}{x-1}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{1}{x-1} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(x-1)} \\
&= \frac{1}{x-1}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x-1}\left(-\frac{y}{x-1} - x^2\right) \\
&= \frac{-x^3 + x^2 - y}{(x-1)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x-1}(1) \\
&= \frac{1}{x-1}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x^3 + x^2 - y}{(x-1)^2}\right) + \left(\frac{1}{x-1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-x^3 + x^2 - y}{(x-1)^2} \mathrm{d}x$$

$$\phi = -\frac{x^2}{2} - x + \frac{y}{x-1} - \ln(x-1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x-1} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x-1}$. Therefore equation (4) becomes

$$\frac{1}{x-1} = \frac{1}{x-1} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - x + \frac{y}{x-1} - \ln(x-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - x + \frac{y}{x-1} - \ln(x-1)$$

The solution becomes

$$y = \frac{(x-1)\left(x^2+2\ln(x-1)+2c_1+2x\right)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -i\pi - c_1$$

$$c_1 = -i\pi - 1$$

Substituting c_1 found above in the general solution gives

$$y = -i\pi x + \frac{x^3}{2} + \ln(x-1)x + i\pi + \frac{x^2}{2} - \ln(x-1) - 2x + 1$$

Summary

The solution(s) found are the following

$$y = -i\pi x + \frac{x^3}{2} + \ln(x-1)x + i\pi + \frac{x^2}{2} - \ln(x-1) - 2x + 1 \tag{1}$$

Verification of solutions

$$y = -i\pi x + \frac{x^3}{2} + \ln(x-1)x + i\pi + \frac{x^2}{2} - \ln(x-1) - 2x + 1$$

Verified OK.

7.4.5 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{x-1} = x^2, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x-1} + x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x-1} = x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'-\frac{y}{x-1}\right)=\mu(x)\,x^2$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'-\frac{y}{x-1}\right)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{\mu(x)}{x-1}$
- Solve to find the integrating factor

  $\mu(x)=\frac{1}{x-1}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\,x^2dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\mu(x)\,x^2dx+c_1$
- Solve for y

  $y=\frac{\int\mu(x)x^2dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\frac{1}{x-1}$

  $y=(x-1)\left(\int\frac{x^2}{x-1}dx+c_1\right)$
- Evaluate the integrals on the rhs

  $y=(x-1)\left(\frac{x^2}{2}+x+\ln(x-1)+c_1\right)$
- Use initial condition $y(0)=1$

  $1=-\mathrm{I}\pi-c_1$
- Solve for c_1

  $c_1=-1-\mathrm{I}\pi$
- Substitute $c_1=-1-\mathrm{I}\pi$ into general solution and simplify

  $y=\left(\frac{x^2}{2}+x+\ln(x-1)-1-\mathrm{I}\pi\right)(x-1)$
- Solution to the IVP

  $y=\left(\frac{x^2}{2}+x+\ln(x-1)-1-\mathrm{I}\pi\right)(x-1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve([diff(y(x),x)=y(x)/(x-1)+x^2,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \left(\frac{x^2}{2} + x + \ln\left(-1 + x\right) - 1 - i\pi\right)\left(-1 + x\right)$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 31

```
DSolve[{y'[x]==y[x]/(x-1)+x^2,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(x-1)\left(x^2 + 2x + 2\log(x-1) - 2i\pi - 2\right)$$

7.5 problem 5

Internal problem ID [12690]
Internal file name [`OUTPUT/11342_Friday_November_03_2023_06_30_40_AM_41951789/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.3, page 71
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y}{x} = \sin\left(x^2\right)}$$

With initial conditions

$$[y(-1) = -1]$$

7.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{1}{x}\\ q(x) &= \sin\left(x^2\right)\end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = \sin\left(x^2\right)$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = \sin\left(x^2\right)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

7.5.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\sin\left(x^2\right)\right) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\sin\left(x^2\right)\right) \\ d\left(\frac{y}{x}\right) &= \left(\frac{\sin\left(x^2\right)}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{\sin\left(x^2\right)}{x} dx \\ \frac{y}{x} &= \frac{\text{Si}\left(x^2\right)}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = \frac{x\,\text{Si}\left(x^2\right)}{2} + c_1 x$$

which simplifies to

$$y = x\left(\frac{\mathrm{Si}\left(x^2\right)}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{\mathrm{Si}\left(1\right)}{2} - c_1$$

$$c_1 = -\frac{\mathrm{Si}\left(1\right)}{2} + 1$$

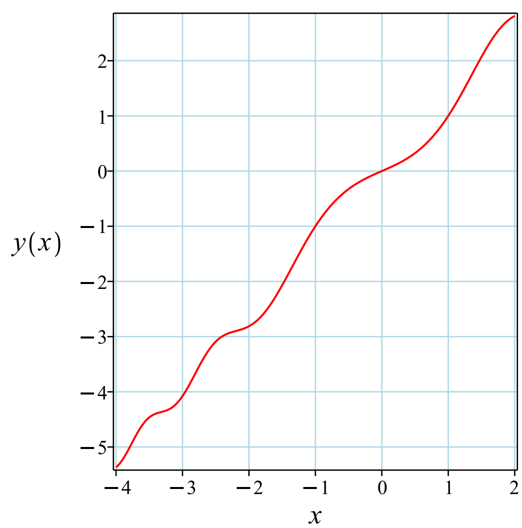
Substituting c_1 found above in the general solution gives

$$y = \frac{x(\mathrm{Si}\left(x^2\right) + 2 - \mathrm{Si}\left(1\right))}{2}$$

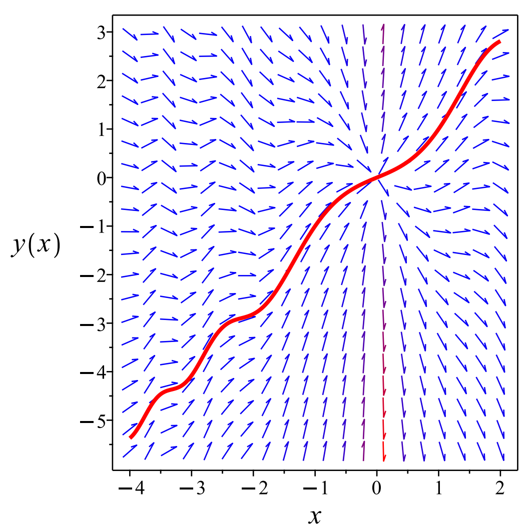
Summary

The solution(s) found are the following

$$y = \frac{x(\mathrm{Si}\left(x^2\right) + 2 - \mathrm{Si}\left(1\right))}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x(\mathrm{Si}\left(x^2\right) + 2 - \mathrm{Si}\left(1\right))}{2}$$

Verified OK.

7.5.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x = \sin\left(x^2\right)$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{\sin\left(x^2\right)}{x} \mathrm{d}x \\ &= \frac{\operatorname{Si}\left(x^2\right)}{2} + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x\left(\frac{\operatorname{Si}\left(x^2\right)}{2} + c_2\right) \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{\operatorname{Si}(1)}{2} - c_2$$

$$c_2 = -\frac{\operatorname{Si}(1)}{2} + 1$$

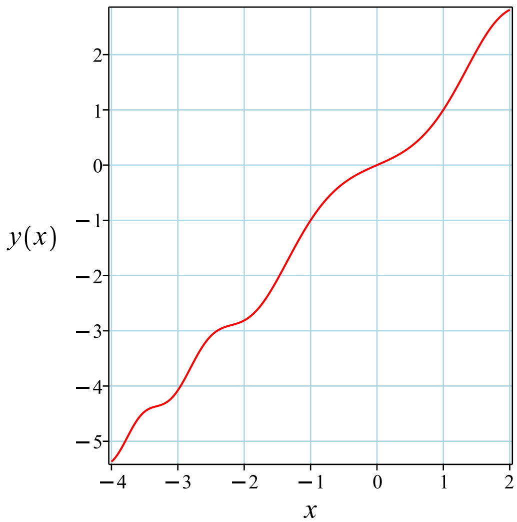
Substituting c_2 found above in the general solution gives

$$y = \frac{x(\operatorname{Si}\left(x^2\right) + 2 - \operatorname{Si}(1))}{2}$$

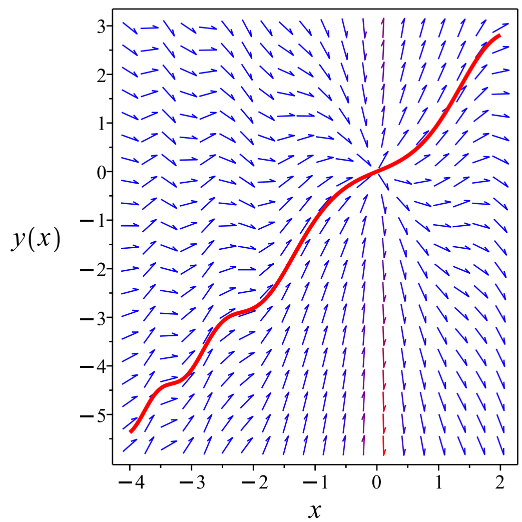
Summary

The solution(s) found are the following

$$y = \frac{x(\operatorname{Si}\left(x^2\right) + 2 - \operatorname{Si}(1))}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x(\operatorname{Si}(x^2) + 2 - \operatorname{Si}(1))}{2}$$

Verified OK.

7.5.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + \sin(x^2)\,x}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 201: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y + \sin(x^2)\, x}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sin(x^2)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\sin(R^2)}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\mathrm{Si}\left(R^2\right)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \frac{\mathrm{Si}\left(x^2\right)}{2} + c_1$$

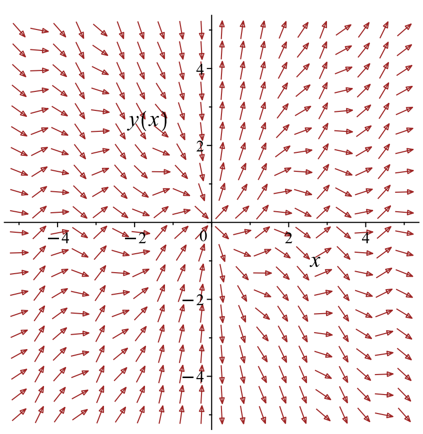
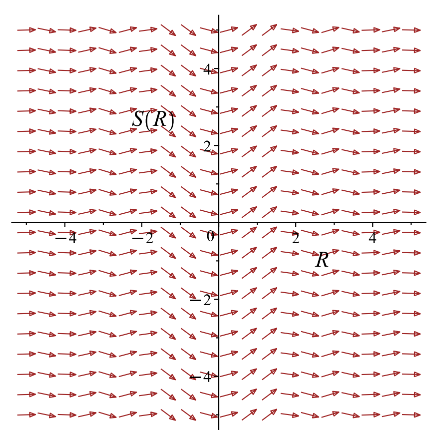
Which simplifies to

$$\frac{y}{x} = \frac{\mathrm{Si}\left(x^2\right)}{2} + c_1$$

Which gives

$$y = \frac{x(\mathrm{Si}\left(x^2\right) + 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\sin(x^2)x}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \frac{\sin(R^2)}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{\mathrm{Si}(1)}{2} - c_1$$

$$c_1 = -\frac{\mathrm{Si}(1)}{2} + 1$$

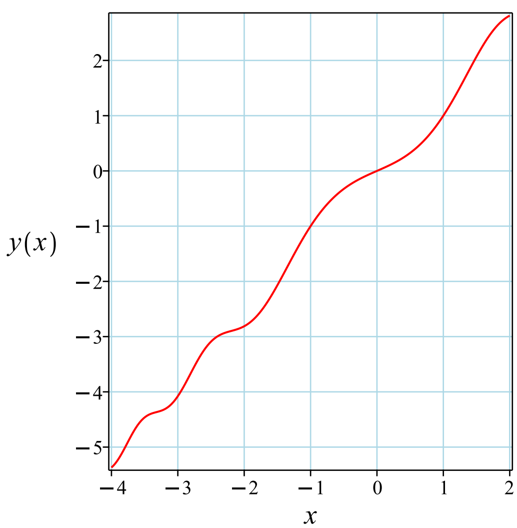
Substituting c_1 found above in the general solution gives

$$y = \frac{x(\mathrm{Si}(x^2) + 2 - \mathrm{Si}(1))}{2}$$

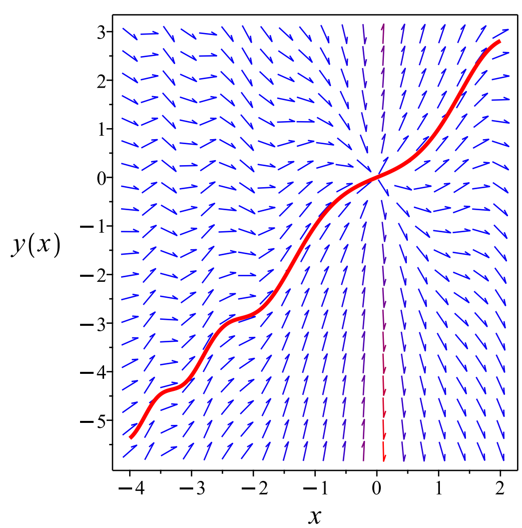
Summary

The solution(s) found are the following

$$y = \frac{x(\mathrm{Si}(x^2) + 2 - \mathrm{Si}(1))}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x(\mathrm{Si}(x^2) + 2 - \mathrm{Si}(1))}{2}$$

Verified OK.

7.5.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(\frac{y}{x}+\sin\left(x^2\right)\right)\mathrm{d}x$$

$$\left(-\frac{y}{x}-\sin\left(x^2\right)\right)\mathrm{d}x+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{y}{x}-\sin\left(x^2\right)\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{x} - \sin\left(x^2\right)\right) \\ &= -\frac{1}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\frac{1}{x}\right) - (0)\right) \\ &= -\frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(x)} \\ &= \frac{1}{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x}\left(-\frac{y}{x} - \sin\left(x^2\right)\right) \\ &= \frac{-y - \sin\left(x^2\right)x}{x^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x}(1) \\
&= \frac{1}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-y - \sin\left(x^{2}\right)x}{x^{2}}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-y - \sin\left(x^{2}\right)x}{x^{2}}\,\mathrm{d}x \\
\phi &= \frac{y}{x} - \frac{\mathrm{Si}\left(x^{2}\right)}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$
\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y}{x} - \frac{\operatorname{Si}(x^2)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y}{x} - \frac{\operatorname{Si}(x^2)}{2}$$

The solution becomes

$$y = \frac{x(\operatorname{Si}(x^2) + 2c_1)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{\operatorname{Si}(1)}{2} - c_1$$

$$c_1 = -\frac{\operatorname{Si}(1)}{2} + 1$$

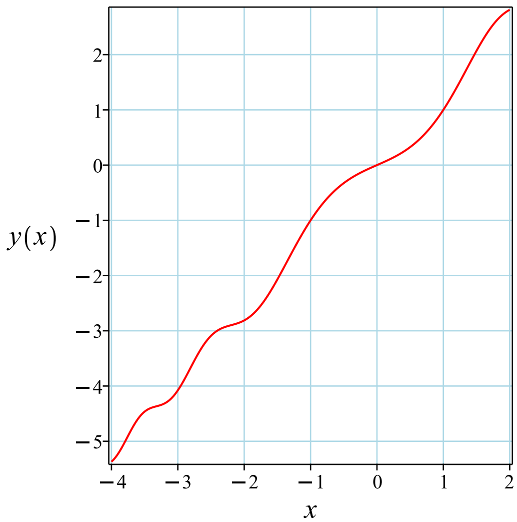
Substituting c_1 found above in the general solution gives

$$y = \frac{x(\operatorname{Si}(x^2) + 2 - \operatorname{Si}(1))}{2}$$

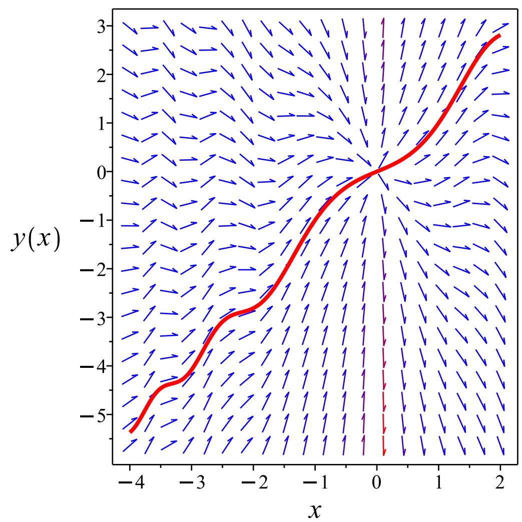
Summary

The solution(s) found are the following

$$y = \frac{x(\operatorname{Si}(x^2) + 2 - \operatorname{Si}(1))}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x(\mathrm{Si}\,(x^2) + 2 - \mathrm{Si}\,(1))}{2}$$

Verified OK.

7.5.6 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{x} = \sin(x^2), y(-1) = -1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + \sin(x^2)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = \sin(x^2)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)\sin(x^2)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x) \sin\left(x^2\right) dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x) \sin\left(x^2\right) dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x) \sin(x^2) dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \frac{\sin(x^2)}{x} dx + c_1\right)$
- Evaluate the integrals on the rhs

  $y = x\left(\frac{\mathrm{Si}(x^2)}{2} + c_1\right)$
- Use initial condition $y(-1) = -1$

  $-1 = -\frac{\mathrm{Si}(1)}{2} - c_1$
- Solve for c_1

  $c_1 = -\frac{\mathrm{Si}(1)}{2} + 1$
- Substitute $c_1 = -\frac{\mathrm{Si}(1)}{2} + 1$ into general solution and simplify

  $y = \frac{x(\mathrm{Si}(x^2) + 2 - \mathrm{Si}(1))}{2}$
- Solution to the IVP

  $y = \frac{x(\mathrm{Si}(x^2) + 2 - \mathrm{Si}(1))}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)=y(x)/x+sin(x^2),y(-1) = -1],y(x), singsol=all)
```

$$y(x) = -\frac{(-\operatorname{Si}(x^2) - 2 + \operatorname{Si}(1))\, x}{2}$$

✓ Solution by Mathematica
Time used: 0.067 (sec). Leaf size: 20

```
DSolve[{y'[x]==y[x]/x+Sin[x^2],{y[-1]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}x(\operatorname{Si}(x^2) - \operatorname{Si}(1) + 2)$$

7.6 problem 6

Internal problem ID [12691]
Internal file name [`OUTPUT/11343_Friday_November_03_2023_06_30_41_AM_1617631/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.3, page 71
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{2y}{x} = e^{x}}$$

With initial conditions

$$\left[y(1) = \frac{1}{2}\right]$$

7.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = e^{x}$$

Hence the ode is

$$y' - \frac{2y}{x} = e^x$$

The domain of $p(x) = -\frac{2}{x}$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

7.6.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(e^x\right) \\ \frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(e^x\right) \\ d\left(\frac{y}{x^2}\right) &= \left(\frac{e^x}{x^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2} &= \int \frac{e^x}{x^2} dx \\ \frac{y}{x^2} &= -\frac{e^x}{x} - \operatorname{expIntegral}_1(-x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(-\frac{e^x}{x} - \operatorname{expIntegral}_1(-x)\right) + c_1 x^2$$

which simplifies to

$$y = x(-\operatorname{expIntegral}_1(-x)\, x + c_1 x - \mathrm{e}^x)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\operatorname{expIntegral}_1(-1) + c_1 - \mathrm{e}$$

$$c_1 = \operatorname{expIntegral}_1(-1) + \mathrm{e} + \frac{1}{2}$$

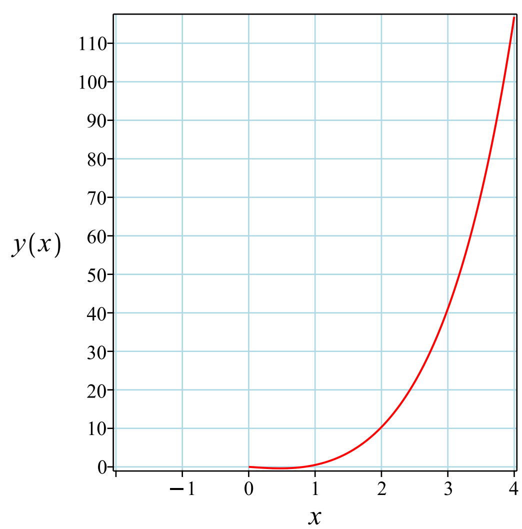
Substituting c_1 found above in the general solution gives

$$y = -\operatorname{expIntegral}_1(-x)\, x^2 + \operatorname{expIntegral}_1(-1)\, x^2 + \mathrm{e}x^2 - x\,\mathrm{e}^x + \frac{x^2}{2}$$

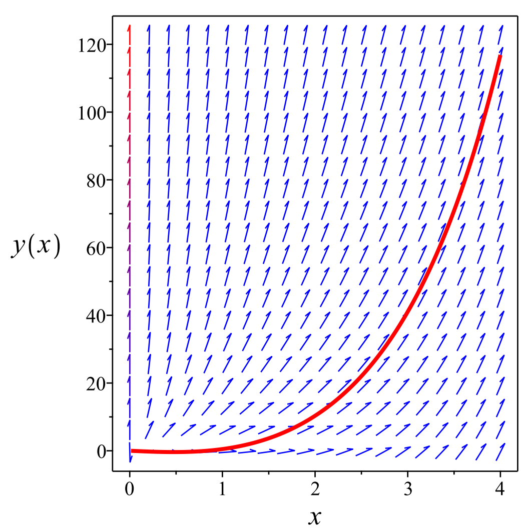
Summary

The solution(s) found are the following

$$y = -\operatorname{expIntegral}_1(-x)\, x^2 + \operatorname{expIntegral}_1(-1)\, x^2 + \mathrm{e}x^2 - x\,\mathrm{e}^x + \frac{x^2}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\operatorname{expIntegral}_1(-x)\, x^2 + \operatorname{expIntegral}_1(-1)\, x^2 + \mathrm{e}x^2 - x\,\mathrm{e}^x + \frac{x^2}{2}$$

Verified OK.

7.6.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x\,\mathrm{e}^{x}+2y}{x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 204: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y'=f(x)y(x)+g(x)$	0	$e^{\int f dx}$
separable ode	$y'=f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y'=f(x)$	0	1
quadrature ode	$y'=g(y)$	1	0
homogeneous ODEs of Class A	$y'=f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y'=(a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y'=\frac{y}{x}+g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y'=g(x)\,e^{h(x)+by}+f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y'=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y'=f(x)\,y+g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y'=f_1(x)\,y+f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = x^2 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x\,\mathrm{e}^{x} + 2y}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2y}{x^3} \\
S_y &= \frac{1}{x^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{e^x}{x^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{e^R}{R^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{e^R}{R} - \text{expIntegral}_1(-R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y}{x^2} = -\frac{e^x}{x} - \text{expIntegral}_1(-x) + c_1
$$

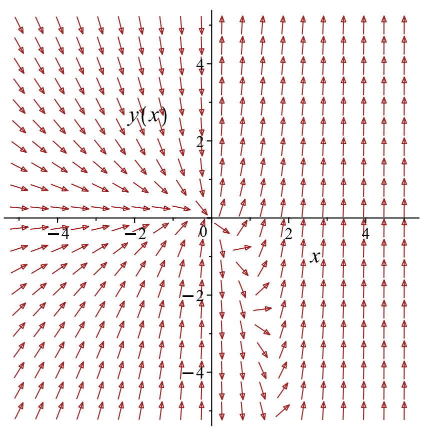
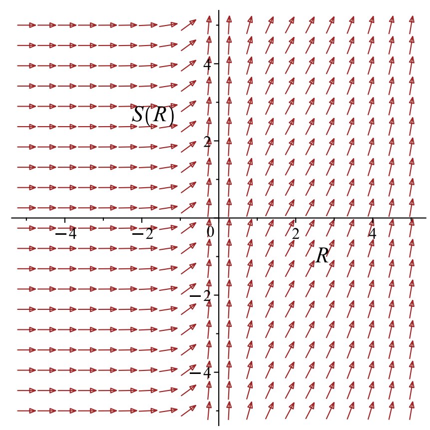
Which simplifies to

$$
\frac{y}{x^2} = -\frac{e^x}{x} - \text{expIntegral}_1(-x) + c_1
$$

Which gives

$$
y = -x(\text{expIntegral}_1(-x)\, x - c_1 x + e^x)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x\,e^{x}+2y}{x}$ 	$R = x$ $S = \frac{y}{x^2}$	$\frac{dS}{dR} = \frac{e^{R}}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\operatorname{expIntegral}_1(-1) + c_1 - e$$

$$c_1 = \operatorname{expIntegral}_1(-1) + e + \frac{1}{2}$$

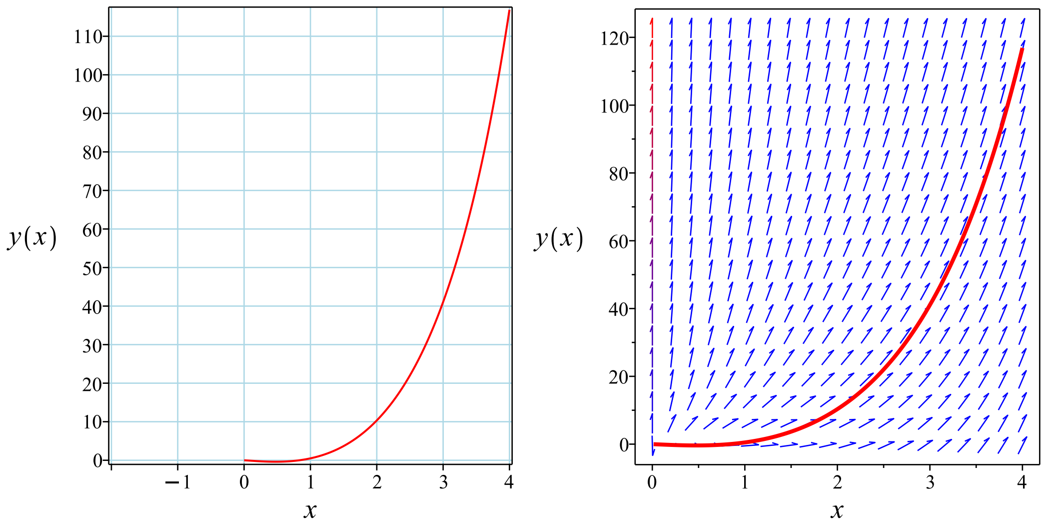
Substituting c_1 found above in the general solution gives

$$y = -\operatorname{expIntegral}_1(-x)\,x^2 + \operatorname{expIntegral}_1(-1)\,x^2 + e x^2 - x\,e^{x} + \frac{x^2}{2}$$

Summary

The solution(s) found are the following

$$y = -\operatorname{expIntegral}_1(-x)\,x^2 + \operatorname{expIntegral}_1(-1)\,x^2 + e x^2 - x\,e^{x} + \frac{x^2}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\operatorname{expIntegral}_1(-x)\,x^2 + \operatorname{expIntegral}_1(-1)\,x^2 + \mathrm{e}x^2 - x\,\mathrm{e}^x + \frac{x^2}{2}$$

Verified OK.

7.6.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{2y}{x} + \mathrm{e}^{x}\right)\mathrm{d}x$$

$$\left(-\frac{2y}{x} - \mathrm{e}^{x}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{2y}{x} - \mathrm{e}^{x} \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2y}{x} - \mathrm{e}^{x}\right) \\ &= -\frac{2}{x} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\frac{2}{x}\right) - (0)\right) \\ &= -\frac{2}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{2}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^2}\left(-\frac{2y}{x} - \mathrm{e}^x\right) \\ &= \frac{-x\,\mathrm{e}^x - 2y}{x^3} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^2}(1) \\ &= \frac{1}{x^2} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-x\,\mathrm{e}^x - 2y}{x^3}\right) + \left(\frac{1}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-x\,\mathrm{e}^{x} - 2y}{x^{3}} \mathrm{d}x$$

$$\phi = \frac{\mathrm{expIntegral}_{1}\left(-x\right) x^{2} + x\,\mathrm{e}^{x} + y}{x^{2}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^{2}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^{2}}$. Therefore equation (4) becomes

$$\frac{1}{x^{2}} = \frac{1}{x^{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\mathrm{expIntegral}_{1}\left(-x\right) x^{2} + x\,\mathrm{e}^{x} + y}{x^{2}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\mathrm{expIntegral}_{1}\left(-x\right) x^{2} + x\,\mathrm{e}^{x} + y}{x^{2}}$$

The solution becomes

$$y = -x(\text{expIntegral}_1(-x)\, x - c_1 x + e^x)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\text{expIntegral}_1(-1) + c_1 - e$$

$$c_1 = \text{expIntegral}_1(-1) + e + \frac{1}{2}$$

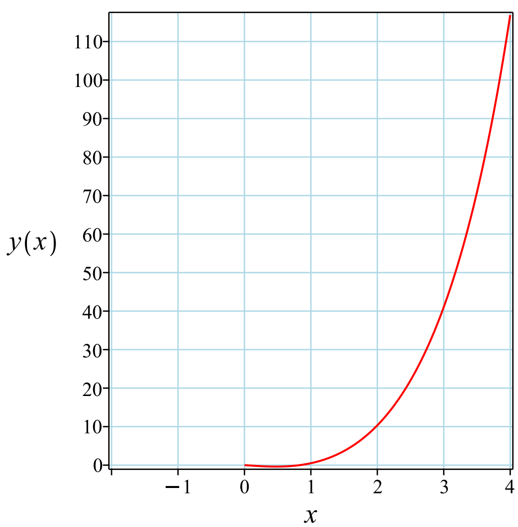
Substituting c_1 found above in the general solution gives

$$y = -\text{expIntegral}_1(-x)\, x^2 + \text{expIntegral}_1(-1)\, x^2 + e x^2 - x\, e^x + \frac{x^2}{2}$$

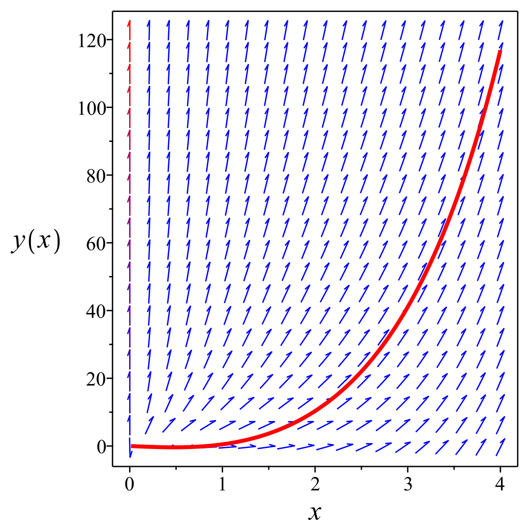
Summary

The solution(s) found are the following

$$y = -\text{expIntegral}_1(-x)\, x^2 + \text{expIntegral}_1(-1)\, x^2 + e x^2 - x\, e^x + \frac{x^2}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\text{expIntegral}_1(-x)\, x^2 + \text{expIntegral}_1(-1)\, x^2 + e x^2 - x\, e^x + \frac{x^2}{2}$$

Verified OK.

7.6.5 Maple step by step solution

Let's solve

$\left[y' - \frac{2y}{x} = \mathrm{e}^{x}, y(1) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{x} + \mathrm{e}^{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{x} = \mathrm{e}^{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = \mu(x)\,\mathrm{e}^{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,\mathrm{e}^{x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,\mathrm{e}^{x} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\mathrm{e}^{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^2}$

  $y = x^2\left(\int \frac{\mathrm{e}^{x}}{x^2} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^2\left(-\frac{\mathrm{e}^{x}}{x} - \mathrm{Ei}_1(-x) + c_1\right)$

- Simplify

  $y = x(-\mathrm{Ei}_1(-x)\,x + c_1 x - \mathrm{e}^{x})$

- Use initial condition $y(1) = \frac{1}{2}$

  $\frac{1}{2} = -\mathrm{Ei}_1(-1) + c_1 - \mathrm{e}$
- Solve for c_1

  $c_1 = \mathrm{Ei}_1(-1) + \mathrm{e} + \frac{1}{2}$
- Substitute $c_1 = \mathrm{Ei}_1(-1) + \mathrm{e} + \frac{1}{2}$ into general solution and simplify

  $y = -\mathrm{Ei}_1(-x)\, x^2 + \mathrm{Ei}_1(-1)\, x^2 + \frac{(2x\,\mathrm{e}+x-2\,\mathrm{e}^x)x}{2}$
- Solution to the IVP

  $y = -\mathrm{Ei}_1(-x)\, x^2 + \mathrm{Ei}_1(-1)\, x^2 + \frac{(2x\,\mathrm{e}+x-2\,\mathrm{e}^x)x}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 29

```
dsolve([diff(y(x),x)=2*y(x)/x+exp(x),y(1) = 1/2],y(x), singsol=all)
```

$$y(x) = -\operatorname{expIntegral}_1(-x)\, x^2 + \operatorname{expIntegral}_1(-1)\, x^2 + \frac{(2x\,\mathrm{e} + x - 2\,\mathrm{e}^x)\, x}{2}$$

✓ Solution by Mathematica

Time used: 0.116 (sec). Leaf size: 31

```
DSolve[{y'[x]==2*y[x]/x+Exp[x],{y[1]==1/2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}x(2x\,\mathrm{ExpIntegralEi}(x) - 2\,\mathrm{ExpIntegralEi}(1)x + 2ex + x - 2e^x)$$

7.7 problem 7

Internal problem ID [12692]
Internal file name [`OUTPUT/11344_Friday_November_03_2023_06_30_47_AM_70439168/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.3, page 71
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - y\cot(x) = \sin(x)}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 0\right]$$

7.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\cot(x)$$
$$q(x) = \sin(x)$$

Hence the ode is

$$y' - y \cot(x) = \sin(x)$$

The domain of $p(x) = -\cot(x)$ is

$$\{x < \pi\_Z118 \lor \pi\_Z118 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The domain of $q(x) = \sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

7.7.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\cot(x)dx} \\ &= \frac{1}{\sin(x)}\end{aligned}$$

Which simplifies to

$$\mu = \csc(x)$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\sin(x)) \\ \frac{d}{dx}(\csc(x)\, y) &= (\csc(x))(\sin(x)) \\ d(\csc(x)\, y) &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\csc(x)\, y &= \int dx \\ \csc(x)\, y &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \csc(x)$ results in

$$y = x\sin(x) + \sin(x)\, c_1$$

which simplifies to

$$y = \sin(x)(x + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\pi}{2} + c_1$$

$$c_1 = -\frac{\pi}{2}$$

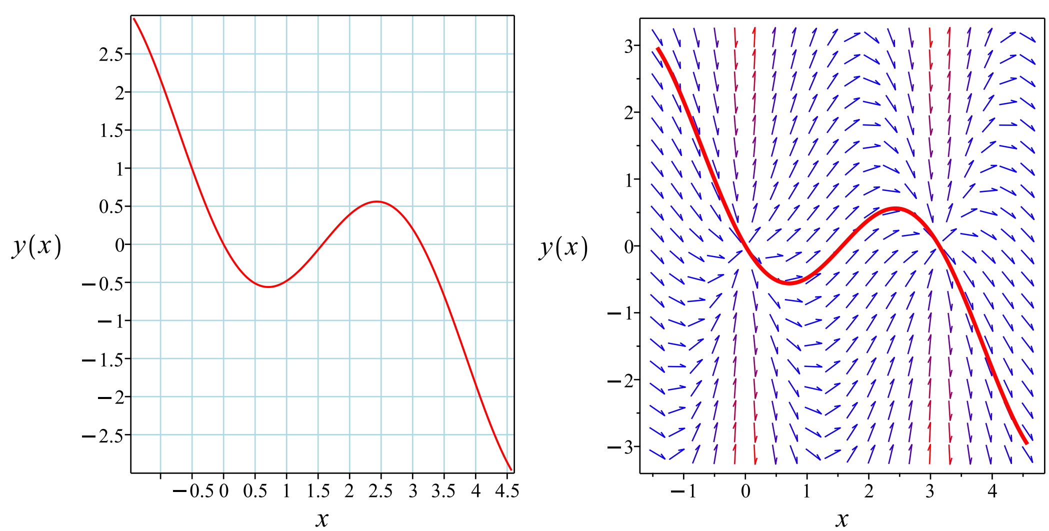
Substituting c_1 found above in the general solution gives

$$y = -\frac{\sin(x)\pi}{2} + x\sin(x)$$

Summary

The solution(s) found are the following

$$y = -\frac{\sin(x)\pi}{2} + x\sin(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{\sin(x)\pi}{2} + x\sin(x)$$

Verified OK.

7.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= y\cot(x) + \sin(x) \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 207: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \sin(x) \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sin(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sin(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = y\cot(x) + \sin(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\csc(x)\cot(x)\, y \\ S_y &= \csc(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\csc(x)\, y = x + c_1$$

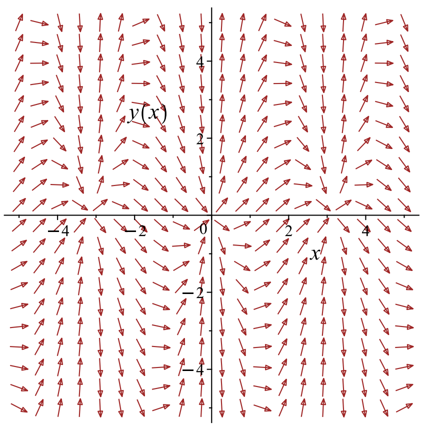
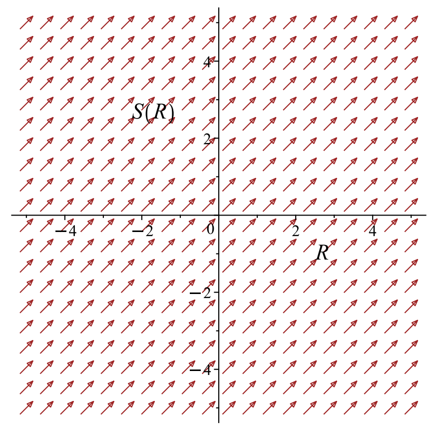
Which simplifies to

$$\csc(x)\, y = x + c_1$$

Which gives

$$y = \frac{x + c_1}{\csc(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y\cot(x) + \sin(x)$ 	$R = x$ $S = \csc(x)\, y$	$\frac{dS}{dR} = 1$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\pi}{2} + c_1$$

$$c_1 = -\frac{\pi}{2}$$

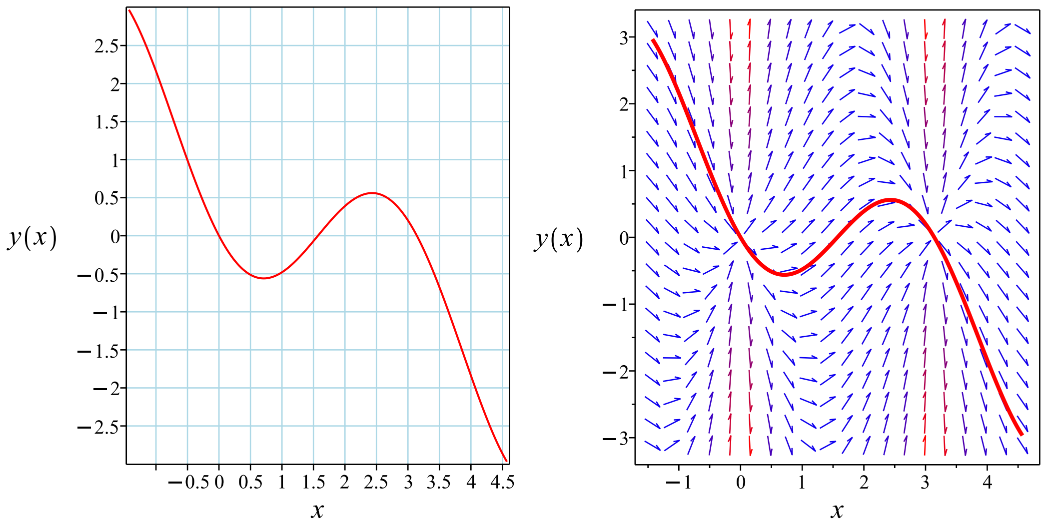
Substituting c_1 found above in the general solution gives

$$y = -\frac{\sin(x)\,\pi}{2} + x\sin(x)$$

Summary

The solution(s) found are the following

$$y = -\frac{\sin(x)\,\pi}{2} + x\sin(x) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{\sin(x)\,\pi}{2} + x\sin(x)$$

Verified OK.

7.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (y\cot(x) + \sin(x))\,\mathrm{d}x\\ (-y\cot(x) - \sin(x))\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y\cot(x) - \sin(x)\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y\cot(x) - \sin(x))\\ &= -\cot(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((-\cot(x)) - (0))\\ &= -\cot(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\cot(x)\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(\sin(x))}\\ &= \csc(x)\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \csc(x)\left(-y\cot(x) - \sin(x)\right)\\ &= -1 - \csc(x)\cot(x)\,y\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \csc(x)\,(1)\\ &= \csc(x)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-1 - \csc(x)\cot(x)\,y\right) + \left(\csc(x)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x\\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int -1 - \csc(x)\cot(x)\,y\,\mathrm{d}x\\ \phi &= -x + \csc(x)\,y + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \csc(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \csc(x)$. Therefore equation (4) becomes

$$\csc(x) = \csc(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \csc(x)\, y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \csc(x)\, y$$

The solution becomes

$$y = \frac{x + c_1}{\csc(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\pi}{2} + c_1$$

$$c_1 = -\frac{\pi}{2}$$

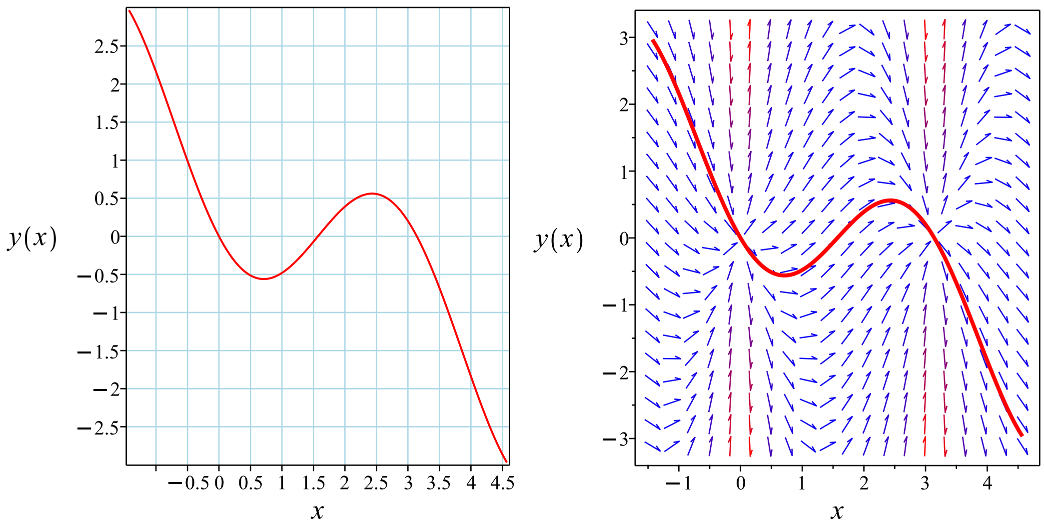
Substituting c_1 found above in the general solution gives

$$y = -\frac{\sin(x)\, \pi}{2} + x \sin(x)$$

Summary

The solution(s) found are the following

$$y = -\frac{\sin(x)\,\pi}{2} + x\sin(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{\sin(x)\,\pi}{2} + x\sin(x)$$

Verified OK.

7.7.5 Maple step by step solution

Let's solve

$\left[y' - y\cot(x) = \sin(x), y\left(\frac{\pi}{2}\right) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y\cot(x) + \sin(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y\cot(x) = \sin(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'-y\cot\left(x\right)\right)=\mu(x)\sin\left(x\right)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y'-y\cot\left(x\right)\right)=\mu'(x)\, y+\mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x)=-\mu(x)\cot\left(x\right)$
- Solve to find the integrating factor

  $\mu(x)=\frac{1}{\sin(x)}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\, y)\right)dx=\int\mu(x)\sin\left(x\right)dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y=\int\mu(x)\sin\left(x\right)dx+c_1$
- Solve for y

  $y=\frac{\int\mu(x)\sin(x)dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\frac{1}{\sin(x)}$

  $y=\sin\left(x\right)\left(\int 1dx+c_1\right)$
- Evaluate the integrals on the rhs

  $y=\sin\left(x\right)\left(x+c_1\right)$
- Use initial condition $y\left(\frac{\pi}{2}\right)=0$

  $0=\frac{\pi}{2}+c_1$
- Solve for c_1

  $c_1=-\frac{\pi}{2}$
- Substitute $c_1=-\frac{\pi}{2}$ into general solution and simplify

  $y=\left(x-\frac{\pi}{2}\right)\sin\left(x\right)$
- Solution to the IVP

  $y=\left(x-\frac{\pi}{2}\right)\sin\left(x\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([diff(y(x),x)=cot(x)*y(x)+sin(x),y(1/2*Pi) = 0],y(x), singsol=all)
```

$$y(x) = \left(-\frac{\pi}{2} + x\right) \sin\left(x\right)$$

✓ Solution by Mathematica

Time used: 0.088 (sec). Leaf size: 16

```
DSolve[{y'[x]==Cot[x]*y[x]+Sin[x],{y[Pi/2]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}(\pi - 2x)\sin(x)$$

7.8 problem 12

Internal problem ID [12693]
Internal file name [`OUTPUT/11345_Friday_November_03_2023_06_30_48_AM_74629381/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.3, page 71
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{-yy' = -x}$$

7.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x}{y} \end{aligned}$$

Where $f(x) = x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = x\,dx$$

$$\int \frac{1}{\frac{1}{y}}\, dy = \int x\, dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c_1$$

Which results in

$$y = \sqrt{x^2 + 2c_1}$$

$$y = -\sqrt{x^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x^2 + 2c_1} \tag{1}$$

$$y = -\sqrt{x^2 + 2c_1} \tag{2}$$

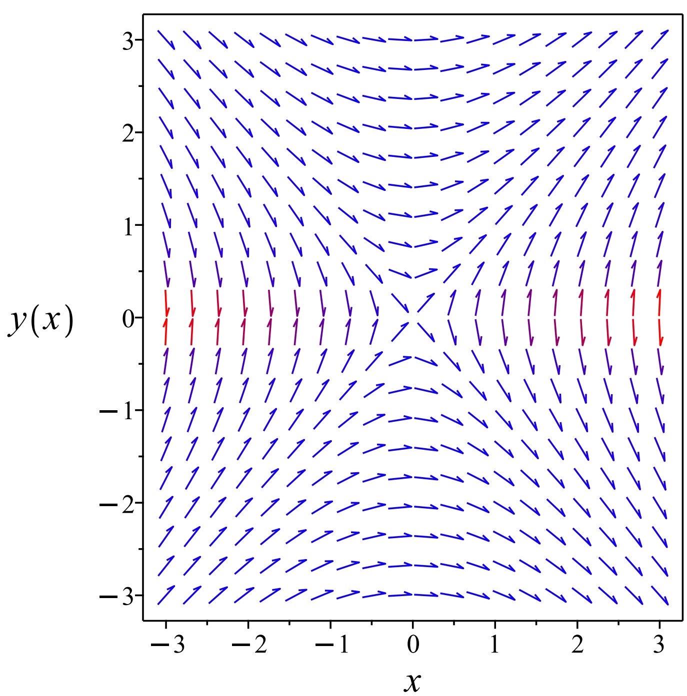


Figure 249: Slope field plot

Verification of solutions

$$y = \sqrt{x^2 + 2c_1}$$

Verified OK.

$$y = -\sqrt{x^2 + 2c_1}$$

Verified OK.

7.8.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-u(x)\,x(u'(x)\,x + u(x)) = -x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2-1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2-1}{u}}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{\frac{u^2-1}{u}}\,du &= \int -\frac{1}{x}\,dx \\ \frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} &= -\ln(x) + c_2 \end{aligned}$$

The above can be written as

$$\begin{aligned} \left(\frac{1}{2}\right)(\ln(u-1) + \ln(u+1)) &= -\ln(x) + 2c_2 \\ \ln(u-1) + \ln(u+1) &= (2)\,(-\ln(x) + 2c_2) \\ &= -2\ln(x) + 4c_2 \end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(u-1)+\ln(u+1)} = \mathrm{e}^{-2\ln(x)+2c_2}$$

Which simplifies to

$$
\begin{aligned}
u^2 - 1 &= \frac{2c_2}{x^2} \\
&= \frac{c_3}{x^2}
\end{aligned}
$$

The solution is

$$
u(x)^2 - 1 = \frac{c_3}{x^2}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\frac{y^2}{x^2} - 1 &= \frac{c_3}{x^2} \\
\frac{y^2}{x^2} - 1 &= \frac{c_3}{x^2}
\end{aligned}
$$

Which simplifies to

$$
-(-y + x)(y + x) = c_3
$$

Summary

The solution(s) found are the following

$$
-(-y + x)(y + x) = c_3 \tag{1}
$$

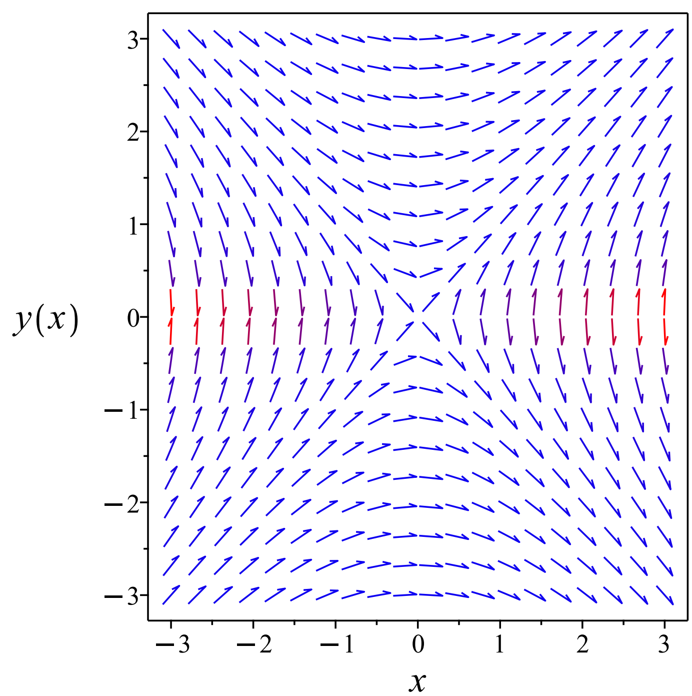


Figure 250: Slope field plot

Verification of solutions

$$-(-y+x)(y+x)=c_3$$

Verified OK.

7.8.3 Solving as differentialType ode

Writing the ode as

$$y'=\frac{x}{y} \tag{1}$$

Which becomes

$$(y)\,dy=(x)\,dx \tag{2}$$

But the RHS is complete differential because

$$(x)\,dx=d\left(\frac{x^2}{2}\right)$$

Hence (2) becomes

$$(y)\,dy=d\left(\frac{x^2}{2}\right)$$

Integrating both sides gives gives these solutions

$$y=\sqrt{x^2+2c_1}+c_1$$
$$y=-\sqrt{x^2+2c_1}+c_1$$

Summary
The solution(s) found are the following

$$y=\sqrt{x^2+2c_1}+c_1 \tag{1}$$
$$y=-\sqrt{x^2+2c_1}+c_1 \tag{2}$$

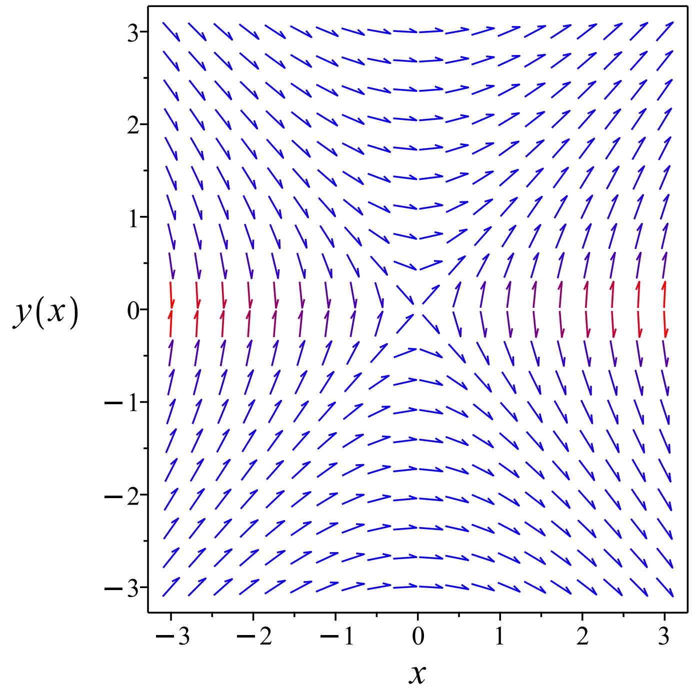


Figure 251: Slope field plot

Verification of solutions

$$y = \sqrt{x^2 + 2c_1} + c_1$$

Verified OK.

$$y = -\sqrt{x^2 + 2c_1} + c_1$$

Verified OK.

7.8.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 210: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{x} \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

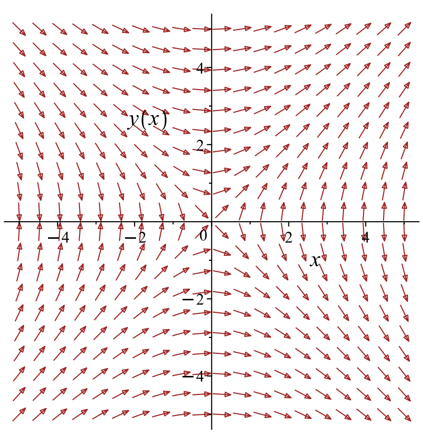
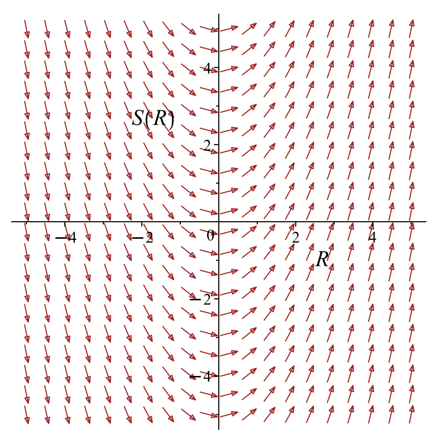
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x}{y}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = R$ 

Summary
The solution(s) found are the following

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1 \tag{1}$$

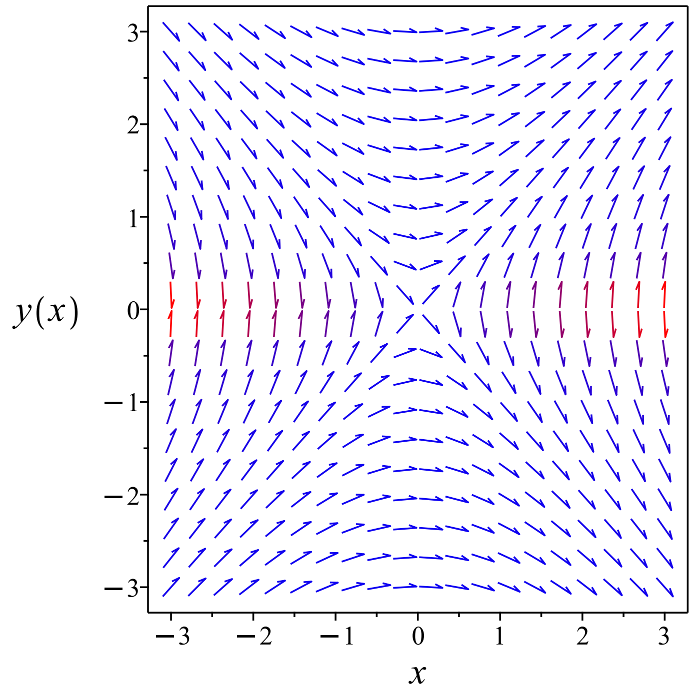


Figure 252: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Verified OK.

7.8.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(y)\, \mathrm{d}y = (x)\, \mathrm{d}x$$
$$(-x)\, \mathrm{d}x + (y)\, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(y)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y$. Therefore equation (4) becomes

$$y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y) \, \mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} + \frac{y^2}{2} = c_1 \tag{1}$$

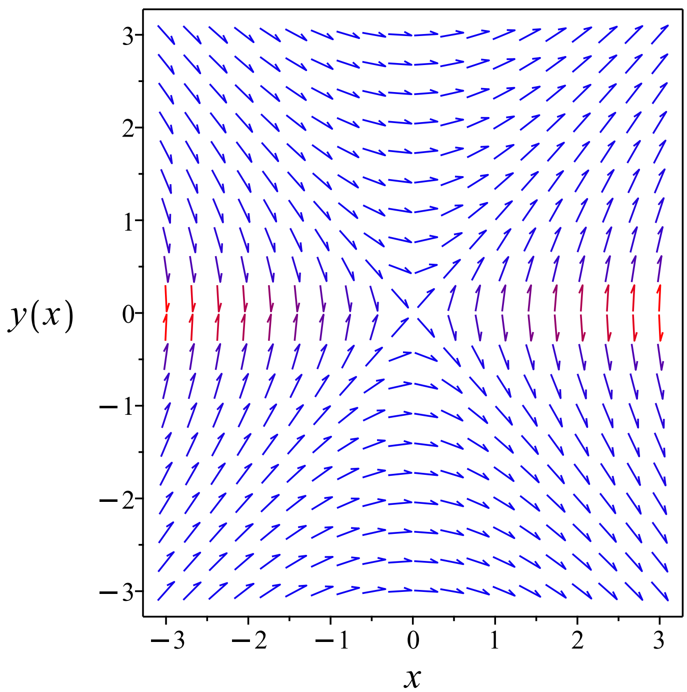


Figure 253: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} + \frac{y^2}{2} = c_1$$

Verified OK.

7.8.6 Maple step by step solution

Let's solve

$-yy' = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int -yy' dx = \int -x dx + c_1$

- Evaluate integral

  $-\frac{y^2}{2} = -\frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = \sqrt{x^2 - 2c_1}, y = -\sqrt{x^2 - 2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x-y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x^2 + c_1}$$
$$y(x) = -\sqrt{x^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.132 (sec). Leaf size: 35

```
DSolve[x-y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\sqrt{x^2 + 2c_1}$$
$$y(x) \rightarrow \sqrt{x^2 + 2c_1}$$

7.9 problem 13

Internal problem ID [12694]
Internal file name [`OUTPUT/11346_Friday_November_03_2023_06_30_49_AM_53449137/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.3, page 71
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y - y'x = 0}$$

7.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\,dy = \frac{1}{x}\,dx$$
$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$
$$\ln(y) = \ln(x) + c_1$$
$$y = e^{\ln(x)+c_1}$$
$$= c_1 x$$

Summary

The solution(s) found are the following

$$y = c_1 x \tag{1}$$

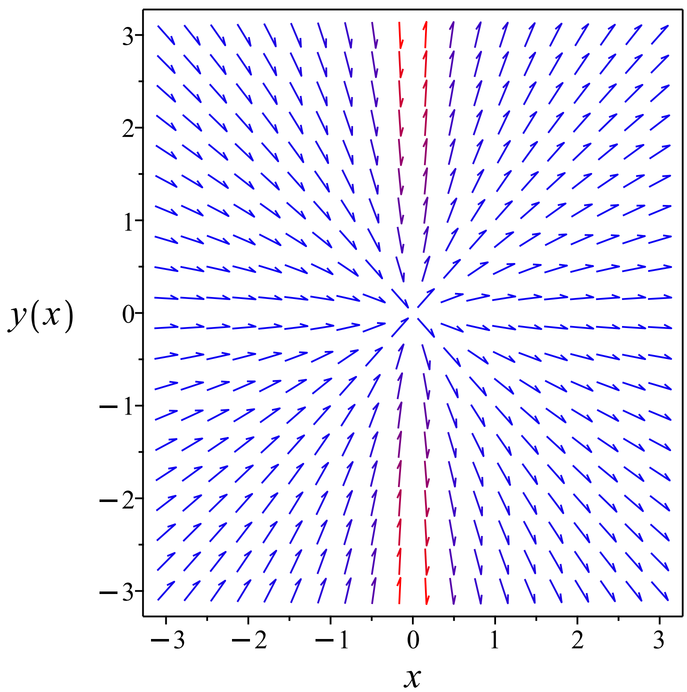


Figure 254: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

7.9.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{1}{x}\\ q(x) &= 0\end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{x}dx}\\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x \tag{1}$$

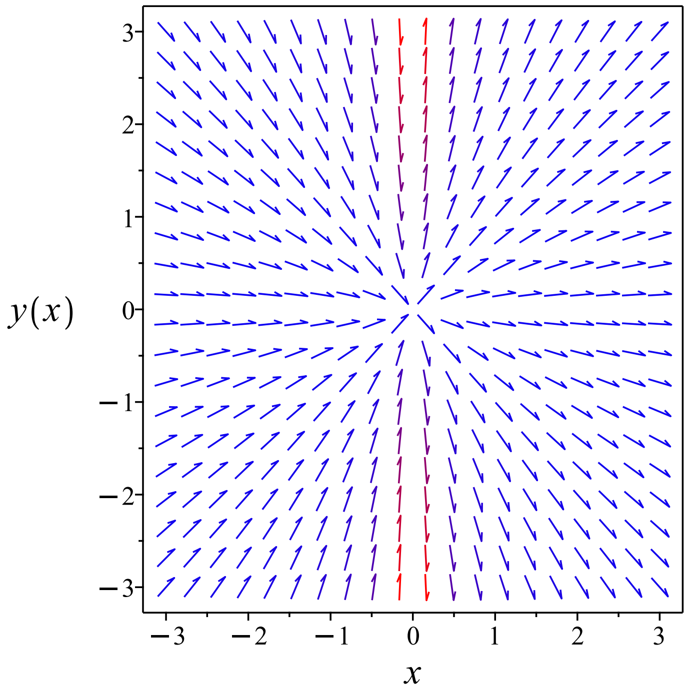


Figure 255: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

7.9.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x - (u'(x)\,x + u(x))\,x = 0$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0\,\mathrm{d}x \\ &= c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= c_2 x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x \tag{1}$$

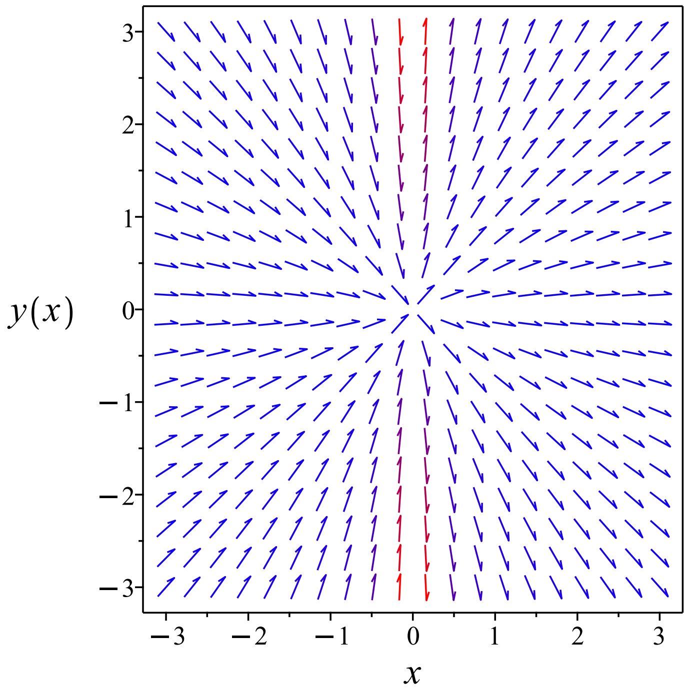


Figure 256: Slope field plot

Verification of solutions

$$y = c_2 x$$

Verified OK.

7.9.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 213: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = c_1$$

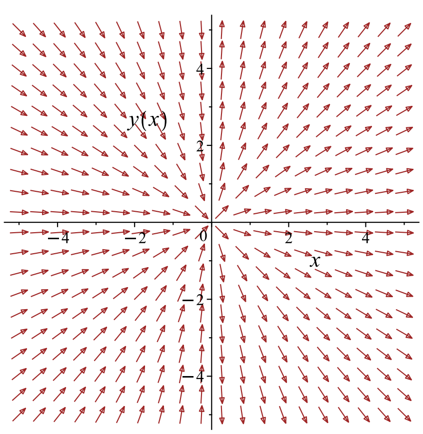
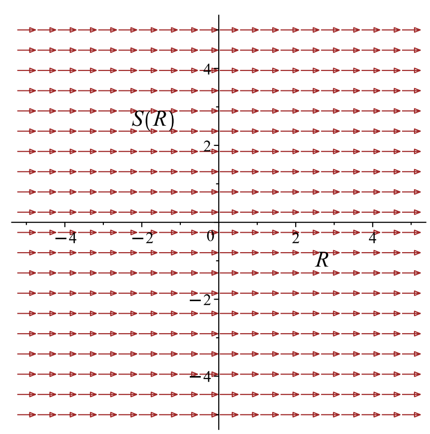
Which simplifies to

$$\frac{y}{x} = c_1$$

Which gives

$$y = c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 x \tag{1}$$

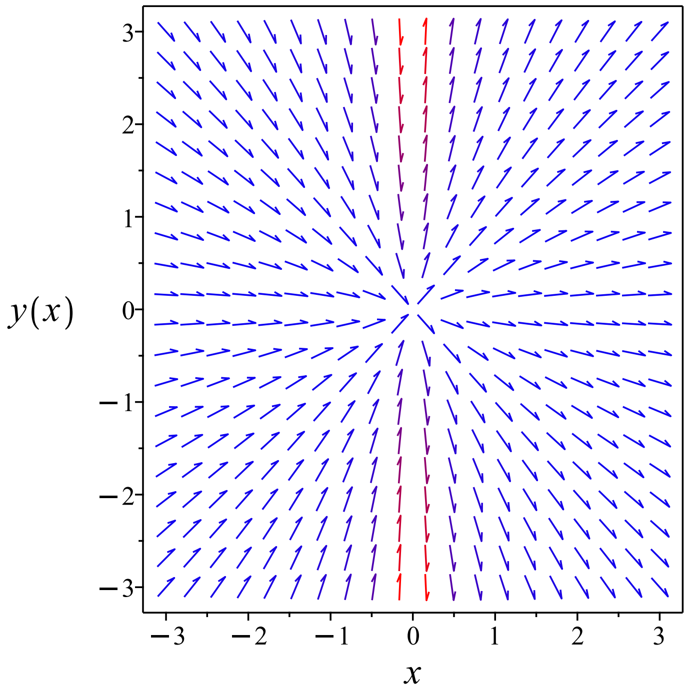


Figure 257: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

7.9.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y}\right) \mathrm{d}y = \left(\frac{1}{x}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{1}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = \frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{x} \, \mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y)$$

The solution becomes

$$y = e^{c_1} x$$

Summary

The solution(s) found are the following

$$y = e^{c_1} x \tag{1}$$

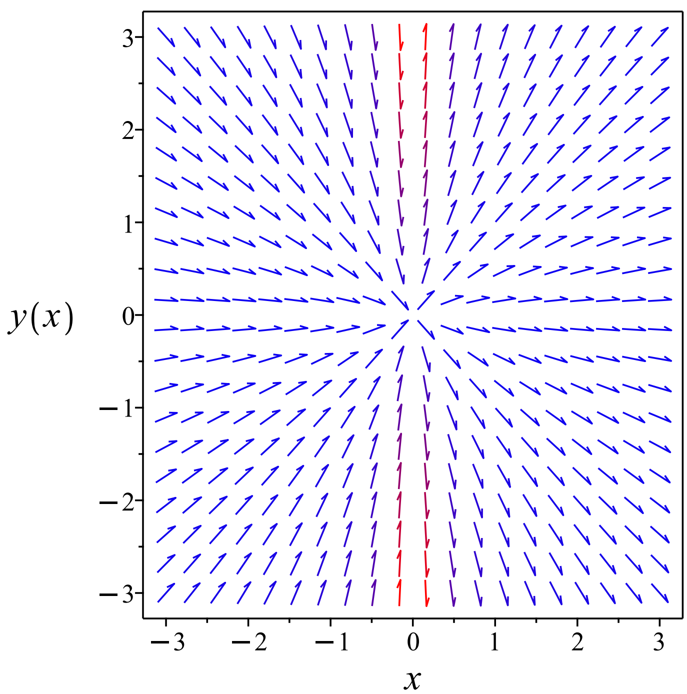


Figure 258: Slope field plot

Verification of solutions

$$y = e^{c_1} x$$

Verified OK.

7.9.6 Maple step by step solution

Let's solve

$y - y'x = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = e^{c_1}x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve(y(x)-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 14

```
DSolve[y[x]-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x$$
$$y(x) \to 0$$

7.10 problem 14

Internal problem ID [12695]
Internal file name [`OUTPUT/11347_Friday_November_03_2023_06_30_50_AM_28944007/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.3, page 71
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x - y = -x^2}$$

7.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = -x$$

Hence the ode is

$$y' - \frac{y}{x} = -x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(-x) \\
\frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)(-x) \\
d\left(\frac{y}{x}\right) &= -1\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int -1\, dx \\
\frac{y}{x} &= -x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
y = c_1 x - x^2
$$

which simplifies to

$$
y = x(-x + c_1)
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = x(-x + c_1) \tag{1}
$$

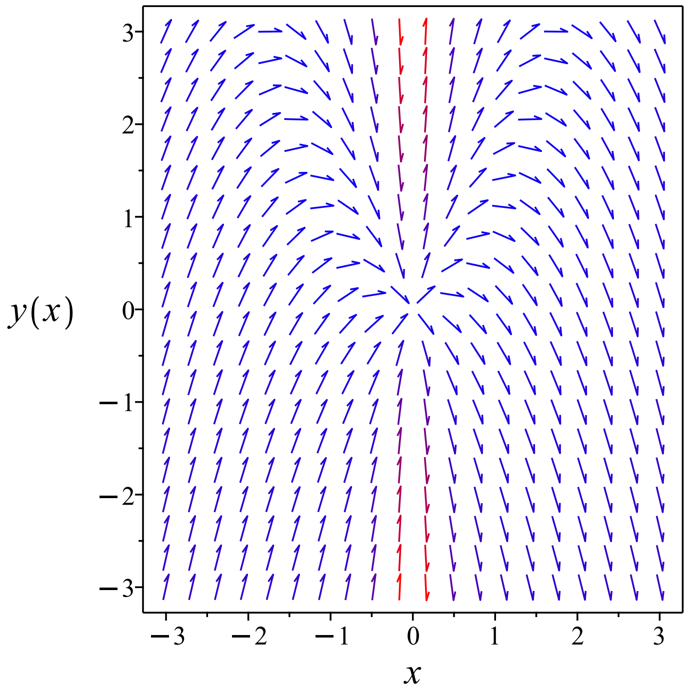


Figure 259: Slope field plot

Verification of solutions

$$y = x(-x + c_1)$$

Verified OK.

7.10.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x - u(x)\,x = -x^2$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int -1\,\mathrm{d}x \\ &= -x + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x(-x + c_2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x(-x + c_2) \tag{1}$$

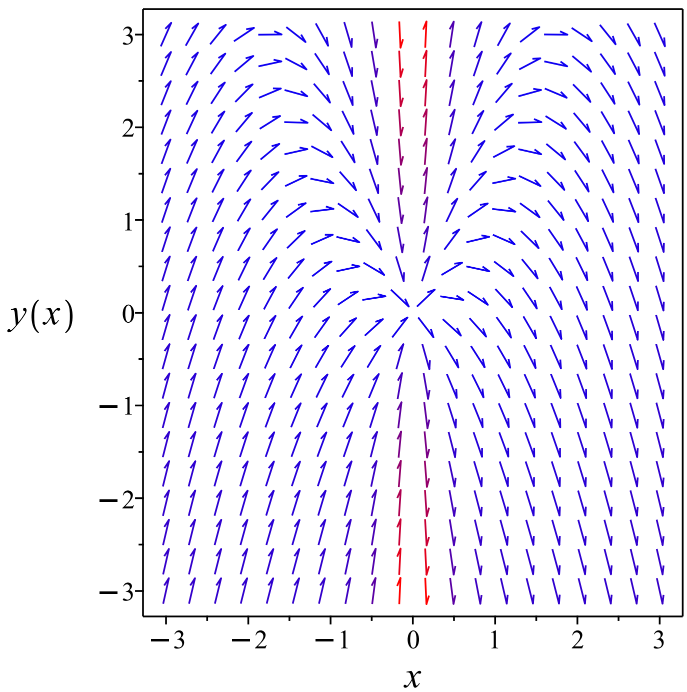


Figure 260: Slope field plot

Verification of solutions

$$y = x(-x + c_2)$$

Verified OK.

7.10.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-x^2 + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 216: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy\end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-x^2+y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = -x + c_1$$

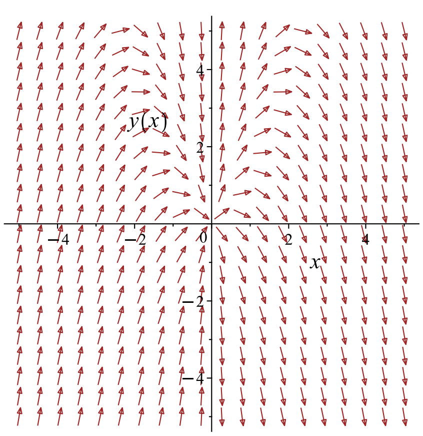
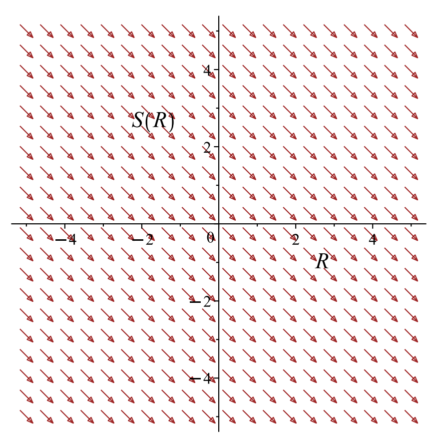
Which simplifies to

$$\frac{y}{x} = -x + c_1$$

Which gives

$$y = x(-x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x^2+y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = x(-x + c_1) \tag{1}$$

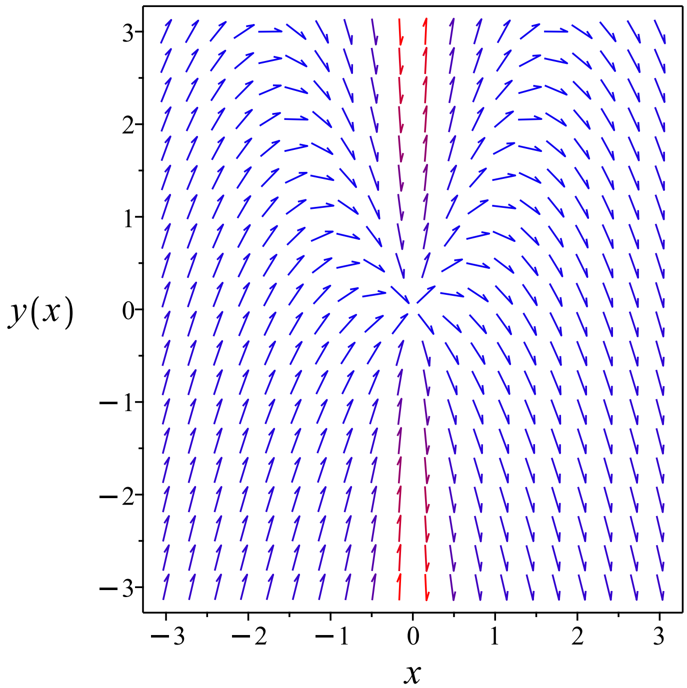


Figure 261: Slope field plot

Verification of solutions

$$y = x(-x + c_1)$$

Verified OK.

7.10.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x) \, \mathrm{d}y &= \left(-x^2 + y\right) \mathrm{d}x \\
\left(x^2 - y\right) \mathrm{d}x + (x) \, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= x^2 - y \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(x^2 - y\right) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((-1) - (1)) \\
&= -\frac{2}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}(x^2 - y) \\
&= \frac{x^2 - y}{x^2}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(x) \\
&= \frac{1}{x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^2 - y}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{x^2 - y}{x^2} \, \mathrm{d}x$$
$$\phi = x + \frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x + \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x + \frac{y}{x}$$

The solution becomes

$$y = x(-x + c_1)$$

Summary

The solution(s) found are the following

$$y = x(-x + c_1) \tag{1}$$

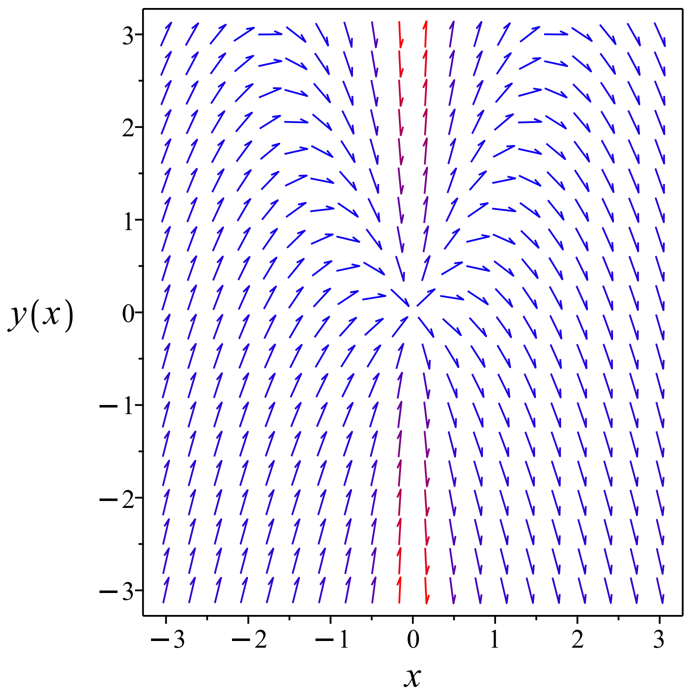


Figure 262: Slope field plot

Verification of solutions

$$y = x(-x + c_1)$$

Verified OK.

7.10.5 Maple step by step solution

Let's solve

$y'x - y = -x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} - x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = -x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = -\mu(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\mu(x)\, x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -\mu(x)\, x dx + c_1$

- Solve for y

  $y = \frac{\int -\mu(x) x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int (-1)\, dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(-x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve((x^2-y(x))+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x(c_1 - x)$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 13

```
DSolve[(x^2-y[x])+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(-x + c_1)$$

7.11 problem 15

Internal problem ID [12696]
Internal file name [`OUTPUT/11348_Friday_November_03_2023_06_30_51_AM_15510492/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.3, page 71
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{xy(1-y)-2y'=0}$$

7.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{xy(y-1)}{2} \end{aligned}$$

Where $f(x)=-\frac{x}{2}$ and $g(y)=y(y-1)$. Integrating both sides gives

$$\frac{1}{y\,(y-1)}\,dy = -\frac{x}{2}\,dx$$

$$\int \frac{1}{y\,(y-1)}\,dy = \int -\frac{x}{2}\,dx$$

$$
\ln (y-1)-\ln (y)=-\frac{x^{2}}{4}+c_{1}
$$

Raising both side to exponential gives

$$
e^{\ln(y-1)-\ln(y)}=e^{-\frac{x^{2}}{4}+c_{1}}
$$

Which simplifies to

$$
\frac{y-1}{y}=c_{2}e^{-\frac{x^{2}}{4}}
$$

Summary

The solution(s) found are the following

$$
y=-\frac{1}{-1+c_{2}e^{-\frac{x^{2}}{4}}} \tag{1}
$$

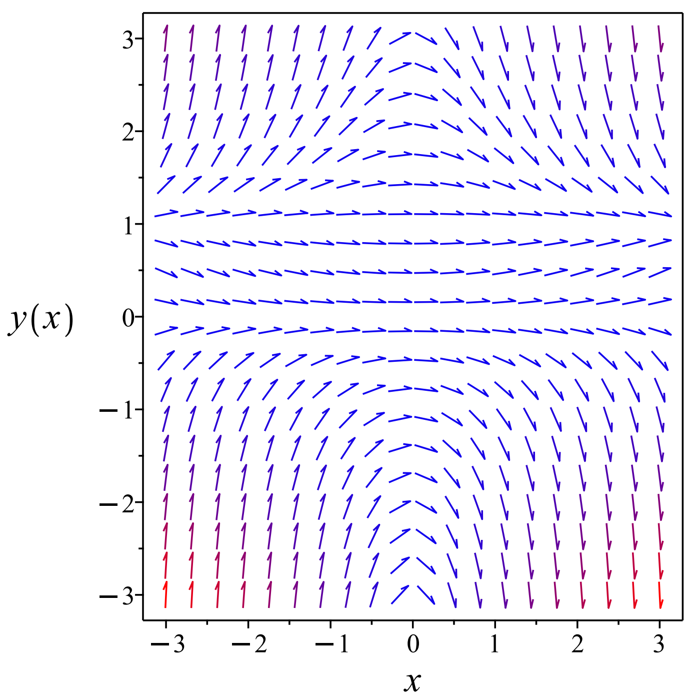


Figure 263: Slope field plot

Verification of solutions

$$
y=-\frac{1}{-1+c_{2}e^{-\frac{x^{2}}{4}}}
$$

Verified OK.

7.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{xy(y-1)}{2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 219: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = -\frac{2}{x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{2}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^2}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{xy(y-1)}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{x}{2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y\left(y-1\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R\left(R-1\right)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln\left(R-1\right) - \ln\left(R\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{4} = \ln\left(y-1\right) - \ln\left(y\right) + c_1$$

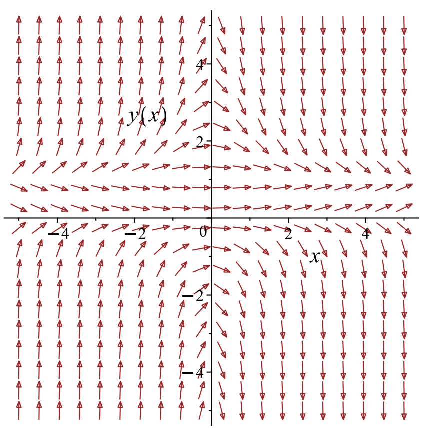
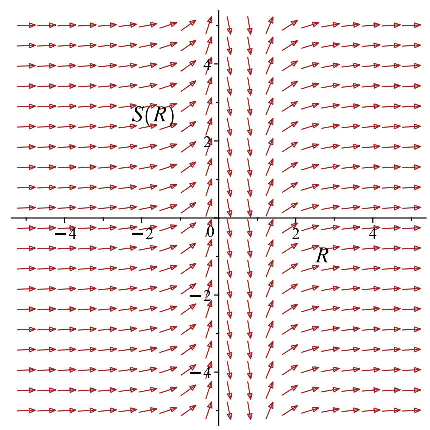
Which simplifies to

$$-\frac{x^2}{4} = \ln\left(y-1\right) - \ln\left(y\right) + c_1$$

Which gives

$$y = \frac{\mathrm{e}^{\frac{x^2}{4}+c_1}}{-1+\mathrm{e}^{\frac{x^2}{4}+c_1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{xy(y-1)}{2}$ 	$R = y$ $S = -\frac{x^2}{4}$	$\frac{dS}{dR} = \frac{1}{R(R-1)}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{\frac{x^2}{4}+c_1}}{-1 + e^{\frac{x^2}{4}+c_1}} \tag{1}$$

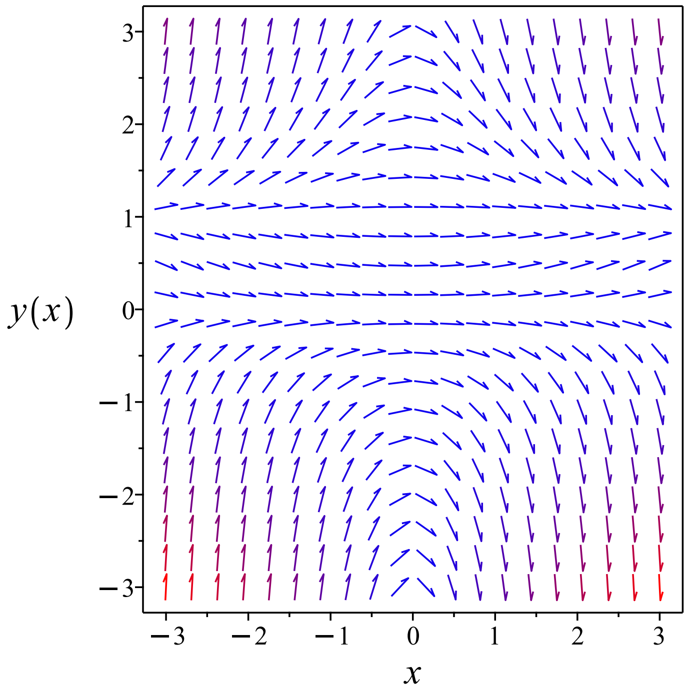


Figure 264: Slope field plot

Verification of solutions

$$y = \frac{e^{\frac{x^2}{4}+c_1}}{-1 + e^{\frac{x^2}{4}+c_1}}$$

Verified OK.

7.11.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{xy(y-1)}{2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{x}{2}y - \frac{x}{2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{x}{2} \\
f_1(x) &= -\frac{x}{2} \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y'\frac{1}{y^2} = \frac{x}{2y} - \frac{x}{2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{y^2}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= \frac{w(x)\,x}{2} - \frac{x}{2} \\
w' &= -\frac{1}{2}xw + \frac{1}{2}x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{x}{2} \\
q(x) &= \frac{x}{2}
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{w(x)\,x}{2} = \frac{x}{2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{x}{2}dx} \\ &= e^{\frac{x^2}{4}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{x}{2}\right) \\ \frac{d}{dx}\left(e^{\frac{x^2}{4}}w\right) &= \left(e^{\frac{x^2}{4}}\right)\left(\frac{x}{2}\right) \\ d\left(e^{\frac{x^2}{4}}w\right) &= \left(\frac{x\,e^{\frac{x^2}{4}}}{2}\right)dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{\frac{x^2}{4}}w &= \int \frac{x\,e^{\frac{x^2}{4}}}{2}\,dx \\ e^{\frac{x^2}{4}}w &= e^{\frac{x^2}{4}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x^2}{4}}$ results in

$$w(x) = e^{-\frac{x^2}{4}}e^{\frac{x^2}{4}} + c_1 e^{-\frac{x^2}{4}}$$

which simplifies to

$$w(x) = 1 + c_1 e^{-\frac{x^2}{4}}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = 1 + c_1 e^{-\frac{x^2}{4}}$$

Or

$$y = \frac{1}{1 + c_1 e^{-\frac{x^2}{4}}}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{1 + c_1 \mathrm{e}^{-\frac{x^2}{4}}} \tag{1}$$

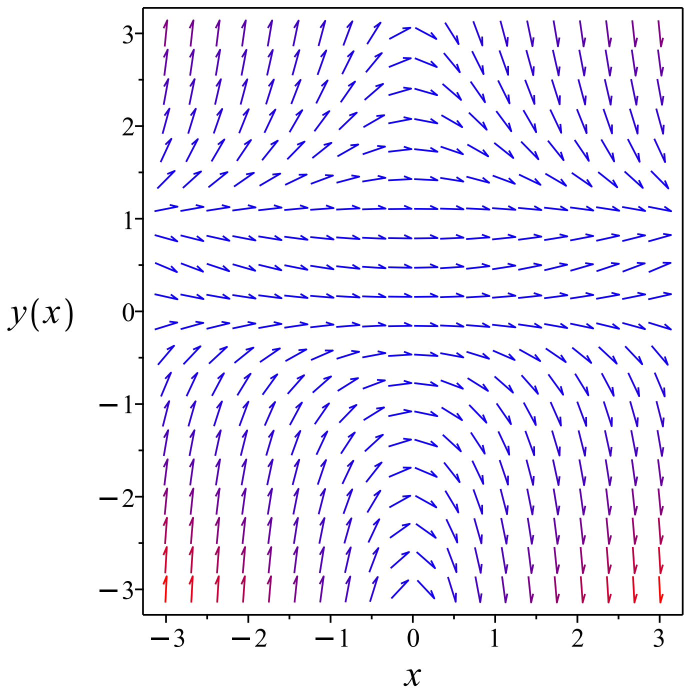


Figure 265: Slope field plot

Verification of solutions

$$y = \frac{1}{1 + c_1 \mathrm{e}^{-\frac{x^2}{4}}}$$

Verified OK.

7.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{2}{y\,(y-1)}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+\left(-\frac{2}{y\,(y-1)}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\\ N(x,y)&=-\frac{2}{y\,(y-1)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{2}{y\,(y-1)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -x\, \mathrm{d}x\end{aligned}$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2}{y(y-1)}$. Therefore equation (4) becomes

$$-\frac{2}{y\,(y-1)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{2}{y\,(y-1)}$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{2}{y\,(y-1)}\right)\mathrm{d}y \\
f(y) &= -2\ln(y-1) + 2\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{x^2}{2} - 2\ln(y-1) + 2\ln(y) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{x^2}{2} - 2\ln(y-1) + 2\ln(y)
$$

The solution becomes

$$
y = \frac{\mathrm{e}^{\frac{x^2}{4}+\frac{c_1}{2}}}{-1+\mathrm{e}^{\frac{x^2}{4}+\frac{c_1}{2}}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\mathrm{e}^{\frac{x^2}{4}+\frac{c_1}{2}}}{-1+\mathrm{e}^{\frac{x^2}{4}+\frac{c_1}{2}}} \tag{1}
$$

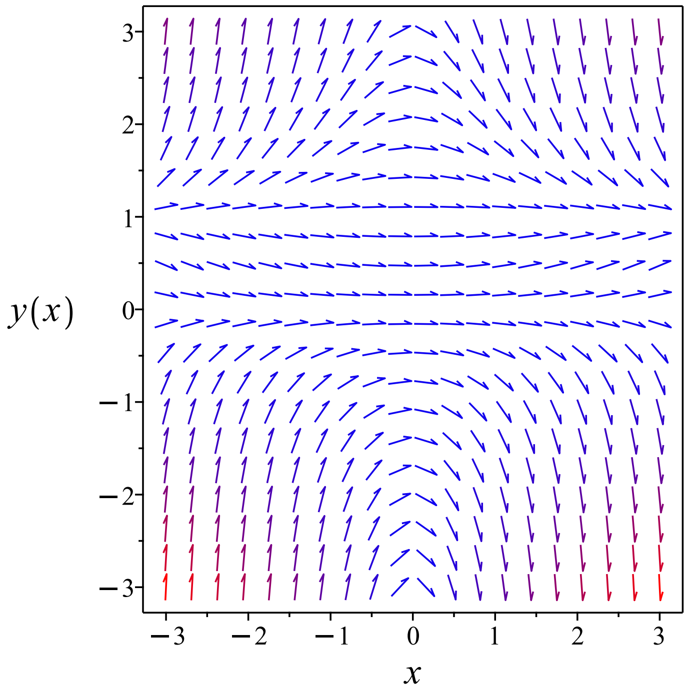


Figure 266: Slope field plot

Verification of solutions

$$y = \frac{e^{\frac{x^2}{4}+\frac{c_1}{2}}}{-1+e^{\frac{x^2}{4}+\frac{c_1}{2}}}$$

Verified OK.

7.11.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{xy(y-1)}{2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{1}{2}x\,y^2 + \frac{1}{2}xy$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{x}{2}$ and $f_2(x) = -\frac{x}{2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{xu}{2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{2} \\ f_1 f_2 &= -\frac{x^2}{4} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{x u''(x)}{2} - \left(-\frac{1}{2} - \frac{x^2}{4}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \mathrm{e}^{\frac{x^2}{4}} c_2$$

The above shows that

$$u'(x) = \frac{x\,\mathrm{e}^{\frac{x^2}{4}} c_2}{2}$$

Using the above in (1) gives the solution

$$y = \frac{\mathrm{e}^{\frac{x^2}{4}} c_2}{c_1 + \mathrm{e}^{\frac{x^2}{4}} c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{e^{\frac{x^2}{4}}}{c_3 + e^{\frac{x^2}{4}}}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{\frac{x^2}{4}}}{c_3 + e^{\frac{x^2}{4}}} \tag{1}$$

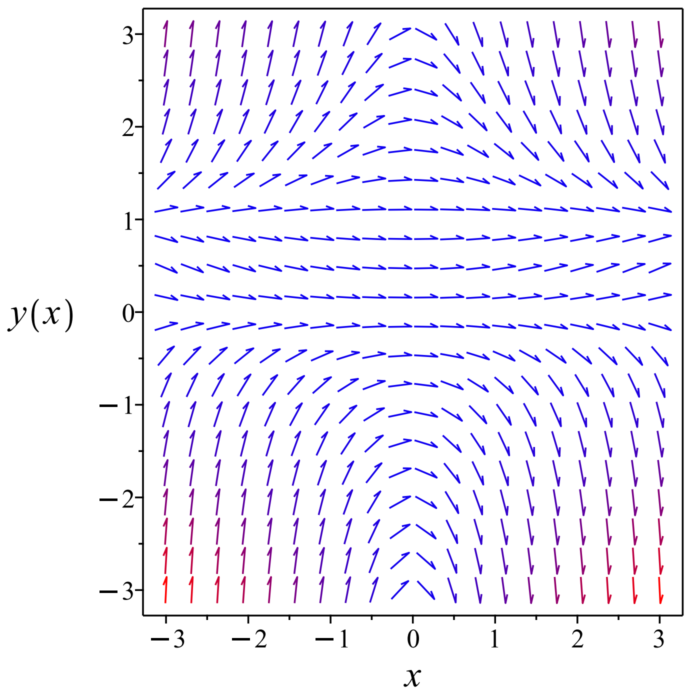


Figure 267: Slope field plot

Verification of solutions

$$y = \frac{e^{\frac{x^2}{4}}}{c_3 + e^{\frac{x^2}{4}}}$$

Verified OK.

7.11.6 Maple step by step solution

Let's solve

$xy(1-y)-2y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(1-y)}=\frac{x}{2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(1-y)}dx=\int \frac{x}{2}dx+c_1$

- Evaluate integral

  $-\ln(y-1)+\ln(y)=\frac{x^2}{4}+c_1$

- Solve for y

  $y=\frac{e^{\frac{x^2}{4}+c_1}}{-1+e^{\frac{x^2}{4}+c_1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*y(x)*(1-y(x))-2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x)=\frac{1}{1+e^{-\frac{x^2}{4}}c_1}$$

✓ Solution by Mathematica

Time used: 0.392 (sec). Leaf size: 41

```
DSolve[x*y[x]*(1-y[x])-2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{\frac{x^2}{4}}}{e^{\frac{x^2}{4}} + e^{c_1}}$$
$$y(x) \to 0$$
$$y(x) \to 1$$

7.12 problem 16

Internal problem ID [12697]
Internal file name [`OUTPUT/11349_Friday_November_03_2023_06_30_51_AM_42738902/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.3, page 71
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x(1-y^3)-3y'y^2=0}$$

7.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x(y^3-1)}{3y^2} \end{aligned}$$

Where $f(x)=-\frac{x}{3}$ and $g(y)=\frac{y^3-1}{y^2}$. Integrating both sides gives

$$\frac{1}{\frac{y^3-1}{y^2}}\,dy=-\frac{x}{3}\,dx$$

$$\int \frac{1}{\frac{y^3-1}{y^2}}\,dy=\int -\frac{x}{3}\,dx$$

$$\frac{\ln(y^3-1)}{3} = -\frac{x^2}{6} + c_1$$

Raising both side to exponential gives

$$(y^3-1)^{\frac{1}{3}} = e^{-\frac{x^2}{6}+c_1}$$

Which simplifies to

$$(y^3-1)^{\frac{1}{3}} = c_2 e^{-\frac{x^2}{6}}$$

The solution is

$$(y^3-1)^{\frac{1}{3}} = c_2 e^{-\frac{x^2}{6}+c_1}$$

<u>Summary</u>

The solution(s) found are the following

$$(y^3-1)^{\frac{1}{3}} = c_2 e^{-\frac{x^2}{6}+c_1} \tag{1}$$

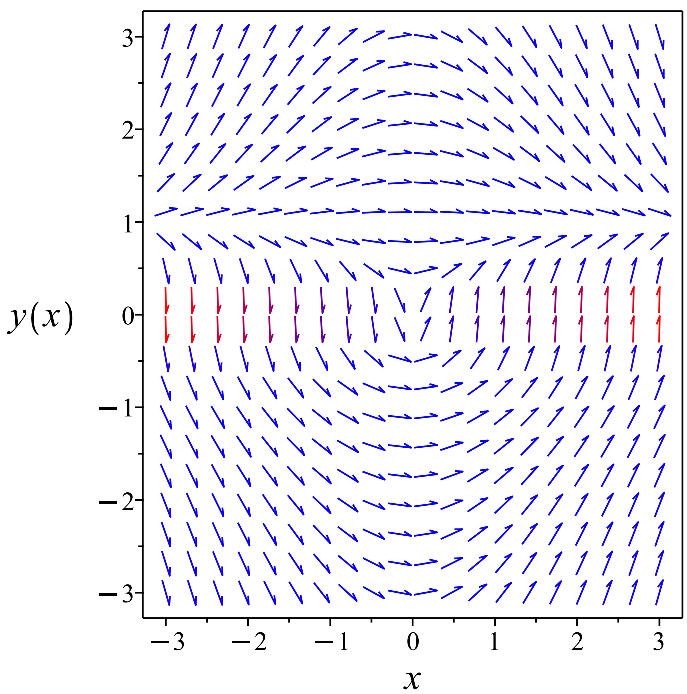


Figure 268: Slope field plot

<u>Verification of solutions</u>

$$(y^3-1)^{\frac{1}{3}} = c_2 e^{-\frac{x^2}{6}+c_1}$$

Verified OK.

7.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x(y^3 - 1)}{3y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 222: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = -\frac{3}{x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{3}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^2}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x(y^3-1)}{3y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{x}{3} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y^2}{y^3 - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^2}{R^3 - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln\left((R-1)\left(R^2 + R + 1\right)\right)}{3} + c_1 \tag{4}$$

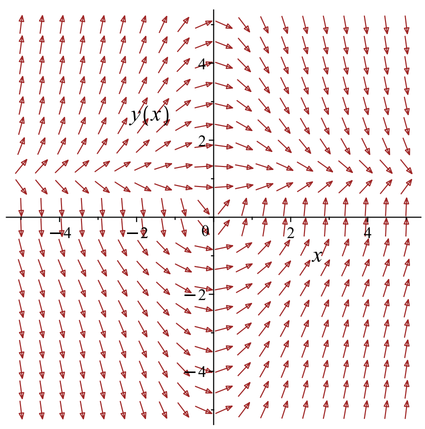
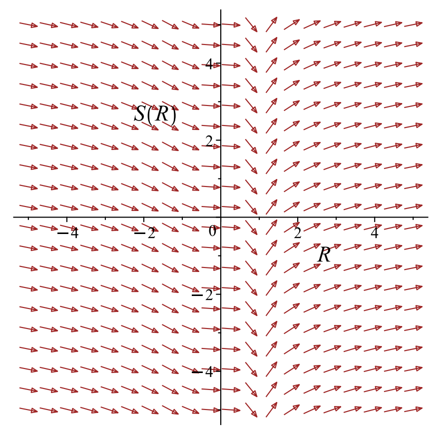
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{6} = \frac{\ln\left((y-1)\left(y^2 + y + 1\right)\right)}{3} + c_1$$

Which simplifies to

$$-\frac{x^2}{6} = \frac{\ln\left((y-1)\left(y^2 + y + 1\right)\right)}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x(y^3-1)}{3y^2}$ 	$R = y$ $S = -\frac{x^2}{6}$	$\frac{dS}{dR} = \frac{R^2}{R^3-1}$ 

Summary

The solution(s) found are the following

$$-\frac{x^2}{6} = \frac{\ln\left((y-1)\left(y^2+y+1\right)\right)}{3} + c_1 \tag{1}$$

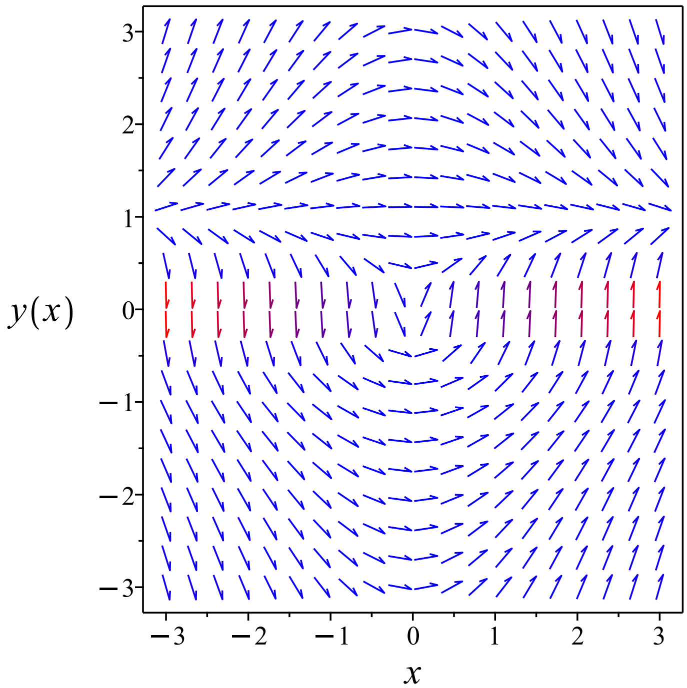


Figure 269: Slope field plot

Verification of solutions

$$-\frac{x^2}{6} = \frac{\ln\left(\left(y-1\right)\left(y^2+y+1\right)\right)}{3} + c_1$$

Verified OK.

7.12.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x(y^3-1)}{3y^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{x}{3}y + \frac{x}{3}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{x}{3} \\ f_1(x) &= \frac{x}{3} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = -\frac{x\,y^3}{3} + \frac{x}{3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= -\frac{w(x)\,x}{3} + \frac{x}{3} \\ w' &= -xw + x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= x \\ q(x) &= x \end{aligned}$$

Hence the ode is

$$w'(x) + w(x)\,x = x$$

The integrating factor μ is

$$\mu = e^{\int x dx} \\ = e^{\frac{x^2}{2}}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)(x)$$
$$\frac{d}{dx}\left(e^{\frac{x^2}{2}} w\right) = \left(e^{\frac{x^2}{2}}\right)(x)$$
$$d\left(e^{\frac{x^2}{2}} w\right) = \left(x\, e^{\frac{x^2}{2}}\right) dx$$

Integrating gives

$$e^{\frac{x^2}{2}} w = \int x\, e^{\frac{x^2}{2}}\, dx$$
$$e^{\frac{x^2}{2}} w = e^{\frac{x^2}{2}} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x^2}{2}}$ results in

$$w(x) = e^{\frac{x^2}{2}} e^{-\frac{x^2}{2}} + c_1 e^{-\frac{x^2}{2}}$$

which simplifies to

$$w(x) = 1 + c_1 e^{-\frac{x^2}{2}}$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = 1 + c_1 e^{-\frac{x^2}{2}}$$

Solving for y gives

$$y(x) = \left(1 + c_1 e^{-\frac{x^2}{2}}\right)^{\frac{1}{3}}$$
$$y(x) = \frac{\left(1 + c_1 e^{-\frac{x^2}{2}}\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2}$$
$$y(x) = -\frac{\left(1 + c_1 e^{-\frac{x^2}{2}}\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \left(1 + c_1 \mathrm{e}^{-\frac{x^2}{2}}\right)^{\frac{1}{3}} \tag{1}$$

$$y = \frac{\left(1 + c_1 \mathrm{e}^{-\frac{x^2}{2}}\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2} \tag{2}$$

$$y = -\frac{\left(1 + c_1 \mathrm{e}^{-\frac{x^2}{2}}\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2} \tag{3}$$

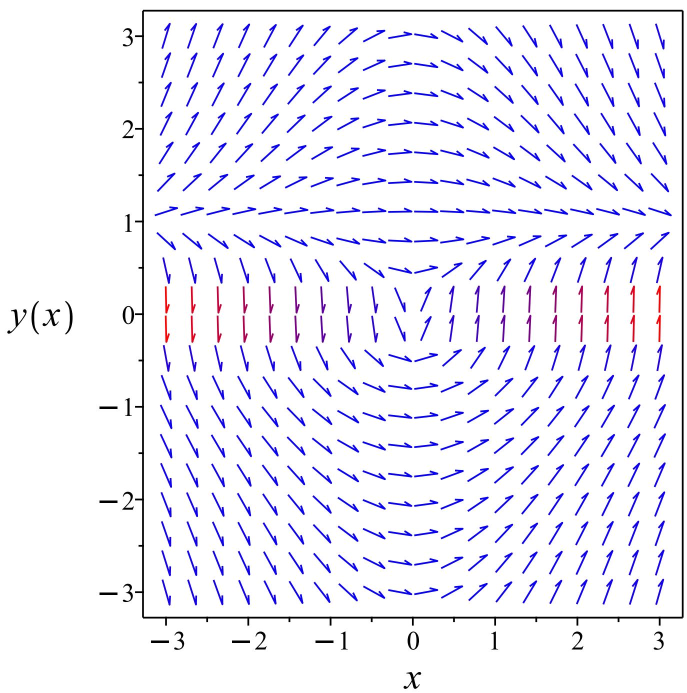


Figure 270: Slope field plot

Verification of solutions

$$y = \left(1+c_1 \mathrm{e}^{-\frac{x^2}{2}}\right)^{\frac{1}{3}}$$

Verified OK.

$$y = \frac{\left(1+c_1 \mathrm{e}^{-\frac{x^2}{2}}\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right)}{2}$$

Verified OK.

$$y = -\frac{\left(1+c_1 \mathrm{e}^{-\frac{x^2}{2}}\right)^{\frac{1}{3}} \left(1+i\sqrt{3}\right)}{2}$$

Verified OK.

7.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{3y^2}{y^3-1}\right)\mathrm{d}y = (x)\,\mathrm{d}x$$
$$(-x)\,\mathrm{d}x + \left(-\frac{3y^2}{y^3-1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -x$$
$$N(x,y) = -\frac{3y^2}{y^3-1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{3y^2}{y^3-1}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{3y^2}{y^3-1}$. Therefore equation (4) becomes

$$-\frac{3y^2}{y^3 - 1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{3y^2}{y^3 - 1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{3y^2}{y^3 - 1}\right) \mathrm{d}y$$

$$f(y) = -\ln\left(y^3 - 1\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \ln\left(y^3 - 1\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \ln\left(y^3 - 1\right)$$

Summary
The solution(s) found are the following

$$-\frac{x^2}{2} - \ln(y^3 - 1) = c_1 \tag{1}$$

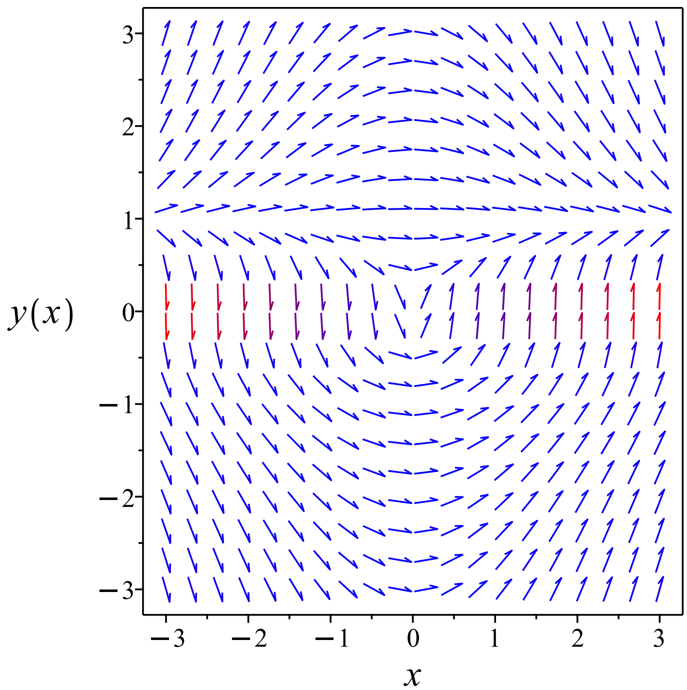


Figure 271: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} - \ln(y^3 - 1) = c_1$$

Verified OK.

7.12.5 Maple step by step solution

Let's solve

$x(1 - y^3) - 3y'y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'y^2}{1-y^3} = \frac{x}{3}$

- Integrate both sides with respect to x

  $\int \frac{y'y^2}{1-y^3}dx = \int \frac{x}{3}dx + c_1$

- Evaluate integral

  $-\frac{\ln(y^3-1)}{3} = \frac{x^2}{6} + c_1$

- Solve for y

  $y = \left(1 + e^{-\frac{x^2}{2}-3c_1}\right)^{\frac{1}{3}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 66

```
dsolve(x*(1-y(x)^3)-3*y(x)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \left(e^{-\frac{x^2}{2}}c_1 + 1\right)^{\frac{1}{3}}$$

$$y(x) = -\frac{\left(e^{-\frac{x^2}{2}}c_1 + 1\right)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2}$$

$$y(x) = \frac{\left(e^{-\frac{x^2}{2}}c_1 + 1\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2}$$

✓ Solution by Mathematica

Time used: 2.121 (sec). Leaf size: 111

```
DSolve[x*(1-y[x]^3)-3*y[x]^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \sqrt[3]{1+e^{-\frac{x^2}{2}+3c_1}} \\
y(x) &\to -\sqrt[3]{-1}\sqrt[3]{1+e^{-\frac{x^2}{2}+3c_1}} \\
y(x) &\to (-1)^{2/3}\sqrt[3]{1+e^{-\frac{x^2}{2}+3c_1}} \\
y(x) &\to 1 \\
y(x) &\to -\sqrt[3]{-1} \\
y(x) &\to (-1)^{2/3}
\end{aligned}
$$

7.13 problem 17

Internal problem ID [12698]
Internal file name [`OUTPUT/11350_Friday_November_03_2023_06_30_53_AM_43134312/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.3.3, page 71
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(2x-1)\, y+x(x+1)\, y'=0}$$

7.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= -\frac{(2x-1)\, y}{x\,(x+1)}
\end{aligned}$$

Where $f(x) = -\frac{2x-1}{x(x+1)}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= -\frac{2x-1}{x\left(x+1\right)}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{2x-1}{x\left(x+1\right)}\,dx \\
\ln\left(y\right) &= -3\ln\left(x+1\right)+\ln\left(x\right)+c_1 \\
y &= \mathrm{e}^{-3\ln(x+1)+\ln(x)+c_1} \\
&= c_1\mathrm{e}^{-3\ln(x+1)+\ln(x)}
\end{aligned}$$

Which simplifies to

$$y = \frac{c_1 x}{\left(x+1\right)^3}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x}{\left(x+1\right)^3} \tag{1}$$

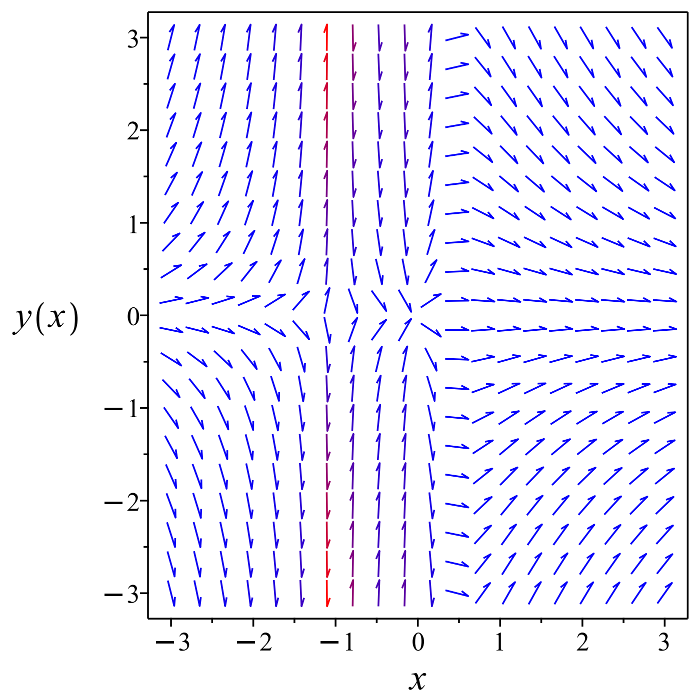


Figure 272: Slope field plot

Verification of solutions

$$y = \frac{c_1 x}{(x+1)^3}$$

Verified OK.

7.13.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{-2x+1}{x(x+1)} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{(-2x+1)\,y}{x(x+1)} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{-2x+1}{x(x+1)}dx} \\ &= \mathrm{e}^{3\ln(x+1)-\ln(x)} \end{aligned}$$

Which simplifies to

$$\mu = \frac{(x+1)^3}{x}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(x+1)^3\, y}{x}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{(x+1)^3\, y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{(x+1)^3}{x}$ results in

$$y = \frac{c_1 x}{(x+1)^3}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x}{(x+1)^3} \tag{1}$$

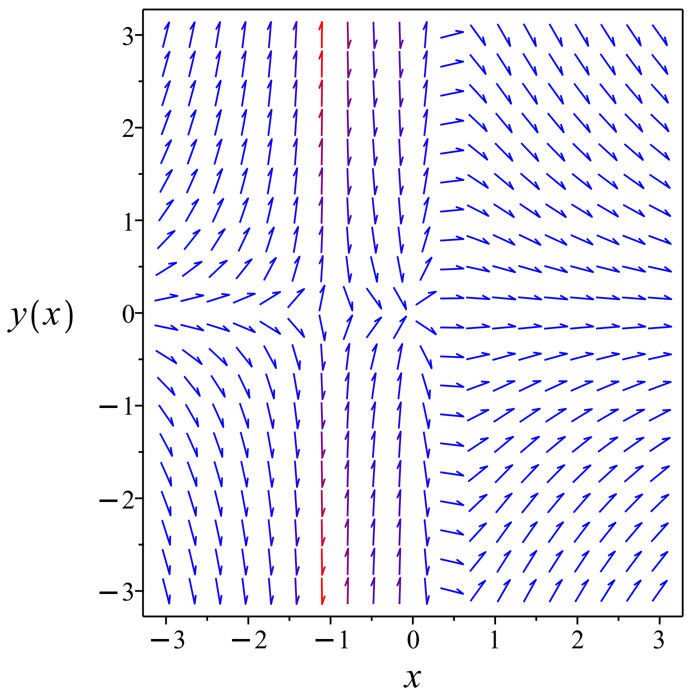


Figure 273: Slope field plot

Verification of solutions

$$y = \frac{c_1 x}{(x+1)^3}$$

Verified OK.

7.13.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$(2x-1)\, u(x)\, x + x(x+1)\,(u'(x)\, x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x+1} \end{aligned}$$

Where $f(x) = -\frac{3}{x+1}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{x+1}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3}{x+1}\, dx \\ \ln(u) &= -3\ln(x+1) + c_2 \\ u &= e^{-3\ln(x+1)+c_2} \\ &= \frac{c_2}{(x+1)^3} \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= \frac{x c_2}{(x+1)^3} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x c_2}{(x+1)^3} \tag{1}$$

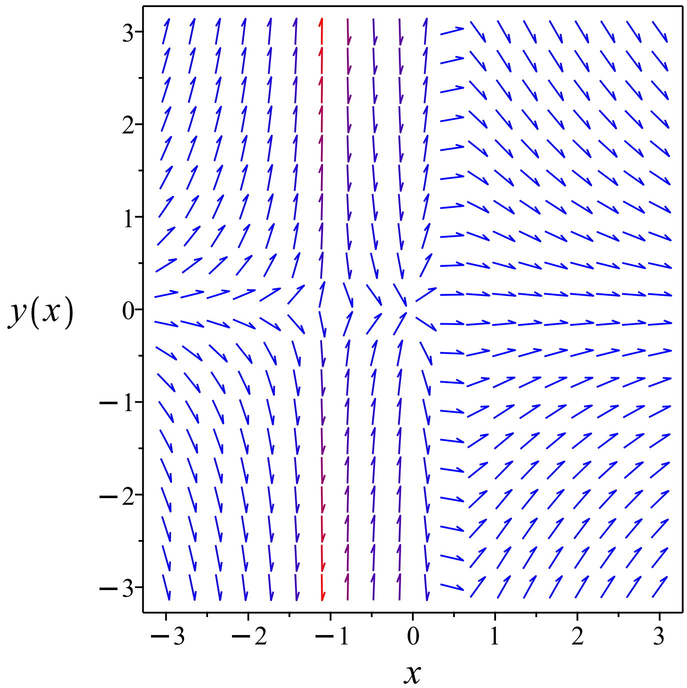


Figure 274: Slope field plot

Verification of solutions

$$y = \frac{x c_2}{(x+1)^3}$$

Verified OK.

7.13.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{(2x-1)\,y}{x\,(x+1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 225: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-3\ln(x+1)+\ln(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-3\ln(x+1)+\ln(x)}} dy \end{aligned}$$

Which results in

$$S = \frac{(x+1)^3 y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{(2x-1)\,y}{x\,(x+1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{(x+1)^2 y(2x-1)}{x^2} \\ S_y &= \frac{(x+1)^3}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(x+1)^3 y}{x} = c_1$$

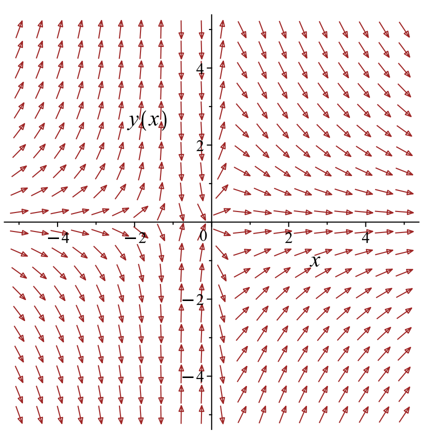
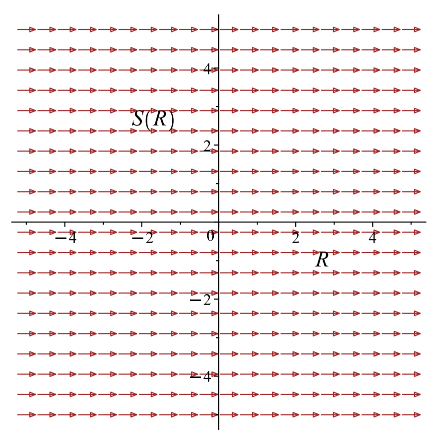
Which simplifies to

$$\frac{(x+1)^3 y}{x} = c_1$$

Which gives

$$y = \frac{c_1 x}{(x+1)^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{(2x-1)y}{x(x+1)}$ 	$R = x$ $S = \frac{(x+1)^3 y}{x}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = \frac{c_1 x}{(x+1)^3} \tag{1}$$

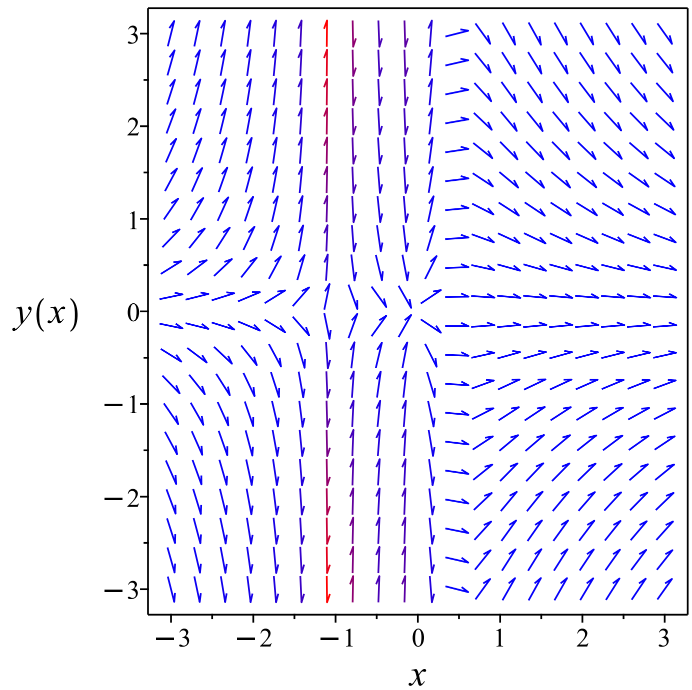


Figure 275: Slope field plot

Verification of solutions

$$y = \frac{c_1 x}{(x+1)^3}$$

Verified OK.

7.13.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y}\right)\mathrm{d}y&=\left(\frac{2x-1}{x\,(x+1)}\right)\mathrm{d}x\\ \left(-\frac{2x-1}{x\,(x+1)}\right)\mathrm{d}x+\left(-\frac{1}{y}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{2x-1}{x\,(x+1)}\\ N(x,y)&=-\frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{2x-1}{x\,(x+1)}\right)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \quad (1)\\
\frac{\partial \phi}{\partial y} &= N \quad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{2x-1}{x(x+1)} \mathrm{d}x\\
\phi &= -3\ln(x+1) + \ln(x) + f(y) \quad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \quad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$
-\frac{1}{y} = 0 + f'(y) \quad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y\\
f(y) &= -\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -3\ln(x+1) + \ln(x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -3\ln(x+1) + \ln(x) - \ln(y)$$

The solution becomes

$$y = \frac{x\,\mathrm{e}^{-c_1}}{(x+1)^3}$$

Summary

The solution(s) found are the following

$$y = \frac{x\,\mathrm{e}^{-c_1}}{(x+1)^3} \tag{1}$$

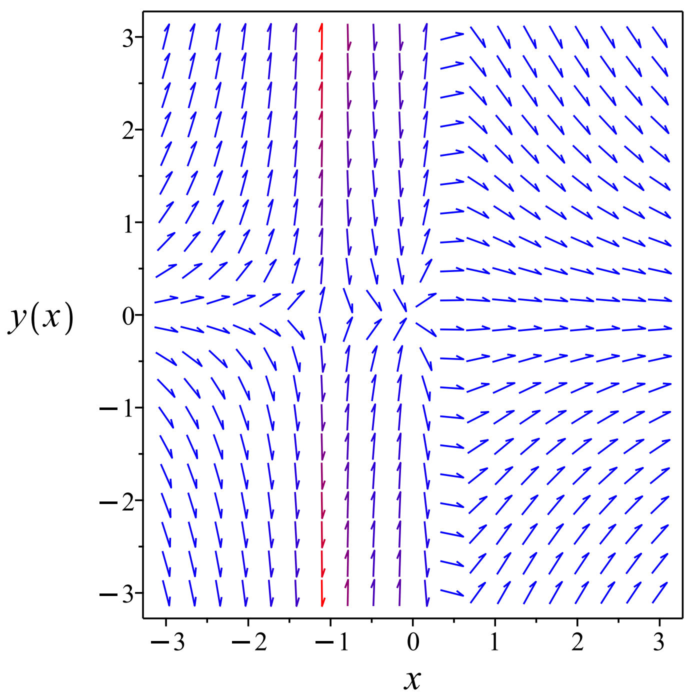


Figure 276: Slope field plot

Verification of solutions

$$y = \frac{x\,\mathrm{e}^{-c_1}}{(x+1)^3}$$

Verified OK.

7.13.6 Maple step by step solution

Let's solve

$(2x-1)\,y + x(x+1)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{2x-1}{x(x+1)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\frac{2x-1}{x(x+1)}dx + c_1$

- Evaluate integral

  $\ln(y) = -3\ln(x+1) + \ln(x) + c_1$

- Solve for y

  $y = \frac{\mathrm{e}^{c_1}x}{x^3+3x^2+3x+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(y(x)*(2*x-1)+x*(x+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x}{(1+x)^3}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 19

```
DSolve[y[x]*(2*x-1)+x*(x+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 x}{(x+1)^3}$$
$$y(x) \to 0$$

8 Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115

8.1 problem 1

Internal problem ID [12699]
Internal file name [`OUTPUT/11351_Friday_November_03_2023_06_30_53_AM_27644608/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = \frac{1}{x-1}}$$

With initial conditions

$$[y(0) = 1]$$

8.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= \frac{1}{x-1} \end{aligned}$$

Hence the ode is

$$y' = \frac{1}{x-1}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1}{x-1}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.1.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{x-1} \, \mathrm{d}x \\ &= \ln(x-1) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = i\pi + c_1$$

$$c_1 = -i\pi + 1$$

Substituting c_1 found above in the general solution gives

$$y = \ln(x-1) + 1 - i\pi$$

Summary
The solution(s) found are the following

$$y = \ln(x-1) + 1 - i\pi \tag{1}$$

Verification of solutions

$$y = \ln(x-1) + 1 - i\pi$$

Verified OK.

8.1.3 Maple step by step solution

Let's solve

$\left[y' = \frac{1}{x-1}, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{1}{x-1} dx + c_1$

- Evaluate integral

  $y = \ln(x-1) + c_1$

- Solve for y

  $y = \ln(x-1) + c_1$

- Use initial condition $y(0) = 1$

  $1 = I\pi + c_1$

- Solve for c_1

  $c_1 = 1 - I\pi$

- Substitute $c_1 = 1 - I\pi$ into general solution and simplify

  $y = \ln(x-1) + 1 - I\pi$

- Solution to the IVP

  $y = \ln(x-1) + 1 - I\pi$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)=1/(x-1),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \ln(-1 + x) + 1 - i\pi$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 16

```
DSolve[{y'[x]==1/(x-1),{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(x - 1) - i\pi + 1$$

8.2 problem 2

Internal problem ID [12700]
Internal file name [`OUTPUT/11352_Friday_November_03_2023_06_30_54_AM_67993084/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = x}$$

With initial conditions

$$[y(0) = 0]$$

8.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = x$$

Hence the ode is

$$y' - y = x$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.2.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-1)dx} \\ &= e^{-x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(x) \\ \frac{d}{dx}(e^{-x} y) &= (e^{-x})(x) \\ d(e^{-x} y) &= (x\, e^{-x})\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-x} y &= \int x\, e^{-x}\, dx \\ e^{-x} y &= -(x+1)\, e^{-x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = -e^{x}(x+1)\, e^{-x} + e^{x} c_1$$

which simplifies to

$$y = e^{x} c_1 - x - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -1 + c_1$$

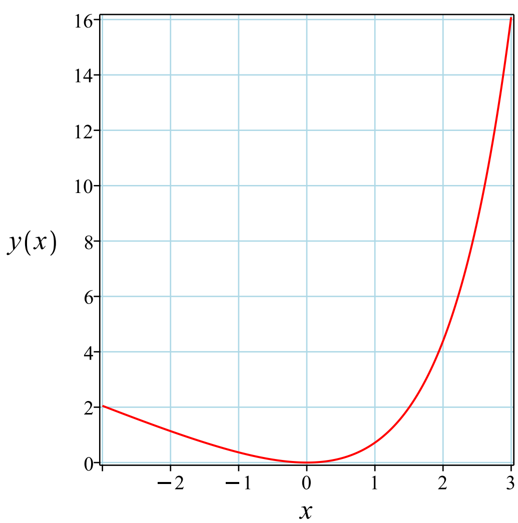
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

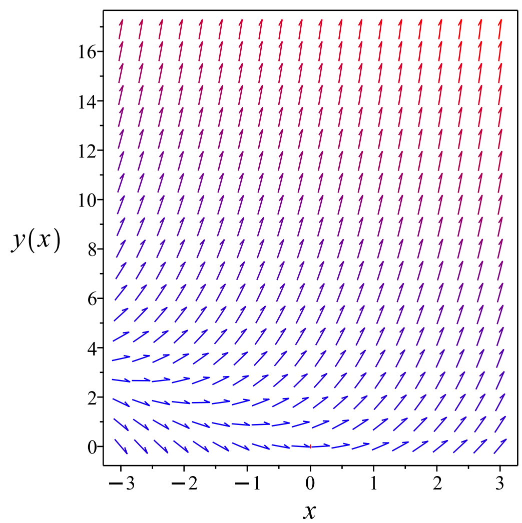
$$y = -1 + e^x - x$$

Summary
The solution(s) found are the following

$$y = -1 + e^x - x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -1 + e^x - x$$

Verified OK.

8.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= y + x \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 229: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{e^{x}} dy
\end{aligned}
$$

Which results in

$$
S = e^{-x} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = y + x
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -e^{-x} y \\
S_y &= e^{-x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x\,\mathrm{e}^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\,\mathrm{e}^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -(R+1)\,\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-x} y = -(x+1)\,\mathrm{e}^{-x} + c_1$$

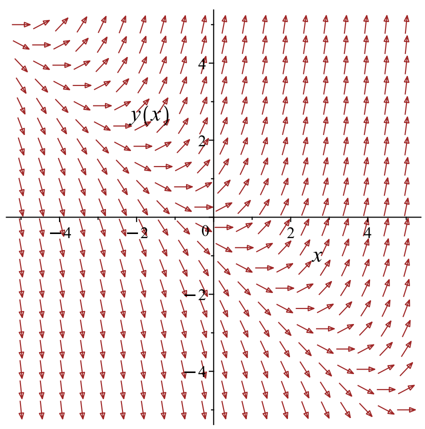
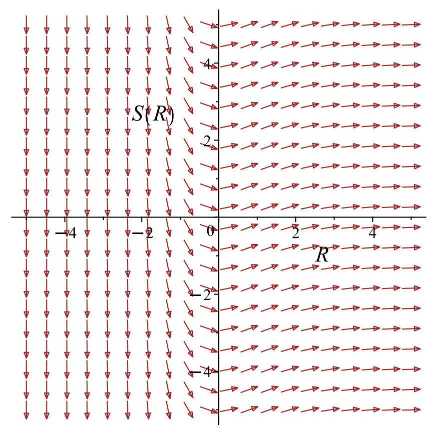
Which simplifies to

$$(x+y+1)\,\mathrm{e}^{-x} - c_1 = 0$$

Which gives

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y + x$ 	$R = x$ $S = \mathrm{e}^{-x} y$	$\frac{dS}{dR} = R\,\mathrm{e}^{-R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -1 + c_1$$

$$c_1 = 1$$

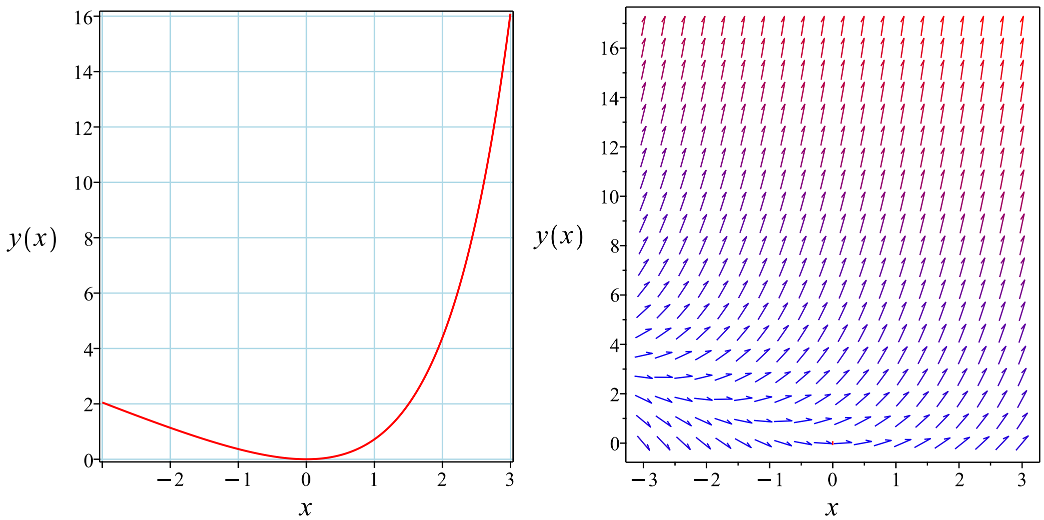
Substituting c_1 found above in the general solution gives

$$y = -1 + \mathrm{e}^{x} - x$$

<u>Summary</u>

The solution(s) found are the following

$$y = -1 + \mathrm{e}^{x} - x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -1 + e^{x} - x$$

Verified OK.

8.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\mathrm{d}y &= (y + x)\,\mathrm{d}x \\
(-y - x)\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -y - x \\
N(x, y) &= 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y - x) \\
&= -1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-1) - (0)) \\
&= -1
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -1 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-x} \\ &= \mathrm{e}^{-x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-x}(-y - x) \\ &= -\mathrm{e}^{-x}(y + x)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{-x}(y + x)\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\mathrm{e}^{-x}(y + x) \, \mathrm{d}x\end{aligned}$$

$$\phi = (x + y + 1)\,\mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (x + y + 1)\,\mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (x + y + 1)\,\mathrm{e}^{-x}$$

The solution becomes

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -1 + c_1$$

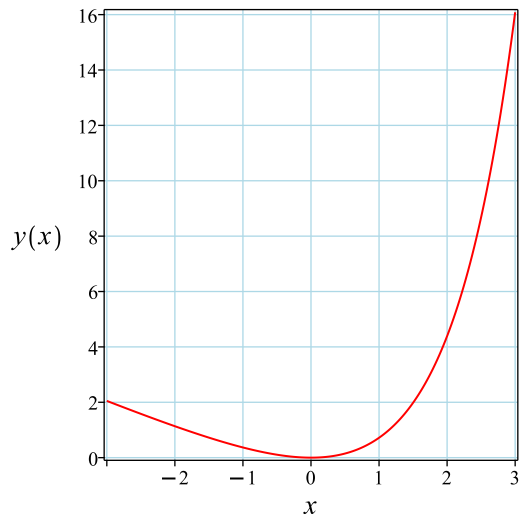
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

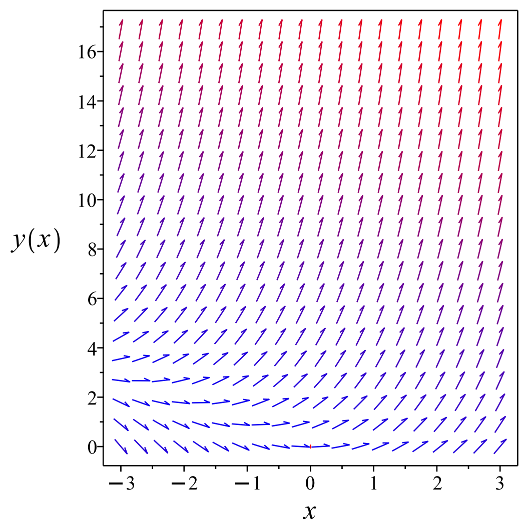
$$y = -1 + \mathrm{e}^{x} - x$$

Summary
The solution(s) found are the following

$$y = -1 + e^x - x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -1 + e^x - x$$

Verified OK.

8.2.5 Maple step by step solution

Let's solve

$[y' - y = x, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y'-y\right)=\mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'-y\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=-\mu(x)$

- Solve to find the integrating factor

  $\mu(x)=\mathrm{e}^{-x}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\,xdx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\mu(x)\,xdx+c_1$

- Solve for y

  $y=\frac{\int\mu(x)xdx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\mathrm{e}^{-x}$

  $y=\frac{\int x\,\mathrm{e}^{-x}dx+c_1}{\mathrm{e}^{-x}}$

- Evaluate the integrals on the rhs

  $y=\frac{-(x+1)\mathrm{e}^{-x}+c_1}{\mathrm{e}^{-x}}$

- Simplify

  $y=\mathrm{e}^{x}c_1-x-1$

- Use initial condition $y(0)=0$

  $0=-1+c_1$

- Solve for c_1

  $c_1=1$

- Substitute $c_1=1$ into general solution and simplify

  $y=-1+\mathrm{e}^{x}-x$

- Solution to the IVP

  $y=-1+\mathrm{e}^{x}-x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve([diff(y(x),x)=y(x)+x,y(0) = 0],y(x), singsol=all)
```

$$y(x) = -x - 1 + e^{x}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 13

```
DSolve[{y'[x]==y[x]+x,{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x + e^{x} - 1$$

8.3 problem 3 (a)

Internal problem ID [12701]
Internal file name [`OUTPUT/11353_Friday_November_03_2023_06_30_54_AM_25771254/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 3 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y}{x} = 0}$$

With initial conditions

$$[y(-1) = 1]$$

8.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{y}{x} = 0$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -1$ is inside this domain. Hence solution exists and is unique.

8.3.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\, dy &= \frac{1}{x}\, dx \\ \int \frac{1}{y}\, dy &= \int \frac{1}{x}\, dx \\ \ln(y) &= \ln(x) + c_1 \\ y &= e^{\ln(x)+c_1} \\ &= c_1 x \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -c_1$$

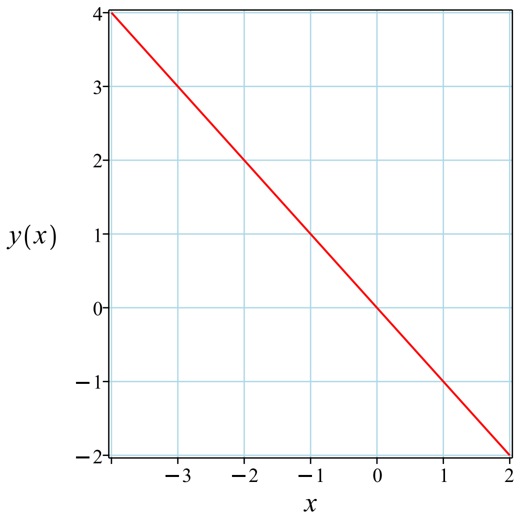
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

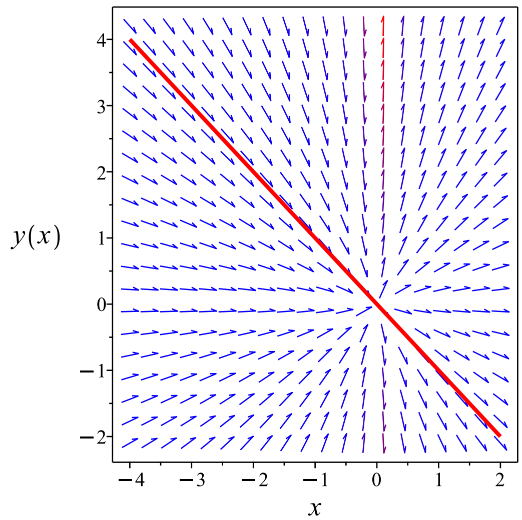
$$y = -x$$

Summary
The solution(s) found are the following

$$y = -x \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -x$$

Verified OK.

8.3.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -c_1$$

$$c_1 = -1$$

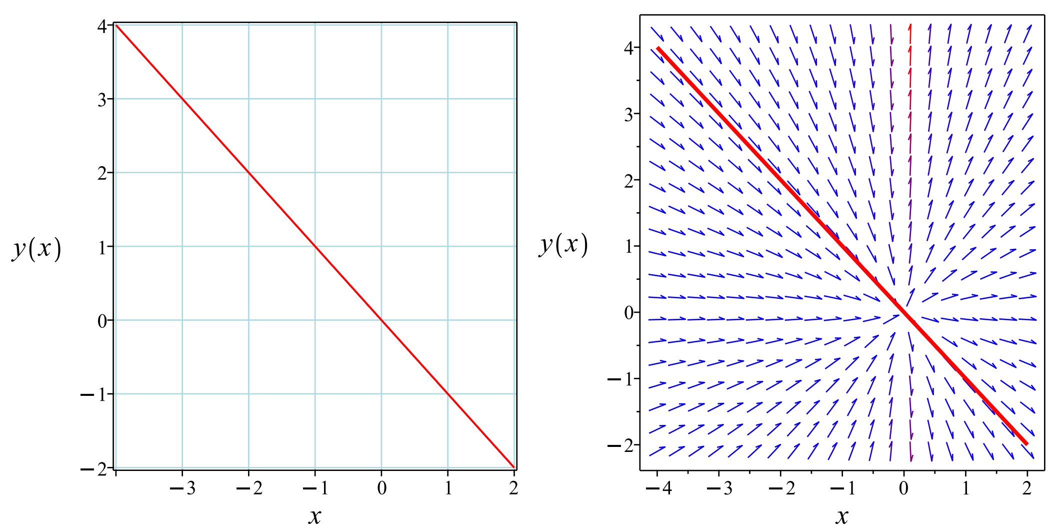
Substituting c_1 found above in the general solution gives

$$y = -x$$

Summary

The solution(s) found are the following

$$y = -x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -x$$

Verified OK.

8.3.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x = 0$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0\,\mathrm{d}x \\ &= c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= c_2 x \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -c_2$$

$$c_2 = -1$$

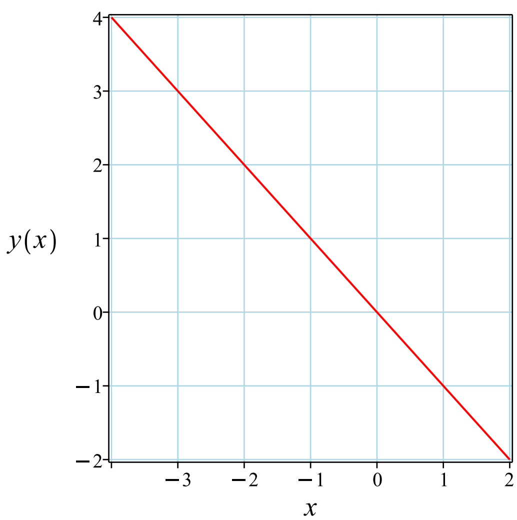
Substituting c_2 found above in the general solution gives

$$y = -x$$

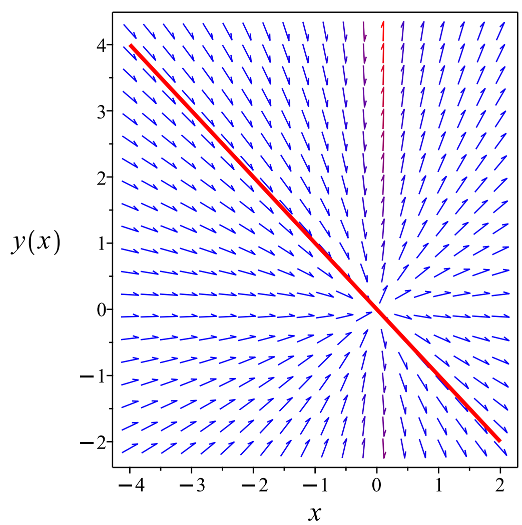
Summary

The solution(s) found are the following

$$y = -x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -x$$

Verified OK.

8.3.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y}{x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 232: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = c_1$$

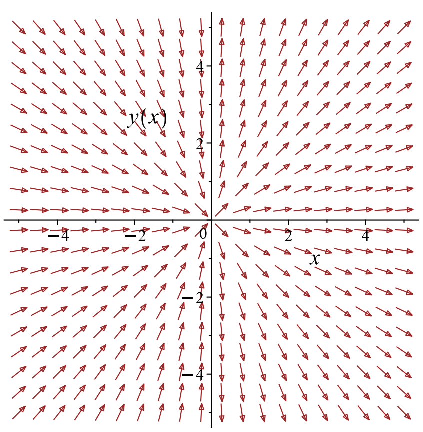
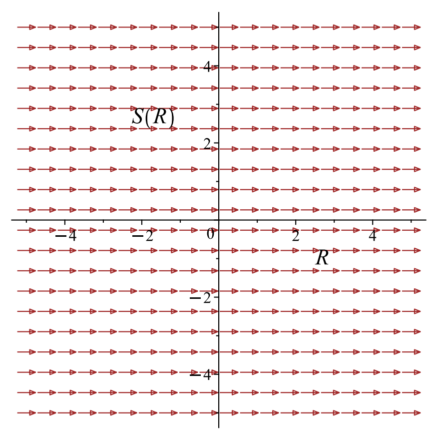
Which simplifies to

$$\frac{y}{x} = c_1$$

Which gives

$$y = c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -c_1$$

$$c_1 = -1$$

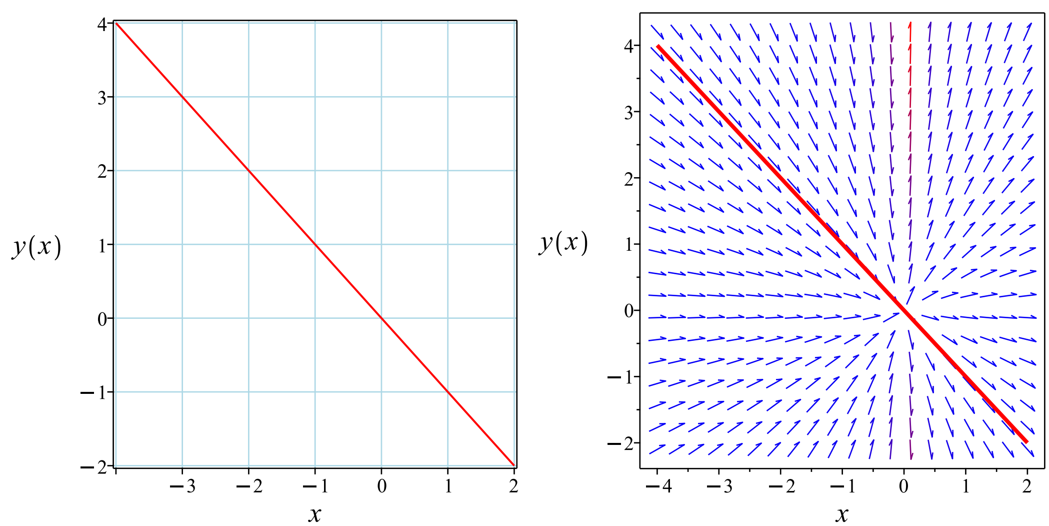
Substituting c_1 found above in the general solution gives

$$y = -x$$

Summary

The solution(s) found are the following

$$y = -x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -x$$

Verified OK.

8.3.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y&=\left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x+\left(\frac{1}{y}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{x}\\ N(x,y)&=\frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &=0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{x} \, \mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y)$$

The solution becomes

$$y = e^{c_1} x$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -e^{c_1}$$

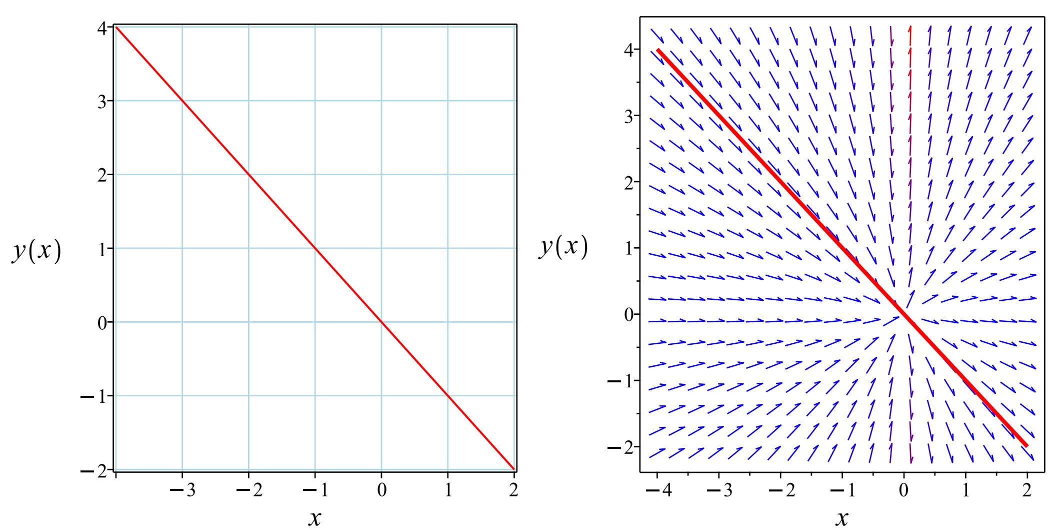
$$c_1 = i\pi$$

Substituting c_1 found above in the general solution gives

$$y = -x$$

Summary
The solution(s) found are the following

$$y = -x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -x$$

Verified OK.

8.3.7 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{x} = 0, y(-1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = e^{c_1}x$

- Use initial condition $y(-1) = 1$

  $1 = -e^{c_1}$

- Solve for c_1

  $c_1 = I\pi$

- Substitute $c_1 = I\pi$ into general solution and simplify

  $y = -x$

- Solution to the IVP

  $y = -x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve([diff(y(x),x)=y(x)/x,y(-1) = 1],y(x), singsol=all)
```

$$y(x) = -x$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 8

```
DSolve[{y'[x]==y[x]/x,{y[-1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x$$

8.4 problem 3 (b)

Internal problem ID [12702]
Internal file name [`OUTPUT/11354_Friday_November_03_2023_06_30_55_AM_48827140/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 3 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y}{x} = 0}$$

With initial conditions

$$[y(-1) = -1]$$

8.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{y}{x} = 0$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -1$ is inside this domain. Hence solution exists and is unique.

8.4.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\, dy &= \frac{1}{x}\, dx \\ \int \frac{1}{y}\, dy &= \int \frac{1}{x}\, dx \\ \ln(y) &= \ln(x) + c_1 \\ y &= \mathrm{e}^{\ln(x)+c_1} \\ &= c_1 x \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -c_1$$

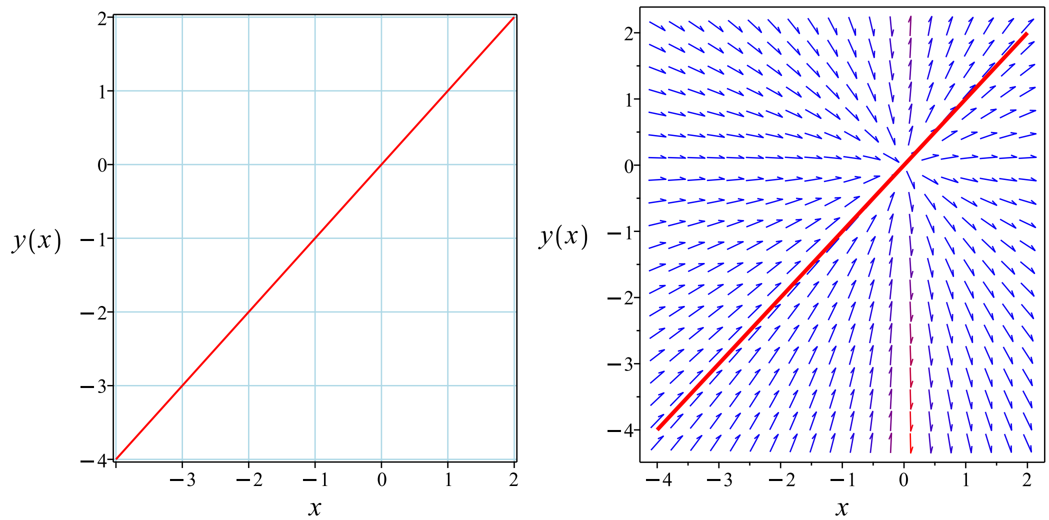
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = x$$

Summary
The solution(s) found are the following

$$y = x \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = x$$

Verified OK.

8.4.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = 0$$

Integrating gives

$$\frac{y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -c_1$$

$$c_1 = 1$$

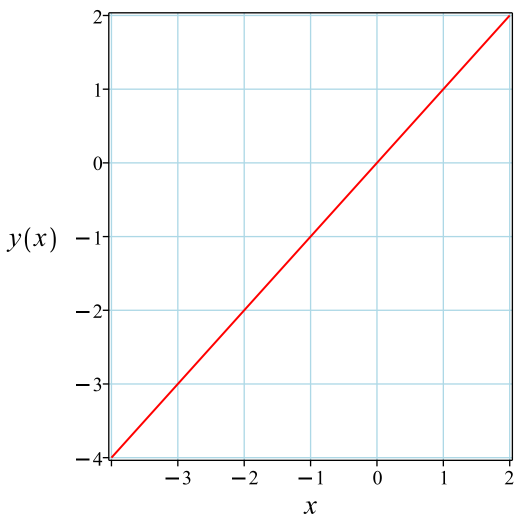
Substituting c_1 found above in the general solution gives

$$y = x$$

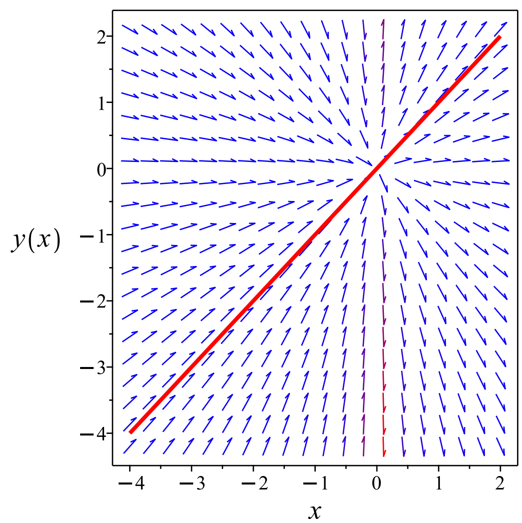
Summary

The solution(s) found are the following

$$y = x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x$$

Verified OK.

8.4.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x = 0$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0\,\mathrm{d}x \\ &= c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= c_2 x \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -c_2$$

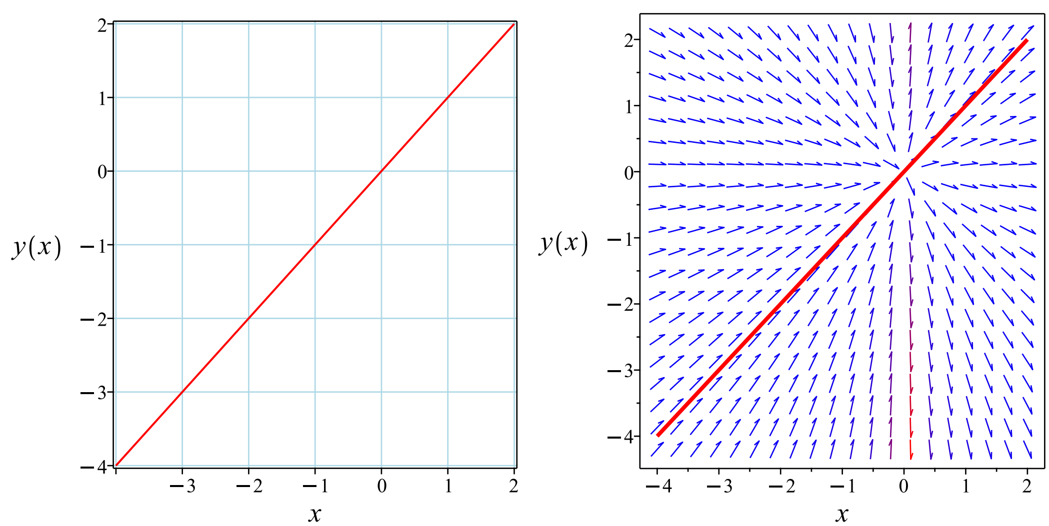
$$c_2 = 1$$

Substituting c_2 found above in the general solution gives

$$y = x$$

Summary
The solution(s) found are the following

$$y = x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x$$

Verified OK.

8.4.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 235: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = c_1$$

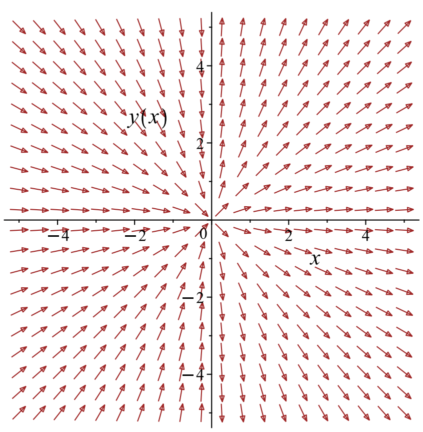
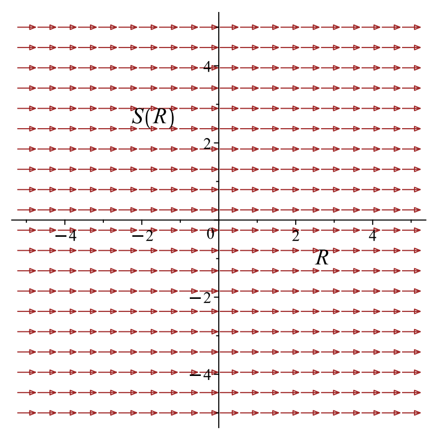
Which simplifies to

$$\frac{y}{x} = c_1$$

Which gives

$$y = c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -c_1$$

$$c_1 = 1$$

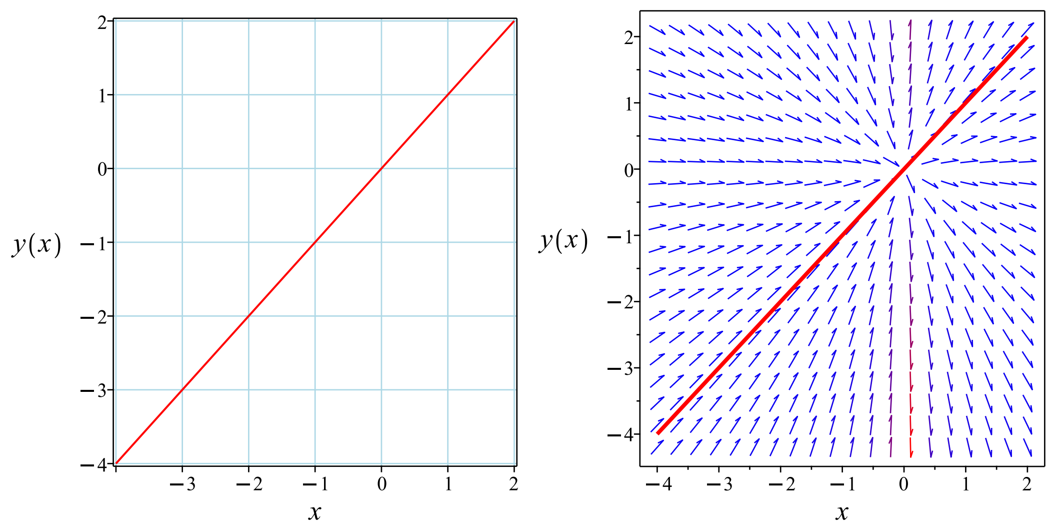
Substituting c_1 found above in the general solution gives

$$y = x$$

Summary

The solution(s) found are the following

$$y = x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x$$

Verified OK.

8.4.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{1}{y}\right) \mathrm{d} y &= \left(\frac{1}{x}\right) \mathrm{d} x \\
\left(-\frac{1}{x}\right) \mathrm{d} x+\left(\frac{1}{y}\right) \mathrm{d} y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= \frac{1}{y}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y)$$

The solution becomes

$$y = e^{c_1} x$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -e^{c_1}$$

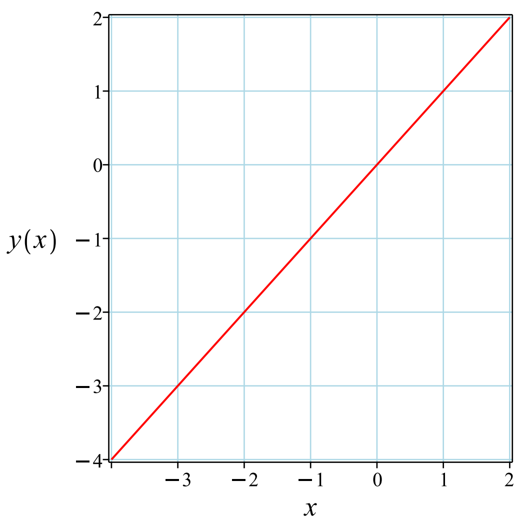
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

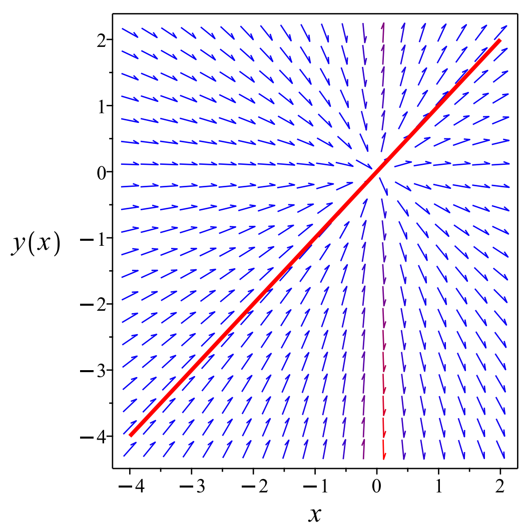
$$y = x$$

Summary
The solution(s) found are the following

$$y = x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x$$

Verified OK.

8.4.7 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{x} = 0, y(-1) = -1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = e^{c_1} x$

- Use initial condition $y(-1) = -1$

  $-1 = -e^{c_1}$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = x$

- Solution to the IVP

  $y = x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=y(x)/x,y(-1) = -1],y(x), singsol=all)
```

$$y(x) = x$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 6

```
DSolve[{y'[x]==y[x]/x,{y[-1]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x$$

8.5 problem 4 (a)

Internal problem ID [12703]
Internal file name [`OUTPUT/11355_Friday_November_03_2023_06_30_56_AM_16141346/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 4 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' - \frac{y}{-x^2 + 1} = \sqrt{x}}$$

With initial conditions

$$\left[y\left(\frac{1}{2}\right) = 1\right]$$

8.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x^2 - 1}$$
$$q(x) = \sqrt{x}$$

Hence the ode is

$$y' + \frac{y}{x^2 - 1} = \sqrt{x}$$

The domain of $p(x) = \frac{1}{x^2-1}$ is

$$\{-\infty \le x < -1, -1 < x < 1, 1 < x \le \infty\}$$

And the point $x_0 = \frac{1}{2}$ is inside this domain. The domain of $q(x) = \sqrt{x}$ is

$$\{0 \le x\}$$

And the point $x_0 = \frac{1}{2}$ is also inside this domain. Hence solution exists and is unique.

8.5.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x^2-1}dx} \\ &= \frac{\sqrt{-x^2+1}}{x+1}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\sqrt{x}\right) \\ \frac{d}{dx}\left(\frac{\sqrt{-x^2+1}\,y}{x+1}\right) &= \left(\frac{\sqrt{-x^2+1}}{x+1}\right)\left(\sqrt{x}\right) \\ d\left(\frac{\sqrt{-x^2+1}\,y}{x+1}\right) &= \left(\frac{\sqrt{x}\,\sqrt{-x^2+1}}{x+1}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{\sqrt{-x^2+1}\,y}{x+1} &= \int \frac{\sqrt{x}\,\sqrt{-x^2+1}}{x+1}\,dx \\ \frac{\sqrt{-x^2+1}\,y}{x+1} &= \frac{\frac{2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \mathrm{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)}{3} - 2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \mathrm{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) - 2}{\sqrt{x}\,\sqrt{-x^2+1}}\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{-x^2+1}}{x+1}$ results in

$$y = \frac{2(x+1)\left(\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) - 3\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticE}\left(\sqrt{x}\right.\right.}{3\left(-x^2+1\right)\sqrt{x}}$$

which simplifies to

$$y = \frac{-2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) + 6\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{}}{2}\right.}{\sqrt{x}\,(-3+3x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\sqrt{3}\left(4i\sqrt{2}\ \text{EllipticF}\left(\frac{\sqrt{2}\,\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right) - 12i\sqrt{2}\ \text{EllipticE}\left(\frac{\sqrt{2}\,\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right) + \sqrt{2}\,\sqrt{3} + 6c_1\right)}{6}$$

$$c_1 = -\frac{\left(4i\sqrt{3}\,\sqrt{2}\ \text{EllipticF}\left(\frac{\sqrt{2}\,\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right) - 12i\sqrt{3}\,\sqrt{2}\ \text{EllipticE}\left(\frac{\sqrt{2}\,\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right) + 3\sqrt{2} - 6\right)\sqrt{3}}{18}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{4x^3\sqrt{-x^2+1} - 4ix^{\frac{5}{2}}\sqrt{2}\ \text{EllipticF}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right) + 12ix^{\frac{5}{2}}\sqrt{2}\ \text{EllipticE}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right) - x^{\frac{5}{2}}\sqrt{3}\,\sqrt{2} - 4\sqrt{-x^2+}}{}$$

Summary

The solution(s) found are the following

$$y = \frac{4x^3\sqrt{-x^2+1} - 4ix^{\frac{5}{2}}\sqrt{2}\ \text{EllipticF}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right) + 12ix^{\frac{5}{2}}\sqrt{2}\ \text{EllipticE}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right) - x^{\frac{5}{2}}\sqrt{3}\,\sqrt{2} - 4\sqrt{-x^2+1}}{} \tag{1}$$

Verification of solutions

$$y = \frac{4x^3\sqrt{-x^2+1} - 4ix^{\frac{5}{2}}\sqrt{2}\ \text{EllipticF}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right) + 12ix^{\frac{5}{2}}\sqrt{2}\ \text{EllipticE}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right) - x^{\frac{5}{2}}\sqrt{3}\,\sqrt{2} - 4\sqrt{-x^2+1}}{}$$

Verified OK.

8.5.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x^{\frac{5}{2}} + \sqrt{x} + y}{x^2 - 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 238: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f(\frac{y}{x})$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F(\frac{y}{x})$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x+1}{\sqrt{-x^2+1}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x+1}{\sqrt{-x^2+1}}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\sqrt{-x^2+1}\, y}{x+1}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{-x^{\frac{5}{2}} + \sqrt{x} + y}{x^2 - 1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y}{(x+1)\sqrt{-x^2+1}} \\
S_y &= \frac{\sqrt{-x^2+1}}{x+1}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{(1-x)\sqrt{x}}{\sqrt{-x^2+1}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{(1-R)\sqrt{R}}{\sqrt{-R^2+1}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{\frac{2\sqrt{R+1}\,\sqrt{2-2R}\,\sqrt{-R}\,\operatorname{EllipticF}\left(\sqrt{R+1}, \frac{\sqrt{2}}{2}\right)}{3} - 2\sqrt{R+1}\,\sqrt{2-2R}\,\sqrt{-R}\,\operatorname{EllipticE}\left(\sqrt{R+1}, \frac{\sqrt{2}}{2}\right) - \frac{2R^3}{3} + \frac{2I}{3}}{\sqrt{-R^2+1}\,\sqrt{R}} \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\sqrt{-x^2+1}\,y}{x+1} = \frac{\frac{2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\operatorname{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)}{3} - 2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\operatorname{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) - 2}{\sqrt{x}\,\sqrt{-x^2+1}}
$$

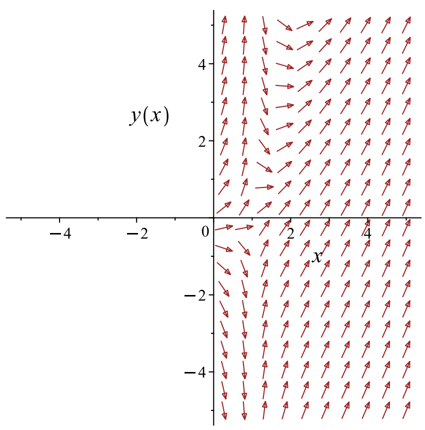
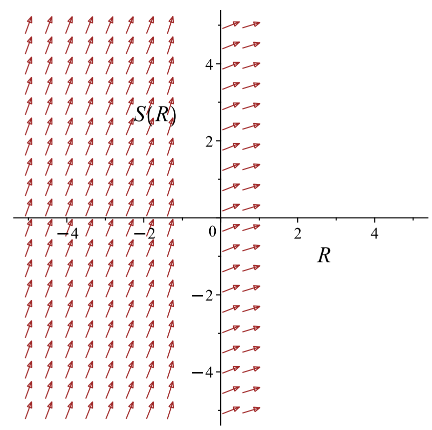
Which simplifies to

$$
\frac{\sqrt{-x^2+1}\,y}{x+1} = \frac{\frac{2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\operatorname{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)}{3} - 2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\operatorname{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) - 2}{\sqrt{x}\,\sqrt{-x^2+1}}
$$

Which gives

$$
y = -\frac{2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\operatorname{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)x - 6\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\operatorname{EllipticE}\left(\sqrt{x+1},\right.}{}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^{\frac{5}{2}}+\sqrt{x}+y}{x^2-1}$ 	$R = x$ $S = \frac{\sqrt{-x^2+1}\,y}{x+1}$	$\frac{dS}{dR} = \frac{(1-R)\sqrt{R}}{\sqrt{-R^2+1}}$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\sqrt{3}\left(4i\sqrt{2}\,\text{EllipticF}\left(\frac{\sqrt{2}\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right) - 12i\sqrt{2}\,\text{EllipticE}\left(\frac{\sqrt{2}\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right) + \sqrt{2}\sqrt{3} + 6c_1\right)}{6}$$

$$c_1 = -\frac{\left(4i\sqrt{3}\sqrt{2}\,\text{EllipticF}\left(\frac{\sqrt{2}\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right) - 12i\sqrt{3}\sqrt{2}\,\text{EllipticE}\left(\frac{\sqrt{2}\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right) + 3\sqrt{2} - 6\right)\sqrt{3}}{18}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{4i\sqrt{2}\,x^{\frac{3}{2}}\sqrt{-x^2+1}\,\text{EllipticF}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right) + 4i\sqrt{2}\,\sqrt{x}\,\sqrt{-x^2+1}\,\text{EllipticF}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right) + 4\,\text{EllipticF}\left(\sqrt{x+\right.}{}$$

Summary

The solution(s) found are the following

$$y = \frac{4i\sqrt{2}\,x^{\frac{3}{2}}\sqrt{-x^2+1}\,\text{EllipticF}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right) + 4i\sqrt{2}\,\sqrt{x}\,\sqrt{-x^2+1}\,\text{EllipticF}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right) + 4\,\text{EllipticF}\left(\sqrt{x+1,\right.}{} \tag{1}$$

Verification of solutions

$$y = \frac{4i\sqrt{2}\,x^{\frac{3}{2}}\sqrt{-x^2+1}\,\text{EllipticF}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right) + 4i\sqrt{2}\,\sqrt{x}\,\sqrt{-x^2+1}\,\text{EllipticF}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right) + 4\,\text{EllipticF}\left(\sqrt{x+1}\right.}{}$$

Verified OK.

8.5.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$dy = \left(\frac{y}{-x^2+1} + \sqrt{x}\right) dx$$

$$\left(-\frac{y}{-x^2+1} - \sqrt{x}\right) dx + dy = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{y}{-x^2+1} - \sqrt{x}$$
$$N(x,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{-x^2+1} - \sqrt{x}\right)\\ &= \frac{1}{x^2-1}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(-\frac{1}{-x^2+1}\right) - (0)\right)\\ &= \frac{1}{x^2-1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\, dx}\\ &= e^{\int \frac{1}{x^2-1}\, dx}\end{aligned}$$

The result of integrating gives

$$\mu = e^{-\operatorname{arctanh}(x)}$$
$$= \frac{\sqrt{-x^2+1}}{x+1}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{\sqrt{-x^2+1}}{x+1}\left(-\frac{y}{-x^2+1} - \sqrt{x}\right) \\ &= -\frac{-x^{\frac{5}{2}} + \sqrt{x} + y}{(x+1)\sqrt{-x^2+1}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{\sqrt{-x^2+1}}{x+1}(1) \\ &= \frac{\sqrt{-x^2+1}}{x+1}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-\frac{-x^{\frac{5}{2}} + \sqrt{x} + y}{(x+1)\sqrt{-x^2+1}}\right) + \left(\frac{\sqrt{-x^2+1}}{x+1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{-x^{\frac{5}{2}} + \sqrt{x} + y}{(x+1)\sqrt{-x^2+1}}\,\mathrm{d}x$$
$$\phi = \int_{\frac{1}{2}}^{x} -\frac{-\_a^{\frac{5}{2}} + \sqrt{\_a} + y}{(\_a+1)\sqrt{-\_a^2+1}} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=\int_{\frac{1}{2}}^{x}-\frac{1}{(\_a+1)\sqrt{-\_a^{2}+1}}d\_a+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\frac{\sqrt{-x^2+1}}{x+1}$. Therefore equation (4) becomes

$$\frac{\sqrt{-x^{2}+1}}{x+1}=-\left(\int_{\frac{1}{2}}^{x}\frac{1}{(\_a+1)\sqrt{-\_a^{2}+1}}d\_a\right)+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y)&=\frac{\left(\int_{\frac{1}{2}}^{x}\frac{1}{(\_a+1)\sqrt{-\_a^{2}+1}}d\_a\right)x+\int_{\frac{1}{2}}^{x}\frac{1}{(\_a+1)\sqrt{-\_a^{2}+1}}d\_a+\sqrt{-x^{2}+1}}{x+1}\\&=\frac{(x+1)\left(\int_{\frac{1}{2}}^{x}\frac{1}{(\_a+1)\sqrt{-\_a^{2}+1}}d\_a\right)+\sqrt{-x^{2}+1}}{x+1}\end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned}\int f'(y)\,\mathrm{d}y&=\int\left(\frac{(x+1)\left(\int_{\frac{1}{2}}^{x}\frac{1}{(\_a+1)\sqrt{-\_a^{2}+1}}d\_a\right)+\sqrt{-x^{2}+1}}{x+1}\right)\mathrm{d}y\\f(y)&=\frac{\left((x+1)\left(\int_{\frac{1}{2}}^{x}\frac{1}{(\_a+1)\sqrt{-\_a^{2}+1}}d\_a\right)+\sqrt{-x^{2}+1}\right)y}{x+1}+c_{1}\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\begin{aligned}\phi=&\int_{\frac{1}{2}}^{x}-\frac{-\_a^{\frac{5}{2}}+\sqrt{\_a}+y}{(\_a+1)\sqrt{-\_a^{2}+1}}d\_a\\&+\frac{\left((x+1)\left(\int_{\frac{1}{2}}^{x}\frac{1}{(\_a+1)\sqrt{-\_a^{2}+1}}d\_a\right)+\sqrt{-x^{2}+1}\right)y}{x+1}+c_{1}\end{aligned}$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int_{\frac{1}{2}}^{x} -\frac{-\_a^{\frac{5}{2}} + \sqrt{\_a} + y}{(\_a+1)\sqrt{-\_a^2+1}} d\_a + \frac{\left((x+1)\left(\int_{\frac{1}{2}}^{x} \frac{1}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + \sqrt{-x^2+1}\right) y}{x+1}$$

The solution becomes

$$y = \frac{c_1 x - x\left(\int_{\frac{1}{2}}^{x} \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + x\left(\int_{\frac{1}{2}}^{x} \frac{\sqrt{\_a}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + c_1 - \left(\int_{\frac{1}{2}}^{x} \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + \int}{\sqrt{-x^2+1}}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{2}$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \sqrt{3}\, c_1$$

$$c_1 = \frac{\sqrt{3}}{3}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-3x\left(\int_{\frac{1}{2}}^{x} \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + 3x\left(\int_{\frac{1}{2}}^{x} \frac{\sqrt{\_a}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + \sqrt{3}\, x - 3\left(\int_{\frac{1}{2}}^{x} \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right)}{3\sqrt{-x^2+1}}$$

Summary

The solution(s) found are the following

$$y = \frac{-3x\left(\int_{\frac{1}{2}}^{x} \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + 3x\left(\int_{\frac{1}{2}}^{x} \frac{\sqrt{\_a}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + \sqrt{3}\, x - 3\left(\int_{\frac{1}{2}}^{x} \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) +}{3\sqrt{-x^2+1}} \quad (1)$$

Verification of solutions

$$y = \frac{-3x\left(\int_{\frac{1}{2}}^{x} \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + 3x\left(\int_{\frac{1}{2}}^{x} \frac{\sqrt{\_a}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + \sqrt{3}\, x - 3\left(\int_{\frac{1}{2}}^{x} \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) +}{3\sqrt{-x^2+1}}$$

Verified OK.

8.5.5 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{-x^2+1} = \sqrt{x}, y\left(\frac{1}{2}\right) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x^2-1} + \sqrt{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x^2-1} = \sqrt{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x^2-1}\right) = \mu(x)\sqrt{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x^2-1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x^2-1}$

- Solve to find the integrating factor

  $\mu(x) = \frac{\sqrt{-(x-1)(x+1)}}{x+1}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\sqrt{x}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\sqrt{x}dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\sqrt{x}dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{\sqrt{-(x-1)(x+1)}}{x+1}$

  $y = \frac{(x+1)\left(\int \frac{\sqrt{-(x-1)(x+1)}\,\sqrt{x}}{x+1}dx + c_1\right)}{\sqrt{-(x-1)(x+1)}}$

- Evaluate the integrals on the rhs

$$y = \frac{(x+1)\left(-\frac{2\sqrt{-(x-1)(x+1)}\left(\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\, EllipticF\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)-3\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\, EllipticE\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)-x^3+x\right)}{3\sqrt{x}\left(x^2-1\right)}+c_1\right)}{\sqrt{-(x-1)(x+1)}}$$

- Simplify

$$y = \frac{-2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\, EllipticF\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)+6\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\, EllipticE\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)-3c_1\sqrt{x}\,\sqrt{-x^2+1}+2x^3-2x}{3(x-1)\sqrt{x}}$$

- Use initial condition $y\left(\frac{1}{2}\right) = 1$

$$1 = -\frac{2\left(-\mathrm{I} EllipticF\left(\frac{\sqrt{2}\,\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right)\sqrt{3}+3\,\mathrm{I} EllipticE\left(\frac{\sqrt{2}\,\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right)\sqrt{3}-\frac{3c_1\sqrt{2}\,\sqrt{3}\,\sqrt{4}}{8}-\frac{3}{4}\right)\sqrt{2}}{3}$$

- Solve for c_1

$$c_1 = -\frac{\left(4\,\mathrm{I}\sqrt{3}\,\sqrt{2}\, EllipticF\left(\frac{\sqrt{2}\,\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right)-12\,\mathrm{I}\sqrt{3}\,\sqrt{2}\, EllipticE\left(\frac{\sqrt{2}\,\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right)+3\sqrt{2}-6\right)\sqrt{3}\,\sqrt{4}}{36}$$

- Substitute $c_1 = -\frac{\left(4\,\mathrm{I}\sqrt{3}\,\sqrt{2}\, EllipticF\left(\frac{\sqrt{2}\,\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right)-12\,\mathrm{I}\sqrt{3}\,\sqrt{2}\, EllipticE\left(\frac{\sqrt{2}\,\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right)+3\sqrt{2}-6\right)\sqrt{3}\,\sqrt{4}}{36}$ into general solut

$$y = \frac{2\left(-\sqrt{x+1}\left(-3 EllipticE\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)+EllipticF\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)\right)\sqrt{-x}\,\sqrt{-2x+2}+\frac{\sqrt{-x^2+1}\left(-12\,\mathrm{I}\sqrt{2}\, EllipticE\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right)+8\,\mathrm{I} EllipticF\left(\right.\right.}{4}}{3\sqrt{x}\,(x-1)}$$

- Solution to the IVP

$$y = \frac{2\left(-\sqrt{x+1}\left(-3 EllipticE\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)+EllipticF\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)\right)\sqrt{-x}\,\sqrt{-2x+2}+\frac{\sqrt{-x^2+1}\left(-12\,\mathrm{I}\sqrt{2}\, EllipticE\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right)+8\,\mathrm{I} EllipticF\left(\right.\right.}{4}}{3\sqrt{x}\,(x-1)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 1.969 (sec). Leaf size: 141

```
dsolve([diff(y(x),x)=y(x)/(1-x^2)+sqrt(x),y(1/2) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\left(12i\sqrt{2}\,\text{EllipticE}\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right) - \sqrt{3}\,\sqrt{2} - 8i\,\text{EllipticF}\left(\frac{\sqrt{3}}{2}, \sqrt{2}\right) + 2\sqrt{3}\right)(1+x)}{6\sqrt{-x^2+1}} + \frac{-2\sqrt{1+x}\,\sqrt{2-2x}\,\sqrt{-x}\,\text{EllipticF}\left(\sqrt{1+x}, \frac{\sqrt{2}}{2}\right) + 6\sqrt{1+x}\,\sqrt{2-2x}\,\sqrt{-x}\,\text{EllipticE}\left(\sqrt{1+x}, \frac{\sqrt{2}}{2}\right) +}{\sqrt{x}\,(3x-3)}$$

✓ Solution by Mathematica

Time used: 1.562 (sec). Leaf size: 215

```
DSolve[{y'[x]==y[x]/(1-x^2)+Sqrt[x],{y[1/2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4\sqrt{1-x^2}x^2\,\text{Hypergeometric2F1}\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, x^2\right) - 4\sqrt{1-x^2}x\,\text{Hypergeometric2F1}\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, x^2\right) - \sqrt{2}\,\text{Hype}}{}$$

8.6 problem 4 (b)

Internal problem ID [12704]
Internal file name [`OUTPUT/11356_Friday_November_03_2023_06_30_59_AM_65093150/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 4 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - \frac{y}{-x^2+1} = \sqrt{x}}$$

8.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x^2-1}$$
$$q(x) = \sqrt{x}$$

Hence the ode is

$$y' + \frac{y}{x^2-1} = \sqrt{x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x^2-1}dx} \\
&= \frac{\sqrt{-x^2+1}}{x+1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\sqrt{x}\right) \\
\frac{d}{dx}\left(\frac{\sqrt{-x^2+1}\,y}{x+1}\right) &= \left(\frac{\sqrt{-x^2+1}}{x+1}\right)\left(\sqrt{x}\right) \\
d\left(\frac{\sqrt{-x^2+1}\,y}{x+1}\right) &= \left(\frac{\sqrt{x}\,\sqrt{-x^2+1}}{x+1}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\sqrt{-x^2+1}\,y}{x+1} &= \int \frac{\sqrt{x}\,\sqrt{-x^2+1}}{x+1}\, dx \\
\frac{\sqrt{-x^2+1}\,y}{x+1} &= \frac{\frac{2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)}{3} - 2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) - 2}{\sqrt{x}\,\sqrt{-x^2+1}}
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{-x^2+1}}{x+1}$ results in

$$
y = \frac{2(x+1)\left(\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) - 3\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticE}\left(\sqrt{x\right.\right.}}{3\left(-x^2+1\right)\sqrt{x}}
$$

which simplifies to

$$
y = \frac{-2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) + 6\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right.}{\sqrt{x}\,(-3+3x)}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{-2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) + 6\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)}{\sqrt{x}\,(-3+3x)} \\
&\quad + \frac{c_1(x+1)}{\sqrt{-x^2+1}}
\end{aligned}
\tag{1}
$$

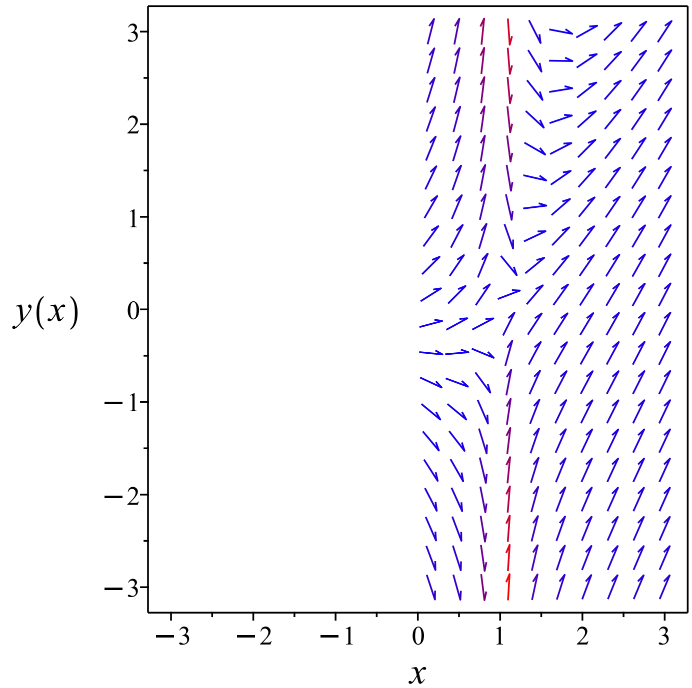


Figure 290: Slope field plot

Verification of solutions

$$
y = \frac{-2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \mathrm{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) + 6\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \mathrm{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)}{\sqrt{x}\,(-3+3x)} + \frac{c_1(x+1)}{\sqrt{-x^2+1}}
$$

Verified OK.

8.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x^{\frac{5}{2}} - \sqrt{x} - y}{x^2 - 1} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 241: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x+1}{\sqrt{-x^2+1}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}\right) S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x+1}{\sqrt{-x^2+1}}} dy \end{aligned}$$

Which results in

$$S=\frac{\sqrt{-x^2+1}\, y}{x+1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y) S_y}{R_x+\omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{x^{\frac{5}{2}}-\sqrt{x}-y}{x^2-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{(x+1)\sqrt{-x^2+1}} \\ S_y &= \frac{\sqrt{-x^2+1}}{x+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=\frac{(1-x)\sqrt{x}}{\sqrt{-x^2+1}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{(1-R)\sqrt{R}}{\sqrt{-R^2+1}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\frac{2\sqrt{R+1}\,\sqrt{2-2R}\,\sqrt{-R}\,\text{EllipticF}\left(\sqrt{R+1}, \frac{\sqrt{2}}{2}\right)}{3} - 2\sqrt{R+1}\,\sqrt{2-2R}\,\sqrt{-R}\,\text{EllipticE}\left(\sqrt{R+1}, \frac{\sqrt{2}}{2}\right) - \frac{2R^3}{3} + \frac{2}{3}}{\sqrt{-R^2+1}\,\sqrt{R}} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{-x^2+1}\,y}{x+1} = \frac{\frac{2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\text{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)}{3} - 2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\text{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) - 2}{\sqrt{x}\,\sqrt{-x^2+1}}$$

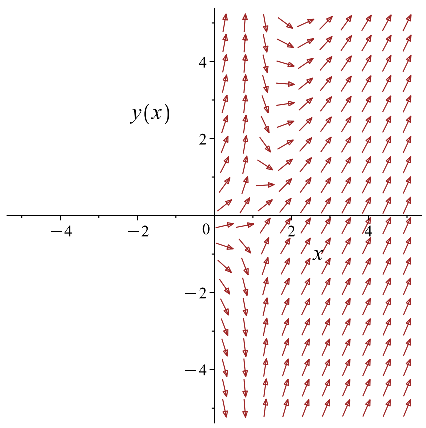
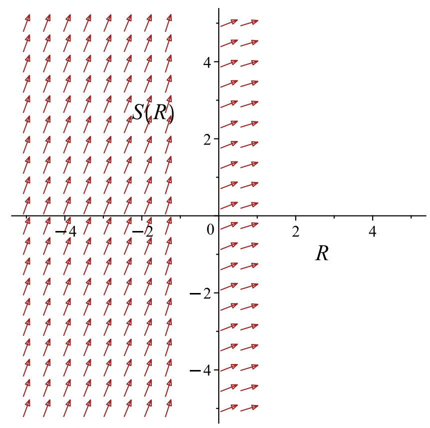
Which simplifies to

$$\frac{\sqrt{-x^2+1}\,y}{x+1} = \frac{\frac{2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\text{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)}{3} - 2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\text{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) - 2}{\sqrt{x}\,\sqrt{-x^2+1}}$$

Which gives

$$y = -\frac{2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\text{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)x - 6\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\text{EllipticE}\left(\sqrt{x+1},\right.}{}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^{\frac{5}{2}} - \sqrt{x} - y}{x^2 - 1}$ 	$R = x$ $S = \frac{\sqrt{-x^2 + 1}\, y}{x + 1}$	$\frac{dS}{dR} = \frac{(1 - R)\sqrt{R}}{\sqrt{-R^2 + 1}}$ 

Summary

The solution(s) found are the following

$$y = \tag{1}$$

$$-\frac{2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) x - 6\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right.}{}$$

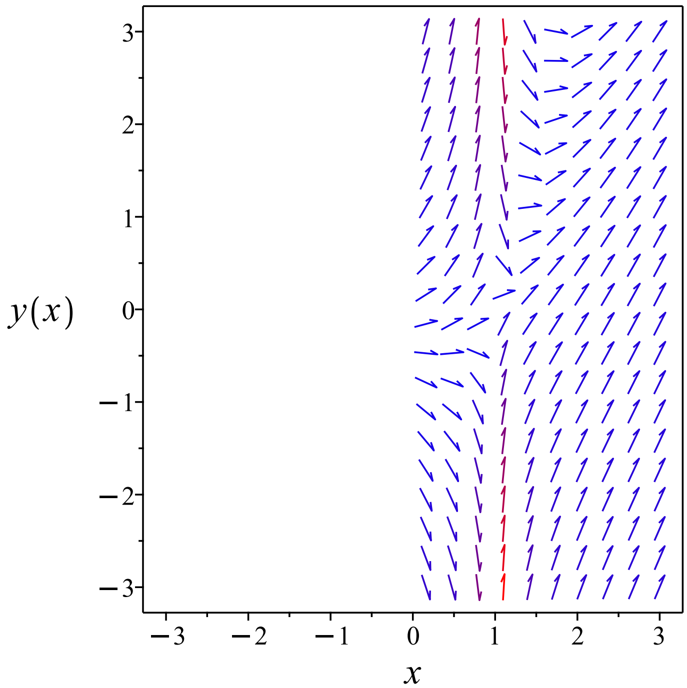


Figure 291: Slope field plot

Verification of solutions

$y =$

$$-\frac{2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \mathrm{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) x - 6\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \mathrm{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right.}{}$$

Verified OK.

8.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{y}{-x^2 + 1} + \sqrt{x}\right) \mathrm{d}x$$
$$\left(-\frac{y}{-x^2 + 1} - \sqrt{x}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{y}{-x^2 + 1} - \sqrt{x}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{y}{-x^2 + 1} - \sqrt{x}\right)$$
$$= \frac{1}{x^2 - 1}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{1}{-x^2+1}\right) - (0)\right) \\
&= \frac{1}{x^2-1}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{x^2-1} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\operatorname{arctanh}(x)} \\
&= \frac{\sqrt{-x^2+1}}{x+1}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\sqrt{-x^2+1}}{x+1}\left(-\frac{y}{-x^2+1} - \sqrt{x}\right) \\
&= -\frac{-x^{\frac{5}{2}} + \sqrt{x} + y}{(x+1)\sqrt{-x^2+1}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\sqrt{-x^2+1}}{x+1}(1) \\
&= \frac{\sqrt{-x^2+1}}{x+1}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(-\frac{-x^{\frac{5}{2}} + \sqrt{x} + y}{(x+1)\sqrt{-x^2+1}}\right) + \left(\frac{\sqrt{-x^2+1}}{x+1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{-x^{\frac{5}{2}} + \sqrt{x} + y}{(x+1)\sqrt{-x^2+1}}\,\mathrm{d}x$$

$$\phi = \int^{x} -\frac{-\_a^{\frac{5}{2}} + \sqrt{\_a} + y}{(\_a+1)\sqrt{-\_a^2+1}} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x-1}{\sqrt{-x^2+1}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\sqrt{-x^2+1}}{x+1}$. Therefore equation (4) becomes

$$\frac{\sqrt{-x^2+1}}{x+1} = -\frac{x-1}{\sqrt{-x^2+1}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^x -\frac{-\_a^{\frac{5}{2}} + \sqrt{\_a} + y}{(\_a + 1)\sqrt{-\_a^2 + 1}} d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^x -\frac{-\_a^{\frac{5}{2}} + \sqrt{\_a} + y}{(\_a + 1)\sqrt{-\_a^2 + 1}} d\_a$$

Summary
The solution(s) found are the following

$$\int^x -\frac{-\_a^{\frac{5}{2}} + \sqrt{\_a} + y}{(\_a + 1)\sqrt{-\_a^2 + 1}} d\_a = c_1 \tag{1}$$

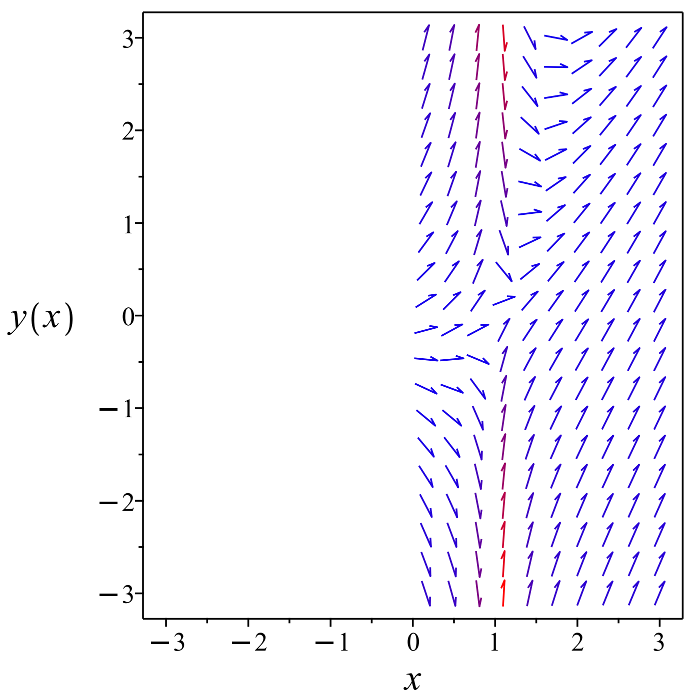


Figure 292: Slope field plot

Verification of solutions

$$\int^x -\frac{-\_a^{\frac{5}{2}} + \sqrt{\_a} + y}{(\_a + 1)\sqrt{-\_a^2 + 1}} d\_a = c_1$$

Verified OK.

8.6.4 Maple step by step solution

Let's solve

$y' - \frac{y}{-x^2+1} = \sqrt{x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x^2-1} + \sqrt{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x^2-1} = \sqrt{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x^2-1}\right) = \mu(x)\sqrt{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x^2-1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x^2-1}$

- Solve to find the integrating factor

  $\mu(x) = \frac{\sqrt{-(x-1)(x+1)}}{x+1}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\sqrt{x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\sqrt{x} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\sqrt{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{\sqrt{-(x-1)(x+1)}}{x+1}$

$$y = \frac{(x+1)\left(\int \frac{\sqrt{-(x-1)(x+1)}\,\sqrt{x}}{x+1}dx+c_1\right)}{\sqrt{-(x-1)(x+1)}}$$

- Evaluate the integrals on the rhs

$$y = \frac{(x+1)\left(-\frac{2\sqrt{-(x-1)(x+1)}\,\left(\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\mathit{EllipticF}\left(\sqrt{x+1},\frac{\sqrt{2}}{2}\right)-3\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\mathit{EllipticE}\left(\sqrt{x+1},\frac{\sqrt{2}}{2}\right)-x^3+x\right)}{3\sqrt{x}\,(x^2-1)}+c_1\right)}{\sqrt{-(x-1)(x+1)}}$$

- Simplify

$$y = \frac{-2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\mathit{EllipticF}\left(\sqrt{x+1},\frac{\sqrt{2}}{2}\right)+6\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\mathit{EllipticE}\left(\sqrt{x+1},\frac{\sqrt{2}}{2}\right)-3c_1\sqrt{x}\,\sqrt{-x^2+1}+2x^3-2x}{3(x-1)\sqrt{x}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 98

```
dsolve(diff(y(x),x)=y(x)/(1-x^2)+sqrt(x),y(x), singsol=all)
```

$$y(x) = \frac{(1+x)\,c_1}{\sqrt{-x^2+1}} + \frac{-2\sqrt{1+x}\,\sqrt{2-2x}\,\sqrt{-x}\,\mathrm{EllipticF}\left(\sqrt{1+x},\frac{\sqrt{2}}{2}\right)+6\sqrt{1+x}\,\sqrt{2-2x}\,\sqrt{-x}\,\mathrm{EllipticE}\left(\sqrt{1+x},\frac{\sqrt{2}}{2}\right)+}{\sqrt{x}\,(3x-3)}$$

✓ Solution by Mathematica

Time used: 1.157 (sec). Leaf size: 100

```
DSolve[y'[x]==y[x]/(1-x^2)+Sqrt[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-\frac{2x\left(-\sqrt{1-x^2}x\,\mathrm{Hypergeometric2F1}\left(\frac{1}{2},\frac{3}{4},\frac{7}{4},x^2\right)+\sqrt{1-x^2}\,\mathrm{Hypergeometric2F1}\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},x^2\right)+x^2-1\right)}{\sqrt{-((x-1)x)}}+3c_1\sqrt{x+1}}{3\sqrt{1-x}}$$

8.7 problem 4 (c)

Internal problem ID [12705]
Internal file name [`OUTPUT/11357_Friday_November_03_2023_06_31_01_AM_6823237/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 4 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y' - \frac{y}{-x^2 + 1} = \sqrt{x}}$$

With initial conditions

$$[y(2) = 1]$$

8.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x^2 - 1}$$
$$q(x) = \sqrt{x}$$

Hence the ode is

$$y' + \frac{y}{x^2-1} = \sqrt{x}$$

The domain of $p(x) = \frac{1}{x^2-1}$ is

$$\{-\infty \leq x < -1, -1 < x < 1, 1 < x \leq \infty\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = \sqrt{x}$ is

$$\{0 \leq x\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

8.7.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x^2-1}dx} \\ &= \frac{\sqrt{-x^2+1}}{x+1}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\sqrt{x}\right) \\ \frac{d}{dx}\left(\frac{\sqrt{-x^2+1}\, y}{x+1}\right) &= \left(\frac{\sqrt{-x^2+1}}{x+1}\right)\left(\sqrt{x}\right) \\ d\left(\frac{\sqrt{-x^2+1}\, y}{x+1}\right) &= \left(\frac{\sqrt{x}\,\sqrt{-x^2+1}}{x+1}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{\sqrt{-x^2+1}\, y}{x+1} &= \int \frac{\sqrt{x}\,\sqrt{-x^2+1}}{x+1}\, dx \\ \frac{\sqrt{-x^2+1}\, y}{x+1} &= \frac{\frac{2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)}{3} - 2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) - 2}{\sqrt{x}\,\sqrt{-x^2+1}}\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{-x^2+1}}{x+1}$ results in

$$y = \frac{2(x+1)\left(\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) - 3\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticE}\left(\sqrt{x\right.}\right.}{3\left(-x^2+1\right)\sqrt{x}}$$

which simplifies to

$$y = \frac{-2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) + 6\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \text{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{}}{2}\right.}{\sqrt{x}\,(-3+3x)}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\sqrt{2}\,\sqrt{3}\left(-3i\sqrt{2}\,c_1 + 4\sqrt{3} + 4\,\text{EllipticF}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right) - 12\,\text{EllipticE}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\right)}{6}$$

$$c_1 = -\frac{i\left(2\sqrt{3}\ \text{EllipticF}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} - 6\sqrt{3}\ \text{EllipticE}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} + 6\sqrt{2} - 3\right)\sqrt{3}}{9}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{2x^3\sqrt{-x^2+1} - 2ix^{\frac{5}{2}}\sqrt{3}\,\sqrt{2} - 2ix^{\frac{5}{2}}\ \text{EllipticF}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} + 6ix^{\frac{5}{2}}\ \text{EllipticE}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} + ix^{\frac{5}{2}}\sqrt{3} -}{}$$

Summary

The solution(s) found are the following

$$y = \frac{2x^3\sqrt{-x^2+1} - 2ix^{\frac{5}{2}}\sqrt{3}\,\sqrt{2} - 2ix^{\frac{5}{2}}\ \text{EllipticF}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} + 6ix^{\frac{5}{2}}\ \text{EllipticE}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} + ix^{\frac{5}{2}}\sqrt{3} - 2}{} \tag{1}$$

Verification of solutions

$$y = \frac{2x^3\sqrt{-x^2+1} - 2ix^{\frac{5}{2}}\sqrt{3}\,\sqrt{2} - 2ix^{\frac{5}{2}}\ \text{EllipticF}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} + 6ix^{\frac{5}{2}}\ \text{EllipticE}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} + ix^{\frac{5}{2}}\sqrt{3} - 2}{}$$

Verified OK.

8.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x^{\frac{5}{2}} + \sqrt{x} + y}{x^2 - 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 244: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x+1}{\sqrt{-x^2+1}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x+1}{\sqrt{-x^2+1}}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\sqrt{-x^2+1}\, y}{x+1}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{-x^{\frac{5}{2}} + \sqrt{x} + y}{x^2 - 1}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{(x+1)\sqrt{-x^2+1}} \\ S_y &= \frac{\sqrt{-x^2+1}}{x+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{(1-x)\sqrt{x}}{\sqrt{-x^2+1}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{(1-R)\sqrt{R}}{\sqrt{-R^2+1}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\frac{2\sqrt{R+1}\,\sqrt{2-2R}\,\sqrt{-R}\ \mathrm{EllipticF}\left(\sqrt{R+1}, \frac{\sqrt{2}}{2}\right)}{3} - 2\sqrt{R+1}\,\sqrt{2-2R}\,\sqrt{-R}\ \mathrm{EllipticE}\left(\sqrt{R+1}, \frac{\sqrt{2}}{2}\right) - \frac{2R^3}{3} + \frac{2I}{3}}{\sqrt{-R^2+1}\,\sqrt{R}} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{-x^2+1}\,y}{x+1} = \frac{\frac{2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \mathrm{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)}{3} - 2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \mathrm{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) - 2}{\sqrt{x}\,\sqrt{-x^2+1}}$$

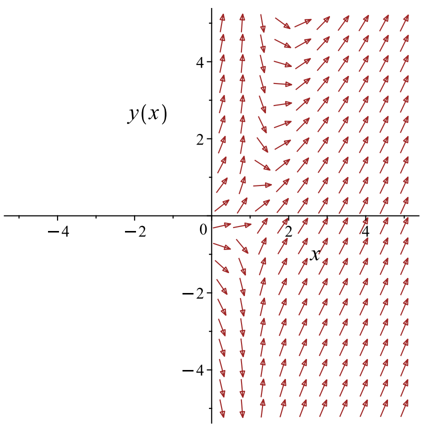
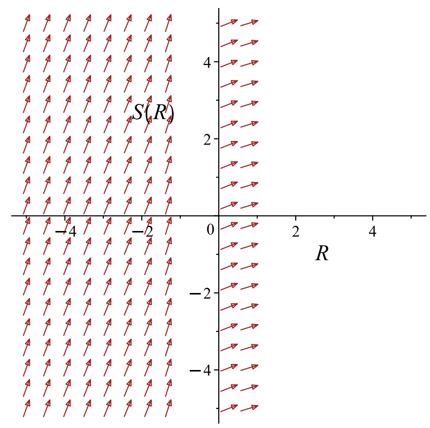
Which simplifies to

$$\frac{\sqrt{-x^2+1}\,y}{x+1} = \frac{\frac{2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \mathrm{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)}{3} - 2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \mathrm{EllipticE}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right) - 2}{\sqrt{x}\,\sqrt{-x^2+1}}$$

Which gives

$$y = -\frac{2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \mathrm{EllipticF}\left(\sqrt{x+1}, \frac{\sqrt{2}}{2}\right)x - 6\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\ \mathrm{EllipticE}\left(\sqrt{x+1},\right.}{}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{-x^{\frac{5}{2}}+\sqrt{x}+y}{x^2-1}$
</td><td>$R = x$
$S = \frac{\sqrt{-x^2+1}\, y}{x+1}$</td><td>$\frac{dS}{dR} = \frac{(1-R)\sqrt{R}}{\sqrt{-R^2+1}}$
</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{\sqrt{2}\,\sqrt{3}\left(-4\,\text{EllipticF}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right) + 12\,\text{EllipticE}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right) - 4\sqrt{3} + 3i\sqrt{2}\,c_1\right)}{6}$$

$$c_1 = -\frac{i\left(2\sqrt{3}\ \text{EllipticF}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} - 6\sqrt{3}\ \text{EllipticE}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} + 6\sqrt{2} - 3\right)\sqrt{3}}{9}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-6ix^{\frac{3}{2}}\sqrt{-x^2+1}\ \text{EllipticE}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} + 2ix^{\frac{3}{2}}\sqrt{-x^2+1}\ \text{EllipticF}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} - 6i\sqrt{x}\,\sqrt{-x^2+1}}{}$$

Summary

The solution(s) found are the following

$$y = \frac{-6ix^{\frac{3}{2}}\sqrt{-x^2+1}\ \text{EllipticE}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} + 2ix^{\frac{3}{2}}\sqrt{-x^2+1}\ \text{EllipticF}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} - 6i\sqrt{x}\,\sqrt{-x^2+1}\ \text{E}}{} \tag{1}$$

Verification of solutions

$$y = \frac{-6ix^{\frac{3}{2}}\sqrt{-x^2+1}\,\text{EllipticE}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} + 2ix^{\frac{3}{2}}\sqrt{-x^2+1}\,\text{EllipticF}\left(\sqrt{3}, \frac{\sqrt{2}}{2}\right)\sqrt{2} - 6i\sqrt{x}\sqrt{-x^2+1}\,\text{E}}{}$$

Verified OK.

8.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$dy = \left(\frac{y}{-x^2+1} + \sqrt{x}\right) dx$$

$$\left(-\frac{y}{-x^2+1} - \sqrt{x}\right) dx + dy = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{y}{-x^2+1} - \sqrt{x} \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{-x^2+1} - \sqrt{x}\right) \\ &= \frac{1}{x^2-1} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\frac{1}{-x^2+1}\right) - (0)\right) \\ &= \frac{1}{x^2-1} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, dx} \\ &= e^{\int \frac{1}{x^2-1} \, dx} \end{aligned}$$

The result of integrating gives

$$\mu = e^{-\operatorname{arctanh}(x)}$$
$$= \frac{\sqrt{-x^2+1}}{x+1}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\overline{M} = \mu M$$
$$= \frac{\sqrt{-x^2+1}}{x+1}\left(-\frac{y}{-x^2+1} - \sqrt{x}\right)$$
$$= -\frac{-x^{\frac{5}{2}} + \sqrt{x} + y}{(x+1)\sqrt{-x^2+1}}$$

And

$$\overline{N} = \mu N$$
$$= \frac{\sqrt{-x^2+1}}{x+1}(1)$$
$$= \frac{\sqrt{-x^2+1}}{x+1}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-\frac{-x^{\frac{5}{2}} + \sqrt{x} + y}{(x+1)\sqrt{-x^2+1}}\right) + \left(\frac{\sqrt{-x^2+1}}{x+1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{-x^{\frac{5}{2}} + \sqrt{x} + y}{(x+1)\sqrt{-x^2+1}}\,\mathrm{d}x$$
$$\phi = \int_2^x -\frac{-\_a^{\frac{5}{2}} + \sqrt{\_a} + y}{(\_a+1)\sqrt{-\_a^2+1}} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \int_2^x -\frac{1}{(\_a+1)\sqrt{-\_a^2+1}} d\_a + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{\sqrt{-x^2+1}}{x+1}$. Therefore equation (4) becomes

$$\frac{\sqrt{-x^2+1}}{x+1} = -\left(\int_2^x \frac{1}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= \frac{\left(\int_2^x \frac{1}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) x + \sqrt{-x^2+1} + \int_2^x \frac{1}{(\_a+1)\sqrt{-\_a^2+1}} d\_a}{x+1} \\ &= \frac{(x+1)\left(\int_2^x \frac{1}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + \sqrt{-x^2+1}}{x+1} \end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned} \int f'(y)\,\mathrm{d}y &= \int \left(\frac{(x+1)\left(\int_2^x \frac{1}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + \sqrt{-x^2+1}}{x+1}\right) \mathrm{d}y \\ f(y) &= \frac{\left((x+1)\left(\int_2^x \frac{1}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + \sqrt{-x^2+1}\right) y}{x+1} + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\begin{aligned} \phi = &\int_2^x -\frac{-\_a^{\frac{5}{2}} + \sqrt{\_a} + y}{(\_a+1)\sqrt{-\_a^2+1}} d\_a \\ &+ \frac{\left((x+1)\left(\int_2^x \frac{1}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + \sqrt{-x^2+1}\right) y}{x+1} + c_1 \end{aligned}$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int_2^x -\frac{-\_a^{\frac{5}{2}} + \sqrt{\_a} + y}{(\_a+1)\sqrt{-\_a^2+1}} d\_a + \frac{\left((x+1)\left(\int_2^x \frac{1}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + \sqrt{-x^2+1}\right) y}{x+1}$$

The solution becomes

$$y = \frac{c_1 x - x\left(\int_2^x \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + x\left(\int_2^x \frac{\sqrt{\_a}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + c_1 - \left(\int_2^x \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + \int}{\sqrt{-x^2+1}}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -i\sqrt{3}\, c_1$$

$$c_1 = \frac{i\sqrt{3}}{3}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{i\sqrt{3}\, x + i\sqrt{3} - 3x\left(\int_2^x \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + 3x\left(\int_2^x \frac{\sqrt{\_a}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) - 3\left(\int_2^x \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}}\right.}{3\sqrt{-x^2+1}}$$

Summary

The solution(s) found are the following

$$y = \frac{i\sqrt{3}\, x + i\sqrt{3} - 3x\left(\int_2^x \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + 3x\left(\int_2^x \frac{\sqrt{\_a}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) - 3\left(\int_2^x \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_\right.}{3\sqrt{-x^2+1}} \tag{1}$$

Verification of solutions

$$y = \frac{i\sqrt{3}\, x + i\sqrt{3} - 3x\left(\int_2^x \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) + 3x\left(\int_2^x \frac{\sqrt{\_a}}{(\_a+1)\sqrt{-\_a^2+1}} d\_a\right) - 3\left(\int_2^x \frac{\_a^{\frac{5}{2}}}{(\_a+1)\sqrt{-\_a^2+1}} d\_\right.}{3\sqrt{-x^2+1}}$$

Verified OK.

8.7.5 Maple step by step solution

Let's solve

$\left[y' - \frac{y}{-x^2+1} = \sqrt{x}, y(2) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x^2-1} + \sqrt{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x^2-1} = \sqrt{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x^2-1}\right) = \mu(x)\sqrt{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x^2-1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x^2-1}$

- Solve to find the integrating factor

  $\mu(x) = \frac{\sqrt{-(x-1)(x+1)}}{x+1}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\sqrt{x}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\sqrt{x}dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\sqrt{x}dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{\sqrt{-(x-1)(x+1)}}{x+1}$

  $y = \frac{(x+1)\left(\int \frac{\sqrt{-(x-1)(x+1)}\,\sqrt{x}}{x+1}dx + c_1\right)}{\sqrt{-(x-1)(x+1)}}$

- Evaluate the integrals on the rhs

$$y = \frac{(x+1)\left(-\frac{2\sqrt{-(x-1)(x+1)}\left(\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\mathit{EllipticF}\left(\sqrt{x+1},\frac{\sqrt{2}}{2}\right)-3\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\mathit{EllipticE}\left(\sqrt{x+1},\frac{\sqrt{2}}{2}\right)-x^3+x\right)}{3\sqrt{x}\,(x^2-1)}+c_1\right)}{\sqrt{-(x-1)(x+1)}}$$

- Simplify

$$y = \frac{-2\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\mathit{EllipticF}\left(\sqrt{x+1},\frac{\sqrt{2}}{2}\right)+6\sqrt{x+1}\,\sqrt{-2x+2}\,\sqrt{-x}\,\mathit{EllipticE}\left(\sqrt{x+1},\frac{\sqrt{2}}{2}\right)-3c_1\sqrt{x}\,\sqrt{-x^2+1}+2x^3-2x}{3(x-1)\sqrt{x}}$$

- Use initial condition $y(2) = 1$

$$1 = \frac{\left(4\sqrt{3}\,\mathit{EllipticF}\left(\sqrt{3},\frac{\sqrt{2}}{2}\right)-12\sqrt{3}\,\mathit{EllipticE}\left(\sqrt{3},\frac{\sqrt{2}}{2}\right)-3c_1\sqrt{2}\,\sqrt{-3}+12\right)\sqrt{2}}{6}$$

- Solve for c_1

$$c_1 = -\frac{\left(2\sqrt{3}\,\mathit{EllipticF}\left(\sqrt{3},\frac{\sqrt{2}}{2}\right)\sqrt{2}-6\sqrt{3}\,\mathit{EllipticE}\left(\sqrt{3},\frac{\sqrt{2}}{2}\right)\sqrt{2}+6\sqrt{2}-3\right)\sqrt{-3}}{9}$$

- Substitute $c_1 = -\frac{\left(2\sqrt{3}\,\mathit{EllipticF}\left(\sqrt{3},\frac{\sqrt{2}}{2}\right)\sqrt{2}-6\sqrt{3}\,\mathit{EllipticE}\left(\sqrt{3},\frac{\sqrt{2}}{2}\right)\sqrt{2}+6\sqrt{2}-3\right)\sqrt{-3}}{9}$ into general solution and

$$y = -\frac{6\left(\frac{\sqrt{x+1}\left(-3\mathit{EllipticE}\left(\sqrt{x+1},\frac{\sqrt{2}}{2}\right)+\mathit{EllipticF}\left(\sqrt{x+1},\frac{\sqrt{2}}{2}\right)\right)\sqrt{-x}\,\sqrt{-2x+2}}{3}+\mathrm{I}\left(\left(\mathit{EllipticE}\left(\sqrt{3},\frac{\sqrt{2}}{2}\right)-\frac{\mathit{EllipticF}\left(\sqrt{3},\frac{\sqrt{2}}{2}\right)}{3}\right)\sqrt{2}+\frac{\sqrt{3}}{6}-\right.\right.}{\sqrt{x}\,(-3+3x)}$$

- Solution to the IVP

$$y = -\frac{6\left(\frac{\sqrt{x+1}\left(-3\mathit{EllipticE}\left(\sqrt{x+1},\frac{\sqrt{2}}{2}\right)+\mathit{EllipticF}\left(\sqrt{x+1},\frac{\sqrt{2}}{2}\right)\right)\sqrt{-x}\,\sqrt{-2x+2}}{3}+\mathrm{I}\left(\left(\mathit{EllipticE}\left(\sqrt{3},\frac{\sqrt{2}}{2}\right)-\frac{\mathit{EllipticF}\left(\sqrt{3},\frac{\sqrt{2}}{2}\right)}{3}\right)\sqrt{2}+\frac{\sqrt{3}}{6}-\right.\right.}{\sqrt{x}\,(-3+3x)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.922 (sec). Leaf size: 136

```
dsolve([diff(y(x),x)=y(x)/(1-x^2)+sqrt(x),y(2) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{i(1+x)\left(-1+\frac{2\left(\sqrt{3}\ \mathrm{EllipticF}\left(\sqrt{3},\frac{\sqrt{2}}{2}\right)-3\sqrt{3}\ \mathrm{EllipticE}\left(\sqrt{3},\frac{\sqrt{2}}{2}\right)+3\right)\sqrt{2}}{3}\right)\sqrt{3}}{3\sqrt{-x^2+1}}$$
$$+\frac{-2\sqrt{1+x}\,\sqrt{2-2x}\,\sqrt{-x}\ \mathrm{EllipticF}\left(\sqrt{1+x},\frac{\sqrt{2}}{2}\right)+6\sqrt{1+x}\,\sqrt{2-2x}\,\sqrt{-x}\ \mathrm{EllipticE}\left(\sqrt{1+x},\frac{\sqrt{2}}{2}\right)\cdots}{\sqrt{x}\,(3x-3)}$$

✓ Solution by Mathematica

Time used: 0.121 (sec). Leaf size: 215

```
DSolve[{y'[x]==y[x]/(1-x^2)+Sqrt[x],{y[2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2\sqrt{1-x^2}x^2\,\mathrm{Hypergeometric2F1}\left(\frac{1}{2},\frac{3}{4},\frac{7}{4},x^2\right)-2\sqrt{1-x^2}x\,\mathrm{Hypergeometric2F1}\left(\frac{1}{4},\frac{1}{2},\frac{5}{4},x^2\right)-4\sqrt{2}\,\mathrm{Hyp}\cdots}{\cdots}$$

8.8 problem 5 (a)

Internal problem ID [12706]
Internal file name [`OUTPUT/11358_Friday_November_03_2023_06_31_03_AM_23906316/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 5 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 = 0}$$

With initial conditions

$$[y(-1) = 1]$$

8.8.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^2 \end{aligned}$$

The y domain of $f(x, y)$ when $x = -1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2) \\ &= 2y \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x=-1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.8.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2} dy = x + c_1$$
$$-\frac{1}{y} = x + c_1$$

Solving for y gives these solutions

$$y_1 = -\frac{1}{x + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{-1 + c_1}$$

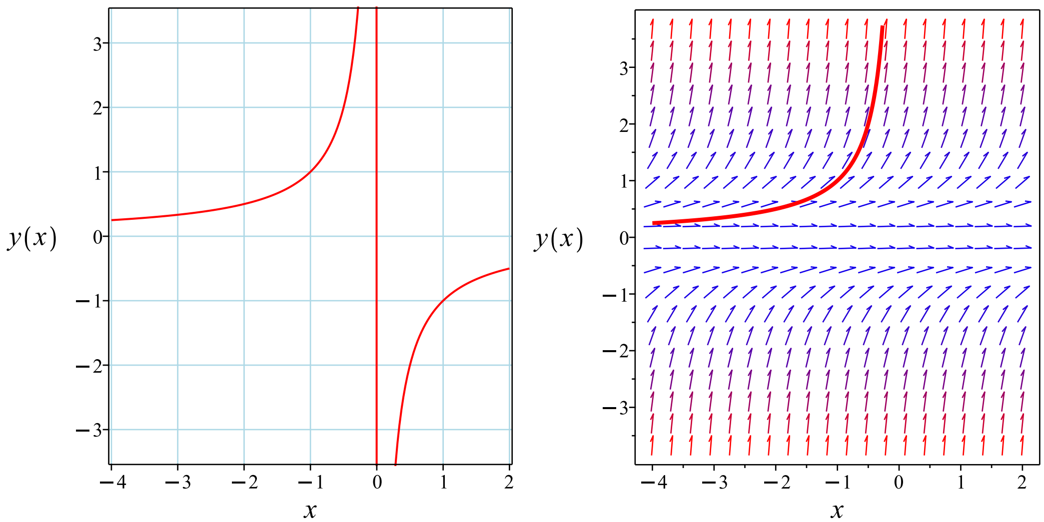
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{x}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{1}{x}$$

Verified OK.

8.8.3 Maple step by step solution

Let's solve

$[y' - y^2 = 0, y(-1) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = x + c_1$

- Solve for y

$y = -\frac{1}{x+c_1}$

- Use initial condition $y(-1) = 1$

  $1 = -\frac{1}{-1+c_1}$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = -\frac{1}{x}$

- Solution to the IVP

  $y = -\frac{1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 9

```
dsolve([diff(y(x),x)=y(x)^2,y(-1) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{1}{x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 10

```
DSolve[{y'[x]==y[x]^2,{y[-1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{x}$$

8.9 problem 5 (b)

Internal problem ID [12707]
Internal file name [`OUTPUT/11359_Friday_November_03_2023_06_31_04_AM_20545278/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 5 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 = 0}$$

With initial conditions

$$[y(-1) = 0]$$

8.9.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^2 \end{aligned}$$

The y domain of $f(x, y)$ when $x = -1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2) \\ &= 2y \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x=-1$ is

$$\{-\infty<y<\infty\}$$

And the point $y_0=0$ is inside this domain. Therefore solution exists and is unique.

8.9.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2}dy = x+c_1$$
$$-\frac{1}{y} = x+c_1$$

Solving for y gives these solutions

$$y_1 = -\frac{1}{x+c_1}$$

Initial conditions are used to solve for c_1. Substituting $x=-1$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

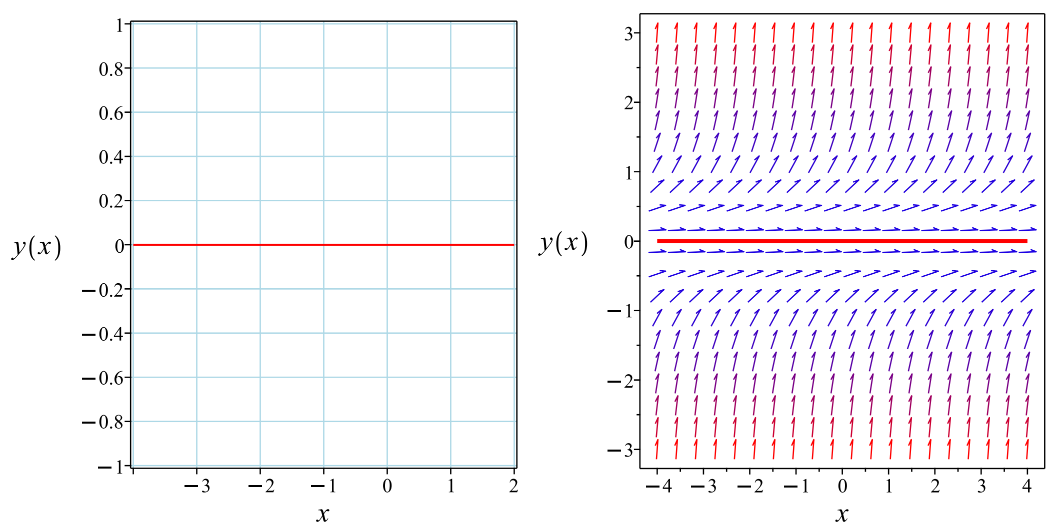
$$0 = -\frac{1}{-1+c_1}$$

Unable to solve for constant of integration. Since $\lim_{c_1\to\infty}$ gives $y=-\frac{1}{x+c_1}=y=0$ and this result satisfies the given initial condition.

Summary

The solution(s) found are the following

$$y = 0$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 0$$

Verified OK.

8.9.3 Maple step by step solution

Let's solve

$[y' - y^2 = 0, y(-1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = x + c_1$

- Solve for y

  $y = -\frac{1}{x + c_1}$

- Use initial condition $y(-1) = 0$

  $0 = -\frac{1}{-1 + c_1}$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=y(x)^2,y(-1) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 6

```
DSolve[{y'[x]==y[x]^2,{y[-1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

8.10 problem 5 (c)

Internal problem ID [12708]
Internal file name [`OUTPUT/11360_Friday_November_03_2023_06_31_04_AM_71305882/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 5 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^2 = 0}$$

With initial conditions

$$\left[y(1) = \frac{1}{2}\right]$$

8.10.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^2 \end{aligned}$$

The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^2) \\ &= 2y \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

8.10.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^2} dy = x + c_1$$
$$-\frac{1}{y} = x + c_1$$

Solving for y gives these solutions

$$y_1 = -\frac{1}{x + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\frac{1}{1 + c_1}$$

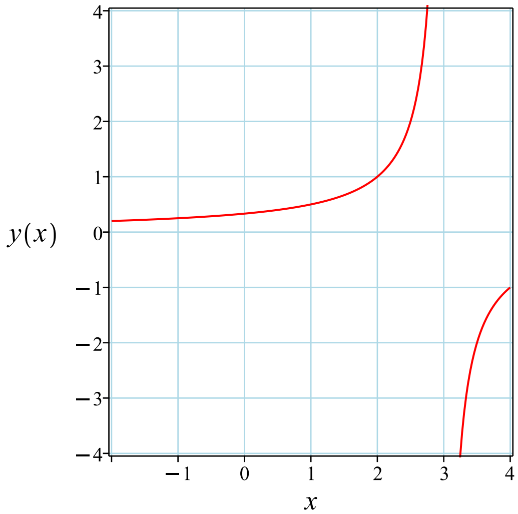
$$c_1 = -3$$

Substituting c_1 found above in the general solution gives

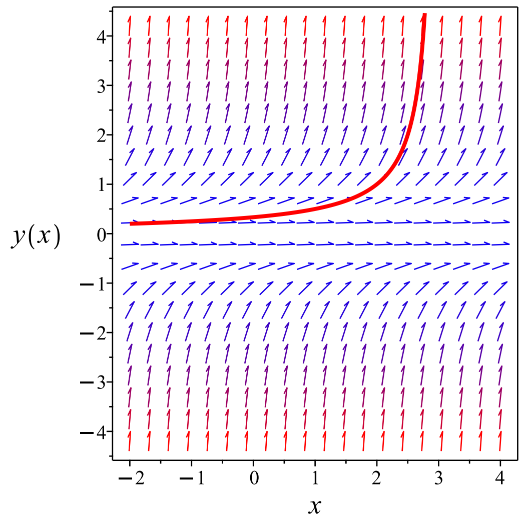
$$y = -\frac{1}{x - 3}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{x - 3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{x-3}$$

Verified OK.

8.10.3 Maple step by step solution

Let's solve

$\left[y' - y^2 = 0, y(1) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = x + c_1$

- Solve for y

$y = -\frac{1}{x+c_1}$

- Use initial condition $y(1) = \frac{1}{2}$

  $\frac{1}{2} = -\frac{1}{1+c_1}$

- Solve for c_1

  $c_1 = -3$

- Substitute $c_1 = -3$ into general solution and simplify

  $y = -\frac{1}{x-3}$

- Solution to the IVP

  $y = -\frac{1}{x-3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve([diff(y(x),x)=y(x)^2,y(1) = 1/2],y(x), singsol=all)
```

$$y(x) = -\frac{1}{-3+x}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 12

```
DSolve[{y'[x]==y[x]^2,{y[1]==1/2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3-x}$$

8.11 problem 6 (a)

Internal problem ID [12709]
Internal file name [`OUTPUT/11361_Friday_November_03_2023_06_31_05_AM_38277058/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 6 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^3 = 0}$$

With initial conditions

$$[y(-1) = 1]$$

8.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^3 \end{aligned}$$

The y domain of $f(x, y)$ when $x = -1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^3\right) \\ &= 3y^2 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = -1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.11.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^3} dy = x + c_1$$
$$-\frac{1}{2y^2} = x + c_1$$

Solving for y gives these solutions

$$y_1 = \frac{1}{\sqrt{-2c_1 - 2x}}$$
$$y_2 = -\frac{1}{\sqrt{-2c_1 - 2x}}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{\sqrt{2 - 2c_1}}$$

Warning: Unable to solve for c_1. No particular solution can be found using given initial conditions for this solution. removing this solution as not valid. Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{\sqrt{2 - 2c_1}}$$

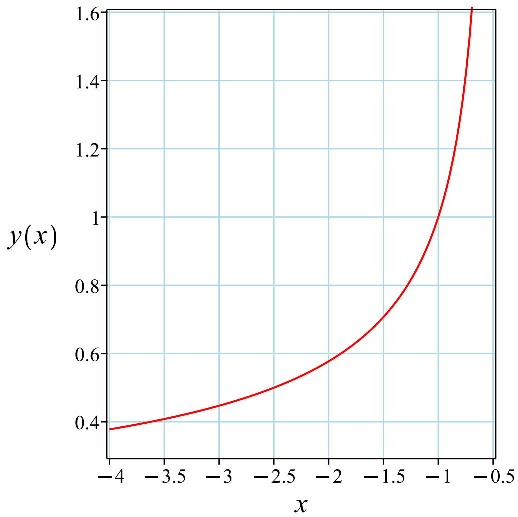
$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

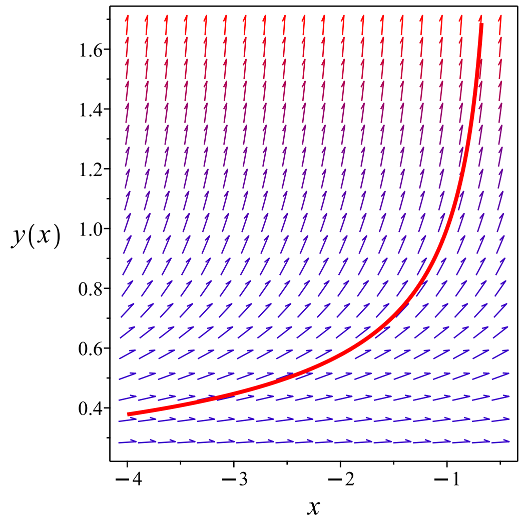
$$y = \frac{1}{\sqrt{-2x - 1}}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{\sqrt{-2x-1}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{-2x-1}}$$

Verified OK.

8.11.3 Maple step by step solution

Let's solve

$[y' - y^3 = 0, y(-1) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\frac{1}{2y^2} = x + c_1$

- Solve for y

  $\left\{y = \frac{1}{\sqrt{-2c_1 - 2x}}, y = -\frac{1}{\sqrt{-2c_1 - 2x}}\right\}$

- Use initial condition $y(-1) = 1$

  $1 = \frac{1}{\sqrt{2 - 2c_1}}$

- Solve for c_1

  $c_1 = \frac{1}{2}$

- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

  $y = \frac{1}{\sqrt{-2x - 1}}$

- Use initial condition $y(-1) = 1$

  $1 = -\frac{1}{\sqrt{2 - 2c_1}}$

- Solution does not satisfy initial condition
- Solution to the IVP

  $y = \frac{1}{\sqrt{-2x - 1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 11

```
dsolve([diff(y(x),x)=y(x)^3,y(-1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{-2x-1}}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 14

```
DSolve[{y'[x]==y[x]^3,{y[-1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{\sqrt{-2x-1}}$$

8.12 problem 6 (b)

Internal problem ID [12710]
Internal file name [`OUTPUT/11362_Friday_November_03_2023_06_31_06_AM_42323688/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 6 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^3 = 0}$$

With initial conditions

$$[y(-1) = 0]$$

8.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^3 \end{aligned}$$

The y domain of $f(x, y)$ when $x = -1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^3) \\ &= 3y^2 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = -1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

8.12.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^3} dy = x + c_1$$
$$-\frac{1}{2y^2} = x + c_1$$

Solving for y gives these solutions

$$y_1 = \frac{1}{\sqrt{-2c_1 - 2x}}$$
$$y_2 = -\frac{1}{\sqrt{-2c_1 - 2x}}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{\sqrt{2 - 2c_1}}$$

Unable to solve for constant of integration. Since $\lim_{c_1 \to \infty}$ gives $y = -\frac{1}{\sqrt{-2c_1 - 2x}} = y = 0$ and this result satisfies the given initial condition. Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

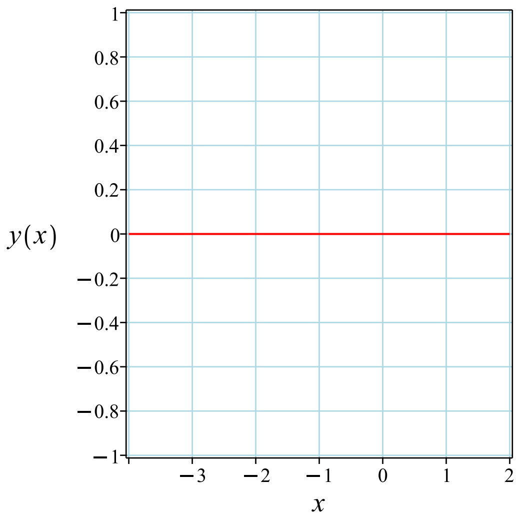
$$0 = \frac{1}{\sqrt{2 - 2c_1}}$$

Unable to solve for constant of integration. Since $\lim_{c_1 \to \infty}$ gives $y = \frac{1}{\sqrt{-2c_1 - 2x}} = y = 0$ and this result satisfies the given initial condition.

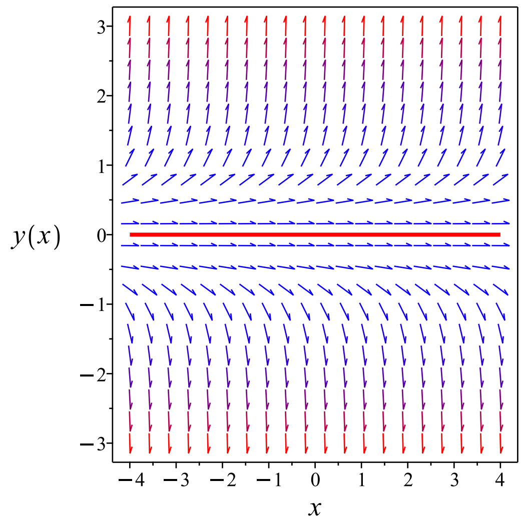
Summary

The solution(s) found are the following

$$y = 0$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 0$$

Verified OK.

8.12.3 Maple step by step solution

Let's solve

$[y' - y^3 = 0, y(-1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\frac{1}{2y^2} = x + c_1$

- Solve for y

  $\left\{y = \frac{1}{\sqrt{-2c_1 - 2x}}, y = -\frac{1}{\sqrt{-2c_1 - 2x}}\right\}$

- Use initial condition $y(-1) = 0$

  $0 = \frac{1}{\sqrt{2-2c_1}}$

- Solution does not satisfy initial condition

- Use initial condition $y(-1) = 0$

  $0 = -\frac{1}{\sqrt{2-2c_1}}$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=y(x)^3,y(-1) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 6

```
DSolve[{y'[x]==y[x]^3,{y[-1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

8.13 problem 6 (c)

Internal problem ID [12711]
Internal file name [`OUTPUT/11363_Friday_November_03_2023_06_31_07_AM_71430434/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 6 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^3 = 0}$$

With initial conditions

$$[y(-1) = -1]$$

8.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^3 \end{aligned}$$

The y domain of $f(x, y)$ when $x = -1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y^3) \\ &= 3y^2 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = -1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

8.13.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^3} dy = x + c_1$$
$$-\frac{1}{2y^2} = x + c_1$$

Solving for y gives these solutions

$$y_1 = \frac{1}{\sqrt{-2c_1 - 2x}}$$
$$y_2 = -\frac{1}{\sqrt{-2c_1 - 2x}}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{1}{\sqrt{2 - 2c_1}}$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{\sqrt{-2x - 1}}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{1}{\sqrt{2 - 2c_1}}$$

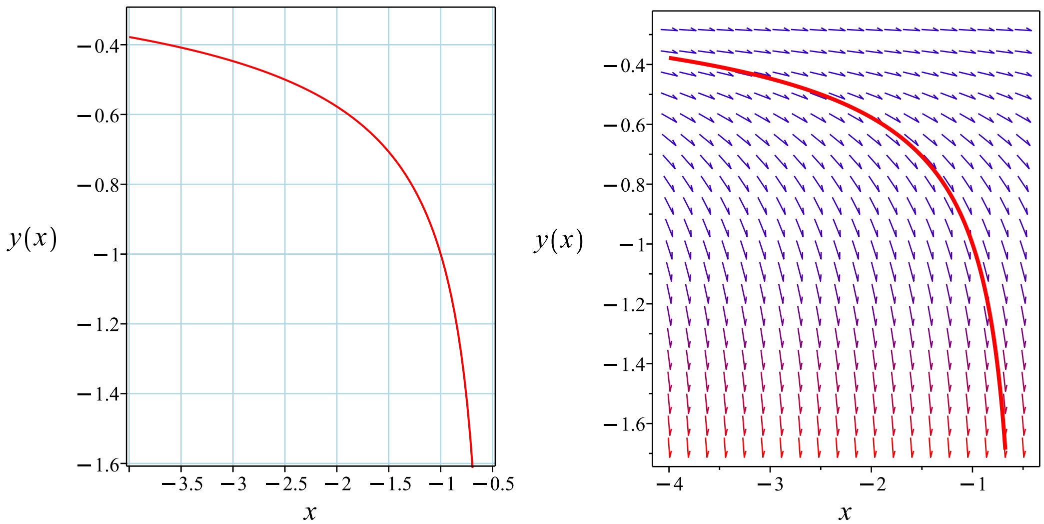
Warning: Unable to solve for c_1. No particular solution can be found using given initial

Summary

The solution(s) found are the following

conditions for this solution. removing this solution as not valid.

$y = -$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{1}{\sqrt{-2x-1}}$$

Verified OK.

8.13.3 Maple step by step solution

Let's solve

$[y' - y^3 = 0, y(-1) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\frac{1}{2y^2} = x + c_1$

- Solve for y

$$\left\{y=\frac{1}{\sqrt{-2c_1-2x}}, y=-\frac{1}{\sqrt{-2c_1-2x}}\right\}$$

- Use initial condition $y(-1)=-1$

  $$-1=\frac{1}{\sqrt{2-2c_1}}$$

- Solution does not satisfy initial condition
- Use initial condition $y(-1)=-1$

  $$-1=-\frac{1}{\sqrt{2-2c_1}}$$

- Solve for c_1

  $$c_1=\frac{1}{2}$$

- Substitute $c_1=\frac{1}{2}$ into general solution and simplify

  $$y=-\frac{1}{\sqrt{-2x-1}}$$

- Solution to the IVP

  $$y=-\frac{1}{\sqrt{-2x-1}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)=y(x)^3,y(-1) = -1],y(x), singsol=all)
```

$$y(x) = -\frac{1}{\sqrt{-2x-1}}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 16

```
DSolve[{y'[x]==y[x]^3,{y[-1]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{-2x-1}}$$

8.14 problem 7 (a)

Internal problem ID [12712]
Internal file name [`OUTPUT/11364_Friday_November_03_2023_06_31_07_AM_48639018/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 7 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \frac{3x^2}{2y} = 0}$$

With initial conditions

$$[y(-1) = 1]$$

8.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{3x^2}{2y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $f(x, y)$ when $x = -1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{3x^2}{2y}\right) \\ &= \frac{3x^2}{2y^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.14.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{3x^2}{2y}\end{aligned}$$

Where $f(x) = -\frac{3x^2}{2}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{y}}\,dy &= -\frac{3x^2}{2}\,dx \\ \int \frac{1}{\frac{1}{y}}\,dy &= \int -\frac{3x^2}{2}\,dx \\ \frac{y^2}{2} &= -\frac{x^3}{2} + c_1\end{aligned}$$

Which results in

$$y = \sqrt{-x^3 + 2c_1}$$
$$y = -\sqrt{-x^3 + 2c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\sqrt{2c_1 + 1}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \sqrt{2c_1 + 1}$$

$$c_1 = 0$$

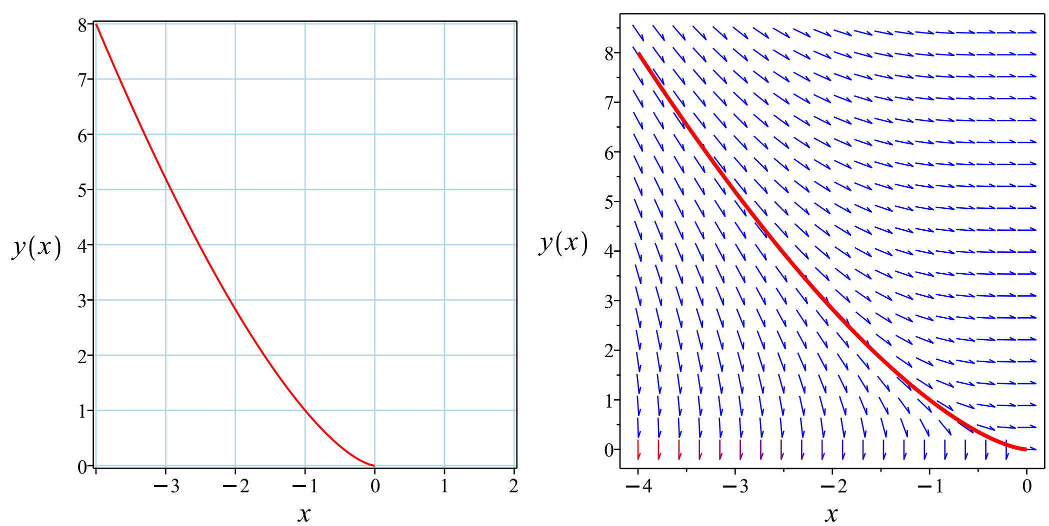
Substituting c_1 found above in the general solution gives

$$y = \sqrt{-x^3}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-x^3} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \sqrt{-x^3}$$

Verified OK.

8.14.3 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{3x^2}{2y} \tag{1}$$

Which becomes

$$(2y)\, dy = \left(-3x^2\right) dx \tag{2}$$

But the RHS is complete differential because

$$\left(-3x^2\right) dx = d\left(-x^3\right)$$

Hence (2) becomes

$$(2y)\, dy = d\left(-x^3\right)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{-x^3 + c_1} + c_1$$
$$y = -\sqrt{-x^3 + c_1} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\sqrt{1 + c_1} + c_1$$

$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

$$y = -\sqrt{-x^3 + 3} + 3$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \sqrt{1 + c_1} + c_1$$

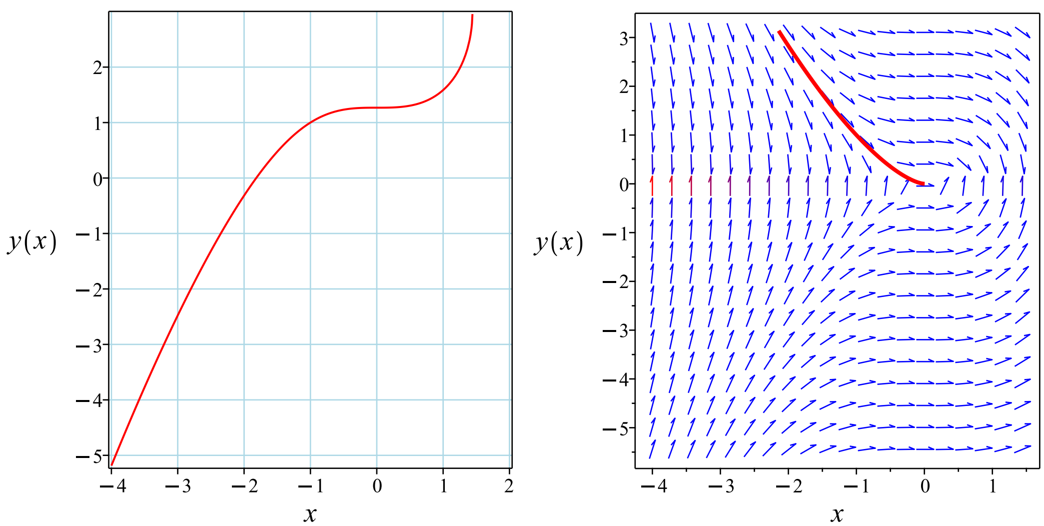
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \sqrt{-x^3}$$

Summary
The solution(s) found are the following

$$y = \sqrt{-x^3} \tag{1}$$
$$y = -\sqrt{-x^3 + 3} + 3 \tag{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sqrt{-x^3}$$

Verified OK.

$$y = -\sqrt{-x^3 + 3} + 3$$

Verified OK.

8.14.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3x^2}{2y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 253: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{2}{3x^2} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{2}{3x^2}} dx
\end{aligned}
$$

Which results in

$$
S = -\frac{x^3}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{3x^2}{2y}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{3x^2}{2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = y \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R^2}{2} + c_1 \tag{4}
$$

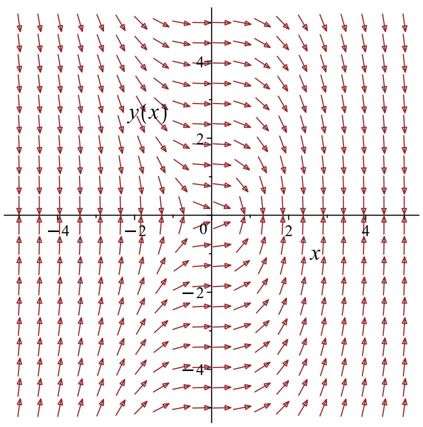
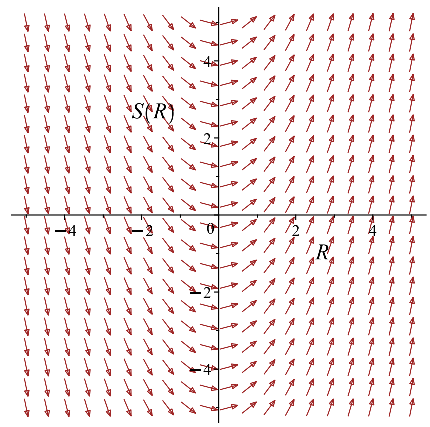
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{x^3}{2} = \frac{y^2}{2} + c_1
$$

Which simplifies to

$$
-\frac{x^3}{2} = \frac{y^2}{2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3x^2}{2y}$ 	$R = y$ $S = -\frac{x^3}{2}$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{2} + c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^3}{2} = \frac{y^2}{2}$$

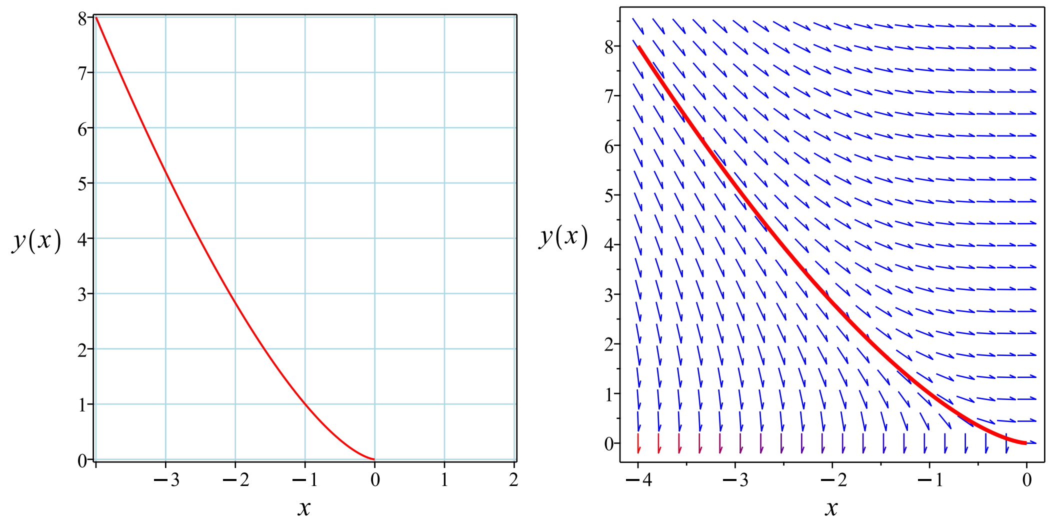
Solving for y from the above gives

$$y = (-x)^{\frac{3}{2}}$$

Summary

The solution(s) found are the following

$$y = (-x)^{\frac{3}{2}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = (-x)^{\frac{3}{2}}$$

Verified OK.

8.14.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(-\frac{2y}{3}\right) \mathrm{d}y &= \left(x^2\right) \mathrm{d}x \\
\left(-x^2\right) \mathrm{d}x + \left(-\frac{2y}{3}\right) \mathrm{d}y &= 0 \qquad (2\mathrm{A})
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -x^2 \\
N(x, y) &= -\frac{2y}{3}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2\right) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{2y}{3}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x^2 \, \mathrm{d}x$$
$$\phi = -\frac{x^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{3}$. Therefore equation (4) becomes

$$-\frac{2y}{3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{2y}{3}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{2y}{3}\right) \mathrm{d}y$$
$$f(y) = -\frac{y^2}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} - \frac{y^2}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} - \frac{y^2}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^3}{3} - \frac{y^2}{3} = 0$$

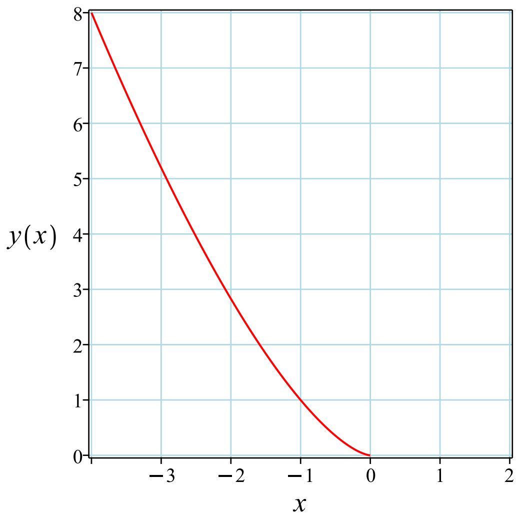
Solving for y from the above gives

$$y = (-x)^{\frac{3}{2}}$$

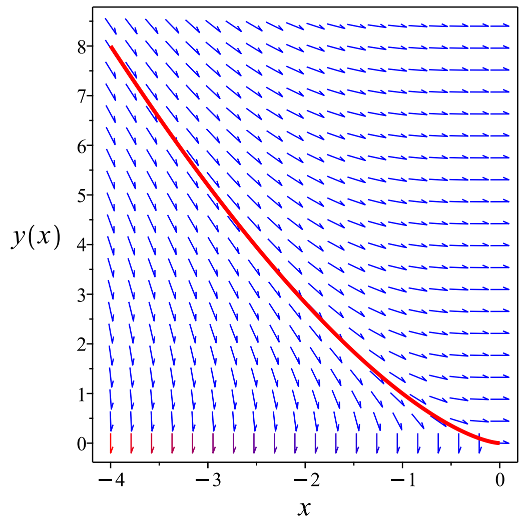
Summary

The solution(s) found are the following

$$y = (-x)^{\frac{3}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (-x)^{\frac{3}{2}}$$

Verified OK.

8.14.6 Maple step by step solution

Let's solve

$\left[y' + \frac{3x^2}{2y} = 0, y(-1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = -\frac{3x^2}{2}$

- Integrate both sides with respect to x

  $\int yy'dx = \int -\frac{3x^2}{2}dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = -\frac{x^3}{2} + c_1$

- Solve for y

  $\{y = \sqrt{-x^3 + 2c_1}, y = -\sqrt{-x^3 + 2c_1}\}$

- Use initial condition $y(-1) = 1$

  $1 = \sqrt{2c_1 + 1}$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = \sqrt{-x^3}$
- Use initial condition $y(-1) = 1$

  $1 = -\sqrt{2c_1 + 1}$
- Solution does not satisfy initial condition
- Solution to the IVP

  $y = \sqrt{-x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([diff(y(x),x)=-3*x^2/(2*y(x)),y(-1) = 1],y(x), singsol=all)
```

$$y(x) = (-x)^{\frac{3}{2}}$$

✓ Solution by Mathematica

Time used: 0.144 (sec). Leaf size: 14

```
DSolve[{y'[x]==-3*x^2/(2*y[x]),{y[-1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{-x^3}$$

8.15 problem 7 (b)

Internal problem ID [12713]
Internal file name [`OUTPUT/11365_Friday_November_03_2023_06_31_08_AM_2515014/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 7 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \frac{3x^2}{2y} = 0}$$

With initial conditions

$$\left[y(-1) = \frac{1}{2}\right]$$

8.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{3x^2}{2y} \end{aligned}$$

The x domain of $f(x,y)$ when $y=\frac{1}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0=-1$ is inside this domain. The y domain of $f(x,y)$ when $x=-1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0=\frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{3x^2}{2y}\right) \\
&= \frac{3x^2}{2y^2}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=\frac{1}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0=-1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=-1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0=\frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

8.15.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= -\frac{3x^2}{2y}
\end{aligned}$$

Where $f(x)=-\frac{3x^2}{2}$ and $g(y)=\frac{1}{y}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{1}{y}}\,dy &= -\frac{3x^2}{2}\,dx \\
\int \frac{1}{\frac{1}{y}}\,dy &= \int -\frac{3x^2}{2}\,dx \\
\frac{y^2}{2} &= -\frac{x^3}{2} + c_1
\end{aligned}$$

Which results in

$$y = \sqrt{-x^3 + 2c_1}$$
$$y = -\sqrt{-x^3 + 2c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\sqrt{2c_1 + 1}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \sqrt{2c_1 + 1}$$

$$c_1 = -\frac{3}{8}$$

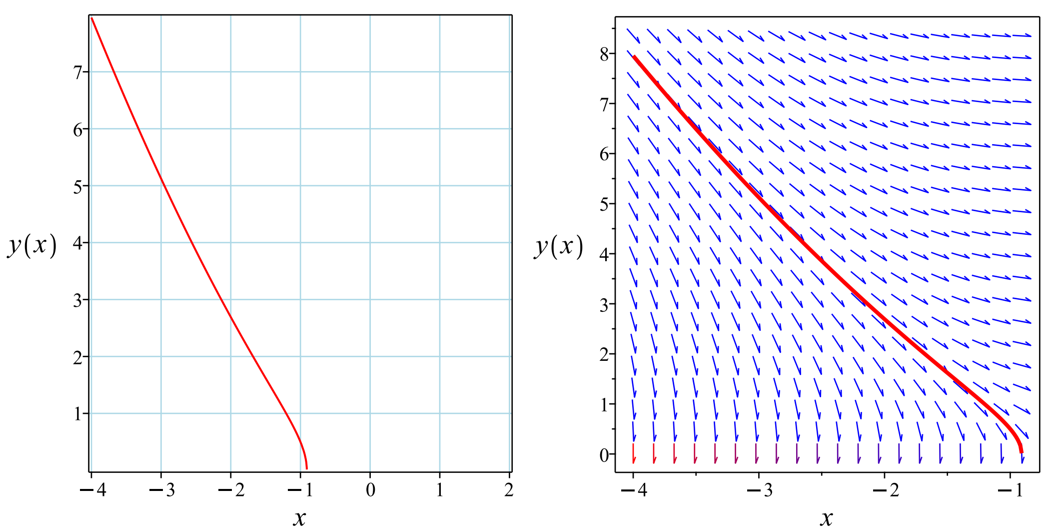
Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{-4x^3 - 3}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-4x^3 - 3}}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-4x^3 - 3}}{2}$$

Verified OK.

8.15.3 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{3x^2}{2y} \tag{1}$$

Which becomes

$$(2y)\,dy = \left(-3x^2\right)dx \tag{2}$$

But the RHS is complete differential because

$$\left(-3x^2\right)dx = d\left(-x^3\right)$$

Hence (2) becomes

$$(2y)\,dy = d\left(-x^3\right)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{-x^3 + c_1} + c_1$$
$$y = -\sqrt{-x^3 + c_1} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\sqrt{1 + c_1} + c_1$$

$$c_1 = 1 + \frac{\sqrt{7}}{2}$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{\sqrt{-4x^3 + 4 + 2\sqrt{7}}}{2} + 1 + \frac{\sqrt{7}}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \sqrt{1 + c_1} + c_1$$

$$c_1 = 1 - \frac{\sqrt{7}}{2}$$

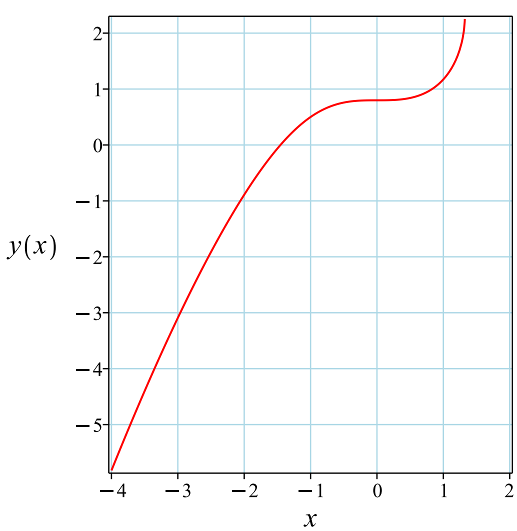
Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{-4x^3 + 4 - 2\sqrt{7}}}{2} + 1 - \frac{\sqrt{7}}{2}$$

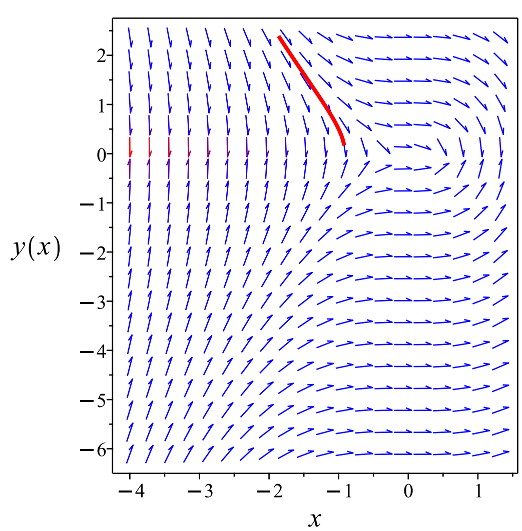
<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\sqrt{-4x^3 + 4 - 2\sqrt{7}}}{2} + 1 - \frac{\sqrt{7}}{2} \tag{1}$$

$$y = -\frac{\sqrt{-4x^3 + 4 + 2\sqrt{7}}}{2} + 1 + \frac{\sqrt{7}}{2} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-4x^3 + 4 - 2\sqrt{7}}}{2} + 1 - \frac{\sqrt{7}}{2}$$

Verified OK.

$$y = -\frac{\sqrt{-4x^3 + 4 + 2\sqrt{7}}}{2} + 1 + \frac{\sqrt{7}}{2}$$

Verified OK.

8.15.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{3x^2}{2y} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 256: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{2}{3x^2} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{2}{3x^2}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^3}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3x^2}{2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{3x^2}{2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

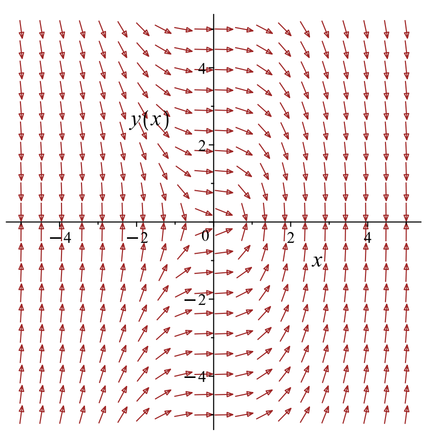
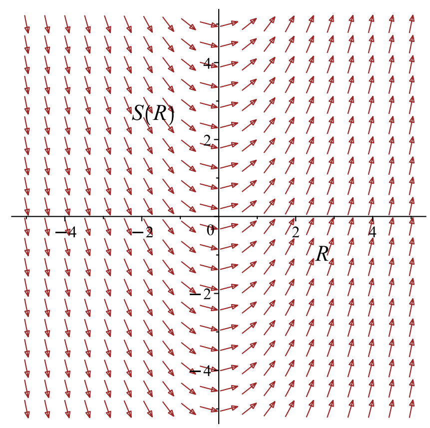
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^3}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$-\frac{x^3}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3x^2}{2y}$ 	$R = y$ $S = -\frac{x^3}{2}$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{8} + c_1$$

$$c_1 = \frac{3}{8}$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^3}{2} = \frac{y^2}{2} + \frac{3}{8}$$

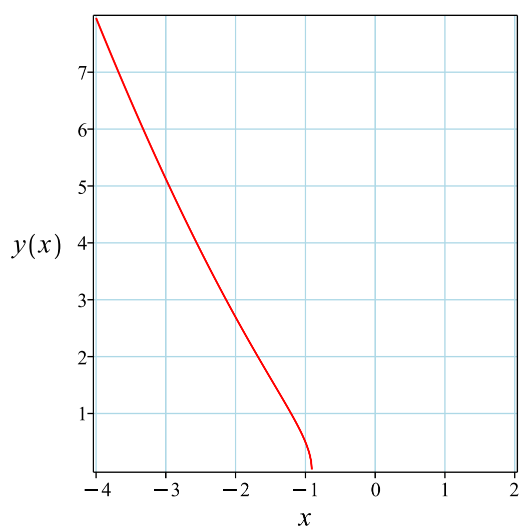
Solving for y from the above gives

$$y = \frac{\sqrt{-4x^3 - 3}}{2}$$

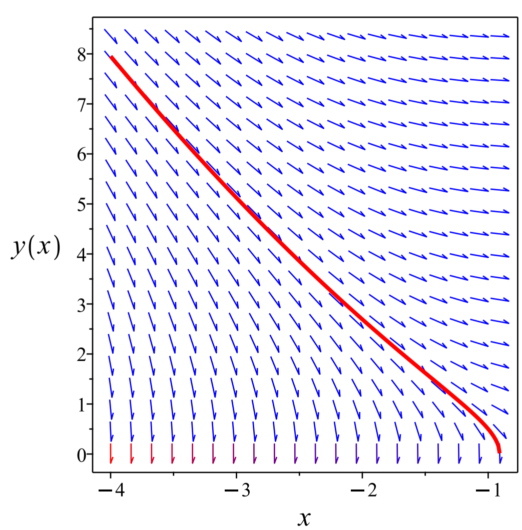
Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-4x^3 - 3}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-4x^3 - 3}}{2}$$

Verified OK.

8.15.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{2y}{3}\right)\mathrm{d}y&=\left(x^2\right)\mathrm{d}x\\ \left(-x^2\right)\mathrm{d}x+\left(-\frac{2y}{3}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -x^2 \\
N(x,y) &= -\frac{2y}{3}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^2) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{2y}{3}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x^2 \, \mathrm{d}x
\end{aligned}
$$

$$\phi = -\frac{x^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = -\frac{2y}{3}$. Therefore equation (4) becomes

$$-\frac{2y}{3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{2y}{3}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{2y}{3}\right)\mathrm{d}y$$

$$f(y) = -\frac{y^2}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} - \frac{y^2}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} - \frac{y^2}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{4} = c_1$$

$$c_1 = \frac{1}{4}$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^3}{3} - \frac{y^2}{3} = \frac{1}{4}$$

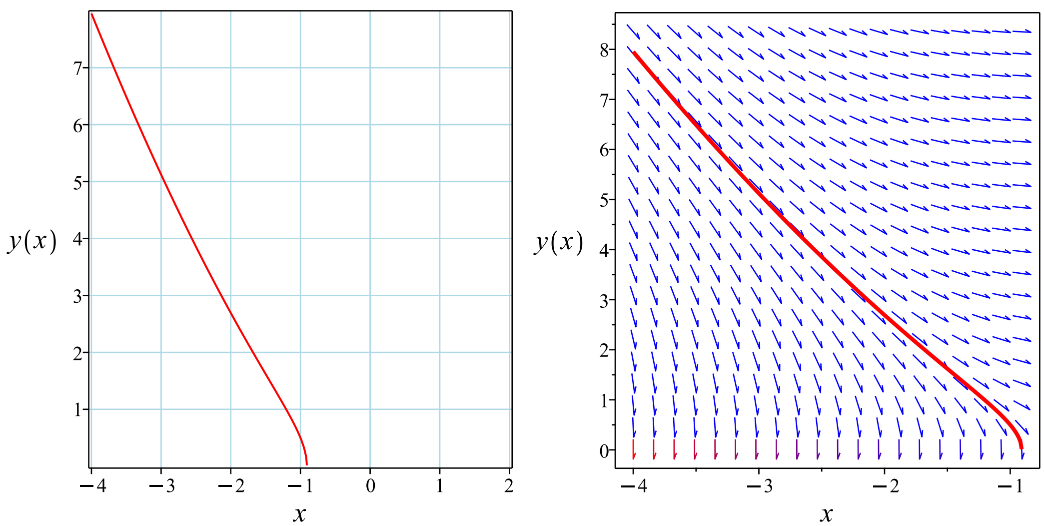
Solving for y from the above gives

$$y = \frac{\sqrt{-4x^3 - 3}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-4x^3 - 3}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-4x^3 - 3}}{2}$$

Verified OK.

8.15.6 Maple step by step solution

Let's solve

$\left[y' + \frac{3x^2}{2y} = 0, y(-1) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = -\frac{3x^2}{2}$

- Integrate both sides with respect to x

  $\int yy'dx = \int -\frac{3x^2}{2}dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = -\frac{x^3}{2} + c_1$

- Solve for y

  $\left\{y = \sqrt{-x^3 + 2c_1}, y = -\sqrt{-x^3 + 2c_1}\right\}$

- Use initial condition $y(-1) = \frac{1}{2}$

  $\frac{1}{2} = \sqrt{2c_1 + 1}$

- Solve for c_1

  $c_1 = -\frac{3}{8}$

- Substitute $c_1 = -\frac{3}{8}$ into general solution and simplify

  $y = \frac{\sqrt{-4x^3-3}}{2}$

- Use initial condition $y(-1) = \frac{1}{2}$

  $\frac{1}{2} = -\sqrt{2c_1 + 1}$

- Solution does not satisfy initial condition
- Solution to the IVP

  $y = \frac{\sqrt{-4x^3-3}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve([diff(y(x),x)=-3*x^2/(2*y(x)),y(-1) = 1/2],y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{-4x^3 - 3}}{2}$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 20

```
DSolve[{y'[x]==-3*x^2/(2*y[x]),{y[-1]==1/2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\sqrt{-4x^3 - 3}$$

8.16 problem 7 (c)

Internal problem ID [12714]
Internal file name [`OUTPUT/11366_Friday_November_03_2023_06_31_09_AM_64424057/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 7 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \frac{3x^2}{2y} = 0}$$

With initial conditions

$$[y(-1) = 0]$$

8.16.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{3x^2}{2y} \end{aligned}$$

$f(x,y)$ is not defined at $y = 0$ therefore existence and uniqueness theorem do not apply.

8.16.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= -\frac{3x^2}{2y}
\end{aligned}
$$

Where $f(x) = -\frac{3x^2}{2}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{y}}\, dy &= -\frac{3x^2}{2}\, dx \\
\int \frac{1}{\frac{1}{y}}\, dy &= \int -\frac{3x^2}{2}\, dx \\
\frac{y^2}{2} &= -\frac{x^3}{2} + c_1
\end{aligned}
$$

Which results in

$$
\begin{aligned}
y &= \sqrt{-x^3 + 2c_1} \\
y &= -\sqrt{-x^3 + 2c_1}
\end{aligned}
$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = -\sqrt{2c_1 + 1}
$$

$$
c_1 = -\frac{1}{2}
$$

Substituting c_1 found above in the general solution gives

$$
y = -\sqrt{-x^3 - 1}
$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = \sqrt{2c_1 + 1}
$$

$$c_1 = -\frac{1}{2}$$

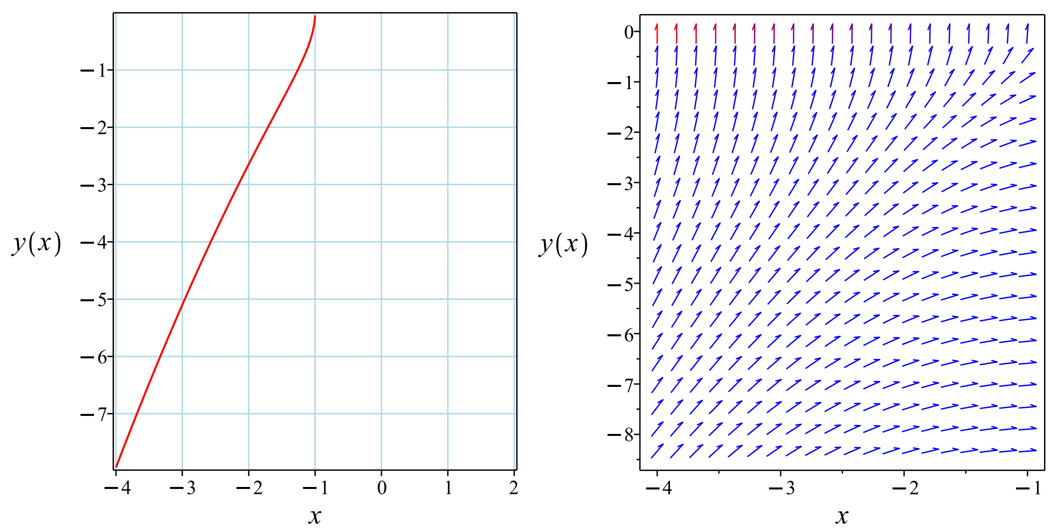
Substituting c_1 found above in the general solution gives

$$y = \sqrt{-x^3 - 1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-x^3 - 1} \tag{1}$$
$$y = -\sqrt{-x^3 - 1} \tag{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sqrt{-x^3 - 1}$$

Verified OK.

$$y = -\sqrt{-x^3 - 1}$$

Verified OK.

8.16.3 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{3x^2}{2y} \tag{1}$$

Which becomes

$$(2y)\,dy = \left(-3x^2\right)dx \tag{2}$$

But the RHS is complete differential because

$$\left(-3x^2\right)dx = d\left(-x^3\right)$$

Hence (2) becomes

$$(2y)\,dy = d\left(-x^3\right)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{-x^3 + c_1} + c_1$$
$$y = -\sqrt{-x^3 + c_1} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\sqrt{1 + c_1} + c_1$$

$$c_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{\sqrt{-4x^3 + 2 + 2\sqrt{5}}}{2} + \frac{1}{2} + \frac{\sqrt{5}}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \sqrt{1 + c_1} + c_1$$

$$c_1 = \frac{1}{2} - \frac{\sqrt{5}}{2}$$

Substituting c_1 found above in the general solution gives

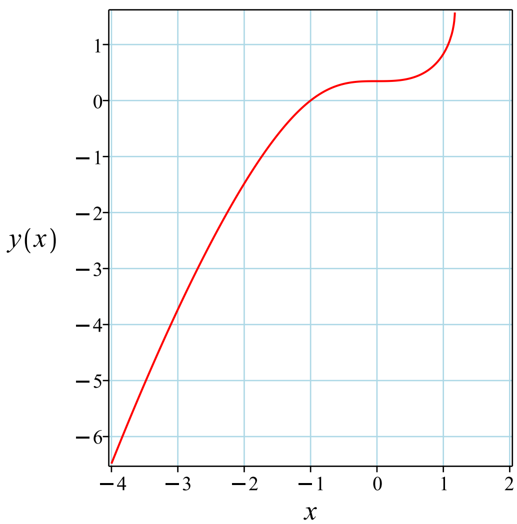
$$y = \frac{\sqrt{-4x^3 + 2 - 2\sqrt{5}}}{2} + \frac{1}{2} - \frac{\sqrt{5}}{2}$$

Summary

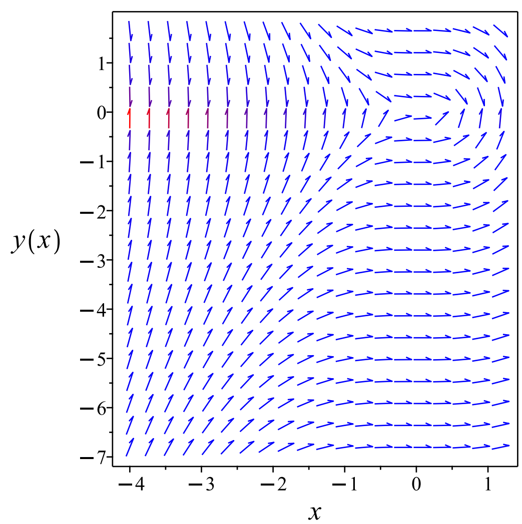
The solution(s) found are the following

$$y = \frac{\sqrt{-4x^3 + 2 - 2\sqrt{5}}}{2} + \frac{1}{2} - \frac{\sqrt{5}}{2} \tag{1}$$

$$y = -\frac{\sqrt{-4x^3 + 2 + 2\sqrt{5}}}{2} + \frac{1}{2} + \frac{\sqrt{5}}{2} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-4x^3 + 2 - 2\sqrt{5}}}{2} + \frac{1}{2} - \frac{\sqrt{5}}{2}$$

Verified OK.

$$y = -\frac{\sqrt{-4x^3 + 2 + 2\sqrt{5}}}{2} + \frac{1}{2} + \frac{\sqrt{5}}{2}$$

Verified OK.

8.16.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{3x^2}{2y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 259: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{2}{3x^2} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{2}{3x^2}} dx
\end{aligned}
$$

Which results in

$$
S = -\frac{x^3}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{3x^2}{2y}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{3x^2}{2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

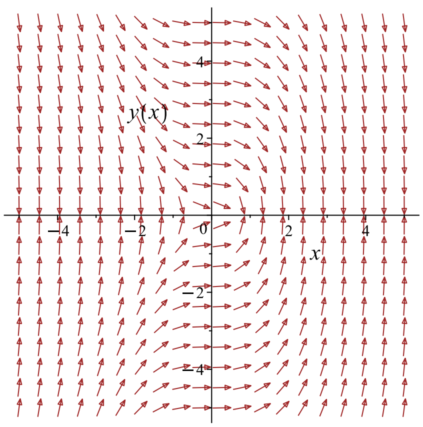
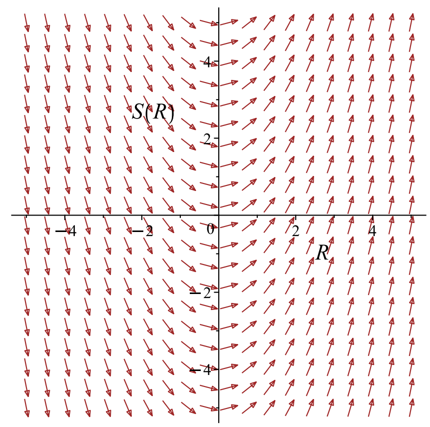
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^3}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$-\frac{x^3}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3x^2}{2y}$ 	$R = y$ $S = -\frac{x^3}{2}$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_1$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^3}{2} = \frac{y^2}{2} + \frac{1}{2}$$

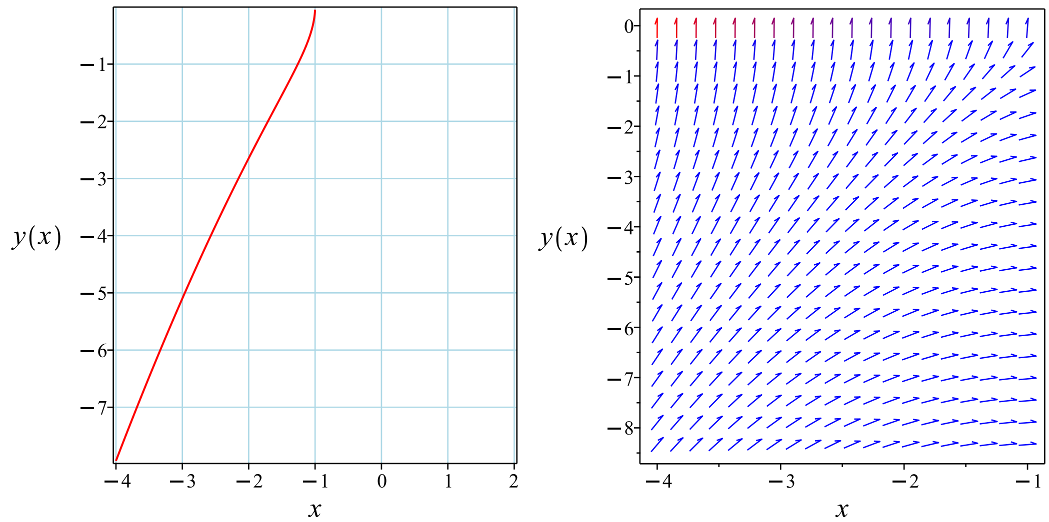
Solving for y from the above gives

$$y = \sqrt{-x^3 - 1}$$
$$y = -\sqrt{-x^3 - 1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-x^3 - 1} \tag{1}$$
$$y = -\sqrt{-x^3 - 1} \tag{2}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \sqrt{-x^3 - 1}$$

Verified OK.

$$y = -\sqrt{-x^3 - 1}$$

Verified OK.

8.16.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{2y}{3}\right) \mathrm{d}y = \left(x^2\right) \mathrm{d}x$$
$$\left(-x^2\right) \mathrm{d}x + \left(-\frac{2y}{3}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^2$$
$$N(x, y) = -\frac{2y}{3}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2\right) \\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{2y}{3}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -x^2 \,\mathrm{d}x$$
$$\phi = -\frac{x^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{3}$. Therefore equation (4) becomes

$$-\frac{2y}{3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{2y}{3}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(-\frac{2y}{3}\right) \mathrm{d}y$$
$$f(y) = -\frac{y^2}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} - \frac{y^2}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} - \frac{y^2}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{3} = c_1$$

$$c_1 = \frac{1}{3}$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^3}{3} - \frac{y^2}{3} = \frac{1}{3}$$

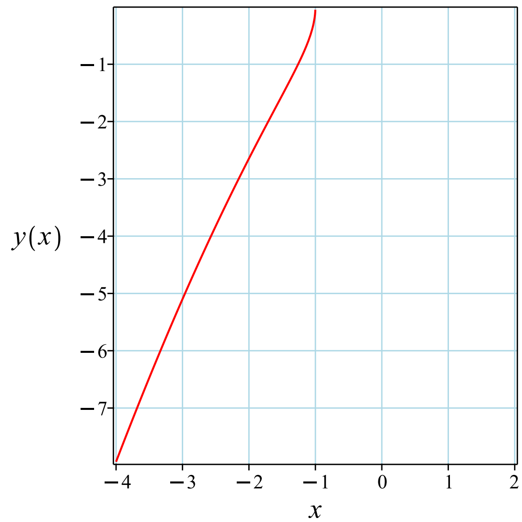
Solving for y from the above gives

$$y = \sqrt{-x^3 - 1}$$
$$y = -\sqrt{-x^3 - 1}$$

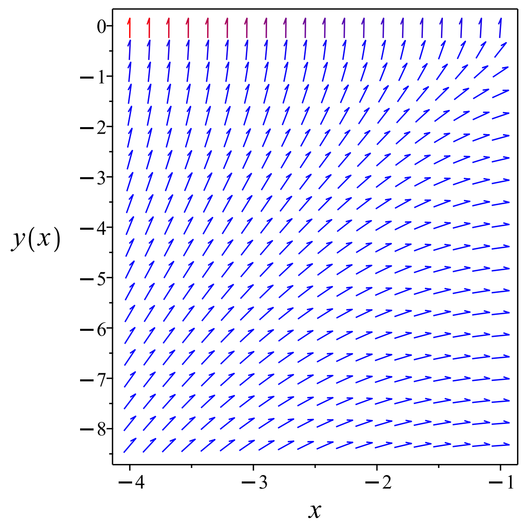
Summary

The solution(s) found are the following

$$y = \sqrt{-x^3 - 1} \tag{1}$$
$$y = -\sqrt{-x^3 - 1} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{-x^3 - 1}$$

Verified OK.

$$y = -\sqrt{-x^3 - 1}$$

Verified OK.

8.16.6 Maple step by step solution

Let's solve

$\left[y' + \frac{3x^2}{2y} = 0, y(-1) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = -\frac{3x^2}{2}$

- Integrate both sides with respect to x

  $\int yy'dx = \int -\frac{3x^2}{2}dx + c_1$

- Evaluate integral

$\frac{y^2}{2} = -\frac{x^3}{2} + c_1$

- Solve for y

  $\left\{y = \sqrt{-x^3 + 2c_1}, y = -\sqrt{-x^3 + 2c_1}\right\}$

- Use initial condition $y(-1) = 0$

  $0 = \sqrt{2c_1 + 1}$

- Solve for c_1

  $c_1 = -\frac{1}{2}$

- Substitute $c_1 = -\frac{1}{2}$ into general solution and simplify

  $y = \sqrt{-x^3 - 1}$

- Use initial condition $y(-1) = 0$

  $0 = -\sqrt{2c_1 + 1}$

- Solve for c_1

  $c_1 = -\frac{1}{2}$

- Substitute $c_1 = -\frac{1}{2}$ into general solution and simplify

  $y = -\sqrt{-x^3 - 1}$

- Solutions to the IVP

  $\left\{y = \sqrt{-x^3 - 1}, y = -\sqrt{-x^3 - 1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 27

```
dsolve([diff(y(x),x)=-3*x^2/(2*y(x)),y(-1) = 0],y(x), singsol=all)
```

$$y(x) = \sqrt{-x^3 - 1}$$
$$y(x) = -\sqrt{-x^3 - 1}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 33

```
DSolve[{y'[x]==-3*x^2/(2*y[x]),{y[-1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x^3 - 1}$$
$$y(x) \to \sqrt{-x^3 - 1}$$

8.17 problem 7 (d)

Internal problem ID [12715]
Internal file name [`OUTPUT/11367_Friday_November_03_2023_06_31_11_AM_38011253/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 7 (d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \frac{3x^2}{2y} = 0}$$

With initial conditions

$$[y(-1) = -1]$$

8.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{3x^2}{2y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $f(x, y)$ when $x = -1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{3x^2}{2y}\right)\\ &= \frac{3x^2}{2y^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

8.17.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= -\frac{3x^2}{2y}\end{aligned}$$

Where $f(x) = -\frac{3x^2}{2}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{y}}\,dy &= -\frac{3x^2}{2}\,dx\\ \int \frac{1}{\frac{1}{y}}\,dy &= \int -\frac{3x^2}{2}\,dx\\ \frac{y^2}{2} &= -\frac{x^3}{2} + c_1\end{aligned}$$

Which results in

$$y = \sqrt{-x^3 + 2c_1}$$
$$y = -\sqrt{-x^3 + 2c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\sqrt{2c_1 + 1}$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = -\sqrt{-x^3}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

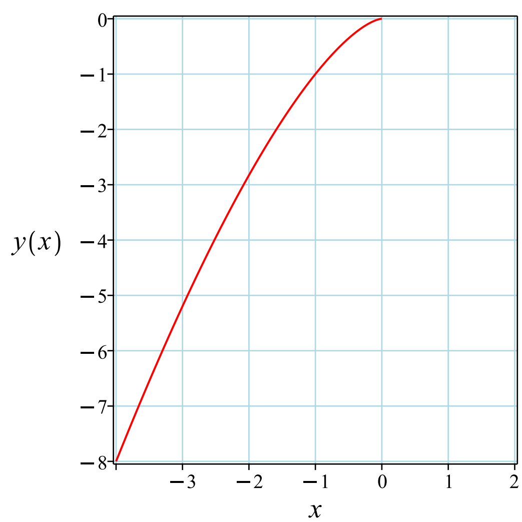
$$-1 = \sqrt{2c_1 + 1}$$

Warning: Unable to solve for constant of integration.

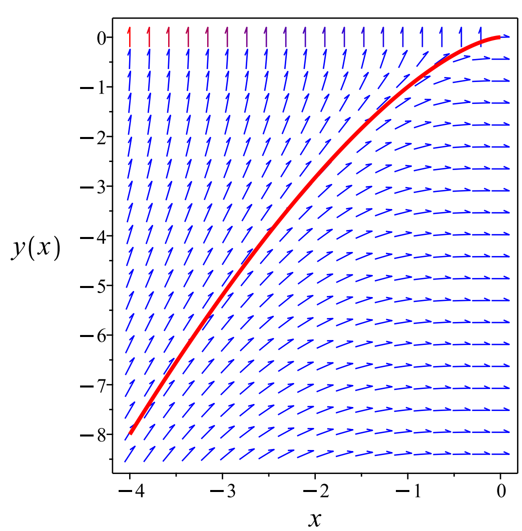
Summary

The solution(s) found are the following

$$y = -\sqrt{-x^3}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\sqrt{-x^3}$$

Verified OK.

8.17.3 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{3x^2}{2y} \tag{1}$$

Which becomes

$$(2y)\,dy = \left(-3x^2\right)dx \tag{2}$$

But the RHS is complete differential because

$$\left(-3x^2\right)dx = d\left(-x^3\right)$$

Hence (2) becomes

$$(2y)\,dy = d\left(-x^3\right)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{-x^3 + c_1} + c_1$$
$$y = -\sqrt{-x^3 + c_1} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\sqrt{1 + c_1} + c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = -\sqrt{-x^3}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \sqrt{1 + c_1} + c_1$$

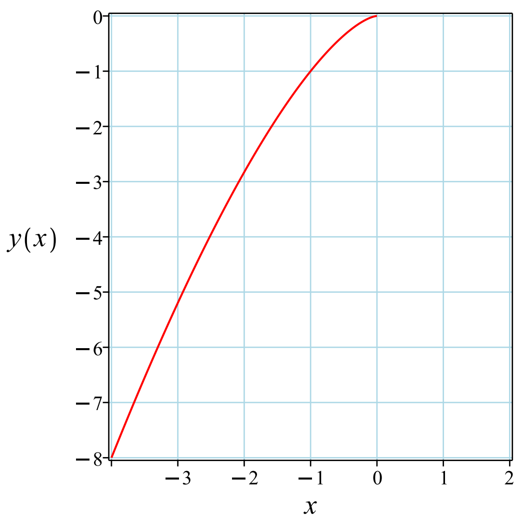
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

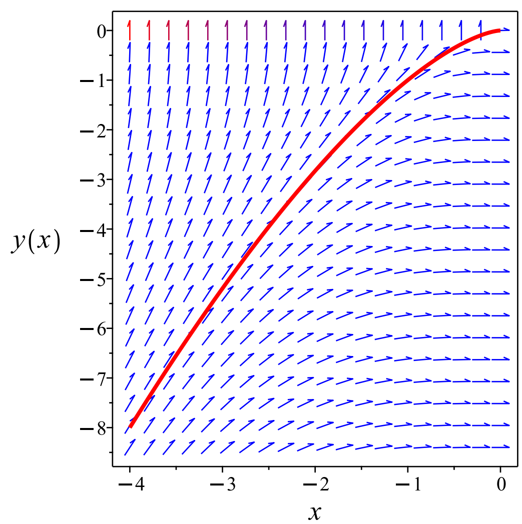
$$y = \sqrt{-x^3 - 1} - 1$$

Summary
The solution(s) found are the following

$$y = \sqrt{-x^3 - 1} - 1 \tag{1}$$
$$y = -\sqrt{-x^3} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{-x^3 - 1} - 1$$

Verified OK.

$$y = -\sqrt{-x^3}$$

Verified OK.

8.17.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3x^2}{2y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 262: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{2}{3x^2} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{2}{3x^2}} dx
\end{aligned}
$$

Which results in

$$
S = -\frac{x^3}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{3x^2}{2y}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{3x^2}{2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = y \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R^2}{2} + c_1 \tag{4}
$$

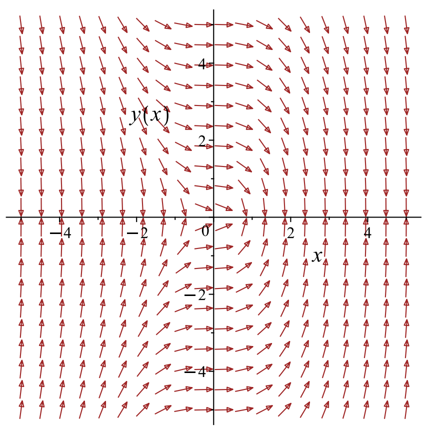
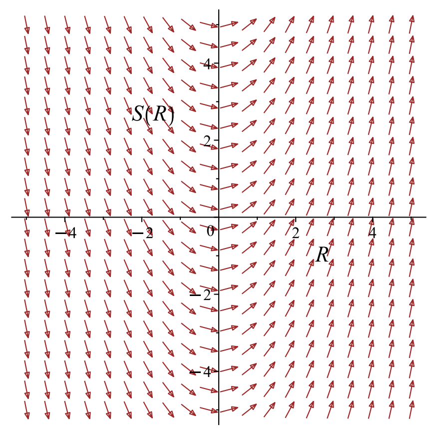
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{x^3}{2} = \frac{y^2}{2} + c_1
$$

Which simplifies to

$$
-\frac{x^3}{2} = \frac{y^2}{2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3x^2}{2y}$ 	$R = y$ $S = -\frac{x^3}{2}$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{1}{2} + c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^3}{2} = \frac{y^2}{2}$$

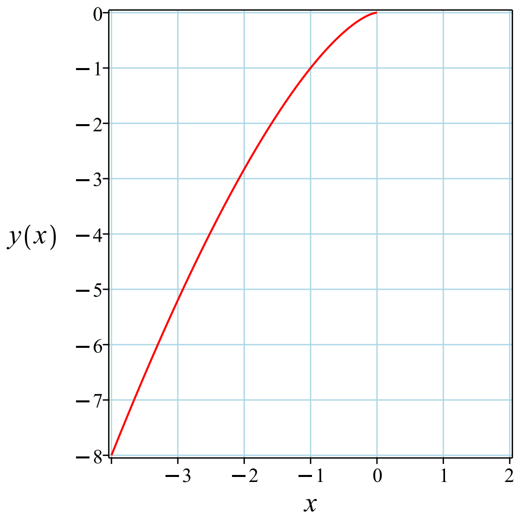
Solving for y from the above gives

$$y = -(-x)^{\frac{3}{2}}$$

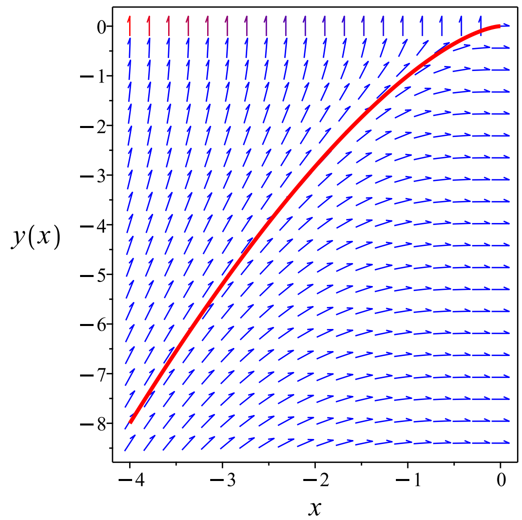
Summary

The solution(s) found are the following

$$y = -(-x)^{\frac{3}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -(-x)^{\frac{3}{2}}$$

Verified OK.

8.17.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{2y}{3}\right)\mathrm{d}y &= (x^2)\,\mathrm{d}x\\ (-x^2)\,\mathrm{d}x + \left(-\frac{2y}{3}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^2\\ N(x, y) &= -\frac{2y}{3}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^2)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{2y}{3}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x^2 \, \mathrm{d}x$$

$$\phi = -\frac{x^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{3}$. Therefore equation (4) becomes

$$-\frac{2y}{3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{2y}{3}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{2y}{3}\right) \mathrm{d}y$$

$$f(y) = -\frac{y^2}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} - \frac{y^2}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} - \frac{y^2}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^3}{3} - \frac{y^2}{3} = 0$$

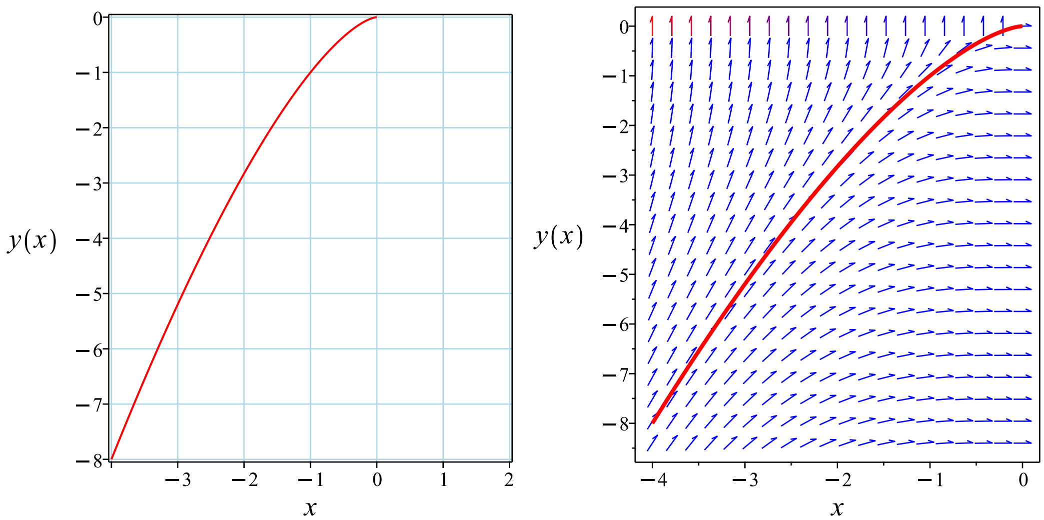
Solving for y from the above gives

$$y = -(-x)^{\frac{3}{2}}$$

Summary

The solution(s) found are the following

$$y = -(-x)^{\frac{3}{2}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -(-x)^{\frac{3}{2}}$$

Verified OK.

8.17.6 Maple step by step solution

Let's solve

$\left[y' + \frac{3x^2}{2y} = 0, y(-1) = -1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $yy' = -\frac{3x^2}{2}$

- Integrate both sides with respect to x

  $\int yy'dx = \int -\frac{3x^2}{2}dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = -\frac{x^3}{2} + c_1$

- Solve for y

  $\{y = \sqrt{-x^3 + 2c_1}, y = -\sqrt{-x^3 + 2c_1}\}$

- Use initial condition $y(-1) = -1$

  $-1 = \sqrt{2c_1 + 1}$
- Solution does not satisfy initial condition
- Use initial condition $y(-1) = -1$

  $-1 = -\sqrt{2c_1 + 1}$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = -\sqrt{-x^3}$
- Solution to the IVP

  $y = -\sqrt{-x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 11

```
dsolve([diff(y(x),x)=-3*x^2/(2*y(x)),y(-1) = -1],y(x), singsol=all)
```

$$y(x) = -(-x)^{\frac{3}{2}}$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 16

```
DSolve[{y'[x]==-3*x^2/(2*y[x]),{y[-1]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x^3}$$

8.18 problem 8 (a)

Internal problem ID [12716]
Internal file name [`OUTPUT/11368_Friday_November_03_2023_06_31_12_AM_81644714/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 8 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{\sqrt{y}}{x} = 0}$$

With initial conditions

$$[y(-1) = 1]$$

8.18.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{\sqrt{y}}{x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $f(x, y)$ when $x = -1$ is

$$\{0 \leq y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\sqrt{y}}{x}\right)\\ &= \frac{1}{2\sqrt{y}\, x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -1$ is

$$\{0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.18.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= \frac{\sqrt{y}}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \sqrt{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\sqrt{y}}\, dy &= \frac{1}{x}\, dx\\ \int \frac{1}{\sqrt{y}}\, dy &= \int \frac{1}{x}\, dx\\ 2\sqrt{y} &= \ln(x) + c_1\end{aligned}$$

The solution is

$$2\sqrt{y} - \ln(x) - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-i\pi - c_1 + 2 = 0$$

$$c_1 = -i\pi + 2$$

Substituting c_1 found above in the general solution gives

$$2\sqrt{y} - \ln(x) - 2 + i\pi = 0$$

Solving for y from the above gives

$$y = -\frac{i\ln(x)\,\pi}{2} - i\pi - \frac{\pi^2}{4} + \frac{\ln(x)^2}{4} + \ln(x) + 1$$

Summary

The solution(s) found are the following

$$y = -\frac{i\ln(x)\,\pi}{2} - i\pi - \frac{\pi^2}{4} + \frac{\ln(x)^2}{4} + \ln(x) + 1 \tag{1}$$

Verification of solutions

$$y = -\frac{i\ln(x)\,\pi}{2} - i\pi - \frac{\pi^2}{4} + \frac{\ln(x)^2}{4} + \ln(x) + 1$$

Verified OK.

8.18.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\sqrt{y}}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 265: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= x \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x} dx \end{aligned}$$

Which results in

$$S = \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\sqrt{y}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{y}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{R}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 2\sqrt{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = 2\sqrt{y} + c_1$$

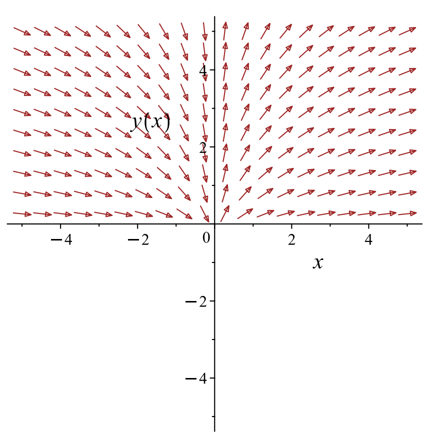
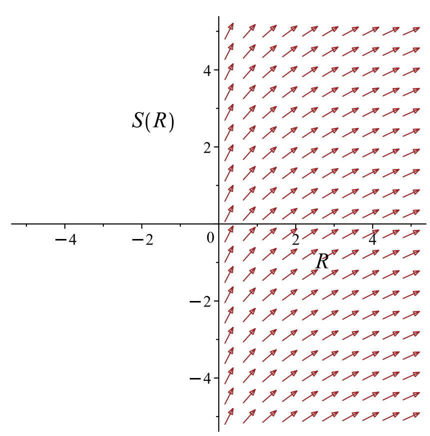
Which simplifies to

$$\ln(x) = 2\sqrt{y} + c_1$$

Which gives

$$y = \frac{\ln(x)^2}{4} - \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{y}}{x}$ 	$R = y$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{1}{\sqrt{R}}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{4}\pi^2 - \frac{1}{2}ic_1\pi + \frac{1}{4}c_1^2$$

$$
c_1 = i\pi - 2
$$

Substituting c_1 found above in the general solution gives

$$
y = -\frac{i \ln(x) \pi}{2} - i\pi - \frac{\pi^2}{4} + \frac{\ln(x)^2}{4} + \ln(x) + 1
$$

Summary

The solution(s) found are the following

$$
y = -\frac{i \ln(x) \pi}{2} - i\pi - \frac{\pi^2}{4} + \frac{\ln(x)^2}{4} + \ln(x) + 1 \tag{1}
$$

Verification of solutions

$$
y = -\frac{i \ln(x) \pi}{2} - i\pi - \frac{\pi^2}{4} + \frac{\ln(x)^2}{4} + \ln(x) + 1
$$

Verified OK.

8.18.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{\sqrt{y}}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{\sqrt{y}}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{1}{x}\\ N(x,y) &= \frac{1}{\sqrt{y}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{y}}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{x} \,\mathrm{d}x \\
\phi &= -\ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{y}}$. Therefore equation (4) becomes

$$
\frac{1}{\sqrt{y}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{\sqrt{y}}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{1}{\sqrt{y}}\right) \mathrm{d}y \\
f(y) &= 2\sqrt{y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = 2\sqrt{y} - \ln(x) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = 2\sqrt{y} - \ln(x)
$$

The solution becomes

$$y=\frac{\ln(x)^2}{4}+\frac{c_1\ln(x)}{2}+\frac{c_1^2}{4}$$

Initial conditions are used to solve for c_1. Substituting $x=-1$ and $y=1$ in the above solution gives an equation to solve for the constant of integration.

$$1=-\frac{1}{4}\pi^2+\frac{1}{2}ic_1\pi+\frac{1}{4}c_1^2$$

$$c_1=-i\pi-2$$

Substituting c_1 found above in the general solution gives

$$y=\frac{\ln(x)^2}{4}-\frac{i\ln(x)\pi}{2}-\ln(x)-\frac{\pi^2}{4}+i\pi+1$$

Summary
The solution(s) found are the following

$$y=\frac{\ln(x)^2}{4}-\frac{i\ln(x)\pi}{2}-\ln(x)-\frac{\pi^2}{4}+i\pi+1 \tag{1}$$

Verification of solutions

$$y=\frac{\ln(x)^2}{4}-\frac{i\ln(x)\pi}{2}-\ln(x)-\frac{\pi^2}{4}+i\pi+1$$

Verified OK.

8.18.5 Maple step by step solution

Let's solve

$\left[y'-\frac{\sqrt{y}}{x}=0, y(-1)=1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{y}}=\frac{1}{x}$

- Integrate both sides with respect to x

  $\int\frac{y'}{\sqrt{y}}dx=\int\frac{1}{x}dx+c_1$

- Evaluate integral

  $2\sqrt{y} = \ln(x) + c_1$

- Solve for y

  $y = \frac{\ln(x)^2}{4} + \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$

- Use initial condition $y(-1) = 1$

  $1 = -\frac{\pi^2}{4} + \frac{\mathrm{I} c_1 \pi}{2} + \frac{c_1^2}{4}$

- Solve for c_1

  $c_1 = (-\mathrm{I}\pi - 2, 2 - \mathrm{I}\pi)$

- Substitute $c_1 = (-\mathrm{I}\pi - 2, 2 - \mathrm{I}\pi)$ into general solution and simplify

  $y = \frac{\ln(x)^2}{4} - \frac{\mathrm{I}\ln(x)\pi}{2} - \ln(x) - \frac{\pi^2}{4} + \mathrm{I}\pi + 1$

- Solution to the IVP

  $y = \frac{\ln(x)^2}{4} - \frac{\mathrm{I}\ln(x)\pi}{2} - \ln(x) - \frac{\pi^2}{4} + \mathrm{I}\pi + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 29

```
dsolve([diff(y(x),x)=sqrt(y(x))/x,y(-1) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{i\pi \ln(x)}{2} - i\pi - \frac{\pi^2}{4} + \frac{\ln(x)^2}{4} + \ln(x) + 1$$

✓ Solution by Mathematica

Time used: 0.235 (sec). Leaf size: 43

```
DSolve[{y'[x]==Sqrt[y[x]]/x,{y[-1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{4}(i\log(x) + \pi - 2i)^2$$
$$y(x) \to -\frac{1}{4}(i\log(x) + \pi + 2i)^2$$

8.19 problem 8 (b)

Internal problem ID [12717]
Internal file name [`OUTPUT/11369_Friday_November_03_2023_06_31_13_AM_5762966/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 8 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{\sqrt{y}}{x} = 0}$$

With initial conditions

$$[y(-1) = 0]$$

8.19.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{\sqrt{y}}{x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $f(x, y)$ when $x = -1$ is

$$\{0 \leq y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\sqrt{y}}{x}\right)\\ &= \frac{1}{2\sqrt{y}\,x}\end{aligned}$$

$\frac{\partial f}{\partial y}$ is not defined at $y = 0$ therefore existence and uniqueness theorem do not apply. Solution exist but not guaranteed to be unique.

8.19.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= \frac{\sqrt{y}}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \sqrt{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\sqrt{y}}\,dy &= \frac{1}{x}\,dx\\ \int \frac{1}{\sqrt{y}}\,dy &= \int \frac{1}{x}\,dx\\ 2\sqrt{y} &= \ln(x) + c_1\end{aligned}$$

The solution is

$$2\sqrt{y} - \ln(x) - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-i\pi - c_1 = 0$$

$$c_1 = -i\pi$$

Substituting c_1 found above in the general solution gives

$$2\sqrt{y}-\ln(x)+i\pi=0$$

Solving for y from the above gives

$$y=-\frac{i\ln(x)\,\pi}{2}+\frac{\ln(x)^2}{4}-\frac{\pi^2}{4}$$

Summary
The solution(s) found are the following

$$y=-\frac{i\ln(x)\,\pi}{2}+\frac{\ln(x)^2}{4}-\frac{\pi^2}{4} \tag{1}$$

Verification of solutions

$$y=-\frac{i\ln(x)\,\pi}{2}+\frac{\ln(x)^2}{4}-\frac{\pi^2}{4}$$

Verified OK.

8.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=\frac{\sqrt{y}}{x}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 268: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x} dx
\end{aligned}$$

Which results in

$$S = \ln (x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\sqrt{y}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{y}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{R}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 2\sqrt{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = 2\sqrt{y} + c_1$$

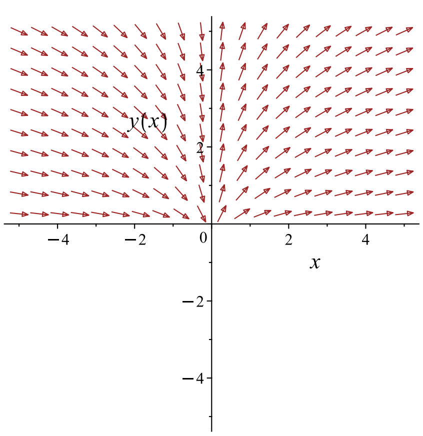
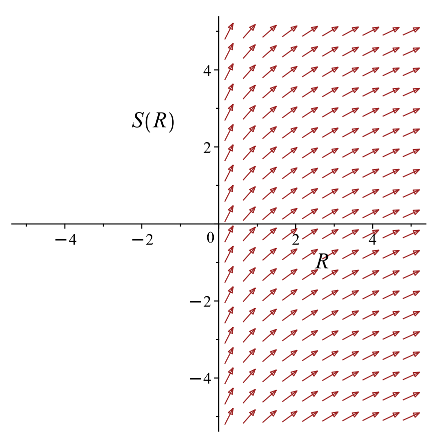
Which simplifies to

$$\ln(x) = 2\sqrt{y} + c_1$$

Which gives

$$y = \frac{\ln(x)^2}{4} - \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{y}}{x}$ 	$R = y$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{1}{\sqrt{R}}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{4}\pi^2 - \frac{1}{2} i c_1 \pi + \frac{1}{4} c_1^2$$

$$c_1 = i\pi$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{i\ln(x)\pi}{2} + \frac{\ln(x)^2}{4} - \frac{\pi^2}{4}$$

Summary

The solution(s) found are the following

$$y = -\frac{i\ln(x)\pi}{2} + \frac{\ln(x)^2}{4} - \frac{\pi^2}{4} \tag{1}$$

Verification of solutions

$$y = -\frac{i\ln(x)\pi}{2} + \frac{\ln(x)^2}{4} - \frac{\pi^2}{4}$$

Verified OK.

8.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{\sqrt{y}}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{\sqrt{y}}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{1}{x}\\ N(x,y) &= \frac{1}{\sqrt{y}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{y}}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{x} \, \mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{y}}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{y}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{y}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{\sqrt{y}}\right) \mathrm{d}y$$
$$f(y) = 2\sqrt{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = 2\sqrt{y} - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2\sqrt{y} - \ln(x)$$

The solution becomes

$$y = \frac{\ln(x)^2}{4} + \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{1}{4}\pi^2 + \frac{1}{2}ic_1\pi + \frac{1}{4}c_1^2$$

$$c_1 = -i\pi$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{i\ln(x)\pi}{2} + \frac{\ln(x)^2}{4} - \frac{\pi^2}{4}$$

Summary

The solution(s) found are the following

$$y = -\frac{i\ln(x)\pi}{2} + \frac{\ln(x)^2}{4} - \frac{\pi^2}{4} \tag{1}$$

Verification of solutions

$$y = -\frac{i\ln(x)\pi}{2} + \frac{\ln(x)^2}{4} - \frac{\pi^2}{4}$$

Verified OK.

8.19.5 Maple step by step solution

Let's solve

$\left[y' - \frac{\sqrt{y}}{x} = 0, y(-1) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{y}} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y}}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $2\sqrt{y} = \ln(x) + c_1$
- Solve for y

  $y = \frac{\ln(x)^2}{4} + \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$
- Use initial condition $y(-1) = 0$

  $0 = -\frac{\pi^2}{4} + \frac{\mathrm{I} c_1 \pi}{2} + \frac{c_1^2}{4}$
- Solve for c_1

  $c_1 = (-\mathrm{I}\pi, -\mathrm{I}\pi)$
- Substitute $c_1 = (-\mathrm{I}\pi, -\mathrm{I}\pi)$ into general solution and simplify

  $y = -\frac{\mathrm{I}\ln(x)\pi}{2} + \frac{\ln(x)^2}{4} - \frac{\pi^2}{4}$
- Solution to the IVP

  $y = -\frac{\mathrm{I}\ln(x)\pi}{2} + \frac{\ln(x)^2}{4} - \frac{\pi^2}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=sqrt(y(x))/x,y(-1) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.157 (sec). Leaf size: 24

```
DSolve[{y'[x]==Sqrt[y[x]]/x,{y[-1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$
$$y(x) \to -\frac{1}{4}(\pi + i\log(x))^2$$

8.20 problem 8 (c)

Internal problem ID [12718]
Internal file name [`OUTPUT/11370_Friday_November_03_2023_06_31_14_AM_37716840/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 8 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{\sqrt{y}}{x} = 0}$$

With initial conditions

$$[y(-1) = -1]$$

8.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{\sqrt{y}}{x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -1$ is

$$\{0 < x\}$$

But the point $x_0 = -1$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

8.20.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\sqrt{y}}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \sqrt{y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\sqrt{y}}\, dy &= \frac{1}{x}\, dx \\ \int \frac{1}{\sqrt{y}}\, dy &= \int \frac{1}{x}\, dx \\ 2\sqrt{y} &= \ln(x) + c_1 \end{aligned}$$

The solution is

$$2\sqrt{y} - \ln(x) - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-i\pi - c_1 + 2i = 0$$

$$c_1 = -i\pi + 2i$$

Substituting c_1 found above in the general solution gives

$$2\sqrt{y} - \ln(x) - 2i + i\pi = 0$$

Solving for y from the above gives

$$y = \frac{\ln(x)^2}{4} + \frac{i(2-\pi)\ln(x)}{2} - \frac{(-2+\pi)^2}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^2}{4} + \frac{i(2-\pi)\ln(x)}{2} - \frac{(-2+\pi)^2}{4} \tag{1}$$

Verification of solutions

$$y = \frac{\ln(x)^2}{4} + \frac{i(2-\pi)\ln(x)}{2} - \frac{(-2+\pi)^2}{4}$$

Verified OK.

8.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\sqrt{y}}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 271: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= x \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x} dx \end{aligned}$$

Which results in

$$S = \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\sqrt{y}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{y}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{R}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\sqrt{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = 2\sqrt{y} + c_1$$

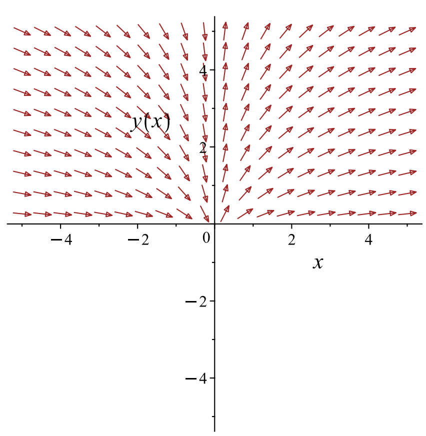
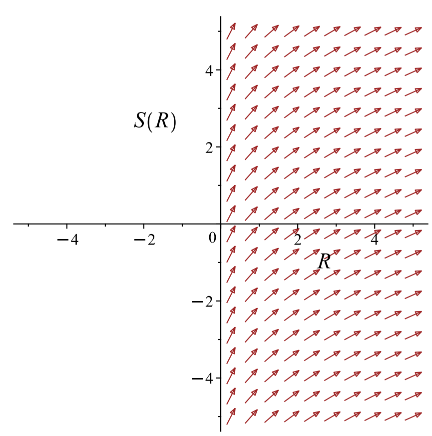
Which simplifies to

$$\ln(x) = 2\sqrt{y} + c_1$$

Which gives

$$y = \frac{\ln(x)^2}{4} - \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{y}}{x}$ 	$R = y$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{1}{\sqrt{R}}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{1}{4}\pi^2 - \frac{1}{2}ic_1\pi + \frac{1}{4}c_1^2$$

$$c_1 = i\pi - 2i$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{i \ln(x) \pi}{2} + \frac{\ln(x)^2}{4} + i \ln(x) - \frac{\pi^2}{4} + \pi - 1$$

Summary

The solution(s) found are the following

$$y = -\frac{i \ln(x) \pi}{2} + \frac{\ln(x)^2}{4} + i \ln(x) - \frac{\pi^2}{4} + \pi - 1 \tag{1}$$

Verification of solutions

$$y = -\frac{i \ln(x) \pi}{2} + \frac{\ln(x)^2}{4} + i \ln(x) - \frac{\pi^2}{4} + \pi - 1$$

Verified OK.

8.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{\sqrt{y}}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{\sqrt{y}}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{1}{x}\\ N(x,y) &= \frac{1}{\sqrt{y}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{y}}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{1}{x}\,\mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{y}}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{y}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{y}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{\sqrt{y}}\right)\mathrm{d}y$$
$$f(y) = 2\sqrt{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = 2\sqrt{y} - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2\sqrt{y} - \ln(x)$$

The solution becomes

$$y = \frac{\ln(x)^2}{4} + \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{1}{4}\pi^2 + \frac{1}{2}ic_1\pi + \frac{1}{4}c_1^2$$

$$c_1 = -i\pi - 2i$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(x)^2}{4} - \frac{i\ln(x)\pi}{2} - i\ln(x) - \frac{\pi^2}{4} - \pi - 1$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^2}{4} - \frac{i\ln(x)\pi}{2} - i\ln(x) - \frac{\pi^2}{4} - \pi - 1 \tag{1}$$

Verification of solutions

$$y = \frac{\ln(x)^2}{4} - \frac{i\ln(x)\pi}{2} - i\ln(x) - \frac{\pi^2}{4} - \pi - 1$$

Verified OK.

8.20.5 Maple step by step solution

Let's solve

$\left[y' - \frac{\sqrt{y}}{x} = 0, y(-1) = -1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{y}} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y}}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $2\sqrt{y} = \ln(x) + c_1$
- Solve for y

  $y = \frac{\ln(x)^2}{4} + \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$
- Use initial condition $y(-1) = -1$

  $-1 = -\frac{\pi^2}{4} + \frac{\mathrm{I} c_1 \pi}{2} + \frac{c_1^2}{4}$
- Solve for c_1

  $c_1 = (-\mathrm{I}\pi - 2\,\mathrm{I}, 2\,\mathrm{I} - \mathrm{I}\pi)$
- Substitute $c_1 = (-\mathrm{I}\pi - 2\,\mathrm{I}, 2\,\mathrm{I} - \mathrm{I}\pi)$ into general solution and simplify

  $y = \frac{\ln(x)^2}{4} + \frac{\mathrm{I}(-\pi-2)\ln(x)}{2} - \frac{(\pi+2)^2}{4}$
- Solution to the IVP

  $y = \frac{\ln(x)^2}{4} + \frac{\mathrm{I}(-\pi-2)\ln(x)}{2} - \frac{(\pi+2)^2}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.093 (sec). Leaf size: 28

```
dsolve([diff(y(x),x)=sqrt(y(x))/x,y(-1) = -1],y(x), singsol=all)
```

$$y(x) = \frac{\ln(x)^2}{4} + \frac{i(2-\pi)\ln(x)}{2} - \frac{(-2+\pi)^2}{4}$$

✓ Solution by Mathematica

Time used: 0.151 (sec). Leaf size: 39

```
DSolve[{y'[x]==Sqrt[y[x]]/x,{y[-1]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{1}{4}(i \log(x) + \pi + 2)^2$$
$$y(x) \rightarrow -\frac{1}{4}(i \log(x) + \pi - 2)^2$$

8.21 problem 8 (d)

Internal problem ID [12719]
Internal file name [`OUTPUT/11371_Friday_November_03_2023_06_31_16_AM_28112914/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 8 (d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{\sqrt{y}}{x} = 0}$$

With initial conditions

$$[y(1) = 1]$$

8.21.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{\sqrt{y}}{x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{0 \leq y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\sqrt{y}}{x}\right)\\ &= \frac{1}{2\sqrt{y}\,x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.21.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= \frac{\sqrt{y}}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \sqrt{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\sqrt{y}}\,dy &= \frac{1}{x}\,dx\\ \int \frac{1}{\sqrt{y}}\,dy &= \int \frac{1}{x}\,dx\\ 2\sqrt{y} &= \ln(x) + c_1\end{aligned}$$

The solution is

$$2\sqrt{y} - \ln(x) - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$2 - c_1 = 0$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$2\sqrt{y} - \ln(x) - 2 = 0$$

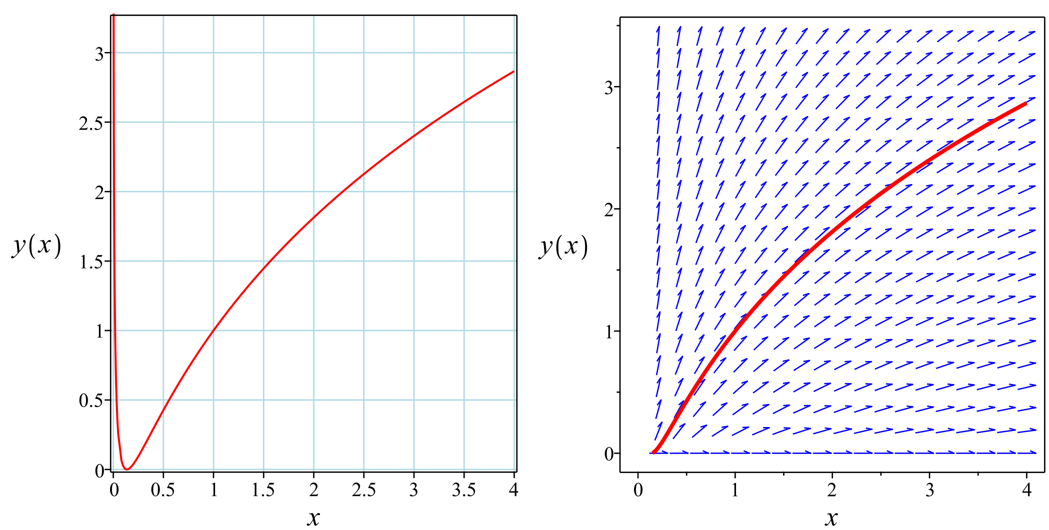
Solving for y from the above gives

$$y = \frac{(\ln(x) + 2)^2}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{(\ln(x) + 2)^2}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{(\ln(x) + 2)^2}{4}$$

Verified OK.

8.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\sqrt{y}}{x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 274: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x \\
\eta(x, y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x} dx
\end{aligned}
$$

Which results in

$$
S = \ln(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \qquad (2)
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{\sqrt{y}}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{y}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{R}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\sqrt{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = 2\sqrt{y} + c_1$$

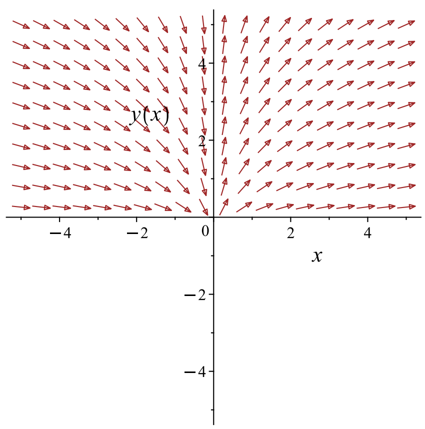
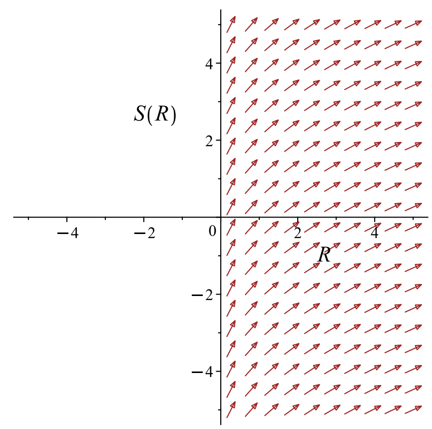
Which simplifies to

$$\ln(x) = 2\sqrt{y} + c_1$$

Which gives

$$y = \frac{\ln(x)^2}{4} - \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{y}}{x}$ 	$R = y$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{1}{\sqrt{R}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1^2}{4}$$

$$c_1 = -2$$

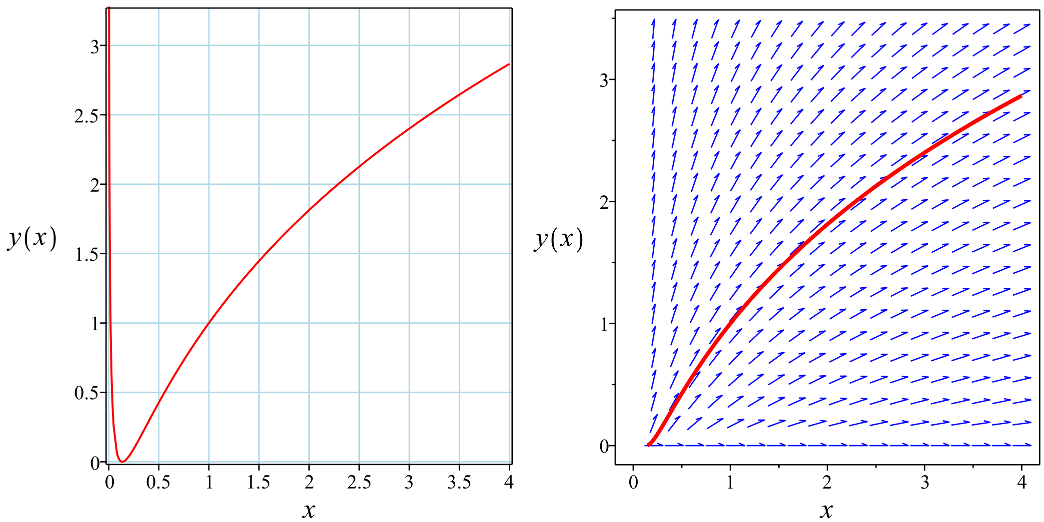
Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(x)^2}{4} + \ln(x) + 1$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^2}{4} + \ln(x) + 1 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\ln(x)^2}{4} + \ln(x) + 1$$

Verified OK.

8.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{\sqrt{y}}\right)\mathrm{d}y&=\left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x+\left(\frac{1}{\sqrt{y}}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{x}\\ N(x,y)&=\frac{1}{\sqrt{y}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{y}}\right)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{x} \, \mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{y}}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{y}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{y}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{\sqrt{y}}\right) \mathrm{d}y$$
$$f(y) = 2\sqrt{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = 2\sqrt{y} - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2\sqrt{y} - \ln(x)$$

The solution becomes

$$y = \frac{\ln(x)^2}{4} + \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1^2}{4}$$

$$c_1 = -2$$

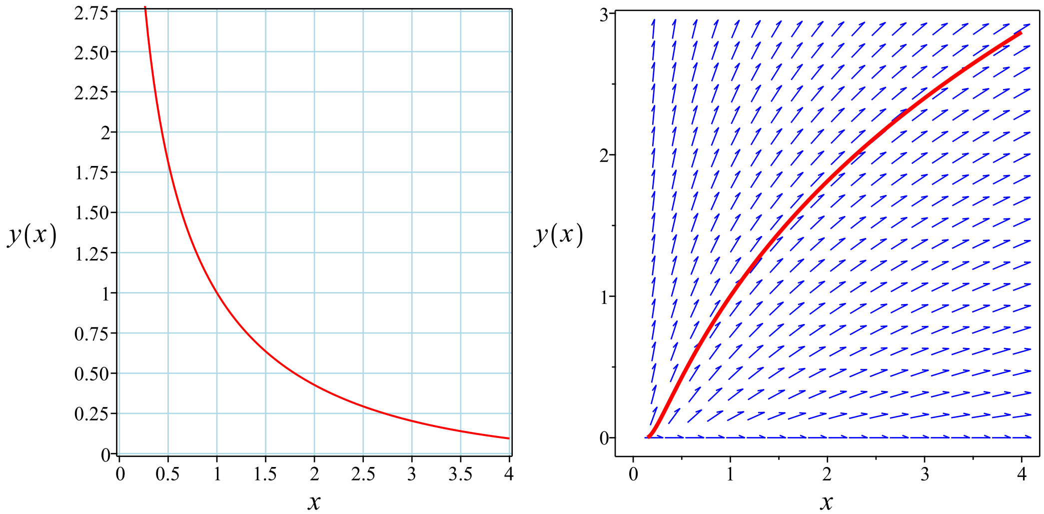
Substituting c_1 found above in the general solution gives

$$y = \frac{\ln(x)^2}{4} - \ln(x) + 1$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^2}{4} - \ln(x) + 1 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\ln (x)^2}{4} - \ln (x) + 1$$

Verified OK.

8.21.5 Maple step by step solution

Let's solve

$\left[y' - \frac{\sqrt{y}}{x} = 0, y(1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{y}} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y}} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $2\sqrt{y} = \ln (x) + c_1$

- Solve for y

$y = \frac{\ln(x)^2}{4} + \frac{c_1 \ln(x)}{2} + \frac{c_1^2}{4}$

- Use initial condition $y(1) = 1$

  $1 = \frac{c_1^2}{4}$

- Solve for c_1

  $c_1 = (-2, 2)$

- Substitute $c_1 = (-2, 2)$ into general solution and simplify

  $y = \frac{(\ln(x)-2)^2}{4}$

- Solution to the IVP

  $y = \frac{(\ln(x)-2)^2}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([diff(y(x),x)=sqrt(y(x))/x,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{(\ln(x) + 2)^2}{4}$$

✓ Solution by Mathematica

Time used: 0.151 (sec). Leaf size: 29

```
DSolve[{y'[x]==Sqrt[y[x]]/x,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}(\log(x) - 2)^2$$
$$y(x) \to \frac{1}{4}(\log(x) + 2)^2$$

8.22 problem 9 (a)

Internal problem ID [12720]
Internal file name [`OUTPUT/11372_Friday_November_03_2023_06_31_17_AM_30341911/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 9 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - 3xy^{\frac{1}{3}} = 0}$$

With initial conditions

$$\left[y(-1) = \frac{3}{2}\right]$$

8.22.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= 3x\, y^{\frac{1}{3}} \end{aligned}$$

The x domain of $f(x, y)$ when $y = \frac{3}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $f(x, y)$ when $x = -1$ is

$$\{0 \leq y\}$$

And the point $y_0 = \frac{3}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(3x\, y^{\frac{1}{3}}\right)\\ &= \frac{x}{y^{\frac{2}{3}}}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{3}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -1$ is

$$\{0 < y\}$$

And the point $y_0 = \frac{3}{2}$ is inside this domain. Therefore solution exists and is unique.

8.22.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= 3x\, y^{\frac{1}{3}}\end{aligned}$$

Where $f(x) = 3x$ and $g(y) = y^{\frac{1}{3}}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^{\frac{1}{3}}}\, dy &= 3x\, dx\\ \int \frac{1}{y^{\frac{1}{3}}}\, dy &= \int 3x\, dx\\ \frac{3y^{\frac{2}{3}}}{2} &= \frac{3x^2}{2} + c_1\end{aligned}$$

The solution is

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \frac{3}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3\,2^{\frac{1}{3}}3^{\frac{2}{3}}}{4} - \frac{3}{2} - c_1 = 0$$

$$c_1 = \frac{3\,2^{\frac{1}{3}}3^{\frac{2}{3}}}{4} - \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - \frac{3\,2^{\frac{1}{3}}3^{\frac{2}{3}}}{4} + \frac{3}{2} = 0$$

Summary

The solution(s) found are the following

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - \frac{3\,2^{\frac{1}{3}}3^{\frac{2}{3}}}{4} + \frac{3}{2} = 0 \tag{1}$$

Verification of solutions

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - \frac{3\,2^{\frac{1}{3}}3^{\frac{2}{3}}}{4} + \frac{3}{2} = 0$$

Verified OK.

8.22.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 3x\,y^{\frac{1}{3}}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 277: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{3x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{3x}} dx \end{aligned}$$

Which results in

$$S = \frac{3x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 3x\, y^{\frac{1}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 3x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^{\frac{1}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^{\frac{1}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{3R^{\frac{2}{3}}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3x^2}{2} = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

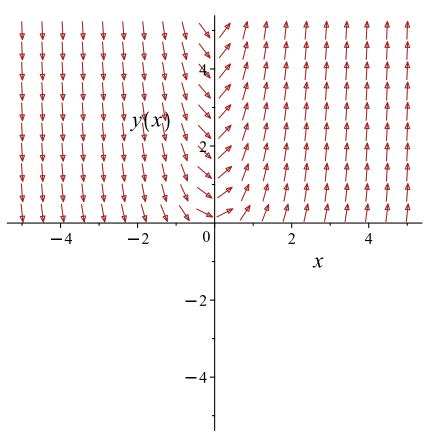
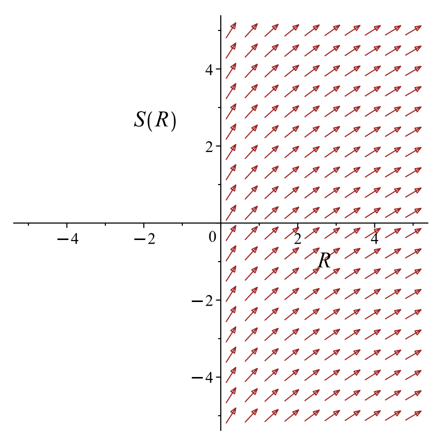
Which simplifies to

$$\frac{3x^2}{2} = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

Which gives

$$y = \frac{(9x^2 - 6c_1)^{\frac{3}{2}}}{27}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3x\, y^{\frac{1}{3}}$  	$R = y$ $S = \frac{3x^2}{2}$	$\frac{dS}{dR} = \frac{1}{R^{\frac{1}{3}}}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \frac{3}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{2} = \frac{(9-6c_1)^{\frac{3}{2}}}{27}$$

$$c_1 = \frac{3}{2} - \frac{3\,2^{\frac{1}{3}}3^{\frac{2}{3}}}{4}$$

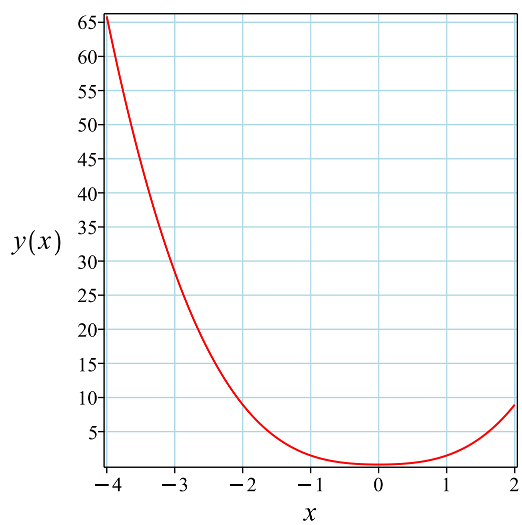
Substituting c_1 found above in the general solution gives

$$y = \frac{\left(2\,2^{\frac{1}{3}}3^{\frac{2}{3}} + 4x^2 - 4\right)^{\frac{3}{2}}}{8}$$

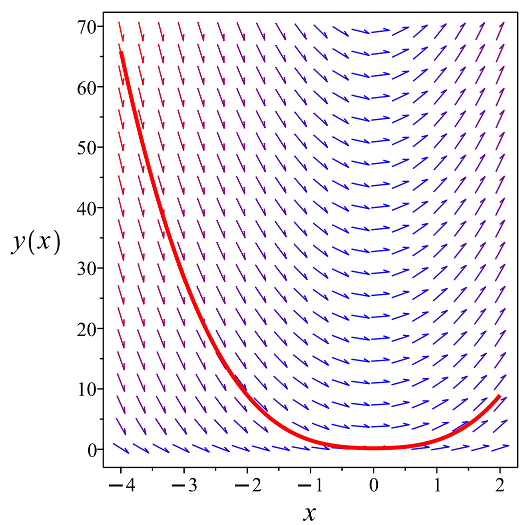
Summary

The solution(s) found are the following

$$y = \frac{\left(2\,2^{\frac{1}{3}}3^{\frac{2}{3}} + 4x^2 - 4\right)^{\frac{3}{2}}}{8} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\left(2\,2^{\frac{1}{3}}3^{\frac{2}{3}} + 4x^2 - 4\right)^{\frac{3}{2}}}{8}$$

Verified OK.

8.22.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{3y^{\frac{1}{3}}}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+\left(\frac{1}{3y^{\frac{1}{3}}}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x \\
N(x, y) &= \frac{1}{3y^{\frac{1}{3}}}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{3y^{\frac{1}{3}}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{3y^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$\frac{1}{3y^{\frac{1}{3}}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{3y^{\frac{1}{3}}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{3y^{\frac{1}{3}}}\right) \mathrm{d}y$$

$$f(y) = \frac{y^{\frac{2}{3}}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^{\frac{2}{3}}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^{\frac{2}{3}}}{2}$$

The solution becomes

$$y = \left(x^2 + 2c_1\right)^{\frac{3}{2}}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \frac{3}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{2} = (2c_1 + 1)^{\frac{3}{2}}$$

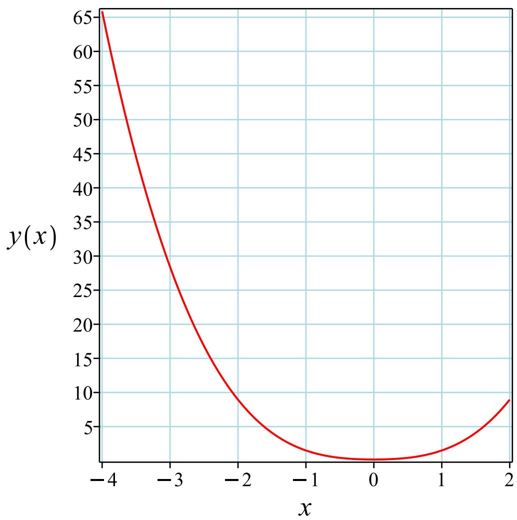
$$c_1 = \frac{2^{\frac{1}{3}} 3^{\frac{2}{3}}}{4} - \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

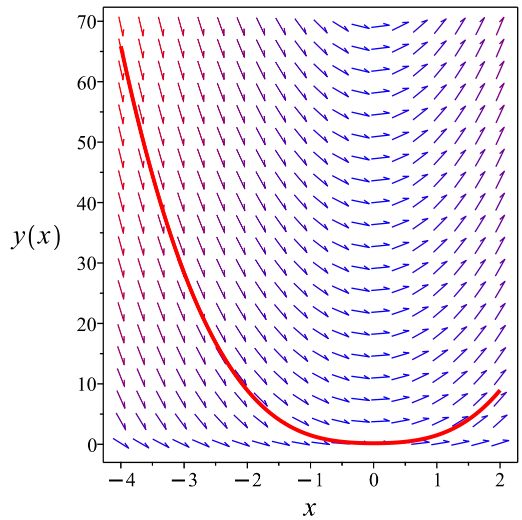
$$y = \frac{\left(2\,2^{\frac{1}{3}}3^{\frac{2}{3}} + 4x^2 - 4\right)^{\frac{3}{2}}}{8}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(2\,2^{\frac{1}{3}}3^{\frac{2}{3}} + 4x^2 - 4\right)^{\frac{3}{2}}}{8} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\left(2\,2^{\frac{1}{3}}3^{\frac{2}{3}} + 4x^2 - 4\right)^{\frac{3}{2}}}{8}$$

Verified OK.

8.22.5 Maple step by step solution

Let's solve

$\left[y' - 3xy^{\frac{1}{3}} = 0, y(-1) = \frac{3}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{1}{3}}} = 3x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{1}{3}}} dx = \int 3x dx + c_1$

- Evaluate integral

  $\frac{3y^{\frac{2}{3}}}{2} = \frac{3x^2}{2} + c_1$

- Solve for y

  $y = \frac{(9x^2+6c_1)^{\frac{3}{2}}}{27}$

- Use initial condition $y(-1) = \frac{3}{2}$

  $\frac{3}{2} = \frac{(6c_1+9)^{\frac{3}{2}}}{27}$

- Solve for c_1

  $c_1 = \frac{3\,2^{\frac{1}{3}}3^{\frac{2}{3}}}{4} - \frac{3}{2}$

- Substitute $c_1 = \frac{3\,2^{\frac{1}{3}}3^{\frac{2}{3}}}{4} - \frac{3}{2}$ into general solution and simplify

  $y = \frac{\left(2^{\frac{1}{3}}3^{\frac{2}{3}}+2x^2-2\right)\sqrt{2\,2^{\frac{1}{3}}3^{\frac{2}{3}}+4x^2-4}}{4}$

- Solution to the IVP

  $y = \frac{\left(2^{\frac{1}{3}}3^{\frac{2}{3}}+2x^2-2\right)\sqrt{2\,2^{\frac{1}{3}}3^{\frac{2}{3}}+4x^2-4}}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.64 (sec). Leaf size: 23

```
dsolve([diff(y(x),x)=3*x*y(x)^(1/3),y(-1) = 3/2],y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{2\,3^{\frac{2}{3}}2^{\frac{1}{3}} + 4x^2 - 4}\,\left(3^{\frac{2}{3}}2^{\frac{1}{3}} + 2x^2 - 2\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.374 (sec). Leaf size: 36

```
DSolve[{y'[x]==3*x*y[x]^(1/3),{y[-1]==3/2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\left(2x^2 + \sqrt[3]{2}3^{2/3} - 2\right)^{3/2}}{2\sqrt{2}}$$

8.23 problem 9 (b)

Internal problem ID [12721]
Internal file name [`OUTPUT/11373_Friday_November_03_2023_06_31_21_AM_53930963/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 9 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - 3xy^{\frac{1}{3}} = 0}$$

With initial conditions

$$[y(-1) = 1]$$

8.23.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= 3x\, y^{\frac{1}{3}} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $f(x, y)$ when $x = -1$ is

$$\{0 \leq y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(3x\, y^{\frac{1}{3}}\right)\\ &= \frac{x}{y^{\frac{2}{3}}}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -1$ is

$$\{0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.23.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= 3x\, y^{\frac{1}{3}}\end{aligned}$$

Where $f(x) = 3x$ and $g(y) = y^{\frac{1}{3}}$. Integrating both sides gives

$$\frac{1}{y^{\frac{1}{3}}}\, dy = 3x\, dx$$

$$\int \frac{1}{y^{\frac{1}{3}}}\, dy = \int 3x\, dx$$

$$\frac{3y^{\frac{2}{3}}}{2} = \frac{3x^2}{2} + c_1$$

The solution is

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-c_1 = 0$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{3x^2}{2} = 0$$

Summary
The solution(s) found are the following

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{3x^2}{2} = 0 \tag{1}$$

Verification of solutions

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{3x^2}{2} = 0$$

Verified OK.

8.23.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 3x\, y^{\frac{1}{3}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 280: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= \frac{1}{3x} \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{3x}} dx \end{aligned}$$

Which results in

$$S = \frac{3x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 3x\, y^{\frac{1}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 3x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^{\frac{1}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^{\frac{1}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{3R^{\frac{2}{3}}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3x^2}{2} = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

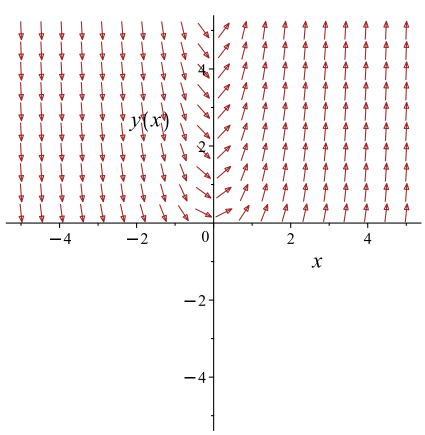
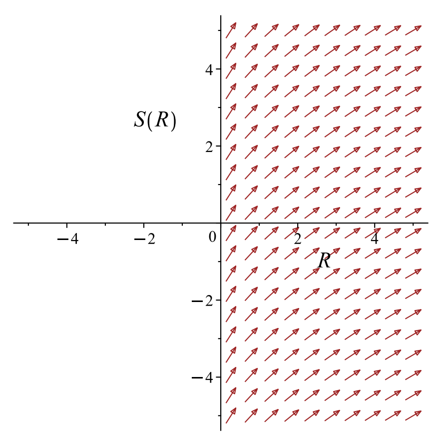
Which simplifies to

$$\frac{3x^2}{2} = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

Which gives

$$y = \frac{(9x^2 - 6c_1)^{\frac{3}{2}}}{27}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3x\, y^{\frac{1}{3}}$ 	$R = y$ $S = \frac{3x^2}{2}$	$\frac{dS}{dR} = \frac{1}{R^{\frac{1}{3}}}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{(9 - 6c_1)^{\frac{3}{2}}}{27}$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = x^3$$

But this does not satisfy the initial conditions. Hence no solution can be found.

Verification of solutions N/A

8.23.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{3y^{\frac{1}{3}}}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{1}{3y^{\frac{1}{3}}}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\\ N(x,y) &= \frac{1}{3y^{\frac{1}{3}}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{3y^{\frac{1}{3}}}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int -x \, dx$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{3y^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$\frac{1}{3y^{\frac{1}{3}}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{3y^{\frac{1}{3}}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int \left(\frac{1}{3y^{\frac{1}{3}}} \right) dy$$
$$f(y) = \frac{y^{\frac{2}{3}}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^{\frac{2}{3}}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^{\frac{2}{3}}}{2}$$

The solution becomes

$$y = \left(x^2 + 2c_1\right)^{\frac{3}{2}}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = (2c_1 + 1)^{\frac{3}{2}}$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = x^3$$

But this does not satisfy the initial conditions. Hence no solution can be found.

Verification of solutions N/A

8.23.5 Maple step by step solution

Let's solve

$\left[y' - 3xy^{\frac{1}{3}} = 0, y(-1) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{1}{3}}} = 3x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{1}{3}}} dx = \int 3x dx + c_1$

- Evaluate integral

$$\frac{3y^{\frac{2}{3}}}{2} = \frac{3x^2}{2} + c_1$$

- Solve for y

$$y = \frac{(9x^2+6c_1)^{\frac{3}{2}}}{27}$$

- Use initial condition $y(-1) = 1$

$$1 = \frac{(6c_1+9)^{\frac{3}{2}}}{27}$$

- Solve for c_1

$$c_1 = 0$$

- Substitute $c_1 = 0$ into general solution and simplify

$$y = \operatorname{csgn}(x)\, x^3$$

- Solution to the IVP

$$y = \operatorname{csgn}(x)\, x^3$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 9

```
dsolve([diff(y(x),x)=3*x*y(x)^(1/3),y(-1) = 1],y(x), singsol=all)
```

$$y(x) = -x^3$$

✓ Solution by Mathematica
Time used: 0.214 (sec). Leaf size: 12

```
DSolve[{y'[x]==3*x*y[x]^(1/3),{y[-1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x^2)^{3/2}$$

8.24 problem 9 (c)

Internal problem ID [12722]
Internal file name [`OUTPUT/11374_Friday_November_03_2023_06_31_22_AM_48526193/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 9 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - 3xy^{\frac{1}{3}} = 0}$$

With initial conditions

$$\left[y(-1) = \frac{1}{2}\right]$$

8.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= 3x\,y^{\frac{1}{3}} \end{aligned}$$

The x domain of $f(x,y)$ when $y = \frac{1}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $f(x, y)$ when $x = -1$ is

$$\{0 \leq y\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(3x\, y^{\frac{1}{3}}\right) \\
&= \frac{x}{y^{\frac{2}{3}}}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{1}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -1$ is

$$\{0 < y\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

8.24.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x) g(y) \\
&= 3x\, y^{\frac{1}{3}}
\end{aligned}$$

Where $f(x) = 3x$ and $g(y) = y^{\frac{1}{3}}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y^{\frac{1}{3}}}\, dy &= 3x\, dx \\
\int \frac{1}{y^{\frac{1}{3}}}\, dy &= \int 3x\, dx \\
\frac{3y^{\frac{2}{3}}}{2} &= \frac{3x^2}{2} + c_1
\end{aligned}$$

The solution is

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3\,2^{\frac{1}{3}}}{4} - \frac{3}{2} - c_1 = 0$$

$$c_1 = \frac{3\,2^{\frac{1}{3}}}{4} - \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

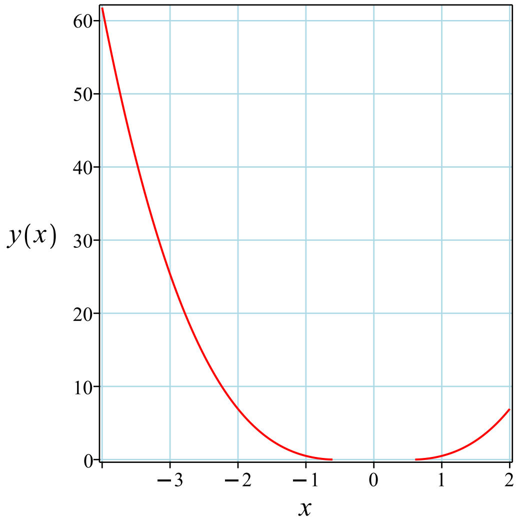
$$\frac{3y^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - \frac{3\,2^{\frac{1}{3}}}{4} + \frac{3}{2} = 0$$

Solving for y from the above gives

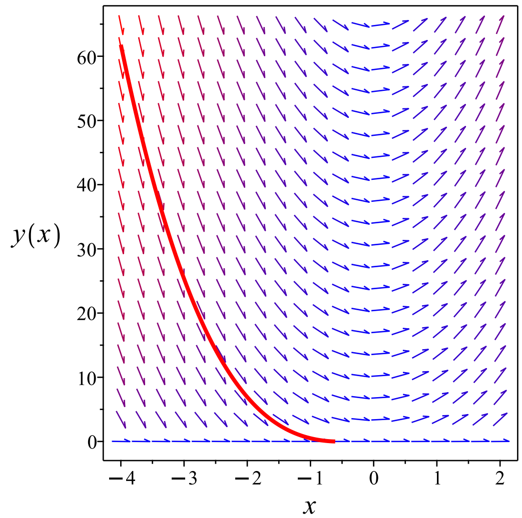
$$y = \frac{\left(2x^2 + 2^{\frac{1}{3}} - 2\right)\sqrt{4x^2 + 2\,2^{\frac{1}{3}} - 4}}{4}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(2x^2 + 2^{\frac{1}{3}} - 2\right)\sqrt{4x^2 + 2\,2^{\frac{1}{3}} - 4}}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\left(2x^2 + 2^{\frac{1}{3}} - 2\right)\sqrt{4x^2 + 2\,2^{\frac{1}{3}} - 4}}{4}$$

Verified OK.

8.24.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 3x\,y^{\frac{1}{3}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 283: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{3x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{3x}} dx \end{aligned}$$

Which results in

$$S = \frac{3x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 3x\, y^{\frac{1}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 3x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^{\frac{1}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^{\frac{1}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{3R^{\frac{2}{3}}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3x^2}{2} = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

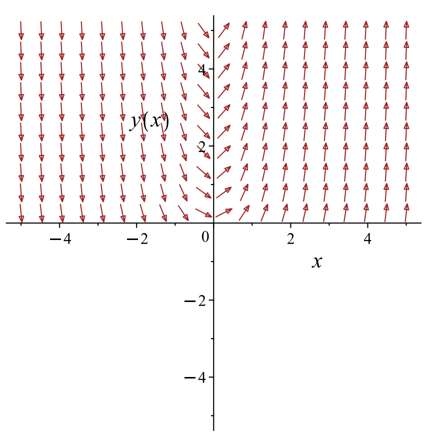
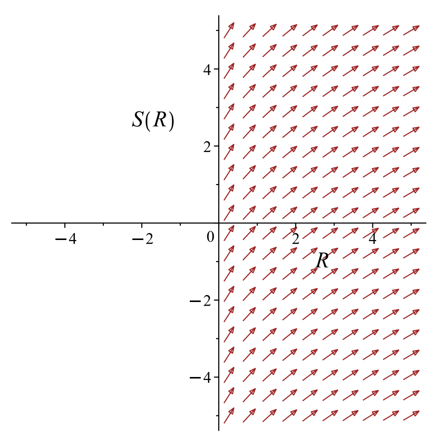
Which simplifies to

$$\frac{3x^2}{2} = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

Which gives

$$y = \frac{(9x^2 - 6c_1)^{\frac{3}{2}}}{27}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3x\, y^{\frac{1}{3}}$ 	$R = y$ $S = \frac{3x^2}{2}$	$\frac{dS}{dR} = \frac{1}{R^{\frac{1}{3}}}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \frac{(9 - 6c_1)^{\frac{3}{2}}}{27}$$

$$c_1 = \frac{3}{2} - \frac{3\,2^{\frac{1}{3}}}{4}$$

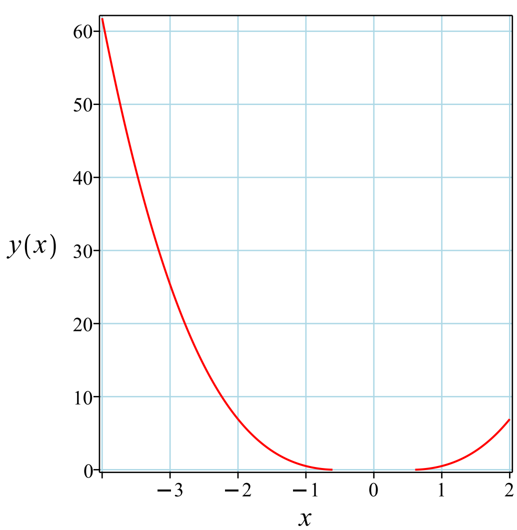
Substituting c_1 found above in the general solution gives

$$y = \frac{\left(4x^2 + 2\,2^{\frac{1}{3}} - 4\right)^{\frac{3}{2}}}{8}$$

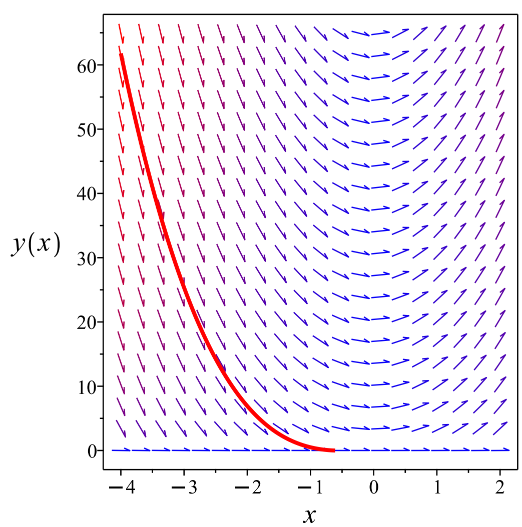
<u>Summary</u>

The solution(s) found are the following

$$y = \frac{\left(4x^2 + 2\,2^{\frac{1}{3}} - 4\right)^{\frac{3}{2}}}{8} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\left(4x^2 + 2\,2^{\frac{1}{3}} - 4\right)^{\frac{3}{2}}}{8}$$

Verified OK.

8.24.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{3y^{\frac{1}{3}}}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+\left(\frac{1}{3y^{\frac{1}{3}}}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -x \\ N(x, y) &= \frac{1}{3y^{\frac{1}{3}}} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{3y^{\frac{1}{3}}}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -x \mathrm{d}x$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{3y^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$\frac{1}{3y^{\frac{1}{3}}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{3y^{\frac{1}{3}}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{3y^{\frac{1}{3}}}\right) \mathrm{d}y$$

$$f(y) = \frac{y^{\frac{2}{3}}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^{\frac{2}{3}}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^{\frac{2}{3}}}{2}$$

The solution becomes

$$y = \left(x^2 + 2c_1\right)^{\frac{3}{2}}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = (2c_1 + 1)^{\frac{3}{2}}$$

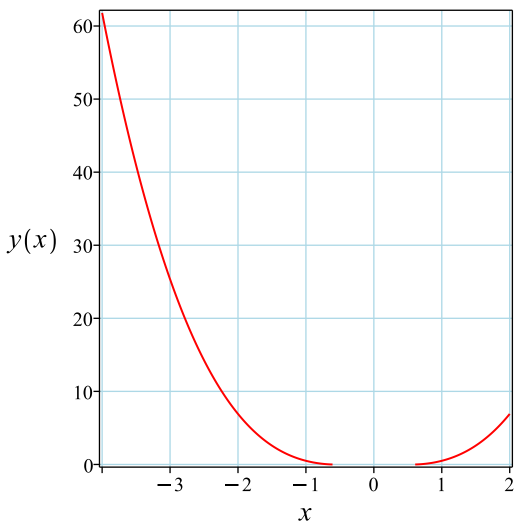
$$c_1 = \frac{2^{\frac{1}{3}}}{4} - \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

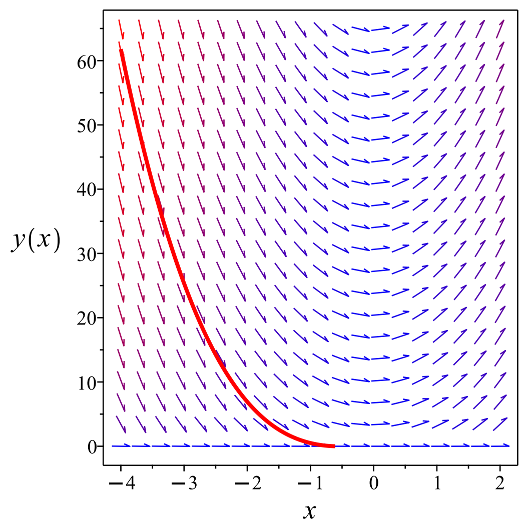
$$y = \frac{\left(4x^2 + 2\,2^{\frac{1}{3}} - 4\right)^{\frac{3}{2}}}{8}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(4x^2 + 2\,2^{\frac{1}{3}} - 4\right)^{\frac{3}{2}}}{8} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\left(4x^2 + 2\,2^{\frac{1}{3}} - 4\right)^{\frac{3}{2}}}{8}$$

Verified OK.

8.24.5 Maple step by step solution

Let's solve

$\left[y' - 3xy^{\frac{1}{3}} = 0, y(-1) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{1}{3}}} = 3x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{1}{3}}}dx = \int 3xdx + c_1$

- Evaluate integral

  $\frac{3y^{\frac{2}{3}}}{2} = \frac{3x^2}{2} + c_1$

- Solve for y

  $y = \frac{(9x^2+6c_1)^{\frac{3}{2}}}{27}$

- Use initial condition $y(-1) = \frac{1}{2}$

  $\frac{1}{2} = \frac{(6c_1+9)^{\frac{3}{2}}}{27}$

- Solve for c_1

  $c_1 = \frac{3\,2^{\frac{1}{3}}}{4} - \frac{3}{2}$

- Substitute $c_1 = \frac{3\,2^{\frac{1}{3}}}{4} - \frac{3}{2}$ into general solution and simplify

  $y = \frac{\left(2x^2+2^{\frac{1}{3}}-2\right)\sqrt{4x^2+2\,2^{\frac{1}{3}}-4}}{4}$

- Solution to the IVP

  $y = \frac{\left(2x^2+2^{\frac{1}{3}}-2\right)\sqrt{4x^2+2\,2^{\frac{1}{3}}-4}}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.266 (sec). Leaf size: 20

```
dsolve([diff(y(x),x)=3*x*y(x)^(1/3),y(-1) = 1/2],y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{4x^2 + 2\,2^{\frac{1}{3}} - 4}\left(2x^2 + 2^{\frac{1}{3}} - 2\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.175 (sec). Leaf size: 30

```
DSolve[{y'[x]==3*x*y[x]^(1/3),{y[-1]==1/2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\left(2x^2 + \sqrt[3]{2} - 2\right)^{3/2}}{2\sqrt{2}}$$

8.25 problem 9 (d)

Internal problem ID [12723]
Internal file name [`OUTPUT/11375_Friday_November_03_2023_06_31_25_AM_2293591/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 9 (d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - 3xy^{\frac{1}{3}} = 0}$$

With initial conditions

$$[y(-1) = 0]$$

8.25.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= 3x\, y^{\frac{1}{3}} \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $f(x, y)$ when $x = -1$ is

$$\{0 \leq y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(3x\, y^{\frac{1}{3}}\right)\\ &= \frac{x}{y^{\frac{2}{3}}}\end{aligned}$$

$\frac{\partial f}{\partial y}$ is not defined at $y = 0$ therefore existence and uniqueness theorem do not apply. Solution exist but not guaranteed to be unique.

8.25.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= 3x\, y^{\frac{1}{3}}\end{aligned}$$

Where $f(x) = 3x$ and $g(y) = y^{\frac{1}{3}}$. Integrating both sides gives

$$\frac{1}{y^{\frac{1}{3}}}\, dy = 3x\, dx$$

$$\int \frac{1}{y^{\frac{1}{3}}}\, dy = \int 3x\, dx$$

$$\frac{3y^{\frac{2}{3}}}{2} = \frac{3x^2}{2} + c_1$$

The solution is

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{3}{2} - c_1 = 0$$

$$c_1 = -\frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{3x^2}{2} + \frac{3}{2} = 0$$

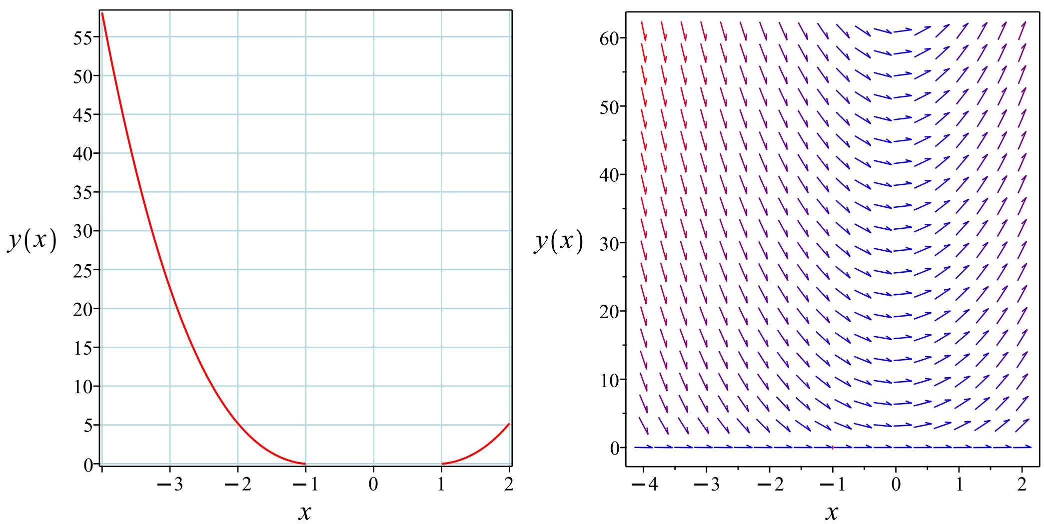
Solving for y from the above gives

$$y = \left(x^2 - 1\right)^{\frac{3}{2}}$$

Summary

The solution(s) found are the following

$$y = \left(x^2 - 1\right)^{\frac{3}{2}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \left(x^2 - 1\right)^{\frac{3}{2}}$$

Verified OK.

8.25.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 3x\,y^{\frac{1}{3}} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 286: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{3x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{3x}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{3x^2}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = 3x\, y^{\frac{1}{3}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= 3x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^{\frac{1}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^{\frac{1}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{3R^{\frac{2}{3}}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3x^2}{2} = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

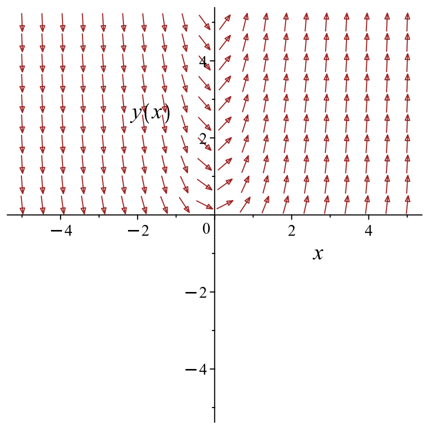
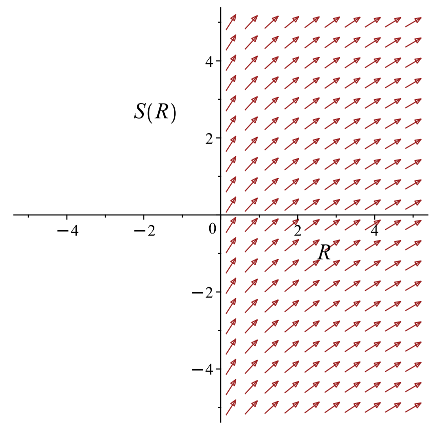
Which simplifies to

$$\frac{3x^2}{2} = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

Which gives

$$y = \frac{(9x^2 - 6c_1)^{\frac{3}{2}}}{27}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3x\, y^{\frac{1}{3}}$ 	$R = y$ $S = \frac{3x^2}{2}$	$\frac{dS}{dR} = \frac{1}{R^{\frac{1}{3}}}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{(9 - 6c_1)^{\frac{3}{2}}}{27}$$

$$c_1 = \frac{3}{2}$$

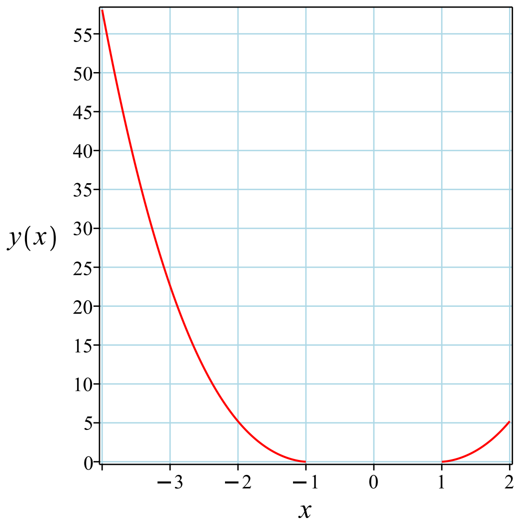
Substituting c_1 found above in the general solution gives

$$y = \left(x^2 - 1\right)^{\frac{3}{2}}$$

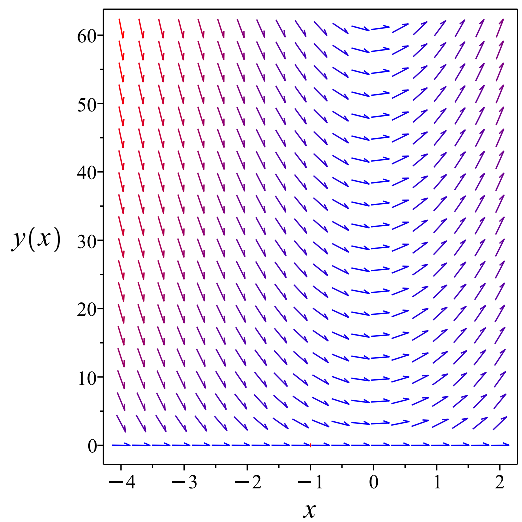
Summary

The solution(s) found are the following

$$y = \left(x^2 - 1\right)^{\frac{3}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \left(x^2 - 1\right)^{\frac{3}{2}}$$

Verified OK.

8.25.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{3y^{\frac{1}{3}}}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{1}{3y^{\frac{1}{3}}}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\\ N(x,y) &= \frac{1}{3y^{\frac{1}{3}}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{3y^{\frac{1}{3}}}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{3y^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$\frac{1}{3y^{\frac{1}{3}}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{3y^{\frac{1}{3}}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{3y^{\frac{1}{3}}}\right) \mathrm{d}y$$
$$f(y) = \frac{y^{\frac{2}{3}}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^{\frac{2}{3}}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^{\frac{2}{3}}}{2}$$

The solution becomes

$$y = \left(x^2 + 2c_1\right)^{\frac{3}{2}}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = (2c_1 + 1)^{\frac{3}{2}}$$

$$c_1 = -\frac{1}{2}$$

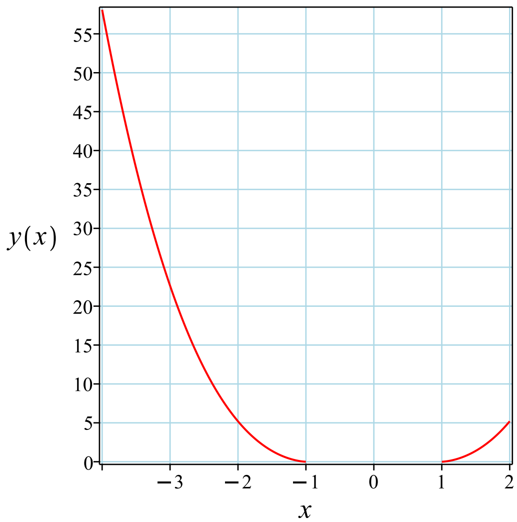
Substituting c_1 found above in the general solution gives

$$y = \left(x^2 - 1\right)^{\frac{3}{2}}$$

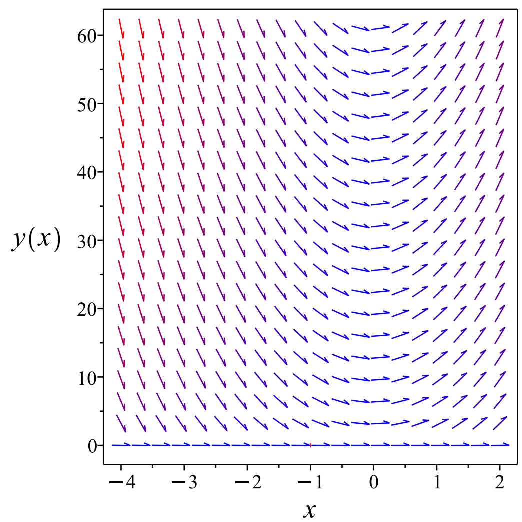
Summary

The solution(s) found are the following

$$y = \left(x^2 - 1\right)^{\frac{3}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (x^2 - 1)^{\frac{3}{2}}$$

Verified OK.

8.25.5 Maple step by step solution

Let's solve

$\left[y' - 3xy^{\frac{1}{3}} = 0, y(-1) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{1}{3}}} = 3x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{1}{3}}} dx = \int 3x dx + c_1$

- Evaluate integral

  $\frac{3y^{\frac{2}{3}}}{2} = \frac{3x^2}{2} + c_1$

- Solve for y

$y = \frac{(9x^2+6c_1)^{\frac{3}{2}}}{27}$

- Use initial condition $y(-1) = 0$

  $0 = \frac{(6c_1+9)^{\frac{3}{2}}}{27}$

- Solve for c_1

  $c_1 = -\frac{3}{2}$

- Substitute $c_1 = -\frac{3}{2}$ into general solution and simplify

  $y = (x^2 - 1)^{\frac{3}{2}}$

- Solution to the IVP

  $y = (x^2 - 1)^{\frac{3}{2}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=3*x*y(x)^(1/3),y(-1) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 19

```
DSolve[{y'[x]==3*x*y[x]^(1/3),{y[-1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$
$$y(x) \to (x^2 - 1)^{3/2}$$

8.26 problem 9 (e)

Internal problem ID [12724]
Internal file name [`OUTPUT/11376_Friday_November_03_2023_06_31_27_AM_46204666/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 9 (e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - 3xy^{\frac{1}{3}} = 0}$$

With initial conditions

$$[y(-1) = -1]$$

8.26.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= 3x\, y^{\frac{1}{3}} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The y domain of $f(x, y)$ when $x = -1$ is

$$\{0 \leq y\}$$

But the point $y_0 = -1$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

8.26.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= 3x\, y^{\frac{1}{3}} \end{aligned}$$

Where $f(x) = 3x$ and $g(y) = y^{\frac{1}{3}}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y^{\frac{1}{3}}}\, dy &= 3x\, dx \\ \int \frac{1}{y^{\frac{1}{3}}}\, dy &= \int 3x\, dx \\ \frac{3y^{\frac{2}{3}}}{2} &= \frac{3x^2}{2} + c_1 \end{aligned}$$

The solution is

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{3x^2}{2} - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{9}{4} + \frac{3i\sqrt{3}}{4} - c_1 = 0$$

$$c_1 = -\frac{9}{4} + \frac{3i\sqrt{3}}{4}$$

Substituting c_1 found above in the general solution gives

$$\frac{3y^{\frac{2}{3}}}{2} - \frac{3x^2}{2} + \frac{9}{4} - \frac{3i\sqrt{3}}{4} = 0$$

Solving for y from the above gives

$$y = \frac{\left(i\sqrt{3} + 2x^2 - 3\right)\sqrt{2i\sqrt{3} + 4x^2 - 6}}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(i\sqrt{3} + 2x^2 - 3\right)\sqrt{2i\sqrt{3} + 4x^2 - 6}}{4} \tag{1}$$

Verification of solutions

$$y = \frac{\left(i\sqrt{3} + 2x^2 - 3\right)\sqrt{2i\sqrt{3} + 4x^2 - 6}}{4}$$

Verified OK.

8.26.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= 3x\, y^{\frac{1}{3}} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 289: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{3x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{3x}} dx \end{aligned}$$

Which results in

$$S = \frac{3x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 3x\, y^{\frac{1}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 3x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^{\frac{1}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^{\frac{1}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{3R^{\frac{2}{3}}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3x^2}{2} = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

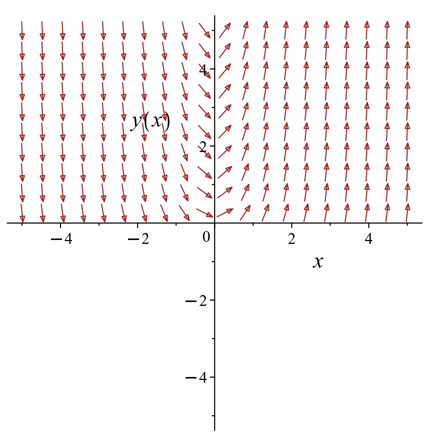
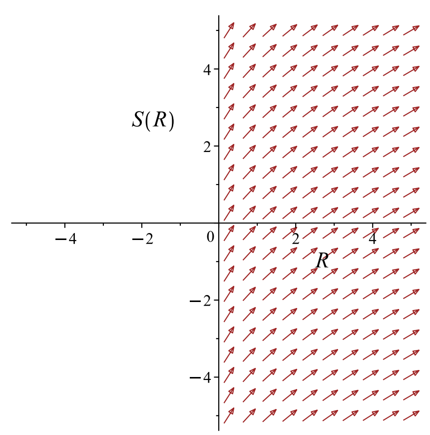
Which simplifies to

$$\frac{3x^2}{2} = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

Which gives

$$y = \frac{\left(9x^2 - 6c_1\right)^{\frac{3}{2}}}{27}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3x\, y^{\frac{1}{3}}$ 	$R = y$ $S = \frac{3x^2}{2}$	$\frac{dS}{dR} = \frac{1}{R^{\frac{1}{3}}}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{(9 - 6c_1)^{\frac{3}{2}}}{27}$$

$$c_1 = \frac{9}{4} - \frac{3i\sqrt{3}}{4}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\left(2i\sqrt{3} + 4x^2 - 6\right)^{\frac{3}{2}}}{8}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(2i\sqrt{3} + 4x^2 - 6\right)^{\frac{3}{2}}}{8} \tag{1}$$

Verification of solutions

$$y = \frac{\left(2i\sqrt{3} + 4x^2 - 6\right)^{\frac{3}{2}}}{8}$$

Verified OK.

8.26.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{3y^{\frac{1}{3}}}\right) \mathrm{d}y &= (x)\, \mathrm{d}x \\
(-x)\, \mathrm{d}x + \left(\frac{1}{3y^{\frac{1}{3}}}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x \\
N(x, y) &= \frac{1}{3y^{\frac{1}{3}}}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{3y^{\frac{1}{3}}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{3y^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$
\frac{1}{3y^{\frac{1}{3}}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{3y^{\frac{1}{3}}}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{3y^{\frac{1}{3}}}\right) \mathrm{d}y$$

$$f(y) = \frac{y^{\frac{2}{3}}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{y^{\frac{2}{3}}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{y^{\frac{2}{3}}}{2}$$

The solution becomes

$$y = \left(x^2 + 2c_1\right)^{\frac{3}{2}}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = (2c_1 + 1)^{\frac{3}{2}}$$

$$c_1 = -\frac{3}{4} - \frac{i\sqrt{3}}{4}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\left(4x^2 - 6 - 2i\sqrt{3}\right)^{\frac{3}{2}}}{8}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(4x^2 - 6 - 2i\sqrt{3}\right)^{\frac{3}{2}}}{8} \tag{1}$$

Verification of solutions

$$y = \frac{\left(4x^2 - 6 - 2i\sqrt{3}\right)^{\frac{3}{2}}}{8}$$

Verified OK.

8.26.5 Maple step by step solution

Let's solve

$\left[y' - 3xy^{\frac{1}{3}} = 0, y(-1) = -1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{1}{3}}} = 3x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{1}{3}}} dx = \int 3x dx + c_1$

- Evaluate integral

  $\frac{3y^{\frac{2}{3}}}{2} = \frac{3x^2}{2} + c_1$

- Solve for y

  $y = \frac{(9x^2+6c_1)^{\frac{3}{2}}}{27}$

- Use initial condition $y(-1) = -1$

  $-1 = \frac{(6c_1+9)^{\frac{3}{2}}}{27}$

- Solve for c_1

  $c_1 = \left(-\frac{9}{4} + \frac{3\,\mathrm{I}\sqrt{3}}{4}, -\frac{9}{4} - \frac{3\,\mathrm{I}\sqrt{3}}{4}\right)$

- Substitute $c_1 = \left(-\frac{9}{4} + \frac{3\,\mathrm{I}\sqrt{3}}{4}, -\frac{9}{4} - \frac{3\,\mathrm{I}\sqrt{3}}{4}\right)$ into general solution and simplify

  $y = \frac{\left(\mathrm{I}\sqrt{3}+2x^2-3\right)\sqrt{2\,\mathrm{I}\sqrt{3}+4x^2-6}}{4}$

- Solution to the IVP

  $y = \frac{\left(\mathrm{I}\sqrt{3}+2x^2-3\right)\sqrt{2\,\mathrm{I}\sqrt{3}+4x^2-6}}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve([diff(y(x),x)=3*x*y(x)^(1/3),y(-1) = -1],y(x), singsol=all)
```

$$y(x) = x^3$$

✓ Solution by Mathematica

Time used: 0.175 (sec). Leaf size: 67

```
DSolve[{y'[x]==3*x*y[x]^(1/3),{y[-1]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\left(2x^2 - i\sqrt{3} - 3\right)^{3/2}}{2\sqrt{2}}$$
$$y(x) \to \frac{\left(2x^2 + i\sqrt{3} - 3\right)^{3/2}}{2\sqrt{2}}$$

8.27 problem 10 (a)

Internal problem ID [12725]
Internal file name [`OUTPUT/11377_Friday_November_03_2023_06_31_29_AM_539436/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 10 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \sqrt{(y+2)(y-1)} = 0}$$

With initial conditions

$$[y(0) = 0]$$

8.27.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \sqrt{(y+2)(y-1)} \end{aligned}$$

The y domain of $f(x,y)$ when $x=0$ is

$$\{1 \leq y \leq \infty, -\infty \leq y \leq -2\}$$

But the point $y_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

8.27.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\sqrt{(y+2)(y-1)}}dy = \int dx$$
$$\ln\left(y+\frac{1}{2}+\sqrt{y^2+y-2}\right) = x+c_1$$

Raising both side to exponential gives

$$y+\frac{1}{2}+\sqrt{y^2+y-2} = \mathrm{e}^{x+c_1}$$

Which simplifies to

$$y+\frac{1}{2}+\sqrt{y^2+y-2} = c_2\mathrm{e}^{x}$$

Initial conditions are used to solve for c_2. Substituting $x=0$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{4c_2^2-4c_2+9}{8c_2}$$

$$c_2 = \frac{1}{2}-i\sqrt{2}$$

Substituting c_2 found above in the general solution gives

$$y = \frac{4i\mathrm{e}^{-x}\mathrm{e}^{2x}\sqrt{2}-4i\mathrm{e}^{-x}\mathrm{e}^{x}\sqrt{2}+7\,\mathrm{e}^{-x}\mathrm{e}^{2x}+2\,\mathrm{e}^{-x}\mathrm{e}^{x}-9\,\mathrm{e}^{-x}}{-4+8i\sqrt{2}}$$

Summary

The solution(s) found are the following

$$y = \frac{4i\mathrm{e}^{-x}\mathrm{e}^{2x}\sqrt{2}-4i\mathrm{e}^{-x}\mathrm{e}^{x}\sqrt{2}+7\,\mathrm{e}^{-x}\mathrm{e}^{2x}+2\,\mathrm{e}^{-x}\mathrm{e}^{x}-9\,\mathrm{e}^{-x}}{-4+8i\sqrt{2}} \tag{1}$$

Verification of solutions

$$y = \frac{4i\mathrm{e}^{-x}\mathrm{e}^{2x}\sqrt{2}-4i\mathrm{e}^{-x}\mathrm{e}^{x}\sqrt{2}+7\,\mathrm{e}^{-x}\mathrm{e}^{2x}+2\,\mathrm{e}^{-x}\mathrm{e}^{x}-9\,\mathrm{e}^{-x}}{-4+8i\sqrt{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 1.031 (sec). Leaf size: 34

```
dsolve([diff(y(x),x)=sqrt( (y(x)+2)*( y(x)-1)),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{i e^{x}\sqrt{2}}{2} - \frac{i\sqrt{2}\, e^{-x}}{2} + \frac{e^{x}}{4} - \frac{1}{2} + \frac{e^{-x}}{4}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 45

```
DSolve[{y'[x]==Sqrt[ (y[x]+2)*( y[x]-1)],{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4} e^{-x}(e^{x}-1)\left(\left(1+2i\sqrt{2}\right)e^{x}-1+2i\sqrt{2}\right)$$

8.28 problem 10 (b)

Internal problem ID [12726]
Internal file name [`OUTPUT/11378_Friday_November_03_2023_06_31_31_AM_61785116/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 10 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \sqrt{(y+2)(y-1)} = 0}$$

With initial conditions

$$[y(0) = 1]$$

8.28.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \sqrt{(y+2)(y-1)} \end{aligned}$$

The y domain of $f(x,y)$ when $x=0$ is

$$\{1 \leq y \leq \infty, -\infty \leq y \leq -2\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\sqrt{(y+2)(y-1)}\right) \\ &= \frac{2y+1}{2\sqrt{(y+2)(y-1)}} \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty \leq y < -2, -2 < y < 1, 1 < y \leq \infty\}$$

But the point $y_0 = 1$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

8.28.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\sqrt{(y+2)(y-1)}} dy = \int dx$$
$$\ln\left(y + \frac{1}{2} + \sqrt{y^2 + y - 2}\right) = x + c_1$$

Raising both side to exponential gives

$$y + \frac{1}{2} + \sqrt{y^2 + y - 2} = e^{x+c_1}$$

Which simplifies to

$$y + \frac{1}{2} + \sqrt{y^2 + y - 2} = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{4c_2^2 - 4c_2 + 9}{8c_2}$$

$$c_2 = \frac{3}{2}$$

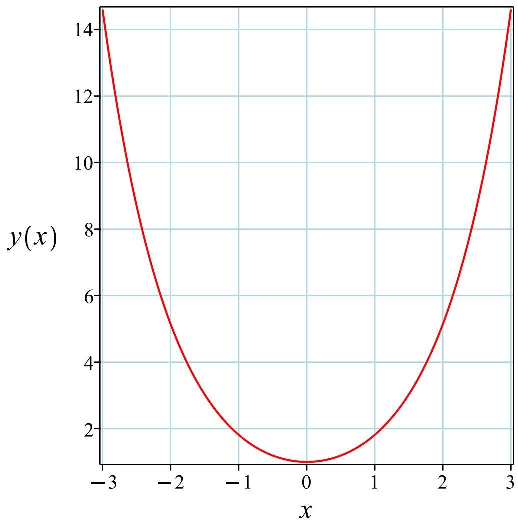
Substituting c_2 found above in the general solution gives

$$y = \frac{3\,e^x}{4} - \frac{1}{2} + \frac{3\,e^{-x}}{4}$$

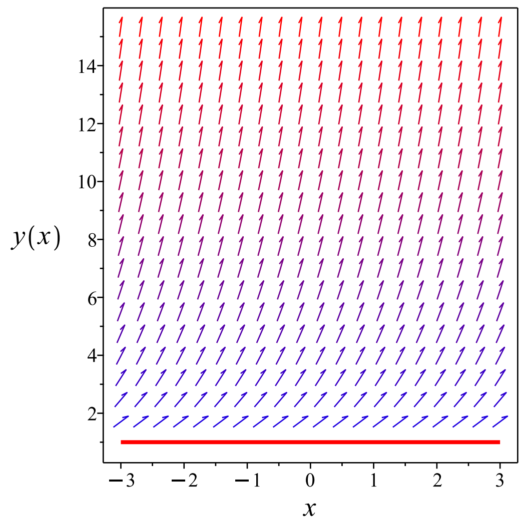
Summary

The solution(s) found are the following

$$y = \frac{3\,e^x}{4} - \frac{1}{2} + \frac{3\,e^{-x}}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{3\,e^{x}}{4} - \frac{1}{2} + \frac{3\,e^{-x}}{4}$$

Verified OK.

8.28.3 Maple step by step solution

Let's solve

$$\left[y' - \sqrt{(y+2)\,(y-1)} = 0, y(0) = 1\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{(y+2)(y-1)}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{(y+2)(y-1)}}dx = \int 1dx + c_1$

- Evaluate integral

  $\ln\left(y + \frac{1}{2} + \sqrt{-2 + y^2 + y}\right) = x + c_1$

- Solve for y

$$y = \frac{4(e^{x+c_1})^2 - 4e^{x+c_1} + 9}{8e^{x+c_1}}$$

- Use initial condition $y(0) = 1$

$$1 = \frac{4(e^{c_1})^2 - 4e^{c_1} + 9}{8e^{c_1}}$$

- Solve for c_1

$$c_1 = \ln\left(\frac{3}{2}\right)$$

- Substitute $c_1 = \ln\left(\frac{3}{2}\right)$ into general solution and simplify

$$y = \frac{3e^x}{4} - \frac{1}{2} + \frac{3e^{-x}}{4}$$

- Solution to the IVP

$$y = \frac{3e^x}{4} - \frac{1}{2} + \frac{3e^{-x}}{4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=sqrt( (y(x)+2)*( y(x)-1)),y(0) = 1],y(x), singsol=all)
```

$$y(x) = 1$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 23

```
DSolve[{y'[x]==Sqrt[ (y[x]+2)*( y[x]-1)],{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}(3e^{-x} + 3e^x - 2)$$

8.29 problem 10 (c)

Internal problem ID [12727]
Internal file name [`OUTPUT/11379_Friday_November_03_2023_06_31_32_AM_295468/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 10 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \sqrt{(y+2)(y-1)} = 0}$$

With initial conditions

$$[y(0) = -3]$$

8.29.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \sqrt{(y+2)(y-1)} \end{aligned}$$

The y domain of $f(x,y)$ when $x=0$ is

$$\{1 \leq y \leq \infty, -\infty \leq y \leq -2\}$$

And the point $y_0 = -3$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\sqrt{(y+2)(y-1)}\right) \\ &= \frac{2y+1}{2\sqrt{(y+2)(y-1)}} \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty \leq y < -2, -2 < y < 1, 1 < y \leq \infty\}$$

And the point $y_0 = -3$ is inside this domain. Therefore solution exists and is unique.

8.29.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\sqrt{(y+2)(y-1)}} dy = \int dx$$
$$\ln\left(y + \frac{1}{2} + \sqrt{y^2 + y - 2}\right) = x + c_1$$

Raising both side to exponential gives

$$y + \frac{1}{2} + \sqrt{y^2 + y - 2} = e^{x+c_1}$$

Which simplifies to

$$y + \frac{1}{2} + \sqrt{y^2 + y - 2} = c_2 e^{x}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = \frac{4c_2^2 - 4c_2 + 9}{8c_2}$$

$$c_2 = -\frac{9}{2}$$

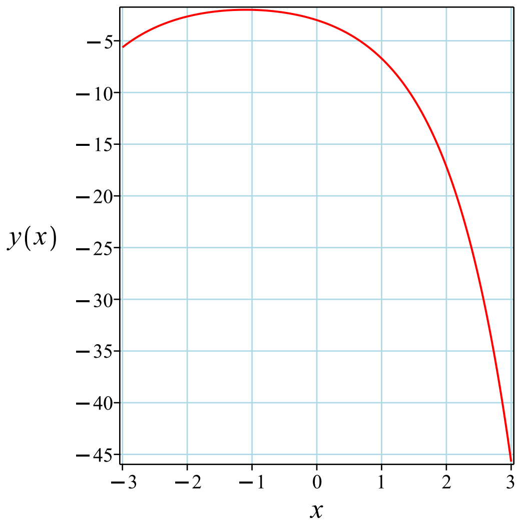
Substituting c_2 found above in the general solution gives

$$y = -\frac{9\,e^{x}}{4} - \frac{1}{2} - \frac{e^{-x}}{4}$$

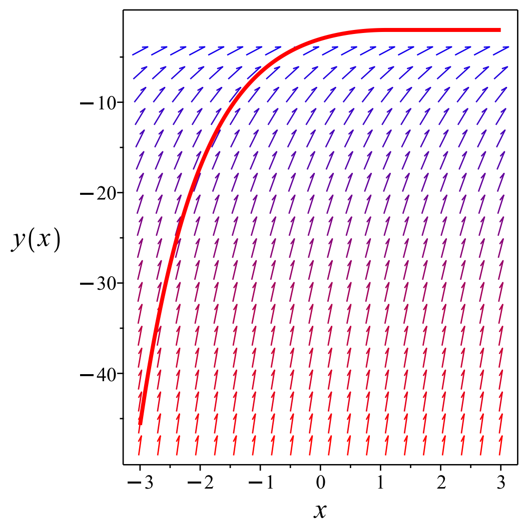
Summary

The solution(s) found are the following

$$y = -\frac{9\,e^{x}}{4} - \frac{1}{2} - \frac{e^{-x}}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{9\,e^{x}}{4} - \frac{1}{2} - \frac{e^{-x}}{4}$$

Verified OK.

8.29.3 Maple step by step solution

Let's solve

$\left[y' - \sqrt{(y+2)(y-1)} = 0, y(0) = -3\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{(y+2)(y-1)}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{(y+2)(y-1)}}dx = \int 1dx + c_1$

- Evaluate integral

  $\ln\left(y + \frac{1}{2} + \sqrt{-2 + y^2 + y}\right) = x + c_1$

- Solve for y

$$y = \frac{4(e^{x+c_1})^2 - 4e^{x+c_1} + 9}{8e^{x+c_1}}$$

- Use initial condition $y(0) = -3$

$$-3 = \frac{4(e^{c_1})^2 - 4e^{c_1} + 9}{8e^{c_1}}$$

- Solve for c_1

$$c_1 = \left(\ln\left(\frac{9}{2}\right) + I\pi, I\pi - \ln(2)\right)$$

- Substitute $c_1 = \left(\ln\left(\frac{9}{2}\right) + I\pi, I\pi - \ln(2)\right)$ into general solution and simplify

$$y = -\frac{9e^x}{4} - \frac{1}{2} - \frac{e^{-x}}{4}$$

- Solution to the IVP

$$y = -\frac{9e^x}{4} - \frac{1}{2} - \frac{e^{-x}}{4}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.234 (sec). Leaf size: 16

```
dsolve([diff(y(x),x)=sqrt( (y(x)+2)*( y(x)-1)),y(0) = -3],y(x), singsol=all)
```

$$y(x) = -\frac{e^x}{4} - \frac{1}{2} - \frac{9e^{-x}}{4}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 23

```
DSolve[{y'[x]==Sqrt[ (y[x]+2)*( y[x]-1)],{y[0]==-3}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{1}{4}\left(-9e^{-x} - e^x - 2\right)$$

8.30 problem 11 (a)

Internal problem ID [12728]
Internal file name [`OUTPUT/11380_Friday_November_03_2023_06_31_33_AM_788875/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 11 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{y}{y - x} = 0}$$

With initial conditions

$$[y(1) = 2]$$

8.30.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y}{y - x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 2$ is

$$\{x < 2 \lor 2 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{y-x}\right)\\ &= \frac{1}{y-x} - \frac{y}{(y-x)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{x < 2 \lor 2 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

8.30.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)\,x}{u\,(x)\,x - x} = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u)\\ &= f(x)g(u)\\ &= -\frac{u(u-2)}{x\,(u-1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u-2)}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u-2)}{u-1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u(u-2)}{u-1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln\left(u(u-2)\right)}{2} = -\ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$\sqrt{u\,(u-2)} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u\,(u-2)} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u\,(x)\,(u\,(x)-2)} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{u\,(x)\,(u\,(x)-2)} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y\left(\frac{y}{x}-2\right)}{x}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$
$$\sqrt{\frac{y\,(-2x+y)}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Initial conditions are used to solve for c_2. Substituting $x=1$ and $y=2$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_3 \mathrm{e}^{c_2}$$

Unable to solve for constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

8.30.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{y}{y-x} \tag{1}$$

Which becomes

$$(-y)\,dy = (-x)\,dy + (-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (-y)\,dx = d(-xy)$$

Hence (2) becomes

$$(-y)\,dy = d(-xy)$$

Integrating both sides gives gives these solutions

$$\begin{aligned} y &= x + \sqrt{x^2 - 2c_1} + c_1 \\ y &= x - \sqrt{x^2 - 2c_1} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 1 - \sqrt{-2c_1 + 1} + c_1$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 1 + \sqrt{-2c_1 + 1} + c_1$$

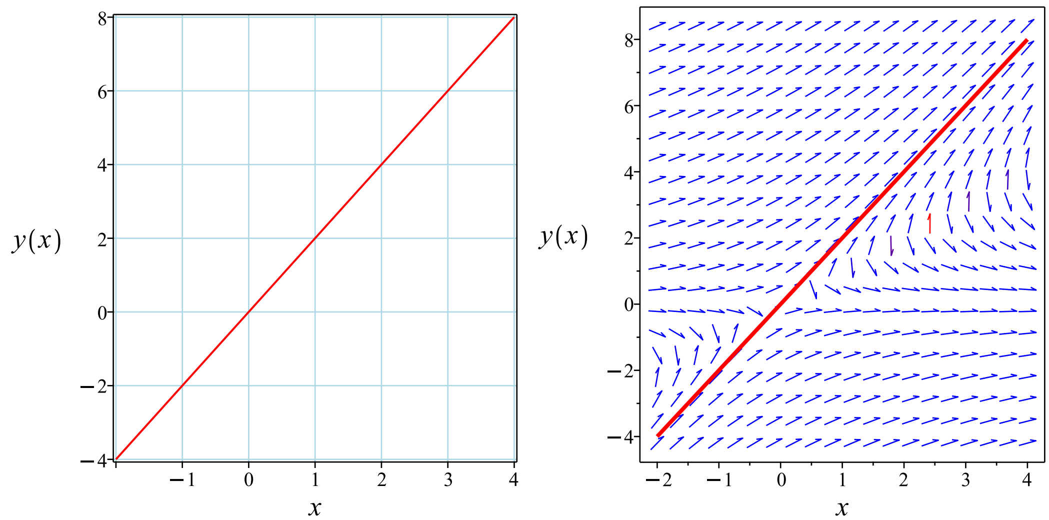
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = 2x$$

Summary
The solution(s) found are the following

$$y = 2x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2x$$

Verified OK. {positive}

8.30.4 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y}{y - x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 + \frac{y(b_3 - a_2)}{y - x} - \frac{y^2 a_3}{(y - x)^2} - \frac{y(x a_2 + y a_3 + a_1)}{(y - x)^2} \\
& - \left(\frac{1}{y - x} - \frac{y}{(y - x)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{2x^2 b_2 - 2xy b_2 - y^2 a_2 - 2y^2 a_3 + y^2 b_2 + y^2 b_3 + x b_1 - y a_1}{(-y + x)^2} = 0
$$

Setting the numerator to zero gives

$$
2x^2 b_2 - 2xy b_2 - y^2 a_2 - 2y^2 a_3 + y^2 b_2 + y^2 b_3 + x b_1 - y a_1 = 0 \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
-a_2 v_2^2 - 2a_3 v_2^2 + 2b_2 v_1^2 - 2b_2 v_1 v_2 + b_2 v_2^2 + b_3 v_2^2 - a_1 v_2 + b_1 v_1 = 0 \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
2b_2 v_1^2 - 2b_2 v_1 v_2 + b_1 v_1 + (-a_2 - 2a_3 + b_2 + b_3) v_2^2 - a_1 v_2 = 0 \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -a_1 &= 0 \\ -2b_2 &= 0 \\ 2b_2 &= 0 \\ -a_2 - 2a_3 + b_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -2a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\,\xi \\ &= y - \left(\frac{y}{y - x}\right)(x) \\ &= \frac{2xy - y^2}{-y + x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S&=\int\frac{1}{\eta}dy\\&=\int\frac{1}{\frac{2xy-y^2}{-y+x}}dy\end{aligned}$$

Which results in

$$S=\frac{\ln\left(y(-2x+y)\right)}{2}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y}\tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{y}{y-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=1\\R_y&=0\\S_x&=\frac{1}{2x-y}\\S_y&=\frac{-y+x}{y\left(2x-y\right)}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=0\tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

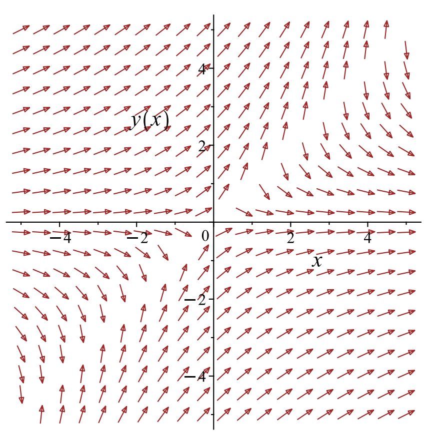
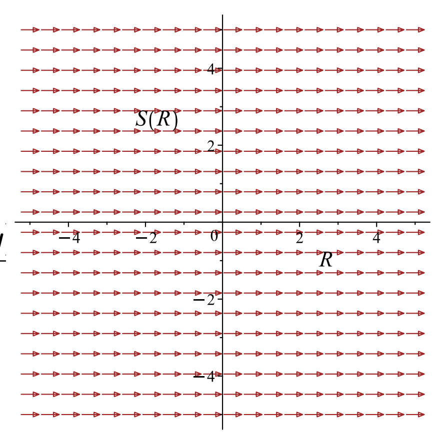
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{2} + \frac{\ln(-2x + y)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{2} + \frac{\ln(-2x + y)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{y-x}$ 	$R = x$ $S = \frac{\ln(y)}{2} + \frac{\ln(-2x + y)}{2}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$-\infty = c_1$$

Unable to solve for constant of integration. Warning: Unable to solve for constant of integration.

<u>Verification of solutions</u> N/A

8.30.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(y-x)\,\mathrm{d}y&=(y)\,\mathrm{d}x\\ (-y)\,\mathrm{d}x+(y-x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y\\ N(x,y)&=y-x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y - x) \\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -y \, \mathrm{d}x$$

$$\phi = -xy + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y - x$. Therefore equation (4) becomes

$$y - x = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y)\,\mathrm{d}y$$

$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -xy + \frac{1}{2}y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -xy + \frac{1}{2}y^2$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$-xy + \frac{1}{2}y^2 = 0$$

Summary
The solution(s) found are the following

$$-yx + \frac{y^2}{2} = 0 \tag{1}$$

Verification of solutions

$$-yx + \frac{y^2}{2} = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 7

```
dsolve([diff(y(x),x)=y(x)/(y(x)-x),y(1) = 2],y(x), singsol=all)
```

$$y(x) = 2x$$

✓ Solution by Mathematica

Time used: 0.838 (sec). Leaf size: 14

```
DSolve[{y'[x]==y[x]/(y[x]-x),{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x^2} + x$$

8.31 problem 11 (b)

Internal problem ID [12729]
Internal file name [`OUTPUT/11381_Friday_November_03_2023_06_31_35_AM_3677167/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 11 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{y}{y - x} = 0}$$

With initial conditions

$$[y(1) = 1]$$

8.31.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y}{y - x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 1 \vee 1 < x\}$$

But the point $x_0 = 1$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

8.31.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)\,x}{u(x)\,x - x} = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u(u-2)}{x\,(u-1)}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u-2)}{u-1}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{u(u-2)}{u-1}}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{\frac{u(u-2)}{u-1}}\,du &= \int -\frac{1}{x}\,dx \\
\frac{\ln\left(u(u-2)\right)}{2} &= -\ln\left(x\right) + c_2
\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{u\,(u-2)} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u\,(u-2)} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u\,(x)\,(u\,(x)-2)} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{u\,(x)\,(u\,(x)-2)} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y\left(\frac{y}{x}-2\right)}{x}} = \frac{c_3 e^{c_2}}{x}$$
$$\sqrt{\frac{y\left(-2x+y\right)}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Substituting initial conditions and solving for c_2 gives $c_2 = \frac{\ln\left(-\frac{1}{c_3^2}\right)}{2}$. Hence the solution becomes Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$i = c_3\sqrt{-\frac{1}{c_3^2}}$$

Since $\lim_{c_1\to\infty}$ gives $\sqrt{\frac{y(-2x+y)}{x^2}} = \frac{c_3\sqrt{-\frac{1}{c_3^2}}}{x} = \sqrt{\frac{y(-2x+y)}{x^2}} = \frac{i}{x}$ and this result satisfies the
Summary
The solution(s) found are the following
given initial condition.

$$\sqrt{\frac{y\left(-2x+y\right)}{x^2}} = \frac{i}{x}$$

Verification of solutions

$$\sqrt{\frac{y\left(-2x+y\right)}{x^2}} = \frac{i}{x}$$

Verified OK.

8.31.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{y}{y-x} \tag{1}$$

Which becomes

$$(-y)\,dy = (-x)\,dy + (-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (-y)\,dx = d(-xy)$$

Hence (2) becomes

$$(-y)\,dy = d(-xy)$$

Integrating both sides gives gives these solutions

$$y = x + \sqrt{x^2 - 2c_1} + c_1$$
$$y = x - \sqrt{x^2 - 2c_1} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 1 - \sqrt{-2c_1 + 1} + c_1$$

$$c_1 = \sqrt{2} - 1$$

Substituting c_1 found above in the general solution gives

$$y = x - \sqrt{x^2 - 2\sqrt{2} + 2} + \sqrt{2} - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = 1 + \sqrt{-2c_1 + 1} + c_1$$

$$c_1 = -\sqrt{2} - 1$$

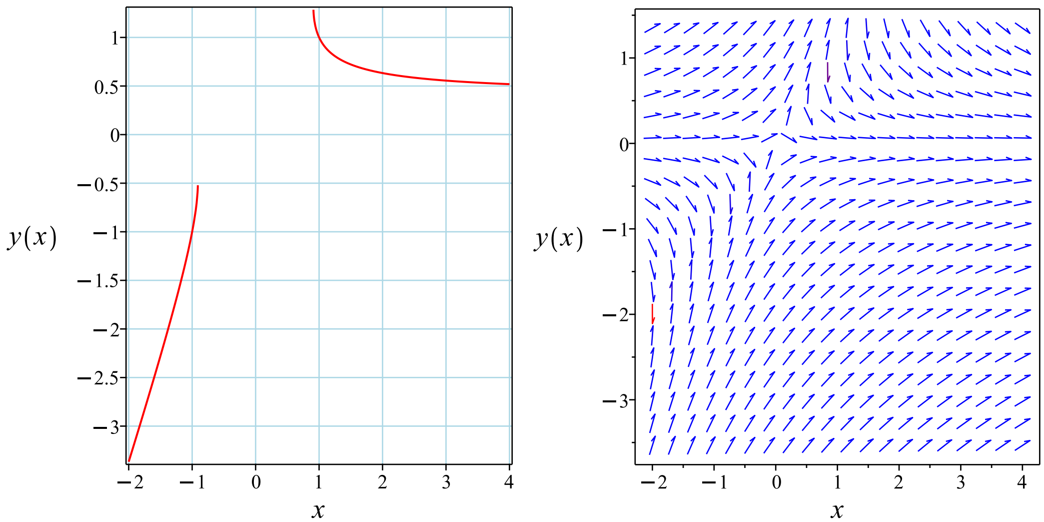
Substituting c_1 found above in the general solution gives

$$y = x + \sqrt{x^2 + 2\sqrt{2} + 2} - \sqrt{2} - 1$$

Summary

The solution(s) found are the following

$$y = x + \sqrt{x^2 + 2\sqrt{2} + 2} - \sqrt{2} - 1 \tag{1}$$
$$y = x - \sqrt{x^2 - 2\sqrt{2} + 2} + \sqrt{2} - 1 \tag{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x + \sqrt{x^2 + 2\sqrt{2} + 2} - \sqrt{2} - 1$$

Verified OK.

$$y = x - \sqrt{x^2 - 2\sqrt{2} + 2} + \sqrt{2} - 1$$

Verified OK.

8.31.4 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y}{y - x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{y(b_3 - a_2)}{y - x} - \frac{y^2 a_3}{(y - x)^2} - \frac{y(x a_2 + y a_3 + a_1)}{(y - x)^2} - \left(\frac{1}{y - x} - \frac{y}{(y - x)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2 b_2 - 2xy b_2 - y^2 a_2 - 2y^2 a_3 + y^2 b_2 + y^2 b_3 + x b_1 - y a_1}{(-y + x)^2} = 0$$

Setting the numerator to zero gives

$$2x^2 b_2 - 2xy b_2 - y^2 a_2 - 2y^2 a_3 + y^2 b_2 + y^2 b_3 + x b_1 - y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2 v_2^2 - 2a_3 v_2^2 + 2b_2 v_1^2 - 2b_2 v_1 v_2 + b_2 v_2^2 + b_3 v_2^2 - a_1 v_2 + b_1 v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$2b_2 v_1^2 - 2b_2 v_1 v_2 + b_1 v_1 + (-a_2 - 2a_3 + b_2 + b_3) v_2^2 - a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_1 &= 0 \\
-a_1 &= 0 \\
-2b_2 &= 0 \\
2b_2 &= 0 \\
-a_2 - 2a_3 + b_2 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= -2a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y}{y - x}\right)(x) \\
&= \frac{2xy - y^2}{-y + x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2xy - y^2}{-y + x}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(y(-2x + y))}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y}{y - x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{2x - y} \\ S_y &= \frac{-y + x}{y(2x - y)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

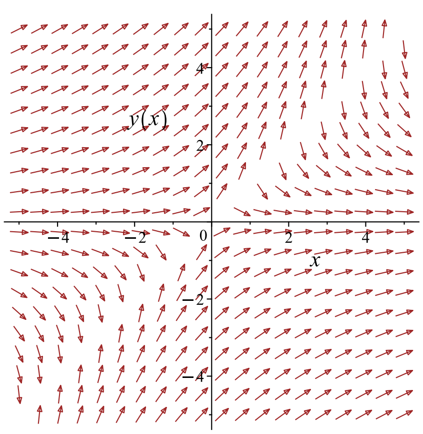
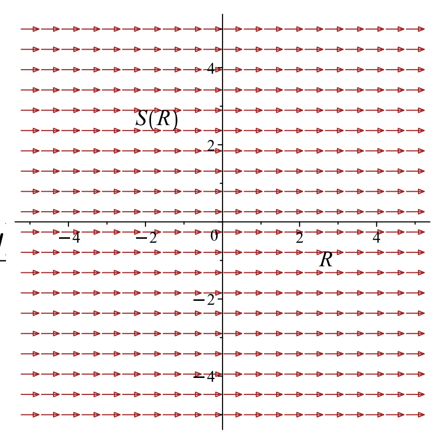
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{2} + \frac{\ln(-2x+y)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{2} + \frac{\ln(-2x+y)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{y-x}$ 	$R = x$ $S = \frac{\ln(y)}{2} + \frac{\ln(-2x+y)}{2}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{i\pi}{2} = c_1$$

$$
c_1 = \frac{i\pi}{2}
$$

Substituting c_1 found above in the general solution gives

$$
\frac{\ln(y)}{2} + \frac{\ln(-2x+y)}{2} = \frac{i\pi}{2}
$$

Summary

The solution(s) found are the following

$$
\frac{\ln(y)}{2} + \frac{\ln(-2x+y)}{2} = \frac{i\pi}{2} \tag{1}
$$

Verification of solutions

$$
\frac{\ln(y)}{2} + \frac{\ln(-2x+y)}{2} = \frac{i\pi}{2}
$$

Verified OK.

8.31.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\frac{\partial \phi}{\partial x} = M
$$
$$
\frac{\partial \phi}{\partial y} = N
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y-x)\,\mathrm{d}y &= (y)\,\mathrm{d}x\\ (-y)\,\mathrm{d}x + (y-x)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y\\ N(x,y) &= y-x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y-x)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -y \, \mathrm{d}x$$
$$\phi = -xy + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y - x$. Therefore equation (4) becomes

$$y - x = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y) \, \mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -xy + \frac{1}{2}y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -xy + \frac{1}{2}y^2$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = c_1$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$-xy + \frac{1}{2}y^2 = -\frac{1}{2}$$

Summary

The solution(s) found are the following

$$-yx + \frac{y^2}{2} = -\frac{1}{2} \tag{1}$$

Verification of solutions

$$-yx + \frac{y^2}{2} = -\frac{1}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve([diff(y(x),x)=y(x)/(y(x)-x),y(1) = 1],y(x), singsol=all)
```

$$y(x) = x - \sqrt{x^2 - 1}$$
$$y(x) = x + \sqrt{x^2 - 1}$$

✓ Solution by Mathematica

Time used: 0.129 (sec). Leaf size: 33

```
DSolve[{y'[x]==y[x]/(y[x]-x),{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - \sqrt{x^2 - 1}$$
$$y(x) \to \sqrt{x^2 - 1} + x$$

8.32 problem 11 (c)

Internal problem ID [12730]
Internal file name [`OUTPUT/11382_Friday_November_03_2023_06_31_37_AM_29318325/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 11 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{y}{y - x} = 0}$$

With initial conditions

$$[y(1) = 0]$$

8.32.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y}{y - x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{y-x}\right)\\ &= \frac{1}{y-x} - \frac{y}{(y-x)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

8.32.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{u(x)\,x}{u(x)\,x - x} = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{u(u-2)}{x\,(u-1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{(u-2)u}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{(u-2)u}{u-1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{(u-2)u}{u-1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u(u-2))}{2} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$\sqrt{u\left(u-2\right)}=\mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u\left(u-2\right)}=\frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u\left(x\right)\left(u\left(x\right)-2\right)}=\frac{c_3\mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{u\left(x\right)\left(u\left(x\right)-2\right)}=\frac{c_3\mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y\left(\frac{y}{x}-2\right)}{x}}=\frac{c_3\mathrm{e}^{c_2}}{x}$$
$$\sqrt{\frac{y\left(-2x+y\right)}{x^2}}=\frac{c_3\mathrm{e}^{c_2}}{x}$$

Initial conditions are used to solve for c_2. Substituting $x=1$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$0=c_3\mathrm{e}^{c_2}$$

Unable to solve for constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

8.32.3 Solving as differentialType ode

Writing the ode as

$$y'=\frac{y}{y-x} \tag{1}$$

Which becomes

$$(-y)\,dy=(-x)\,dy+(-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (-y)\,dx = d(-xy)$$

Hence (2) becomes

$$(-y)\,dy = d(-xy)$$

Integrating both sides gives gives these solutions

$$y = x + \sqrt{x^2 - 2c_1} + c_1$$
$$y = x - \sqrt{x^2 - 2c_1} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 1 - \sqrt{-2c_1 + 1} + c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 1 + \sqrt{-2c_1 + 1} + c_1$$

$$c_1 = -4$$

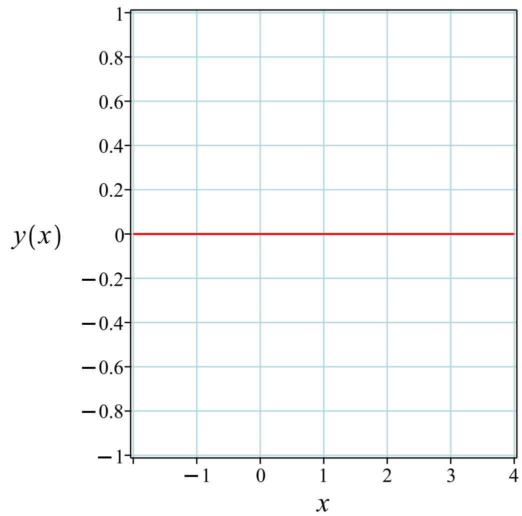
Substituting c_1 found above in the general solution gives

$$y = x + \sqrt{x^2 + 8} - 4$$

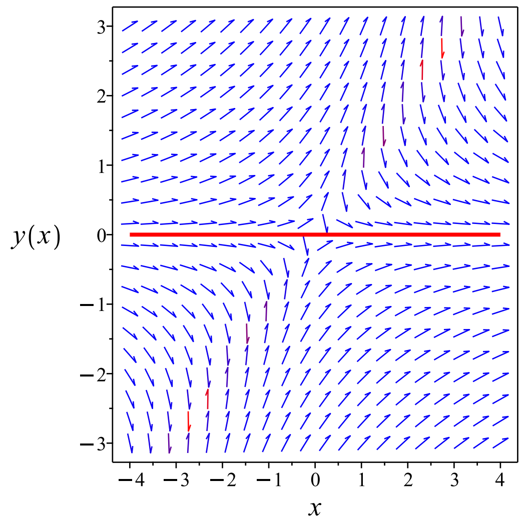
Summary

The solution(s) found are the following

$$y = x + \sqrt{x^2 + 8} - 4 \tag{1}$$
$$y = 0 \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x + \sqrt{x^2 + 8} - 4$$

Verified OK. `{positive}`

$$y = 0$$

Verified OK. `{positive}`

8.32.4 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y}{y - x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{y(b_3 - a_2)}{y - x} - \frac{y^2 a_3}{(y - x)^2} - \frac{y(x a_2 + y a_3 + a_1)}{(y - x)^2} - \left(\frac{1}{y - x} - \frac{y}{(y - x)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2 b_2 - 2xy b_2 - y^2 a_2 - 2y^2 a_3 + y^2 b_2 + y^2 b_3 + x b_1 - y a_1}{(-y + x)^2} = 0$$

Setting the numerator to zero gives

$$2x^2 b_2 - 2xy b_2 - y^2 a_2 - 2y^2 a_3 + y^2 b_2 + y^2 b_3 + x b_1 - y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2 v_2^2 - 2a_3 v_2^2 + 2b_2 v_1^2 - 2b_2 v_1 v_2 + b_2 v_2^2 + b_3 v_2^2 - a_1 v_2 + b_1 v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$2b_2 v_1^2 - 2b_2 v_1 v_2 + b_1 v_1 + (-a_2 - 2a_3 + b_2 + b_3) v_2^2 - a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_1 &= 0 \\
-a_1 &= 0 \\
-2b_2 &= 0 \\
2b_2 &= 0 \\
-a_2 - 2a_3 + b_2 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= -2a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y}{y - x}\right)(x) \\
&= \frac{2xy - y^2}{-y + x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2xy - y^2}{-y+x}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln\left(y(-2x+y)\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{y-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{2x-y} \\ S_y &= \frac{-y+x}{y\,(2x-y)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

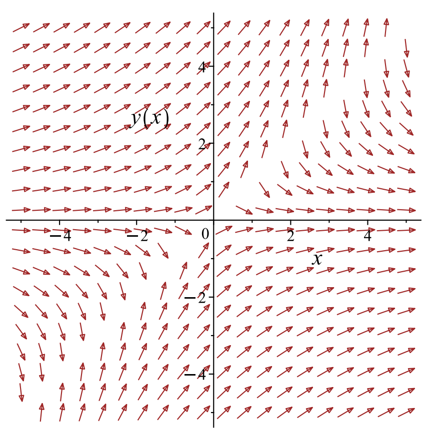
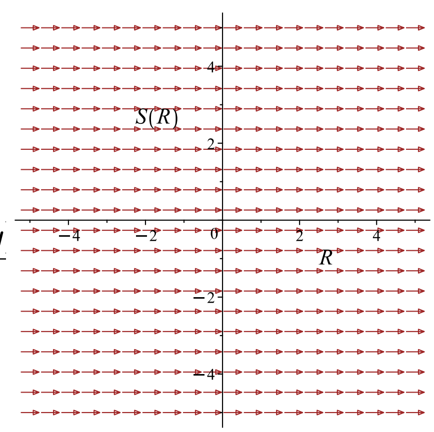
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)}{2} + \frac{\ln(-2x + y)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(y)}{2} + \frac{\ln(-2x + y)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{y-x}$ 	$R = x$ $S = \frac{\ln(y)}{2} + \frac{\ln(-2x+y)}{2}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\infty = c_1$$

Unable to solve for constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

8.32.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(y-x)\,\mathrm{d}y&=(y)\,\mathrm{d}x\\ (-y)\,\mathrm{d}x+(y-x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y\\ N(x,y)&=y-x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y - x) \\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -y \, \mathrm{d}x$$

$$\phi = -xy + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y - x$. Therefore equation (4) becomes

$$y - x = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int (y) \, dy$$

$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -xy + \frac{1}{2}y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -xy + \frac{1}{2}y^2$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$-xy + \frac{1}{2}y^2 = 0$$

Summary

The solution(s) found are the following

$$-yx + \frac{y^2}{2} = 0 \tag{1}$$

Verification of solutions

$$-yx + \frac{y^2}{2} = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=y(x)/(y(x)-x),y(1) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 6

```
DSolve[{y'[x]==y[x]/(y[x]-x),{y[1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

8.33 problem 11 (d)

Internal problem ID [12731]
Internal file name [`OUTPUT/11383_Friday_November_03_2023_06_31_39_AM_99472337/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 11 (d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{y}{y - x} = 0}$$

With initial conditions

$$[y(1) = -1]$$

8.33.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y}{y - x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -1$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{y-x}\right)\\ &= \frac{1}{y-x} - \frac{y}{(y-x)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

8.33.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - \frac{u(x)\, x}{u(x)\, x - x} = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u)\\ &= f(x)g(u)\\ &= -\frac{u(u-2)}{x(u-1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{(u-2)u}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{(u-2)u}{u-1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{(u-2)u}{u-1}}\, du = \int -\frac{1}{x}\, dx$$

$$\frac{\ln (u(u-2))}{2} = -\ln (x) + c_2$$

Raising both side to exponential gives

$$\sqrt{u\left(u-2\right)} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u\left(u-2\right)} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u\left(x\right)\left(u\left(x\right)-2\right)} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{u\left(x\right)\left(u\left(x\right)-2\right)} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y\left(\frac{y}{x}-2\right)}{x}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$
$$\sqrt{\frac{y\left(-2x+y\right)}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Substituting initial conditions and solving for c_2 gives $c_2 = \frac{\ln\left(\frac{3}{c_3^2}\right)}{2}$. Hence the solution becomes Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$\sqrt{3} = c_3\sqrt{3}\,\sqrt{\frac{1}{c_3^2}}$$

This solution is valid for any c_3. Hence there are infinite number of solutions.

Summary

The solution(s) found are the following

$$\sqrt{\frac{y\left(-2x+y\right)}{x^2}} = \frac{c_3\sqrt{3}\,\sqrt{\frac{1}{c_3^2}}}{x} \tag{1}$$

Verification of solutions

$$\sqrt{\frac{y\left(-2x+y\right)}{x^2}} = \frac{c_3\sqrt{3}\,\sqrt{\frac{1}{c_3^2}}}{x}$$

Verified OK.

8.33.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{y}{y - x} \tag{1}$$

Which becomes

$$(-y)\,dy = (-x)\,dy + (-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (-y)\,dx = d(-xy)$$

Hence (2) becomes

$$(-y)\,dy = d(-xy)$$

Integrating both sides gives gives these solutions

$$y = x + \sqrt{x^2 - 2c_1} + c_1$$
$$y = x - \sqrt{x^2 - 2c_1} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = 1 - \sqrt{-2c_1 + 1} + c_1$$

$$c_1 = \sqrt{6} - 3$$

Substituting c_1 found above in the general solution gives

$$y = x - \sqrt{x^2 - 2\sqrt{6} + 6} + \sqrt{6} - 3$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = 1 + \sqrt{-2c_1 + 1} + c_1$$

$$c_1 = -\sqrt{6} - 3$$

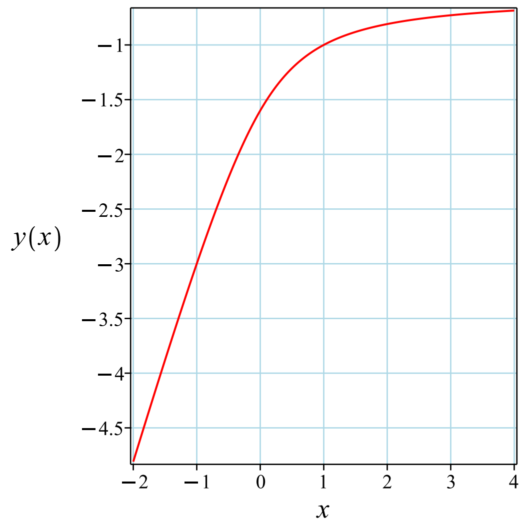
Substituting c_1 found above in the general solution gives

$$y = x + \sqrt{x^2 + 2\sqrt{6} + 6} - \sqrt{6} - 3$$

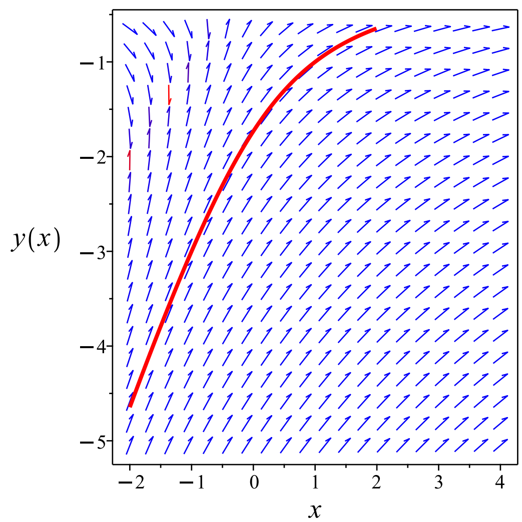
Summary
The solution(s) found are the following

$$y = x + \sqrt{x^2 + 2\sqrt{6} + 6} - \sqrt{6} - 3 \tag{1}$$

$$y = x - \sqrt{x^2 - 2\sqrt{6} + 6} + \sqrt{6} - 3 \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x + \sqrt{x^2 + 2\sqrt{6} + 6} - \sqrt{6} - 3$$

Verified OK. `{positive}`

$$y = x - \sqrt{x^2 - 2\sqrt{6} + 6} + \sqrt{6} - 3$$

Verified OK. `{positive}`

8.33.4 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y}{y-x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 + \frac{y(b_3 - a_2)}{y-x} - \frac{y^2 a_3}{(y-x)^2} - \frac{y(x a_2 + y a_3 + a_1)}{(y-x)^2} \\
- \left(\frac{1}{y-x} - \frac{y}{(y-x)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{2x^2 b_2 - 2xy b_2 - y^2 a_2 - 2y^2 a_3 + y^2 b_2 + y^2 b_3 + x b_1 - y a_1}{(-y+x)^2} = 0
$$

Setting the numerator to zero gives

$$
2x^2 b_2 - 2xy b_2 - y^2 a_2 - 2y^2 a_3 + y^2 b_2 + y^2 b_3 + x b_1 - y a_1 = 0 \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_2^2 - 2a_3v_2^2 + 2b_2v_1^2 - 2b_2v_1v_2 + b_2v_2^2 + b_3v_2^2 - a_1v_2 + b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$2b_2v_1^2 - 2b_2v_1v_2 + b_1v_1 + (-a_2 - 2a_3 + b_2 + b_3) v_2^2 - a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -a_1 &= 0 \\ -2b_2 &= 0 \\ 2b_2 &= 0 \\ -a_2 - 2a_3 + b_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -2a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y}{y-x}\right)(x) \\
&= \frac{2xy - y^2}{-y+x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2xy-y^2}{-y+x}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(y(-2x+y)\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{y-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{2x - y} \\
S_y &= \frac{-y + x}{y\,(2x - y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

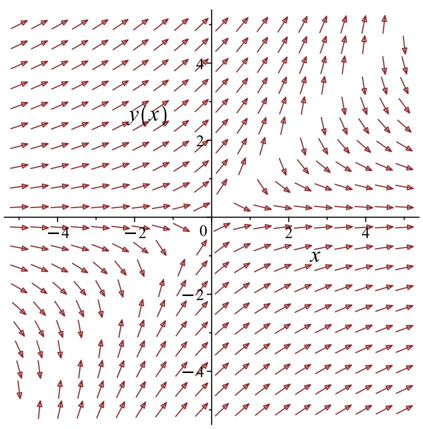
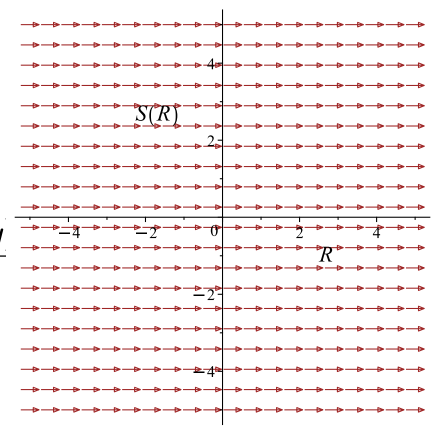
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(y\right)}{2} + \frac{\ln\left(-2x + y\right)}{2} = c_1$$

Which simplifies to

$$\frac{\ln\left(y\right)}{2} + \frac{\ln\left(-2x + y\right)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{y-x}$ 	$R = x$ $S = \frac{\ln(y)}{2} + \frac{\ln(-2x+y)}{2}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$i\pi + \frac{\ln(3)}{2} = c_1$$

$$c_1 = i\pi + \frac{\ln(3)}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(y)}{2} + \frac{\ln(-2x+y)}{2} = i\pi + \frac{\ln(3)}{2}$$

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{2} + \frac{\ln(-2x+y)}{2} = i\pi + \frac{\ln(3)}{2} \tag{1}$$

Verification of solutions

$$\frac{\ln(y)}{2} + \frac{\ln(-2x+y)}{2} = i\pi + \frac{\ln(3)}{2}$$

Verified OK.

8.33.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(y-x)\,\mathrm{d}y&=(y)\,\mathrm{d}x\\ (-y)\,\mathrm{d}x+(y-x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y\\ N(x,y)&=y-x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y - x) \\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -y \,\mathrm{d}x\end{aligned}$$

$$\phi = -xy + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y - x$. Therefore equation (4) becomes

$$y - x = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (y) \,\mathrm{d}y$$

$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -xy + \frac{1}{2}y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -xy + \frac{1}{2}y^2$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{2} = c_1$$

$$c_1 = \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$-xy + \frac{1}{2}y^2 = \frac{3}{2}$$

Summary
The solution(s) found are the following

$$-yx + \frac{y^2}{2} = \frac{3}{2} \tag{1}$$

Verification of solutions

$$-yx+\frac{y^2}{2}=\frac{3}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([diff(y(x),x)=y(x)/(y(x)-x),y(1) = -1],y(x), singsol=all)
```

$$y(x)=x-\sqrt{x^2+3}$$

✓ Solution by Mathematica

Time used: 0.127 (sec). Leaf size: 18

```
DSolve[{y'[x]==y[x]/(y[x]-x),{y[1]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to x-\sqrt{x^2+3}$$

8.34 problem 12 (a)

Internal problem ID [12732]
Internal file name [`OUTPUT/11384_Friday_November_03_2023_06_31_41_AM_99448235/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 12 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y' - \frac{xy}{x^2 + y^2} = 0}$$

With initial conditions

$$[y(0) = 1]$$

8.34.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{xy}{x^2 + y^2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{xy}{x^2+y^2}\right)\\ &= \frac{x}{x^2+y^2} - \frac{2x\,y^2}{\left(x^2+y^2\right)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.34.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x^2 u(x)}{x^2 + u(x)^2\,x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{u^3}{x\,(u^2+1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^3}{u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^3}{u^2+1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^3}{u^2+1}}\,du = \int -\frac{1}{x}\,dx$$

$$-\frac{1}{2u^2} + \ln(u) = -\ln(x) + c_2$$

The solution is

$$-\frac{1}{2u(x)^2}+\ln(u(x))+\ln(x)-c_2=0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x^2}{2y^2}+\ln\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$
$$-\frac{x^2}{2y^2}+\ln\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Initial conditions are used to solve for c_2. Substituting $x=0$ and $y=1$ in the above solution gives an equation to solve for the constant of integration.

$$undefined=0$$

This shows that no solution exist.

Summary

The solution(s) found are the following

$$-\frac{x^2}{2y^2}+\ln\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Verification of solutions

$$-\frac{x^2}{2y^2}+\ln\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Warning, solution could not be verified

8.34.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{xy}{x^2+y^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{xy(b_3 - a_2)}{x^2 + y^2} - \frac{x^2y^2a_3}{(x^2 + y^2)^2} - \left(\frac{y}{x^2 + y^2} - \frac{2x^2y}{(x^2 + y^2)^2}\right)(xa_2 + ya_3 + a_1) - \left(\frac{x}{x^2 + y^2} - \frac{2x\,y^2}{(x^2 + y^2)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-3x^2y^2b_2 + 2x\,y^3a_2 - 2x\,y^3b_3 + y^4a_3 - y^4b_2 + x^3b_1 - x^2ya_1 - x\,y^2b_1 + y^3a_1}{(x^2 + y^2)^2} = 0$$

Setting the numerator to zero gives

$$3x^2y^2b_2 - 2x\,y^3a_2 + 2x\,y^3b_3 - y^4a_3 + y^4b_2 - x^3b_1 + x^2ya_1 + x\,y^2b_1 - y^3a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2v_1v_2^3 - a_3v_2^4 + 3b_2v_1^2v_2^2 + b_2v_2^4 + 2b_3v_1v_2^3 + a_1v_1^2v_2 - a_1v_2^3 - b_1v_1^3 + b_1v_1v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_1v_1^3 + 3b_2v_1^2v_2^2 + a_1v_1^2v_2 + (-2a_2 + 2b_3)\,v_1v_2^3 + b_1v_1v_2^2 + (-a_3 + b_2)\,v_2^4 - a_1v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0\\
b_1 &= 0\\
-a_1 &= 0\\
-b_1 &= 0\\
3b_2 &= 0\\
-2a_2 + 2b_3 &= 0\\
-a_3 + b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x\\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi\\
&= y - \left(\frac{xy}{x^2 + y^2}\right)(x)\\
&= \frac{y^3}{x^2 + y^2}\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S&=\int\frac{1}{\eta}dy\\&=\int\frac{1}{\frac{y^3}{x^2+y^2}}dy\end{aligned}$$

Which results in

$$S=-\frac{x^2}{2y^2}+\ln(y)$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y}\tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{xy}{x^2+y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=1\\R_y&=0\\S_x&=-\frac{x}{y^2}\\S_y&=\frac{x^2+y^2}{y^3}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=0\tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln(y)\, y^2 - x^2}{2y^2} = c_1$$

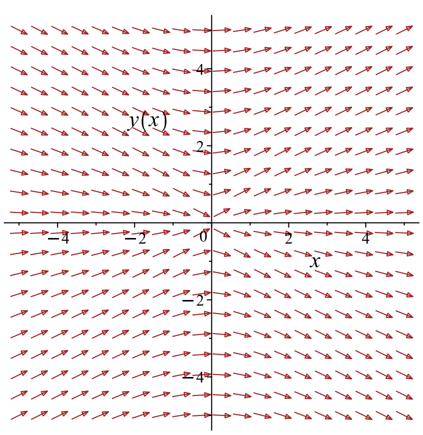
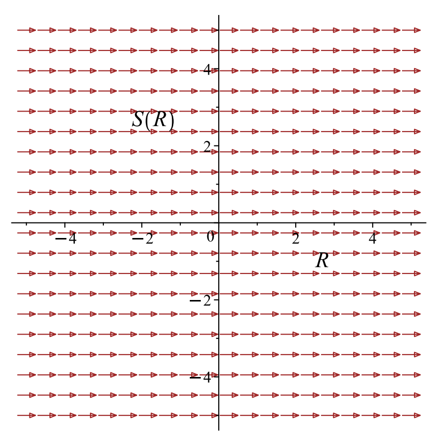
Which simplifies to

$$\frac{2\ln(y)\, y^2 - x^2}{2y^2} = c_1$$

Which gives

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{-2c_1} x^2\right)}{2} + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$  	$R = x$ $S = \frac{2\ln(y)\, y^2 - x^2}{2y^2}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = e^{c_1}$$

$$c_1 = 0$$

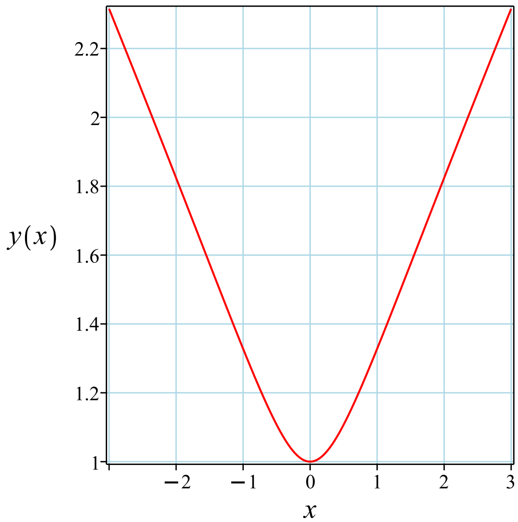
Substituting c_1 found above in the general solution gives

$$y = \sqrt{\frac{x^2}{\text{LambertW}(x^2)}}$$

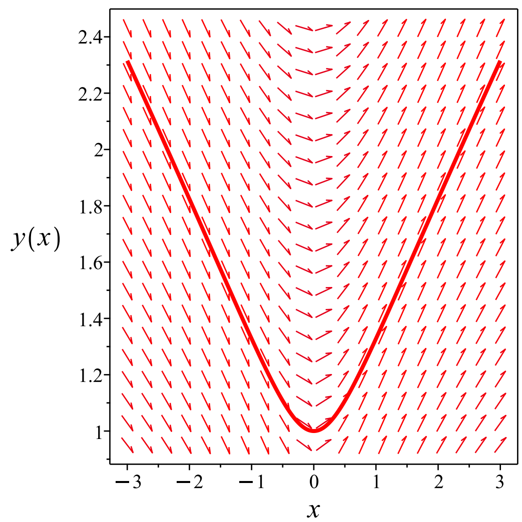
Summary

The solution(s) found are the following

$$y = \sqrt{\frac{x^2}{\text{LambertW}(x^2)}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{\frac{x^2}{\text{LambertW}(x^2)}}$$

Verified OK.

8.34.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2+y^2\right)\mathrm{d}y&=(xy)\,\mathrm{d}x\\ (-xy)\,\mathrm{d}x+\left(x^2+y^2\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-xy\\ N(x,y)&=x^2+y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-xy) \\ &= -x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 + y^2\right) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x^2 + y^2}((-x) - (2x)) \\ &= -\frac{3x}{x^2 + y^2}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{xy}((2x) - (-x)) \\ &= -\frac{3}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}y} \\ &= e^{\int -\frac{3}{y} \,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(y)} \\
&= \frac{1}{y^3}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^3}(-xy) \\
&= -\frac{x}{y^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^3}\left(x^2+y^2\right) \\
&= \frac{x^2+y^2}{y^3}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{x}{y^2}\right) + \left(\frac{x^2+y^2}{y^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{y^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2y^2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x^2}{y^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2+y^2}{y^3}$. Therefore equation (4) becomes

$$\frac{x^2 + y^2}{y^3} = \frac{x^2}{y^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2y^2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2y^2} + \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{-2c_1}x^2\right)}{2} + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \mathrm{e}^{c_1}$$

$$c_1 = 0$$

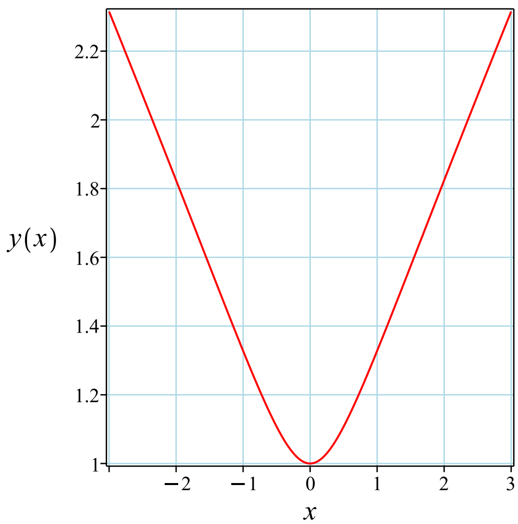
Substituting c_1 found above in the general solution gives

$$y = \sqrt{\frac{x^2}{\operatorname{LambertW}\left(x^2\right)}}$$

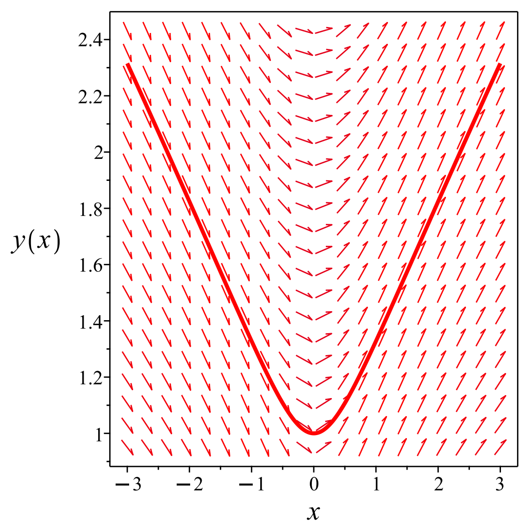
Summary

The solution(s) found are the following

$$y = \sqrt{\frac{x^2}{\operatorname{LambertW}\left(x^2\right)}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{\frac{x^2}{\operatorname{LambertW}\left(x^2\right)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.75 (sec). Leaf size: 11

```
dsolve([diff(y(x),x)=x*y(x)/(x^2+y(x)^2),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \sqrt{\frac{x^2}{\operatorname{LambertW}(x^2)}}$$

✓ Solution by Mathematica

Time used: 10.851 (sec). Leaf size: 15

```
DSolve[{y'[x]==x*y[x]/(x^2+y[x]^2),{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{\sqrt{W(x^2)}}$$

8.35 problem 12 (b)

Internal problem ID [12733]
Internal file name [`OUTPUT/11385_Friday_November_03_2023_06_31_48_AM_49211186/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 12 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y' - \frac{xy}{x^2 + y^2} = 0}$$

With initial conditions

$$[y(0) = 0]$$

8.35.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{xy}{x^2 + y^2} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{xy}{x^2+y^2}\right)\\ &= \frac{x}{x^2+y^2} - \frac{2x\,y^2}{\left(x^2+y^2\right)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{x < 0 \lor 0 < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

8.35.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x^2 u(x)}{x^2 + u(x)^2\,x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u)\\ &= f(x)g(u)\\ &= -\frac{u^3}{x\,(u^2+1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^3}{u^2+1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u^3}{u^2+1}}\,du &= -\frac{1}{x}\,dx\\ \int \frac{1}{\frac{u^3}{u^2+1}}\,du &= \int -\frac{1}{x}\,dx\\ -\frac{1}{2u^2} + \ln(u) &= -\ln(x) + c_2\end{aligned}$$

The solution is

$$-\frac{1}{2u(x)^2}+\ln(u(x))+\ln(x)-c_2=0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x^2}{2y^2}+\ln\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$
$$-\frac{x^2}{2y^2}+\ln\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Initial conditions are used to solve for c_2. Substituting $x=0$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$-\infty=0$$

This shows that no solution exist.

Summary

The solution(s) found are the following

$$-\frac{x^2}{2y^2}+\ln\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Verification of solutions

$$-\frac{x^2}{2y^2}+\ln\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Verified OK.

8.35.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{xy}{x^2+y^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{xy(b_3 - a_2)}{x^2 + y^2} - \frac{x^2 y^2 a_3}{(x^2 + y^2)^2} - \left(\frac{y}{x^2 + y^2} - \frac{2x^2 y}{(x^2 + y^2)^2}\right)(xa_2 + ya_3 + a_1) - \left(\frac{x}{x^2 + y^2} - \frac{2x\, y^2}{(x^2 + y^2)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-3x^2y^2b_2 + 2x\, y^3 a_2 - 2x\, y^3 b_3 + y^4 a_3 - y^4 b_2 + x^3 b_1 - x^2 y a_1 - x\, y^2 b_1 + y^3 a_1}{(x^2 + y^2)^2} = 0$$

Setting the numerator to zero gives

$$3x^2y^2b_2 - 2x\, y^3 a_2 + 2x\, y^3 b_3 - y^4 a_3 + y^4 b_2 - x^3 b_1 + x^2 y a_1 + x\, y^2 b_1 - y^3 a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2 v_1 v_2^3 - a_3 v_2^4 + 3b_2 v_1^2 v_2^2 + b_2 v_2^4 + 2b_3 v_1 v_2^3 + a_1 v_1^2 v_2 - a_1 v_2^3 - b_1 v_1^3 + b_1 v_1 v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_1 v_1^3 + 3b_2 v_1^2 v_2^2 + a_1 v_1^2 v_2 + (-2a_2 + 2b_3)\, v_1 v_2^3 + b_1 v_1 v_2^2 + (-a_3 + b_2)\, v_2^4 - a_1 v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
b_1 &= 0 \\
-a_1 &= 0 \\
-b_1 &= 0 \\
3b_2 &= 0 \\
-2a_2 + 2b_3 &= 0 \\
-a_3 + b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{xy}{x^2 + y^2}\right)(x) \\
&= \frac{y^3}{x^2 + y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}\right) S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^3}{x^2+y^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^2}{2y^2} + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{xy}{x^2+y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x}{y^2} \\ S_y &= \frac{x^2+y^2}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln(y)\,y^2 - x^2}{2y^2} = c_1$$

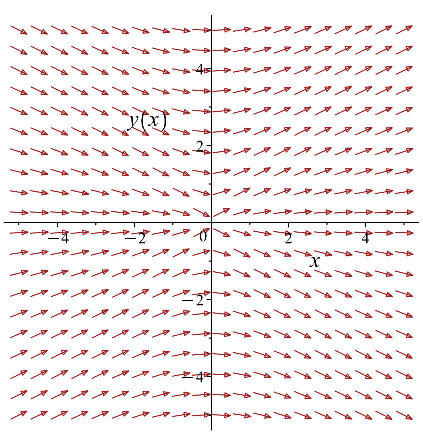
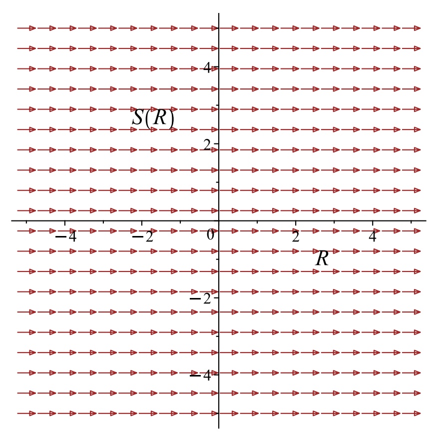
Which simplifies to

$$\frac{2\ln(y)\,y^2 - x^2}{2y^2} = c_1$$

Which gives

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{-2c_1}x^2\right)}{2} + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$ 	$R = x$ $S = \frac{2\ln(y)\,y^2 - x^2}{2y^2}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \mathrm{e}^{c_1}$$

Unable to solve for constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

8.35.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
\left(x^{2}+y^{2}\right) \mathrm{d}y &= (xy)\,\mathrm{d}x \\
(-xy)\,\mathrm{d}x+\left(x^{2}+y^{2}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -xy \\
N(x,y) &= x^{2}+y^{2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-xy) \\
&= -x
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^{2}+y^{2}\right) \\
&= 2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^{2}+y^{2}}((-x)-(2x)) \\
&= -\frac{3x}{x^{2}+y^{2}}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{xy}((2x)-(-x)) \\
&= -\frac{3}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{3}{y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(y)} \\ &= \frac{1}{y^3}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{y^3}(-xy) \\ &= -\frac{x}{y^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{y^3}\left(x^2+y^2\right) \\ &= \frac{x^2+y^2}{y^3}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\frac{x}{y^2}\right) + \left(\frac{x^2+y^2}{y^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{x}{y^2} \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2y^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x^2}{y^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2+y^2}{y^3}$. Therefore equation (4) becomes

$$\frac{x^2 + y^2}{y^3} = \frac{x^2}{y^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2y^2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2y^2} + \ln(y)$$

The solution becomes

$$y = e^{\frac{\mathrm{LambertW}\left(e^{-2c_1}x^2\right)}{2} + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = e^{c_1}$$

Unable to solve for constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=x*y(x)/(x^2+y(x)^2),y(0) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 6

```
DSolve[{y'[x]==x*y[x]/(x^2+y[x]^2),{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

8.36 problem 12 (c)

Internal problem ID [12734]
Internal file name [`OUTPUT/11386_Friday_November_03_2023_06_31_54_AM_18794178/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 12 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y' - \frac{xy}{x^2 + y^2} = 0}$$

With initial conditions

$$[y(0) = -1]$$

8.36.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{xy}{x^2 + y^2} \end{aligned}$$

The x domain of $f(x,y)$ when $y = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{xy}{x^2+y^2}\right)\\ &= \frac{x}{x^2+y^2} - \frac{2x\, y^2}{\left(x^2+y^2\right)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

8.36.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - \frac{x^2 u(x)}{x^2 + u\left(x\right)^2 x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u)\\ &= f(x) g(u)\\ &= -\frac{u^3}{x\left(u^2+1\right)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^3}{u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^3}{u^2+1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u^3}{u^2+1}}\, du = \int -\frac{1}{x}\, dx$$

$$-\frac{1}{2u^2} + \ln\left(u\right) = -\ln\left(x\right) + c_2$$

The solution is

$$-\frac{1}{2u(x)^2}+\ln(u(x))+\ln(x)-c_2=0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x^2}{2y^2}+\ln\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$
$$-\frac{x^2}{2y^2}+\ln\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Initial conditions are used to solve for c_2. Substituting $x=0$ and $y=-1$ in the above solution gives an equation to solve for the constant of integration.

$$i\pi-c_2=0$$

$$c_2=i\pi$$

Substituting c_2 found above in the general solution gives

$$\frac{-2i\pi\,y^2+2\ln(x)\,y^2+2\ln\left(\frac{y}{x}\right)y^2-x^2}{2y^2}=0$$

The above simplifies to

$$-2i\pi\,y^2+2\ln(x)\,y^2+2\ln\left(\frac{y}{x}\right)y^2-x^2=0$$

Summary

The solution(s) found are the following

$$-2i\pi y^2+2y^2\ln(x)+2\ln\left(\frac{y}{x}\right)y^2-x^2=0 \tag{1}$$

Verification of solutions

$$-2i\pi y^2+2y^2\ln(x)+2\ln\left(\frac{y}{x}\right)y^2-x^2=0$$

Warning, solution could not be verified

8.36.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{xy}{x^2+y^2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 + \frac{xy(b_3-a_2)}{x^2+y^2} - \frac{x^2 y^2 a_3}{(x^2+y^2)^2} - \left(\frac{y}{x^2+y^2} - \frac{2x^2 y}{(x^2+y^2)^2}\right)(x a_2 + y a_3 + a_1) \\
& - \left(\frac{x}{x^2+y^2} - \frac{2x\,y^2}{(x^2+y^2)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-3x^2y^2b_2 + 2x\,y^3a_2 - 2x\,y^3b_3 + y^4a_3 - y^4b_2 + x^3b_1 - x^2ya_1 - x\,y^2b_1 + y^3a_1}{(x^2+y^2)^2} = 0
$$

Setting the numerator to zero gives

$$
3x^2y^2b_2 - 2x\,y^3a_2 + 2x\,y^3b_3 - y^4a_3 + y^4b_2 - x^3b_1 + x^2ya_1 + x\,y^2b_1 - y^3a_1 = 0 \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2v_1v_2^3 - a_3v_2^4 + 3b_2v_1^2v_2^2 + b_2v_2^4 + 2b_3v_1v_2^3 + a_1v_1^2v_2 - a_1v_2^3 - b_1v_1^3 + b_1v_1v_2^2 = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_1v_1^3 + 3b_2v_1^2v_2^2 + a_1v_1^2v_2 + (-2a_2 + 2b_3)\, v_1v_2^3 + b_1v_1v_2^2 + (-a_3 + b_2)\, v_2^4 - a_1v_2^3 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ -a_1 &= 0 \\ -b_1 &= 0 \\ 3b_2 &= 0 \\ -2a_2 + 2b_3 &= 0 \\ -a_3 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{xy}{x^2+y^2}\right)(x) \\
&= \frac{y^3}{x^2+y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y^3}{x^2+y^2}} dy
\end{aligned}$$

Which results in

$$S = -\frac{x^2}{2y^2} + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{xy}{x^2+y^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{x}{y^2} \\
S_y &= \frac{x^2+y^2}{y^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{2\ln(y)\, y^2 - x^2}{2y^2} = c_1
$$

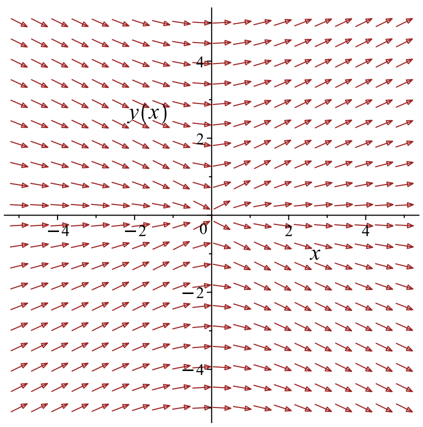
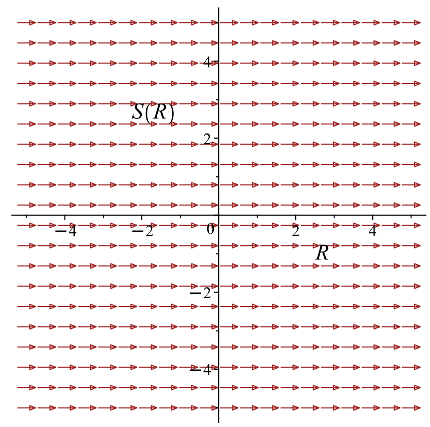
Which simplifies to

$$
\frac{2\ln(y)\, y^2 - x^2}{2y^2} = c_1
$$

Which gives

$$
y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{-2c_1} x^2\right)}{2} + c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$ 	$R = x$ $S = \frac{2\ln(y)\, y^2 - x^2}{2y^2}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = e^{c_1}$$

$$c_1 = i\pi$$

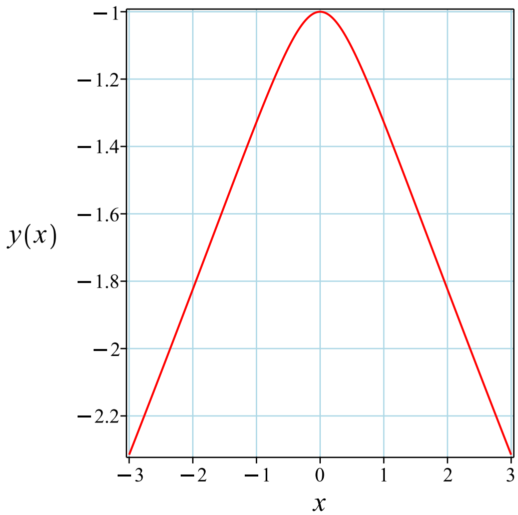
Substituting c_1 found above in the general solution gives

$$y = -\sqrt{\frac{x^2}{\mathrm{LambertW}(x^2)}}$$

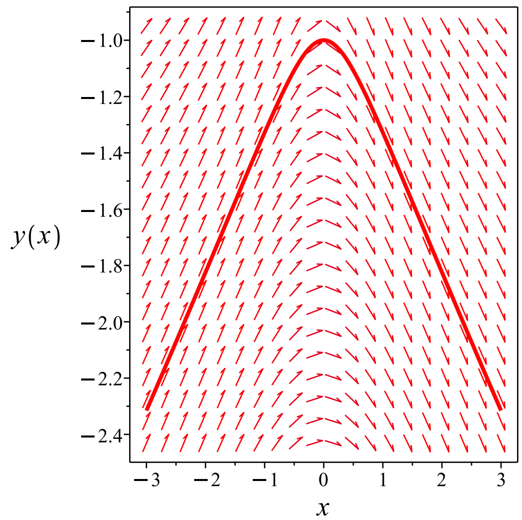
Summary

The solution(s) found are the following

$$y = -\sqrt{\frac{x^2}{\mathrm{LambertW}(x^2)}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\sqrt{\frac{x^2}{\operatorname{LambertW}(x^2)}}$$

Verified OK.

8.36.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 + y^2)\,\mathrm{d}y &= (xy)\,\mathrm{d}x\\ (-xy)\,\mathrm{d}x + (x^2 + y^2)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -xy\\ N(x, y) &= x^2 + y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-xy)\\ &= -x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + y^2)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^2+y^2}((-x)-(2x)) \\
&= -\frac{3x}{x^2+y^2}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{xy}((2x)-(-x)) \\
&= -\frac{3}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{3}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(y)} \\
&= \frac{1}{y^3}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^3}(-xy) \\
&= -\frac{x}{y^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^3}(x^2+y^2) \\
&= \frac{x^2+y^2}{y^3}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-\frac{x}{y^2}\right) + \left(\frac{x^2+y^2}{y^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{x}{y^2}\,\mathrm{d}x$$
$$\phi = -\frac{x^2}{2y^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x^2}{y^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2+y^2}{y^3}$. Therefore equation (4) becomes

$$\frac{x^2+y^2}{y^3} = \frac{x^2}{y^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right)\mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2y^2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2y^2} + \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(\mathrm{e}^{-2c_1}x^2\right)}{2}+c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \mathrm{e}^{c_1}$$

$$c_1 = i\pi$$

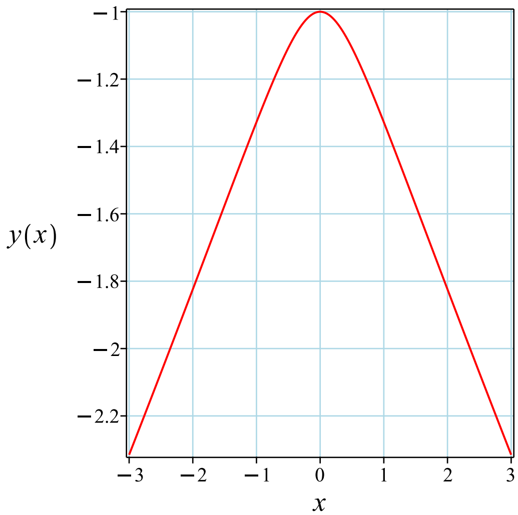
Substituting c_1 found above in the general solution gives

$$y = -\sqrt{\frac{x^2}{\mathrm{LambertW}(x^2)}}$$

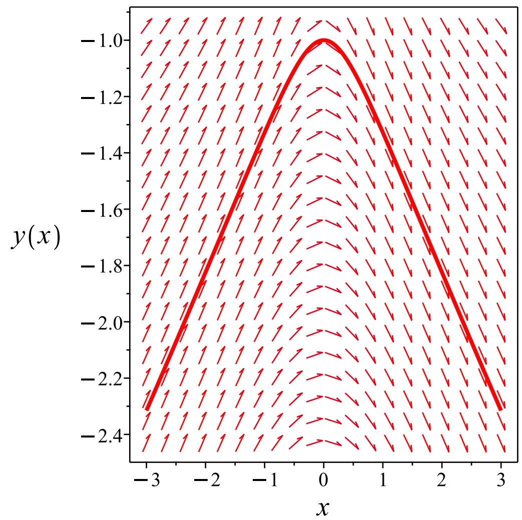
Summary

The solution(s) found are the following

$$y = -\sqrt{\frac{x^2}{\mathrm{LambertW}(x^2)}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\sqrt{\frac{x^2}{\operatorname{LambertW}(x^2)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.875 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)=x*y(x)/(x^2+y(x)^2),y(0) = -1],y(x), singsol=all)
```

$$y(x) = -\sqrt{\frac{x^2}{\text{LambertW}\left(x^2\right)}}$$

✓ Solution by Mathematica

Time used: 0.443 (sec). Leaf size: 16

```
DSolve[{y'[x]==x*y[x]/(x^2+y[x]^2),{y[0]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{x}{\sqrt{W\left(x^2\right)}}$$

8.37 problem 13 (a)

Internal problem ID [12735]
Internal file name [`OUTPUT/11387_Friday_November_03_2023_06_31_58_AM_6006130/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 13 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - x\sqrt{1 - y^2} = 0}$$

With initial conditions

$$[y(0) = 1]$$

8.37.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= x\sqrt{-y^2 + 1} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x\sqrt{-y^2+1}\right) \\ &= -\frac{xy}{\sqrt{-y^2+1}}\end{aligned}$$

$\frac{\partial f}{\partial y}$ is not defined at $y = 1$ therefore existence and uniqueness theorem do not apply. Solution exist but not guaranteed to be unique.

8.37.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= x\sqrt{-y^2+1}\end{aligned}$$

Where $f(x) = x$ and $g(y) = \sqrt{-y^2+1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\sqrt{-y^2+1}}\,dy &= x\,dx \\ \int \frac{1}{\sqrt{-y^2+1}}\,dy &= \int x\,dx \\ \arcsin(y) &= \frac{x^2}{2} + c_1\end{aligned}$$

Which results in

$$y = \sin\left(\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \sin(c_1)$$

$$c_1 = \frac{\pi}{2}$$

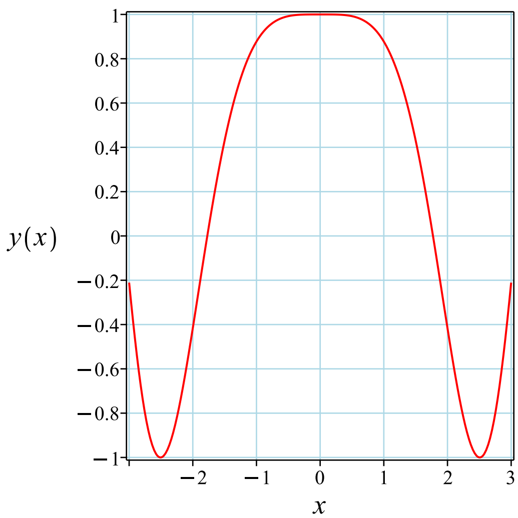
Substituting c_1 found above in the general solution gives

$$y = \cos\left(\frac{x^2}{2}\right)$$

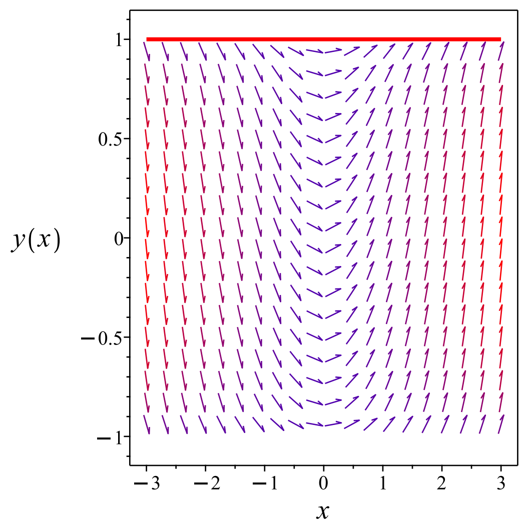
Summary

The solution(s) found are the following

$$y = \cos\left(\frac{x^2}{2}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cos\left(\frac{x^2}{2}\right)$$

Verified OK.

8.37.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x\sqrt{-y^2+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 294: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{x}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{x^2}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = x\sqrt{-y^2+1}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{-y^2+1}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arcsin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \arcsin(y) + c_1$$

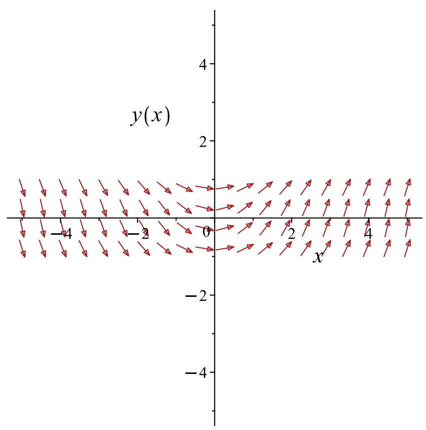
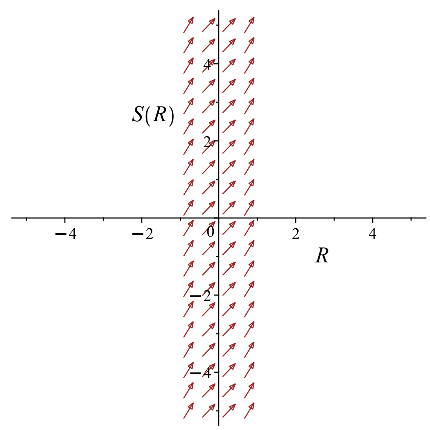
Which simplifies to

$$\frac{x^2}{2} = \arcsin(y) + c_1$$

Which gives

$$y = -\sin\left(-\frac{x^2}{2} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x\sqrt{-y^2+1}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\sin(c_1)$$

$$c_1 = -\frac{\pi}{2}$$

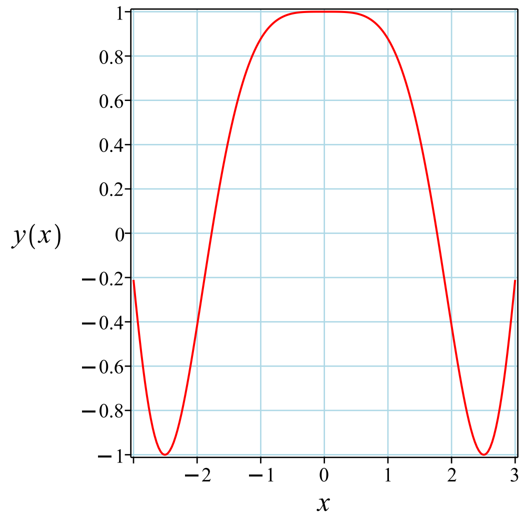
Substituting c_1 found above in the general solution gives

$$y = \cos\left(\frac{x^2}{2}\right)$$

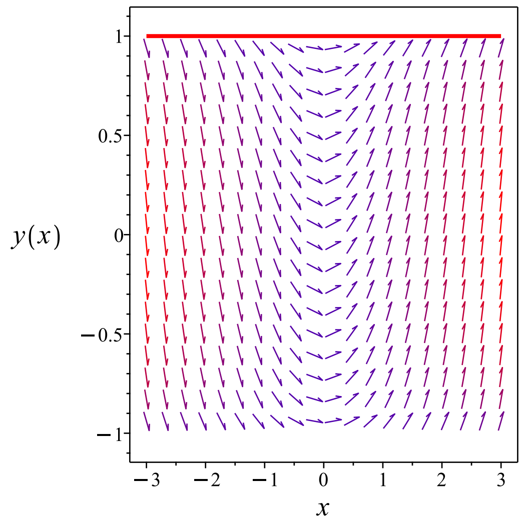
Summary

The solution(s) found are the following

$$y = \cos\left(\frac{x^2}{2}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cos\left(\frac{x^2}{2}\right)$$

Verified OK.

8.37.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{\sqrt{-y^2+1}}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+\left(\frac{1}{\sqrt{-y^2+1}}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\\ N(x,y)&=\frac{1}{\sqrt{-y^2+1}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-x)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{-y^2+1}}\right)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{-y^2+1}}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{-y^2 + 1}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{-y^2 + 1}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{\sqrt{-y^2 + 1}} \right) \mathrm{d}y$$

$$f(y) = \arcsin(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \arcsin(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \arcsin(y)$$

The solution becomes

$$y = \sin\left(\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \sin(c_1)$$

$$c_1 = \frac{\pi}{2}$$

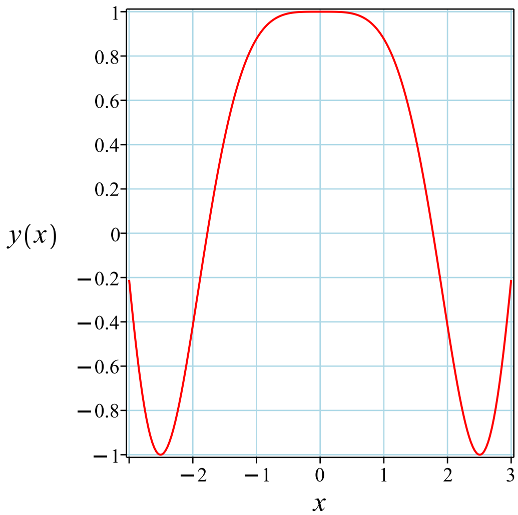
Substituting c_1 found above in the general solution gives

$$y = \cos\left(\frac{x^2}{2}\right)$$

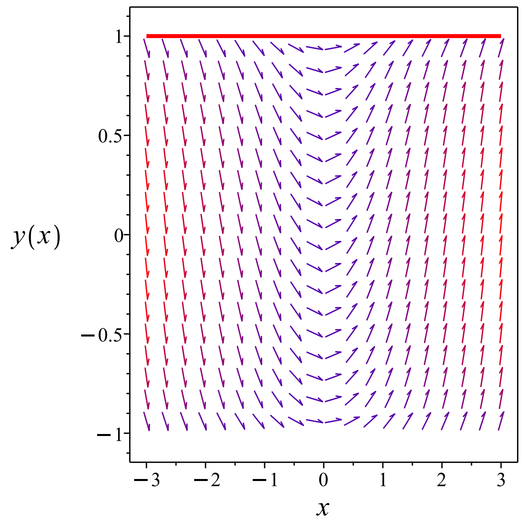
Summary

The solution(s) found are the following

$$y = \cos\left(\frac{x^2}{2}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cos\left(\frac{x^2}{2}\right)$$

Verified OK.

8.37.5 Maple step by step solution

Let's solve

$\left[y' - x\sqrt{1-y^2} = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y^2}} dx = \int x dx + c_1$

- Evaluate integral

  $\arcsin(y) = \frac{x^2}{2} + c_1$

- Solve for y

$$y = \sin\left(\frac{x^2}{2} + c_1\right)$$

- Use initial condition $y(0) = 1$

$$1 = \sin(c_1)$$

- Solve for c_1

$$c_1 = \frac{\pi}{2}$$

- Substitute $c_1 = \frac{\pi}{2}$ into general solution and simplify

$$y = \cos\left(\frac{x^2}{2}\right)$$

- Solution to the IVP

$$y = \cos\left(\frac{x^2}{2}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=x*sqrt(1-y(x)^2),y(0) = 1],y(x), singsol=all)
```

$$y(x) = 1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 6

```
DSolve[{y'[x]==x*Sqrt[1-y[x]^2],{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1$$

8.38 problem 13 (b)

Internal problem ID [12736]
Internal file name [`OUTPUT/11388_Friday_November_03_2023_06_31_59_AM_72333973/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 13 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - x\sqrt{1-y^2} = 0}$$

With initial conditions

$$\left[y(0) = \frac{9}{10}\right]$$

8.38.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= x\sqrt{-y^2+1} \end{aligned}$$

The x domain of $f(x,y)$ when $y = \frac{9}{10}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{9}{10}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x\sqrt{-y^2+1}\right)\\ &= -\frac{xy}{\sqrt{-y^2+1}}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{9}{10}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{9}{10}$ is inside this domain. Therefore solution exists and is unique.

8.38.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= x\sqrt{-y^2+1}\end{aligned}$$

Where $f(x) = x$ and $g(y) = \sqrt{-y^2+1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\sqrt{-y^2+1}}\,dy &= x\,dx\\ \int \frac{1}{\sqrt{-y^2+1}}\,dy &= \int x\,dx\\ \arcsin(y) &= \frac{x^2}{2} + c_1\end{aligned}$$

Which results in

$$y = \sin\left(\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x=0$ and $y=\frac{9}{10}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{9}{10}=\sin(c_1)$$

$$c_1=\arcsin\left(\frac{9}{10}\right)$$

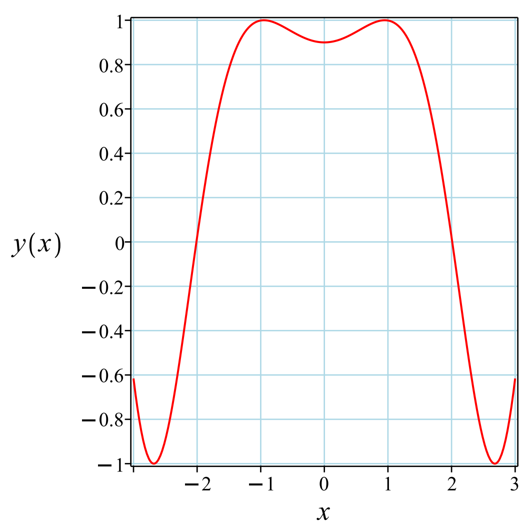
Substituting c_1 found above in the general solution gives

$$y=\sin\left(\frac{x^2}{2}+\arcsin\left(\frac{9}{10}\right)\right)$$

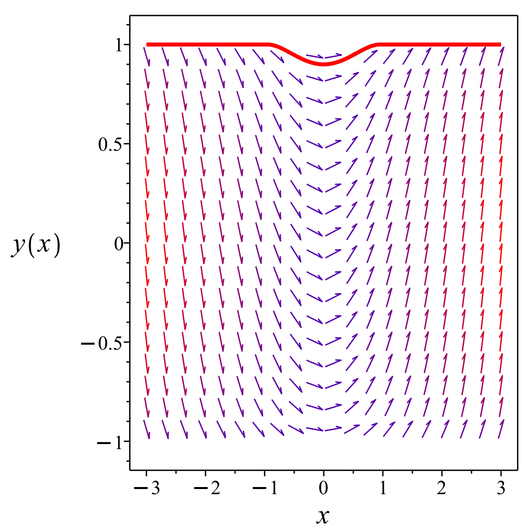
Summary

The solution(s) found are the following

$$y=\sin\left(\frac{x^2}{2}+\arcsin\left(\frac{9}{10}\right)\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y=\sin\left(\frac{x^2}{2}+\arcsin\left(\frac{9}{10}\right)\right)$$

Verified OK.

8.38.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= x\sqrt{-y^2+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 297: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{1}{x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$S = \int \frac{1}{\xi} dx$$
$$= \int \frac{1}{\frac{1}{x}} dx$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x\sqrt{-y^2+1}$$

Evaluating all the partial derivatives gives

$$R_x = 0$$
$$R_y = 1$$
$$S_x = x$$
$$S_y = 0$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{-y^2 + 1}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2 + 1}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arcsin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \arcsin(y) + c_1$$

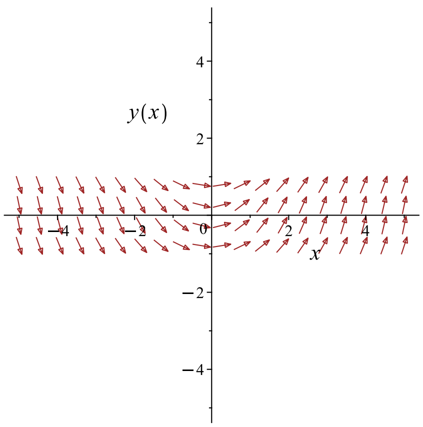
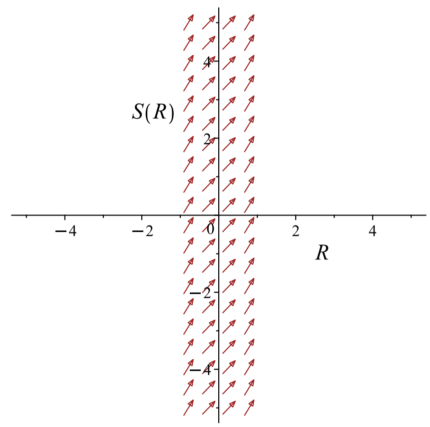
Which simplifies to

$$\frac{x^2}{2} = \arcsin(y) + c_1$$

Which gives

$$y = -\sin\left(-\frac{x^2}{2} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x\sqrt{-y^2+1}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{9}{10}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{9}{10} = -\sin(c_1)$$

$$c_1 = -\arcsin\left(\frac{9}{10}\right)$$

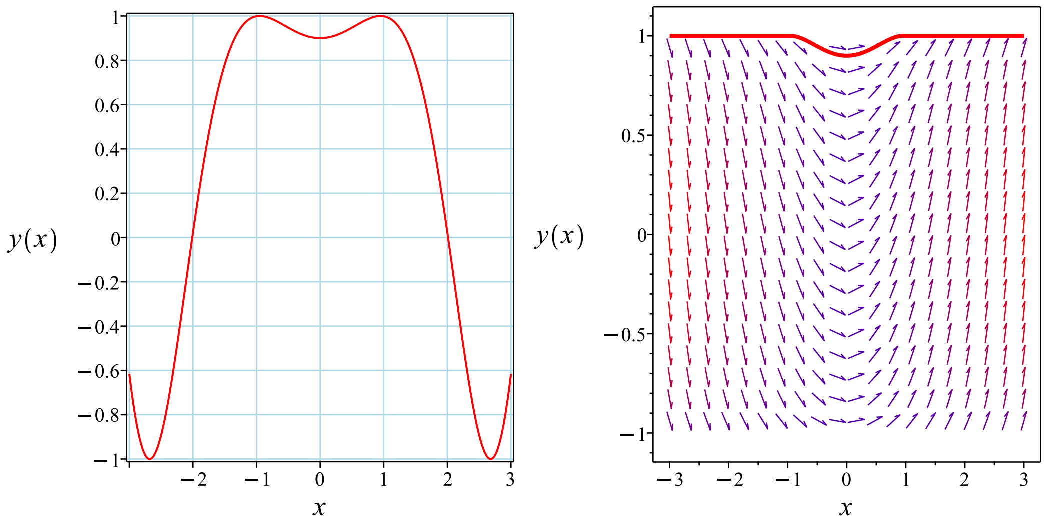
Substituting c_1 found above in the general solution gives

$$y = \sin\left(\frac{x^2}{2} + \arcsin\left(\frac{9}{10}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \sin\left(\frac{x^2}{2} + \arcsin\left(\frac{9}{10}\right)\right) \quad (1)$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \sin\left(\frac{x^2}{2} + \arcsin\left(\frac{9}{10}\right)\right)$$

Verified OK.

8.38.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{\sqrt{-y^2+1}}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\(-x)\,\mathrm{d}x+\left(\frac{1}{\sqrt{-y^2+1}}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\\N(x,y)&=\frac{1}{\sqrt{-y^2+1}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-x)\\&=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{-y^2+1}}\right)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{-y^2+1}}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{-y^2+1}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{-y^2+1}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{\sqrt{-y^2+1}}\right) \mathrm{d}y$$
$$f(y) = \arcsin(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \arcsin(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \arcsin(y)$$

The solution becomes

$$y = \sin\left(\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{9}{10}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{9}{10} = \sin(c_1)$$

$$c_1 = \arcsin\left(\frac{9}{10}\right)$$

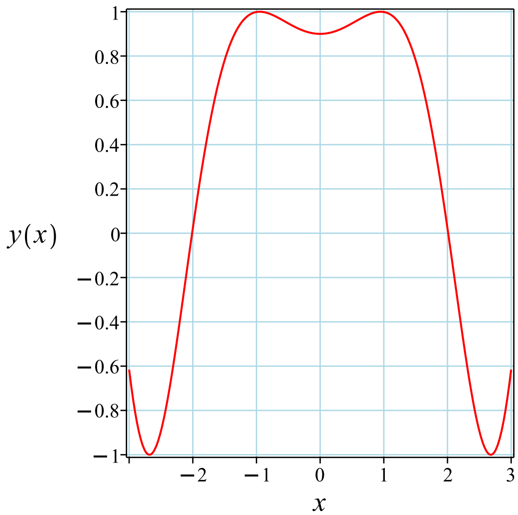
Substituting c_1 found above in the general solution gives

$$y = \sin\left(\frac{x^2}{2} + \arcsin\left(\frac{9}{10}\right)\right)$$

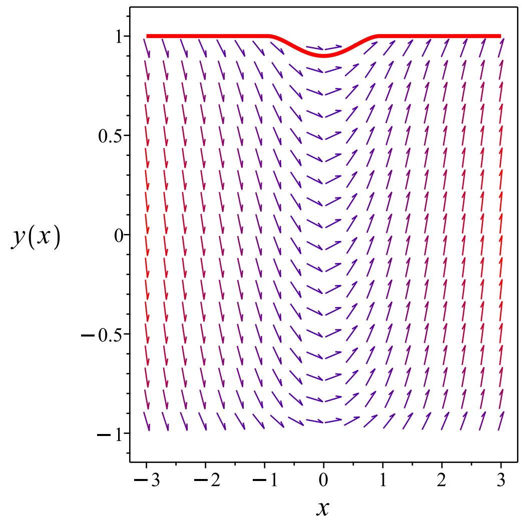
Summary

The solution(s) found are the following

$$y = \sin\left(\frac{x^2}{2} + \arcsin\left(\frac{9}{10}\right)\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sin\left(\frac{x^2}{2} + \arcsin\left(\frac{9}{10}\right)\right)$$

Verified OK.

8.38.5 Maple step by step solution

Let's solve

$\left[y' - x\sqrt{1-y^2} = 0, y(0) = \frac{9}{10}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y^2}}dx = \int x dx + c_1$

- Evaluate integral

  $\arcsin(y) = \frac{x^2}{2} + c_1$

- Solve for y

$$y = \sin\left(\frac{x^2}{2} + c_1\right)$$

- Use initial condition $y(0) = \frac{9}{10}$

$$\frac{9}{10} = \sin(c_1)$$

- Solve for c_1

$$c_1 = \arcsin\left(\frac{9}{10}\right)$$

- Substitute $c_1 = \arcsin\left(\frac{9}{10}\right)$ into general solution and simplify

$$y = \sin\left(\frac{x^2}{2} + \arcsin\left(\frac{9}{10}\right)\right)$$

- Solution to the IVP

$$y = \sin\left(\frac{x^2}{2} + \arcsin\left(\frac{9}{10}\right)\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)=x*sqrt(1-y(x)^2),y(0) = 9/10],y(x), singsol=all)
```

$$y(x) = \sin\left(\frac{x^2}{2} + \arcsin\left(\frac{9}{10}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.368 (sec). Leaf size: 43

```
DSolve[{y'[x]==x*Sqrt[1-y[x]^2],{y[0]==9/10}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos\left(\frac{1}{2}\left(4\arctan\left(\frac{1}{\sqrt{19}}\right) + x^2\right)\right)$$
$$y(x) \to \cos\left(\frac{1}{2}\left(x^2 - 4\arctan\left(\frac{1}{\sqrt{19}}\right)\right)\right)$$

8.39 problem 13 (c)

Internal problem ID [12737]
Internal file name [`OUTPUT/11389_Friday_November_03_2023_06_32_05_AM_78469741/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 13 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - x\sqrt{1-y^2} = 0}$$

With initial conditions

$$\left[y(0) = \frac{1}{2}\right]$$

8.39.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= x\sqrt{-y^2+1} \end{aligned}$$

The x domain of $f(x,y)$ when $y = \frac{1}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x\sqrt{-y^2+1}\right) \\ &= -\frac{xy}{\sqrt{-y^2+1}}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{1}{2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

8.39.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= x\sqrt{-y^2+1}\end{aligned}$$

Where $f(x) = x$ and $g(y) = \sqrt{-y^2+1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\sqrt{-y^2+1}}\, dy &= x\, dx \\ \int \frac{1}{\sqrt{-y^2+1}}\, dy &= \int x\, dx \\ \arcsin(y) &= \frac{x^2}{2} + c_1\end{aligned}$$

Which results in

$$y = \sin\left(\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \sin(c_1)$$

$$c_1 = \frac{\pi}{6}$$

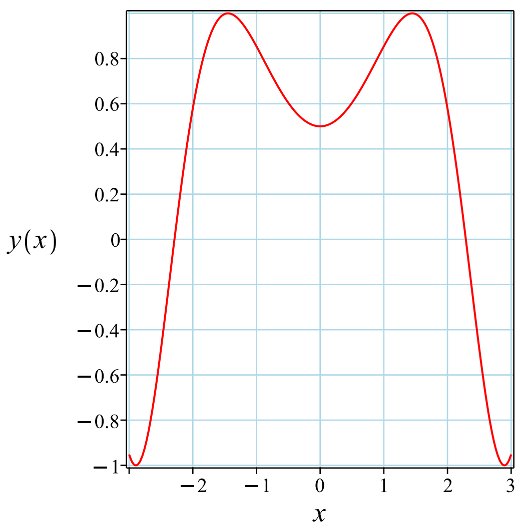
Substituting c_1 found above in the general solution gives

$$y = \sin\left(\frac{x^2}{2} + \frac{\pi}{6}\right)$$

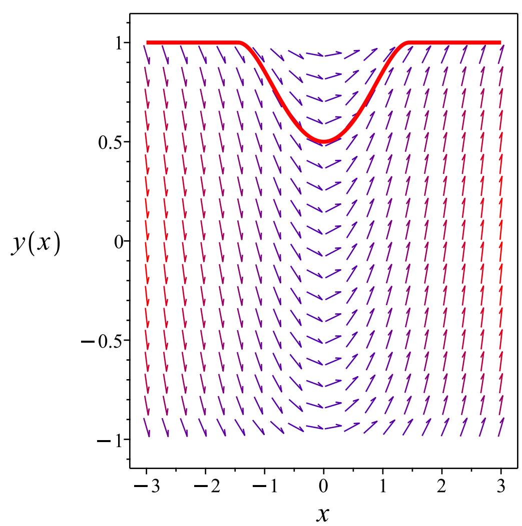
<u>Summary</u>

The solution(s) found are the following

$$y = \sin\left(\frac{x^2}{2} + \frac{\pi}{6}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \sin\left(\frac{x^2}{2} + \frac{\pi}{6}\right)$$

Verified OK.

8.39.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= x\sqrt{-y^2+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 300: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = \frac{1}{x}$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x\sqrt{-y^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{-y^2+1}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arcsin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \arcsin(y) + c_1$$

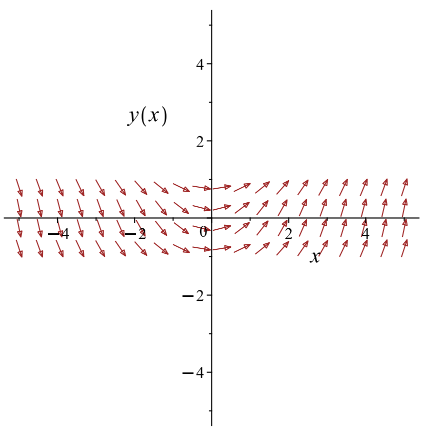
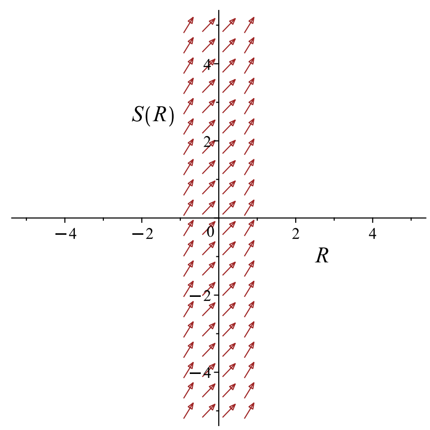
Which simplifies to

$$\frac{x^2}{2} = \arcsin(y) + c_1$$

Which gives

$$y = -\sin\left(-\frac{x^2}{2} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x\sqrt{-y^2+1}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\sin(c_1)$$

$$c_1 = -\frac{\pi}{6}$$

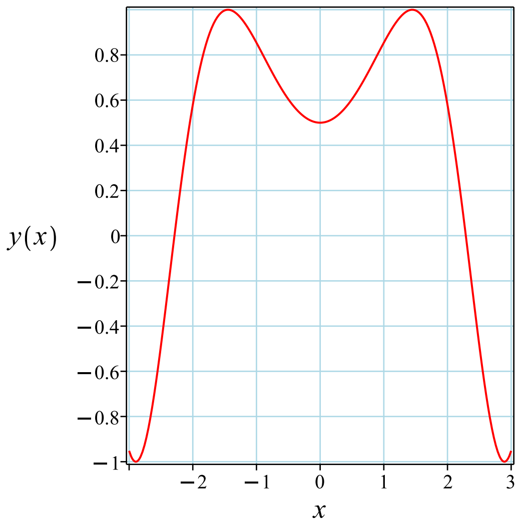
Substituting c_1 found above in the general solution gives

$$y = \sin\left(\frac{x^2}{2} + \frac{\pi}{6}\right)$$

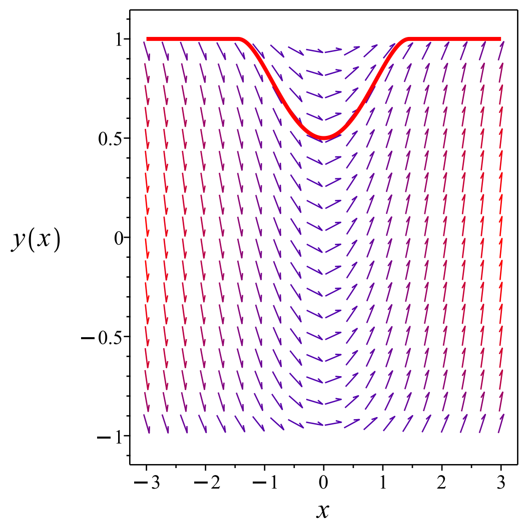
Summary

The solution(s) found are the following

$$y = \sin\left(\frac{x^2}{2} + \frac{\pi}{6}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \sin\left(\frac{x^2}{2} + \frac{\pi}{6}\right)$$

Verified OK.

8.39.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{\sqrt{-y^2+1}}\right)\mathrm{d}y = (x)\,\mathrm{d}x$$
$$(-x)\,\mathrm{d}x + \left(\frac{1}{\sqrt{-y^2+1}}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -x$$
$$N(x,y) = \frac{1}{\sqrt{-y^2+1}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{-y^2+1}}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -x\,\mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{1}{\sqrt{-y^2+1}}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{-y^2+1}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{-y^2+1}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{\sqrt{-y^2+1}}\right)\mathrm{d}y$$
$$f(y) = \arcsin(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \arcsin(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \arcsin(y)$$

The solution becomes

$$y = \sin\left(\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = \sin(c_1)$$

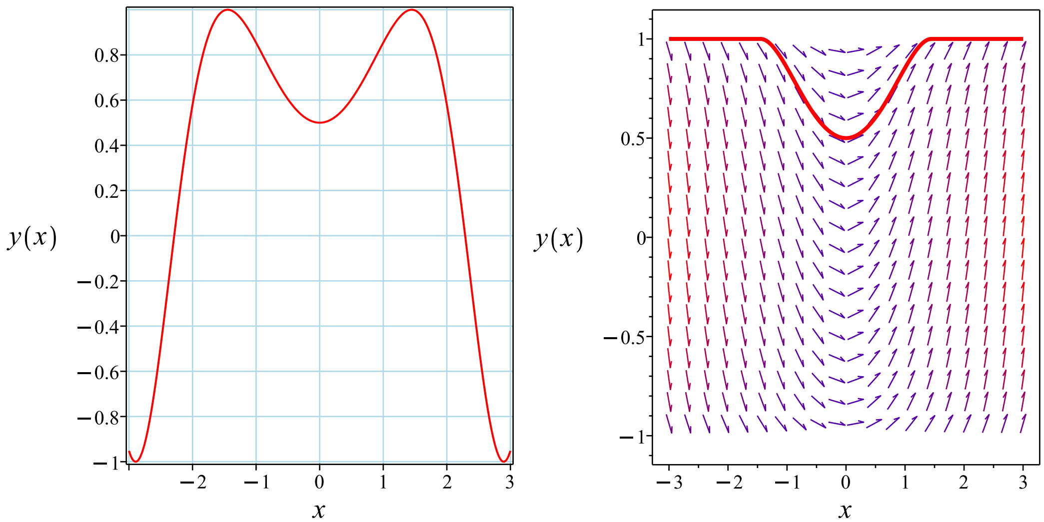
$$c_1 = \frac{\pi}{6}$$

Substituting c_1 found above in the general solution gives

$$y = \sin\left(\frac{x^2}{2} + \frac{\pi}{6}\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \sin\left(\frac{x^2}{2} + \frac{\pi}{6}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \sin\left(\frac{x^2}{2} + \frac{\pi}{6}\right)$$

Verified OK.

8.39.5 Maple step by step solution

Let's solve

$\left[y' - x\sqrt{1 - y^2} = 0, y(0) = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y^2}} dx = \int x dx + c_1$

- Evaluate integral

  $\arcsin(y) = \frac{x^2}{2} + c_1$

- Solve for y

$$y = \sin\left(\frac{x^2}{2} + c_1\right)$$

- Use initial condition $y(0) = \frac{1}{2}$

$$\frac{1}{2} = \sin(c_1)$$

- Solve for c_1

$$c_1 = \frac{\pi}{6}$$

- Substitute $c_1 = \frac{\pi}{6}$ into general solution and simplify

$$y = \sin\left(\frac{x^2}{2} + \frac{\pi}{6}\right)$$

- Solution to the IVP

$$y = \sin\left(\frac{x^2}{2} + \frac{\pi}{6}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)=x*sqrt(1-y(x)^2),y(0) = 1/2],y(x), singsol=all)
```

$$y(x) = \sin\left(\frac{x^2}{2} + \frac{\pi}{6}\right)$$

✓ Solution by Mathematica

Time used: 0.215 (sec). Leaf size: 33

```
DSolve[{y'[x]==x*Sqrt[1-y[x]^2],{y[0]==1/2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin\left(\frac{1}{6}(\pi - 3x^2)\right)$$
$$y(x) \to \sin\left(\frac{1}{6}(3x^2 + \pi)\right)$$

8.40 problem 13 (d)

Internal problem ID [12738]
Internal file name [`OUTPUT/11390_Friday_November_03_2023_06_32_09_AM_1705554/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 13 (d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - x\sqrt{1 - y^2} = 0}$$

With initial conditions

$$[y(0) = 0]$$

8.40.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= x\sqrt{-y^2 + 1} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x\sqrt{-y^2+1}\right) \\ &= -\frac{xy}{\sqrt{-y^2+1}}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

8.40.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= x\sqrt{-y^2+1}\end{aligned}$$

Where $f(x) = x$ and $g(y) = \sqrt{-y^2+1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\sqrt{-y^2+1}}\,dy &= x\,dx \\ \int \frac{1}{\sqrt{-y^2+1}}\,dy &= \int x\,dx \\ \arcsin(y) &= \frac{x^2}{2} + c_1\end{aligned}$$

Which results in

$$y = \sin\left(\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \sin(c_1)$$

$$c_1 = 0$$

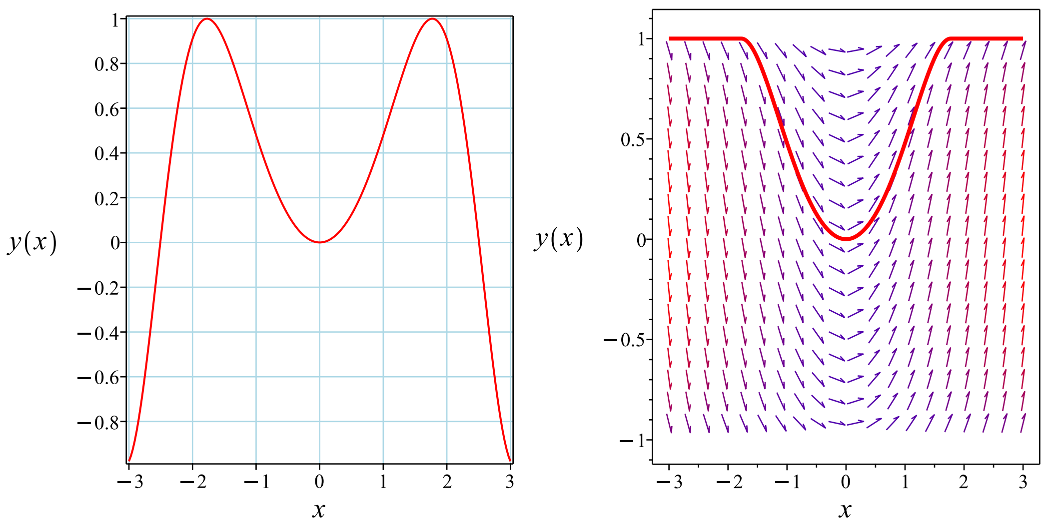
Substituting c_1 found above in the general solution gives

$$y = \sin\left(\frac{x^2}{2}\right)$$

<u>Summary</u>

The solution(s) found are the following

$$y = \sin\left(\frac{x^2}{2}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \sin\left(\frac{x^2}{2}\right)$$

Verified OK.

8.40.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= x\sqrt{-y^2+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 303: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= \frac{1}{x} \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x\sqrt{-y^2 + 1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{-y^2 + 1}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2 + 1}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arcsin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \arcsin(y) + c_1$$

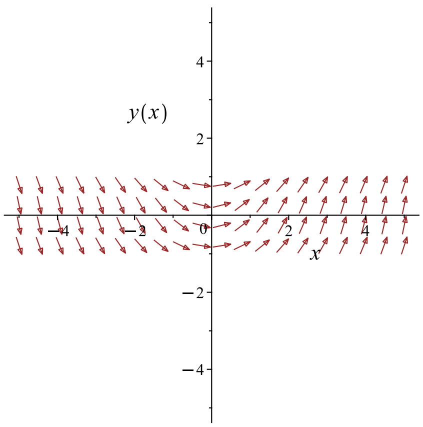
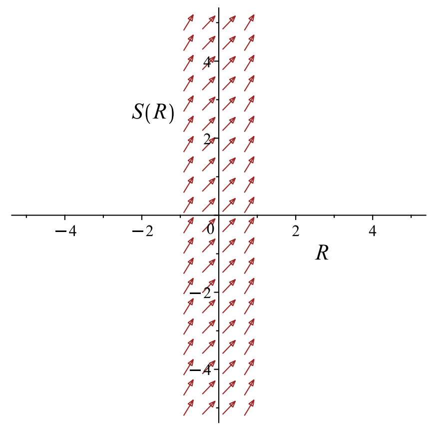
Which simplifies to

$$\frac{x^2}{2} = \arcsin(y) + c_1$$

Which gives

$$y = -\sin\left(-\frac{x^2}{2} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x\sqrt{-y^2+1}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\sin(c_1)$$

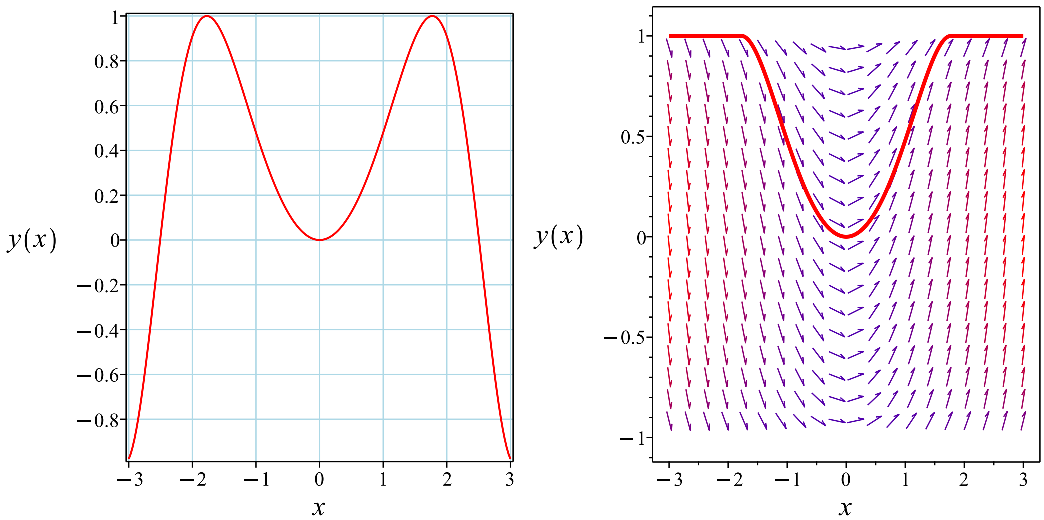
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \sin\left(\frac{x^2}{2}\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \sin\left(\frac{x^2}{2}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sin\left(\frac{x^2}{2}\right)$$

Verified OK.

8.40.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{\sqrt{-y^2+1}}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{1}{\sqrt{-y^2+1}}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x\\ N(x, y) &= \frac{1}{\sqrt{-y^2+1}}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{-y^2+1}}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{-y^2+1}}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{-y^2+1}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{-y^2+1}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{\sqrt{-y^2+1}}\right) \mathrm{d}y$$
$$f(y) = \arcsin(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \arcsin(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \arcsin(y)$$

The solution becomes

$$y = \sin\left(\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \sin(c_1)$$

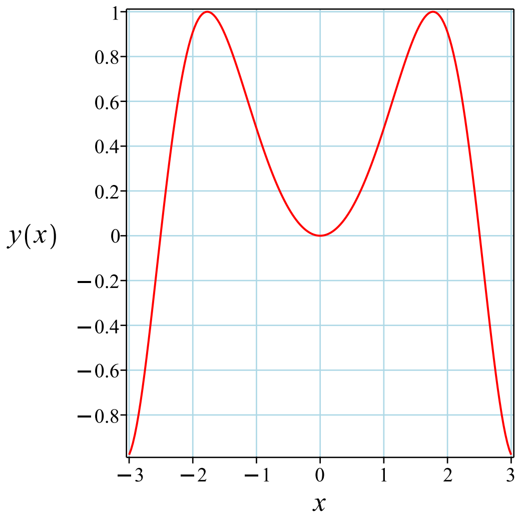
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

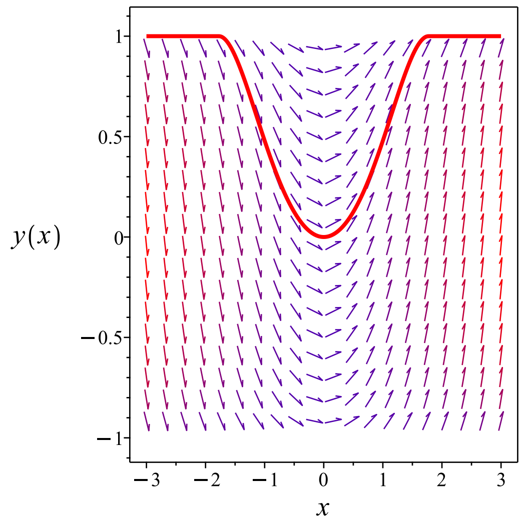
$$y = \sin\left(\frac{x^2}{2}\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \sin\left(\frac{x^2}{2}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sin\left(\frac{x^2}{2}\right)$$

Verified OK.

8.40.5 Maple step by step solution

Let's solve

$\left[y' - x\sqrt{1-y^2} = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y^2}}dx = \int x dx + c_1$

- Evaluate integral

  $\arcsin(y) = \frac{x^2}{2} + c_1$

- Solve for y

$$y = \sin\left(\frac{x^2}{2} + c_1\right)$$

- Use initial condition $y(0) = 0$

$$0 = \sin(c_1)$$

- Solve for c_1

$$c_1 = 0$$

- Substitute $c_1 = 0$ into general solution and simplify

$$y = \sin\left(\frac{x^2}{2}\right)$$

- Solution to the IVP

$$y = \sin\left(\frac{x^2}{2}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 10

```
dsolve([diff(y(x),x)=x*sqrt(1-y(x)^2),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \sin\left(\frac{x^2}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.21 (sec). Leaf size: 27

```
DSolve[{y'[x]==x*Sqrt[1-y[x]^2],{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sin\left(\frac{x^2}{2}\right)$$
$$y(x) \to \sin\left(\frac{x^2}{2}\right)$$

8.41 problem 14 (a)

Internal problem ID [12739]
Internal file name [`OUTPUT/11391_Friday_November_03_2023_06_32_10_AM_36940351/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 14 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Clairaut]
```

$$\boxed{y' - \frac{\sqrt{x^2+4y}}{2} = -\frac{x}{2}}$$

With initial conditions

$$[y(0) = 1]$$

8.41.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{0 \leq y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{2}+\frac{\sqrt{x^2+4y}}{2}\right) \\
&= \frac{1}{\sqrt{x^2+4y}}
\end{aligned}
$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$
\{0 < y\}
$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

8.41.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{x}{2}+\frac{\sqrt{x^2+4y}}{2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \left(-\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}\right)(b_3 - a_2) - \left(-\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}\right)^2 a_3 \\
&- \left(-\frac{1}{2} + \frac{x}{2\sqrt{x^2+4y}}\right)(xa_2 + ya_3 + a_1) - \frac{xb_2 + yb_3 + b_1}{\sqrt{x^2+4y}} = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{(x^2+4y)^{\frac{3}{2}} a_3 + \sqrt{x^2+4y}\, x^2 a_3 - 2x^3 a_3 - 4\sqrt{x^2+4y}\, x a_2 + 2\sqrt{x^2+4y}\, x b_3 - 2\sqrt{x^2+4y}\, y a_3 + 4x^2 a_2 -}{4\sqrt{x^2+4y}}
$$
$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-(x^2+4y)^{\frac{3}{2}} a_3 - \sqrt{x^2+4y}\, x^2 a_3 + 2x^3 a_3 + 4\sqrt{x^2+4y}\, x a_2 \\
&- 2\sqrt{x^2+4y}\, x b_3 + 2\sqrt{x^2+4y}\, y a_3 - 4x^2 a_2 + 2x^2 b_3 + 6xya_3 \\
&+ 2\sqrt{x^2+4y}\, a_1 + 4b_2\sqrt{x^2+4y} - 2xa_1 - 4xb_2 - 8ya_2 + 4yb_3 - 4b_1 = 0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-(x^2+4y)^{\frac{3}{2}} a_3 + 2(x^2+4y)\, x a_3 - \sqrt{x^2+4y}\, x^2 a_3 - 2(x^2+4y)\, a_2 \\
&+ 2(x^2+4y)\, b_3 + 4\sqrt{x^2+4y}\, x a_2 - 2\sqrt{x^2+4y}\, x b_3 + 2\sqrt{x^2+4y}\, y a_3 - 2x^2 a_2 \\
&- 2xya_3 + 2\sqrt{x^2+4y}\, a_1 + 4b_2\sqrt{x^2+4y} - 2xa_1 - 4xb_2 - 4yb_3 - 4b_1 = 0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&2x^3 a_3 - 2\sqrt{x^2+4y}\, x^2 a_3 - 4x^2 a_2 + 2x^2 b_3 + 4\sqrt{x^2+4y}\, x a_2 \\
&- 2\sqrt{x^2+4y}\, x b_3 + 6xya_3 - 2\sqrt{x^2+4y}\, y a_3 - 2xa_1 - 4xb_2 \\
&+ 2\sqrt{x^2+4y}\, a_1 + 4b_2\sqrt{x^2+4y} - 8ya_2 + 4yb_3 - 4b_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^2+4y}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2+4y} = v_3\right\}$$

The above PDE (6E) now becomes

$$2v_1^3 a_3 - 2v_3 v_1^2 a_3 - 4v_1^2 a_2 + 4v_3 v_1 a_2 + 6v_1 v_2 a_3 - 2v_3 v_2 a_3 + 2v_1^2 b_3 \\ - 2v_3 v_1 b_3 - 2v_1 a_1 + 2v_3 a_1 - 8v_2 a_2 - 4v_1 b_2 + 4b_2 v_3 + 4v_2 b_3 - 4b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$2v_1^3 a_3 - 2v_3 v_1^2 a_3 + (-4a_2 + 2b_3) v_1^2 + 6v_1 v_2 a_3 + (4a_2 - 2b_3) v_1 v_3 \\ + (-2a_1 - 4b_2) v_1 - 2v_3 v_2 a_3 + (-8a_2 + 4b_3) v_2 + (2a_1 + 4b_2) v_3 - 4b_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_3 &= 0 \\ 2a_3 &= 0 \\ 6a_3 &= 0 \\ -4b_1 &= 0 \\ -2a_1 - 4b_2 &= 0 \\ 2a_1 + 4b_2 &= 0 \\ -8a_2 + 4b_3 &= 0 \\ -4a_2 + 2b_3 &= 0 \\ 4a_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -2b_2 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= 2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2 \\
\eta &= x
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= x - \left(-\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}\right)(-2) \\
&= \sqrt{x^2+4y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\sqrt{x^2+4y}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\sqrt{x^2+4y}}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x}{2} + \frac{\sqrt{x^2 + 4y}}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x}{2\sqrt{x^2 + 4y}} \\ S_y &= \frac{1}{\sqrt{x^2 + 4y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{x^2 + 4y}}{2} = \frac{x}{2} + c_1$$

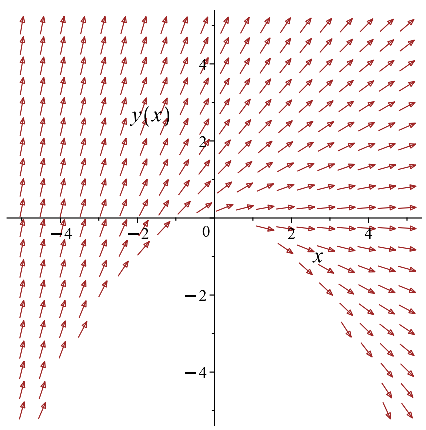
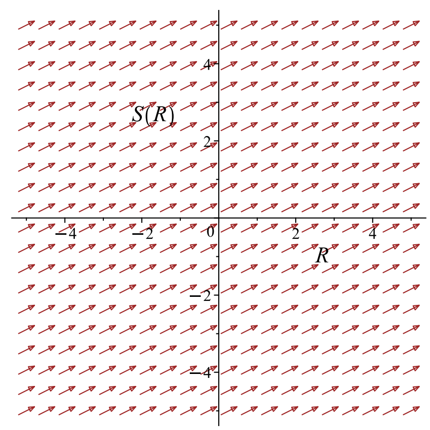
Which simplifies to

$$\frac{\sqrt{x^2 + 4y}}{2} = \frac{x}{2} + c_1$$

Which gives

$$y = c_1^2 + c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}$ 	$R = x$ $S = \frac{\sqrt{x^2+4y}}{2}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1^2$$

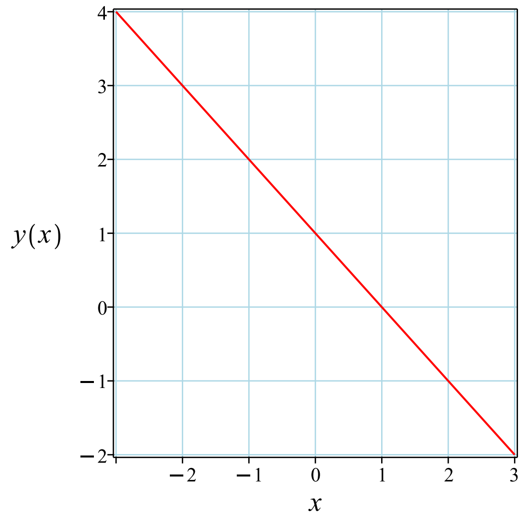
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

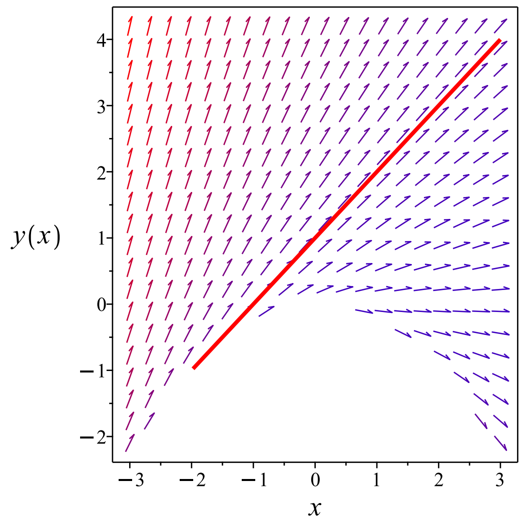
$$y = 1 - x$$

Summary
The solution(s) found are the following

$$y = 1 - x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 - x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
differential order: 1; found: 2 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.359 (sec). Leaf size: 15

```
dsolve([diff(y(x),x)=(-x+sqrt(x^2+4*y(x)))/2,y(0) = 1],y(x), singsol=all)
```

$$y(x) = 1 - x$$
$$y(x) = 1 + x$$

✓ Solution by Mathematica

Time used: 0.443 (sec). Leaf size: 17

```
DSolve[{y'[x]==(-x+Sqrt[x^2+4*y[x]])/2,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 - x$$
$$y(x) \to x + 1$$

8.42 problem 14 (b)

Internal problem ID [12740]
Internal file name [OUTPUT/11392_Friday_November_03_2023_06_32_12_AM_32103521/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 14 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Clairaut]
```

$$\boxed{y' - \frac{\sqrt{x^2+4y}}{2} = -\frac{x}{2}}$$

With initial conditions

$$[y(0) = 0]$$

8.42.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2} \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{x < 0 \lor 0 < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

8.42.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 &+ \left(-\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}\right)(b_3 - a_2) - \left(-\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}\right)^2 a_3 \\
&- \left(-\frac{1}{2} + \frac{x}{2\sqrt{x^2+4y}}\right)(x a_2 + y a_3 + a_1) - \frac{x b_2 + y b_3 + b_1}{\sqrt{x^2+4y}} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{(x^2+4y)^{\frac{3}{2}} a_3 + \sqrt{x^2+4y}\, x^2 a_3 - 2x^3 a_3 - 4\sqrt{x^2+4y}\, x a_2 + 2\sqrt{x^2+4y}\, x b_3 - 2\sqrt{x^2+4y}\, y a_3 + 4x^2 a_2 - \cdots}{4\sqrt{x^2+4y}} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-(x^2+4y)^{\frac{3}{2}} a_3 - \sqrt{x^2+4y}\, x^2 a_3 + 2x^3 a_3 + 4\sqrt{x^2+4y}\, x a_2 \\
&- 2\sqrt{x^2+4y}\, x b_3 + 2\sqrt{x^2+4y}\, y a_3 - 4x^2 a_2 + 2x^2 b_3 + 6xy a_3 \\
&+ 2\sqrt{x^2+4y}\, a_1 + 4b_2\sqrt{x^2+4y} - 2x a_1 - 4x b_2 - 8y a_2 + 4y b_3 - 4b_1 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$-(x^2+4y)^{\frac{3}{2}} a_3 + 2(x^2+4y)\, x a_3 - \sqrt{x^2+4y}\, x^2 a_3 - 2(x^2+4y)\, a_2 + 2(x^2+4y)\, b_3 + 4\sqrt{x^2+4y}\, x a_2 - 2\sqrt{x^2+4y}\, x b_3 + 2\sqrt{x^2+4y}\, y a_3 - 2x^2 a_2 - 2xy a_3 + 2\sqrt{x^2+4y}\, a_1 + 4b_2\sqrt{x^2+4y} - 2x a_1 - 4x b_2 - 4y b_3 - 4b_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$2x^3 a_3 - 2\sqrt{x^2+4y}\, x^2 a_3 - 4x^2 a_2 + 2x^2 b_3 + 4\sqrt{x^2+4y}\, x a_2 - 2\sqrt{x^2+4y}\, x b_3 + 6xy a_3 - 2\sqrt{x^2+4y}\, y a_3 - 2x a_1 - 4x b_2 + 2\sqrt{x^2+4y}\, a_1 + 4b_2\sqrt{x^2+4y} - 8y a_2 + 4y b_3 - 4b_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2+4y}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2+4y} = v_3\right\}$$

The above PDE (6E) now becomes

$$2v_1^3 a_3 - 2v_3 v_1^2 a_3 - 4v_1^2 a_2 + 4v_3 v_1 a_2 + 6v_1 v_2 a_3 - 2v_3 v_2 a_3 + 2v_1^2 b_3 - 2v_3 v_1 b_3 - 2v_1 a_1 + 2v_3 a_1 - 8v_2 a_2 - 4v_1 b_2 + 4b_2 v_3 + 4v_2 b_3 - 4b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$2v_1^3 a_3 - 2v_3 v_1^2 a_3 + (-4a_2 + 2b_3)\, v_1^2 + 6v_1 v_2 a_3 + (4a_2 - 2b_3)\, v_1 v_3 + (-2a_1 - 4b_2)\, v_1 - 2v_3 v_2 a_3 + (-8a_2 + 4b_3)\, v_2 + (2a_1 + 4b_2)\, v_3 - 4b_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_3 &= 0 \\
2a_3 &= 0 \\
6a_3 &= 0 \\
-4b_1 &= 0 \\
-2a_1 - 4b_2 &= 0 \\
2a_1 + 4b_2 &= 0 \\
-8a_2 + 4b_3 &= 0 \\
-4a_2 + 2b_3 &= 0 \\
4a_2 - 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -2b_2 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2 \\
\eta &= x
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= x - \left(-\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}\right)(-2) \\
&= \sqrt{x^2+4y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{x^2 + 4y}} dy \end{aligned}$$

Which results in

$$S = \frac{\sqrt{x^2 + 4y}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x}{2} + \frac{\sqrt{x^2 + 4y}}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x}{2\sqrt{x^2 + 4y}} \\ S_y &= \frac{1}{\sqrt{x^2 + 4y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{x^2+4y}}{2} = \frac{x}{2} + c_1$$

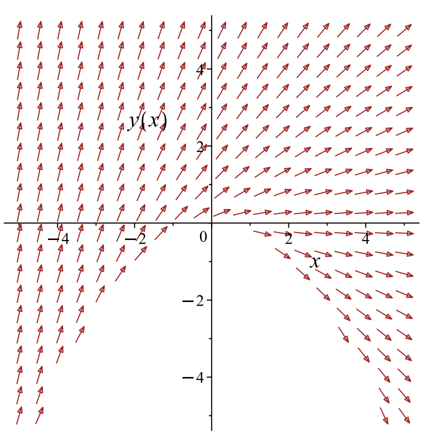
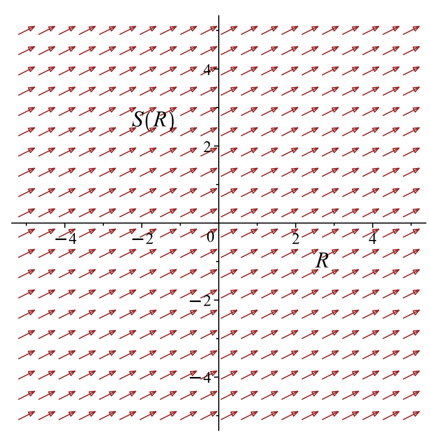
Which simplifies to

$$\frac{\sqrt{x^2+4y}}{2} = \frac{x}{2} + c_1$$

Which gives

$$y = c_1^2 + c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}$ 	$R = x$ $S = \frac{\sqrt{x^2+4y}}{2}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1^2$$

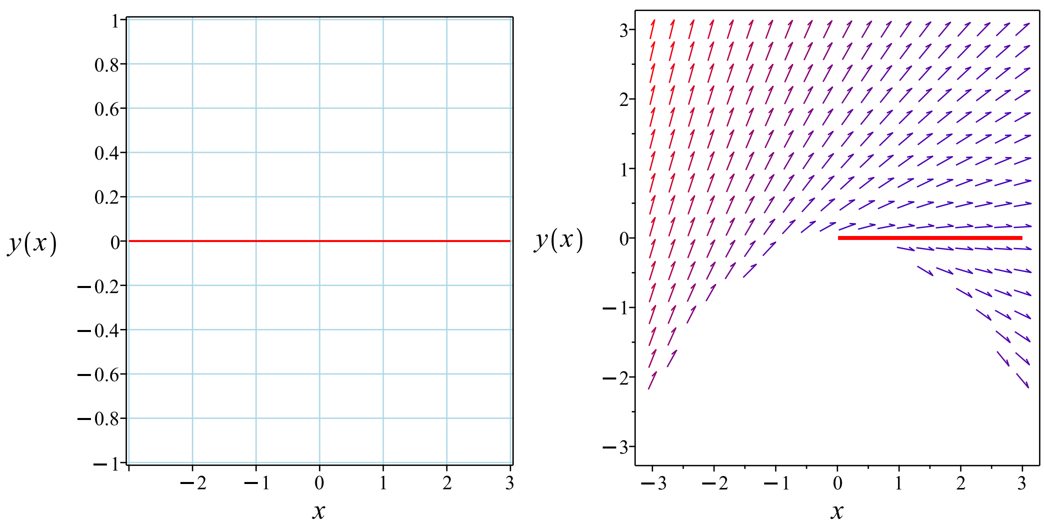
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = 0$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
differential order: 1; found: 2 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)=(-x+sqrt(x^2+4*y(x)))/2,y(0) = 0],y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = -\frac{x^2}{4}$$

✓ Solution by Mathematica

Time used: 0.287 (sec). Leaf size: 6

```
DSolve[{y'[x]==(-x+Sqrt[x^2+4*y[x]])/2,{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

8.43 problem 14 (c)

Internal problem ID [12741]
Internal file name [`OUTPUT/11393_Friday_November_03_2023_06_32_13_AM_7445606/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 14 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Clairaut]
```

$$\boxed{y' - \frac{\sqrt{x^2+4y}}{2} = -\frac{x}{2}}$$

With initial conditions

$$[y(0) = -1]$$

8.43.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2} \end{aligned}$$

The x domain of $f(x,y)$ when $y=-1$ is

$$\{2 \le x \le \infty, -\infty \le x \le -2\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

8.43.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{x}{2}+\frac{\sqrt{x^2+4y}}{2}\\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2+ya_3+a_1 \tag{1E}
$$

$$
\eta = xb_2+yb_3+b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1,a_2,a_3,b_1,b_2,b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2+\left(-\frac{x}{2}+\frac{\sqrt{x^2+4y}}{2}\right)(b_3-a_2)-\left(-\frac{x}{2}+\frac{\sqrt{x^2+4y}}{2}\right)^2 a_3\\
&-\left(-\frac{1}{2}+\frac{x}{2\sqrt{x^2+4y}}\right)(xa_2+ya_3+a_1)-\frac{xb_2+yb_3+b_1}{\sqrt{x^2+4y}}=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{(x^2+4y)^{\frac{3}{2}}a_3+\sqrt{x^2+4y}\,x^2a_3-2x^3a_3-4\sqrt{x^2+4y}\,xa_2+2\sqrt{x^2+4y}\,xb_3-2\sqrt{x^2+4y}\,ya_3+4x^2a_2-}{4\sqrt{x^2+4y}}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
&-(x^2+4y)^{\frac{3}{2}}a_3-\sqrt{x^2+4y}\,x^2a_3+2x^3a_3+4\sqrt{x^2+4y}\,xa_2\\
&-2\sqrt{x^2+4y}\,xb_3+2\sqrt{x^2+4y}\,ya_3-4x^2a_2+2x^2b_3+6xya_3\\
&+2\sqrt{x^2+4y}\,a_1+4b_2\sqrt{x^2+4y}-2xa_1-4xb_2-8ya_2+4yb_3-4b_1=0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-\left(x^{2}+4y\right)^{\frac{3}{2}} a_{3}+2\left(x^{2}+4y\right) x a_{3}-\sqrt{x^{2}+4y}\, x^{2} a_{3}-2\left(x^{2}+4y\right) a_{2} \\
&+2\left(x^{2}+4y\right) b_{3}+4\sqrt{x^{2}+4y}\, x a_{2}-2\sqrt{x^{2}+4y}\, x b_{3}+2\sqrt{x^{2}+4y}\, y a_{3}-2x^{2} a_{2} \\
&-2xya_{3}+2\sqrt{x^{2}+4y}\, a_{1}+4b_{2}\sqrt{x^{2}+4y}-2xa_{1}-4xb_{2}-4yb_{3}-4b_{1}=0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&2x^{3} a_{3}-2\sqrt{x^{2}+4y}\, x^{2} a_{3}-4x^{2} a_{2}+2x^{2} b_{3}+4\sqrt{x^{2}+4y}\, x a_{2} \\
&-2\sqrt{x^{2}+4y}\, x b_{3}+6xya_{3}-2\sqrt{x^{2}+4y}\, y a_{3}-2xa_{1}-4xb_{2} \\
&+2\sqrt{x^{2}+4y}\, a_{1}+4b_{2}\sqrt{x^{2}+4y}-8ya_{2}+4yb_{3}-4b_{1}=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^{2}+4y}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_{1}, y=v_{2}, \sqrt{x^{2}+4y}=v_{3}\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2v_{1}^{3} a_{3}-2v_{3} v_{1}^{2} a_{3}-4v_{1}^{2} a_{2}+4v_{3} v_{1} a_{2}+6v_{1} v_{2} a_{3}-2v_{3} v_{2} a_{3}+2v_{1}^{2} b_{3} \\
&-2v_{3} v_{1} b_{3}-2v_{1} a_{1}+2v_{3} a_{1}-8v_{2} a_{2}-4v_{1} b_{2}+4b_{2} v_{3}+4v_{2} b_{3}-4b_{1}=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_{1}, v_{2}, v_{3}\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&2v_{1}^{3} a_{3}-2v_{3} v_{1}^{2} a_{3}+\left(-4a_{2}+2b_{3}\right) v_{1}^{2}+6v_{1} v_{2} a_{3}+\left(4a_{2}-2b_{3}\right) v_{1} v_{3} \\
&+\left(-2a_{1}-4b_{2}\right) v_{1}-2v_{3} v_{2} a_{3}+\left(-8a_{2}+4b_{3}\right) v_{2}+\left(2a_{1}+4b_{2}\right) v_{3}-4b_{1}=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_3 &= 0 \\
2a_3 &= 0 \\
6a_3 &= 0 \\
-4b_1 &= 0 \\
-2a_1 - 4b_2 &= 0 \\
2a_1 + 4b_2 &= 0 \\
-8a_2 + 4b_3 &= 0 \\
-4a_2 + 2b_3 &= 0 \\
4a_2 - 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -2b_2 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2 \\
\eta &= x
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= x - \left(-\frac{x}{2} + \frac{\sqrt{x^2 + 4y}}{2}\right)(-2) \\
&= \sqrt{x^2 + 4y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{x^2 + 4y}} dy \end{aligned}$$

Which results in

$$S = \frac{\sqrt{x^2 + 4y}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x}{2} + \frac{\sqrt{x^2 + 4y}}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x}{2\sqrt{x^2 + 4y}} \\ S_y &= \frac{1}{\sqrt{x^2 + 4y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{x^2+4y}}{2} = \frac{x}{2} + c_1$$

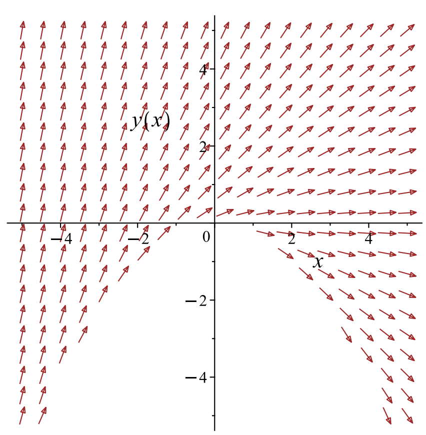
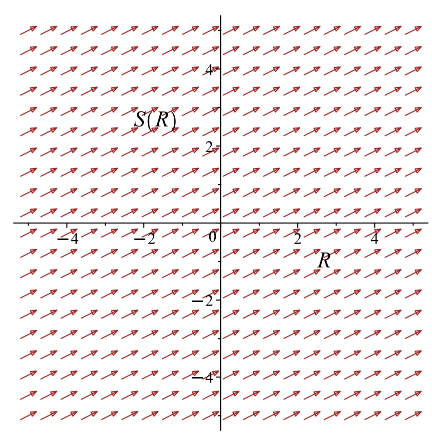
Which simplifies to

$$\frac{\sqrt{x^2+4y}}{2} = \frac{x}{2} + c_1$$

Which gives

$$y = c_1^2 + c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}$ 	$R = x$ $S = \frac{\sqrt{x^2+4y}}{2}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1^2$$

$$c_1 = -i$$

Substituting c_1 found above in the general solution gives

$$y = -ix - 1$$

Summary

The solution(s) found are the following

$$y = -ix - 1 \tag{1}$$

Verification of solutions

$$y = -ix - 1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
differential order: 1; found: 2 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.25 (sec). Leaf size: 19

```
dsolve([diff(y(x),x)=(-x+sqrt(x^2+4*y(x)))/2,y(0) = -1],y(x), singsol=all)
```

$$y(x) = -ix - 1$$
$$y(x) = ix - 1$$

✓ Solution by Mathematica

Time used: 0.293 (sec). Leaf size: 23

```
DSolve[{y'[x]==(-x+Sqrt[x^2+4*y[x]])/2,{y[0]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -1 - ix$$
$$y(x) \to -1 + ix$$

8.44 problem 14 (d)

Internal problem ID [12742]
Internal file name [`OUTPUT/11394_Friday_November_03_2023_06_32_14_AM_95329720/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 14 (d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{y' - \frac{\sqrt{x^2+4y}}{2} = -\frac{x}{2}}$$

With initial conditions

$$\left[y(1) = -\frac{1}{5}\right]$$

8.44.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \left(-\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}\right)(b_3 - a_2) - \left(-\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}\right)^2 a_3 - \left(-\frac{1}{2} + \frac{x}{2\sqrt{x^2+4y}}\right)(xa_2 + ya_3 + a_1) - \frac{xb_2 + yb_3 + b_1}{\sqrt{x^2+4y}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(x^2+4y)^{\frac{3}{2}} a_3 + \sqrt{x^2+4y}\, x^2 a_3 - 2x^3 a_3 - 4\sqrt{x^2+4y}\, x a_2 + 2\sqrt{x^2+4y}\, x b_3 - 2\sqrt{x^2+4y}\, y a_3 + 4x^2 a_2 - }{4\sqrt{x^2+4y}} = 0$$

Setting the numerator to zero gives

$$-(x^2+4y)^{\frac{3}{2}} a_3 - \sqrt{x^2+4y}\, x^2 a_3 + 2x^3 a_3 + 4\sqrt{x^2+4y}\, x a_2 - 2\sqrt{x^2+4y}\, x b_3 + 2\sqrt{x^2+4y}\, y a_3 - 4x^2 a_2 + 2x^2 b_3 + 6xya_3 + 2\sqrt{x^2+4y}\, a_1 + 4b_2\sqrt{x^2+4y} - 2xa_1 - 4xb_2 - 8ya_2 + 4yb_3 - 4b_1 = 0 \tag{6E}$$

Simplifying the above gives

$$-(x^2+4y)^{\frac{3}{2}} a_3 + 2(x^2+4y)\, x a_3 - \sqrt{x^2+4y}\, x^2 a_3 - 2(x^2+4y)\, a_2 + 2(x^2+4y)\, b_3 + 4\sqrt{x^2+4y}\, x a_2 - 2\sqrt{x^2+4y}\, x b_3 + 2\sqrt{x^2+4y}\, y a_3 - 2x^2 a_2 - 2xya_3 + 2\sqrt{x^2+4y}\, a_1 + 4b_2\sqrt{x^2+4y} - 2xa_1 - 4xb_2 - 4yb_3 - 4b_1 = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$2x^3 a_3 - 2\sqrt{x^2+4y}\, x^2 a_3 - 4x^2 a_2 + 2x^2 b_3 + 4\sqrt{x^2+4y}\, x a_2 - 2\sqrt{x^2+4y}\, x b_3 + 6xya_3 - 2\sqrt{x^2+4y}\, y a_3 - 2xa_1 - 4xb_2 + 2\sqrt{x^2+4y}\, a_1 + 4b_2\sqrt{x^2+4y} - 8ya_2 + 4yb_3 - 4b_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2+4y}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2+4y} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &2v_1^3a_3 - 2v_3v_1^2a_3 - 4v_1^2a_2 + 4v_3v_1a_2 + 6v_1v_2a_3 - 2v_3v_2a_3 + 2v_1^2b_3 \\ &- 2v_3v_1b_3 - 2v_1a_1 + 2v_3a_1 - 8v_2a_2 - 4v_1b_2 + 4b_2v_3 + 4v_2b_3 - 4b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &2v_1^3a_3 - 2v_3v_1^2a_3 + (-4a_2 + 2b_3)\, v_1^2 + 6v_1v_2a_3 + (4a_2 - 2b_3)\, v_1v_3 \\ &+ (-2a_1 - 4b_2)\, v_1 - 2v_3v_2a_3 + (-8a_2 + 4b_3)\, v_2 + (2a_1 + 4b_2)\, v_3 - 4b_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_3 &= 0 \\ 2a_3 &= 0 \\ 6a_3 &= 0 \\ -4b_1 &= 0 \\ -2a_1 - 4b_2 &= 0 \\ 2a_1 + 4b_2 &= 0 \\ -8a_2 + 4b_3 &= 0 \\ -4a_2 + 2b_3 &= 0 \\ 4a_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -2b_2 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2 \\
\eta &= x
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= x - \left(-\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}\right)(-2) \\
&= \sqrt{x^2+4y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\sqrt{x^2+4y}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\sqrt{x^2 + 4y}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x}{2} + \frac{\sqrt{x^2 + 4y}}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x}{2\sqrt{x^2 + 4y}} \\ S_y &= \frac{1}{\sqrt{x^2 + 4y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{x^2 + 4y}}{2} = \frac{x}{2} + c_1$$

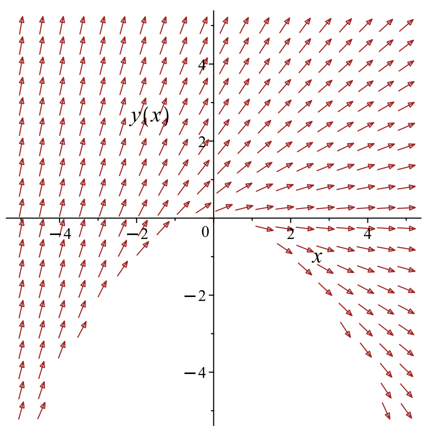
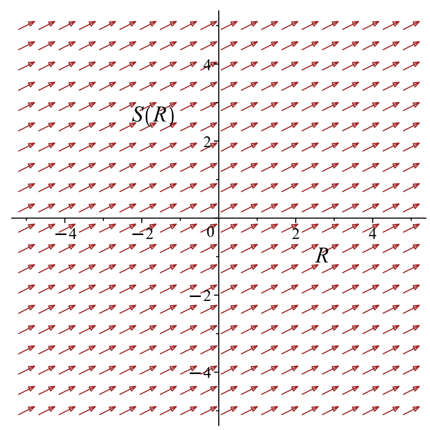
Which simplifies to

$$\frac{\sqrt{x^2+4y}}{2} = \frac{x}{2} + c_1$$

Which gives

$$y = c_1^2 + c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}$ 	$R = x$ $S = \frac{\sqrt{x^2+4y}}{2}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -\frac{1}{5}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{5} = c_1^2 + c_1$$

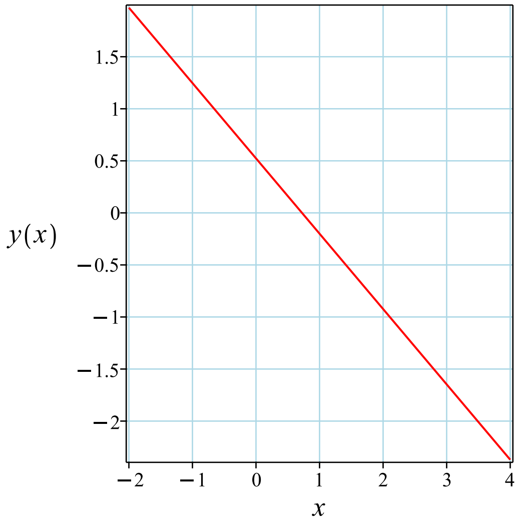
$$c_1 = -\frac{1}{2} - \frac{\sqrt{5}}{10}$$

Substituting c_1 found above in the general solution gives

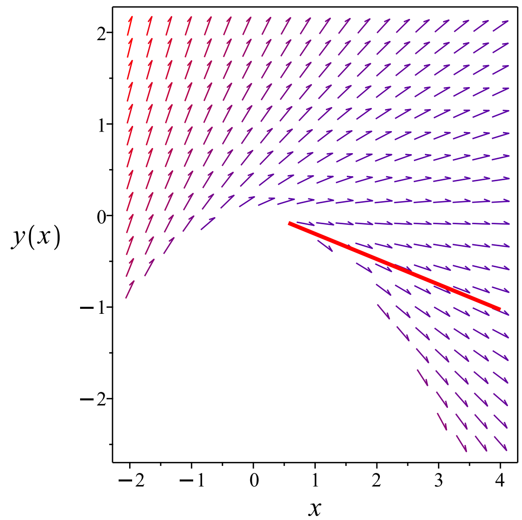
$$y = \frac{3}{10} + \frac{\sqrt{5}}{10} - \frac{x}{2} - \frac{\sqrt{5}\,x}{10}$$

Summary
The solution(s) found are the following

$$y = \frac{3}{10} + \frac{\sqrt{5}}{10} - \frac{x}{2} - \frac{\sqrt{5}\,x}{10} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{3}{10} + \frac{\sqrt{5}}{10} - \frac{x}{2} - \frac{\sqrt{5}\,x}{10}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
differential order: 1; found: 2 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.969 (sec). Leaf size: 69

```
dsolve([diff(y(x),x)=(-x+sqrt(x^2+4*y(x)))/2,y(1) = -1/5],y(x), singsol=all)
```

$$y(x) = \frac{(-1+x)\sqrt{5}}{10} - \frac{x}{2} + \frac{3}{10}$$
$$y(x) = \frac{\left(\sqrt{5}-5\right)\left(-5+\sqrt{5}+10x\right)}{100}$$
$$y(x) = -\frac{2^{\frac{1}{3}}\left(50+20\sqrt{5}\right)^{\frac{1}{3}}\left(2^{\frac{1}{3}}x - \frac{\left(50+20\sqrt{5}\right)^{\frac{1}{3}}}{5}\right)}{10}$$

✓ Solution by Mathematica

Time used: 0.301 (sec). Leaf size: 51

```
DSolve[{y'[x]==(-x+Sqrt[x^2+4*y[x]])/2,{y[1]==-2/10}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{100}\left(5+\sqrt{5}\right)\left(-10x+\sqrt{5}+5\right)$$
$$y(x) \to \frac{1}{100}\left(\sqrt{5}-5\right)\left(10x+\sqrt{5}-5\right)$$

8.45 problem 14 (e)

Internal problem ID [12743]
Internal file name [`OUTPUT/11395_Friday_November_03_2023_06_32_17_AM_92906540/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 2. The Initial Value Problem. Exercises 2.4.4, page 115
Problem number: 14 (e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Clairaut]
```

$$\boxed{y' - \frac{\sqrt{x^2+4y}}{2} = -\frac{x}{2}}$$

With initial conditions

$$\left[y(1) = -\frac{1}{4}\right]$$

8.45.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2} \end{aligned}$$

The x domain of $f(x,y)$ when $y = -\frac{1}{4}$ is

$$\{1 \leq x \leq \infty, -\infty \leq x \leq -1\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\left\{-\frac{1}{4} \leq y\right\}$$

And the point $y_0 = -\frac{1}{4}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}\right) \\
&= \frac{1}{\sqrt{x^2+4y}}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -\frac{1}{4}$ is

$$\{-\infty \leq x < -1, 1 < x \leq \infty\}$$

But the point $x_0 = 1$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

8.45.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2} \\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\left(-\frac{x}{2}+\frac{\sqrt{x^2+4y}}{2}\right)(b_3-a_2)-\left(-\frac{x}{2}+\frac{\sqrt{x^2+4y}}{2}\right)^2 a_3 \\ -\left(-\frac{1}{2}+\frac{x}{2\sqrt{x^2+4y}}\right)(xa_2+ya_3+a_1)-\frac{xb_2+yb_3+b_1}{\sqrt{x^2+4y}}=0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(x^2+4y)^{\frac{3}{2}}a_3+\sqrt{x^2+4y}\,x^2a_3-2x^3a_3-4\sqrt{x^2+4y}\,xa_2+2\sqrt{x^2+4y}\,xb_3-2\sqrt{x^2+4y}\,ya_3+4x^2a_2-}{4\sqrt{x^2+4y}} \\ =0$$

Setting the numerator to zero gives

$$-(x^2+4y)^{\frac{3}{2}}a_3-\sqrt{x^2+4y}\,x^2a_3+2x^3a_3+4\sqrt{x^2+4y}\,xa_2 \\ -2\sqrt{x^2+4y}\,xb_3+2\sqrt{x^2+4y}\,ya_3-4x^2a_2+2x^2b_3+6xya_3 \\ +2\sqrt{x^2+4y}\,a_1+4b_2\sqrt{x^2+4y}-2xa_1-4xb_2-8ya_2+4yb_3-4b_1=0 \tag{6E}$$

Simplifying the above gives

$$-(x^2+4y)^{\frac{3}{2}}a_3+2(x^2+4y)\,xa_3-\sqrt{x^2+4y}\,x^2a_3-2(x^2+4y)\,a_2 \\ +2(x^2+4y)\,b_3+4\sqrt{x^2+4y}\,xa_2-2\sqrt{x^2+4y}\,xb_3+2\sqrt{x^2+4y}\,ya_3-2x^2a_2 \\ -2xya_3+2\sqrt{x^2+4y}\,a_1+4b_2\sqrt{x^2+4y}-2xa_1-4xb_2-4yb_3-4b_1=0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$2x^3a_3-2\sqrt{x^2+4y}\,x^2a_3-4x^2a_2+2x^2b_3+4\sqrt{x^2+4y}\,xa_2 \\ -2\sqrt{x^2+4y}\,xb_3+6xya_3-2\sqrt{x^2+4y}\,ya_3-2xa_1-4xb_2 \\ +2\sqrt{x^2+4y}\,a_1+4b_2\sqrt{x^2+4y}-8ya_2+4yb_3-4b_1=0$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,\sqrt{x^2+4y}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2 + 4y} = v_3\right\}$$

The above PDE (6E) now becomes

$$2v_1^3 a_3 - 2v_3 v_1^2 a_3 - 4v_1^2 a_2 + 4v_3 v_1 a_2 + 6v_1 v_2 a_3 - 2v_3 v_2 a_3 + 2v_1^2 b_3 \\ - 2v_3 v_1 b_3 - 2v_1 a_1 + 2v_3 a_1 - 8v_2 a_2 - 4v_1 b_2 + 4b_2 v_3 + 4v_2 b_3 - 4b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$2v_1^3 a_3 - 2v_3 v_1^2 a_3 + (-4a_2 + 2b_3) v_1^2 + 6v_1 v_2 a_3 + (4a_2 - 2b_3) v_1 v_3 \\ + (-2a_1 - 4b_2) v_1 - 2v_3 v_2 a_3 + (-8a_2 + 4b_3) v_2 + (2a_1 + 4b_2) v_3 - 4b_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -2a_3 &= 0 \\ 2a_3 &= 0 \\ 6a_3 &= 0 \\ -4b_1 &= 0 \\ -2a_1 - 4b_2 &= 0 \\ 2a_1 + 4b_2 &= 0 \\ -8a_2 + 4b_3 &= 0 \\ -4a_2 + 2b_3 &= 0 \\ 4a_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -2b_2 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= 2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -2 \\
\eta &= x
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= x - \left(-\frac{x}{2} + \frac{\sqrt{x^2 + 4y}}{2}\right)(-2) \\
&= \sqrt{x^2 + 4y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\sqrt{x^2 + 4y}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\sqrt{x^2 + 4y}}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x}{2\sqrt{x^2+4y}} \\ S_y &= \frac{1}{\sqrt{x^2+4y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{x^2+4y}}{2} = \frac{x}{2} + c_1$$

Which simplifies to

$$\frac{\sqrt{x^2+4y}}{2} = \frac{x}{2} + c_1$$

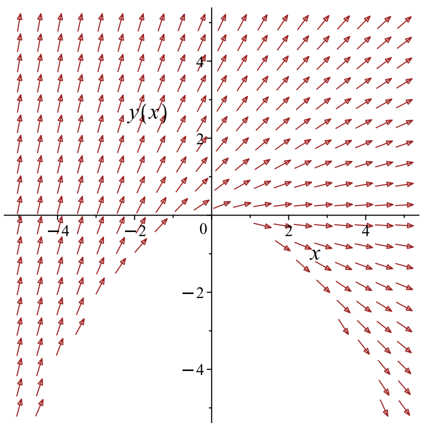
Which gives

$$y = c_1^2 + c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{x}{2} + \frac{\sqrt{x^2+4y}}{2}$



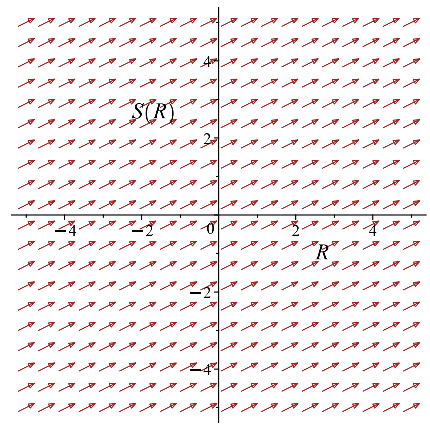
</td><td>

$R = x$

$S = \frac{\sqrt{x^2+4y}}{2}$

</td><td>

$\frac{dS}{dR} = \frac{1}{2}$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -\frac{1}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{4} = c_1^2 + c_1$$

$$c_1 = -\frac{1}{2}$$

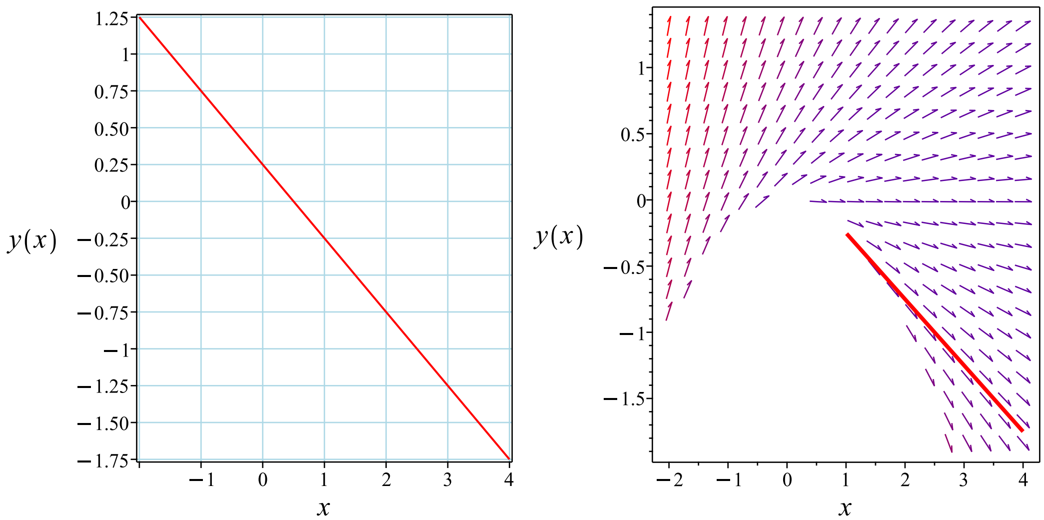
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{4} - \frac{x}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{4} - \frac{x}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{1}{4} - \frac{x}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
differential order: 1; found: 2 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 8.516 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)=(-x+sqrt(x^2+4*y(x)))/2,y(1) = -1/4],y(x), singsol=all)
```

$$y(x) = -\frac{x^2}{4}$$
$$y(x) = \frac{1}{4} - \frac{x}{2}$$

✓ Solution by Mathematica

Time used: 0.282 (sec). Leaf size: 14

```
DSolve[{y'[x]==(-x+Sqrt[x^2+4*y[x]])/2,{y[1]==-1/4}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{1}{4}(1 - 2x)$$

9 Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186

9.1 problem 1

Internal problem ID [12744]
Internal file name [`OUTPUT/11396_Friday_November_03_2023_06_32_19_AM_14196107/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3y'' - 2y' + 4y = x}$$

With initial conditions

$$[y(-1) = 2, y'(-1) = 3]$$

9.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{3} \\ q(x) &= \frac{4}{3} \\ F &= \frac{x}{3} \end{aligned}$$

Hence the ode is

$$y'' - \frac{2y'}{3} + \frac{4y}{3} = \frac{x}{3}$$

The domain of $p(x) = -\frac{2}{3}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = \frac{4}{3}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is also inside this domain. The domain of $F = \frac{x}{3}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

9.1.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 3, B = -2, C = 4, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$3y'' - 2y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 3, B = -2, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$3\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$3\lambda^2 - 2\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 3, B = -2, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{2}{(2)(3)} \pm \frac{1}{(2)(3)}\sqrt{-2^2 - (4)(3)(4)} \\
&= \frac{1}{3} \pm \frac{i\sqrt{11}}{3}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{1}{3} + \frac{i\sqrt{11}}{3} \\
\lambda_2 &= \frac{1}{3} - \frac{i\sqrt{11}}{3}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= \frac{1}{3} + \frac{i\sqrt{11}}{3} \\
\lambda_2 &= \frac{1}{3} - \frac{i\sqrt{11}}{3}
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{3}$ and $\beta = \frac{\sqrt{11}}{3}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{\frac{x}{3}}\left(c_1 \cos\left(\frac{\sqrt{11}\, x}{3}\right) + c_2 \sin\left(\frac{\sqrt{11}\, x}{3}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{\frac{x}{3}}\left(c_1 \cos\left(\frac{\sqrt{11}\, x}{3}\right) + c_2 \sin\left(\frac{\sqrt{11}\, x}{3}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\, x}{3}\right), e^{\frac{x}{3}} \sin\left(\frac{\sqrt{11}\, x}{3}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_2 x + 4A_1 - 2A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x}{4} + \frac{1}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{\frac{x}{3}} \left(c_1 \cos\left(\frac{\sqrt{11}\, x}{3}\right) + c_2 \sin\left(\frac{\sqrt{11}\, x}{3}\right)\right)\right) + \left(\frac{x}{4} + \frac{1}{8}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{\frac{x}{3}} \left(c_1 \cos\left(\frac{\sqrt{11}\, x}{3}\right) + c_2 \sin\left(\frac{\sqrt{11}\, x}{3}\right)\right) + \frac{x}{4} + \frac{1}{8} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = -1$ in the above gives

$$2 = \mathrm{e}^{-\frac{1}{3}} \cos\left(\frac{\sqrt{11}}{3}\right) c_1 - \mathrm{e}^{-\frac{1}{3}} \sin\left(\frac{\sqrt{11}}{3}\right) c_2 - \frac{1}{8} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\mathrm{e}^{\frac{x}{3}} \left(c_1 \cos\left(\frac{\sqrt{11}\, x}{3}\right) + c_2 \sin\left(\frac{\sqrt{11}\, x}{3}\right)\right)}{3} + \mathrm{e}^{\frac{x}{3}} \left(-\frac{c_1 \sqrt{11}\, \sin\left(\frac{\sqrt{11}\, x}{3}\right)}{3} + \frac{c_2 \sqrt{11}\, \cos\left(\frac{\sqrt{11}\, x}{3}\right)}{3}\right) + \frac{1}{4}$$

substituting $y' = 3$ and $x = -1$ in the above gives

$$3 = \frac{\mathrm{e}^{-\frac{1}{3}} \left(c_2 \sqrt{11} + c_1\right) \cos\left(\frac{\sqrt{11}}{3}\right)}{3} + \frac{1}{4} + \frac{\mathrm{e}^{-\frac{1}{3}} \left(\sqrt{11}\, c_1 - c_2\right) \sin\left(\frac{\sqrt{11}}{3}\right)}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{\left(17\sqrt{11}\, \cos\left(\frac{\sqrt{11}}{3}\right) + 49 \sin\left(\frac{\sqrt{11}}{3}\right)\right) \mathrm{e}^{\frac{1}{3}} \sqrt{11}}{88}$$

$$c_2 = \frac{\left(-17\sqrt{11}\, \sin\left(\frac{\sqrt{11}}{3}\right) + 49 \cos\left(\frac{\sqrt{11}}{3}\right)\right) \mathrm{e}^{\frac{1}{3}} \sqrt{11}}{88}$$

Substituting these values back in above solution results in

$$y = \frac{1}{8} + \frac{x}{4} + \frac{17 \cos\left(\frac{\sqrt{11}\, x}{3}\right) \cos\left(\frac{\sqrt{11}}{3}\right) \mathrm{e}^{\frac{x}{3}+\frac{1}{3}}}{8} + \frac{49 \cos\left(\frac{\sqrt{11}\, x}{3}\right) \sin\left(\frac{\sqrt{11}}{3}\right) \sqrt{11}\, \mathrm{e}^{\frac{x}{3}+\frac{1}{3}}}{88} - \frac{17 \sin\left(\frac{\sqrt{11}\, x}{3}\right) \mathrm{e}^{\frac{x}{3}+\frac{1}{3}} \text{si}}{8}$$

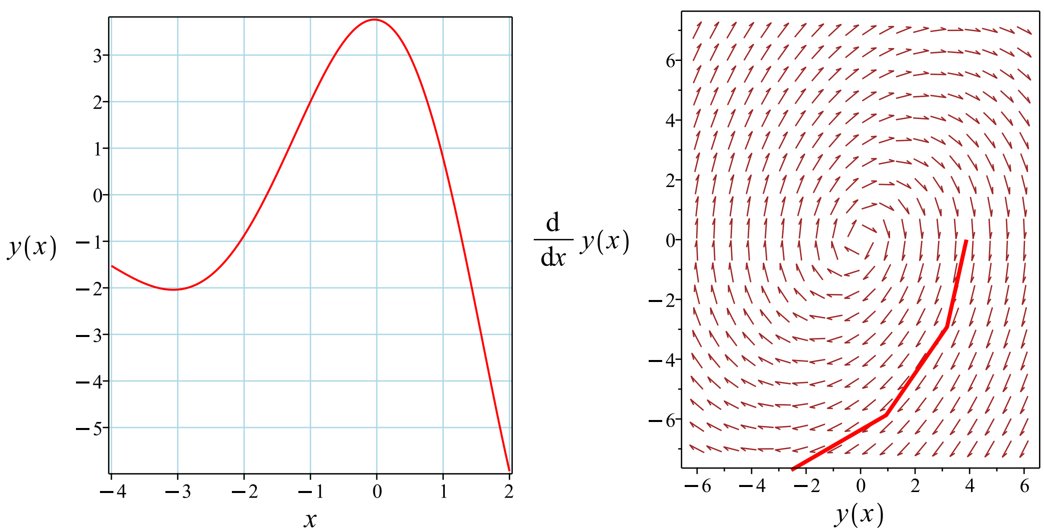
Which simplifies to

$$y = \frac{\left(\left(49\sqrt{11}\, \sin\left(\frac{\sqrt{11}}{3}\right) + 187 \cos\left(\frac{\sqrt{11}}{3}\right)\right) \cos\left(\frac{\sqrt{11}\, x}{3}\right) + 49\left(\sqrt{11}\, \cos\left(\frac{\sqrt{11}}{3}\right) - \frac{187 \sin\left(\frac{\sqrt{11}}{3}\right)}{49}\right) \sin\left(\frac{\sqrt{11}\, x}{3}\right)\right) \mathrm{e}}{88} + \frac{x}{4} + \frac{1}{8}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\left(49\sqrt{11}\,\sin\left(\frac{\sqrt{11}}{3}\right)+187\cos\left(\frac{\sqrt{11}}{3}\right)\right)\cos\left(\frac{\sqrt{11}\,x}{3}\right)+49\left(\sqrt{11}\,\cos\left(\frac{\sqrt{11}}{3}\right)-\frac{187\sin\left(\frac{\sqrt{11}}{3}\right)}{49}\right)\sin\left(\frac{\sqrt{11}\,x}{3}\right)\right)\mathrm{e}}{88} + \frac{x}{4}+\frac{1}{8} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\left(\left(49\sqrt{11}\,\sin\left(\frac{\sqrt{11}}{3}\right)+187\cos\left(\frac{\sqrt{11}}{3}\right)\right)\cos\left(\frac{\sqrt{11}\,x}{3}\right)+49\left(\sqrt{11}\,\cos\left(\frac{\sqrt{11}}{3}\right)-\frac{187\sin\left(\frac{\sqrt{11}}{3}\right)}{49}\right)\sin\left(\frac{\sqrt{11}\,x}{3}\right)\right)\mathrm{e}}{88} + \frac{x}{4}+\frac{1}{8}$$

Verified OK.

9.1.3 Solving using Kovacic algorithm

Writing the ode as

$$3y'' - 2y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3 \\ B &= -2 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-11}{9} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -11 \\ t &= 9 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{11z(x)}{9} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 306: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{11}{9}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\frac{\sqrt{11}\, x}{3}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{3}\,dx} \\
&= z_1 e^{\frac{x}{3}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{3}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\,x}{3}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{3}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{2x}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{3\sqrt{11}\,\tan\left(\frac{\sqrt{11}\,x}{3}\right)}{11}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\,x}{3}\right)\right) + c_2 \left(\mathrm{e}^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\,x}{3}\right)\left(\frac{3\sqrt{11}\,\tan\left(\frac{\sqrt{11}\,x}{3}\right)}{11}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$3y''-2y'+4y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\, x}{3}\right)+\frac{3c_2 \sin\left(\frac{\sqrt{11}\, x}{3}\right) \mathrm{e}^{\frac{x}{3}} \sqrt{11}}{11}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\, x}{3}\right), \frac{3\sin\left(\frac{\sqrt{11}\, x}{3}\right) \mathrm{e}^{\frac{x}{3}} \sqrt{11}}{11}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_2 x + 4A_1 - 2A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x}{4} + \frac{1}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\, x}{3}\right) + \frac{3c_2 \sin\left(\frac{\sqrt{11}\, x}{3}\right) \mathrm{e}^{\frac{x}{3}} \sqrt{11}}{11}\right) + \left(\frac{x}{4} + \frac{1}{8}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\, x}{3}\right) + \frac{3c_2 \sin\left(\frac{\sqrt{11}\, x}{3}\right) \mathrm{e}^{\frac{x}{3}} \sqrt{11}}{11} + \frac{x}{4} + \frac{1}{8} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = -1$ in the above gives

$$2 = \mathrm{e}^{-\frac{1}{3}} \cos\left(\frac{\sqrt{11}}{3}\right) c_1 - \frac{3\,\mathrm{e}^{-\frac{1}{3}} \sin\left(\frac{\sqrt{11}}{3}\right) c_2 \sqrt{11}}{11} - \frac{1}{8} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 \mathrm{e}^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\, x}{3}\right)}{3} - \frac{c_1 \mathrm{e}^{\frac{x}{3}} \sqrt{11} \sin\left(\frac{\sqrt{11}\, x}{3}\right)}{3} + c_2 \cos\left(\frac{\sqrt{11}\, x}{3}\right) \mathrm{e}^{\frac{x}{3}} + \frac{c_2 \sin\left(\frac{\sqrt{11}\, x}{3}\right) \mathrm{e}^{\frac{x}{3}} \sqrt{11}}{11} + \frac{1}{4}$$

substituting $y' = 3$ and $x = -1$ in the above gives

$$3 = \frac{\mathrm{e}^{-\frac{1}{3}} (c_1 + 3c_2) \cos\left(\frac{\sqrt{11}}{3}\right)}{3} + \frac{1}{4} + \frac{\sqrt{11}\,\mathrm{e}^{-\frac{1}{3}} \left(c_1 - \frac{3c_2}{11}\right) \sin\left(\frac{\sqrt{11}}{3}\right)}{3} \tag{2A}$$

Equations $\{1\text{A}, 2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{\left(49\sqrt{11} \sin\left(\frac{\sqrt{11}}{3}\right) + 187 \cos\left(\frac{\sqrt{11}}{3}\right)\right) \mathrm{e}^{\frac{1}{3}}}{88} \\ c_2 &= \frac{\left(-17\sqrt{11} \sin\left(\frac{\sqrt{11}}{3}\right) + 49 \cos\left(\frac{\sqrt{11}}{3}\right)\right) \mathrm{e}^{\frac{1}{3}}}{24} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{1}{8}+\frac{x}{4}+\frac{17\cos\left(\frac{\sqrt{11}\,x}{3}\right)\cos\left(\frac{\sqrt{11}}{3}\right)\mathrm{e}^{\frac{x}{3}+\frac{1}{3}}}{8}+\frac{49\cos\left(\frac{\sqrt{11}\,x}{3}\right)\sin\left(\frac{\sqrt{11}}{3}\right)\sqrt{11}\,\mathrm{e}^{\frac{x}{3}+\frac{1}{3}}}{88}-\frac{17\sin\left(\frac{\sqrt{11}\,x}{3}\right)\mathrm{e}^{\frac{x}{3}+\frac{1}{3}}\,\mathrm{si}}{8}$$

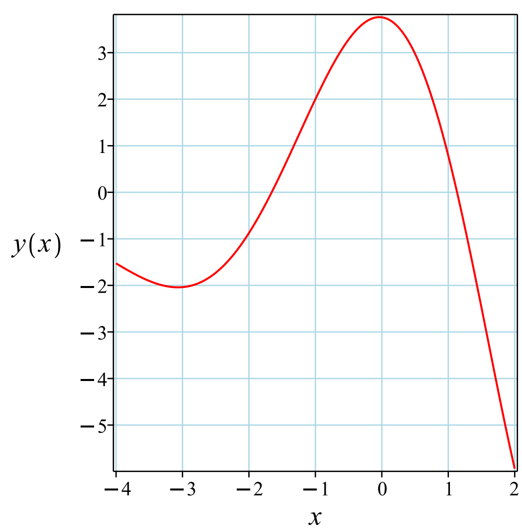
Which simplifies to

$$y = \frac{\left(\left(49\sqrt{11}\,\sin\left(\frac{\sqrt{11}}{3}\right)+187\cos\left(\frac{\sqrt{11}}{3}\right)\right)\cos\left(\frac{\sqrt{11}\,x}{3}\right)+49\left(\sqrt{11}\,\cos\left(\frac{\sqrt{11}}{3}\right)-\frac{187\sin\left(\frac{\sqrt{11}}{3}\right)}{49}\right)\sin\left(\frac{\sqrt{11}\,x}{3}\right)\right)\mathrm{e}}{88}+\frac{x}{4}+\frac{1}{8}$$

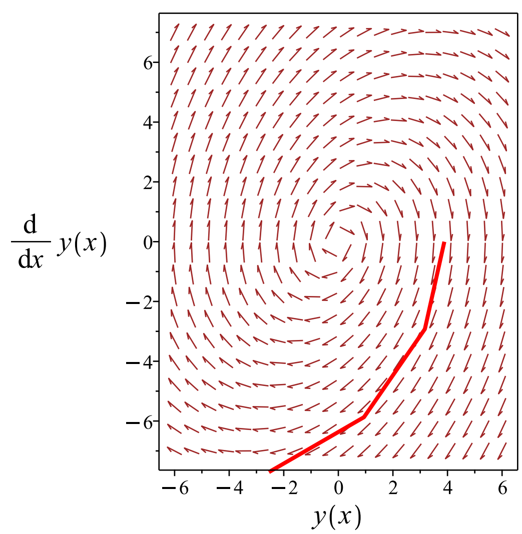
Summary

The solution(s) found are the following

$$y = \frac{\left(\left(49\sqrt{11}\,\sin\left(\frac{\sqrt{11}}{3}\right)+187\cos\left(\frac{\sqrt{11}}{3}\right)\right)\cos\left(\frac{\sqrt{11}\,x}{3}\right)+49\left(\sqrt{11}\,\cos\left(\frac{\sqrt{11}}{3}\right)-\frac{187\sin\left(\frac{\sqrt{11}}{3}\right)}{49}\right)\sin\left(\frac{\sqrt{11}\,x}{3}\right)\right)\mathrm{e}}{88}+\frac{x}{4}+\frac{1}{8} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

y

$$= \frac{\left(\left(49\sqrt{11}\,\sin\left(\frac{\sqrt{11}}{3}\right)+187\cos\left(\frac{\sqrt{11}}{3}\right)\right)\cos\left(\frac{\sqrt{11}\,x}{3}\right)+49\left(\sqrt{11}\,\cos\left(\frac{\sqrt{11}}{3}\right)-\frac{187\sin\left(\frac{\sqrt{11}}{3}\right)}{49}\right)\sin\left(\frac{\sqrt{11}\,x}{3}\right)\right)\mathrm{e}^{\frac{x}{3}}}{88}$$

$$+\frac{x}{4}+\frac{1}{8}$$

Verified OK.

9.1.4 Maple step by step solution

Let's solve

$$\left[3y''-2y'+4y=x, y(-1)=2, y'\Big|_{\{x=-1\}}=3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2y'}{3}-\frac{4y}{3}+\frac{x}{3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2y'}{3}+\frac{4y}{3}=\frac{x}{3}$

- Characteristic polynomial of homogeneous ODE

  $r^2-\frac{2}{3}r+\frac{4}{3}=0$

- Use quadratic formula to solve for r

  $r=\frac{\left(\frac{2}{3}\right)\pm\left(\sqrt{-\frac{44}{9}}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(\frac{1}{3}-\frac{\mathrm{I}\sqrt{11}}{3}, \frac{1}{3}+\frac{\mathrm{I}\sqrt{11}}{3}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{\frac{x}{3}}\cos\left(\frac{\sqrt{11}\,x}{3}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\mathrm{e}^{\frac{x}{3}}\sin\left(\frac{\sqrt{11}\,x}{3}\right)$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\, x}{3}\right) + e^{\frac{x}{3}} c_2 \sin\left(\frac{\sqrt{11}\, x}{3}\right) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{x}{3}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\, x}{3}\right) & e^{\frac{x}{3}} \sin\left(\frac{\sqrt{11}\, x}{3}\right) \\ \frac{e^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\, x}{3}\right)}{3} - \frac{\sin\left(\frac{\sqrt{11}\, x}{3}\right) e^{\frac{x}{3}} \sqrt{11}}{3} & \frac{e^{\frac{x}{3}} \sin\left(\frac{\sqrt{11}\, x}{3}\right)}{3} + \frac{e^{\frac{x}{3}} \sqrt{11} \cos\left(\frac{\sqrt{11}\, x}{3}\right)}{3} \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{11}\, e^{\frac{2x}{3}}}{3}$

- ○ Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\frac{e^{\frac{x}{3}} \sqrt{11}\left(\cos\left(\frac{\sqrt{11}\, x}{3}\right)\left(\int x\, e^{-\frac{x}{3}} \sin\left(\frac{\sqrt{11}\, x}{3}\right) dx\right) - \sin\left(\frac{\sqrt{11}\, x}{3}\right)\left(\int x\, e^{-\frac{x}{3}} \cos\left(\frac{\sqrt{11}\, x}{3}\right) dx\right)\right)}{11}$$

- ○ Compute integrals

  $y_p(x) = \frac{x}{4} + \frac{1}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\, x}{3}\right) + e^{\frac{x}{3}} c_2 \sin\left(\frac{\sqrt{11}\, x}{3}\right) + \frac{x}{4} + \frac{1}{8}$

☐ Check validity of solution $y = c_1 e^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\, x}{3}\right) + e^{\frac{x}{3}} c_2 \sin\left(\frac{\sqrt{11}\, x}{3}\right) + \frac{x}{4} + \frac{1}{8}$

- ○ Use initial condition $y(-1) = 2$

  $2 = e^{-\frac{1}{3}} \cos\left(\frac{\sqrt{11}}{3}\right) c_1 - e^{-\frac{1}{3}} \sin\left(\frac{\sqrt{11}}{3}\right) c_2 - \frac{1}{8}$

- ○ Compute derivative of the solution

  $$y' = \frac{c_1 e^{\frac{x}{3}} \cos\left(\frac{\sqrt{11}\, x}{3}\right)}{3} - \frac{c_1 e^{\frac{x}{3}} \sqrt{11} \sin\left(\frac{\sqrt{11}\, x}{3}\right)}{3} + \frac{e^{\frac{x}{3}} c_2 \sin\left(\frac{\sqrt{11}\, x}{3}\right)}{3} + \frac{e^{\frac{x}{3}} c_2 \sqrt{11} \cos\left(\frac{\sqrt{11}\, x}{3}\right)}{3} + \frac{1}{4}$$

- ○ Use the initial condition $y'\Big|_{\{x=-1\}} = 3$

  $$3 = \frac{1}{4} + \frac{e^{-\frac{1}{3}} c_1 \sin\left(\frac{\sqrt{11}}{3}\right) \sqrt{11}}{3} + \frac{e^{-\frac{1}{3}} \cos\left(\frac{\sqrt{11}}{3}\right) c_1}{3} + \frac{c_2 e^{-\frac{1}{3}} \cos\left(\frac{\sqrt{11}}{3}\right) \sqrt{11}}{3} - \frac{e^{-\frac{1}{3}} \sin\left(\frac{\sqrt{11}}{3}\right) c_2}{3}$$

- ○ Solve for c_1 and c_2

$$\left\{c_1 = \frac{\left(17\sqrt{11}\,\cos\left(\frac{\sqrt{11}}{3}\right)+49\sin\left(\frac{\sqrt{11}}{3}\right)\right)\sqrt{11}}{88\,\mathrm{e}^{-\frac{1}{3}}\left(\cos\left(\frac{\sqrt{11}}{3}\right)^2+\sin\left(\frac{\sqrt{11}}{3}\right)^2\right)}, c_2 = \frac{\sqrt{11}\left(-17\sqrt{11}\,\sin\left(\frac{\sqrt{11}}{3}\right)+49\cos\left(\frac{\sqrt{11}}{3}\right)\right)}{88\,\mathrm{e}^{-\frac{1}{3}}\left(\cos\left(\frac{\sqrt{11}}{3}\right)^2+\sin\left(\frac{\sqrt{11}}{3}\right)^2\right)}\right\}$$

- Substitute constant values into general solution and simplify

$$y = \frac{\left(\left(49\sqrt{11}\,\sin\left(\frac{\sqrt{11}}{3}\right)+187\cos\left(\frac{\sqrt{11}}{3}\right)\right)\cos\left(\frac{\sqrt{11}\,x}{3}\right)+49\left(\sqrt{11}\,\cos\left(\frac{\sqrt{11}}{3}\right)-\frac{187\sin\left(\frac{\sqrt{11}}{3}\right)}{49}\right)\sin\left(\frac{\sqrt{11}\,x}{3}\right)\right)\mathrm{e}^{\frac{x}{3}+\frac{1}{3}}}{88} + \frac{x}{4} + \frac{1}{8}$$

- Solution to the IVP

$$y = \frac{\left(\left(49\sqrt{11}\,\sin\left(\frac{\sqrt{11}}{3}\right)+187\cos\left(\frac{\sqrt{11}}{3}\right)\right)\cos\left(\frac{\sqrt{11}\,x}{3}\right)+49\left(\sqrt{11}\,\cos\left(\frac{\sqrt{11}}{3}\right)-\frac{187\sin\left(\frac{\sqrt{11}}{3}\right)}{49}\right)\sin\left(\frac{\sqrt{11}\,x}{3}\right)\right)\mathrm{e}^{\frac{x}{3}+\frac{1}{3}}}{88} + \frac{x}{4} + \frac{1}{8}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.266 (sec). Leaf size: 85

```
dsolve([3*diff(y(x),x$2)-2*diff(y(x),x)+4*y(x)=x,y(-1) = 2, D(y)(-1) = 3],y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(49\sin\left(\frac{\sqrt{11}}{3}\right)\sqrt{11}+187\cos\left(\frac{\sqrt{11}}{3}\right)\right)\cos\left(\frac{\sqrt{11}\,x}{3}\right)+49\sin\left(\frac{\sqrt{11}\,x}{3}\right)\left(\cos\left(\frac{\sqrt{11}}{3}\right)\sqrt{11}-\frac{187\sin\left(\frac{\sqrt{11}}{3}\right)}{49}\right)\right)\mathrm{e}^{\frac{x}{3}+\frac{1}{3}}}{88} + \frac{x}{4} + \frac{1}{8}$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 67

```
DSolve[{3*y''[x]-2*y'[x]+4*y[x]==x,{y[-1]==2,y'[-1]==3}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{88}\left(22x + 49\sqrt{11}e^{\frac{x+1}{3}}\sin\left(\frac{1}{3}\sqrt{11}(x+1)\right) + 187e^{\frac{x+1}{3}}\cos\left(\frac{1}{3}\sqrt{11}(x+1)\right) + 11\right)$$

9.2 problem 2

Internal problem ID [12745]
Internal file name [OUTPUT/11397_Friday_November_03_2023_06_32_22_AM_11601804/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 2.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{xy''' + y'x = 4}$$

With initial conditions

$$[y(1) = 0, y'(1) = 1, y''(1) = -1]$$

Since y is missing from the ode then we can use the substitution $y' = v(x)$ to reduce the order by one. The ODE becomes

$$v''(x)\, x + v(x)\, x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

But since $y' = v(x)$ then we now need to solve the ode $y' = c_1 \cos(x) + c_2 \sin(x)$. Integrating both sides gives

$$\begin{aligned}y &= \int c_1 \cos(x) + c_2 \sin(x)\, \mathrm{d}x \\ &= \sin(x) c_1 - c_2 \cos(x) + c_3\end{aligned}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$xy''' + y'x = 0$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} 1 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \\ 0 & -\cos(x) & -\sin(x) \end{bmatrix}$$
$$|W| = \cos(x)^2 + \sin(x)^2$$

The determinant simplifies to

$$|W| = 1$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix} \\ &= 1 \end{aligned}$$

$$
\begin{aligned}
W_2(x) &= \det \begin{bmatrix} 1 & \sin(x) \\ 0 & \cos(x) \end{bmatrix} \\
&= \cos(x)
\end{aligned}
$$

$$
\begin{aligned}
W_3(x) &= \det \begin{bmatrix} 1 & \cos(x) \\ 0 & -\sin(x) \end{bmatrix} \\
&= -\sin(x)
\end{aligned}
$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\
&= (-1)^2 \int \frac{(4)\,(1)}{(x)\,(1)}\, dx \\
&= \int \frac{4}{x}\, dx \\
&= \int \left(\frac{4}{x}\right) dx \\
&= 4\ln(x)
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\
&= (-1)^1 \int \frac{(4)\,(\cos(x))}{(x)\,(1)}\, dx \\
&= -\int \frac{4\cos(x)}{x}\, dx \\
&= -\int \left(\frac{4\cos(x)}{x}\right) dx \\
&= -4\ \mathrm{Ci}(x)
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^0 \int \frac{(4)\,(-\sin(x))}{(x)\,(1)}\, dx \\
&= \int \frac{-4\sin(x)}{x}\, dx \\
&= \int \left(-\frac{4\sin(x)}{x}\right) dx \\
&= -4\ \mathrm{Si}(x)
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Hence

$$\begin{aligned} y_p &= (4 \ln(x)) \\ &\quad + (-4 \operatorname{Ci}(x))(\cos(x)) \\ &\quad + (-4 \operatorname{Si}(x))(\sin(x)) \end{aligned}$$

Therefore the particular solution is

$$y_p = 4 \ln(x) - 4 \operatorname{Ci}(x) \cos(x) - 4 \operatorname{Si}(x) \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (y \\ &= \sin(x) c_1 - c_2 \cos(x) + c_3) + (4 \ln(x) - 4 \operatorname{Ci}(x) \cos(x) - 4 \operatorname{Si}(x) \sin(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \sin(x) c_1 - c_2 \cos(x) + c_3 + 4 \ln(x) - 4 \operatorname{Ci}(x) \cos(x) - 4 \operatorname{Si}(x) \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = (-c_2 - 4 \operatorname{Ci}(1)) \cos(1) + (c_1 - 4 \operatorname{Si}(1)) \sin(1) + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \cos(x) + c_2 \sin(x) + \frac{4}{x} - \frac{4 \cos(x)^2}{x} + 4 \operatorname{Ci}(x) \sin(x) - \frac{4 \sin(x)^2}{x} - 4 \operatorname{Si}(x) \cos(x)$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = (c_1 - 4 \operatorname{Si}(1)) \cos(1) + \sin(1) (c_2 + 4 \operatorname{Ci}(1)) \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = -\sin(x) c_1 + c_2 \cos(x) - \frac{4}{x^2} + \frac{4 \cos(x)^2}{x^2} + 4 \operatorname{Ci}(x) \cos(x) + \frac{4 \sin(x)^2}{x^2} + 4 \operatorname{Si}(x) \sin(x)$$

substituting $y'' = -1$ and $x = 1$ in the above gives

$$-1 = (c_2 + 4\ \mathrm{Ci}(1)) \cos(1) - (c_1 - 4\ \mathrm{Si}(1)) \sin(1) \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \sin(1) + \cos(1) + 4\ \mathrm{Si}(1) \\ c_2 &= \sin(1) - \cos(1) - 4\ \mathrm{Ci}(1) \\ c_3 &= -1 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -1 + 4\cos(x)\,\mathrm{Ci}(1) + 4\sin(x)\,\mathrm{Si}(1) + 4\ln(x) - 4\ \mathrm{Ci}(x)\cos(x) - 4\ \mathrm{Si}(x)\sin(x) + \cos(x)\cos(1) -$$

Which simplifies to

$$\begin{aligned} y = {} & (4\ \mathrm{Ci}(1) - 4\ \mathrm{Ci}(x) + \cos(1) - \sin(1)) \cos(x) \\ & + (4\ \mathrm{Si}(1) - 4\ \mathrm{Si}(x) + \cos(1) + \sin(1)) \sin(x) + 4\ln(x) - 1 \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & (4\ \mathrm{Ci}(1) - 4\ \mathrm{Ci}(x) + \cos(1) - \sin(1)) \cos(x) \\ & + (4\ \mathrm{Si}(1) - 4\ \mathrm{Si}(x) + \cos(1) + \sin(1)) \sin(x) + 4\ln(x) - 1 \end{aligned} \tag{1}$$

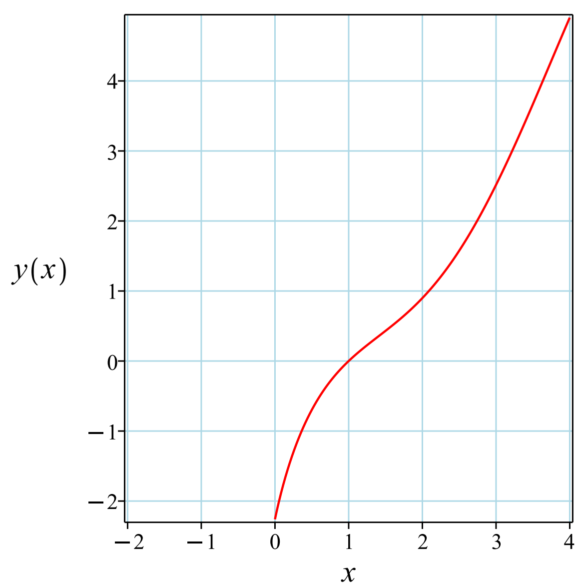


Figure 354: Solution plot

Verification of solutions

$$y = (4 \operatorname{Ci}(1) - 4 \operatorname{Ci}(x) + \cos(1) - \sin(1)) \cos(x) + (4 \operatorname{Si}(1) - 4 \operatorname{Si}(x) + \cos(1) + \sin(1)) \sin(x) + 4 \ln(x) - 1$$

Verified OK.

9.2.1 Maple step by step solution

Let's solve

$$\left[x y''' + y' x = 4, y(1) = 0, y'\Big|_{\{x=1\}} = 1, y''\Big|_{\{x=1\}} = -1\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{y' x - 4}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + y' = \frac{4}{x}$

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -\frac{y_2(x) x - 4}{x}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -\frac{y_2(x) x - 4}{x}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = \begin{bmatrix} c_3\sin(x) - c_2\cos(x) + c_1 \\ c_3\cos(x) + c_2\sin(x) \\ -c_3\sin(x) + c_2\cos(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_3\sin(x) - c_2\cos(x) + c_1$$

- Use the initial condition $y(1) = 0$

$$0 = c_3\sin(1) - c_2\cos(1) + c_1$$

- Calculate the 1st derivative of the solution

$$y' = c_3\cos(x) + c_2\sin(x)$$

- Use the initial condition $y'\Big|_{\{x=1\}} = 1$

$1 = c_3 \cos(1) + c_2 \sin(1)$

- Calculate the 2nd derivative of the solution

  $y'' = -c_3 \sin(x) + c_2 \cos(x)$

- Use the initial condition $y''\Big|_{\{x=1\}} = -1$

  $-1 = -c_3 \sin(1) + c_2 \cos(1)$

- Solve for the unknown coefficients

  $\left\{c_1 = -1, c_2 = \frac{\sin(1)-\cos(1)}{\sin(1)^2+\cos(1)^2}, c_3 = \frac{\sin(1)+\cos(1)}{\sin(1)^2+\cos(1)^2}\right\}$

- Solution to the IVP

  $y = -1 + (-\sin(1) + \cos(1)) \cos(x) + (\sin(1) + \cos(1)) \sin(x)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -(_b(_a)*_a-4)/_a, _b(_a)`
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.171 (sec). Leaf size: 49

```
dsolve([x*diff(y(x),x$3)+x*diff(y(x),x)=4,y(1) = 0, D(y)(1) = 1, (D@@2)(y)(1) = -1],y(x), si
```

$$y(x) = (4\ \mathrm{Ci}\,(1) - 4\ \mathrm{Ci}\,(x) + \cos\,(1) - \sin\,(1)) \cos\,(x) \\ + (4\ \mathrm{Si}\,(1) - 4\ \mathrm{Si}\,(x) + \cos\,(1) + \sin\,(1)) \sin\,(x) + 4 \ln\,(x) - 1$$

✓ Solution by Mathematica

Time used: 0.184 (sec). Leaf size: 85

```
DSolve[{x*y'''[x]+x*y'[x]==4,{y[1]==0,y'[1]==1,y''[1]==-1}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to -4\,\mathrm{CosIntegral}(x)\cos(x) + 4\,\mathrm{CosIntegral}(1)\cos(x) - 2\mathrm{sinc}(1)\cos(2-x) \\ - 6\mathrm{sinc}(1)\cos(x) + 8\mathrm{sinc}(1)\cos(1) - 4\mathrm{Si}(x)\sin(x) + 4\mathrm{Si}(1)\sin(x) + 4\log(x) \\ + \sin(1-x) + \sin(3-x) + 3\sin(x+1) + \cos(1-x) - 1 - 4\sin(2)$$

9.3 problem 3

Internal problem ID [12746]
Internal file name [`OUTPUT/11398_Friday_November_03_2023_06_32_23_AM_45606230/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x(x-3)\,y''+3y'=x^2}$$

With initial conditions

$$[y(1)=0, y'(1)=1]$$

9.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= \frac{3}{x^2-3x} \\ q(x) &= 0 \\ F &= \frac{x^2}{x^2-3x} \end{aligned}$$

Hence the ode is

$$y'' + \frac{3y'}{x^2 - 3x} = \frac{x^2}{x^2 - 3x}$$

The domain of $p(x) = \frac{3}{x^2-3x}$ is

$$\{-\infty \le x < 0, 0 < x < 3, 3 < x \le \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $F = \frac{x^2}{x^2-3x}$ is

$$\{-\infty \le x < 0, 0 < x < 3, 3 < x \le \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

9.3.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$\left(x^2 - 3x\right) p'(x) + 3p(x) - x^2 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{3}{x(x-3)} dx} \\ &= e^{\ln(x-3)-\ln(x)} \end{aligned}$$

Which simplifies to

$$\mu = \frac{x-3}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu p) = (\mu)\left(\frac{x}{x-3}\right)$$

$$\frac{d}{dx}\left(\frac{(x-3)\,p}{x}\right) = \left(\frac{x-3}{x}\right)\left(\frac{x}{x-3}\right)$$

$$d\left(\frac{(x-3)\,p}{x}\right) = dx$$

Integrating gives

$$\frac{(x-3)\,p}{x} = \int dx$$

$$\frac{(x-3)\,p}{x} = x + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{x-3}{x}$ results in

$$p(x) = \frac{x^2}{x-3} + \frac{c_1 x}{x-3}$$

which simplifies to

$$p(x) = \frac{x(x+c_1)}{x-3}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{2} - \frac{c_1}{2}$$

$$c_1 = -3$$

Substituting c_1 found above in the general solution gives

$$p(x) = x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = x$$

Integrating both sides gives

$$y = \int x \, dx$$

$$= \frac{x^2}{2} + c_2$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{2} + c_2$$

$$c_2 = -\frac{1}{2}$$

Substituting c_2 found above in the general solution gives

$$y = \frac{x^2}{2} - \frac{1}{2}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} - \frac{1}{2} \tag{1}$$

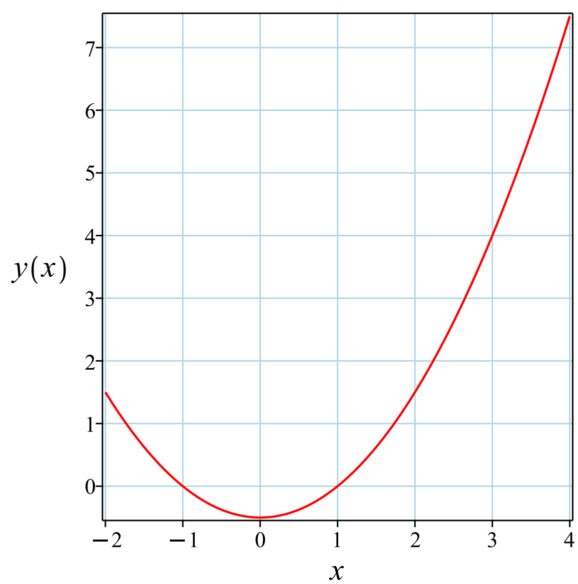


Figure 355: Solution plot

Verification of solutions

$$y = \frac{x^2}{2} - \frac{1}{2}$$

Verified OK.

9.3.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 - 3x \\ B &= 3 \\ C &= 0 \\ F &= x^2 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2 - 3x\right)(0) + (3)(0) + (0)(3) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$3x^2 - 9xv'' + (9)\,v' = 0$$

Now by applying $v' = u$ the above becomes

$$\left(3x^2 - 9x\right)u'(x) + 9u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x\,(x-3)} \end{aligned}$$

Where $f(x) = -\frac{3}{x(x-3)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{3}{x\,(x-3)}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{3}{x\,(x-3)}\,dx \\ \ln\left(u\right) &= -\ln\left(x-3\right) + \ln\left(x\right) + c_1 \\ u &= \mathrm{e}^{-\ln(x-3)+\ln(x)+c_1} \\ &= c_1 \mathrm{e}^{-\ln(x-3)+\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x}{x-3}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1 x}{x-3} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1 x}{x-3}\,\mathrm{d}x \\ &= c_1\left(x + 3\ln\left(x-3\right)\right) + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (3)\left(c_1\left(x + 3\ln\left(x-3\right)\right) + c_2\right) \\ &= 3c_1\left(x + 3\ln\left(x-3\right)\right) + 3c_2 \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 3 \\ y_2 &= 9 \ln(x-3) + 3x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 3 & 9 \ln(x-3) + 3x \\ \frac{d}{dx}(3) & \frac{d}{dx}(9 \ln(x-3) + 3x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 3 & 9 \ln(x-3) + 3x \\ 0 & \frac{9}{x-3} + 3 \end{vmatrix}$$

Therefore

$$W = (3) \left(\frac{9}{x-3} + 3 \right) - (9 \ln(x-3) + 3x)(0)$$

Which simplifies to

$$W = \frac{9x}{x-3}$$

Which simplifies to

$$W = \frac{9x}{x-3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(9\ln(x-3)+3x)\,x^2}{\frac{9(x^2-3x)x}{x-3}}\,dx$$

Which simplifies to

$$u_1 = -\int \left(\frac{x}{3}+\ln(x-3)\right)dx$$

Hence

$$u_1 = -\frac{x^2}{6} - \ln(x-3)(x-3) + x - 3$$

And Eq. (3) becomes

$$u_2 = \int \frac{3x^2}{\frac{9(x^2-3x)x}{x-3}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{3}dx$$

Hence

$$u_2 = \frac{x}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2}{2} - 3\ln(x-3)(x-3) + 3x - 9 + \frac{x(9\ln(x-3)+3x)}{3}$$

Which simplifies to

$$y_p(x) = \frac{x^2}{2} + 9\ln(x-3) + 3x - 9$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (3c_1(x + 3\ln(x-3)) + 3c_2) + \left(\frac{x^2}{2} + 9\ln(x-3) + 3x - 9\right) \\ &= -9 + 9(1+c_1)\ln(x-3) + \frac{x^2}{2} + 3(1+c_1)\,x + 3c_2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -9 + 9(1+c_1)\ln(x-3) + \frac{x^2}{2} + 3(1+c_1)\,x + 3c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = -\frac{11}{2} + 9(1+c_1)\ln(2) + 9ic_1\pi + 9i\pi + 3c_1 + 3c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{9c_1 + 9}{x - 3} + x + 3 + 3c_1$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = -\frac{3c_1}{2} - \frac{1}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -1 \\ c_2 &= \frac{17}{6} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{x^2}{2} - \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} - \frac{1}{2} \tag{1}$$

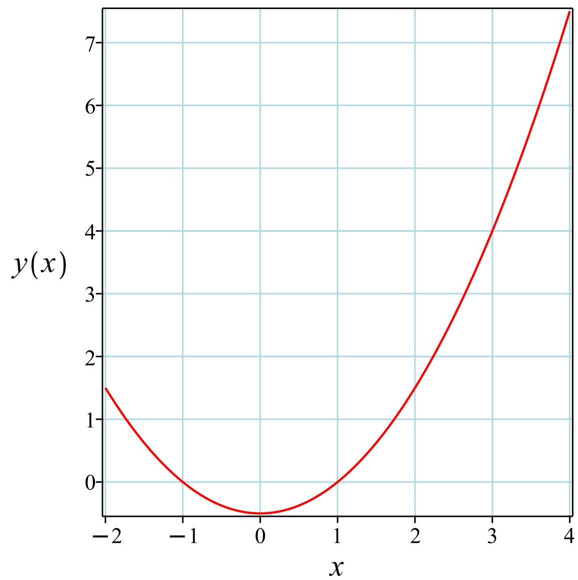


Figure 356: Solution plot

Verification of solutions

$$y = \frac{x^2}{2} - \frac{1}{2}$$

Verified OK.

9.3.4 Solving using Kovacic algorithm

Writing the ode as

$$(x^2 - 3x) y'' + 3y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 - 3x \\ B &= 3 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{27 - 12x}{4\left(x^2 - 3x\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 27 - 12x$$
$$t = 4\left(x^2 - 3x\right)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{27 - 12x}{4\left(x^2 - 3x\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 309: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 4 - 1 \\
&= 3
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 3x)^2$. There is a pole at $x = 3$ of order 2. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2} - \frac{1}{6(x-3)} - \frac{1}{4(x-3)^2} + \frac{1}{6x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = 3$ let b be the coefficient of $\frac{1}{(x-3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{27 - 12x}{4\left(x^2 - 3x\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
3	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 0 - (0) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + \frac{1}{2x - 6} + (0) \\ &= -\frac{1}{2x} + \frac{1}{2x - 6} \\ &= \frac{3}{2x\left(x - 3\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} + \frac{1}{2x-6}\right)(0) + \left(\left(\frac{1}{2x^2} - \frac{1}{2\left(x-3\right)^2}\right) + \left(-\frac{1}{2x} + \frac{1}{2x-6}\right)^2 - \left(\frac{27-12x}{4\left(x^2-3x\right)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2x} + \frac{1}{2x-6}\right) dx} \\ &= \frac{\sqrt{x-3}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{3}{x^2-3x} \, dx} \\ &= z_1 e^{-\frac{\ln(x-3)}{2} + \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{\sqrt{x}}{\sqrt{x-3}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x^2-3x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-3)+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x + 3\ln(x-3))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(x + 3\ln(x-3)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
(x^2 - 3x)\,y'' + 3y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + c_2(x + 3\ln(x-3))
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x + 3\ln(x-3)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & x + 3\ln(x-3) \\ \frac{d}{dx}(1) & \frac{d}{dx}(x + 3\ln(x-3)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & x + 3\ln(x-3) \\ 0 & 1 + \frac{3}{x-3} \end{vmatrix}$$

Therefore

$$W = (1)\left(1 + \frac{3}{x-3}\right) - (x + 3\ln(x-3))(0)$$

Which simplifies to

$$W = \frac{x}{x-3}$$

Which simplifies to

$$W = \frac{x}{x-3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x + 3\ln(x-3))\, x^2}{\frac{(x^2-3x)x}{x-3}}\, dx$$

Which simplifies to

$$u_1 = -\int (x + 3\ln(x-3))\, dx$$

Hence

$$u_1 = -\frac{x^2}{2} - 3\ln(x-3)(x-3) + 3x - 9$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{\frac{(x^2-3x)x}{x-3}}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -9 - \frac{x^2}{2} - 3\ln(x-3)(x-3) + 3x + x(x + 3\ln(x-3))$$

Which simplifies to

$$y_p(x) = \frac{x^2}{2} + 9\ln(x-3) + 3x - 9$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2(x + 3\ln(x-3))) + \left(\frac{x^2}{2} + 9\ln(x-3) + 3x - 9\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + c_2(x + 3\ln(x-3)) + \frac{x^2}{2} + 9\ln(x-3) + 3x - 9 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 1$ in the above gives

$$0 = -\frac{11}{2} + 3(c_2 + 3)\ln(2) + 3ic_2\pi + 9i\pi + c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2\left(1+\frac{3}{x-3}\right)+x+\frac{9}{x-3}+3$$

substituting $y'=1$ and $x=1$ in the above gives

$$1 = -\frac{1}{2}-\frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{17}{2}$$
$$c_2 = -3$$

Substituting these values back in above solution results in

$$y = \frac{x^2}{2}-\frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2}-\frac{1}{2} \tag{1}$$

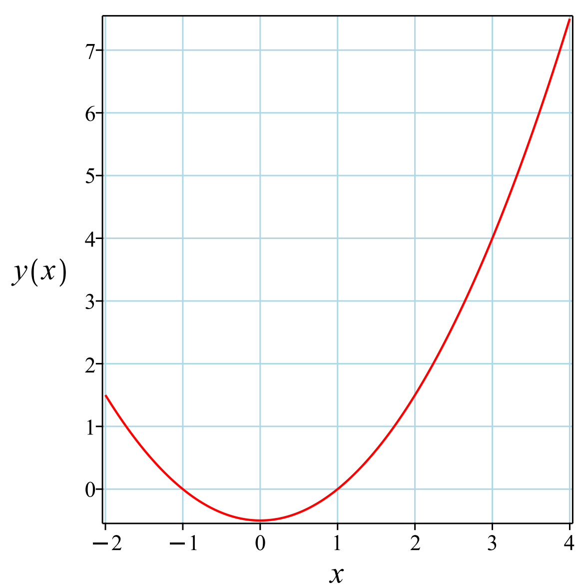


Figure 357: Solution plot

Verification of solutions

$$y = \frac{x^2}{2} - \frac{1}{2}$$

Verified OK.

9.3.5 Maple step by step solution

Let's solve

$\left[(x^2 - 3x)\, y'' + 3y' = x^2, y(1) = 0, y'\Big|_{\{x=1\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $(x^2 - 3x)\, u'(x) + 3u(x) = x^2$

- Isolate the derivative

  $u'(x) = -\frac{3u(x)}{x(x-3)} + \frac{x}{x-3}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) + \frac{3u(x)}{x(x-3)} = \frac{x}{x-3}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(u'(x) + \frac{3u(x)}{x(x-3)}\right) = \frac{\mu(x)x}{x-3}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, u(x))$

  $\mu(x) \left(u'(x) + \frac{3u(x)}{x(x-3)}\right) = \mu'(x)\, u(x) + \mu(x)\, u'(x)$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)}{x(x-3)}$

- Solve to find the integrating factor

  $\mu(x) = \frac{x-3}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, u(x))\right) dx = \int \frac{\mu(x)x}{x-3} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, u(x) = \int \frac{\mu(x)x}{x-3} dx + c_1$

- Solve for $u(x)$

  $u(x) = \frac{\int \frac{\mu(x)x}{x-3}dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \frac{x-3}{x}$

  $u(x) = \frac{x(\int 1dx + c_1)}{x-3}$
- Evaluate the integrals on the rhs

  $u(x) = \frac{x(x+c_1)}{x-3}$
- Solve 1st ODE for $u(x)$

  $u(x) = \frac{x(x+c_1)}{x-3}$
- Make substitution $u = y'$

  $y' = \frac{x(x+c_1)}{x-3}$
- Integrate both sides to solve for y

  $\int y'dx = \int \frac{x(x+c_1)}{x-3}dx + c_2$
- Compute integrals

  $y = \frac{x^2}{2} + c_1 x + 3x + (3c_1 + 9)\ln(x-3) + c_2$

☐ Check validity of solution $y = \frac{x^2}{2} + c_1 x + 3x + (3c_1 + 9)\ln(x-3) + c_2$

  - Use initial condition $y(1) = 0$

    $0 = \frac{7}{2} + c_1 + (3c_1 + 9)(\ln(2) + \mathrm{I}\pi) + c_2$
  - Compute derivative of the solution

    $y' = x + c_1 + 3 + \frac{3c_1+9}{x-3}$
  - Use the initial condition $y'\Big|_{\{x=1\}} = 1$

    $1 = -\frac{1}{2} - \frac{c_1}{2}$
  - Solve for c_1 and c_2

    $\left\{c_1 = -3, c_2 = -\frac{1}{2}\right\}$
  - Substitute constant values into general solution and simplify

    $y = \frac{x^2}{2} - \frac{1}{2}$
- Solution to the IVP

  $y = \frac{x^2}{2} - \frac{1}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(-_a^2+3*_b(_a))/(_a*(-3+_a)), _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve([x*(x-3)*diff(y(x),x$2)+3*diff(y(x),x)=x^2,y(1) = 0, D(y)(1) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{1}{2} + \frac{x^2}{2}$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 14

```
DSolve[{x*(x-3)*y''[x]+3*y'[x]==x^2,{y[1]==0,y'[1]==1}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{2}(x^2 - 1)$$

9.4 problem 4

Internal problem ID [12747]
Internal file name [`OUTPUT/11399_Friday_November_03_2023_06_32_24_AM_67661546/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x(x-3)\,y''+3y'=x^2}$$

With initial conditions

$$[y(5)=0, y'(5)=1]$$

9.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=\frac{3}{x^2-3x}\\ q(x)&=0\\ F&=\frac{x^2}{x^2-3x}\end{aligned}$$

Hence the ode is

$$y'' + \frac{3y'}{x^2 - 3x} = \frac{x^2}{x^2 - 3x}$$

The domain of $p(x) = \frac{3}{x^2 - 3x}$ is

$$\{-\infty \leq x < 0, 0 < x < 3, 3 < x \leq \infty\}$$

And the point $x_0 = 5$ is inside this domain. The domain of $F = \frac{x^2}{x^2 - 3x}$ is

$$\{-\infty \leq x < 0, 0 < x < 3, 3 < x \leq \infty\}$$

And the point $x_0 = 5$ is also inside this domain. Hence solution exists and is unique.

9.4.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$\left(x^2 - 3x\right) p'(x) + 3p(x) - x^2 = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{x(x-3)} dx} \\ &= e^{\ln(x-3) - \ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{x - 3}{x}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu p) &= (\mu)\left(\frac{x}{x-3}\right) \\
\frac{d}{dx}\left(\frac{(x-3)\,p}{x}\right) &= \left(\frac{x-3}{x}\right)\left(\frac{x}{x-3}\right) \\
d\left(\frac{(x-3)\,p}{x}\right) &= dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{(x-3)\,p}{x} &= \int dx \\
\frac{(x-3)\,p}{x} &= x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{x-3}{x}$ results in

$$
p(x) = \frac{x^2}{x-3} + \frac{c_1 x}{x-3}
$$

which simplifies to

$$
p(x) = \frac{x(x+c_1)}{x-3}
$$

Initial conditions are used to solve for c_1. Substituting $x = 5$ and $p = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = \frac{25}{2} + \frac{5c_1}{2}
$$

$$
c_1 = -\frac{23}{5}
$$

Substituting c_1 found above in the general solution gives

$$
p(x) = \frac{x(5x-23)}{5x-15}
$$

Since $p = y'$ then the new first order ode to solve is

$$
y' = \frac{x(5x-23)}{5x-15}
$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{x(5x-23)}{5x-15}\, dx \\
&= \frac{x^2}{2} - \frac{8x}{5} - \frac{24\ln(x-3)}{5} + c_2
\end{aligned}
$$

Initial conditions are used to solve for c_2. Substituting $x = 5$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{9}{2} - \frac{24\ln(2)}{5} + c_2$$

$$c_2 = -\frac{9}{2} + \frac{24\ln(2)}{5}$$

Substituting c_2 found above in the general solution gives

$$y = \frac{x^2}{2} - \frac{8x}{5} - \frac{24\ln(x-3)}{5} - \frac{9}{2} + \frac{24\ln(2)}{5}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} - \frac{8x}{5} - \frac{24\ln(x-3)}{5} - \frac{9}{2} + \frac{24\ln(2)}{5} \tag{1}$$

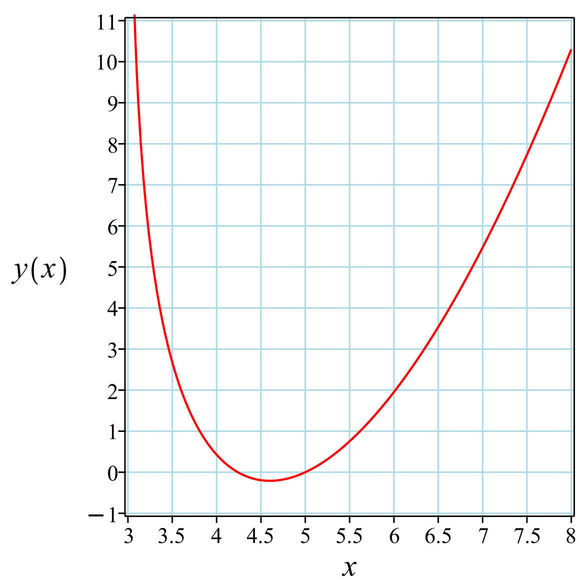


Figure 358: Solution plot

Verification of solutions

$$y = \frac{x^2}{2} - \frac{8x}{5} - \frac{24\ln(x-3)}{5} - \frac{9}{2} + \frac{24\ln(2)}{5}$$

Verified OK.

9.4.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \qquad (1)
\end{aligned}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2 - 3x \\
B &= 3 \\
C &= 0 \\
F &= x^2
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left(x^2 - 3x\right)(0) + (3)(0) + (0)(3) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$3x^2 - 9xv'' + (9)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$(3x^2 - 9x)\, u'(x) + 9u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x\,(x-3)} \end{aligned}$$

Where $f(x) = -\frac{3}{x(x-3)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{x\,(x-3)}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3}{x\,(x-3)}\, dx \\ \ln(u) &= -\ln(x-3) + \ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x-3)+\ln(x)+c_1} \\ &= c_1 \mathrm{e}^{-\ln(x-3)+\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x}{x-3}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1 x}{x-3} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1 x}{x-3}\, \mathrm{d}x \\ &= c_1(x + 3\ln(x-3)) + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (3)\,(c_1(x + 3\ln(x-3)) + c_2) \\ &= 3c_1(x + 3\ln(x-3)) + 3c_2 \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 3 \\ y_2 &= 9\ln(x-3) + 3x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 3 & 9\ln(x-3)+3x \\ \frac{d}{dx}(3) & \frac{d}{dx}(9\ln(x-3)+3x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 3 & 9\ln(x-3)+3x \\ 0 & \frac{9}{x-3}+3 \end{vmatrix}$$

Therefore

$$W = (3)\left(\frac{9}{x-3}+3\right) - (9\ln(x-3)+3x)(0)$$

Which simplifies to

$$W = \frac{9x}{x-3}$$

Which simplifies to

$$W = \frac{9x}{x-3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(9\ln(x-3)+3x)\,x^2}{\frac{9(x^2-3x)x}{x-3}}\,dx$$

Which simplifies to

$$u_1 = -\int \left(\frac{x}{3}+\ln(x-3)\right) dx$$

Hence

$$u_1 = -\frac{x^2}{6} - \ln(x-3)(x-3) + x - 3$$

And Eq. (3) becomes

$$u_2 = \int \frac{3x^2}{\frac{9(x^2-3x)x}{x-3}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{3} dx$$

Hence

$$u_2 = \frac{x}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2}{2} - 3\ln(x-3)(x-3) + 3x - 9 + \frac{x(9\ln(x-3)+3x)}{3}$$

Which simplifies to

$$y_p(x) = \frac{x^2}{2} + 9\ln(x-3) + 3x - 9$$

Hence the complete solution is

$$
\begin{aligned}
y(x) &= y_h + y_p \\
&= (3c_1(x + 3\ln(x-3)) + 3c_2) + \left(\frac{x^2}{2} + 9\ln(x-3) + 3x - 9\right) \\
&= -9 + 9(1+c_1)\ln(x-3) + \frac{x^2}{2} + 3(1+c_1)x + 3c_2
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = -9 + 9(1+c_1)\ln(x-3) + \frac{x^2}{2} + 3(1+c_1)x + 3c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 5$ in the above gives

$$
0 = \frac{37}{2} + 9(1+c_1)\ln(2) + 15c_1 + 3c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{9c_1 + 9}{x - 3} + x + 3 + 3c_1
$$

substituting $y' = 1$ and $x = 5$ in the above gives

$$
1 = \frac{15c_1}{2} + \frac{25}{2} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{23}{15} \\
c_2 &= \frac{8\ln(2)}{5} + \frac{3}{2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{x^2}{2} - \frac{8x}{5} - \frac{24\ln(x-3)}{5} - \frac{9}{2} + \frac{24\ln(2)}{5}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^2}{2} - \frac{8x}{5} - \frac{24\ln(x-3)}{5} - \frac{9}{2} + \frac{24\ln(2)}{5} \tag{1}
$$

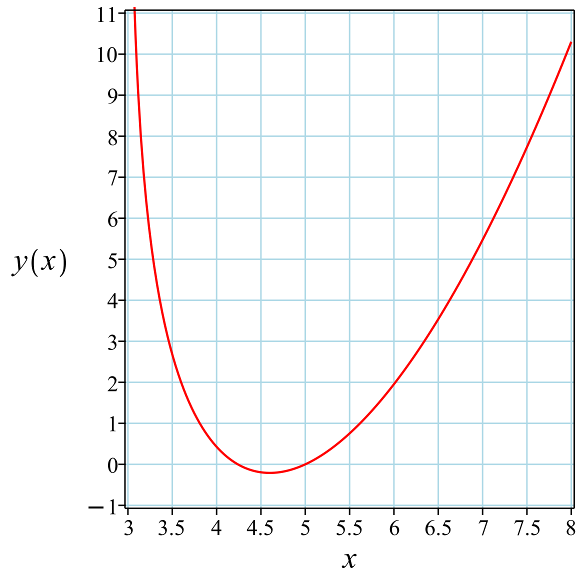


Figure 359: Solution plot

Verification of solutions

$$y = \frac{x^2}{2} - \frac{8x}{5} - \frac{24\ln(x-3)}{5} - \frac{9}{2} + \frac{24\ln(2)}{5}$$

Verified OK.

9.4.4 Solving using Kovacic algorithm

Writing the ode as

$$(x^2 - 3x)\, y'' + 3y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 - 3x \\ B &= 3 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{27 - 12x}{4\left(x^2 - 3x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 27 - 12x \\
t &= 4\left(x^2 - 3x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{27 - 12x}{4\left(x^2 - 3x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 311: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 3x)^2$. There is a pole at $x = 3$ of order 2. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 3 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2} - \frac{1}{6(x-3)} - \frac{1}{4(x-3)^2} + \frac{1}{6x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = 3$ let b be the coefficient of $\frac{1}{(x-3)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{27 - 12x}{4 \left(x^2 - 3x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
3	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 0 - (0) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + \frac{1}{2x - 6} + (0) \\
&= -\frac{1}{2x} + \frac{1}{2x - 6} \\
&= \frac{3}{2x \left(x - 3\right)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}$$

Let

$$p(x)=1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(-\frac{1}{2x}+\frac{1}{2x-6}\right)(0)+\left(\left(\frac{1}{2x^2}-\frac{1}{2\left(x-3\right)^2}\right)+\left(-\frac{1}{2x}+\frac{1}{2x-6}\right)^2-\left(\frac{27-12x}{4\left(x^2-3x\right)^2}\right)\right)=0$$

$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\,dx} \\ &= e^{\int \left(-\frac{1}{2x}+\frac{1}{2x-6}\right)dx} \\ &= \frac{\sqrt{x-3}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{x^2-3x}\,dx} \\ &= z_1 e^{-\frac{\ln(x-3)}{2}+\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{\sqrt{x}}{\sqrt{x-3}}\right) \end{aligned}$$

Which simplifies to

$$y_1=1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{x^2-3x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x-3)+\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x + 3\ln(x-3))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(x + 3\ln(x-3)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
(x^2-3x)\,y'' + 3y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + c_2(x + 3\ln(x-3))
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x + 3\ln(x-3)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & x + 3\ln(x-3) \\ \frac{d}{dx}(1) & \frac{d}{dx}(x + 3\ln(x-3)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & x + 3\ln(x-3) \\ 0 & 1 + \frac{3}{x-3} \end{vmatrix}$$

Therefore

$$W = (1)\left(1 + \frac{3}{x-3}\right) - (x + 3\ln(x-3))(0)$$

Which simplifies to

$$W = \frac{x}{x-3}$$

Which simplifies to

$$W = \frac{x}{x-3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x + 3\ln(x-3))\,x^2}{\frac{(x^2-3x)x}{x-3}}\,dx$$

Which simplifies to

$$u_1 = -\int (x + 3\ln(x-3))\,dx$$

Hence

$$u_1 = -\frac{x^2}{2} - 3\ln(x-3)(x-3) + 3x - 9$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{\frac{(x^2-3x)x}{x-3}} \, dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -9 - \frac{x^2}{2} - 3\ln(x-3)(x-3) + 3x + x(x + 3\ln(x-3))$$

Which simplifies to

$$y_p(x) = \frac{x^2}{2} + 9\ln(x-3) + 3x - 9$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2(x + 3\ln(x-3))) + \left(\frac{x^2}{2} + 9\ln(x-3) + 3x - 9\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + c_2(x + 3\ln(x-3)) + \frac{x^2}{2} + 9\ln(x-3) + 3x - 9 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 5$ in the above gives

$$0 = \frac{37}{2} + 3(c_2 + 3)\ln(2) + c_1 + 5c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2\left(1+\frac{3}{x-3}\right)+x+\frac{9}{x-3}+3$$

substituting $y' = 1$ and $x = 5$ in the above gives

$$1 = \frac{5c_2}{2}+\frac{25}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{24\ln(2)}{5}+\frac{9}{2}$$
$$c_2 = -\frac{23}{5}$$

Substituting these values back in above solution results in

$$y = \frac{x^2}{2}-\frac{8x}{5}-\frac{24\ln(x-3)}{5}-\frac{9}{2}+\frac{24\ln(2)}{5}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2}-\frac{8x}{5}-\frac{24\ln(x-3)}{5}-\frac{9}{2}+\frac{24\ln(2)}{5} \tag{1}$$

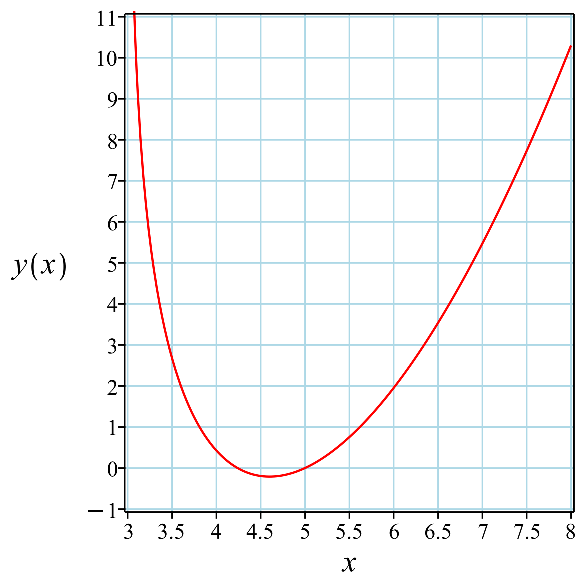


Figure 360: Solution plot

Verification of solutions

$$y = \frac{x^2}{2} - \frac{8x}{5} - \frac{24\ln(x-3)}{5} - \frac{9}{2} + \frac{24\ln(2)}{5}$$

Verified OK.

9.4.5 Maple step by step solution

Let's solve

$$\left[(x^2-3x)\,y''+3y'=x^2, y(5)=0, y'\Big|_{\{x=5\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $(x^2-3x)\,u'(x)+3u(x)=x^2$

- Isolate the derivative

  $u'(x)=-\frac{3u(x)}{x(x-3)}+\frac{x}{x-3}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x)+\frac{3u(x)}{x(x-3)}=\frac{x}{x-3}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x)+\frac{3u(x)}{x(x-3)}\right)=\frac{\mu(x)x}{x-3}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x)+\frac{3u(x)}{x(x-3)}\right)=\mu'(x)\,u(x)+\mu(x)\,u'(x)$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{3\mu(x)}{x(x-3)}$

- Solve to find the integrating factor

  $\mu(x)=\frac{x-3}{x}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,u(x))\right)dx=\int\frac{\mu(x)x}{x-3}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,u(x)=\int\frac{\mu(x)x}{x-3}dx+c_1$

- Solve for $u(x)$

  $u(x) = \frac{\int \frac{\mu(x)x}{x-3}dx+c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{x-3}{x}$

  $u(x) = \frac{x\left(\int 1dx+c_1\right)}{x-3}$

- Evaluate the integrals on the rhs

  $u(x) = \frac{x(x+c_1)}{x-3}$

- Solve 1st ODE for $u(x)$

  $u(x) = \frac{x(x+c_1)}{x-3}$

- Make substitution $u = y'$

  $y' = \frac{x(x+c_1)}{x-3}$

- Integrate both sides to solve for y

  $\int y'dx = \int \frac{x(x+c_1)}{x-3}dx + c_2$

- Compute integrals

  $y = \frac{x^2}{2} + c_1x + 3x + (3c_1 + 9)\ln(x-3) + c_2$

- □ Check validity of solution $y = \frac{x^2}{2} + c_1x + 3x + (3c_1 + 9)\ln(x-3) + c_2$
  - Use initial condition $y(5) = 0$

    $0 = \frac{55}{2} + 5c_1 + (3c_1 + 9)\ln(2) + c_2$

  - Compute derivative of the solution

    $y' = x + c_1 + 3 + \frac{3c_1+9}{x-3}$

  - Use the initial condition $y'\Big|_{\{x=5\}} = 1$

    $1 = \frac{25}{2} + \frac{5c_1}{2}$

  - Solve for c_1 and c_2

    $\left\{c_1 = -\frac{23}{5}, c_2 = -\frac{9}{2} + \frac{24\ln(2)}{5}\right\}$

  - Substitute constant values into general solution and simplify

    $y = \frac{x^2}{2} - \frac{8x}{5} - \frac{24\ln(x-3)}{5} - \frac{9}{2} + \frac{24\ln(2)}{5}$

- Solution to the IVP

  $y = \frac{x^2}{2} - \frac{8x}{5} - \frac{24\ln(x-3)}{5} - \frac{9}{2} + \frac{24\ln(2)}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(-_a^2+3*_b(_a))/(_a*(-3+_a)), _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 24

```
dsolve([x*(x-3)*diff(y(x),x$2)+3*diff(y(x),x)=x^2,y(5) = 0, D(y)(5) = 1],y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} - \frac{8x}{5} - \frac{24\ln(-3+x)}{5} + \frac{24\ln(2)}{5} - \frac{9}{2}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 29

```
DSolve[{x*(x-3)*y''[x]+3*y'[x]==x^2,{y[5]==0,y'[5]==1}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{10}\left(5x^2 - 16x - 48\log(x-3) - 45 + 48\log(2)\right)$$

9.5 problem 5

Internal problem ID [12748]
Internal file name [`OUTPUT/11400_Friday_November_03_2023_06_32_25_AM_18586947/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{\sqrt{1-x}\, y'' - 4y = \sin(x)}$$

With initial conditions

$$[y(-2) = 3, y'(-2) = -1]$$

9.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -\frac{4}{\sqrt{1-x}} \\ F &= \frac{\sin(x)}{\sqrt{1-x}} \end{aligned}$$

Hence the ode is

$$y'' - \frac{4y}{\sqrt{1-x}} = \frac{\sin(x)}{\sqrt{1-x}}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -2$ is inside this domain. The domain of $q(x) = -\frac{4}{\sqrt{1-x}}$ is

$$\{x < 1\}$$

And the point $x_0 = -2$ is also inside this domain. The domain of $F = \frac{\sin(x)}{\sqrt{1-x}}$ is

$$\{x < 1\}$$

And the point $x_0 = -2$ is also inside this domain. Hence solution exists and is unique.

9.5.2 Solving as second order bessel ode ode

Writing the ode as

$$y''x^2 - 4yx^{\frac{3}{2}} = x^{\frac{3}{2}} \sin(x) \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$y''x^2 + y'x + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$y''x^2 + (1 - 2\alpha)\, xy' + (\beta^2\gamma^2 x^{2\gamma} - n^2\gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = \frac{8i}{3}$$
$$n = \frac{2}{3}$$
$$\gamma = \frac{3}{4}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x}\,\mathrm{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)$$

$$y_2 = \sqrt{x}\,\mathrm{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) & \sqrt{x}\,\mathrm{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) \\ \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\right) & \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x}\, \text{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) & \sqrt{x}\, \text{BesselY} \\ \frac{\text{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)}{2\sqrt{x}} + 2ix^{\frac{1}{4}}\left(\text{BesselJ}\left(-\frac{1}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) + \frac{i\,\text{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)}{4x^{\frac{3}{4}}}\right) & \frac{\text{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)}{2\sqrt{x}} + 2ix^{\frac{1}{4}}\left(\text{BesselY}\right. \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\sqrt{x}\, \text{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\right)\left(\frac{\text{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)}{2\sqrt{x}}\right. \\ & \left. + 2ix^{\frac{1}{4}}\left(\text{BesselY}\left(-\frac{1}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) + \frac{i\,\text{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)}{4x^{\frac{3}{4}}}\right)\right) \\ & - \left(\sqrt{x}\, \text{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\right)\left(\frac{\text{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)}{2\sqrt{x}}\right. \\ & \left. + 2ix^{\frac{1}{4}}\left(\text{BesselJ}\left(-\frac{1}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) + \frac{i\,\text{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)}{4x^{\frac{3}{4}}}\right)\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} W = 2ix^{\frac{3}{4}}\Bigg(& \text{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\text{BesselY}\left(-\frac{1}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) \\ & - \text{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\text{BesselJ}\left(-\frac{1}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\Bigg) \end{aligned}$$

Which simplifies to

$$W = \frac{3}{2\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2\, \text{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\sin(x)}{\frac{3x^2}{2\pi}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{2 \,\mathrm{BesselY}\left(\frac{2}{3}, \frac{8 i x^{\frac{3}{4}}}{3}\right) \sin \left(x\right) \pi}{3}d x$$

Hence

$$u_1 = -\left(\int_{0}^{x}\frac{2 \,\mathrm{BesselY}\left(\frac{2}{3}, \frac{8 i \alpha^{\frac{3}{4}}}{3}\right) \sin \left(\alpha \right) \pi}{3}d \alpha \right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{2} \mathrm{BesselJ}\left(\frac{2}{3}, \frac{8 i x^{\frac{3}{4}}}{3}\right) \sin \left(x\right)}{\frac{3 x^{2}}{2 \pi}}d x$$

Which simplifies to

$$u_2 = \int \frac{2 \,\mathrm{BesselJ}\left(\frac{2}{3}, \frac{8 i x^{\frac{3}{4}}}{3}\right) \sin \left(x\right) \pi}{3}d x$$

Hence

$$u_2 = \int_{0}^{x}\frac{2 \,\mathrm{BesselJ}\left(\frac{2}{3}, \frac{8 i \alpha^{\frac{3}{4}}}{3}\right) \sin \left(\alpha \right) \pi}{3}d \alpha$$

Which simplifies to

$$u_1 = -\frac{2 \pi \left(\int_{0}^{x}\mathrm{BesselY}\left(\frac{2}{3}, \frac{8 i \alpha^{\frac{3}{4}}}{3}\right) \sin \left(\alpha \right)d \alpha \right)}{3}$$

$$u_2 = \frac{2 \pi \left(\int_{0}^{x}\mathrm{BesselJ}\left(\frac{2}{3}, \frac{8 i \alpha^{\frac{3}{4}}}{3}\right) \sin \left(\alpha \right)d \alpha \right)}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{2 \pi \left(\int_{0}^{x}\mathrm{BesselY}\left(\frac{2}{3}, \frac{8 i \alpha^{\frac{3}{4}}}{3}\right) \sin \left(\alpha \right)d \alpha \right) \sqrt{x}\, \mathrm{BesselJ}\left(\frac{2}{3}, \frac{8 i x^{\frac{3}{4}}}{3}\right)}{3} + \frac{2 \pi \left(\int_{0}^{x}\mathrm{BesselJ}\left(\frac{2}{3}, \frac{8 i \alpha^{\frac{3}{4}}}{3}\right) \sin \left(\alpha \right)d \alpha \right) \sqrt{x}\, \mathrm{BesselY}\left(\frac{2}{3}, \frac{8 i x^{\frac{3}{4}}}{3}\right)}{3}$$

Which simplifies to

$$y_p(x) = -\frac{2\pi\sqrt{x}\left(\left(\int_0^x \operatorname{BesselY}\left(\frac{2}{3}, \frac{8i\alpha^{\frac{3}{4}}}{3}\right)\sin(\alpha)\,d\alpha\right)\operatorname{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) - \left(\int_0^x \operatorname{BesselJ}\left(\frac{2}{3}, \frac{8i\alpha^{\frac{3}{4}}}{3}\right)\sin(\alpha)\,d\alpha\right)\operatorname{Bess}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\sqrt{x}\operatorname{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) + c_2\sqrt{x}\operatorname{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\right) \\ &\quad + \left(-\frac{2\pi\sqrt{x}\left(\left(\int_0^x \operatorname{BesselY}\left(\frac{2}{3}, \frac{8i\alpha^{\frac{3}{4}}}{3}\right)\sin(\alpha)\,d\alpha\right)\operatorname{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) - \left(\int_0^x \operatorname{BesselJ}\left(\frac{2}{3}, \frac{8i\alpha^{\frac{3}{4}}}{3}\right)\sin(\alpha)\,d\alpha\right)}{3}\right. \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\sqrt{x}\operatorname{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) + c_2\sqrt{x}\operatorname{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) - \frac{2\pi\sqrt{x}\left(\left(\int_0^x \operatorname{BesselY}\left(\frac{2}{3}, \frac{8i\alpha^{\frac{3}{4}}}{3}\right)\sin(\alpha)\,d\alpha\right)\operatorname{B}\right.}{} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = -2$ in the above gives

$$3 = \frac{2i\sqrt{2}\left(\pi\operatorname{BesselJ}\left(\frac{2}{3}, \left(-\frac{8}{3} - \frac{8i}{3}\right)2^{\frac{1}{4}}\right)\left(\int_{-2}^0 \operatorname{BesselY}\left(\frac{2}{3}, \left(-\frac{4}{3} - \frac{4i}{3}\right)(-\alpha)^{\frac{3}{4}}\sqrt{2}\right)\sin(\alpha)\,d\alpha\right) - \pi\operatorname{BesselY}\left(\right.\right.}{} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1\operatorname{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)}{2\sqrt{x}} + 2ic_1x^{\frac{1}{4}}\left(\operatorname{BesselJ}\left(-\frac{1}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) + \frac{i\operatorname{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)}{4x^{\frac{3}{4}}}\right) + \frac{c_2\operatorname{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)}{2\sqrt{x}}$$

substituting $y' = -1$ and $x = -2$ in the above gives

$$-1 = \left(-\frac{2}{3}+\frac{2i}{3}\right) 2^{\frac{3}{4}} \left(\pi \left(\int_{-2}^{0} \text{BesselY}\left(\frac{2}{3}, \left(-\frac{4}{3}-\frac{4i}{3}\right) (-\alpha)^{\frac{3}{4}} \sqrt{2}\right) \sin(\alpha)\, d\alpha\right) \text{BesselJ}\left(-\frac{1}{3}, \left(-\frac{8}{3}-\frac{8i}{3}\right.\right.\right. \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{2\pi\left(3i2^{\frac{3}{4}} - 3\,2^{\frac{3}{4}}\right) \text{BesselY}\left(-\frac{1}{3}, \left(-\frac{8}{3}-\frac{8i}{3}\right) 2^{\frac{1}{4}}\right)}{3} + \frac{2i\pi\sqrt{2}\, \text{BesselY}\left(\frac{2}{3}, \left(-\frac{8}{3}-\frac{8i}{3}\right) 2^{\frac{1}{4}}\right)}{3} - \frac{2\pi\left(\int_{-2}^{0} \text{Besse}\right.}{}$$

$$c_2 = \frac{2\pi\left(-3i2^{\frac{3}{4}} + 3\,2^{\frac{3}{4}}\right) \text{BesselJ}\left(-\frac{1}{3}, \left(-\frac{8}{3}-\frac{8i}{3}\right) 2^{\frac{1}{4}}\right)}{3} - \frac{2i\pi\sqrt{2}\, \text{BesselJ}\left(\frac{2}{3}, \left(-\frac{8}{3}-\frac{8i}{3}\right) 2^{\frac{1}{4}}\right)}{3} + \frac{2\pi\left(\int_{-2}^{0} \text{Bess}\right.}{}$$

Substituting these values back in above solution results in

$$y = -2i\sqrt{x}\, \text{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) \pi 2^{\frac{3}{4}}\, \text{BesselJ}\left(-\frac{1}{3}, \left(-\frac{8}{3}-\frac{8i}{3}\right) 2^{\frac{1}{4}}\right) + 2i\sqrt{x}\, \text{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) \pi 2^{\frac{3}{4}}\, \text{Bess}$$

Summary

The solution(s) found are the following

$$y = 2\left(\frac{\operatorname{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\left(\int_{-2}^{0} \operatorname{BesselJ}\left(\frac{2}{3}, \left(-\frac{4}{3}-\frac{4i}{3}\right)(-\alpha)^{\frac{3}{4}}\sqrt{2}\right)\sin(\alpha)\, d\alpha\right)}{3} - \frac{\operatorname{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\left(\int_{-2}^{0} \operatorname{BesselY}\left(\frac{2}{3}, \left(-\frac{4}{3}-\frac{4i}{3}\right)(-\alpha)^{\frac{3}{4}}\sqrt{2}\right)\sin(\alpha)\, d\alpha\right)}{3} + \frac{\left(\int_{0}^{x} \operatorname{BesselJ}\left(\frac{2}{3}, \frac{8i\alpha^{\frac{3}{4}}}{3}\right)\sin(\alpha)\, d\alpha\right)\operatorname{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)}{3} - \frac{\left(\int_{0}^{x} \operatorname{BesselY}\left(\frac{2}{3}, \frac{8i\alpha^{\frac{3}{4}}}{3}\right)\sin(\alpha)\, d\alpha\right)\operatorname{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)}{3} + \left((-1+i)\, 2^{\frac{3}{4}} \operatorname{BesselY}\left(-\frac{1}{3}, \left(-\frac{8}{3}-\frac{8i}{3}\right)2^{\frac{1}{4}}\right) + \frac{i\sqrt{2}\,\operatorname{BesselY}\left(\frac{2}{3}, \left(-\frac{8}{3}-\frac{8i}{3}\right)2^{\frac{1}{4}}\right)}{3}\right)\operatorname{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) - \operatorname{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\left((-1+i)\, 2^{\frac{3}{4}} \operatorname{BesselJ}\left(-\frac{1}{3}, \left(-\frac{8}{3}-\frac{8i}{3}\right)2^{\frac{1}{4}}\right) + \frac{i\sqrt{2}\,\operatorname{BesselJ}\left(\frac{2}{3}, \left(-\frac{8}{3}-\frac{8i}{3}\right)2^{\frac{1}{4}}\right)}{3}\right)\right)\sqrt{x}\,\pi \tag{1}$$

Verification of solutions

$$
\begin{aligned}
y = 2\Bigg(& \frac{\text{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\left(\int_{-2}^{0}\text{BesselJ}\left(\frac{2}{3}, \left(-\frac{4}{3}-\frac{4i}{3}\right)(-\alpha)^{\frac{3}{4}}\sqrt{2}\right)\sin(\alpha)\,d\alpha\right)}{3} \\
& - \frac{\text{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\left(\int_{-2}^{0}\text{BesselY}\left(\frac{2}{3}, \left(-\frac{4}{3}-\frac{4i}{3}\right)(-\alpha)^{\frac{3}{4}}\sqrt{2}\right)\sin(\alpha)\,d\alpha\right)}{3} \\
& + \frac{\left(\int_{0}^{x}\text{BesselJ}\left(\frac{2}{3}, \frac{8i\alpha^{\frac{3}{4}}}{3}\right)\sin(\alpha)\,d\alpha\right)\text{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)}{3} \\
& - \frac{\left(\int_{0}^{x}\text{BesselY}\left(\frac{2}{3}, \frac{8i\alpha^{\frac{3}{4}}}{3}\right)\sin(\alpha)\,d\alpha\right)\text{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)}{3} \\
& + \Bigg((-1+i)\,2^{\frac{3}{4}}\,\text{BesselY}\left(-\frac{1}{3}, \left(-\frac{8}{3}-\frac{8i}{3}\right)2^{\frac{1}{4}}\right) \\
& + \frac{i\sqrt{2}\,\text{BesselY}\left(\frac{2}{3}, \left(-\frac{8}{3}-\frac{8i}{3}\right)2^{\frac{1}{4}}\right)}{3}\Bigg)\text{BesselJ}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right) \\
& - \text{BesselY}\left(\frac{2}{3}, \frac{8ix^{\frac{3}{4}}}{3}\right)\Bigg((-1+i)\,2^{\frac{3}{4}}\,\text{BesselJ}\left(-\frac{1}{3}, \left(-\frac{8}{3}-\frac{8i}{3}\right)2^{\frac{1}{4}}\right) \\
& + \frac{i\sqrt{2}\,\text{BesselJ}\left(\frac{2}{3}, \left(-\frac{8}{3}-\frac{8i}{3}\right)2^{\frac{1}{4}}\right)}{3}\Bigg)\Bigg)\sqrt{x}\,\pi
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
         <- hyper3 successful: received ODE is equivalent to the 0F1 ODE
      <- Whittaker successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.843 (sec). Leaf size: 185

```
dsolve([sqrt(1-x)*diff(y(x),x$2)-4*y(x)=sin(x),y(-2) = 3, D(y)(-2) = -1],y(x), singsol=all)
```

$$y(x) = \frac{4\left(\left((1-x)^{\frac{3}{2}}\right)^{\frac{2}{3}}\left(\left(\left(\int_{-2}^{x}\frac{\operatorname{BesselI}\left(\frac{2}{3},\frac{8\sqrt{\left(-\_z1+1\right)^{\frac{3}{2}}}}{3}\right)\sqrt{-\_z1+1}\,\sin\left(\_z1\right)}{\left(\left(-\_z1+1\right)^{\frac{3}{2}}\right)^{\frac{1}{3}}}d\_z1\right)\sqrt{3}+6\,3^{\frac{3}{4}}\operatorname{BesselI}\left(-\frac{1}{3},\frac{8\,3^{\frac{3}{4}}}{3}\right)\cdots\right.\right.\right.}{\cdots}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{Sqrt[1-x]*y''[x]-4*y[x]==Sin[x],{y[-2]==3,y'[-2]==-1}},y[x],x,IncludeSingularSolutio
```

Not solved

9.6 problem 6

Internal problem ID [12749]
Internal file name [`OUTPUT/11401_Friday_November_03_2023_06_32_33_AM_59301967/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

Unable to solve or complete the solution.

$$\boxed{(x^2-4)\, y''+y\ln(x)=x\,\mathrm{e}^{x}}$$

With initial conditions

$$[y(1)=1, y'(1)=2]$$

9.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned}p(x)&=0\\ q(x)&=\frac{\ln(x)}{x^2-4}\\ F&=\frac{x\,\mathrm{e}^{x}}{x^2-4}\end{aligned}$$

Hence the ode is

$$y''+\frac{\ln(x)\,y}{x^2-4}=\frac{x\,\mathrm{e}^{x}}{x^2-4}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{\ln(x)}{x^2-4}$ is

$$\{0 < x < 2, 2 < x \leq \infty\}$$

And the point $x_0 = 1$ is also inside this domain. The domain of $F = \frac{x\,e^x}{x^2-4}$ is

$$\{-\infty \leq x < -2, -2 < x < 2, 2 < x \leq \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
   <- unable to find a useful change of variables
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
      -> trying reduction of order to Riccati
         trying Riccati sub-methods:
            trying Riccati_symmetries
            -> trying a symmetry pattern of the form [F(x)*G(y), 0]
            -> trying a symmetry pattern of the form [0, F(x)*G(y)]
            -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5`
```

✗ Solution by Maple

```
dsolve([(x^2-4)*diff(y(x),x$2)+ln(x)*y(x)=x*exp(x),y(1) = 1, D(y)(1) = 2],y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{(x^2-4)*y''[x]+Log[x]*y[x]==x*Exp[x],{y[1]==1,y'[1]==2}},y[x],x,IncludeSingularSolut
```

Not solved

9.7 problem 7

Internal problem ID [12750]
Internal file name [`OUTPUT/11402_Friday_November_03_2023_06_32_34_AM_69342231/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - y = 0}$$

9.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\lambda_{1,2} = \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}$$
$$= \pm 1$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = e^x c_1 + c_2 e^{-x}$$

Summary
The solution(s) found are the following

$$y = e^x c_1 + c_2 e^{-x} \tag{1}$$

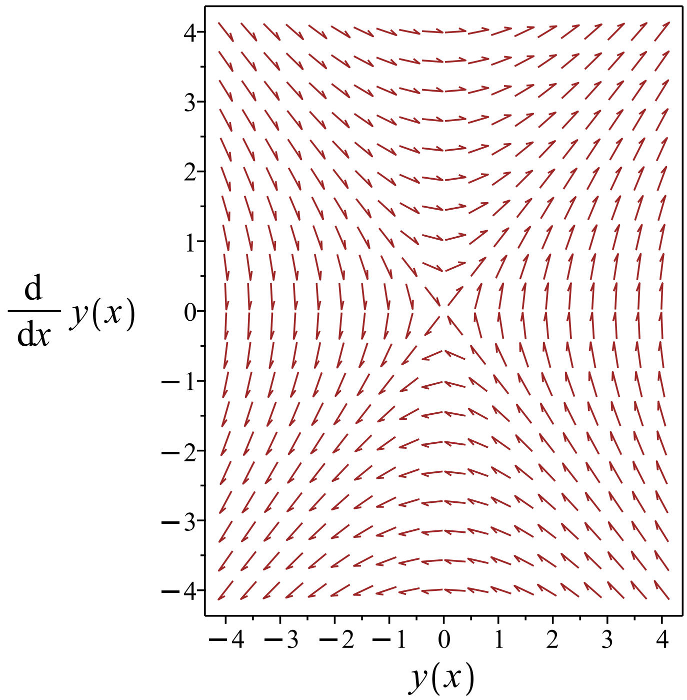


Figure 361: Slope field plot

Verification of solutions

$$y = e^{x} c_1 + c_2 e^{-x}$$

Verified OK.

9.7.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - yy')\,dx = 0$$
$$\frac{y'^2}{2} - \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{y^2+2c_1}}dy = \int dx$$
$$\ln\left(y+\sqrt{y^2+2c_1}\right) = x+c_2$$

Raising both side to exponential gives

$$y+\sqrt{y^2+2c_1} = e^{x+c_2}$$

Which simplifies to

$$y+\sqrt{y^2+2c_1} = c_3 e^{x}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{y^2+2c_1}}dy = \int dx$$
$$-\ln\left(y+\sqrt{y^2+2c_1}\right) = x+c_4$$

Raising both side to exponential gives

$$\frac{1}{y+\sqrt{y^2+2c_1}} = e^{x+c_4}$$

Which simplifies to

$$\frac{1}{y+\sqrt{y^2+2c_1}} = c_5 e^{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(e^{2x}c_3^2-2c_1\right)e^{-x}}{2c_3} \tag{1}$$
$$y = -\frac{\left(2c_1c_5^2e^{2x}-1\right)e^{-x}}{2c_5} \tag{2}$$

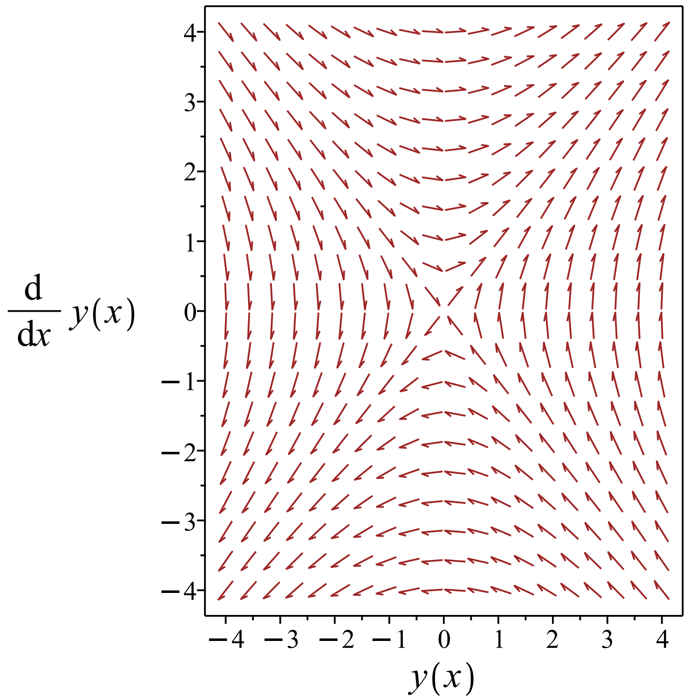


Figure 362: Slope field plot

Verification of solutions

$$y = \frac{\left(e^{2x} c_3^2 - 2c_1\right) e^{-x}}{2c_3}$$

Verified OK.

$$y = -\frac{\left(2c_1 c_5^2 e^{2x} - 1\right) e^{-x}}{2c_5}$$

Verified OK.

9.7.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 313: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \mathrm{e}^{-x}\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\ &= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2 \mathrm{e}^{x}}{2} + c_1 \mathrm{e}^{-x} \tag{1}$$

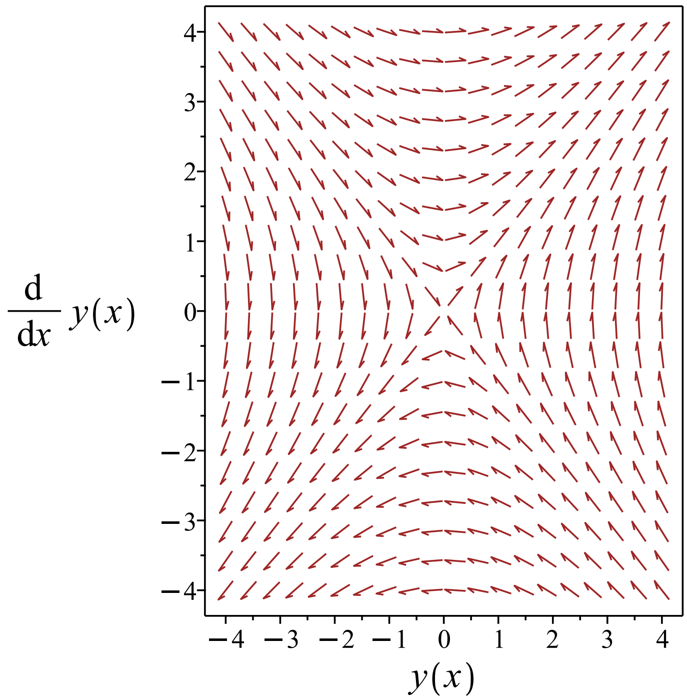


Figure 363: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{c_2 e^x}{2} + c_1 e^{-x}$$

Verified OK.

9.7.4 Maple step by step solution

Let's solve

$y'' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the ODE

$$y_2(x) = e^{x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{-x} + c_2 e^{x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x} + c_2 e^{x}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 20

```
DSolve[y''[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x} + c_2 e^{-x}$$

9.8 problem 8

Internal problem ID [12751]
Internal file name [`OUTPUT/11403_Friday_November_03_2023_06_32_34_AM_64520764/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' + y = 0}$$

9.8.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

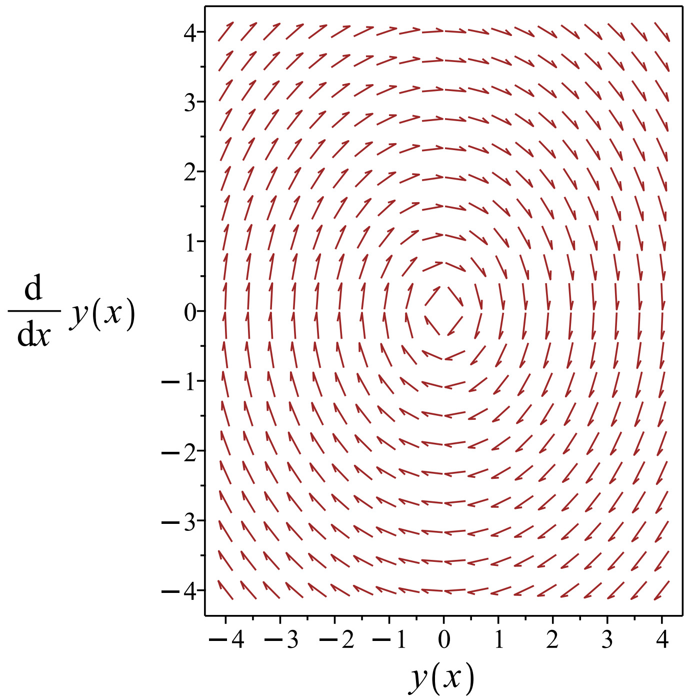


Figure 364: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Verified OK.

9.8.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + yy')\, dx = 0$$
$$\frac{y'^2}{2} + \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2 + 2c_1}} dy = \int dx$$
$$\arctan\left(\frac{y}{\sqrt{-y^2 + 2c_1}}\right) = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-y^2 + 2c_1}} dy = \int dx$$
$$-\arctan\left(\frac{y}{\sqrt{-y^2 + 2c_1}}\right) = c_3 + x$$

Summary

The solution(s) found are the following

$$\arctan\left(\frac{y}{\sqrt{-y^2 + 2c_1}}\right) = x + c_2 \tag{1}$$
$$-\arctan\left(\frac{y}{\sqrt{-y^2 + 2c_1}}\right) = c_3 + x \tag{2}$$

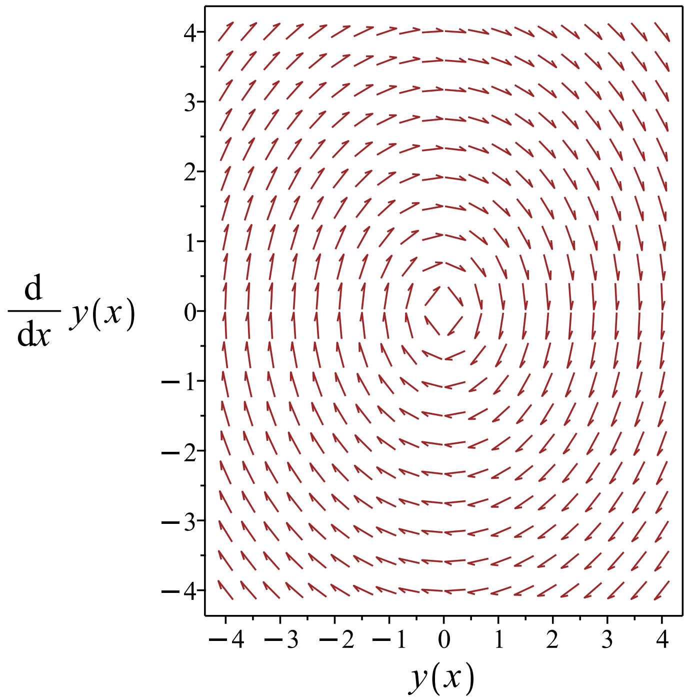


Figure 365: Slope field plot

<u>Verification of solutions</u>

$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_2$$

Verified OK.

$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = c_3 + x$$

Verified OK.

9.8.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 315: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x) (\tan(x))) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

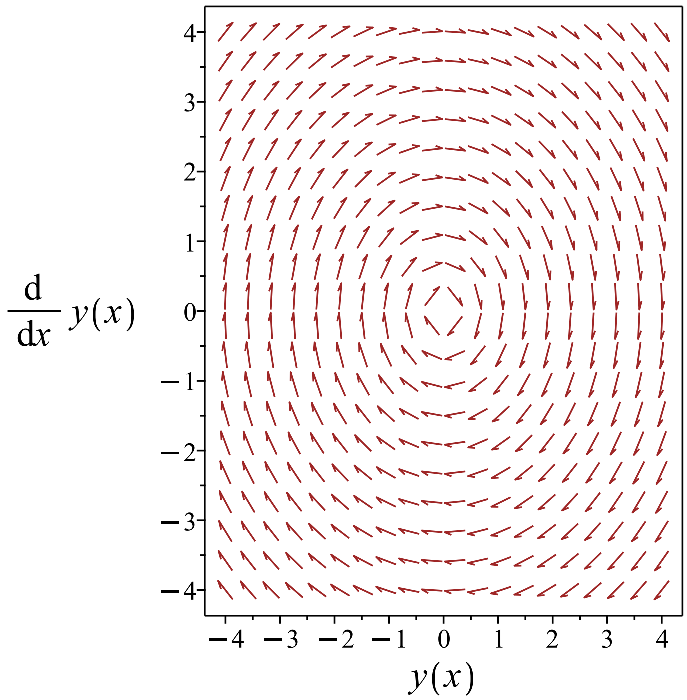


Figure 366: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Verified OK.

9.8.4 Maple step by step solution

Let's solve

$y'' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-I, I)$

- 1st solution of the ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(x) + c_2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x$2)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(x) + c_2 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 16

```
DSolve[y''[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(x) + c_2 \sin(x)$$

9.9 problem 9

Internal problem ID [12752]
Internal file name [`OUTPUT/11404_Friday_November_03_2023_06_32_35_AM_34093328/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y''x^2 + 2y'x - 2y = 0}$$

9.9.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 2xrx^{r-1} - 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 2r\,x^r - 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 2r - 2 = 0$$

Or

$$r^2 + r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2 \\ r_2 &= 1 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + c_2 x$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1}{x^2} + c_2 x \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1}{x^2} + c_2 x$$

Verified OK.

9.9.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$y''x^2 + 2y'x - 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= -\frac{2}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{2}{x}dx\right)}dx \\
&= \int e^{-2\ln(x)}\,dx \\
&= \int \frac{1}{x^2}dx \\
&= -\frac{1}{x}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2}{x^2}}{\frac{1}{x^4}} \\
&= -2x^2
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - 2x^2 y(\tau) = 0$$

But in terms of τ

$$-2x^2 = -\frac{2}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{2y(\tau)}{\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 2\tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r - 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= 2 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau} + c_2\tau^2$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{-c_1 x^3 + c_2}{x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{-c_1 x^3 + c_2}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{-c_1 x^3 + c_2}{x^2}$$

Verified OK.

9.9.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$y''x^2 + 2y'x - 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = -\frac{2}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{2n}{x^2} - \frac{2}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \frac{4v'(x)}{x} &= 0 \\
v''(x) + \frac{4v'(x)}{x} &= 0 \qquad (7)
\end{aligned}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \frac{4u(x)}{x} = 0 \qquad (8)
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{4u}{x}
\end{aligned}
$$

Where $f(x) = -\frac{4}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{4}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{4}{x}\, dx \\
\ln(u) &= -4\ln(x) + c_1 \\
u &= e^{-4\ln(x)+c_1} \\
&= \frac{c_1}{x^4}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{3x^3} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{3x^3} + c_2\right) x \\
&= \left(-\frac{c_1}{3x^3} + c_2\right) x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{3x^3} + c_2\right) x \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{3x^3} + c_2\right) x
$$

Verified OK.

9.9.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0
\end{aligned} \tag{1}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$AB u' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x \\ C &= -2 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2\right)(0) + (2x)(2) + (-2)(2x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$2x^3 v'' + \left(8x^2\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$2x^2 (u'(x)\, x + 4u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{4u}{x} \end{aligned}$$

Where $f(x) = -\frac{4}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{4}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{4}{x}\, dx \\ \ln(u) &= -4\ln(x) + c_1 \\ u &= \mathrm{e}^{-4\ln(x) + c_1} \\ &= \frac{c_1}{x^4} \end{aligned}$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= \frac{c_1}{x^4}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1}{x^4}\, \mathrm{d}x \\
&= -\frac{c_1}{3x^3} + c_2
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (2x)\left(-\frac{c_1}{3x^3} + c_2\right) \\
&= \frac{6c_2x^3 - 2c_1}{3x^2}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{6c_2x^3 - 2c_1}{3x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{6c_2x^3 - 2c_1}{3x^2}
$$

Verified OK.

9.9.5 Solving using Kovacic algorithm

Writing the ode as

$$
y''x^2 + 2y'x - 2y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= 2x \\
C &= -2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 317: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -1$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^3}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2}\right) + c_2 \left(\frac{1}{x^2}\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{c_1}{x^2} + \frac{c_2 x}{3} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1}{x^2} + \frac{c_2 x}{3}
$$

Verified OK.

9.9.6 Maple step by step solution

Let's solve

$y''x^2 + 2y'x - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} + \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} - \frac{2y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 + 2y'x - 2y = 0$

- Make a change of variables

  $t = \ln(x)$

- □ Substitute the change of variables back into the ODE

  - ○ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - ○ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - ○ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  - ○ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^2 + 2\frac{d}{dt}y(t) - 2y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) - 2y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-1)=0$

- Roots of the characteristic polynomial

  $r=(-2,1)$

- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-2t}$

- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{t}$

- General solution of the ODE

  $y(t)=c_1y_1(t)+c_2y_2(t)$

- Substitute in solutions

  $y(t)=c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{t}$

- Change variables back using $t=\ln(x)$

  $y=\frac{c_1}{x^2}+c_2x$

- Simplify

  $y=\frac{c_1}{x^2}+c_2x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1x^3+c_2}{x^2}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 16

```
DSolve[x^2*y''[x]+2*x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x^2} + c_2 x$$

9.10 problem 10

Internal problem ID [12753]
Internal file name [`OUTPUT/11405_Friday_November_03_2023_06_32_36_AM_25067696/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{2yy'' - y'^2 = 0}$$

9.10.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$2yp(y)\left(\frac{d}{dy}p(y)\right) - p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= \frac{p}{2y}
\end{aligned}$$

Where $f(y) = \frac{1}{2y}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= \frac{1}{2y}\,dy \\
\int \frac{1}{p}\,dp &= \int \frac{1}{2y}\,dy \\
\ln(p) &= \frac{\ln(y)}{2} + c_1 \\
p &= e^{\frac{\ln(y)}{2}+c_1} \\
&= c_1\sqrt{y}
\end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1\sqrt{y}$$

Integrating both sides gives

$$\int \frac{1}{c_1\sqrt{y}}dy = \int dx$$

$$\frac{2\sqrt{y}}{c_1} = x + c_2$$

Summary

The solution(s) found are the following

$$y = \frac{1}{4}c_1^2c_2^2 + \frac{1}{2}c_2c_1^2x + \frac{1}{4}c_1^2x^2 \tag{1}$$

Verification of solutions

$$y = \frac{1}{4}c_1^2c_2^2 + \frac{1}{2}c_2c_1^2x + \frac{1}{4}c_1^2x^2$$

Verified OK.

9.10.2 Maple step by step solution

Let's solve

$2yy'' - y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $2yu(y)\left(\frac{d}{dy}u(y)\right) - u(y)^2 = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)} = \frac{1}{2y}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)}dy = \int \frac{1}{2y}dy + c_1$

- Evaluate integral

  $\ln(u(y)) = \frac{\ln(y)}{2} + c_1$

- Solve for $u(y)$

  $\left\{u(y) = \frac{\sqrt{e^{-2c_1}y}}{e^{-2c_1}}, u(y) = -\frac{\sqrt{e^{-2c_1}y}}{e^{-2c_1}}\right\}$

- Solve 1st ODE for $u(y)$

  $u(y) = \frac{\sqrt{e^{-2c_1}y}}{e^{-2c_1}}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = \frac{\sqrt{e^{-2c_1}y}}{e^{-2c_1}}$

- Separate variables

  $\frac{y'}{\sqrt{e^{-2c_1}y}} = \frac{1}{e^{-2c_1}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{e^{-2c_1}y}}dx = \int \frac{1}{e^{-2c_1}}dx + c_2$

- Evaluate integral

  $\frac{2\sqrt{e^{-2c_1}y}}{e^{-2c_1}} = \frac{x}{e^{-2c_1}} + c_2$

- Solve for y

  $y = \frac{c_2^2(e^{-2c_1})^2+2c_2e^{-2c_1}x+x^2}{4\,e^{-2c_1}}$

- Solve 2nd ODE for $u(y)$

  $u(y) = -\frac{\sqrt{e^{-2c_1}y}}{e^{-2c_1}}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = -\frac{\sqrt{e^{-2c_1}y}}{e^{-2c_1}}$

- Separate variables

  $\frac{y'}{\sqrt{e^{-2c_1}y}} = -\frac{1}{e^{-2c_1}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{e^{-2c_1}y}}dx = \int -\frac{1}{e^{-2c_1}}dx + c_2$

- Evaluate integral

  $\frac{2\sqrt{e^{-2c_1}y}}{e^{-2c_1}} = -\frac{x}{e^{-2c_1}} + c_2$

- Solve for y

  $y = \frac{c_2^2(e^{-2c_1})^2-2c_2e^{-2c_1}x+x^2}{4\,e^{-2c_1}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 17

```
dsolve(2*y(x)*diff(y(x),x$2)-diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \frac{(c_1 x + c_2)^2}{4}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 29

```
DSolve[2*y[x]*y''[x]-(y'[x])^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(c_1 x + 2c_2)^2}{4c_2}$$
$$y(x) \to \text{Indeterminate}$$

9.11 problem 13

Internal problem ID [12754]
Internal file name [`OUTPUT/11406_Friday_November_03_2023_06_32_36_AM_70321041/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

9.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - y = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.11.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = \mathrm{e}^{x} c_1 + c_2 \mathrm{e}^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{x} c_1 + c_2 \mathrm{e}^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \mathrm{e}^{x} c_1 - c_2 \mathrm{e}^{-x}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2}$$
$$c_2 = -\frac{1}{2}$$

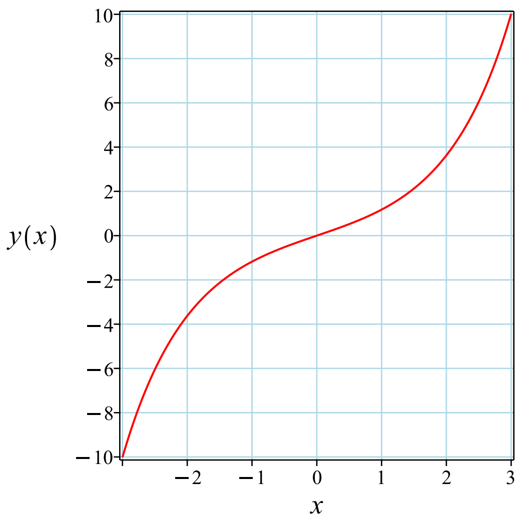
Substituting these values back in above solution results in

$$y = -\frac{\mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^{x}}{2}$$

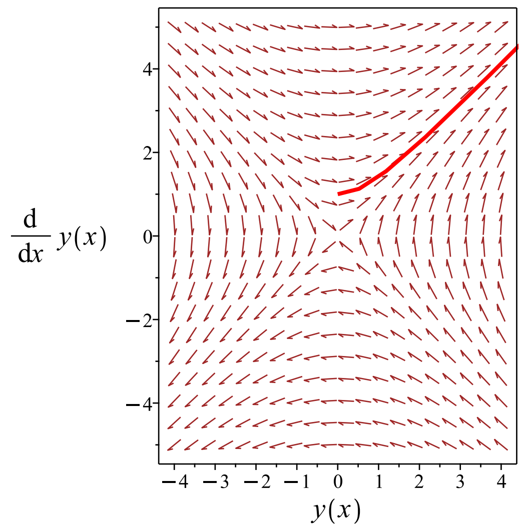
Summary

The solution(s) found are the following

$$y = -\frac{e^{-x}}{2} + \frac{e^{x}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-x}}{2} + \frac{e^{x}}{2}$$

Verified OK.

9.11.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - yy')\, dx = 0$$
$$\frac{y'^2}{2} - \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{y^2 + 2c_1}} dy = \int dx$$
$$\ln\left(y + \sqrt{y^2 + 2c_1}\right) = x + c_2$$

Raising both side to exponential gives

$$y + \sqrt{y^2 + 2c_1} = e^{x + c_2}$$

Which simplifies to

$$y + \sqrt{y^2 + 2c_1} = c_3 e^x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{y^2 + 2c_1}} dy = \int dx$$
$$-\ln\left(y + \sqrt{y^2 + 2c_1}\right) = x + c_4$$

Raising both side to exponential gives

$$\frac{1}{y + \sqrt{y^2 + 2c_1}} = e^{x + c_4}$$

Which simplifies to

$$\frac{1}{y + \sqrt{y^2 + 2c_1}} = c_5 e^x$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$y = \frac{\left(e^{2x} c_3^2 - 2c_1\right) e^{-x}}{2c_3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{c_3^2 - 2c_1}{2c_3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_3 e^x - \frac{(e^{2x} c_3^2 - 2c_1)\, e^{-x}}{2c_3}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \frac{c_3^2 + 2c_1}{2c_3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{1}{2} \\ c_3 &= 1 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{e^{-x}(e^{2x} - 1)}{2}$$

Which simplifies to

$$y = -\frac{e^{-x}}{2} + \frac{e^x}{2}$$

Looking at the Second solution

$$y = -\frac{(2c_1 c_5^2 e^{2x} - 1)\, e^{-x}}{2c_5} \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{-2c_1 c_5^2 + 1}{2c_5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 c_5 e^x + \frac{(2c_1 c_5^2 e^{2x} - 1)\, e^{-x}}{2c_5}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \frac{-2c_1 c_5^2 - 1}{2c_5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_5\}$. Solving for the constants gives

$$c_1 = \frac{1}{2}$$
$$c_5 = -1$$

Substituting these values back in above solution results in

$$y = \frac{e^{-x}(e^{2x} - 1)}{2}$$

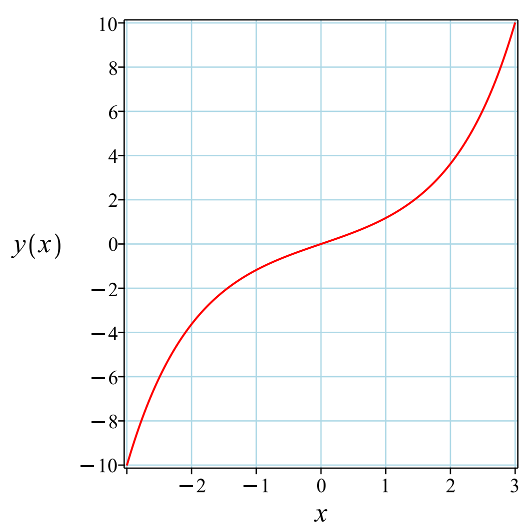
Which simplifies to

$$y = -\frac{e^{-x}}{2} + \frac{e^x}{2}$$

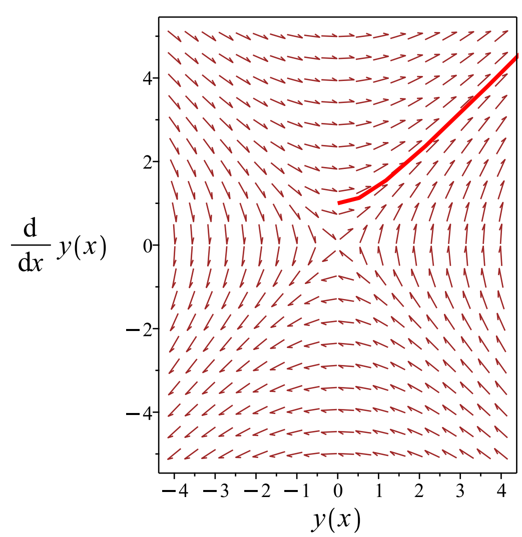
<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{e^{-x}}{2} + \frac{e^x}{2} \tag{1}$$
$$y = -\frac{e^{-x}}{2} + \frac{e^x}{2} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-x}}{2} + \frac{e^{x}}{2}$$

Verified OK.

$$y = -\frac{e^{-x}}{2} + \frac{e^{x}}{2}$$

Verified OK.

9.11.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 320: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\ &= \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_2 \mathrm{e}^{x}}{2} + c_1 \mathrm{e}^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{c_2}{2} + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_2 \mathrm{e}^{x}}{2} - c_1 \mathrm{e}^{-x}$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = \frac{c_2}{2} - c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

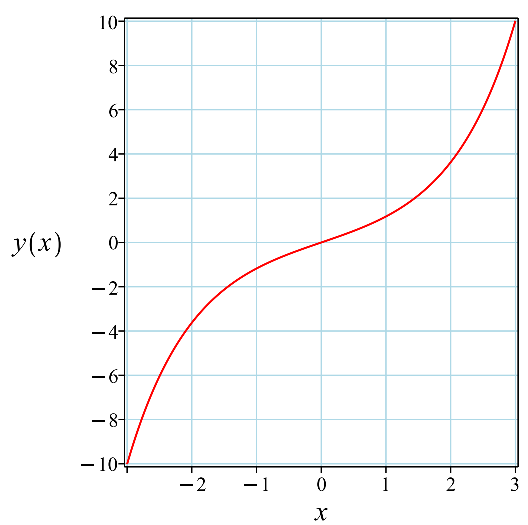
$$c_1 = -\frac{1}{2}$$
$$c_2 = 1$$

Substituting these values back in above solution results in

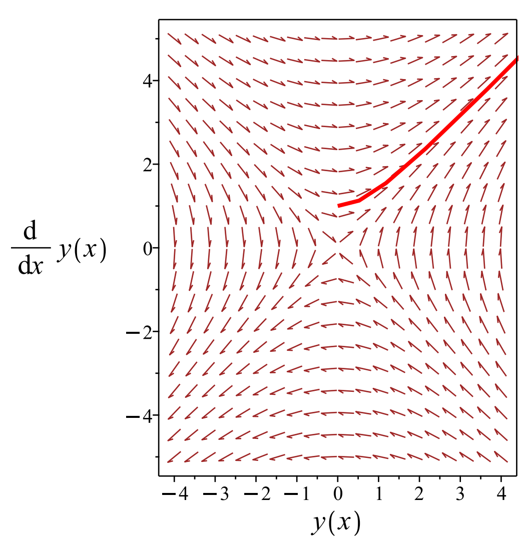
$$y = -\frac{\mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^{x}}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^{x}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^{x}}{2}$$

Verified OK.

9.11.5 Maple step by step solution

Let's solve

$$\left[y'' - y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x}$

□ Check validity of solution $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x}$

  ○ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2$

  ○ Compute derivative of the solution

  $y' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = -c_1 + c_2$
- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}\right\}$
- Substitute constant values into general solution and simplify

  $y = -\frac{e^{-x}}{2} + \frac{e^{x}}{2}$

- Solution to the IVP

  $y = -\frac{e^{-x}}{2} + \frac{e^{x}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([diff(y(x),x$2)-y(x)=0,y(0) = 0, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{e^{-x}}{2} + \frac{e^{x}}{2}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 21

```
DSolve[{y''[x]-y[x]==0,{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(e^{2x} - 1\right)$$

9.12 problem 14

Internal problem ID [12755]
Internal file name [OUTPUT/11407_Friday_November_03_2023_06_32_38_AM_49937730/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 14.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y' = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0, y''(0) = -1]$$

The characteristic equation is

$$\lambda^3 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + \mathrm{e}^{ix} c_2 + \mathrm{e}^{-ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = \mathrm{e}^{ix}$$
$$y_3 = \mathrm{e}^{-ix}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + e^{ix}c_2 + e^{-ix}c_3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = ie^{ix}c_2 - ie^{-ix}c_3$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = i(c_2 - c_3) \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = -e^{ix}c_2 - e^{-ix}c_3$$

substituting $y'' = -1$ and $x = 0$ in the above gives

$$-1 = -c_2 - c_3 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= \frac{1}{2} \\ c_3 &= \frac{1}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \cos(x)$$

Summary

The solution(s) found are the following

$$y = \cos(x) \tag{1}$$

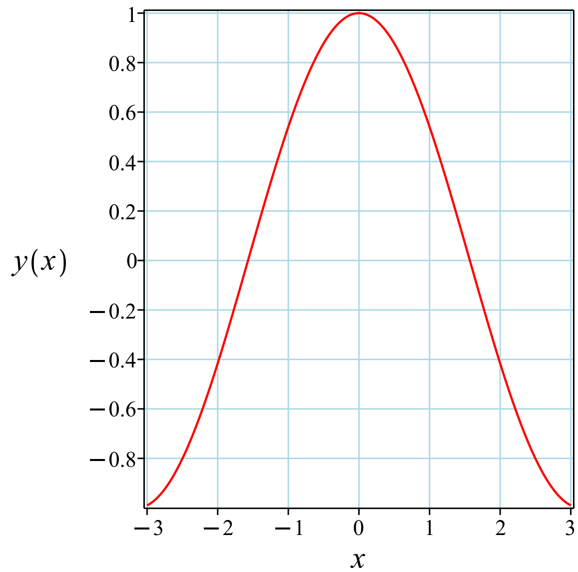


Figure 370: Solution plot

Verification of solutions

$$y = \cos(x)$$

Verified OK.

9.12.1 Maple step by step solution

Let's solve

$$\left[y''' + y' = 0, y(0) = 1, y'\Big|_{\{x=0\}} = 0, y''\Big|_{\{x=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ∘ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ∘ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ∘ Define new variable $y_3(x)$

    $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = -y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}x} \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \overrightarrow{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = \begin{bmatrix} c_3 \sin(x) - c_2 \cos(x) + c_1 \\ c_3 \cos(x) + c_2 \sin(x) \\ -c_3 \sin(x) + c_2 \cos(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = c_3 \sin(x) - c_2 \cos(x) + c_1$
- Use the initial condition $y(0) = 1$

  $1 = c_1 - c_2$
- Calculate the 1st derivative of the solution

  $y' = c_3 \cos(x) + c_2 \sin(x)$
- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = c_3$
- Calculate the 2nd derivative of the solution

  $y'' = -c_3 \sin(x) + c_2 \cos(x)$
- Use the initial condition $y''\Big|_{\{x=0\}} = -1$

  $-1 = c_2$
- Solve for the unknown coefficients

  $\{c_1 = 0, c_2 = -1, c_3 = 0\}$
- Solution to the IVP

  $y = \cos(x)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 6

```
dsolve([diff(y(x),x$3)+diff(y(x),x)=0,y(0) = 1, D(y)(0) = 0, (D@@2)(y)(0) = -1],y(x), singso
```

$$y(x) = \cos(x)$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 7

```
DSolve[{y'''[x]+y'[x]==0,{y[0]==1,y'[0]==0,y''[0]==-1}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \cos(x)$$

9.13 problem 15

Internal problem ID [12756]
Internal file name [`OUTPUT/11408_Friday_November_03_2023_06_32_38_AM_13755880/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y''x^2 - y'x + y = 0}$$

With initial conditions

$$[y(1) = 2, y'(1) = -1]$$

9.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{1}{x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{y'}{x} + \frac{y}{x^2} = 0$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{1}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

9.13.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r + 1 = 0$$

Or

$$r^2 - 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 1$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1 x + c_2 x \ln(x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x + c_2 x \ln(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 + \ln(x) c_2 + c_2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -3$$

Substituting these values back in above solution results in

$$y = -3x \ln(x) + 2x$$

Which simplifies to

$$y = (-3 \ln(x) + 2) x$$

Summary

The solution(s) found are the following

$$y = (-3\ln(x) + 2)x \tag{1}$$

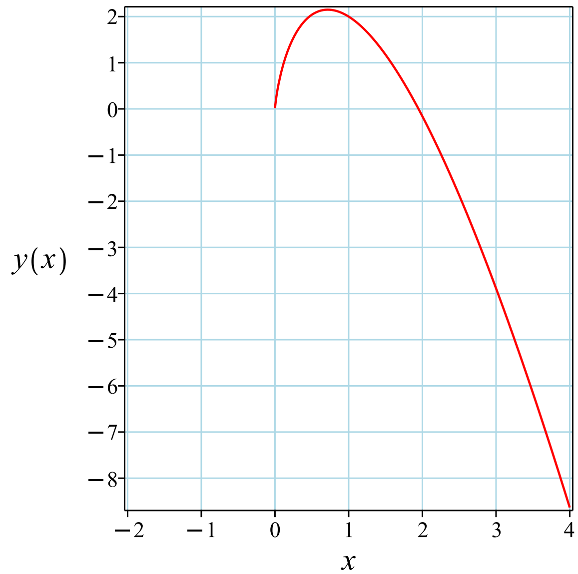


Figure 371: Solution plot

Verification of solutions

$$y = (-3\ln(x) + 2)x$$

Verified OK.

9.13.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$y''x^2 - y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-(\int p(x)dx)} dx \\
&= \int \mathrm{e}^{-(\int -\frac{1}{x} dx)} dx \\
&= \int e^{\ln(x)}\, dx \\
&= \int x dx \\
&= \frac{x^2}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2}}{x^2} \\
&= \frac{1}{x^4}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + \frac{y(\tau)}{x^4} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{1}{x^4} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,x(c_1 - c_2\ln(2) + 2\ln(x)\,c_2)}{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{\sqrt{2}\, x(c_1 - c_2 \ln(2) + 2\ln(x)\, c_2)}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$2 = \frac{(c_1 - c_2 \ln(2))\, \sqrt{2}}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\sqrt{2}\,(c_1 - c_2 \ln(2) + 2\ln(x)\, c_2)}{2} + \sqrt{2}\, c_2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = \frac{(-c_2 \ln(2) + c_1 + 2c_2)\, \sqrt{2}}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{(3\ln(2) - 4)\, \sqrt{2}}{2}$$
$$c_2 = -\frac{3\sqrt{2}}{2}$$

Substituting these values back in above solution results in

$$y = -3x \ln(x) + 2x$$

Which simplifies to

$$y = (-3\ln(x) + 2)\, x$$

Summary

The solution(s) found are the following

$$y = (-3\ln(x) + 2)\, x \tag{1}$$

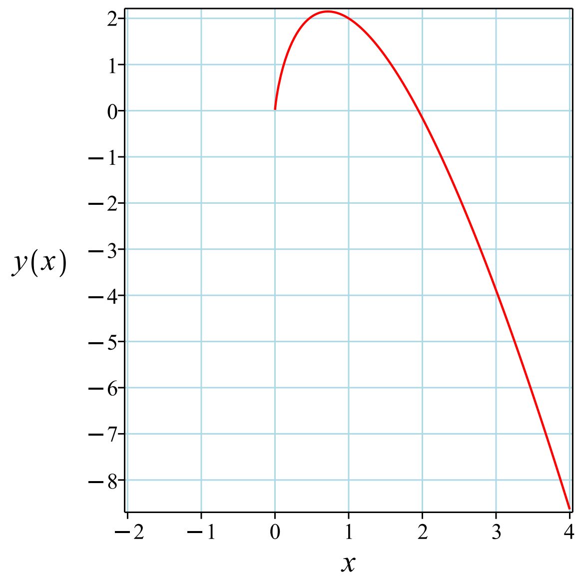


Figure 372: Solution plot

Verification of solutions

$$y = (-3\ln(x) + 2)\, x$$

Verified OK.

9.13.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$y''x^2 - y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{1}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -2c \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{c\tau} c_1$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\ &= \frac{\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

9.13.5 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$y''x^2 - y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= (c_1 \ln(x) + c_2)\,x \\
&= (c_1 \ln(x) + c_2)\,x
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (c_1 \ln(x) + c_2)\,x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$2 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 + c_1 \ln(x) + c_2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -3$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = -3x\ln(x) + 2x$$

Which simplifies to

$$y = (-3\ln(x) + 2)\,x$$

Summary

The solution(s) found are the following

$$y = (-3\ln(x) + 2)\,x \tag{1}$$

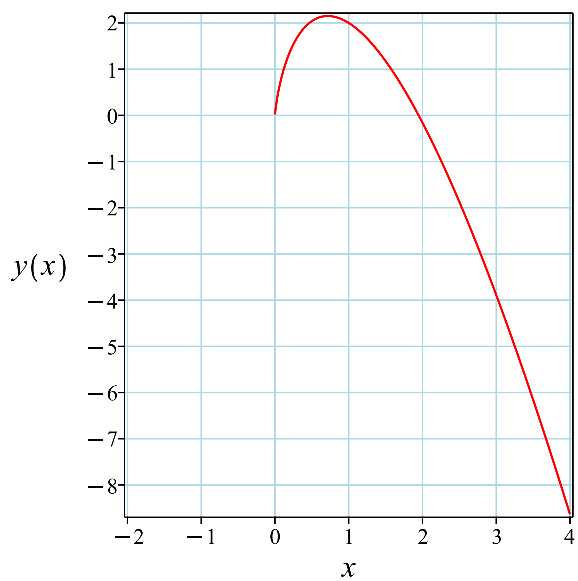


Figure 373: Solution plot

Verification of solutions

$$y = (-3\ln(x) + 2)\,x$$

Verified OK.

9.13.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= 1 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2\right)(0) + (-x)(-1) + (1)(-x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x^3v'' + (-x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$-x^2(u'(x)\, x + u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x}\, \mathrm{d}x \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (-x)\,(c_1 \ln(x) + c_2) \\ &= -(c_1 \ln(x) + c_2)\, x \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -(c_1 \ln(x) + c_2) x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$2 = -c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 - c_1 \ln(x) - c_2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = -c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 3 \\ c_2 &= -2 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -3x \ln(x) + 2x$$

Which simplifies to

$$y = (-3 \ln(x) + 2) x$$

Summary
The solution(s) found are the following

$$y = (-3 \ln(x) + 2) x \tag{1}$$

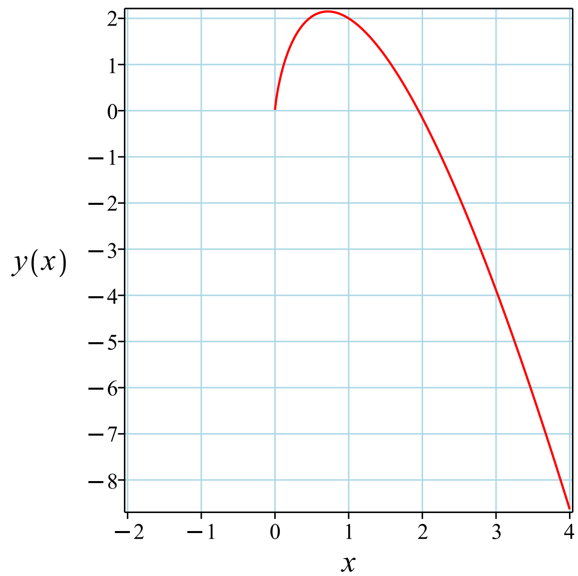


Figure 374: Solution plot

Verification of solutions

$$y = (-3\ln(x) + 2)\, x$$

Verified OK.

9.13.7 Solving using Kovacic algorithm

Writing the ode as

$$y''x^2 - y'x + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 323: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1 \left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2(x(\ln(x)))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 x + c_2 x \ln(x) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$
2 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 + \ln(x)\, c_2 + c_2
$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$
-1 = c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= -3
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -3x \ln(x) + 2x
$$

Which simplifies to

$$y = (-3\ln(x) + 2)\,x$$

Summary

The solution(s) found are the following

$$y = (-3\ln(x) + 2)\,x \tag{1}$$

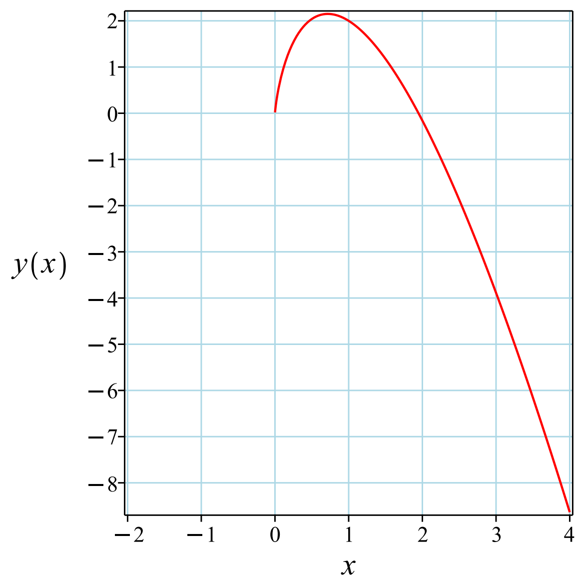


Figure 375: Solution plot

Verification of solutions

$$y = (-3\ln(x) + 2)\,x$$

Verified OK.

9.13.8 Maple step by step solution

Let's solve

$$\left[y''x^2 - y'x + y = 0, y(1) = 2, y'\Big|_{\{x=1\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Isolate 2nd derivative

$y'' = \frac{y'}{x} - \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{y'}{x} + \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

$y''x^2 - y'x + y = 0$

- Make a change of variables

$t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - \frac{d}{dt}y(t) + y(t) = 0$

- Simplify

$\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + y(t) = 0$

- Characteristic polynomial of ODE

$r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

$(r-1)^2 = 0$

- Root of the characteristic polynomial

$r = 1$

- 1st solution of the ODE

$y_1(t) = e^t$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^t$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^t + c_2 t\,\mathrm{e}^t$
- Change variables back using $t = \ln(x)$

  $y = c_1 x + c_2 x \ln(x)$
- Simplify

  $y = x(c_1 + \ln(x)\, c_2)$

☐ Check validity of solution $y = x(c_1 + \ln(x)\, c_2)$

  - Use initial condition $y(1) = 2$

    $2 = c_1$
  - Compute derivative of the solution

    $y' = c_1 + \ln(x)\, c_2 + c_2$
  - Use the initial condition $y'\Big|_{\{x=1\}} = -1$

    $-1 = c_1 + c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 2, c_2 = -3\}$
  - Substitute constant values into general solution and simplify

    $y = (-3\ln(x) + 2)\, x$
- Solution to the IVP

  $y = (-3\ln(x) + 2)\, x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve([x^2*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(1) = 2, D(y)(1) = -1],y(x), singsol=all)
```

$$y(x) = x(2 - 3\ln(x))$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 13

```
DSolve[{x^2*y''[x]-x*y'[x]+y[x]==0,{y[1]==2,y'[1]==-1}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \rightarrow x(2 - 3\log(x))$$

9.14 problem 16

Internal problem ID [12757]
Internal file name [`OUTPUT/11409_Friday_November_03_2023_06_32_40_AM_99045612/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y = 31}$$

With initial conditions

$$[y(0) = -9, y'(0) = 6]$$

9.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -4 \\ F &= 31 \end{aligned}$$

Hence the ode is

$$y'' - 4y = 31$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 31$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.14.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -4, f(x) = 31$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-4)} \\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2 \\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 = 31$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{31}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{31}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{-2x}\right) + \left(-\frac{31}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{31}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -9$ and $x = 0$ in the above gives

$$-9 = c_1 + c_2 - \frac{31}{4} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

substituting $y' = 6$ and $x = 0$ in the above gives

$$6 = 2c_1 - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{7}{8}$$
$$c_2 = -\frac{17}{8}$$

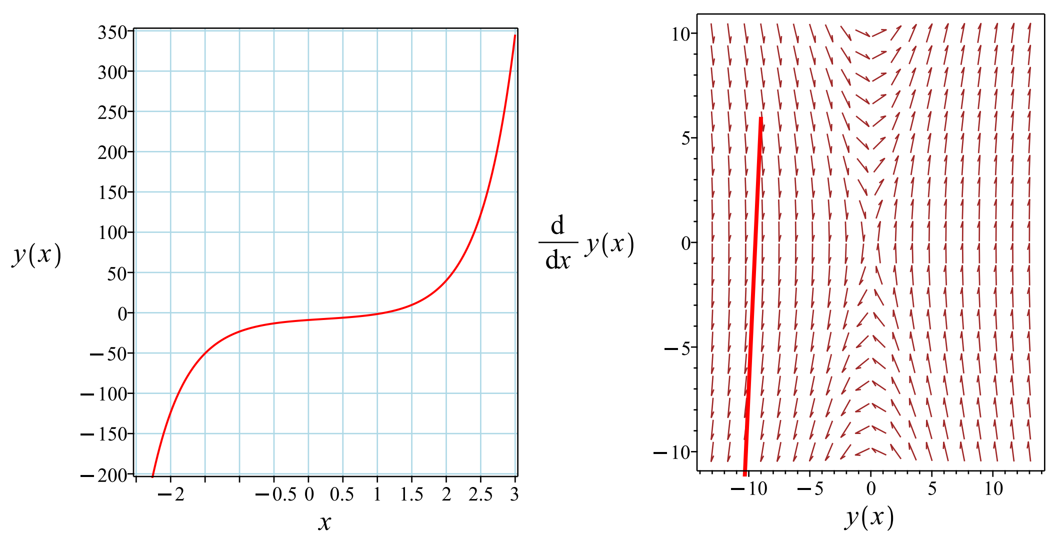
Substituting these values back in above solution results in

$$y = -\frac{31}{4} + \frac{7\,e^{2x}}{8} - \frac{17\,e^{-2x}}{8}$$

Summary

The solution(s) found are the following

$$y = -\frac{31}{4} + \frac{7\,e^{2x}}{8} - \frac{17\,e^{-2x}}{8} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{31}{4} + \frac{7\,e^{2x}}{8} - \frac{17\,e^{-2x}}{8}$$

Verified OK.

9.14.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - 4yy' - 31y' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - 4yy' - 31y')\, dx = 0$$
$$\frac{y'^2}{2} - 2y^2 - 31y = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{4y^2 + 62y + 2c_1} \tag{1}$$
$$y' = -\sqrt{4y^2 + 62y + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{4y^2 + 2c_1 + 62y}} dy = \int dx$$
$$\frac{\ln\left(\frac{(4y+31)\sqrt{4}}{4} + \sqrt{4y^2 + 2c_1 + 62y}\right)\sqrt{4}}{4} = x + c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln\left(\frac{(4y+31)\sqrt{4}}{4} + \sqrt{4y^2 + 2c_1 + 62y}\right)\sqrt{4}}{4}} = e^{x+c_2}$$

Which simplifies to

$$\frac{\sqrt{2}\,\sqrt{4y + 31 + 2\sqrt{4y^2 + 2c_1 + 62y}}}{2} = c_3 e^x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{4y^2 + 2c_1 + 62y}} dy = \int dx$$
$$-\frac{\ln\left(\frac{(4y+31)\sqrt{4}}{4} + \sqrt{4y^2 + 2c_1 + 62y}\right)\sqrt{4}}{4} = x + c_4$$

Raising both side to exponential gives

$$e^{-\frac{\ln\left(\frac{(4y+31)\sqrt{4}}{4}+\sqrt{4y^2+2c_1+62y}\right)\sqrt{4}}{4}} = e^{x+c_4}$$

Which simplifies to

$$\frac{\sqrt{2}}{\sqrt{4y+31+2\sqrt{4y^2+2c_1+62y}}} = c_5 e^x$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$y = \frac{\left(4\,e^{4x} c_3^4 - 124\,e^{2x} c_3^2 - 8c_1 + 961\right) e^{-2x}}{16c_3^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -9$ and $x = 0$ in the above gives

$$-9 = \frac{4c_3^4 - 124c_3^2 - 8c_1 + 961}{16c_3^2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\left(16\,e^{4x} c_3^4 - 248\,e^{2x} c_3^2\right) e^{-2x}}{16c_3^2} - \frac{\left(4\,e^{4x} c_3^4 - 124\,e^{2x} c_3^2 - 8c_1 + 961\right) e^{-2x}}{8c_3^2}$$

substituting $y' = 6$ and $x = 0$ in the above gives

$$6 = \frac{4c_3^4 + 8c_1 - 961}{8c_3^2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

Looking at the Second solution

$$y = -\frac{\left(8c_1 c_5^4 e^{4x} - 961 c_5^4 e^{4x} + 124 c_5^2 e^{2x} - 4\right) e^{-2x}}{16c_5^2} \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -9$ and $x = 0$ in the above gives

$$-9 = \frac{4 + (-8c_1 + 961)\, c_5^4 - 124c_5^2}{16c_5^2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\left(32c_1c_5^4\mathrm{e}^{4x} - 3844c_5^4\mathrm{e}^{4x} + 248c_5^2\mathrm{e}^{2x}\right)\mathrm{e}^{-2x}}{16c_5^2} + \frac{\left(8c_1c_5^4\mathrm{e}^{4x} - 961c_5^4\mathrm{e}^{4x} + 124c_5^2\mathrm{e}^{2x} - 4\right)\mathrm{e}^{-2x}}{8c_5^2}$$

substituting $y' = 6$ and $x = 0$ in the above gives

$$6 = \frac{-4 + (-8c_1 + 961)\,c_5^4}{8c_5^2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_5\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

9.14.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 325: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{-2x} \end{aligned}$$

Which simplifies to

$$y_1 = e^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= e^{-2x} \int \frac{1}{e^{-4x}}\,dx \\ &= e^{-2x}\left(\frac{e^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{-2x}\right) + c_2\left(e^{-2x}\left(\frac{e^{4x}}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{2x}}{4}, e^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 = 31$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{31}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{31}{4}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}\right) + \left(-\frac{31}{4}\right)\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} - \frac{31}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -9$ and $x = 0$ in the above gives

$$-9 = c_1 + \frac{c_2}{4} - \frac{31}{4} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2x} + \frac{c_2 e^{2x}}{2}$$

substituting $y' = 6$ and $x = 0$ in the above gives

$$6 = -2c_1 + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}c_1 &= -\frac{17}{8} \\ c_2 &= \frac{7}{2}\end{aligned}$$

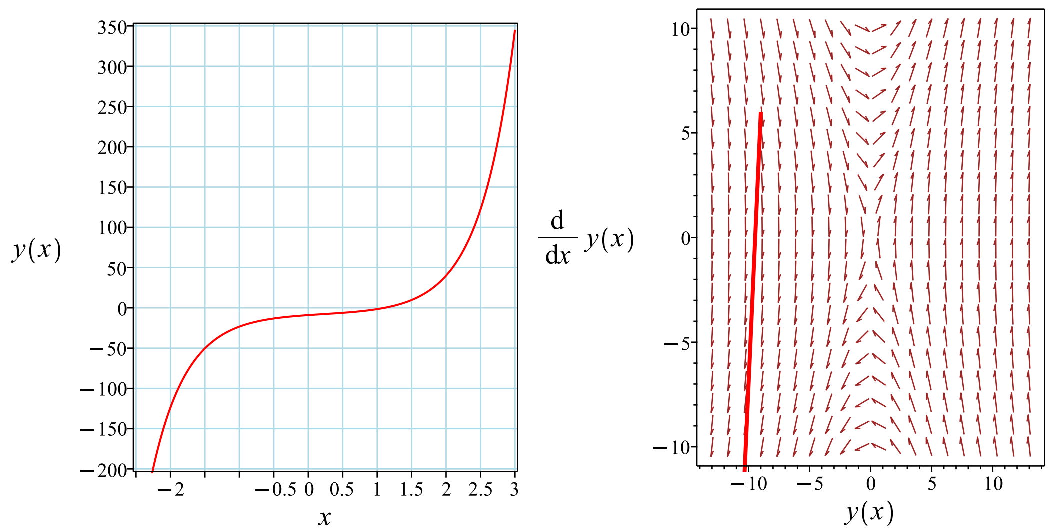
Substituting these values back in above solution results in

$$y = -\frac{31}{4} + \frac{7\,e^{2x}}{8} - \frac{17\,e^{-2x}}{8}$$

Summary

The solution(s) found are the following

$$y = -\frac{31}{4} + \frac{7\,e^{2x}}{8} - \frac{17\,e^{-2x}}{8} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{31}{4} + \frac{7\,e^{2x}}{8} - \frac{17\,e^{-2x}}{8}$$

Verified OK.

9.14.5 Maple step by step solution

Let's solve

$\left[y'' - 4y = 31, y(0) = -9, y'\Big|_{\{x=0\}} = 6\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

$(r-2)(r+2)=0$

- Roots of the characteristic polynomial

  $r=(-2,2)$

- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\mathrm{e}^{2x}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{2x}+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=31\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x),y_2(x))=\begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{2x} \\ -2\,\mathrm{e}^{-2x} & 2\,\mathrm{e}^{2x} \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x),y_2(x))=4$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x)=-\frac{31\,\mathrm{e}^{-2x}\left(\int \mathrm{e}^{2x}dx\right)}{4}+\frac{31\,\mathrm{e}^{2x}\left(\int \mathrm{e}^{-2x}dx\right)}{4}$

  ◦ Compute integrals

    $y_p(x)=-\frac{31}{4}$

- Substitute particular solution into general solution to ODE

  $y=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{2x}-\frac{31}{4}$

□ Check validity of solution $y=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{2x}-\frac{31}{4}$

  ◦ Use initial condition $y(0)=-9$

    $-9=c_1+c_2-\frac{31}{4}$

  ◦ Compute derivative of the solution

$$y' = -2c_1 e^{-2x} + 2c_2 e^{2x}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 6$

$$6 = -2c_1 + 2c_2$$

- Solve for c_1 and c_2

$$\left\{c_1 = -\frac{17}{8}, c_2 = \frac{7}{8}\right\}$$

- Substitute constant values into general solution and simplify

$$y = -\frac{31}{4} + \frac{7\,e^{2x}}{8} - \frac{17\,e^{-2x}}{8}$$

- Solution to the IVP

$$y = -\frac{31}{4} + \frac{7\,e^{2x}}{8} - \frac{17\,e^{-2x}}{8}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve([diff(y(x),x$2)-4*y(x)=31,y(0) = -9, D(y)(0) = 6],y(x), singsol=all)
```

$$y(x) = \frac{7\,e^{2x}}{8} - \frac{17\,e^{-2x}}{8} - \frac{31}{4}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 25

```
DSolve[{y''[x]-4*y[x]==31,{y[0]==-9,y'[0]==6}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}\left(-17e^{-2x} + 7e^{2x} - 62\right)$$

9.15 problem 17

Internal problem ID [12758]
Internal file name [`OUTPUT/11410_Friday_November_03_2023_06_32_42_AM_5773826/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 9y = 27x + 18}$$

With initial conditions

$$[y(0) = 23, y'(0) = 21]$$

9.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 9 \\ F &= 27x + 18 \end{aligned}$$

Hence the ode is

$$y'' + 9y = 27x + 18$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 27x + 18$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.15.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = 27x + 18$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)} \\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i \\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i \\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_2 x + 9A_1 = 27x + 18$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x + 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3x) + c_2 \sin(3x)) + (3x + 2) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(3x) + c_2 \sin(3x) + 3x + 2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 23$ and $x = 0$ in the above gives

$$23 = c_1 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 \sin(3x) + 3c_2 \cos(3x) + 3$$

substituting $y' = 21$ and $x = 0$ in the above gives

$$21 = 3 + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 21$$
$$c_2 = 6$$

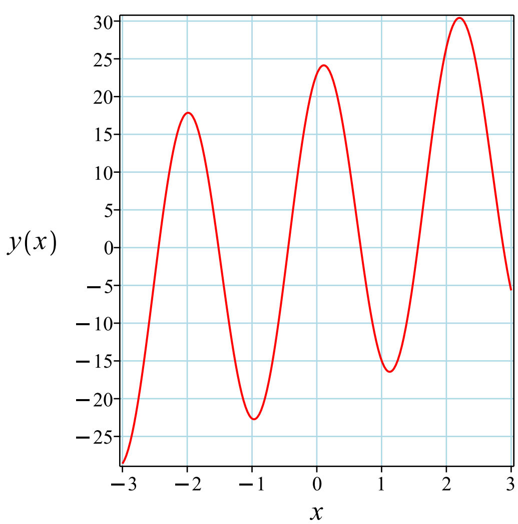
Substituting these values back in above solution results in

$$y = 2 + 21 \cos(3x) + 6 \sin(3x) + 3x$$

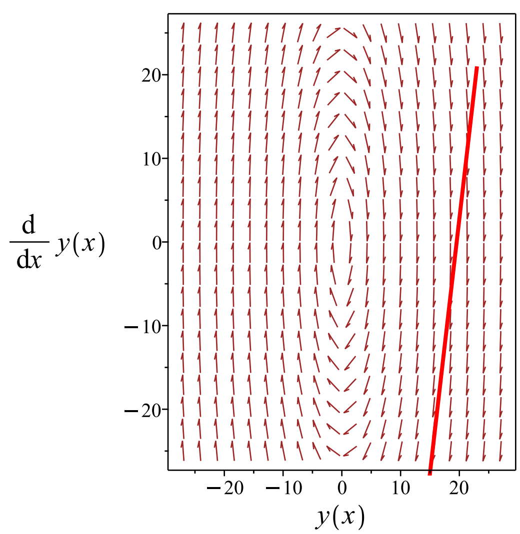
Summary

The solution(s) found are the following

$$y = 2 + 21 \cos(3x) + 6 \sin(3x) + 3x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2 + 21 \cos(3x) + 6 \sin(3x) + 3x$$

Verified OK.

9.15.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 327: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2}\, dx \\ &= \cos(3x) \left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_2 x + 9A_1 = 27x + 18$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x + 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + (3x + 2) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + 3x + 2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 23$ and $x = 0$ in the above gives

$$23 = c_1 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 \sin(3x) + c_2 \cos(3x) + 3$$

substituting $y' = 21$ and $x = 0$ in the above gives

$$21 = c_2 + 3 \tag{2A}$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 21$$
$$c_2 = 18$$

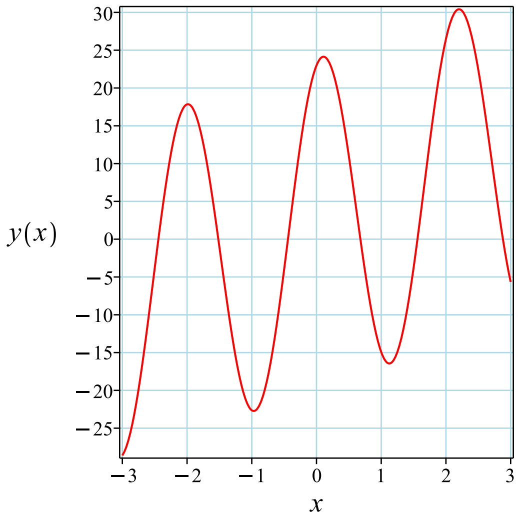
Substituting these values back in above solution results in

$$y = 2 + 21\cos(3x) + 6\sin(3x) + 3x$$

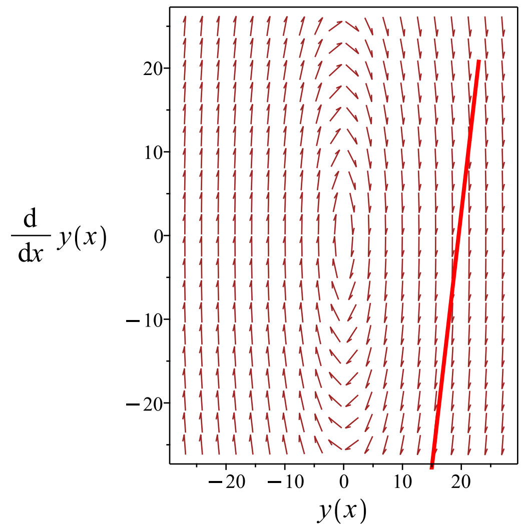
Summary

The solution(s) found are the following

$$y = 2 + 21\cos(3x) + 6\sin(3x) + 3x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2 + 21\cos(3x) + 6\sin(3x) + 3x$$

Verified OK.

9.15.4 Maple step by step solution

Let's solve

$\left[y'' + 9y = 27x + 18, y(0) = 23, y'\Big|_{\{x=0\}} = 21\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 27x + 18\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{array}\right]$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -3\cos(3x)\left(\int \sin(3x)(3x+2)\,dx\right) + 3\sin(3x)\left(\int \cos(3x)(3x+2)\,dx\right)$

- Compute integrals

  $y_p(x) = 3x + 2$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + 3x + 2$

□ Check validity of solution $y = c_1 \cos(3x) + c_2 \sin(3x) + 3x + 2$

- Use initial condition $y(0) = 23$

  $23 = c_1 + 2$

- Compute derivative of the solution

  $y' = -3c_1 \sin(3x) + 3c_2 \cos(3x) + 3$

- Use the initial condition $y'\Big|_{\{x=0\}} = 21$

  $21 = 3 + 3c_2$

- Solve for c_1 and c_2

  $\{c_1 = 21, c_2 = 6\}$

- Substitute constant values into general solution and simplify

  $y = 2 + 21\cos(3x) + 6\sin(3x) + 3x$

- Solution to the IVP

  $y = 2 + 21\cos(3x) + 6\sin(3x) + 3x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$2)+9*y(x)=27*x+18,y(0) = 23, D(y)(0) = 21],y(x), singsol=all)
```

$$y(x) = 6\sin(3x) + 21\cos(3x) + 3x + 2$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 22

```
DSolve[{y''[x]+9*y[x]==27*x+18,{y[0]==23,y'[0]==21}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to 3x + 6\sin(3x) + 21\cos(3x) + 2$$

9.16 problem 18

Internal problem ID [12759]
Internal file name [`OUTPUT/11411_Friday_November_03_2023_06_32_42_AM_32058958/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.1, page 186
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y''x^2 + y'x - 4y = -3x - \frac{3}{x}}$$

With initial conditions

$$[y(1) = 3, y'(1) = -6]$$

9.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \\ F &= \frac{-3x - \frac{3}{x}}{x^2} \end{aligned}$$

Hence the ode is

$$y'' + \frac{y'}{x} - \frac{4y}{x^2} = \frac{-3x - \frac{3}{x}}{x^2}$$

The domain of $p(x) = \frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -\frac{4}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. The domain of $F = \frac{-3x-\frac{3}{x}}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

9.16.2 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -4, f(x) = -3x - \frac{3}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$y''x^2 + y'x - 4y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 4 = 0$$

Or

$$r^2 - 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2 \\ r_2 &= 2 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + c_2x^2$$

Next, we find the particular solution to the ODE

$$y''x^2 + y'x - 4y = -3x - \frac{3}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ -\frac{2}{x^3} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(2x) - (x^2)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{4}{x}$$

Which simplifies to

$$W = \frac{4}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2\left(-3x - \frac{3}{x}\right)}{4x}\, dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{3x^2}{4} - \frac{3}{4}\right) dx$$

Hence

$$u_1 = \frac{1}{4}x^3 + \frac{3}{4}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{-3x - \frac{3}{x}}{x^2}}{4x}\, dx$$

Which simplifies to

$$u_2 = \int \frac{-3x^2 - 3}{4x^4} dx$$

Hence

$$u_2 = \frac{1}{4x^3} + \frac{3}{4x}$$

Which simplifies to

$$u_1 = \frac{1}{4}x^3 + \frac{3}{4}x$$
$$u_2 = \frac{3x^2 + 1}{4x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\frac{1}{4}x^3 + \frac{3}{4}x}{x^2} + \frac{3x^2 + 1}{4x}$$

Which simplifies to

$$y_p(x) = \frac{x^2 + 1}{x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{c_2 x^4 + x^3 + c_1 + x}{x^2}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_2 x^4 + x^3 + c_1 + x}{x^2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$
3 = c_1 + c_2 + 2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{4c_2 x^3 + 3x^2 + 1}{x^2} - \frac{2(c_2 x^4 + x^3 + c_1 + x)}{x^3}
$$

substituting $y' = -6$ and $x = 1$ in the above gives

$$
-6 = -2c_1 + 2c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= -1
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -\frac{x^4 - x^3 - x - 2}{x^2}
$$

Which simplifies to

$$
y = \frac{-x^4 + x^3 + x + 2}{x^2}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{-x^4 + x^3 + x + 2}{x^2} \tag{1}
$$

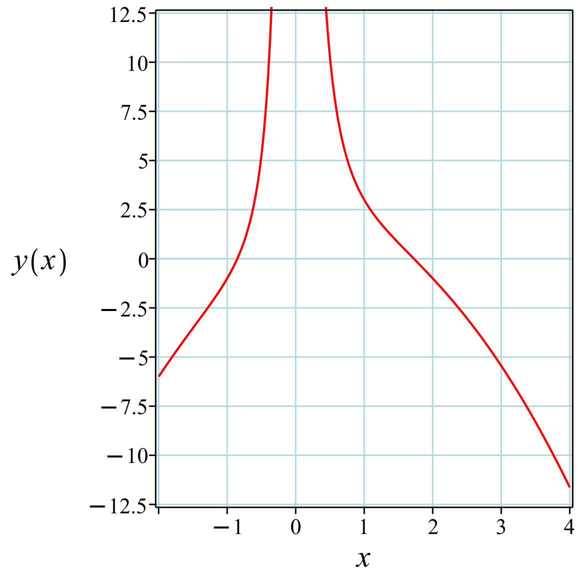


Figure 380: Solution plot

Verification of solutions

$$y = \frac{-x^4 + x^3 + x + 2}{x^2}$$

Verified OK.

9.16.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y''x^2 + y'x - 4y = 0$$

In normal form the ode

$$y''x^2 + y'x - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -\frac{4}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} && (4) \\
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} && (5)
\end{aligned}
$$

Let $p_1 = 0$. Eq (4) simplifies to

$$
\tau''(x) + p(x)\,\tau'(x) = 0
$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\
&= \int e^{-\ln(x)}\,dx \\
&= \int \frac{1}{x}dx \\
&= \ln(x) && (6)
\end{aligned}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{4}{x^2}}{\frac{1}{x^2}} \\
&= -4 && (7)
\end{aligned}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - 4y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} - 4\,e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\lambda_1 = +2$$
$$\lambda_2 = -2$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{(2)\tau} + c_2 e^{(-2)\tau}$$

Or

$$y(\tau) = c_1 e^{2\tau} + c_2 e^{-2\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^4 + c_2}{x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x^4 + c_2}{x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ -\frac{2}{x^3} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(2x) - \left(x^2\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{4}{x}$$

Which simplifies to

$$W = \frac{4}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2\left(-3x - \frac{3}{x}\right)}{4x}\, dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{3x^2}{4} - \frac{3}{4}\right) dx$$

Hence

$$u_1 = \frac{1}{4}x^3 + \frac{3}{4}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{-3x - \frac{3}{x}}{x^2}}{4x}\, dx$$

Which simplifies to

$$u_2 = \int \frac{-3x^2 - 3}{4x^4} dx$$

Hence

$$u_2 = \frac{1}{4x^3} + \frac{3}{4x}$$

Which simplifies to

$$u_1 = \frac{1}{4}x^3 + \frac{3}{4}x$$
$$u_2 = \frac{3x^2+1}{4x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\frac{1}{4}x^3 + \frac{3}{4}x}{x^2} + \frac{3x^2+1}{4x}$$

Which simplifies to

$$y_p(x) = \frac{x^2+1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1x^4+c_2}{x^2}\right) + \left(\frac{x^2+1}{x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1x^4+c_2}{x^2} + \frac{x^2+1}{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = c_1 + c_2 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 4c_1x - \frac{2(c_1x^4+c_2)}{x^3} + 2 - \frac{x^2+1}{x^2}$$

substituting $y' = -6$ and $x = 1$ in the above gives

$$-6 = 2c_1 - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = -\frac{x^4 - x^3 - x - 2}{x^2}$$

Which simplifies to

$$y = \frac{-x^4 + x^3 + x + 2}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-x^4 + x^3 + x + 2}{x^2} \tag{1}$$

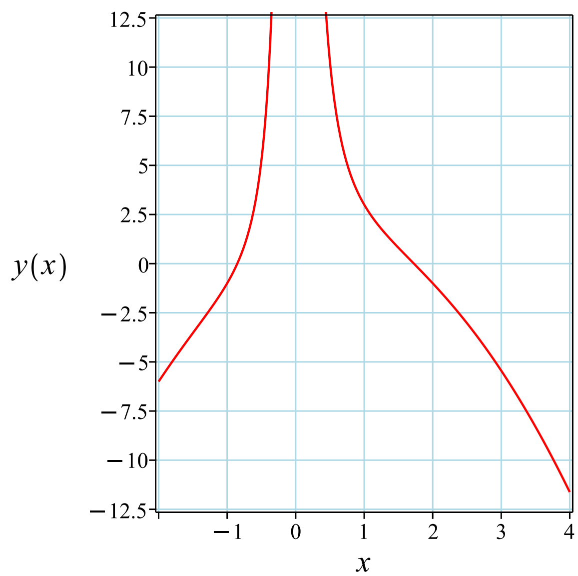


Figure 381: Solution plot

Verification of solutions

$$y = \frac{-x^4 + x^3 + x + 2}{x^2}$$

Verified OK.

9.16.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -4, f(x) = -3x - \frac{3}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$y''x^2 + y'x - 4y = 0$$

In normal form the ode

$$y''x^2 + y'x - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{-\frac{1}{x^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = \frac{2}{c\sqrt{-\frac{1}{x^2}}\, x^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{2}{c\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{2\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{-\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{-\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cosh(2\ln(x)) + i c_2 \sinh(2\ln(x))
$$

Now the particular solution to this ODE is found

$$
y''x^2 + y'x - 4y = -3x - \frac{3}{x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ -\frac{2}{x^3} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(2x) - (x^2)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{4}{x}$$

Which simplifies to

$$W = \frac{4}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2\left(-3x - \frac{3}{x}\right)}{4x}\, dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{3x^2}{4} - \frac{3}{4}\right) dx$$

Hence

$$u_1 = \frac{1}{4}x^3 + \frac{3}{4}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{-3x - \frac{3}{x}}{x^2}}{4x}\, dx$$

Which simplifies to

$$u_2 = \int \frac{-3x^2 - 3}{4x^4} dx$$

Hence

$$u_2 = \frac{1}{4x^3} + \frac{3}{4x}$$

Which simplifies to

$$u_1 = \frac{1}{4}x^3 + \frac{3}{4}x$$

$$u_2 = \frac{3x^2 + 1}{4x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\frac{1}{4}x^3 + \frac{3}{4}x}{x^2} + \frac{3x^2 + 1}{4x}$$

Which simplifies to

$$y_p(x) = \frac{x^2+1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cosh(2\ln(x)) + ic_2 \sinh(2\ln(x))) + \left(\frac{x^2+1}{x}\right) \\ &= \frac{x^2+1}{x} + c_1 \cosh(2\ln(x)) + ic_2 \sinh(2\ln(x)) \end{aligned}$$

Which simplifies to

$$y = \frac{(ic_2+c_1)\,x^4 + 2x^3 + 2x - ic_2 + c_1}{2x^2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{(ic_2+c_1)\,x^4 + 2x^3 + 2x - ic_2 + c_1}{2x^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = c_1 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{4(ic_2+c_1)\,x^3 + 6x^2 + 2}{2x^2} - \frac{(ic_2+c_1)\,x^4 + 2x^3 + 2x - ic_2 + c_1}{x^3}$$

substituting $y' = -6$ and $x = 1$ in the above gives

$$-6 = 2ic_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 3i \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{-x^4 + x^3 + x + 2}{x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{-x^4 + x^3 + x + 2}{x^2} \tag{1}$$

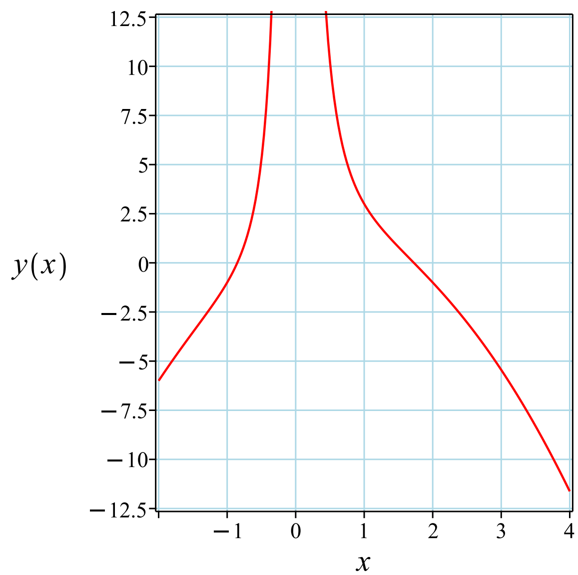


Figure 382: Solution plot

Verification of solutions

$$y = \frac{-x^4 + x^3 + x + 2}{x^2}$$

Verified OK.

9.16.5 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -4, f(x) = -3x - \frac{3}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$y''x^2 + y'x - 4y = 0$$

In normal form the ode

$$y''x^2 + y'x - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= \frac{1}{x}\\ q(x) &= -\frac{4}{x^2}\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(x) + \frac{5v'(x)}{x} &= 0\\ v''(x) + \frac{5v'(x)}{x} &= 0\end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{5u}{x}
\end{aligned}$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{5}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{5}{x}\,dx \\
\ln(u) &= -5\ln(x) + c_1 \\
u &= e^{-5\ln(x)+c_1} \\
&= \frac{c_1}{x^5}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{4x^4} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{4x^4} + c_2\right)x^2 \\
&= \frac{4c_2x^4 - c_1}{4x^2}
\end{aligned}$$

Now the particular solution to this ODE is found

$$y''x^2 + y'x - 4y = -3x - \frac{3}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ -\frac{2}{x^3} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(2x) - \left(x^2\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{4}{x}$$

Which simplifies to

$$W = \frac{4}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2\left(-3x - \frac{3}{x}\right)}{4x}\, dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{3x^2}{4} - \frac{3}{4}\right) dx$$

Hence

$$u_1 = \frac{1}{4}x^3 + \frac{3}{4}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{-3x-\frac{3}{x}}{x^2}}{4x}\, dx$$

Which simplifies to

$$u_2 = \int \frac{-3x^2 - 3}{4x^4} dx$$

Hence

$$u_2 = \frac{1}{4x^3} + \frac{3}{4x}$$

Which simplifies to

$$u_1 = \frac{1}{4}x^3 + \frac{3}{4}x$$
$$u_2 = \frac{3x^2+1}{4x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\frac{1}{4}x^3 + \frac{3}{4}x}{x^2} + \frac{3x^2+1}{4x}$$

Which simplifies to

$$y_p(x) = \frac{x^2+1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{4x^4} + c_2\right)x^2\right) + \left(\frac{x^2+1}{x}\right) \\ &= \frac{x^2+1}{x} + \left(-\frac{c_1}{4x^4} + c_2\right)x^2 \end{aligned}$$

Which simplifies to

$$y = -\frac{-4c_2x^4 - 4x^3 + c_1 - 4x}{4x^2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{-4c_2x^4 - 4x^3 + c_1 - 4x}{4x^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = -\frac{c_1}{4} + c_2 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{-4c_2x^4 - 4x^3 + c_1 - 4x}{2x^3} - \frac{-16c_2x^3 - 12x^2 - 4}{4x^2}$$

substituting $y' = -6$ and $x = 1$ in the above gives

$$-6 = \frac{c_1}{2} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -8$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = -\frac{x^4 - x^3 - x - 2}{x^2}$$

Which simplifies to

$$y = \frac{-x^4 + x^3 + x + 2}{x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{-x^4 + x^3 + x + 2}{x^2} \tag{1}$$

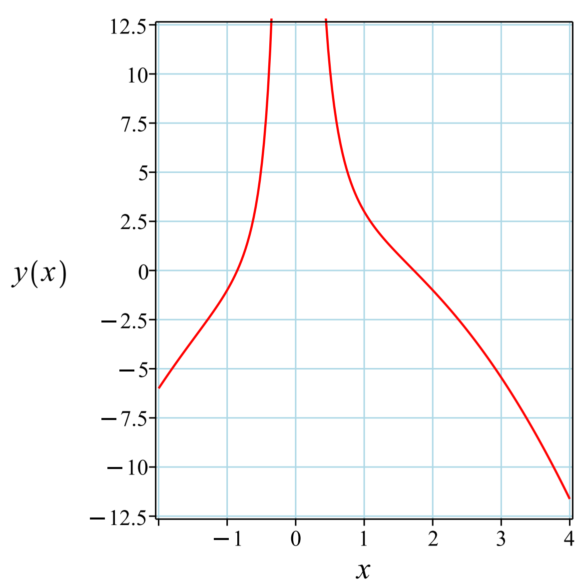


Figure 383: Solution plot

Verification of solutions

$$y = \frac{-x^4 + x^3 + x + 2}{x^2}$$

Verified OK.

9.16.6 Solving using Kovacic algorithm

Writing the ode as

$$y''x^2 + y'x - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 329: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = -\frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{3}{2x} + (-)\,(0) \\ &= -\frac{3}{2x} \\ &= -\frac{3}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{3}{2x}\right)(0) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x}\right)^2 - \left(\frac{15}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{3}{2x} dx} \\
&= \frac{1}{x^{\frac{3}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^4}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2}\right) + c_2 \left(\frac{1}{x^2} \left(\frac{x^4}{4}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y''x^2 + y'x - 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x^2} + \frac{c_2 x^2}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$

$$y_2 = \frac{x^2}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{x^2}{4} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{x^2}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{x^2}{4} \\ -\frac{2}{x^3} & \frac{x}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(\frac{x}{2}\right) - \left(\frac{x^2}{4}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^2\left(-3x-\frac{3}{x}\right)}{4}}{x}\,dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{3x^2}{4} - \frac{3}{4}\right) dx$$

Hence

$$u_1 = \frac{1}{4}x^3 + \frac{3}{4}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{-3x-\frac{3}{x}}{x^2}}{x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{-3x^2 - 3}{x^4} dx$$

Hence

$$u_2 = \frac{1}{x^3} + \frac{3}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\frac{1}{4}x^3 + \frac{3}{4}x}{x^2} + \frac{\left(\frac{1}{x^3} + \frac{3}{x}\right) x^2}{4}$$

Which simplifies to

$$y_p(x) = \frac{x^2 + 1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x^2} + \frac{c_2 x^2}{4}\right) + \left(\frac{x^2 + 1}{x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x^2} + \frac{c_2 x^2}{4} + \frac{x^2 + 1}{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = c_1 + \frac{c_2}{4} + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{2c_1}{x^3} + \frac{c_2 x}{2} + 2 - \frac{x^2 + 1}{x^2}$$

substituting $y' = -6$ and $x = 1$ in the above gives

$$-6 = -2c_1 + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -4$$

Substituting these values back in above solution results in

$$y = -\frac{x^4 - x^3 - x - 2}{x^2}$$

Which simplifies to

$$y = \frac{-x^4 + x^3 + x + 2}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-x^4 + x^3 + x + 2}{x^2} \tag{1}$$

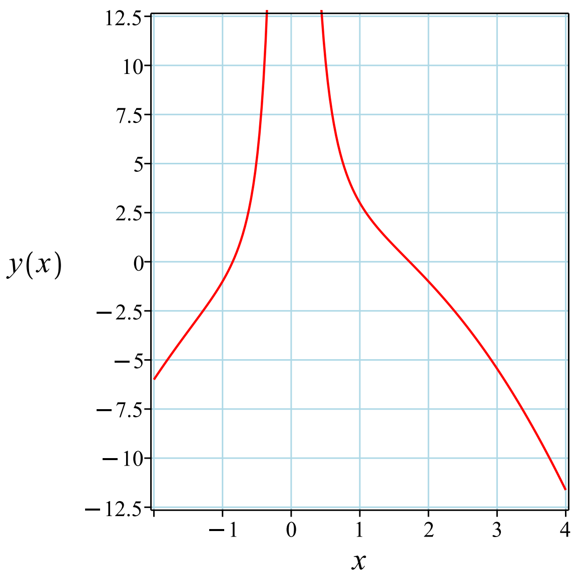


Figure 384: Solution plot

Verification of solutions

$$y = \frac{-x^4 + x^3 + x + 2}{x^2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve([x^2*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=-3*x-3/x,y(1) = 3, D(y)(1) = -6],y(x), sing
```

$$y(x) = \frac{-x^4 + x^3 + x + 2}{x^2}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 20

```
DSolve[{x^2*y''[x]+x*y'[x]-4*y[x]==-3*x-3/x,{y[1]==3,y'[1]==-6}},y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{-x^4 + x^3 + x + 2}{x^2}$$

10 Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210

10.1 problem 1

Internal problem ID [12760]
Internal file name [`OUTPUT/11412_Friday_November_03_2023_06_32_45_AM_28043090/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' + 4y' - 3y = 0}$$

10.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = 4, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} + 4\lambda e^{\lambda x} - 3 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 + 4\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 4, C = -3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(4)} \pm \frac{1}{(2)\,(4)}\sqrt{4^2 - (4)\,(4)\,(-3)} \\ &= -\frac{1}{2} \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + 1 \\ \lambda_2 &= -\frac{1}{2} - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{1}{2} \\ \lambda_2 &= -\frac{3}{2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{\left(-\frac{3}{2}\right)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{3x}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{3x}{2}} \tag{1}$$

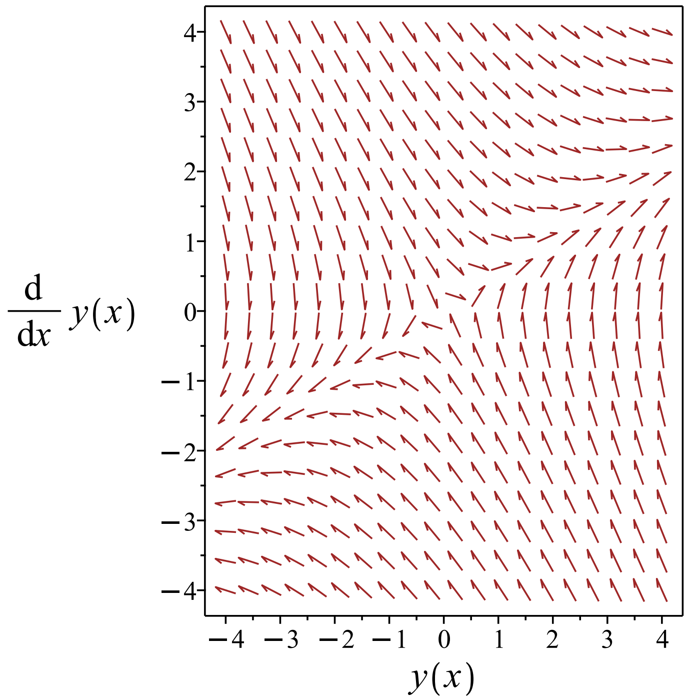


Figure 385: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{3x}{2}}$$

Verified OK.

10.1.2 Solving using Kovacic algorithm

Writing the ode as

$$4y'' + 4y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 4 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 330: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{4}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{3x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{4}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-\frac{3x}{2}}\right) + c_2 \left(e^{-\frac{3x}{2}} \left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-\frac{3x}{2}} + \frac{e^{\frac{x}{2}} c_2}{2} \tag{1}
$$

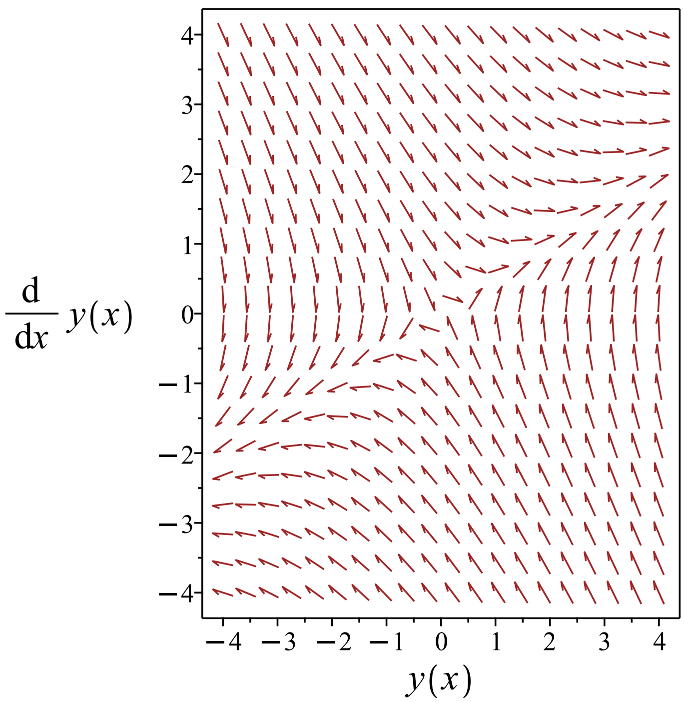


Figure 386: Slope field plot

Verification of solutions

$$
y = c_1 e^{-\frac{3x}{2}} + \frac{e^{\frac{x}{2}} c_2}{2}
$$

Verified OK.

10.1.3 Maple step by step solution

Let's solve

$4y'' + 4y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' + \frac{3y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' - \frac{3y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 + r - \frac{3}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r+3)(2r-1)}{4} = 0$

- Roots of the characteristic polynomial

  $r = \left(-\frac{3}{2}, \frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-\frac{3x}{2}}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-\frac{3x}{2}} + \mathrm{e}^{\frac{x}{2}} c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(4*diff(y(x),x$2)+4*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(e^{2x}c_1 + c_2\right) e^{-\frac{3x}{2}}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 24

```
DSolve[4*y''[x]+4*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x/2}\left(c_2 e^{2x} + c_1\right)$$

10.2 problem 2

Internal problem ID [12761]
Internal file name [`OUTPUT/11413_Friday_November_03_2023_06_32_46_AM_10799578/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210
Problem number: 2.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 4y'' + 6y' - 4y = 0}$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 + 6\lambda - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 1 - i$$
$$\lambda_3 = 1 + i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{2x} + \mathrm{e}^{(1+i)x} c_2 + \mathrm{e}^{(1-i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{2x}$$
$$y_2 = \mathrm{e}^{(1+i)x}$$
$$y_3 = \mathrm{e}^{(1-i)x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + e^{(1+i)x} c_2 + e^{(1-i)x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{2x} + e^{(1+i)x} c_2 + e^{(1-i)x} c_3$$

Verified OK.

10.2.1 Maple step by step solution

Let's solve

$y''' - 4y'' + 6y' - 4y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 4y_3(x) - 6y_2(x) + 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 4y_3(x) - 6y_2(x) + 4y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -6 & 4 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -6 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[1 - \mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{2} \\ \frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1 + \mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{2} \\ \frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - \mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{2} \\ \frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(1-I)x} \cdot \begin{bmatrix} \frac{I}{2} \\ \frac{1}{2}+\frac{I}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{x} \cdot (\cos(x) - I\sin(x)) \cdot \begin{bmatrix} \frac{I}{2} \\ \frac{1}{2}+\frac{I}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{x} \cdot \begin{bmatrix} \frac{I}{2}(\cos(x) - I\sin(x)) \\ \left(\frac{1}{2}+\frac{I}{2}\right)(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{x} \cdot \begin{bmatrix} \frac{\sin(x)}{2} \\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = e^{x} \cdot \begin{bmatrix} \frac{\cos(x)}{2} \\ -\frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{x} \cdot \begin{bmatrix} \frac{\sin(x)}{2} \\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ \cos(x) \end{bmatrix} + c_3 e^{x} \cdot \begin{bmatrix} \frac{\cos(x)}{2} \\ -\frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ -\sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{c_1 e^{2x}}{4} + \frac{e^{x}(c_3\cos(x) + c_2\sin(x))}{2}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x$2)+6*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{x} \sin(x) + c_3 \cos(x) \mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 26

```
DSolve[y'''[x]-4*y''[x]+6*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x} (c_3 e^{x} + c_2 \cos(x) + c_1 \sin(x))$$

10.3 problem 3

Internal problem ID [12762]
Internal file name [`OUTPUT/11414_Friday_November_03_2023_06_32_46_AM_85835499/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210
Problem number: 3.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 16y = 0}$$

The characteristic equation is

$$\lambda^4 - 16 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\\ \lambda_3 &= 2i\\ \lambda_4 &= -2i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{2ix} c_3 + \mathrm{e}^{-2ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= \mathrm{e}^{-2x}\\ y_2 &= \mathrm{e}^{2x}\\ y_3 &= \mathrm{e}^{2ix}\\ y_4 &= \mathrm{e}^{-2ix}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{2x} + e^{2ix} c_3 + e^{-2ix} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{2x} + e^{2ix} c_3 + e^{-2ix} c_4$$

Verified OK.

10.3.1 Maple step by step solution

Let's solve

$y'''' - 16y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 16y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 16y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(2x) - \mathrm{I}\sin(2x)) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\mathrm{I}}{8}(\cos(2x) - \mathrm{I}\sin(2x)) \\ -\frac{\cos(2x)}{4} + \frac{\mathrm{I}\sin(2x)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{c_3\sin(2x)}{8} - \frac{c_4\cos(2x)}{8} \\ -\frac{c_3\cos(2x)}{4} + \frac{c_4\sin(2x)}{4} \\ \frac{c_3\sin(2x)}{2} + \frac{c_4\cos(2x)}{2} \\ c_3\cos(2x) - c_4\sin(2x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{c_1 \mathrm{e}^{-2x}}{8} + \frac{c_2 \mathrm{e}^{2x}}{8} - \frac{c_4\cos(2x)}{8} - \frac{c_3\sin(2x)}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$4)-16*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{2x}c_1 + e^{-2x}c_2 + c_3 \sin(2x) + c_4 \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 36

```
DSolve[y''''[x]-16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{2x} + c_3 e^{-2x} + c_2 \cos(2x) + c_4 \sin(2x)$$

10.4 problem 4

Internal problem ID [12763]
Internal file name [`OUTPUT/11415_Friday_November_03_2023_06_32_46_AM_60972317/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210
Problem number: 4.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 16y = 0}$$

The characteristic equation is

$$\lambda^4 + 16 = 0$$

The roots of the above equation are

$$\lambda_1 = \sqrt{2} + i\sqrt{2}$$
$$\lambda_2 = -\sqrt{2} + i\sqrt{2}$$
$$\lambda_3 = -\sqrt{2} - i\sqrt{2}$$
$$\lambda_4 = \sqrt{2} - i\sqrt{2}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{\left(\sqrt{2}-i\sqrt{2}\right)x} c_1 + \mathrm{e}^{\left(-\sqrt{2}+i\sqrt{2}\right)x} c_2 + \mathrm{e}^{\left(-\sqrt{2}-i\sqrt{2}\right)x} c_3 + \mathrm{e}^{\left(\sqrt{2}+i\sqrt{2}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{\left(\sqrt{2}-i\sqrt{2}\right)x}$$
$$y_2 = \mathrm{e}^{\left(-\sqrt{2}+i\sqrt{2}\right)x}$$
$$y_3 = \mathrm{e}^{\left(-\sqrt{2}-i\sqrt{2}\right)x}$$
$$y_4 = \mathrm{e}^{\left(\sqrt{2}+i\sqrt{2}\right)x}$$

Summary

The solution(s) found are the following

$$y = e^{\left(\sqrt{2}-i\sqrt{2}\right)x} c_1 + e^{\left(-\sqrt{2}+i\sqrt{2}\right)x} c_2 + e^{\left(-\sqrt{2}-i\sqrt{2}\right)x} c_3 + e^{\left(\sqrt{2}+i\sqrt{2}\right)x} c_4 \tag{1}$$

Verification of solutions

$$y = e^{\left(\sqrt{2}-i\sqrt{2}\right)x} c_1 + e^{\left(-\sqrt{2}+i\sqrt{2}\right)x} c_2 + e^{\left(-\sqrt{2}-i\sqrt{2}\right)x} c_3 + e^{\left(\sqrt{2}+i\sqrt{2}\right)x} c_4$$

Verified OK.

10.4.1 Maple step by step solution

Let's solve

$y'''' + 16y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -16y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -16y_1(x)]$

- Define vector

$$\vec{y}(x)=\left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x)=\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & 0 & 0 & 0 \end{array}\right]\cdot\vec{y}(x)$$

- Define the coefficient matrix

$$A=\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & 0 & 0 & 0 \end{array}\right]$$

- Rewrite the system as

$\vec{y}'(x)=A\cdot\vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\sqrt{2}-\mathrm{I}\sqrt{2},\left[\begin{array}{c} \frac{1}{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)^3} \\ \frac{1}{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)^2} \\ \frac{1}{-\sqrt{2}-\mathrm{I}\sqrt{2}} \\ 1 \end{array}\right]\right],\left[-\sqrt{2}+\mathrm{I}\sqrt{2},\left[\begin{array}{c} \frac{1}{\left(-\sqrt{2}+\mathrm{I}\sqrt{2}\right)^3} \\ \frac{1}{\left(-\sqrt{2}+\mathrm{I}\sqrt{2}\right)^2} \\ \frac{1}{-\sqrt{2}+\mathrm{I}\sqrt{2}} \\ 1 \end{array}\right]\right],\left[\sqrt{2}-\mathrm{I}\sqrt{2},\left[\begin{array}{c} \frac{1}{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)^3} \\ \frac{1}{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)^2} \\ \frac{1}{\sqrt{2}-\mathrm{I}\sqrt{2}} \\ 1 \end{array}\right.\right.$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\sqrt{2}-\mathrm{I}\sqrt{2}, \left[\begin{array}{c} \frac{1}{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{3}} \\ \frac{1}{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{2}} \\ \frac{1}{-\sqrt{2}-\mathrm{I}\sqrt{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{3}} \\ \frac{1}{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{2}} \\ \frac{1}{-\sqrt{2}-\mathrm{I}\sqrt{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-\sqrt{2}\,x} \cdot \left(\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{3}} \\ \frac{1}{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{2}} \\ \frac{1}{-\sqrt{2}-\mathrm{I}\sqrt{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-\sqrt{2}\,x} \cdot \left[\begin{array}{c} \frac{\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{3}} \\ \frac{\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{2}} \\ \frac{\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{-\sqrt{2}-\mathrm{I}\sqrt{2}} \\ \cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = e^{-\sqrt{2}\,x} \cdot \begin{bmatrix} \frac{\cos(\sqrt{2}\,x)\sqrt{2}}{16} + \frac{\sin(\sqrt{2}\,x)\sqrt{2}}{16} \\ -\frac{\sin(\sqrt{2}\,x)}{4} \\ -\frac{\cos(\sqrt{2}\,x)\sqrt{2}}{4} + \frac{\sin(\sqrt{2}\,x)\sqrt{2}}{4} \\ \cos(\sqrt{2}\,x) \end{bmatrix}, \vec{y}_2(x) = e^{-\sqrt{2}\,x} \cdot \begin{bmatrix} \frac{\cos(\sqrt{2}\,x)\sqrt{2}}{16} - \frac{\sin(\sqrt{2}\,x)\sqrt{2}}{16} \\ -\frac{\cos(\sqrt{2}\,x)}{4} \\ \frac{\cos(\sqrt{2}\,x)\sqrt{2}}{4} + \frac{\sin(\sqrt{2}\,x)\sqrt{2}}{4} \\ -\sin(\sqrt{2}\,x) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\sqrt{2} - \mathrm{I}\sqrt{2}, \begin{bmatrix} \frac{1}{(\sqrt{2}-\mathrm{I}\sqrt{2})^3} \\ \frac{1}{(\sqrt{2}-\mathrm{I}\sqrt{2})^2} \\ \frac{1}{\sqrt{2}-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(\sqrt{2}-\mathrm{I}\sqrt{2})x} \cdot \begin{bmatrix} \frac{1}{(\sqrt{2}-\mathrm{I}\sqrt{2})^3} \\ \frac{1}{(\sqrt{2}-\mathrm{I}\sqrt{2})^2} \\ \frac{1}{\sqrt{2}-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{\sqrt{2}\,x} \cdot (\cos(\sqrt{2}\,x) - \mathrm{I}\sin(\sqrt{2}\,x)) \cdot \begin{bmatrix} \frac{1}{(\sqrt{2}-\mathrm{I}\sqrt{2})^3} \\ \frac{1}{(\sqrt{2}-\mathrm{I}\sqrt{2})^2} \\ \frac{1}{\sqrt{2}-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$
\mathrm{e}^{\sqrt{2}\,x}\cdot\left[\begin{array}{c}
\frac{\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{3}} \\
\frac{\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{2}} \\
\frac{\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{\sqrt{2}-\mathrm{I}\sqrt{2}} \\
\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)
\end{array}\right]
$$

- Both real and imaginary parts are solutions to the homogeneous system

$$
\left[\vec{y}_{3}(x)=\mathrm{e}^{\sqrt{2}\,x}\cdot\left[\begin{array}{c}
-\frac{\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{16}+\frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{16} \\
\frac{\sin\left(\sqrt{2}\,x\right)}{4} \\
\frac{\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{4}+\frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{4} \\
\cos\left(\sqrt{2}\,x\right)
\end{array}\right], \vec{y}_{4}(x)=\mathrm{e}^{\sqrt{2}\,x}\cdot\left[\begin{array}{c}
\frac{\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{16}+\frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{16} \\
\frac{\cos\left(\sqrt{2}\,x\right)}{4} \\
\frac{\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{4}-\frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{4} \\
-\sin\left(\sqrt{2}\,x\right)
\end{array}\right]\right]
$$

- General solution to the system of ODEs

$$
\vec{y}=c_{1}\vec{y}_{1}(x)+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)+c_{4}\vec{y}_{4}(x)
$$

- Substitute solutions into the general solution

$$
\vec{y}=c_{1}\mathrm{e}^{-\sqrt{2}\,x}\cdot\left[\begin{array}{c}
\frac{\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{16}+\frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{16} \\
-\frac{\sin\left(\sqrt{2}\,x\right)}{4} \\
-\frac{\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{4}+\frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{4} \\
\cos\left(\sqrt{2}\,x\right)
\end{array}\right]+c_{2}\mathrm{e}^{-\sqrt{2}\,x}\cdot\left[\begin{array}{c}
\frac{\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{16}-\frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{16} \\
-\frac{\cos\left(\sqrt{2}\,x\right)}{4} \\
\frac{\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{4}+\frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{4} \\
-\sin\left(\sqrt{2}\,x\right)
\end{array}\right]+c_{3}\mathrm{e}^{\sqrt{2}\,x}
$$

- First component of the vector is the solution to the ODE

$$
y=\frac{\sqrt{2}\left(\left(\left(c_{1}+c_{2}\right)\cos\left(\sqrt{2}\,x\right)+\sin\left(\sqrt{2}\,x\right)\left(c_{1}-c_{2}\right)\right)\mathrm{e}^{-\sqrt{2}\,x}-\left(\left(c_{3}-c_{4}\right)\cos\left(\sqrt{2}\,x\right)-\sin\left(\sqrt{2}\,x\right)\left(c_{3}+c_{4}\right)\right)\mathrm{e}^{\sqrt{2}\,x}\right)}{16}
$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 65

```
dsolve(diff(y(x),x$4)+16*y(x)=0,y(x), singsol=all)
```

$$y(x) = -c_1 e^{-x\sqrt{2}} \sin\left(x\sqrt{2}\right) - c_2 e^{x\sqrt{2}} \sin\left(x\sqrt{2}\right) + c_3 e^{-x\sqrt{2}} \cos\left(x\sqrt{2}\right) + c_4 e^{x\sqrt{2}} \cos\left(x\sqrt{2}\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 67

```
DSolve[y''''[x]+16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\sqrt{2}x}\left(\left(c_1 e^{2\sqrt{2}x} + c_2\right)\cos\left(\sqrt{2}x\right) + \left(c_4 e^{2\sqrt{2}x} + c_3\right)\sin\left(\sqrt{2}x\right)\right)$$

10.5 problem 5

Internal problem ID [12764]
Internal file name [`OUTPUT/11416_Friday_November_03_2023_06_32_46_AM_52274915/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210
Problem number: 5.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 4y''' + 8y'' - 8y' + 4y = 0}$$

The characteristic equation is

$$\lambda^4 - 4\lambda^3 + 8\lambda^2 - 8\lambda + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 1 - i$$
$$\lambda_2 = 1 + i$$
$$\lambda_3 = 1 - i$$
$$\lambda_4 = 1 + i$$

Therefore the homogeneous solution is

$$y_h(x) = e^{(1+i)x}c_1 + x\,e^{(1+i)x}c_2 + e^{(1-i)x}c_3 + x\,e^{(1-i)x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{(1+i)x}$$
$$y_2 = x\,e^{(1+i)x}$$
$$y_3 = e^{(1-i)x}$$
$$y_4 = x\,e^{(1-i)x}$$

Summary

The solution(s) found are the following

$$y = e^{(1+i)x}c_1 + x\,e^{(1+i)x}c_2 + e^{(1-i)x}c_3 + x\,e^{(1-i)x}c_4 \tag{1}$$

Verification of solutions

$$y = e^{(1+i)x}c_1 + x\,e^{(1+i)x}c_2 + e^{(1-i)x}c_3 + x\,e^{(1-i)x}c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$4)-4*diff(y(x),x$3)+8*diff(y(x),x$2)-8*diff(y(x),x)+4*y(x)=0,y(x), singso
```

$$y(x) = ((c_4x + c_2)\cos(x) + \sin(x)(c_3x + c_1))\,e^x$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 30

```
DSolve[y''''[x]-4*y'''[x]+8*y''[x]-8*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^x((c_4x + c_3)\cos(x) + (c_2x + c_1)\sin(x))$$

10.6 problem 6

Internal problem ID [12765]
Internal file name [`OUTPUT/11417_Friday_November_03_2023_06_32_46_AM_81255054/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210
Problem number: 6.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{y'''' - 8y' = 0}$$

The characteristic equation is

$$\lambda^4 - 8\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = i\sqrt{3} - 1$$
$$\lambda_4 = -i\sqrt{3} - 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{2x} + e^{\left(i\sqrt{3}-1\right)x} c_3 + e^{\left(-i\sqrt{3}-1\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{2x}$$
$$y_3 = e^{\left(i\sqrt{3}-1\right)x}$$
$$y_4 = e^{\left(-i\sqrt{3}-1\right)x}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 e^{2x} + e^{\left(i\sqrt{3}-1\right)x} c_3 + e^{\left(-i\sqrt{3}-1\right)x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^{2x} + e^{\left(i\sqrt{3}-1\right)x} c_3 + e^{\left(-i\sqrt{3}-1\right)x} c_4$$

Verified OK.

10.6.1 Maple step by step solution

Let's solve

$y'''' - 8y' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 8y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 8y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 8 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 8 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}\sqrt{3}-1, \begin{bmatrix} \frac{1}{\left(-\mathrm{I}\sqrt{3}-1\right)^3} \\ \frac{1}{\left(-\mathrm{I}\sqrt{3}-1\right)^2} \\ \frac{1}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{3}-1, \begin{bmatrix} \frac{1}{\left(\mathrm{I}\sqrt{3}-1\right)^3} \\ \frac{1}{\left(\mathrm{I}\sqrt{3}-1\right)^2} \\ \frac{1}{\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\begin{bmatrix} -\mathrm{I}\sqrt{3} - 1, \begin{bmatrix} \frac{1}{\left(-\mathrm{I}\sqrt{3}-1\right)^3} \\ \frac{1}{\left(-\mathrm{I}\sqrt{3}-1\right)^2} \\ \frac{1}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution from eigenpair

$$e^{\left(-\mathrm{I}\sqrt{3}-1\right)x} \cdot \begin{bmatrix} \frac{1}{\left(-\mathrm{I}\sqrt{3}-1\right)^3} \\ \frac{1}{\left(-\mathrm{I}\sqrt{3}-1\right)^2} \\ \frac{1}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-x} \cdot \left(\cos\left(\sqrt{3}\, x\right) - \mathrm{I} \sin\left(\sqrt{3}\, x\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\mathrm{I}\sqrt{3}-1\right)^{3}} \\ \frac{1}{\left(-\mathrm{I}\sqrt{3}-1\right)^{2}} \\ \frac{1}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-x} \cdot \left[\begin{array}{c} \frac{\cos\left(\sqrt{3}\, x\right)-\mathrm{I}\sin\left(\sqrt{3}\, x\right)}{\left(-\mathrm{I}\sqrt{3}-1\right)^{3}} \\ \frac{\cos\left(\sqrt{3}\, x\right)-\mathrm{I}\sin\left(\sqrt{3}\, x\right)}{\left(-\mathrm{I}\sqrt{3}-1\right)^{2}} \\ \frac{\cos\left(\sqrt{3}\, x\right)-\mathrm{I}\sin\left(\sqrt{3}\, x\right)}{-\mathrm{I}\sqrt{3}-1} \\ \cos\left(\sqrt{3}\, x\right) - \mathrm{I}\sin\left(\sqrt{3}\, x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{3}(x) = e^{-x} \cdot \left[\begin{array}{c} \frac{\cos\left(\sqrt{3}\, x\right)}{8} \\ -\frac{\cos\left(\sqrt{3}\, x\right)}{8} - \frac{\sqrt{3}\, \sin\left(\sqrt{3}\, x\right)}{8} \\ -\frac{\cos\left(\sqrt{3}\, x\right)}{4} + \frac{\sqrt{3}\, \sin\left(\sqrt{3}\, x\right)}{4} \\ \cos\left(\sqrt{3}\, x\right) \end{array}\right], \vec{y}_{4}(x) = e^{-x} \cdot \left[\begin{array}{c} -\frac{\sin\left(\sqrt{3}\, x\right)}{8} \\ -\frac{\cos\left(\sqrt{3}\, x\right)\sqrt{3}}{8} + \frac{\sin\left(\sqrt{3}\, x\right)}{8} \\ \frac{\cos\left(\sqrt{3}\, x\right)\sqrt{3}}{4} + \frac{\sin\left(\sqrt{3}\, x\right)}{4} \\ -\sin\left(\sqrt{3}\, x\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3(x) + c_4 \vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_2 e^{2x} \cdot \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right] + c_3 e^{-x} \cdot \left[\begin{array}{c} \frac{\cos\left(\sqrt{3}\, x\right)}{8} \\ -\frac{\cos\left(\sqrt{3}\, x\right)}{8} - \frac{\sqrt{3}\, \sin\left(\sqrt{3}\, x\right)}{8} \\ -\frac{\cos\left(\sqrt{3}\, x\right)}{4} + \frac{\sqrt{3}\, \sin\left(\sqrt{3}\, x\right)}{4} \\ \cos\left(\sqrt{3}\, x\right) \end{array}\right] + c_4 e^{-x} \cdot \left[\begin{array}{c} -\frac{\sin\left(\sqrt{3}\, x\right)}{8} \\ -\frac{\cos\left(\sqrt{3}\, x\right)\sqrt{3}}{8} + \frac{\sin\left(\sqrt{3}\, x\right)}{8} \\ \frac{\cos\left(\sqrt{3}\, x\right)\sqrt{3}}{4} + \frac{\sin\left(\sqrt{3}\, x\right)}{4} \\ -\sin\left(\sqrt{3}\, x\right) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{c_2 e^{2x}}{8} + \frac{c_3 e^{-x} \cos\left(\sqrt{3}\, x\right)}{8} - \frac{c_4 e^{-x} \sin\left(\sqrt{3}\, x\right)}{8} + c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$4)-8*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 e^{2x} + c_3 e^{-x} \sin\left(\sqrt{3}\, x\right) + c_4 e^{-x} \cos\left(\sqrt{3}\, x\right)$$

✓ Solution by Mathematica

Time used: 0.658 (sec). Leaf size: 70

```
DSolve[y''''[x]-8*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4} e^{-x} \left(2 c_1 e^{3x} - \left(c_2 + \sqrt{3} c_3\right) \cos\left(\sqrt{3} x\right) + \left(\sqrt{3} c_2 - c_3\right) \sin\left(\sqrt{3} x\right)\right) + c_4$$

10.7 problem 7

Internal problem ID [12766]
Internal file name [OUTPUT/11418_Friday_November_03_2023_06_32_47_AM_53036301/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210
Problem number: 7.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{36y'''' - 12y''' - 11y'' + 2y' + y = 0}$$

The characteristic equation is

$$36\lambda^4 - 12\lambda^3 - 11\lambda^2 + 2\lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{1}{3}$$
$$\lambda_2 = -\frac{1}{3}$$
$$\lambda_3 = \frac{1}{2}$$
$$\lambda_4 = \frac{1}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-\frac{x}{3}} + x\,\mathrm{e}^{-\frac{x}{3}} c_2 + \mathrm{e}^{\frac{x}{2}} c_3 + x\,\mathrm{e}^{\frac{x}{2}} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-\frac{x}{3}} \\
y_2 &= x\,\mathrm{e}^{-\frac{x}{3}} \\
y_3 &= \mathrm{e}^{\frac{x}{2}} \\
y_4 &= \mathrm{e}^{\frac{x}{2}} x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{3}} + x\,\mathrm{e}^{-\frac{x}{3}} c_2 + \mathrm{e}^{\frac{x}{2}} c_3 + x\,\mathrm{e}^{\frac{x}{2}} c_4 \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x}{3}} + x\,\mathrm{e}^{-\frac{x}{3}} c_2 + \mathrm{e}^{\frac{x}{2}} c_3 + x\,\mathrm{e}^{\frac{x}{2}} c_4
$$

Verified OK.

10.7.1 Maple step by step solution

Let's solve

$36y'''' - 12y''' - 11y'' + 2y' + y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- Isolate 4th derivative

  $y'''' = \frac{y'''}{3} + \frac{11y''}{36} - \frac{y'}{18} - \frac{y}{36}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'''' - \frac{y'''}{3} - \frac{11y''}{36} + \frac{y'}{18} + \frac{y}{36} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

$y_4(x) = y'''$

- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = \frac{y_4(x)}{3} + \frac{11y_3(x)}{36} - \frac{y_2(x)}{18} - \frac{y_1(x)}{36}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = \frac{y_4(x)}{3} + \frac{11y_3(x)}{36} - \frac{y_2(x)}{18} - \frac{y_1(x)}{36}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{36} & -\frac{1}{18} & \frac{11}{36} & \frac{1}{3} \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{36} & -\frac{1}{18} & \frac{11}{36} & \frac{1}{3} \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{3}, \begin{bmatrix} -27 \\ 9 \\ -3 \\ 1 \end{bmatrix}\right], \left[-\frac{1}{3}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-\frac{1}{3}, \left[\begin{array}{c}-27 \\ 9 \\ -3 \\ 1\end{array}\right]\right]$$

- First solution from eigenvalue $-\frac{1}{3}$

$$\overrightarrow{y}_1(x) = \mathrm{e}^{-\frac{x}{3}} \cdot \left[\begin{array}{c}-27 \\ 9 \\ -3 \\ 1\end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -\frac{1}{3}$ is the eigenvalue, a

$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$

- Simplify equation

$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue $-\frac{1}{3}$

$$\left(\left[\begin{array}{cccc}0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{36} & -\frac{1}{18} & \frac{11}{36} & \frac{1}{3}\end{array}\right] - -\frac{1}{3} \cdot \left[\begin{array}{cccc}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c}-27 \\ 9 \\ -3 \\ 1\end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} -81 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue $-\frac{1}{3}$

$$\overrightarrow{y}_2(x) = \mathrm{e}^{-\frac{x}{3}} \cdot \left(x \cdot \begin{bmatrix} -27 \\ 9 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -81 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[\frac{1}{2}, \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue $\frac{1}{2}$

$$\overrightarrow{y}_3(x) = \mathrm{e}^{\frac{x}{2}} \cdot \begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = \frac{1}{2}$ is the eigenvalue, an

$$\overrightarrow{y}_4(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_4(x)$ into the homogeneous system

$$\lambda \,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda \,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

  $\lambda \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_4(x)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue $\frac{1}{2}$

  $$\left(\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{36} & -\frac{1}{18} & \frac{11}{36} & \frac{1}{3} \end{array}\right] - \frac{1}{2} \cdot \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c} 8 \\ 4 \\ 2 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p} = \left[\begin{array}{c} -16 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue $\frac{1}{2}$

  $$\overrightarrow{y}_4(x) = e^{\frac{x}{2}} \cdot \left(x \cdot \left[\begin{array}{c} 8 \\ 4 \\ 2 \\ 1 \end{array}\right] + \left[\begin{array}{c} -16 \\ 0 \\ 0 \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

  $\overrightarrow{y} = c_1 \overrightarrow{y}_1(x) + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3(x) + c_4 \overrightarrow{y}_4(x)$

- Substitute solutions into the general solution

  $$\overrightarrow{y} = c_1 e^{-\frac{x}{3}} \cdot \left[\begin{array}{c} -27 \\ 9 \\ -3 \\ 1 \end{array}\right] + c_2 e^{-\frac{x}{3}} \cdot \left(x \cdot \left[\begin{array}{c} -27 \\ 9 \\ -3 \\ 1 \end{array}\right] + \left[\begin{array}{c} -81 \\ 0 \\ 0 \\ 0 \end{array}\right]\right) + e^{\frac{x}{2}} c_3 \cdot \left[\begin{array}{c} 8 \\ 4 \\ 2 \\ 1 \end{array}\right] + c_4 e^{\frac{x}{2}} \cdot \left(x \cdot \left[\begin{array}{c} 8 \\ 4 \\ 2 \\ 1 \end{array}\right]\right.$$

- First component of the vector is the solution to the ODE

$$y = -27\,\mathrm{e}^{-\frac{x}{3}}\left(\frac{8((-x+2)c_4-c_3)\,\mathrm{e}^{\frac{5x}{6}}}{27}+c_2(x+3)+c_1\right)$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 24

```
dsolve(36*diff(y(x),x$4)-12*diff(y(x),x$3)-11*diff(y(x),x$2)+2*diff(y(x),x)+y(x)=0,y(x), sin
```

$$y(x) = \left((c_4x+c_3)\,\mathrm{e}^{\frac{5x}{6}}+c_2x+c_1\right)\mathrm{e}^{-\frac{x}{3}}$$

✓ Solution by Mathematica
Time used: 0.005 (sec). Leaf size: 41

```
DSolve[36*y''''[x]-12*y'''[x]-11*y''[x]+2*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x)\to e^{-x/3}\left(c_3e^{5x/6}+x\left(c_4e^{5x/6}+c_2\right)+c_1\right)$$

10.8 problem 8

Internal problem ID [12767]
Internal file name [`OUTPUT/11419_Friday_November_03_2023_06_32_47_AM_75314869/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210
Problem number: 8.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(5)} - 3y'''' + 3y''' - 3y'' + 2y' = 0}$$

The characteristic equation is

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - 3\lambda^2 + 2\lambda = 0$$

The roots of the above equation are

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 1 \\ \lambda_3 &= 2 \\ \lambda_4 &= i \\ \lambda_5 &= -i \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^x + e^{2x} c_3 + e^{ix} c_4 + e^{-ix} c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^x$$
$$y_3 = e^{2x}$$
$$y_4 = e^{ix}$$
$$y_5 = e^{-ix}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^x + e^{2x} c_3 + e^{ix} c_4 + e^{-ix} c_5 \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^x + e^{2x} c_3 + e^{ix} c_4 + e^{-ix} c_5$$

Verified OK.

10.8.1 Maple step by step solution

Let's solve

$y^{(5)} - 3y'''' + 3y''' - 3y'' + 2y' = 0$

- Highest derivative means the order of the ODE is 5

  $y^{(5)}$

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Define new variable $y_5(x)$

    $y_5(x) = y''''$

- Isolate for $y_5'(x)$ using original ODE

  $y_5'(x) = 3y_5(x) - 3y_4(x) + 3y_3(x) - 2y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = 3y_5(x) - 3y_4(x) + 3y_3(x) -$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 3 & -3 & 3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 3 & -3 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right], \left[1, \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right], \left[2, \begin{bmatrix}\frac{1}{16}\\\frac{1}{8}\\\frac{1}{4}\\\frac{1}{2}\\1\end{bmatrix}\right], \left[-I, \begin{bmatrix}1\\-I\\-1\\I\\1\end{bmatrix}\right], \left[I, \begin{bmatrix}1\\I\\-1\\-I\\1\end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, & \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\begin{bmatrix} -\mathrm{I}, & \begin{bmatrix} 1 \\ -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} 1 \\ -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} 1 \\ -I \\ -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} \cos(x) - I\sin(x) \\ -I(\cos(x) - I\sin(x)) \\ -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_4(x) = \begin{bmatrix} \cos(x) \\ -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_5(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + c_4\vec{y}_4(x) + c_5\vec{y}_5(x)$

- Substitute solutions into the general solution

$$\vec{y} = c_2 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + e^{2x} c_3 \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} c_1 + c_4 \cos(x) - c_5 \sin(x) \\ -c_4 \sin(x) - c_5 \cos(x) \\ -c_4 \cos(x) + c_5 \sin(x) \\ c_4 \sin(x) + c_5 \cos(x) \\ c_4 \cos(x) - c_5 \sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = c_2 e^{x} + \frac{e^{2x} c_3}{16} - c_5 \sin(x) + c_4 \cos(x) + c_1$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$5)-3*diff(y(x),x$4)+3*diff(y(x),x$3)-3*diff(y(x),x$2)+2*diff(y(x),x)=0,y(
```

$$y(x) = c_1 + c_2 e^{x} + c_3 e^{2x} + c_4 \sin(x) + c_5 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 36

```
DSolve[y'''''[x]-3*y''''[x]+3*y'''[x]-3*y''[x]+2*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to c_3 e^{x} + \frac{1}{2} c_4 e^{2x} - c_2 \cos(x) + c_1 \sin(x) + c_5$$

10.9 problem 9

Internal problem ID [12768]
Internal file name [`OUTPUT/11420_Friday_November_03_2023_06_32_47_AM_78862228/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210
Problem number: 9.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(5)} - y'''' + y''' + 35y'' + 16y' - 52y = 0}$$

The characteristic equation is

$$\lambda^5 - \lambda^4 + \lambda^3 + 35\lambda^2 + 16\lambda - 52 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= 2 - 3i\\ \lambda_3 &= 2 + 3i\\ \lambda_4 &= -2\\ \lambda_5 &= -2\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + x\, e^{-2x} c_2 + c_3 e^x + e^{(2-3i)x} c_4 + e^{(2+3i)x} c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = x e^{-2x}$$
$$y_3 = e^{x}$$
$$y_4 = e^{(2-3i)x}$$
$$y_5 = e^{(2+3i)x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + x e^{-2x} c_2 + c_3 e^{x} + e^{(2-3i)x} c_4 + e^{(2+3i)x} c_5 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + x e^{-2x} c_2 + c_3 e^{x} + e^{(2-3i)x} c_4 + e^{(2+3i)x} c_5$$

Verified OK.

10.9.1 Maple step by step solution

Let's solve

$y^{(5)} - y'''' + y''' + 35y'' + 16y' - 52y = 0$

- Highest derivative means the order of the ODE is 5

  $y^{(5)}$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Define new variable $y_5(x)$

    $y_5(x) = y''''$

- Isolate for $y_5'(x)$ using original ODE

  $y_5'(x) = y_5(x) - y_4(x) - 35y_3(x) - 16y_2(x) + 52y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = y_5(x) - y_4(x) - 35y_3(x) -$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 52 & -16 & -35 & -1 & 1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 52 & -16 & -35 & -1 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[-2, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right], \left[2-3\,\mathrm{I}, \left[\begin{array}{c} -\frac{119}{28561}-\frac{120\,\mathrm{I}}{28561} \\ -\frac{46}{2197}+\frac{9\,\mathrm{I}}{2197} \\ -\frac{5}{169}+\frac{12\,\mathrm{I}}{169} \\ \frac{2}{13}+\frac{3\,\mathrm{I}}{13} \\ 1 \end{array}\right]\right], \left[2+3\,\mathrm{I}, \left[\begin{array}{c} -\frac{119}{28561} \\ -\frac{46}{2197} \\ -\frac{5}{169} \\ \frac{2}{13} \end{array}\right.\right.$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-2, \left[\begin{array}{c} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue -2

$$\overrightarrow{y}_1(x) = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -2$ is the eigenvalue, a

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x}A\right)\cdot\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A\cdot\overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A\cdot\overrightarrow{p}$$

- Make use of the identity matrix I

$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -2

$$\left(\left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 52 & -16 & -35 & -1 & 1 \end{array}\right] - (-2) \cdot \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]\right) \cdot \vec{p} = \left[\begin{array}{c} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p} = \left[\begin{array}{c} \frac{1}{32} \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue -2

$$\vec{y}_2(x) = \mathrm{e}^{-2x} \cdot \left(x \cdot \left[\begin{array}{c} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right] + \left[\begin{array}{c} \frac{1}{32} \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2-3\,\mathrm{I}, \begin{bmatrix} -\frac{119}{28561} - \frac{120\,\mathrm{I}}{28561} \\ -\frac{46}{2197} + \frac{9\,\mathrm{I}}{2197} \\ -\frac{5}{169} + \frac{12\,\mathrm{I}}{169} \\ \frac{2}{13} + \frac{3\,\mathrm{I}}{13} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(2-3\,\mathrm{I})x} \cdot \begin{bmatrix} -\frac{119}{28561} - \frac{120\,\mathrm{I}}{28561} \\ -\frac{46}{2197} + \frac{9\,\mathrm{I}}{2197} \\ -\frac{5}{169} + \frac{12\,\mathrm{I}}{169} \\ \frac{2}{13} + \frac{3\,\mathrm{I}}{13} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{2x} \cdot (\cos(3x) - \mathrm{I}\sin(3x)) \cdot \begin{bmatrix} -\frac{119}{28561} - \frac{120\,\mathrm{I}}{28561} \\ -\frac{46}{2197} + \frac{9\,\mathrm{I}}{2197} \\ -\frac{5}{169} + \frac{12\,\mathrm{I}}{169} \\ \frac{2}{13} + \frac{3\,\mathrm{I}}{13} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{2x} \cdot \begin{bmatrix} \left(-\frac{119}{28561} - \frac{120\,I}{28561}\right)(\cos(3x) - I\sin(3x)) \\ \left(-\frac{46}{2197} + \frac{9\,I}{2197}\right)(\cos(3x) - I\sin(3x)) \\ \left(-\frac{5}{169} + \frac{12\,I}{169}\right)(\cos(3x) - I\sin(3x)) \\ \left(\frac{2}{13} + \frac{3\,I}{13}\right)(\cos(3x) - I\sin(3x)) \\ \cos(3x) - I\sin(3x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_4(x) = e^{2x} \cdot \begin{bmatrix} -\frac{119\cos(3x)}{28561} - \frac{120\sin(3x)}{28561} \\ -\frac{46\cos(3x)}{2197} + \frac{9\sin(3x)}{2197} \\ -\frac{5\cos(3x)}{169} + \frac{12\sin(3x)}{169} \\ \frac{2\cos(3x)}{13} + \frac{3\sin(3x)}{13} \\ \cos(3x) \end{bmatrix}, \vec{y}_5(x) = e^{2x} \cdot \begin{bmatrix} \frac{119\sin(3x)}{28561} - \frac{120\cos(3x)}{28561} \\ \frac{46\sin(3x)}{2197} + \frac{9\cos(3x)}{2197} \\ \frac{5\sin(3x)}{169} + \frac{12\cos(3x)}{169} \\ -\frac{2\sin(3x)}{13} + \frac{3\cos(3x)}{13} \\ -\sin(3x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3 + c_4\vec{y}_4(x) + c_5\vec{y}_5(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{-2x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{16} \\ -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{32} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + e^{2x} c_4 \cdot \begin{bmatrix} -\frac{119\cos(3x)}{28561} \\ -\frac{46\cos(3x)}{2197} \\ -\frac{5\cos(3x)}{169} \\ \frac{2\cos(3x)}{13} + \\ \cos(\end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{e^{-2x}\left(\frac{16\left((-119c_4 - 120c_5)\cos(3x) - 120\sin(3x)\left(c_4 - \frac{119c_5}{120}\right)\right)e^{4x}}{28561} + 16c_3e^{3x} + \left(x + \frac{1}{2}\right)c_2 + c_1\right)}{16}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$5)-diff(y(x),x$4)+diff(y(x),x$3)+35*diff(y(x),x$2)+16*diff(y(x),x)-52*y(x
```

$$y(x) = \left(c_4 e^{4x} \sin(3x) + c_5 e^{4x} \cos(3x) + c_1 e^{3x} + c_3 x + c_2\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 50

```
DSolve[y'''''[x]-y''''[x]+y'''[x]+35*y''[x]+16*y'[x]-52*y[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to e^{-2x}\left(c_4 x + c_5 e^{3x} + c_2 e^{4x} \cos(3x) + c_1 e^{4x} \sin(3x) + c_3\right)$$

10.10 problem 10

Internal problem ID [12769]
Internal file name [`OUTPUT/11421_Friday_November_03_2023_06_32_47_AM_79775332/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210
Problem number: 10.
ODE order: 8.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(8)} + 8y'''' + 16y = 0}$$

The characteristic equation is

$$\lambda^8 + 8\lambda^4 + 16 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 1 - i \\
\lambda_2 &= 1 + i \\
\lambda_3 &= -1 - i \\
\lambda_4 &= -1 + i \\
\lambda_5 &= 1 - i \\
\lambda_6 &= 1 + i \\
\lambda_7 &= -1 - i \\
\lambda_8 &= -1 + i
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{(1+i)x}c_1 + x\,\mathrm{e}^{(1+i)x}c_2 + \mathrm{e}^{(1-i)x}c_3 + x\,\mathrm{e}^{(1-i)x}c_4 + \mathrm{e}^{(-1+i)x}c_5 + x\,\mathrm{e}^{(-1+i)x}c_6 + \mathrm{e}^{(-1-i)x}c_7 + x\,\mathrm{e}^{(-1-i)x}c_8$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{(1+i)x} \\
y_2 &= x\,e^{(1+i)x} \\
y_3 &= e^{(1-i)x} \\
y_4 &= x\,e^{(1-i)x} \\
y_5 &= e^{(-1+i)x} \\
y_6 &= x\,e^{(-1+i)x} \\
y_7 &= e^{(-1-i)x} \\
y_8 &= x\,e^{(-1-i)x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & e^{(1+i)x} c_1 + x\,e^{(1+i)x} c_2 + e^{(1-i)x} c_3 + x\,e^{(1-i)x} c_4 \\
& + e^{(-1+i)x} c_5 + x\,e^{(-1+i)x} c_6 + e^{(-1-i)x} c_7 + x\,e^{(-1-i)x} c_8
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & e^{(1+i)x} c_1 + x\,e^{(1+i)x} c_2 + e^{(1-i)x} c_3 + x\,e^{(1-i)x} c_4 \\
& + e^{(-1+i)x} c_5 + x\,e^{(-1+i)x} c_6 + e^{(-1-i)x} c_7 + x\,e^{(-1-i)x} c_8
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(diff(y(x),x$8)+8*diff(y(x),x$4)+16*y(x)=0,y(x), singsol=all)
```

$$
y(x) = ((c_4 x + c_2) \cos(x) + \sin(x)(c_3 x + c_1)) e^{-x} + e^{x}((c_8 x + c_6) \cos(x) + \sin(x)(c_7 x + c_5))
$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 66

```
DSolve[D[y[x],{x,8}]+8*y''''[x]+16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(\left(c_4x + c_7e^{2x} + c_8e^{2x}x + c_3\right)\cos(x) + \left(c_2x + c_5e^{2x} + c_6e^{2x}x + c_1\right)\sin(x)\right)$$

10.11 problem 11

Internal problem ID [12770]
Internal file name [`OUTPUT/11422_Friday_November_03_2023_06_32_48_AM_76405819/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' + \alpha y = 0}$$

10.11.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = \alpha$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \alpha\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \alpha = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = \alpha$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(\alpha)} \\
&= \pm\sqrt{-\alpha}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +\sqrt{-\alpha} \\
\lambda_2 &= -\sqrt{-\alpha}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= \sqrt{-\alpha} \\
\lambda_2 &= -\sqrt{-\alpha}
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{\left(\sqrt{-\alpha}\right)x} + c_2 e^{\left(-\sqrt{-\alpha}\right)x}
\end{aligned}
$$

Or

$$
y = c_1 \mathrm{e}^{\sqrt{-\alpha}\, x} + c_2 \mathrm{e}^{-\sqrt{-\alpha}\, x}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\sqrt{-\alpha}\, x} + c_2 \mathrm{e}^{-\sqrt{-\alpha}\, x} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{\sqrt{-\alpha}\, x} + c_2 \mathrm{e}^{-\sqrt{-\alpha}\, x}
$$

Verified OK.

10.11.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + \alpha y'y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + \alpha y'y) \, dx = 0$$
$$\frac{y'^2}{2} + \frac{\alpha y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-\alpha y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-\alpha y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-\alpha\, y^2 + 2c_1}} dy = \int dx$$
$$\frac{\arctan\left(\frac{\sqrt{\alpha}\, y}{\sqrt{-\alpha y^2 + 2c_1}}\right)}{\sqrt{\alpha}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-\alpha\, y^2 + 2c_1}} dy = \int dx$$
$$-\frac{\arctan\left(\frac{\sqrt{\alpha}\, y}{\sqrt{-\alpha y^2 + 2c_1}}\right)}{\sqrt{\alpha}} = c_3 + x$$

Summary
The solution(s) found are the following

$$
\frac{\arctan\left(\frac{\sqrt{\alpha}\, y}{\sqrt{-\alpha y^{2}+2 c_{1}}}\right)}{\sqrt{\alpha}} = x + c_{2} \tag{1}
$$

$$
-\frac{\arctan\left(\frac{\sqrt{\alpha}\, y}{\sqrt{-\alpha y^{2}+2 c_{1}}}\right)}{\sqrt{\alpha}} = c_{3} + x \tag{2}
$$

Verification of solutions

$$
\frac{\arctan\left(\frac{\sqrt{\alpha}\, y}{\sqrt{-\alpha y^{2}+2 c_{1}}}\right)}{\sqrt{\alpha}} = x + c_{2}
$$

Verified OK.

$$
-\frac{\arctan\left(\frac{\sqrt{\alpha}\, y}{\sqrt{-\alpha y^{2}+2 c_{1}}}\right)}{\sqrt{\alpha}} = c_{3} + x
$$

Verified OK.

10.11.3 Solving using Kovacic algorithm

Writing the ode as

$$
y'' + \alpha y = 0 \tag{1}
$$

$$
A y'' + B y' + C y = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= \alpha
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^{2} - 4AC}{4A^{2}}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-\alpha}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -\alpha \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (-\alpha)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 339: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\alpha$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{\sqrt{-\alpha}\, x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{\sqrt{-\alpha}\, x} \end{aligned}$$

Which simplifies to

$$y_1 = e^{\sqrt{-\alpha}\, x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= e^{\sqrt{-\alpha}\, x} \int \frac{1}{e^{2\sqrt{-\alpha}\, x}}\, dx \\ &= e^{\sqrt{-\alpha}\, x} \left(-\frac{e^{-2\sqrt{-\alpha}\, x}}{2\sqrt{-\alpha}} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\sqrt{-\alpha}\, x}\right) + c_2 \left(\mathrm{e}^{\sqrt{-\alpha}\, x} \left(-\frac{\mathrm{e}^{-2\sqrt{-\alpha}\, x}}{2\sqrt{-\alpha}}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\sqrt{-\alpha}\, x} - \frac{c_2 \mathrm{e}^{-\sqrt{-\alpha}\, x}}{2\sqrt{-\alpha}} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{\sqrt{-\alpha}\, x} - \frac{c_2 \mathrm{e}^{-\sqrt{-\alpha}\, x}}{2\sqrt{-\alpha}}
$$

Verified OK.

10.11.4 Maple step by step solution

Let's solve

$y'' + \alpha y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + \alpha = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{-4\alpha}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(\sqrt{-\alpha}, -\sqrt{-\alpha}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{\sqrt{-\alpha}\, x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\sqrt{-\alpha}\, x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\sqrt{-\alpha}\, x} + c_2 e^{-\sqrt{-\alpha}\, x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+alpha*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin\left(\sqrt{\alpha}\, x\right) + c_2 \cos\left(\sqrt{\alpha}\, x\right)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 28

```
DSolve[y''[x]+a*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(\sqrt{a}x\right) + c_2 \sin\left(\sqrt{a}x\right)$$

10.12 problem 17

Internal problem ID [12771]
Internal file name [`OUTPUT/11423_Friday_November_03_2023_06_32_48_AM_73742763/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210
Problem number: 17.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + (-3 - 4i)\, y'' + (-4 + 12i)\, y' + 12y = 0}$$

The characteristic equation is

$$\lambda^3 - 4i\lambda^2 - 3\lambda^2 + 12i\lambda - 4\lambda + 12 = 0$$

The roots of the above equation are

$$\lambda_1 = 3$$
$$\lambda_2 = 2i$$
$$\lambda_3 = 2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{3x} + \mathrm{e}^{2ix} c_2 + x\, \mathrm{e}^{2ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{3x}$$
$$y_2 = \mathrm{e}^{2ix}$$
$$y_3 = x\, \mathrm{e}^{2ix}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{3x} + e^{2ix} c_2 + x e^{2ix} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{3x} + e^{2ix} c_2 + x e^{2ix} c_3$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$3)-(3+4*I)*diff(y(x),x$2)-(4-12*I)*diff(y(x),x)+12*y(x)=0,y(x), singsol=a
```

$$y(x) = (c_3 x + c_2) e^{2ix} + c_1 e^{3x}$$

✓ Solution by Mathematica
Time used: 0.005 (sec). Leaf size: 29

```
DSolve[y'''[x]-(3+4*I)*y''[x]-(4-12*I)*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{2ix}(c_2 x + c_1) + c_3 e^{3x}$$

10.13 problem 18

Internal problem ID [12772]
Internal file name [`OUTPUT/11424_Friday_November_03_2023_06_32_48_AM_49111732/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210
Problem number: 18.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + (-3 - i)\, y''' + (4 + 3i)\, y'' = 0}$$

The characteristic equation is

$$\lambda^4 - i\lambda^3 - 3\lambda^3 + 3i\lambda^2 + 4\lambda^2 = 0$$

The roots of the above equation are

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 0 \\ \lambda_3 &= 1 + 2i \\ \lambda_4 &= 2 - i \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + \mathrm{e}^{(2-i)x} c_3 + \mathrm{e}^{(1+2i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= 1 \\ y_2 &= x \\ y_3 &= \mathrm{e}^{(2-i)x} \\ y_4 &= \mathrm{e}^{(1+2i)x} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 x + c_1 + e^{(2-i)x} c_3 + e^{(1+2i)x} c_4 \tag{1}$$

Verification of solutions

$$y = c_2 x + c_1 + e^{(2-i)x} c_3 + e^{(1+2i)x} c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$4)-(3+I)*diff(y(x),x$3)+(4+3*I)*diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{(1+2i)x} + c_2 e^{(2-i)x} + c_3 + c_4 x$$

✓ Solution by Mathematica
Time used: 0.156 (sec). Leaf size: 46

```
DSolve[y''''[x]-(3+I)*y'''[x]+(4+3*I)*y''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(-\frac{3}{25} - \frac{4i}{25}\right) c_1 e^{(1+2i)x} + \left(\frac{3}{25} + \frac{4i}{25}\right) c_2 e^{(2-i)x} + c_4 x + c_3$$

10.14 problem 19

Internal problem ID [12773]
Internal file name [`OUTPUT/11425_Friday_November_03_2023_06_32_49_AM_49345908/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.3, page 210
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - iy = 0}$$

With initial conditions

$$[y(0) = 1]$$

10.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -i$$
$$q(x) = 0$$

Hence the ode is

$$y' - iy = 0$$

The domain of $p(x) = -i$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Hence solution exists and is unique.

10.14.2 Solving as quadrature ode

Integrating both sides gives

$$\int -\frac{i}{y} dy = \int dx$$
$$-i \ln(y) = x + c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{-i \ln(y)} = \mathrm{e}^{x+c_1}$$

Which simplifies to

$$y^{-i} = c_2 \mathrm{e}^{x}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_2^{i}$$

$$c_2 = 1$$

Substituting c_2 found above in the general solution gives

$$y = (\mathrm{e}^{x})^{i}$$

Summary

The solution(s) found are the following

$$y = (\mathrm{e}^{x})^{i} \tag{1}$$

Verification of solutions

$$y = (\mathrm{e}^{x})^{i}$$

Verified OK.

10.14.3 Maple step by step solution

Let's solve

$[y' - \mathrm{I}y = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

  $\frac{y'}{y} = \mathrm{I}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \mathrm{I} dx + c_1$

- Evaluate integral

  $\ln (y) = \mathrm{I}x + c_1$

- Use initial condition $y(0) = 1$

  $0 = c_1$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $\ln (y) = \mathrm{I}x$

- Solution to the IVP

  $\ln (y) = \mathrm{I}x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([diff(y(x),x)-I*y(x)=0,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{ix}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 12

```
DSolve[{y'[x]-I*y[x]==0,{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{ix}$$

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11.1 problem 1

Internal problem ID [12774]
Internal file name [`OUTPUT/11426_Friday_November_03_2023_06_32_50_AM_61348493/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.4, page 218
Problem number: 1.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 6y''' + 13y'' - 12y' + 4y = 2\,\mathrm{e}^{x} - 4\,\mathrm{e}^{2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 6y''' + 13y'' - 12y' + 4y = 0$$

The characteristic equation is

$$\lambda^4 - 6\lambda^3 + 13\lambda^2 - 12\lambda + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 2$$
$$\lambda_4 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + c_2 x\, e^x + e^{2x} c_3 + x\, e^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^x \\ y_2 &= x\, e^x \\ y_3 &= e^{2x} \\ y_4 &= e^{2x} x \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 6y''' + 13y'' - 12y' + 4y = 2\, e^x - 4\, e^{2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\, e^x - 4\, e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^x, e^{2x} x, e^x, e^{2x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^x\}, \{e^{2x}\}]$$

Since $x\, e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^x\}, \{e^{2x}\}]$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^x\}, \{e^{2x} x\}]$$

Since $e^{2x} x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^x\}, \{x^2 e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^x + A_2 x^2 e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x + 2A_2 e^{2x} = 2\,e^x - 4\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 e^x - 2x^2 e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + c_2 x\, e^x + e^{2x} c_3 + x\, e^{2x} c_4\right) + \left(x^2 e^x - 2x^2 e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = (c_4 x + c_3)\, e^{2x} + e^x (c_2 x + c_1) + x^2 e^x - 2x^2 e^{2x}$$

Summary

The solution(s) found are the following

$$y = (c_4 x + c_3)\, e^{2x} + e^x (c_2 x + c_1) + x^2 e^x - 2x^2 e^{2x} \tag{1}$$

Verification of solutions

$$y = (c_4 x + c_3)\, e^{2x} + e^x (c_2 x + c_1) + x^2 e^x - 2x^2 e^{2x}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$4)-6*diff(y(x),x$3)+13*diff(y(x),x$2)-12*diff(y(x),x)+4*y(x)=2*exp(x)-4*e
```

$$y(x) = \left(-2x^2 + (c_4 + 8)\,x + c_2 - 12\right) \mathrm{e}^{2x} + \left(x^2 + (c_3 + 4)\,x + c_1 + 6\right) \mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.187 (sec). Leaf size: 41

```
DSolve[y''''[x]-6*y'''[x]+13*y''[x]-12*y'[x]+4*y[x]==2*Exp[x]-4*Exp[2*x],y[x],x,IncludeSingu
```

$$y(x) \to e^x \left(x^2 + e^x \left(-2x^2 + (8 + c_4)x - 12 + c_3\right) + (4 + c_2)x + 6 + c_1\right)$$

11.2 problem 2

Internal problem ID [12775]
Internal file name [`OUTPUT/11427_Friday_November_03_2023_06_32_50_AM_27271063/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.4, page 218
Problem number: 2.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' + 4y'' = 24x^2 - 6x + 14 + 32\cos(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 4y'' = 0$$

The characteristic equation is

$$\lambda^4 + 4\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 2i$$
$$\lambda_4 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_2x + c_1 + e^{2ix}c_3 + e^{-2ix}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= \mathrm{e}^{2ix} \\
y_4 &= \mathrm{e}^{-2ix}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y'''' + 4y'' = 24x^2 - 6x + 14 + 32\cos(2x)
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
\begin{aligned}
W &= \begin{bmatrix} 1 & x & \mathrm{e}^{2ix} & \mathrm{e}^{-2ix} \\ 0 & 1 & 2i\mathrm{e}^{2ix} & -2i\mathrm{e}^{-2ix} \\ 0 & 0 & -4\,\mathrm{e}^{2ix} & -4\,\mathrm{e}^{-2ix} \\ 0 & 0 & -8i\mathrm{e}^{2ix} & 8i\mathrm{e}^{-2ix} \end{bmatrix} \\
|W| &= -64i\mathrm{e}^{2ix}\mathrm{e}^{-2ix}
\end{aligned}
$$

The determinant simplifies to

$$|W| = -64i$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} x & e^{2ix} & e^{-2ix} \\ 1 & 2ie^{2ix} & -2ie^{-2ix} \\ 0 & -4\,e^{2ix} & -4\,e^{-2ix} \end{bmatrix}$$
$$= -16ix$$

$$W_2(x) = \det \begin{bmatrix} 1 & e^{2ix} & e^{-2ix} \\ 0 & 2ie^{2ix} & -2ie^{-2ix} \\ 0 & -4\,e^{2ix} & -4\,e^{-2ix} \end{bmatrix}$$
$$= -16i$$

$$W_3(x) = \det \begin{bmatrix} 1 & x & e^{-2ix} \\ 0 & 1 & -2ie^{-2ix} \\ 0 & 0 & -4\,e^{-2ix} \end{bmatrix}$$
$$= -4\,e^{-2ix}$$

$$W_4(x) = \det \begin{bmatrix} 1 & x & e^{2ix} \\ 0 & 1 & 2ie^{2ix} \\ 0 & 0 & -4\,e^{2ix} \end{bmatrix}$$
$$= -4\,e^{2ix}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^3 \int \frac{(24x^2 - 6x + 14 + 32\cos(2x))\,(-16ix)}{(1)\,(-64i)}\,dx \\ &= -\int \frac{-16i(24x^2 - 6x + 14 + 32\cos(2x))\,x}{-64i}\,dx \\ &= -\int \left(\frac{(7 + 12x^2 - 3x + 16\cos(2x))\,x}{2}\right)dx \\ &= -\frac{3x^4}{2} - 2\cos(2x) - 4\sin(2x)\,x + \frac{x^3}{2} - \frac{7x^2}{4} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{(24x^2 - 6x + 14 + 32\cos(2x))(-16i)}{(1)(-64i)}\,dx \\
&= \int \frac{-16i(24x^2 - 6x + 14 + 32\cos(2x))}{-64i}\,dx \\
&= \int \left(6x^2 - \frac{3x}{2} + \frac{7}{2} + 8\cos(2x)\right)dx \\
&= \frac{7x}{2} - \frac{3x^2}{4} + 2x^3 + 4\sin(2x)
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(24x^2 - 6x + 14 + 32\cos(2x))(-4\,\mathrm{e}^{-2ix})}{(1)(-64i)}\,dx \\
&= -\int \frac{-4(24x^2 - 6x + 14 + 32\cos(2x))\,\mathrm{e}^{-2ix}}{-64i}\,dx \\
&= -\int \left(-\frac{i(7 + 12x^2 - 3x + 16\cos(2x))\,\mathrm{e}^{-2ix}}{8}\right)dx \\
&= -\frac{(24x^2 - 24ix - 6x + 3i + 2)\,\mathrm{e}^{-2ix}}{32} + ix - \frac{\mathrm{e}^{-4ix}}{4} \\
&= -\frac{(24x^2 - 24ix - 6x + 3i + 2)\,\mathrm{e}^{-2ix}}{32} + ix - \frac{\mathrm{e}^{-4ix}}{4}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(24x^2 - 6x + 14 + 32\cos(2x))(-4\,\mathrm{e}^{2ix})}{(1)(-64i)}\,dx \\
&= \int \frac{-4(24x^2 - 6x + 14 + 32\cos(2x))\,\mathrm{e}^{2ix}}{-64i}\,dx \\
&= \int \left(-\frac{i(7 + 12x^2 - 3x + 16\cos(2x))\,\mathrm{e}^{2ix}}{8}\right)dx \\
&= -ix - \frac{\mathrm{e}^{4ix}}{4} - \frac{(24x^2 + 24ix - 6x - 3i + 2)\,\mathrm{e}^{2ix}}{32} \\
&= -ix - \frac{\mathrm{e}^{4ix}}{4} - \frac{(24x^2 + 24ix - 6x - 3i + 2)\,\mathrm{e}^{2ix}}{32}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\frac{3x^4}{2} - 2\cos(2x) - 4\sin(2x)\,x + \frac{x^3}{2} - \frac{7x^2}{4}\right) \\
&+ \left(\frac{7x}{2} - \frac{3x^2}{4} + 2x^3 + 4\sin(2x)\right)(x) \\
&+ \left(-\frac{(24x^2 - 24ix - 6x + 3i + 2)\,\mathrm{e}^{-2ix}}{32} + ix - \frac{\mathrm{e}^{-4ix}}{4}\right)\left(\mathrm{e}^{2ix}\right) \\
&+ \left(-ix - \frac{\mathrm{e}^{4ix}}{4} - \frac{(24x^2 + 24ix - 6x - 3i + 2)\,\mathrm{e}^{2ix}}{32}\right)\left(\mathrm{e}^{-2ix}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{x^2}{4} + \frac{x^4}{2} + \frac{3x}{8} - \frac{x^3}{4} - \frac{1}{8} - \frac{5\cos(2x)}{2} - 2\sin(2x)\,x
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_2x + c_1 + \mathrm{e}^{2ix}c_3 + \mathrm{e}^{-2ix}c_4\right) + \left(\frac{x^2}{4} + \frac{x^4}{2} + \frac{3x}{8} - \frac{x^3}{4} - \frac{1}{8} - \frac{5\cos(2x)}{2} - 2\sin(2x)\,x\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2x + c_1 + \mathrm{e}^{2ix}c_3 + \mathrm{e}^{-2ix}c_4 + \frac{x^2}{4} + \frac{x^4}{2} + \frac{3x}{8} - \frac{x^3}{4} - \frac{1}{8} - \frac{5\cos(2x)}{2} - 2\sin(2x)\,x \quad (1)
$$

Verification of solutions

$$
y = c_2x + c_1 + \mathrm{e}^{2ix}c_3 + \mathrm{e}^{-2ix}c_4 + \frac{x^2}{4} + \frac{x^4}{2} + \frac{3x}{8} - \frac{x^3}{4} - \frac{1}{8} - \frac{5\cos(2x)}{2} - 2\sin(2x)\,x
$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 24*_a^2+32*cos(2*_a)-4*_b(_a)
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(diff(y(x),x$4)+4*diff(y(x),x$2)=24*x^2-6*x+14+32*cos(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(-c_1 - 10)\cos(2x)}{4} + \frac{(-8x - c_2)\sin(2x)}{4} + \frac{x^4}{2} - \frac{x^3}{4} + \frac{x^2}{4} + c_3x + c_4$$

✓ Solution by Mathematica

Time used: 1.052 (sec). Leaf size: 54

```
DSolve[y''''[x]+4*y''[x]==24*x^2-6*x+14+32*Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(2x^4 - x^3 + x^2 + 4c_4x - (12 + c_1)\cos(2x) - (8x + c_2)\sin(2x) + 4c_3\right)$$

11.3 problem 3

Internal problem ID [12776]
Internal file name [`OUTPUT/11428_Friday_November_03_2023_06_32_51_AM_25526768/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.4, page 218
Problem number: 3.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _linear, _nonhomogeneous]]`

$$\boxed{y'''' + 2y'' + y = 3 + \cos(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y'' + y = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^2 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = e^{ix}c_1 + x\,e^{ix}c_2 + e^{-ix}c_3 + x\,e^{-ix}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{ix} \\
y_2 &= x\,\mathrm{e}^{ix} \\
y_3 &= \mathrm{e}^{-ix} \\
y_4 &= x\,\mathrm{e}^{-ix}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y'''' + 2y'' + y = 3 + \cos(2x)
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
3 + \cos(2x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}, \{\cos(2x), \sin(2x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{ix}, x\,\mathrm{e}^{-ix}, \mathrm{e}^{ix}, \mathrm{e}^{-ix}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
9A_2 \cos(2x) + 9A_3 \sin(2x) + A_1 = 3 + \cos(2x)
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = 3, A_2 = \frac{1}{9}, A_3 = 0\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3 + \frac{\cos(2x)}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{ix}c_1 + x\,e^{ix}c_2 + e^{-ix}c_3 + x\,e^{-ix}c_4\right) + \left(3 + \frac{\cos(2x)}{9}\right) \end{aligned}$$

Which simplifies to

$$y = (c_4x + c_3)\,e^{-ix} + (c_2x + c_1)\,e^{ix} + 3 + \frac{\cos(2x)}{9}$$

Summary

The solution(s) found are the following

$$y = (c_4x + c_3)\,e^{-ix} + (c_2x + c_1)\,e^{ix} + 3 + \frac{\cos(2x)}{9} \tag{1}$$

Verification of solutions

$$y = (c_4x + c_3)\,e^{-ix} + (c_2x + c_1)\,e^{ix} + 3 + \frac{\cos(2x)}{9}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$2)+y(x)=3+cos(2*x),y(x), singsol=all)
```

$$y(x) = 3 + \frac{\cos(2x)}{9} + (c_4 x + c_1)\cos(x) + (c_3 x + c_2)\sin(x)$$

✓ Solution by Mathematica

Time used: 0.199 (sec). Leaf size: 36

```
DSolve[y''''[x]+2*y''[x]+y[x]==3+Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9}\cos(2x) + (c_2 x + c_1)\cos(x) + c_3\sin(x) + c_4 x\sin(x) + 3$$

11.4 problem 4

Internal problem ID [12777]
Internal file name [`OUTPUT/11429_Friday_November_03_2023_06_32_51_AM_75044440/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.4, page 218
Problem number: 4.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' - 3y''' + 3y'' - y' = 6x - 20 - 120x^2 e^x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 3y''' + 3y'' - y' = 0$$

The characteristic equation is

$$\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^x + x e^x c_3 + x^2 e^x c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= 1 \\ y_2 &= e^x \\ y_3 &= x e^x \\ y_4 &= x^2 e^x \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 3y''' + 3y'' - y' = 6x - 20 - 120x^2 e^x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$6x - 20 - 120x^2 e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}, \{x e^x, x^2 e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x e^x, x^2 e^x, e^x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}, \{x e^x, x^2 e^x, e^x\}]$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}, \{x e^x, x^2 e^x, x^3 e^x\}]$$

Since $x e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}, \{x^2 e^x, x^3 e^x, x^4 e^x\}]$$

Since $x^2 e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}, \{x^3 e^x, x^4 e^x, e^x x^5\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2x^2 + A_1x + A_3x^3\mathrm{e}^x + A_4x^4\mathrm{e}^x + A_5\mathrm{e}^x x^5$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&120A_5\mathrm{e}^x x + 24A_4x\,\mathrm{e}^x + 60A_5\mathrm{e}^x x^2 - A_1 + 6A_2 - 2A_2x + 6A_3\mathrm{e}^x + 24A_4\mathrm{e}^x \\ &= 6x - 20 - 120x^2\mathrm{e}^x\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = -3, A_3 = -40, A_4 = 10, A_5 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -3x^2 + 2x - 40x^3\mathrm{e}^x + 10x^4\mathrm{e}^x - 2\,\mathrm{e}^x x^5$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(c_1 + c_2\mathrm{e}^x + x\,\mathrm{e}^x c_3 + x^2\mathrm{e}^x c_4\right) + \left(-3x^2 + 2x - 40x^3\mathrm{e}^x + 10x^4\mathrm{e}^x - 2\,\mathrm{e}^x x^5\right)\end{aligned}$$

Which simplifies to

$$y = \left(c_4x^2 + c_3x + c_2\right)\mathrm{e}^x + c_1 - 3x^2 + 2x - 40x^3\mathrm{e}^x + 10x^4\mathrm{e}^x - 2\,\mathrm{e}^x x^5$$

Summary

The solution(s) found are the following

$$y = \left(c_4x^2 + c_3x + c_2\right)\mathrm{e}^x + c_1 - 3x^2 + 2x - 40x^3\mathrm{e}^x + 10x^4\mathrm{e}^x - 2\,\mathrm{e}^x x^5 \tag{1}$$

Verification of solutions

$$y = \left(c_4x^2 + c_3x + c_2\right)\mathrm{e}^x + c_1 - 3x^2 + 2x - 40x^3\mathrm{e}^x + 10x^4\mathrm{e}^x - 2\,\mathrm{e}^x x^5$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(diff(_b(_a), _a), _a), _a) = -120*_a^2*exp(_a)-3
   Methods for third order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
   trying high order linear exact nonhomogeneous
   trying differential order: 3; missing the dependent variable
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 56

```
dsolve(diff(y(x),x$4)-3*diff(y(x),x$3)+3*diff(y(x),x$2)-diff(y(x),x)=6*x-20-120*x^2*exp(x),y
```

$$y(x) = (-2x^5 + 10x^4 - 40x^3 + (c_3 + 120)x^2 + (c_2 - 2c_3 - 240)x + c_1 - c_2 + 2c_3 + 240)e^x - 3x^2 + 2x + c_4$$

✓ Solution by Mathematica

Time used: 0.569 (sec). Leaf size: 65

```
DSolve[y''''[x]-3*y'''[x]+3*y''[x]-y'[x]==6*x-20-120*x^2*Exp[x],y[x],x,IncludeSingularSoluti
```

$$y(x) \to -3x^2 + e^x(-2x^5 + 10x^4 - 40x^3 + (120 + c_3)x^2 + (-240 + c_2 - 2c_3)x + 240 + c_1 - c_2 + 2c_3) + 2x + c_4$$

11.5 problem 5

Internal problem ID [12778]
Internal file name [`OUTPUT/11430_Friday_November_03_2023_06_32_52_AM_71108883/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.4, page 218
Problem number: 5.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''' - 6y'' + 21y' - 26y = 36\,\mathrm{e}^{2x} \sin(3x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 6y'' + 21y' - 26y = 0$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 21\lambda - 26 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 2 - 3i$$
$$\lambda_3 = 2 + 3i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1\mathrm{e}^{2x} + \mathrm{e}^{(2-3i)x}c_2 + \mathrm{e}^{(2+3i)x}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= \mathrm{e}^{2x} \\
y_2 &= \mathrm{e}^{(2-3i)x} \\
y_3 &= \mathrm{e}^{(2+3i)x}
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - 6y'' + 21y' - 26y = 36\,\mathrm{e}^{2x} \sin(3x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$\begin{aligned}
W &= \begin{bmatrix} \mathrm{e}^{2x} & \mathrm{e}^{(2-3i)x} & \mathrm{e}^{(2+3i)x} \\ 2\,\mathrm{e}^{2x} & (2-3i)\,\mathrm{e}^{(2-3i)x} & (2+3i)\,\mathrm{e}^{(2+3i)x} \\ 4\,\mathrm{e}^{2x} & (-5-12i)\,\mathrm{e}^{(2-3i)x} & (-5+12i)\,\mathrm{e}^{(2+3i)x} \end{bmatrix} \\
|W| &= 54i\mathrm{e}^{2x}\mathrm{e}^{(2-3i)x}\mathrm{e}^{(2+3i)x}
\end{aligned}$$

The determinant simplifies to

$$|W| = 54i\mathrm{e}^{6x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} e^{(2-3i)x} & e^{(2+3i)x} \\ (2-3i)\,e^{(2-3i)x} & (2+3i)\,e^{(2+3i)x} \end{bmatrix}$$
$$= 6ie^{4x}$$

$$W_2(x) = \det \begin{bmatrix} e^{2x} & e^{(2+3i)x} \\ 2\,e^{2x} & (2+3i)\,e^{(2+3i)x} \end{bmatrix}$$
$$= 3ie^{(4+3i)x}$$

$$W_3(x) = \det \begin{bmatrix} e^{2x} & e^{(2-3i)x} \\ 2\,e^{2x} & (2-3i)\,e^{(2-3i)x} \end{bmatrix}$$
$$= -3ie^{(4-3i)x}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{\left(36\,e^{2x}\sin(3x)\right)\left(6ie^{4x}\right)}{(1)\left(54ie^{6x}\right)}\,dx \\
&= \int \frac{216ie^{2x}\sin(3x)\,e^{4x}}{54ie^{6x}}\,dx \\
&= \int \left(4\sin(3x)\right)\,dx \\
&= -\frac{4\cos(3x)}{3}
\end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\
&= (-1)^1 \int \frac{\left(36\,\mathrm{e}^{2x} \sin\left(3x\right)\right)\left(3i\mathrm{e}^{(4+3i)x}\right)}{(1)\left(54i\mathrm{e}^{6x}\right)}\, dx \\
&= -\int \frac{108i\mathrm{e}^{2x} \sin\left(3x\right) \mathrm{e}^{(4+3i)x}}{54i\mathrm{e}^{6x}}\, dx \\
&= -\int \left(2\sin\left(3x\right) \mathrm{e}^{3ix}\right)\, dx \\
&= -\frac{-\frac{2i\mathrm{e}^{3ix}\tan\left(\frac{3x}{2}\right)}{3} + ix\,\mathrm{e}^{3ix} + 2x\,\mathrm{e}^{3ix}\tan\left(\frac{3x}{2}\right) - ix\,\mathrm{e}^{3ix}\tan\left(\frac{3x}{2}\right)^2}{1+\tan\left(\frac{3x}{2}\right)^2} \\
&= -\frac{-\frac{2i\mathrm{e}^{3ix}\tan\left(\frac{3x}{2}\right)}{3} + ix\,\mathrm{e}^{3ix} + 2x\,\mathrm{e}^{3ix}\tan\left(\frac{3x}{2}\right) - ix\,\mathrm{e}^{3ix}\tan\left(\frac{3x}{2}\right)^2}{1+\tan\left(\frac{3x}{2}\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^0 \int \frac{\left(36\,\mathrm{e}^{2x} \sin\left(3x\right)\right)\left(-3i\mathrm{e}^{(4-3i)x}\right)}{(1)\left(54i\mathrm{e}^{6x}\right)}\, dx \\
&= \int \frac{-108i\mathrm{e}^{2x} \sin\left(3x\right) \mathrm{e}^{(4-3i)x}}{54i\mathrm{e}^{6x}}\, dx \\
&= \int \left(-2\sin\left(3x\right) \mathrm{e}^{-3ix}\right)\, dx \\
&= \frac{-\frac{2i\mathrm{e}^{-3ix}\tan\left(\frac{3x}{2}\right)}{3} + ix\,\mathrm{e}^{-3ix} - 2x\,\mathrm{e}^{-3ix}\tan\left(\frac{3x}{2}\right) - ix\,\mathrm{e}^{-3ix}\tan\left(\frac{3x}{2}\right)^2}{1+\tan\left(\frac{3x}{2}\right)^2} \\
&= \frac{-\frac{2i\mathrm{e}^{-3ix}\tan\left(\frac{3x}{2}\right)}{3} + ix\,\mathrm{e}^{-3ix} - 2x\,\mathrm{e}^{-3ix}\tan\left(\frac{3x}{2}\right) - ix\,\mathrm{e}^{-3ix}\tan\left(\frac{3x}{2}\right)^2}{1+\tan\left(\frac{3x}{2}\right)^2}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(-\frac{4\cos\left(3x\right)}{3}\right)\left(\mathrm{e}^{2x}\right) \\
&+ \left(-\frac{-\frac{2i\mathrm{e}^{3ix}\tan\left(\frac{3x}{2}\right)}{3} + ix\,\mathrm{e}^{3ix} + 2x\,\mathrm{e}^{3ix}\tan\left(\frac{3x}{2}\right) - ix\,\mathrm{e}^{3ix}\tan\left(\frac{3x}{2}\right)^2}{1+\tan\left(\frac{3x}{2}\right)^2}\right)\left(\mathrm{e}^{(2-3i)x}\right) \\
&+ \left(\frac{-\frac{2i\mathrm{e}^{-3ix}\tan\left(\frac{3x}{2}\right)}{3} + ix\,\mathrm{e}^{-3ix} - 2x\,\mathrm{e}^{-3ix}\tan\left(\frac{3x}{2}\right) - ix\,\mathrm{e}^{-3ix}\tan\left(\frac{3x}{2}\right)^2}{1+\tan\left(\frac{3x}{2}\right)^2}\right)\left(\mathrm{e}^{(2+3i)x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$y_p = -\frac{2\,e^{2x}(3x\sin(3x) + 2\cos(3x))}{3}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{2x} + e^{(2-3i)x}c_2 + e^{(2+3i)x}c_3\right) + \left(-\frac{2\,e^{2x}(3x\sin(3x) + 2\cos(3x))}{3}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + e^{(2-3i)x}c_2 + e^{(2+3i)x}c_3 - \frac{2\,e^{2x}(3x\sin(3x) + 2\cos(3x))}{3} \tag{1}$$

Verification of solutions

$$y = c_1 e^{2x} + e^{(2-3i)x}c_2 + e^{(2+3i)x}c_3 - \frac{2\,e^{2x}(3x\sin(3x) + 2\cos(3x))}{3}$$

Verified OK.

11.5.1 Maple step by step solution

Let's solve

$y''' - 6y'' + 21y' - 26y = 36\,e^{2x}\sin(3x)$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 36\,e^{2x}\sin(3x) + 6y_3(x) - 21y_2(x) + 26y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 36\,\mathrm{e}^{2x} \sin(3x) + 6y_3(x) - 21y_2(x) + 26y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 26 & -21 & 6 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 36\,\mathrm{e}^{2x} \sin(3x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 36\,\mathrm{e}^{2x} \sin(3x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 26 & -21 & 6 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2 - 3\,\mathrm{I}, \begin{bmatrix} -\frac{5}{169} + \frac{12\,\mathrm{I}}{169} \\ \frac{2}{13} + \frac{3\,\mathrm{I}}{13} \\ 1 \end{bmatrix}\right], \left[2 + 3\,\mathrm{I}, \begin{bmatrix} -\frac{5}{169} - \frac{12\,\mathrm{I}}{169} \\ \frac{2}{13} - \frac{3\,\mathrm{I}}{13} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \mathrm{e}^{2x}\cdot\left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2-3\,\mathrm{I}, \left[\begin{array}{c}-\frac{5}{169}+\frac{12\,\mathrm{I}}{169}\\ \frac{2}{13}+\frac{3\,\mathrm{I}}{13}\\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(2-3\,\mathrm{I})x}\cdot\left[\begin{array}{c}-\frac{5}{169}+\frac{12\,\mathrm{I}}{169}\\ \frac{2}{13}+\frac{3\,\mathrm{I}}{13}\\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{2x}\cdot\left(\cos\left(3x\right)-\mathrm{I}\sin\left(3x\right)\right)\cdot\left[\begin{array}{c}-\frac{5}{169}+\frac{12\,\mathrm{I}}{169}\\ \frac{2}{13}+\frac{3\,\mathrm{I}}{13}\\ 1\end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{2x}\cdot\left[\begin{array}{c}\left(-\frac{5}{169}+\frac{12\,\mathrm{I}}{169}\right)\left(\cos\left(3x\right)-\mathrm{I}\sin\left(3x\right)\right)\\ \left(\frac{2}{13}+\frac{3\,\mathrm{I}}{13}\right)\left(\cos\left(3x\right)-\mathrm{I}\sin\left(3x\right)\right)\\ \cos\left(3x\right)-\mathrm{I}\sin\left(3x\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{y}_{2}(x) = \mathrm{e}^{2x}\cdot\left[\begin{array}{c}-\frac{5\cos(3x)}{169}+\frac{12\sin(3x)}{169}\\ \frac{2\cos(3x)}{13}+\frac{3\sin(3x)}{13}\\ \cos\left(3x\right)\end{array}\right], \overrightarrow{y}_{3}(x) = \mathrm{e}^{2x}\cdot\left[\begin{array}{c}\frac{5\sin(3x)}{169}+\frac{12\cos(3x)}{169}\\ -\frac{2\sin(3x)}{13}+\frac{3\cos(3x)}{13}\\ -\sin\left(3x\right)\end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

  $\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3(x) + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{2x}}{4} & e^{2x}\left(-\frac{5\cos(3x)}{169} + \frac{12\sin(3x)}{169}\right) & e^{2x}\left(\frac{5\sin(3x)}{169} + \frac{12\cos(3x)}{169}\right) \\ \frac{e^{2x}}{2} & e^{2x}\left(\frac{2\cos(3x)}{13} + \frac{3\sin(3x)}{13}\right) & e^{2x}\left(-\frac{2\sin(3x)}{13} + \frac{3\cos(3x)}{13}\right) \\ e^{2x} & e^{2x}\cos(3x) & -e^{2x}\sin(3x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{2x}}{4} & e^{2x}\left(-\frac{5\cos(3x)}{169} + \frac{12\sin(3x)}{169}\right) & e^{2x}\left(\frac{5\sin(3x)}{169} + \frac{12\cos(3x)}{169}\right) \\ \frac{e^{2x}}{2} & e^{2x}\left(\frac{2\cos(3x)}{13} + \frac{3\sin(3x)}{13}\right) & e^{2x}\left(-\frac{2\sin(3x)}{13} + \frac{3\cos(3x)}{13}\right) \\ e^{2x} & e^{2x}\cos(3x) & -e^{2x}\sin(3x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & -\frac{5}{169} & \frac{12}{169} \\ \frac{1}{2} & \frac{2}{13} & \frac{3}{13} \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -\frac{e^{2x}(-13+4\cos(3x)+6\sin(3x))}{9} & \frac{e^{2x}(-4+4\cos(3x)+3\sin(3x))}{9} & -\frac{e^{2x}(\cos(3x)-1)}{9} \\ -\frac{26\,e^{2x}(\cos(3x)-1)}{9} & -\frac{e^{2x}(8-17\cos(3x)+6\sin(3x))}{9} & \frac{e^{2x}(2-2\cos(3x)+3\sin(3x))}{9} \\ \frac{26\,e^{2x}(2-2\cos(3x)+3\sin(3x))}{9} & -\frac{e^{2x}(16-16\cos(3x)+63\sin(3x))}{9} & \frac{e^{2x}(4+5\cos(3x)+12\sin(3x))}{9} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

  $$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

  $$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

  $$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{y}_p(x) = \begin{bmatrix} -\frac{2\,e^{2x}(-2+3x\sin(3x)+2\cos(3x))}{3} \\ 2\,e^{2x}\left(\frac{4}{3} + \frac{(-9x-4)\cos(3x)}{3} + (-2x+1)\sin(3x)\right) \\ 2\,e^{2x}\left(\frac{8}{3} + \frac{4(-2-9x)\cos(3x)}{3} + (5x+4)\sin(3x)\right) \end{bmatrix}$$

- Plug particular solution back into general solution

  $$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \begin{bmatrix} -\frac{2\,e^{2x}(-2+3x\sin(3x)+2\cos(3x))}{3} \\ 2\,e^{2x}\left(\frac{4}{3} + \frac{(-9x-4)\cos(3x)}{3} + (-2x+1)\sin(3x)\right) \\ 2\,e^{2x}\left(\frac{8}{3} + \frac{4(-2-9x)\cos(3x)}{3} + (5x+4)\sin(3x)\right) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $$y = -2\left(\left(\frac{5c_2}{338} - \frac{6c_3}{169} + \frac{2}{3}\right)\cos(3x) + \left(x - \frac{6c_2}{169} - \frac{5c_3}{338}\right)\sin(3x) - \frac{c_1}{8} - \frac{2}{3}\right)e^{2x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+21*diff(y(x),x)-26*y(x)=36*exp(2*x)*sin(3*x),y(x), si
```

$$y(x) = \frac{e^{2x}(3c_3 \sin(3x) - 6x \sin(3x) + 3c_2 \cos(3x) - 2\cos(3x) + 3c_1)}{3}$$

✓ Solution by Mathematica

Time used: 0.103 (sec). Leaf size: 34

```
DSolve[y'''[x]-6*y''[x]+21*y'[x]-26*y[x]==36*Exp[2*x]*Sin[3*x],y[x],x,IncludeSingularSolutio
```

$$y(x) \to e^{2x}((-1 + c_2)\cos(3x) + (-2x + c_1)\sin(3x) + c_3)$$

11.6 problem 6

Internal problem ID [12779]
Internal file name [OUTPUT/11431_Friday_November_03_2023_06_32_53_AM_14195515/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.4, page 218
Problem number: 6.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + y'' - y' - y = \left(2x^2 + 4x + 8\right)\cos\left(x\right) + \left(6x^2 + 8x + 12\right)\sin\left(x\right)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' - y' - y = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + c_3 \mathrm{e}^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = x\,\mathrm{e}^{-x}$$
$$y_3 = \mathrm{e}^{x}$$

Now the particular solution to the given ODE is found

$$y''' + y'' - y' - y = \left(2x^2 + 4x + 8\right)\cos(x) + \left(6x^2 + 8x + 12\right)\sin(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\left(2x^2 + 4x + 8\right)\cos(x) + \left(6x^2 + 8x + 12\right)\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\sin(x), \cos(x)\,x, \cos(x)\,x^2, \sin(x)\,x^2, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, \mathrm{e}^{x}, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\sin(x) + A_2\cos(x)\,x + A_3\cos(x)\,x^2 + A_4\sin(x)\,x^2 + A_5\cos(x) + A_6\sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&2A_5\sin(x) - 2A_6\cos(x) + 2A_1\cos(x) - 2A_1 x\sin(x) - 2A_2\cos(x)\,x - 2A_3\cos(x)\,x^2 \\
&- 2A_4\sin(x)\,x^2 - 8A_3\cos(x)\,x - 8A_4\sin(x)\,x - 4A_3\sin(x)\,x + 4A_4\cos(x)\,x \\
&- 2A_5\cos(x) - 2A_6\sin(x) - 4A_1\sin(x) - 2A_1 x\cos(x) - 4A_2\cos(x) + 2A_2\sin(x)\,x \\
&+ 2A_3\sin(x)\,x^2 - 2A_4\cos(x)\,x^2 - 2A_2\sin(x) + 2A_3\cos(x) + 2A_4\sin(x) \\
&- 6A_3\sin(x) + 6A_4\cos(x) = \left(2x^2 + 4x + 8\right)\cos(x) + \left(6x^2 + 8x + 12\right)\sin(x)
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -4, A_2 = -6, A_3 = 1, A_4 = -2, A_5 = -2, A_6 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -4x \sin(x) - 6\cos(x)\, x + \cos(x)\, x^2 - 2\sin(x)\, x^2 - 2\cos(x) + \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + c_3 \mathrm{e}^{x}\right) \\ &\quad + \left(-4x \sin(x) - 6\cos(x)\, x + \cos(x)\, x^2 - 2\sin(x)\, x^2 - 2\cos(x) + \sin(x)\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} y = {} & \mathrm{e}^{-x}(c_2 x + c_1) + c_3 \mathrm{e}^{x} - 4x \sin(x) - 6\cos(x)\, x \\ & + \cos(x)\, x^2 - 2\sin(x)\, x^2 - 2\cos(x) + \sin(x) \end{aligned}$$

Summary
The solution(s) found are the following

$$\begin{aligned} y = {} & \mathrm{e}^{-x}(c_2 x + c_1) + c_3 \mathrm{e}^{x} - 4x \sin(x) - 6\cos(x)\, x \\ & + \cos(x)\, x^2 - 2\sin(x)\, x^2 - 2\cos(x) + \sin(x) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = {} & \mathrm{e}^{-x}(c_2 x + c_1) + c_3 \mathrm{e}^{x} - 4x \sin(x) - 6\cos(x)\, x \\ & + \cos(x)\, x^2 - 2\sin(x)\, x^2 - 2\cos(x) + \sin(x) \end{aligned}$$

Verified OK.

11.6.1 Maple step by step solution

Let's solve

$y''' + y'' - y' - y = (2x^2 + 4x + 8)\cos(x) + (6x^2 + 8x + 12)\sin(x)$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

$$y''' = y + 2\cos(x)\,x^2 + 6\sin(x)\,x^2 + 4\cos(x)\,x + 8x\sin(x) + 8\cos(x) + 12\sin(x) - y'' + y'$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y''' + y'' - y' - y = 6\sin(x)\,x^2 + 2\cos(x)\,x^2 + 8x\sin(x) + 4\cos(x)\,x + 12\sin(x) + 8\cos(x)$$

☐ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

$$y_1(x) = y$$

  - Define new variable $y_2(x)$

$$y_2(x) = y'$$

  - Define new variable $y_3(x)$

$$y_3(x) = y''$$

  - Isolate for $y_3'(x)$ using original ODE

$$y_3'(x) = 2\cos(x)\,x^2 + 6\sin(x)\,x^2 + 4\cos(x)\,x + 8x\sin(x) + 8\cos(x) + 12\sin(x) - y_3(x) + y_2(x$$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = 2\cos(x)\,x^2 + 6\sin(x)\,x^2 + 4\cos(x)\,x + 8x\sin(x) + 8\cos$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 6\sin(x)\,x^2 + 2\cos(x)\,x^2 + 8x\sin(x) + 4\cos(x)\,x + 12\sin(x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 6\sin(x)\,x^2 + 2\cos(x)\,x^2 + 8x\sin(x) + 4\cos(x)\,x + 12\sin(x) + 8\cos(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$
- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$
- First solution from eigenvalue -1

  $$\overrightarrow{y}_1(x) = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$
- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

  $\overrightarrow{y}_2(x) = e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$
- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

  $\lambda e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = \left(e^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$
- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $\lambda e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = e^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$
- Simplify equation

  $\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} - (-1) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

$$\vec{y}_2(x) = \mathrm{e}^{-x} \cdot \left(x \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3 + \vec{y}_p(x)$

☐ Fundamental matrix

  ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{-x} & (x+1)e^{-x} & e^{x} \\ -e^{-x} & -x e^{-x} & e^{x} \\ e^{-x} & x e^{-x} & e^{x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & (x+1)e^{-x} & e^{x} \\ -e^{-x} & -x e^{-x} & e^{x} \\ e^{-x} & x e^{-x} & e^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} (x+1)e^{-x} & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & -\frac{e^{-x}}{2} - x e^{-x} + \frac{e^{x}}{2} \\ -x e^{-x} & \frac{e^{-x}}{2} + \frac{e^{x}}{2} & -\frac{e^{-x}}{2} + x e^{-x} + \frac{e^{x}}{2} \\ x e^{-x} & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{e^{-x}}{2} - x e^{-x} + \frac{e^{x}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \begin{bmatrix} 2(-x-2)\,\mathrm{e}^{-x} + 2(x^2-6x-2)\cos(x) + 2(-2x^2-4x+1)\sin(x) + 8\,\mathrm{e}^{x} \\ (2x+2)\,\mathrm{e}^{-x} + (-4x^2-4x-10)\cos(x) + (-2x^2+4x-4)\sin(x) + 8\,\mathrm{e}^{x} \\ (-2x-2)\,\mathrm{e}^{-x} + (-8x-6)\cos(x) + (-4x+8)\sin(x) + 8\,\mathrm{e}^{x} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1(x) + c_2\overrightarrow{y}_2(x) + c_3\overrightarrow{y}_3 + \begin{bmatrix} 2(-x-2)\,\mathrm{e}^{-x} + 2(x^2-6x-2)\cos(x) + 2(-2x^2-4x+ \\ (2x+2)\,\mathrm{e}^{-x} + (-4x^2-4x-10)\cos(x) + (-2x^2+4x- \\ (-2x-2)\,\mathrm{e}^{-x} + (-8x-6)\cos(x) + (-4x+8)\sin \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = ((c_2-2)\,x + c_1 + c_2 - 4)\,\mathrm{e}^{-x} + 2(x^2-6x-2)\cos(x) + 2(-2x^2-4x+1)\sin(x) + \mathrm{e}^{x}(c_3 +$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)-diff(y(x),x)-y(x)=(2*x^2+4*x+8)*cos(x)+(6*x^2+8*x+12)*s
```

$$y(x) = (c_3x + c_2)\,\mathrm{e}^{-x} + \left(x^2-6x-2\right)\cos(x) + \left(-2x^2-4x+1\right)\sin(x) + c_1\mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 55

```
DSolve[y'''[x]+y''[x]-y'[x]-y[x]==(2*x^2+4*x+8)*Cos[x]+(6*x^2+8*x+12)*Sin[x],y[x],x,IncludeS
```

$$y(x) \to (x^2 - 6x - 2)\cos(x) + e^{-x}\left(-e^x(2x^2 + 4x - 1)\sin(x) + c_2x + c_3e^{2x} + c_1\right)$$

11.7 problem 7

Internal problem ID [12780]
Internal file name [`OUTPUT/11432_Friday_November_03_2023_06_32_53_AM_50427656/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.4, page 218
Problem number: 7.
ODE order: 6.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _linear, _nonhomogeneous]]`

$$
\boxed{y^{(6)} - 12y^{(5)} + 63y'''' - 18y''' + 315y'' - 300y' + 125y = \mathrm{e}^{x}(48\cos(x) + 96\sin(x))}
$$

This is higher order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$
y^{(6)} - 12y^{(5)} + 63y'''' - 18y''' + 315y'' - 300y' + 125y = 0
$$

The characteristic equation is

$$
\lambda^6 - 12\lambda^5 + 63\lambda^4 - 18\lambda^3 + 315\lambda^2 - 300\lambda + 125 = 0
$$

The roots of the above equation are

$$
\begin{aligned}
\lambda_1 &= \mathrm{RootOf}\left(\_Z^6 - 12\_Z^5 + 63\_Z^4 - 18\_Z^3 + 315\_Z^2 - 300\_Z + 125, \mathrm{index} = 1\right) \\
\lambda_2 &= \mathrm{RootOf}\left(\_Z^6 - 12\_Z^5 + 63\_Z^4 - 18\_Z^3 + 315\_Z^2 - 300\_Z + 125, \mathrm{index} = 2\right) \\
\lambda_3 &= \mathrm{RootOf}\left(\_Z^6 - 12\_Z^5 + 63\_Z^4 - 18\_Z^3 + 315\_Z^2 - 300\_Z + 125, \mathrm{index} = 3\right) \\
\lambda_4 &= \mathrm{RootOf}\left(\_Z^6 - 12\_Z^5 + 63\_Z^4 - 18\_Z^3 + 315\_Z^2 - 300\_Z + 125, \mathrm{index} = 4\right) \\
\lambda_5 &= \mathrm{RootOf}\left(\_Z^6 - 12\_Z^5 + 63\_Z^4 - 18\_Z^3 + 315\_Z^2 - 300\_Z + 125, \mathrm{index} = 5\right) \\
\lambda_6 &= \mathrm{RootOf}\left(\_Z^6 - 12\_Z^5 + 63\_Z^4 - 18\_Z^3 + 315\_Z^2 - 300\_Z + 125, \mathrm{index} = 6\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$y_h(x) = e^{\text{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\text{index}=5)x}c_1+e^{\text{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-3}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{\text{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\text{index}=5)x}$$

$$y_2 = e^{\text{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\text{index}=3)x}$$

$$y_3 = e^{\text{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\text{index}=1)x}$$

$$y_4 = e^{\text{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\text{index}=6)x}$$

$$y_5 = e^{\text{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\text{index}=4)x}$$

$$y_6 = e^{\text{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\text{index}=2)x}$$

Now the particular solution to the given ODE is found

$$y^{(6)} - 12y^{(5)} + 63y'''' - 18y''' + 315y'' - 300y' + 125y = e^x(48\cos(x) + 96\sin(x))$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x(48\cos(x) + 96\sin(x))$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\cos(x), e^x\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{\text{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\text{index}=1)x}, e^{\text{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\text{inc}}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x\cos(x) + A_2 e^x\sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-334A_1e^x\sin(x) + 334A_2e^x\cos(x) - 343A_1e^x\cos(x) - 343A_2e^x\sin(x)\\&= e^x(48\cos(x) + 96\sin(x))\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{48528}{229205}, A_2 = -\frac{16896}{229205}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{48528\,\mathrm{e}^{x}\cos(x)}{229205} - \frac{16896\,\mathrm{e}^{x}\sin(x)}{229205}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{\mathrm{RootOf}(_Z^6-12_Z^5+63_Z^4-18_Z^3+315_Z^2-300_Z+125,\mathrm{index}=5)x} c_1 \right.\\
&\quad + \mathrm{e}^{\mathrm{RootOf}(_Z^6-12_Z^5+63_Z^4-18_Z^3+315_Z^2-300_Z+125,\mathrm{index}=3)x} c_2 \\
&\quad + \mathrm{e}^{\mathrm{RootOf}(_Z^6-12_Z^5+63_Z^4-18_Z^3+315_Z^2-300_Z+125,\mathrm{index}=1)x} c_3 \\
&\quad + \mathrm{e}^{\mathrm{RootOf}(_Z^6-12_Z^5+63_Z^4-18_Z^3+315_Z^2-300_Z+125,\mathrm{index}=6)x} c_4 \\
&\quad + \mathrm{e}^{\mathrm{RootOf}(_Z^6-12_Z^5+63_Z^4-18_Z^3+315_Z^2-300_Z+125,\mathrm{index}=4)x} c_5 \\
&\quad \left. + \mathrm{e}^{\mathrm{RootOf}(_Z^6-12_Z^5+63_Z^4-18_Z^3+315_Z^2-300_Z+125,\mathrm{index}=2)x} c_6 \right) \\
&\quad + \left(-\frac{48528\,\mathrm{e}^{x}\cos(x)}{229205} - \frac{16896\,\mathrm{e}^{x}\sin(x)}{229205}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y &= \mathrm{e}^{\mathrm{RootOf}(_Z^6-12_Z^5+63_Z^4-18_Z^3+315_Z^2-300_Z+125,\mathrm{index}=5)x} c_1 \\
&\quad + \mathrm{e}^{\mathrm{RootOf}(_Z^6-12_Z^5+63_Z^4-18_Z^3+315_Z^2-300_Z+125,\mathrm{index}=3)x} c_2 \\
&\quad + \mathrm{e}^{\mathrm{RootOf}(_Z^6-12_Z^5+63_Z^4-18_Z^3+315_Z^2-300_Z+125,\mathrm{index}=1)x} c_3 \\
&\quad + \mathrm{e}^{\mathrm{RootOf}(_Z^6-12_Z^5+63_Z^4-18_Z^3+315_Z^2-300_Z+125,\mathrm{index}=6)x} c_4 \\
&\quad + \mathrm{e}^{\mathrm{RootOf}(_Z^6-12_Z^5+63_Z^4-18_Z^3+315_Z^2-300_Z+125,\mathrm{index}=4)x} c_5 \\
&\quad + \mathrm{e}^{\mathrm{RootOf}(_Z^6-12_Z^5+63_Z^4-18_Z^3+315_Z^2-300_Z+125,\mathrm{index}=2)x} c_6 \\
&\quad - \frac{48528\,\mathrm{e}^{x}\cos(x)}{229205} - \frac{16896\,\mathrm{e}^{x}\sin(x)}{229205}
\end{aligned} \tag{1}$$

Verification of solutions

$$y = e^{\mathrm{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\mathrm{index}=5)x}c_1 + e^{\mathrm{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\mathrm{index}=3)x}c_2 + e^{\mathrm{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\mathrm{index}=1)x}c_3 + e^{\mathrm{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\mathrm{index}=6)x}c_4 + e^{\mathrm{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\mathrm{index}=4)x}c_5 + e^{\mathrm{RootOf}(\_Z^6-12\_Z^5+63\_Z^4-18\_Z^3+315\_Z^2-300\_Z+125,\mathrm{index}=2)x}c_6 - \frac{48528\,e^x\cos(x)}{229205} - \frac{16896\,e^x\sin(x)}{229205}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 6; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 6; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 5468

```
dsolve(diff(y(x),x$6)-12*diff(y(x),x$5)+63*diff(y(x),x$4)-18*diff(y(x),x$3)+315*diff(y(x),x$
```

Expression too large to display

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 292

```
DSolve[y''''''[x]-12*y'''''[x]+63*y''''[x]-18*y'''[x]+315*y''[x]-300*y'[x]+125*y[x]==Exp[x]*
```

$$
\begin{aligned}
y(x) \to\ & c_3 \exp \left(x\text{Root}\left[\#1^6 - 12\#1^5 + 63\#1^4 - 18\#1^3 + 315\#1^2 - 300\#1 + 125\&, 3\right]\right) \\
& + c_4 \exp \left(x\text{Root}\left[\#1^6 - 12\#1^5 + 63\#1^4 - 18\#1^3 + 315\#1^2 - 300\#1 + 125\&, 4\right]\right) \\
& + c_1 \exp \left(x\text{Root}\left[\#1^6 - 12\#1^5 + 63\#1^4 - 18\#1^3 + 315\#1^2 - 300\#1 + 125\&, 1\right]\right) \\
& + c_2 \exp \left(x\text{Root}\left[\#1^6 - 12\#1^5 + 63\#1^4 - 18\#1^3 + 315\#1^2 - 300\#1 + 125\&, 2\right]\right) \\
& + c_5 \exp \left(x\text{Root}\left[\#1^6 - 12\#1^5 + 63\#1^4 - 18\#1^3 + 315\#1^2 - 300\#1 + 125\&, 5\right]\right) \\
& + c_6 \exp \left(x\text{Root}\left[\#1^6 - 12\#1^5 + 63\#1^4 - 18\#1^3 + 315\#1^2 - 300\#1 + 125\&, 6\right]\right) \\
& - \frac{48e^x(352\sin(x) + 1011\cos(x))}{229205}
\end{aligned}
$$

12 Chapter 4. N-th Order Linear Differential Equations. Exercises 4.5, page 221

12.1 problem 1

Internal problem ID [12781]
Internal file name [OUTPUT/11433_Friday_November_03_2023_06_32_54_AM_79193795/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.5, page 221
Problem number: 1.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 3y'' - 4y' + 12y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 5, y''(0) = -1]$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = e^{2x}$$
$$y_3 = e^{3x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^{3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2x} + 2c_2 e^{2x} + 3c_3 e^{3x}$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = -2c_1 + 2c_2 + 3c_3 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = 4c_1 e^{-2x} + 4c_2 e^{2x} + 9c_3 e^{3x}$$

substituting $y'' = -1$ and $x = 0$ in the above gives

$$-1 = 4c_1 + 4c_2 + 9c_3 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -1 \\ c_2 &= 3 \\ c_3 &= -1 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -e^{-2x} + 3\,e^{2x} - e^{3x}$$

Summary

The solution(s) found are the following

$$y = -e^{-2x} + 3\,e^{2x} - e^{3x} \tag{1}$$

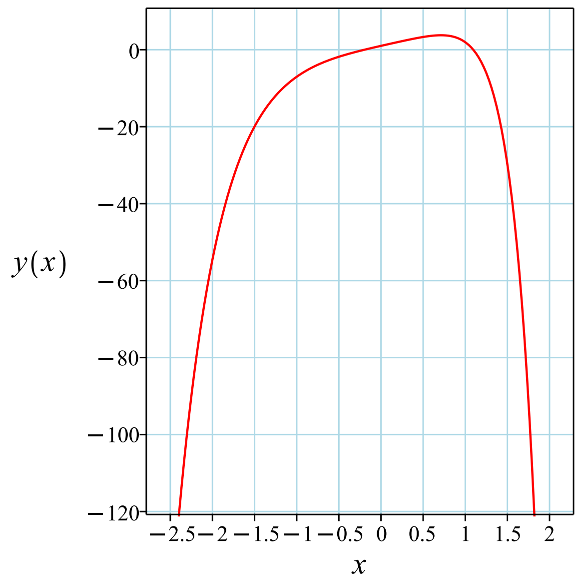


Figure 387: Solution plot

Verification of solutions

$$y = -\mathrm{e}^{-2x} + 3\,\mathrm{e}^{2x} - \mathrm{e}^{3x}$$

Verified OK.

12.1.1 Maple step by step solution

Let's solve

$$\left[y''' - 3y'' - 4y' + 12y = 0, y(0) = 1, y'\Big|_{\{x=0\}} = 5, y''\Big|_{\{x=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 3y_3(x) + 4y_2(x) - 12y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3y_3(x) + 4y_2(x) - 12y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(4c_3 e^{5x} + 9c_2 e^{4x} + 9c_1) e^{-2x}}{36}$$

- Use the initial condition $y(0) = 1$

  $1 = \frac{c_3}{9} + \frac{c_2}{4} + \frac{c_1}{4}$
- Calculate the 1st derivative of the solution

  $y' = \frac{(20c_3\mathrm{e}^{5x}+36c_2\mathrm{e}^{4x})\mathrm{e}^{-2x}}{36} - \frac{(4c_3\mathrm{e}^{5x}+9c_2\mathrm{e}^{4x}+9c_1)\mathrm{e}^{-2x}}{18}$
- Use the initial condition $y'\Big|_{\{x=0\}} = 5$

  $5 = \frac{c_3}{3} + \frac{c_2}{2} - \frac{c_1}{2}$
- Calculate the 2nd derivative of the solution

  $y'' = \frac{(100c_3\mathrm{e}^{5x}+144c_2\mathrm{e}^{4x})\mathrm{e}^{-2x}}{36} - \frac{(20c_3\mathrm{e}^{5x}+36c_2\mathrm{e}^{4x})\mathrm{e}^{-2x}}{9} + \frac{(4c_3\mathrm{e}^{5x}+9c_2\mathrm{e}^{4x}+9c_1)\mathrm{e}^{-2x}}{9}$
- Use the initial condition $y''\Big|_{\{x=0\}} = -1$

  $-1 = c_1 + c_2 + c_3$
- Solve for the unknown coefficients

  $\{c_1 = -4, c_2 = 12, c_3 = -9\}$
- Solution to the IVP

  $y = (-\mathrm{e}^{5x} + 3\,\mathrm{e}^{4x} - 1)\,\mathrm{e}^{-2x}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$3)-3*diff(y(x),x$2)-4*diff(y(x),x)+12*y(x)=0,y(0) = 1, D(y)(0) = 5, (D@@
```

$$y(x) = \left(-\mathrm{e}^{5x} + 3\,\mathrm{e}^{4x} - 1\right)\mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 26

```
DSolve[{y'''[x]-3*y''[x]-4*y'[x]+12*y[x]==0,{y[0]==1,y'[0]==5,y''[0]==-1}},y[x],x,IncludeSin
```

$$y(x) \to -e^{-2x}\left(-3e^{4x} + e^{5x} + 1\right)$$

12.2 problem 2

Internal problem ID [12782]
Internal file name [`OUTPUT/11434_Friday_November_03_2023_06_32_54_AM_22446224/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.5, page 221
Problem number: 2.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 2y''' + 2y' - y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1, y''(0) = -3, y'''(0) = 3]$$

The characteristic equation is

$$\lambda^4 - 2\lambda^3 + 2\lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + x\, e^{x} c_3 + x^2 e^{x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= \mathrm{e}^{x} \\
y_3 &= x\,\mathrm{e}^{x} \\
y_4 &= x^2\mathrm{e}^{x}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{x} + x\,\mathrm{e}^{x}c_3 + x^2\mathrm{e}^{x}c_4 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{x} + c_3\mathrm{e}^{x} + x\,\mathrm{e}^{x}c_3 + 2x\,\mathrm{e}^{x}c_4 + x^2\mathrm{e}^{x}c_4
$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$
-1 = -c_1 + c_2 + c_3 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{x} + 2c_3\mathrm{e}^{x} + x\,\mathrm{e}^{x}c_3 + 2\,\mathrm{e}^{x}c_4 + 4x\,\mathrm{e}^{x}c_4 + x^2\mathrm{e}^{x}c_4
$$

substituting $y'' = -3$ and $x = 0$ in the above gives

$$
-3 = c_1 + c_2 + 2c_3 + 2c_4 \tag{3A}
$$

Taking three derivatives of the solution gives

$$
y''' = -c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{x} + 3c_3\mathrm{e}^{x} + x\,\mathrm{e}^{x}c_3 + 6\,\mathrm{e}^{x}c_4 + 6x\,\mathrm{e}^{x}c_4 + x^2\mathrm{e}^{x}c_4
$$

substituting $y''' = 3$ and $x = 0$ in the above gives

$$
3 = -c_1 + c_2 + 3c_3 + 6c_4 \tag{4A}
$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 2$$
$$c_3 = -4$$
$$c_4 = 2$$

Substituting these values back in above solution results in

$$y = 2x^2 e^x - 4x\, e^x + 2\, e^x - e^{-x}$$

Which simplifies to

$$y = -e^{-x} + (2x^2 - 4x + 2)\, e^x$$

Summary

The solution(s) found are the following

$$y = -e^{-x} + (2x^2 - 4x + 2)\, e^x \tag{1}$$

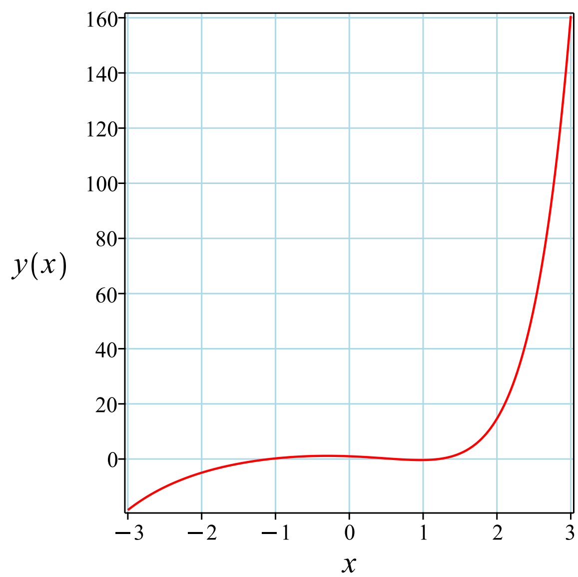


Figure 388: Solution plot

Verification of solutions

$$y = -e^{-x} + (2x^2 - 4x + 2)\, e^x$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve([diff(y(x),x$4)-2*diff(y(x),x$3)+2*diff(y(x),x)-y(x)=0,y(0) = 1, D(y)(0) = -1, (D@@2)
```

$$y(x) = -\mathrm{e}^{-x} + \left(2x^2 - 4x + 2\right)\mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 25

```
DSolve[{y''''[x]-2*y'''[x]+2*y'[x]-y[x]==0,{y[0]==1,y'[0]==-1,y''[0]==-3,y'''[0]==3}},y[x],x
```

$$y(x) \to e^{-x}\left(2e^{2x}(x-1)^2 - 1\right)$$

12.3 problem 3

Internal problem ID [12783]
Internal file name [`OUTPUT/11435_Saturday_November_04_2023_08_47_16_AM_64086885/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.5, page 221
Problem number: 3.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _with_linear_symmetries]]`

$$\boxed{y''' - y'' + y' - y = 2\,\mathrm{e}^{x}}$$

With initial conditions

$$[y(0) = 1, y'(0) = 3, y''(0) = -3]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' + y' - y = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + e^{ix} c_2 + e^{-ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{ix}$$
$$y_3 = e^{-ix}$$

Now the particular solution to the given ODE is found

$$y''' - y'' + y' - y = 2\,e^x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{ix}, e^{-ix}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x = 2\,e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + e^{ix} c_2 + e^{-ix} c_3\right) + \left(x\,e^x\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^x c_1 + e^{ix} c_2 + e^{-ix} c_3 + x\,e^x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 + c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^x c_1 + i e^{ix} c_2 - i e^{-ix} c_3 + x\,e^x + e^x$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = c_2 i - c_3 i + c_1 + 1 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = e^x c_1 - e^{ix} c_2 - e^{-ix} c_3 + x\,e^x + 2\,e^x$$

substituting $y'' = -3$ and $x = 0$ in the above gives

$$-3 = c_1 - c_2 - c_3 + 2 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -2 \\
c_2 &= \frac{3}{2} - 2i \\
c_3 &= \frac{3}{2} + 2i
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = x\,e^{x} - 2\,e^{x} + 3\cos(x) + 4\sin(x)$$

Which simplifies to

$$y = (x-2)\,e^{x} + 3\cos(x) + 4\sin(x)$$

Summary

The solution(s) found are the following

$$y = (x-2)\,e^{x} + 3\cos(x) + 4\sin(x) \tag{1}$$

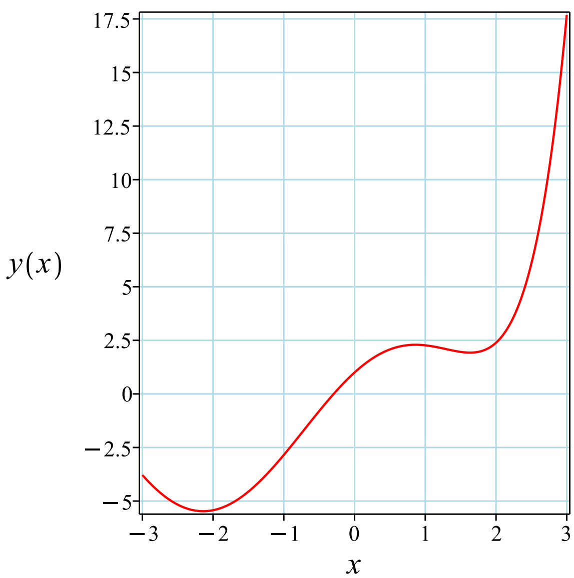


Figure 389: Solution plot

Verification of solutions

$$y = (x-2)\,e^{x} + 3\cos(x) + 4\sin(x)$$

Verified OK.

12.3.1 Maple step by step solution

Let's solve

$$\left[y''' - y'' + y' - y = 2\,e^{x}, y(0) = 1, y'\Big|_{\{x=0\}} = 3, y''\Big|_{\{x=0\}} = -3\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 2\,e^{x} + y_3(x) - y_2(x) + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 2\,e^{x} + y_3(x) - y_2(x) + y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 2\,e^{x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 2\,e^{x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}x} \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \mathrm{e}^x & -\cos(x) & \sin(x) \\ \mathrm{e}^x & \sin(x) & \cos(x) \\ \mathrm{e}^x & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \mathrm{e}^x & -\cos(x) & \sin(x) \\ \mathrm{e}^x & \sin(x) & \cos(x) \\ \mathrm{e}^x & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^x}{2} + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & \sin(x) & \frac{e^x}{2} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \\ \frac{e^x}{2} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & \cos(x) & \frac{e^x}{2} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} \\ \frac{e^x}{2} + \frac{\sin(x)}{2} - \frac{\cos(x)}{2} & -\sin(x) & \frac{e^x}{2} + \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{y}_p(x) = \begin{bmatrix} e^x(x-1) + \cos(x) \\ x\,e^x - \sin(x) \\ x\,e^x + e^x - \cos(x) \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + \begin{bmatrix} e^x(x-1) + \cos(x) \\ x\, e^x - \sin(x) \\ x\, e^x + e^x - \cos(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = (c_1 + x - 1)\, e^x + (-c_2 + 1)\cos(x) + c_3 \sin(x)$
- Use the initial condition $y(0) = 1$

  $1 = c_1 - c_2$
- Calculate the 1st derivative of the solution

  $y' = e^x + (c_1 + x - 1)\, e^x - (-c_2 + 1)\sin(x) + c_3 \cos(x)$
- Use the initial condition $y'\Big|_{\{x=0\}} = 3$

  $3 = c_1 + c_3$
- Calculate the 2nd derivative of the solution

  $y'' = 2\, e^x + (c_1 + x - 1)\, e^x - (-c_2 + 1)\cos(x) - c_3 \sin(x)$
- Use the initial condition $y''\Big|_{\{x=0\}} = -3$

  $-3 = c_1 + c_2$
- Solve for the unknown coefficients

  $\{c_1 = -1, c_2 = -2, c_3 = 4\}$
- Solution to the IVP

  $y = (x - 2)\, e^x + 3\cos(x) + 4\sin(x)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$3)-diff(y(x),x$2)+diff(y(x),x)-y(x)=2*exp(x),y(0) = 1, D(y)(0) = 3, (D@@
```

$$y(x) = (x-2)\,\mathrm{e}^{x} + 3\cos(x) + 4\sin(x)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 21

```
DSolve[{y'''[x]-y''[x]+y'[x]-y[x]==2*Exp[x],{y[0]==1,y'[0]==3,y''[0]==-3}},y[x],x,IncludeSin
```

$$y(x) \to e^{x}(x-2) + 4\sin(x) + 3\cos(x)$$

12.4 problem 4

Internal problem ID [12784]
Internal file name [`OUTPUT/11436_Saturday_November_04_2023_08_47_19_AM_78487314/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 4. N-th Order Linear Differential Equations. Exercises 4.5, page 221
Problem number: 4.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _with_linear_symmetries]]`

$$\boxed{y'''' + 2y'' + y = 4 + 3x}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0, y''(0) = 1, y'''(0) = 1]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y'' + y = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^2 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = e^{ix} c_1 + x\, e^{ix} c_2 + e^{-ix} c_3 + x\, e^{-ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{ix}$$
$$y_2 = x\, e^{ix}$$
$$y_3 = e^{-ix}$$
$$y_4 = x\, e^{-ix}$$

Now the particular solution to the given ODE is found

$$y'''' + 2y'' + y = 4 + 3x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{ix}, x\, e^{-ix}, e^{ix}, e^{-ix}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 = 4 + 3x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 4, A_2 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4 + 3x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{ix} c_1 + x\, e^{ix} c_2 + e^{-ix} c_3 + x\, e^{-ix} c_4\right) + (4 + 3x) \end{aligned}$$

Which simplifies to

$$y = (c_4 x + c_3)\, e^{-ix} + (c_2 x + c_1)\, e^{ix} + 4 + 3x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (c_4 x + c_3)\, e^{-ix} + (c_2 x + c_1)\, e^{ix} + 4 + 3x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_3 + c_1 + 4 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{-ix} c_4 - i(c_4 x + c_3)\, e^{-ix} + e^{ix} c_2 + i(c_2 x + c_1)\, e^{ix} + 3$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_1 i - c_3 i + c_2 + c_4 + 3 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = -2i e^{-ix} c_4 - (c_4 x + c_3)\, e^{-ix} + 2i e^{ix} c_2 - (c_2 x + c_1)\, e^{ix}$$

substituting $y'' = 1$ and $x = 0$ in the above gives

$$1 = 2c_2 i - 2c_4 i - c_1 - c_3 \tag{3A}$$

Taking three derivatives of the solution gives

$$y''' = -3\, e^{-ix} c_4 + i(c_4 x + c_3)\, e^{-ix} - 3\, e^{ix} c_2 - i(c_2 x + c_1)\, e^{ix}$$

substituting $y''' = 1$ and $x = 0$ in the above gives

$$1 = -c_1 i + c_3 i - 3c_2 - 3c_4 \tag{4A}$$

Equations {1A,2A,3A,4A} are now solved for $\{c_1, c_2, c_3, c_4\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -2 + 2i \\ c_2 &= \frac{1}{2} + \frac{3i}{4} \\ c_3 &= -2 - 2i \\ c_4 &= \frac{1}{2} - \frac{3i}{4} \end{aligned}$$

Substituting these values back in above solution results in

$$y = 4 + 3x + \cos(x)\, x - 4\cos(x) - \frac{3x\sin(x)}{2} - 4\sin(x)$$

Which simplifies to

$$y = 4 + (-4 + x)\cos(x) + \frac{(-3x - 8)\sin(x)}{2} + 3x$$

Summary

The solution(s) found are the following

$$y = 4 + (-4 + x)\cos(x) + \frac{(-3x - 8)\sin(x)}{2} + 3x \tag{1}$$

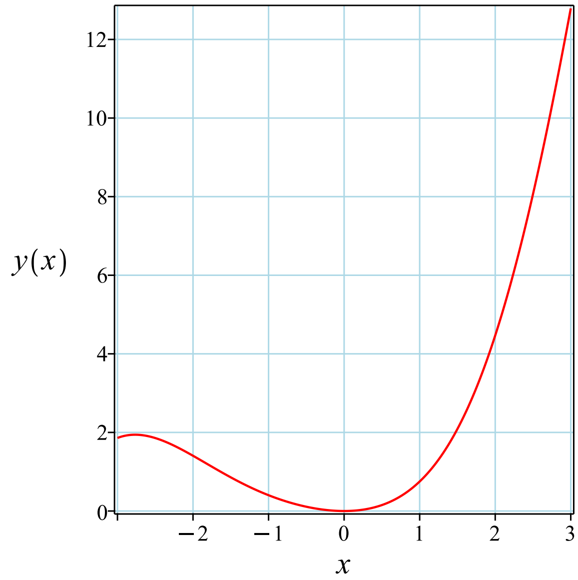


Figure 390: Solution plot

Verification of solutions

$$y = 4 + (-4 + x)\cos(x) + \frac{(-3x - 8)\sin(x)}{2} + 3x$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve([diff(y(x),x$4)+2*diff(y(x),x$2)+y(x)=3*x+4,y(0) = 0, D(y)(0) = 0, (D@@2)(y)(0) = 1,
```

$$y(x) = 4 + (x - 4)\cos(x) + \frac{(-3x - 8)\sin(x)}{2} + 3x$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 27

```
DSolve[{y''''[x]+2*y''[x]+y[x]==3*x+4,{y[0]==0,y'[0]==0,y''[0]==1,y'''[0]==1}},y[x],x,Includ
```

$$y(x) \to 3x - \frac{1}{2}(3x + 8)\sin(x) + (x - 4)\cos(x) + 4$$

13 Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248

13.1 problem 1

Internal problem ID [12785]
Internal file name [`OUTPUT/11437_Saturday_November_04_2023_08_47_19_AM_68303959/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - y = 0}$$

13.1.1 Solving as laplace ode

Since no initial condition is explicitly given, then let

$$y(0) = c_1$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - Y(s) = 0 \tag{1}$$

Replacing initial condition gives

$$sY(s) - c_1 - Y(s) = 0$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{c_1}{s-1}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{c_1}{s-1}\right) \\ &= \mathrm{e}^{x} c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{x} c_1 \tag{1}$$

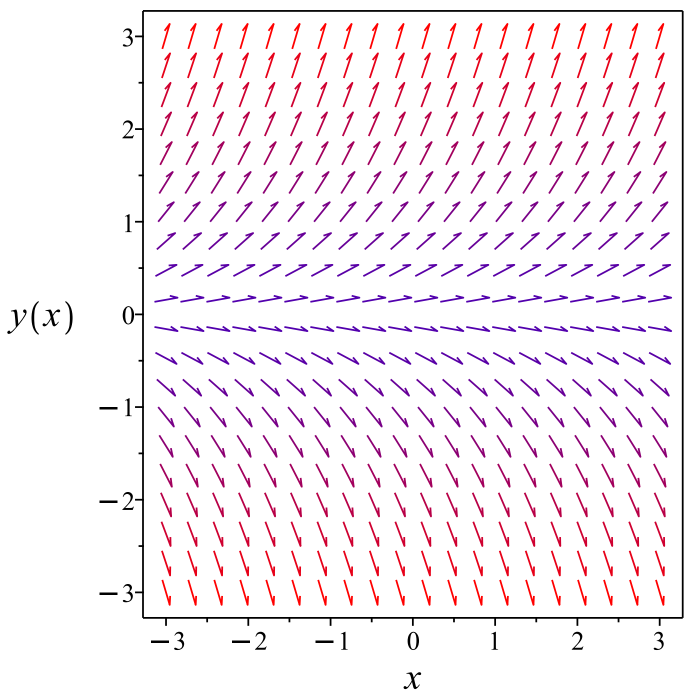


Figure 391: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{x} c_1$$

Verified OK.

13.1.2 Maple step by step solution

Let's solve

$y' - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 1 dx + c_1$

- Evaluate integral

  $\ln(y) = x + c_1$

- Solve for y

  $y = \mathrm{e}^{x + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 4.609 (sec). Leaf size: 9

```
dsolve(diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{x} y(0)$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 16

```
DSolve[y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x$$
$$y(x) \to 0$$

13.2 problem 2

Internal problem ID [12786]
Internal file name [`OUTPUT/11438_Saturday_November_04_2023_08_47_19_AM_72649872/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 5y = 0}$$

Since no initial conditions are explicitly given, then let

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= c_2 \end{aligned}$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) + 5Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= c_2 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - c_2 - sc_1 - 2sY(s) + 2c_1 + 5Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{sc_1 - 2c_1 + c_2}{s^2 - 2s + 5}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{(1+2i)\left(\frac{c_1}{8} - \frac{c_2}{8}\right) + \frac{3c_1}{8} + \frac{c_2}{8}}{s-1-2i} + \frac{(1-2i)\left(\frac{c_1}{8} - \frac{c_2}{8}\right) + \frac{3c_1}{8} + \frac{c_2}{8}}{s-1+2i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{(1+2i)\left(\frac{c_1}{8} - \frac{c_2}{8}\right) + \frac{3c_1}{8} + \frac{c_2}{8}}{s-1-2i}\right) = \frac{e^{(1+2i)x}(-ic_2 + (2+i)\,c_1)}{4}$$

$$\mathcal{L}^{-1}\left(\frac{(1-2i)\left(\frac{c_1}{8} - \frac{c_2}{8}\right) + \frac{3c_1}{8} + \frac{c_2}{8}}{s-1+2i}\right) = \frac{e^{(1-2i)x}(ic_2 + (2-i)\,c_1)}{4}$$

Adding the above results and simplifying gives

$$y = \frac{e^x(2c_1\cos(2x) + \sin(2x)(-c_1 + c_2))}{2}$$

Simplifying the solution gives

$$y = \frac{(-c_1 + c_2)\,e^x \sin(2x)}{2} + c_1\cos(2x)\,e^x$$

Summary

The solution(s) found are the following

$$y = \frac{(-c_1 + c_2)\,e^x \sin(2x)}{2} + c_1\cos(2x)\,e^x \tag{1}$$

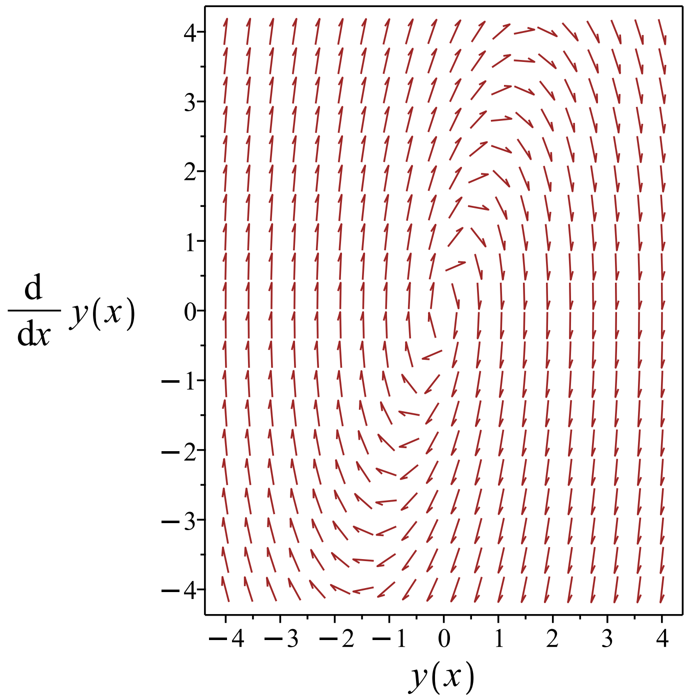


Figure 392: Slope field plot

Verification of solutions

$$y = \frac{(-c_1 + c_2) e^x \sin(2x)}{2} + c_1 \cos(2x) e^x$$

Verified OK.

13.2.1 Maple step by step solution

Let's solve

$y'' - 2y' + 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (1 - 2\,\mathrm{I}, 1 + 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{x} \cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{x} \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(2x)\, \mathrm{e}^{x} + \mathrm{e}^{x} c_2 \sin(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 5.516 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+5*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{x}(2y(0)\cos(2x) + \sin(2x)(D(y)(0) - y(0)))}{2}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 24

```
DSolve[y''[x]-2*y'[x]+5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}(c_2 \cos(2x) + c_1 \sin(2x))$$

13.3 problem 3

Internal problem ID [12787]
Internal file name [`OUTPUT/11439_Saturday_November_04_2023_08_47_19_AM_18391819/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' + 2y = 4}$$

13.3.1 Solving as laplace ode

Since no initial condition is explicitly given, then let

$$y(0) = c_1$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + 2Y(s) = \frac{4}{s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - c_1 + 2Y(s) = \frac{4}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{c_1 s + 4}{s\,(s+2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{2}{s} + \frac{-2 + c_1}{s+2}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{2}{s}\right) = 2$$
$$\mathcal{L}^{-1}\left(\frac{-2 + c_1}{s+2}\right) = (-2 + c_1)\,\mathrm{e}^{-2x}$$

Adding the above results and simplifying gives

$$y = (-2 + c_1)\,\mathrm{e}^{-2x} + 2$$

Summary

The solution(s) found are the following

$$y = (-2 + c_1)\,\mathrm{e}^{-2x} + 2 \tag{1}$$

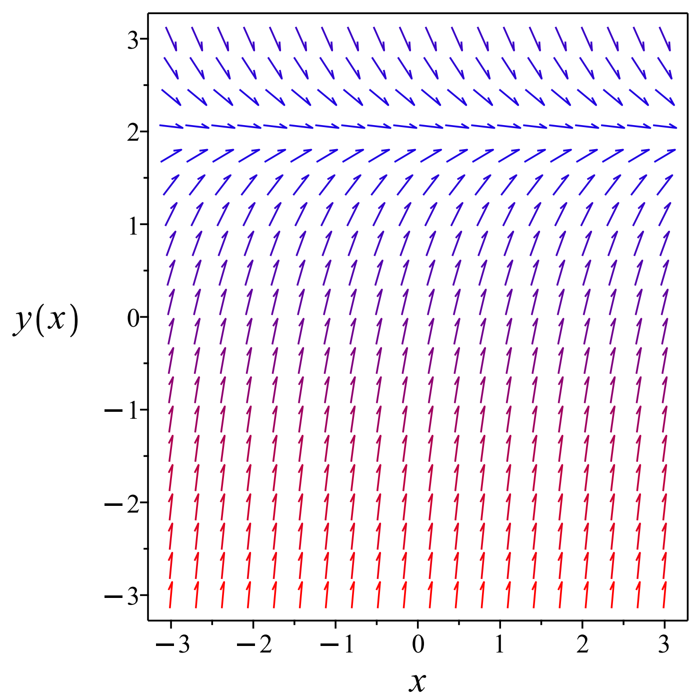


Figure 393: Slope field plot

Verification of solutions

$$y = (-2 + c_1) e^{-2x} + 2$$

Verified OK.

13.3.2 Maple step by step solution

Let's solve

$y' + 2y = 4$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-2y+4} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-2y+4} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\frac{\ln(-y+2)}{2} = x + c_1$

- Solve for y

  $y = -e^{-2c_1-2x} + 2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 5.422 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)+2*y(x)=4,y(x), singsol=all)
```

$$y(x) = (y(0) - 2)\,e^{-2x} + 2$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 20

```
DSolve[y'[x]+2*y[x]==4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2 + c_1 e^{-2x}$$
$$y(x) \to 2$$

13.4 problem 4

Internal problem ID [12788]
Internal file name [`OUTPUT/11440_Saturday_November_04_2023_08_47_20_AM_85677221/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 9y = 2\sin(3x)}$$

Since no initial conditions are explicitly given, then let

$$y(0) = c_1$$
$$y'(0) = c_2$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 9Y(s) = \frac{6}{s^2 + 9} \tag{1}$$

But the initial conditions are

$$y(0) = c_1$$
$$y'(0) = c_2$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - c_2 - sc_1 - 9Y(s) = \frac{6}{s^2+9}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{c_1 s^3 + c_2 s^2 + 9sc_1 + 9c_2 + 6}{(s^2+9)(s^2-9)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{\frac{c_1}{2} + \frac{c_2}{6} + \frac{1}{18}}{s-3} + \frac{i}{18s - 54i} - \frac{i}{18(s+3i)} + \frac{\frac{c_1}{2} - \frac{c_2}{6} - \frac{1}{18}}{s+3}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{\frac{c_1}{2} + \frac{c_2}{6} + \frac{1}{18}}{s-3}\right) = \frac{(9c_1 + 3c_2 + 1)\,e^{3x}}{18}$$

$$\mathcal{L}^{-1}\left(\frac{i}{18s - 54i}\right) = \frac{i e^{3ix}}{18}$$

$$\mathcal{L}^{-1}\left(-\frac{i}{18(s+3i)}\right) = -\frac{i e^{-3ix}}{18}$$

$$\mathcal{L}^{-1}\left(\frac{\frac{c_1}{2} - \frac{c_2}{6} - \frac{1}{18}}{s+3}\right) = \frac{(9c_1 - 3c_2 - 1)\,e^{-3x}}{18}$$

Adding the above results and simplifying gives

$$y = -\frac{\sin(3x)}{9} + c_1 \cosh(3x) + \frac{\sinh(3x)(1+3c_2)}{9}$$

Simplifying the solution gives

$$y = -\frac{\sin(3x)}{9} + c_1 \cosh(3x) + \frac{\sinh(3x)(1+3c_2)}{9}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sin(3x)}{9} + c_1 \cosh(3x) + \frac{\sinh(3x)(1+3c_2)}{9} \tag{1}$$

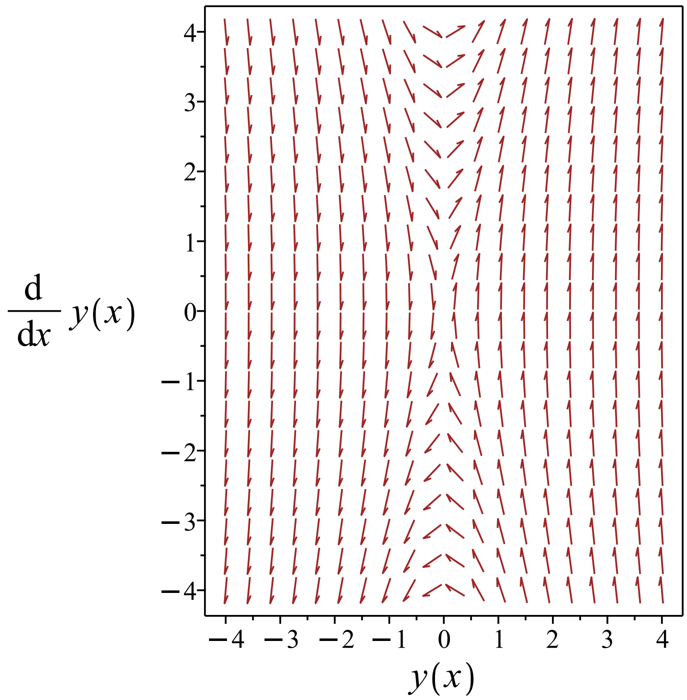


Figure 394: Slope field plot

Verification of solutions

$$y = -\frac{\sin(3x)}{9} + c_1 \cosh(3x) + \frac{\sinh(3x)(1+3c_2)}{9}$$

Verified OK.

13.4.1 Maple step by step solution

Let's solve

$y'' - 9y = 2\sin(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 9 = 0$

- Factor the characteristic polynomial

  $(r-3)(r+3) = 0$

- Roots of the characteristic polynomial

$r = (-3, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3x} + e^{3x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = 2\sin(3x)\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-3x} & e^{3x} \\ -3e^{-3x} & 3e^{3x} \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 6$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{e^{-3x}\left(\int e^{3x}\sin(3x)dx\right)}{3} + \frac{e^{3x}\left(\int e^{-3x}\sin(3x)dx\right)}{3}$

  ○ Compute integrals

  $y_p(x) = -\frac{\sin(3x)}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-3x} + e^{3x} c_2 - \frac{\sin(3x)}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 5.781 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)-9*y(x)=2*sin(3*x),y(x), singsol=all)
```

$$y(x) = -\frac{\sin(3x)}{9} + y(0)\cosh(3x) + \frac{\sinh(3x)(1 + 3D(y)(0))}{9}$$

✓ Solution by Mathematica
Time used: 0.032 (sec). Leaf size: 30

```
DSolve[y''[x]-9*y[x]==2*Sin[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{9}\sin(3x) + c_1e^{3x} + c_2e^{-3x}$$

13.5 problem 5

Internal problem ID [12789]
Internal file name [OUTPUT/11441_Saturday_November_04_2023_08_47_20_AM_58456927/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = 2\sin(3x)}$$

Since no initial conditions are explicitly given, then let

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= c_2 \end{aligned}$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 9Y(s) = \frac{6}{s^2+9} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= c_2 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - c_2 - sc_1 + 9Y(s) = \frac{6}{s^2+9}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{c_1 s^3 + c_2 s^2 + 9sc_1 + 9c_2 + 6}{\left(s^2+9\right)^2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{6\left(s-3i\right)^2} - \frac{1}{6\left(s+3i\right)^2} + \frac{3i\left(-\frac{c_2}{18} - \frac{1}{54}\right) + \frac{c_1}{2}}{s-3i} + \frac{-3i\left(-\frac{c_2}{18} - \frac{1}{54}\right) + \frac{c_1}{2}}{s+3i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{6\left(s-3i\right)^2}\right) = -\frac{x\,\mathrm{e}^{3ix}}{6}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{6\left(s+3i\right)^2}\right) = -\frac{x\,\mathrm{e}^{-3ix}}{6}$$
$$\mathcal{L}^{-1}\left(\frac{3i\left(-\frac{c_2}{18} - \frac{1}{54}\right) + \frac{c_1}{2}}{s-3i}\right) = \frac{\left(-3ic_2 + 9c_1 - i\right)\mathrm{e}^{3ix}}{18}$$
$$\mathcal{L}^{-1}\left(\frac{-3i\left(-\frac{c_2}{18} - \frac{1}{54}\right) + \frac{c_1}{2}}{s+3i}\right) = \frac{\left(3ic_2 + 9c_1 + i\right)\mathrm{e}^{-3ix}}{18}$$

Adding the above results and simplifying gives

$$y = -\frac{\cos\left(3x\right)\left(-3c_1 + x\right)}{3} + \frac{\sin\left(3x\right)\left(1+3c_2\right)}{9}$$

Simplifying the solution gives

$$y = \frac{\left(-3x + 9c_1\right)\cos\left(3x\right)}{9} + \frac{\sin\left(3x\right)\left(1+3c_2\right)}{9}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-3x + 9c_1\right)\cos\left(3x\right)}{9} + \frac{\sin\left(3x\right)\left(1+3c_2\right)}{9} \tag{1}$$

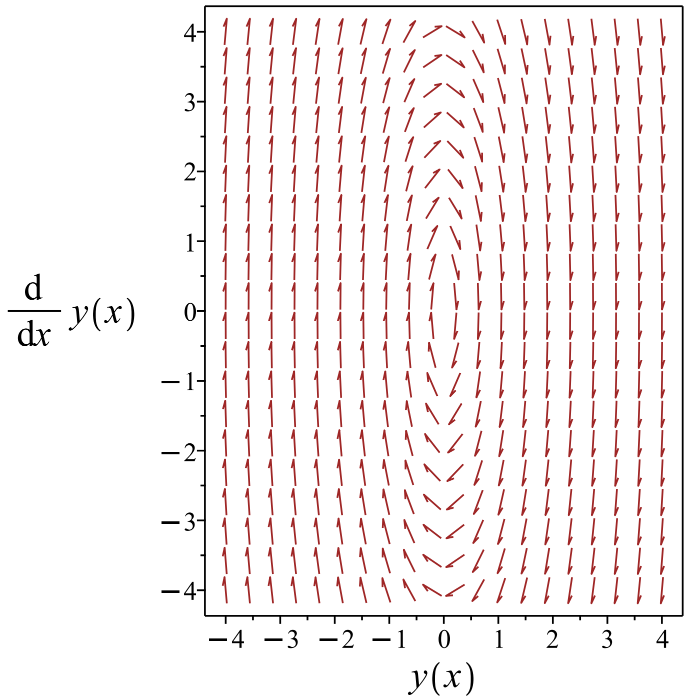


Figure 395: Slope field plot

Verification of solutions

$$y = \frac{(-3x + 9c_1)\cos(3x)}{9} + \frac{\sin(3x)(1 + 3c_2)}{9}$$

Verified OK.

13.5.1 Maple step by step solution

Let's solve

$y'' + 9y = 2\sin(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\sin(3x)\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2\cos(3x)\left(\int \sin(3x)^2 dx\right)}{3} + \frac{\sin(3x)\left(\int \sin(6x)dx\right)}{3}$

  ○ Compute integrals

  $y_p(x) = \frac{\sin(3x)}{18} - \frac{x\cos(3x)}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{\sin(3x)}{18} - \frac{x\cos(3x)}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.437 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+9*y(x)=2*sin(3*x),y(x), singsol=all)
```

$$y(x) = -\frac{\cos(3x)(x - 3y(0))}{3} + \frac{\sin(3x)(1 + 3D(y)(0))}{9}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 33

```
DSolve[y''[x]+9*y[x]==2*Sin[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(-\frac{x}{3} + c_1\right)\cos(3x) + \frac{1}{18}(1 + 18c_2)\sin(3x)$$

13.6 problem 6

Internal problem ID [12790]
Internal file name [`OUTPUT/11442_Saturday_November_04_2023_08_47_20_AM_55985488/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y' - 2y = x\,\mathrm{e}^{x} - 3x^{2}}$$

Since no initial conditions are explicitly given, then let

$$y(0) = c_1$$
$$y'(0) = c_2$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + sY(s) - y(0) - 2Y(s) = \frac{1}{(s-1)^2} - \frac{6}{s^3} \tag{1}$$

But the initial conditions are

$$y(0) = c_1$$
$$y'(0) = c_2$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - c_2 - sc_1 + sY(s) - c_1 - 2Y(s) = \frac{1}{(s-1)^2} - \frac{6}{s^3}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{c_1 s^6 - c_1 s^5 + c_2 s^5 - c_1 s^4 - 2c_2 s^4 + c_1 s^3 + c_2 s^3 + s^3 - 6s^2 + 12s - 6}{(s-1)^2 s^3 (s^2 + s - 2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{3}{s^3} + \frac{3}{2s^2} + \frac{9}{4s} + \frac{\frac{c_1}{3} - \frac{c_2}{3} - \frac{31}{108}}{s+2} - \frac{1}{9(s-1)^2} + \frac{1}{3(s-1)^3} + \frac{\frac{2c_1}{3} + \frac{c_2}{3} - \frac{53}{27}}{s-1}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{3}{s^3}\right) &= \frac{3x^2}{2} \\
\mathcal{L}^{-1}\left(\frac{3}{2s^2}\right) &= \frac{3x}{2} \\
\mathcal{L}^{-1}\left(\frac{9}{4s}\right) &= \frac{9}{4} \\
\mathcal{L}^{-1}\left(\frac{\frac{c_1}{3} - \frac{c_2}{3} - \frac{31}{108}}{s+2}\right) &= \frac{(36c_1 - 36c_2 - 31)\,\mathrm{e}^{-2x}}{108} \\
\mathcal{L}^{-1}\left(-\frac{1}{9(s-1)^2}\right) &= -\frac{x\,\mathrm{e}^{x}}{9} \\
\mathcal{L}^{-1}\left(\frac{1}{3(s-1)^3}\right) &= \frac{x^2\mathrm{e}^{x}}{6} \\
\mathcal{L}^{-1}\left(\frac{\frac{2c_1}{3} + \frac{c_2}{3} - \frac{53}{27}}{s-1}\right) &= \frac{(18c_1 + 9c_2 - 53)\,\mathrm{e}^{x}}{27}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = \frac{9}{4} + \frac{3x^2}{2} + \frac{3x}{2} + \frac{\mathrm{e}^{x}(9x^2 + 36c_1 + 18c_2 - 6x - 106)}{54} + \frac{(36c_1 - 36c_2 - 31)\,\mathrm{e}^{-2x}}{108}$$

Simplifying the solution gives

$$y = \frac{3\left(-\frac{31}{162} + \frac{\left(x^2 - \frac{2}{3}x + 4c_1 + 2c_2 - \frac{106}{9}\right)\mathrm{e}^{3x}}{9} + \left(x^2 + x + \frac{3}{2}\right)\mathrm{e}^{2x} + \frac{2c_1}{9} - \frac{2c_2}{9}\right)\mathrm{e}^{-2x}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{3\left(-\frac{31}{162}+\frac{\left(x^2-\frac{2}{3}x+4c_1+2c_2-\frac{106}{9}\right)e^{3x}}{9}+\left(x^2+x+\frac{3}{2}\right)e^{2x}+\frac{2c_1}{9}-\frac{2c_2}{9}\right)e^{-2x}}{2} \tag{1}$$

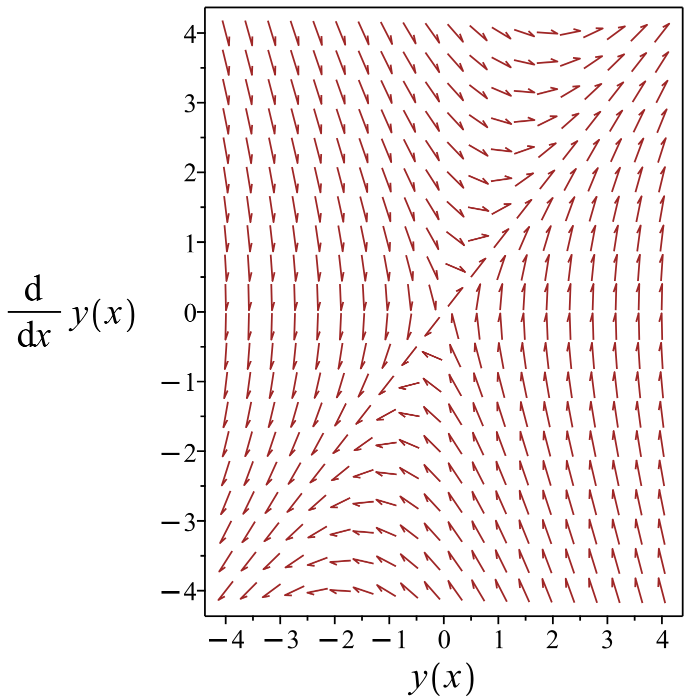


Figure 396: Slope field plot

Verification of solutions

$$y = \frac{3\left(-\frac{31}{162}+\frac{\left(x^2-\frac{2}{3}x+4c_1+2c_2-\frac{106}{9}\right)e^{3x}}{9}+\left(x^2+x+\frac{3}{2}\right)e^{2x}+\frac{2c_1}{9}-\frac{2c_2}{9}\right)e^{-2x}}{2}$$

Verified OK.

13.6.1 Maple step by step solution

Let's solve

$y'' + y' - 2y = x\,e^{x} - 3x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$
- Factor the characteristic polynomial

  $(r+2)(r-1) = 0$
- Roots of the characteristic polynomial

  $r = (-2, 1)$
- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x\,e^{x} - 3x^2\right]$
  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{x} \\ -2\,e^{-2x} & e^{x} \end{bmatrix}$$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3\,e^{-x}$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\left(e^{3x}\left(\int (3x^2 e^{-x} - x) dx\right) + \int e^{2x} x (e^{x} - 3x) dx\right) e^{-2x}}{3}$
  - Compute integrals

    $y_p(x) = \frac{9}{4} + \frac{(9x^2 - 6x + 2) e^{x}}{54} + \frac{3x^2}{2} + \frac{3x}{2}$
- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{x} + \frac{9}{4} + \frac{(9x^2 - 6x + 2) e^{x}}{54} + \frac{3x^2}{2} + \frac{3x}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.812 (sec). Leaf size: 52

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-2*y(x)=x*exp(x)-3*x^2,y(x), singsol=all)
```

$$y(x) = \frac{9}{4} + \frac{3x}{2} + \frac{3x^2}{2} + \frac{e^x(9x^2 + 18D(y)(0) + 36y(0) - 6x - 106)}{54} + \frac{(36y(0) - 36D(y)(0) - 31)e^{-2x}}{108}$$

✓ Solution by Mathematica

Time used: 0.313 (sec). Leaf size: 49

```
DSolve[y''[x]+y'[x]-2*y[x]==x*Exp[x]-3*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3}{4}(2x^2 + 2x + 3) + \frac{1}{54}e^x(9x^2 - 6x + 2 + 54c_2) + c_1e^{-2x}$$

13.7 problem 7

Internal problem ID [12791]
Internal file name [`OUTPUT/11443_Saturday_November_04_2023_08_47_21_AM_22023351/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248
Problem number: 7.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

`[[_high_order, _missing_y]]`

$$\boxed{y'''' - 2y''' + y'' = x\,\mathrm{e}^{x} - 3x^{2}}$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0) \\
\mathcal{L}(y'''') &= s^4Y(s) - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^4Y(s)-y'''(0)-sy''(0)-s^2y'(0)-s^3y(0)-2s^3Y(s)+2y''(0)+2sy'(0)+2s^2y(0)+s^2Y(s)-y'(0)-sy(0) = \frac{}{(s - } \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= c_1 \\
y'(0) &= c_2 \\
y''(0) &= c_3 \\
y'''(0) &= c_4
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^4Y(s)-c_4-sc_3-s^2c_2-s^3c_1-2s^3Y(s)+2c_3+2sc_2+2s^2c_1+s^2Y(s)-c_2-sc_1 = \frac{1}{(s-1)^2}-\frac{6}{s^3}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{c_1s^8-4c_1s^7+c_2s^7+6c_1s^6-4c_2s^6+c_3s^6-4c_1s^5+6c_2s^5-4c_3s^5+c_4s^5+c_1s^4-4c_2s^4+5c_3s^4}{(s-1)^2 s^5 (s^2-2s+1)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{2}{(s-1)^3}+\frac{1}{(s-1)^4}-\frac{18}{s^3}+\frac{-26-3c_3+2c_4+c_1}{s}+\frac{-23+c_2-2c_3+c_4}{s^2}+\frac{-c_3+c_4-3}{(s-1)^2}+\frac{3c_3-2c_4}{s-1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{2}{(s-1)^3}\right) = -x^2\mathrm{e}^x$$
$$\mathcal{L}^{-1}\left(\frac{1}{(s-1)^4}\right) = \frac{x^3\mathrm{e}^x}{6}$$
$$\mathcal{L}^{-1}\left(-\frac{18}{s^3}\right) = -9x^2$$
$$\mathcal{L}^{-1}\left(\frac{-26-3c_3+2c_4+c_1}{s}\right) = -26-3c_3+2c_4+c_1$$
$$\mathcal{L}^{-1}\left(\frac{-23+c_2-2c_3+c_4}{s^2}\right) = (-23+c_2-2c_3+c_4)\,x$$
$$\mathcal{L}^{-1}\left(\frac{-c_3+c_4-3}{(s-1)^2}\right) = (-c_3+c_4-3)\,x\,\mathrm{e}^x$$
$$\mathcal{L}^{-1}\left(\frac{3c_3-2c_4+26}{s-1}\right) = (3c_3-2c_4+26)\,\mathrm{e}^x$$
$$\mathcal{L}^{-1}\left(-\frac{6}{s^5}\right) = -\frac{x^4}{4}$$
$$\mathcal{L}^{-1}\left(-\frac{12}{s^4}\right) = -2x^3$$

Adding the above results and simplifying gives

$$y = -26-2x^3-9x^2-\frac{x^4}{4}-3c_3+2c_4+c_1+\frac{\mathrm{e}^x(x^3-6c_3x+6c_4x-6x^2+18c_3-12c_4-18x+156)}{6}+(-23+c$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &-26 - 2x^3 - 9x^2 - \frac{x^4}{4} - 3c_3 + 2c_4 + c_1 \\
&+ \frac{e^x (x^3 - 6c_3 x + 6c_4 x - 6x^2 + 18c_3 - 12c_4 - 18x + 156)}{6} \\
&+ (-23 + c_2 - 2c_3 + c_4)\, x
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = &-26 - 2x^3 - 9x^2 - \frac{x^4}{4} - 3c_3 + 2c_4 + c_1 \\
&+ \frac{e^x (x^3 - 6c_3 x + 6c_4 x - 6x^2 + 18c_3 - 12c_4 - 18x + 156)}{6} + (-23 + c_2 - 2c_3 + c_4)\, x
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _a*exp(_a)-3*_a^2+2*(diff(_b(
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 5.985 (sec). Leaf size: 79

```
dsolve(diff(y(x),x$4)-2*diff(y(x),x$3)+diff(y(x),x$2)=x*exp(x)-3*x^2,y(x), singsol=all)
```

$$y(x) = -26 - \frac{x^4}{4} - 9x^2 - 2x^3 + y(0) + \frac{e^x \left(x^3 + 6x D^{(3)}(y)(0) - 6x D^{(2)}(y)(0) - 6x^2 - 12 D^{(3)}(y)(0) + 18 D^{(2)}(y)(0) - 18x + 156\right)}{6} - D^{(2)}(y)(0)(3+2x) + D^{(3)}(y)(0)(x+2) + x(-23 + D(y)(0))$$

✓ Solution by Mathematica

Time used: 0.812 (sec). Leaf size: 59

```
DSolve[y''''[x]-2*y'''[x]+y''[x]==x*Exp[x]-3*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^4}{4} - 2x^3 - 9x^2 + e^x \left(\frac{x^3}{6} - x^2 + (3 + c_2)x - 4 + c_1 - 2c_2\right) + c_4 x + c_3$$

13.8 problem 8

Internal problem ID [12792]
Internal file name [`OUTPUT/11444_Saturday_November_04_2023_08_47_21_AM_50415031/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = e^x}$$

With initial conditions

$$[y(0) = -1]$$

13.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = e^x$$

Hence the ode is

$$y' = e^x$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

13.8.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) = \frac{1}{s-1} \tag{1}$$

Replacing initial condition gives

$$sY(s) + 1 = \frac{1}{s-1}$$

Solving for $Y(s)$ gives

$$Y(s) = -\frac{-2+s}{(s-1)\,s}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{2}{s} + \frac{1}{s-1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{2}{s}\right) = -2$$
$$\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^x$$

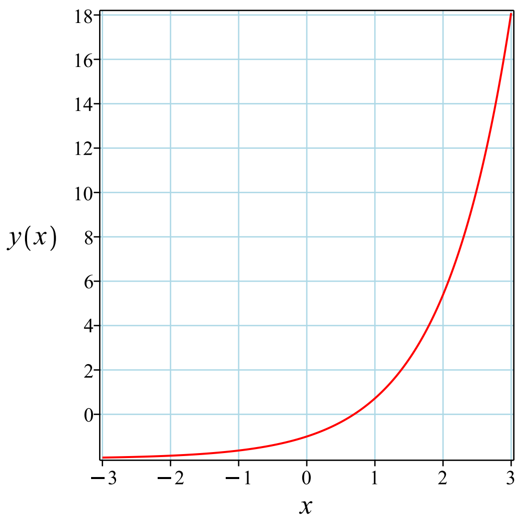
Adding the above results and simplifying gives

$$y = e^x - 2$$

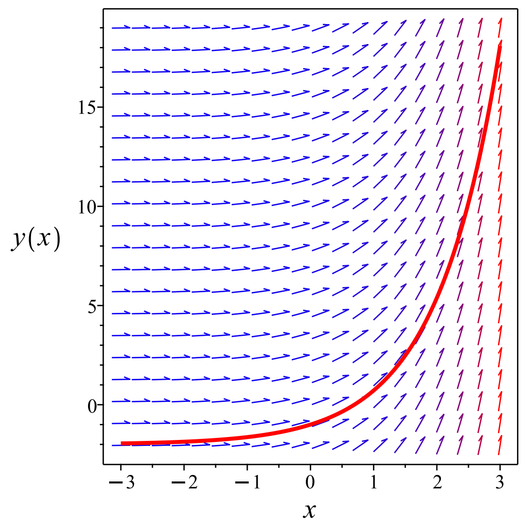
Summary

The solution(s) found are the following

$$y = e^x - 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^x - 2$$

Verified OK.

13.8.3 Maple step by step solution

Let's solve

$[y' = e^x, y(0) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int e^x dx + c_1$

- Evaluate integral

  $y = e^x + c_1$
- Solve for y

  $y = e^x + c_1$
- Use initial condition $y(0) = -1$

  $-1 = 1 + c_1$
- Solve for c_1

  $c_1 = -2$
- Substitute $c_1 = -2$ into general solution and simplify

  $y = e^x - 2$
- Solution to the IVP

  $y = e^x - 2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 5.328 (sec). Leaf size: 8

```
dsolve([diff(y(x),x)=exp(x),y(0) = -1],y(x), singsol=all)
```

$$y(x) = e^x - 2$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 10

```
DSolve[{y'[x]==Exp[x],{y[0]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x - 2$$

13.9 problem 9

Internal problem ID [12793]
Internal file name [`OUTPUT/11445_Saturday_November_04_2023_08_47_21_AM_30044945/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = 2\,\mathrm{e}^{x}}$$

With initial conditions

$$[y(0) = 1]$$

13.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = 2\,\mathrm{e}^{x}$$

Hence the ode is

$$y' - y = 2\,\mathrm{e}^{x}$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2\,\mathrm{e}^{x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

13.9.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - Y(s) = \frac{2}{s-1} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 1 - Y(s) = \frac{2}{s-1}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{1+s}{(s-1)^2}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s-1} + \frac{2}{(s-1)^2}$$

The inverse Laplace of each term above is now found, which gives

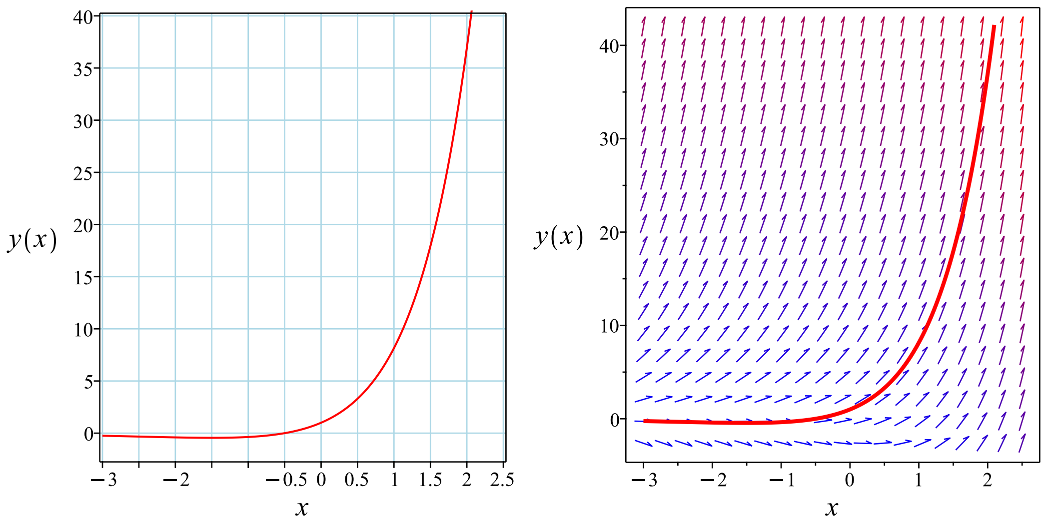
$$\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = \mathrm{e}^{x}$$
$$\mathcal{L}^{-1}\left(\frac{2}{(s-1)^2}\right) = 2x\,\mathrm{e}^{x}$$

Adding the above results and simplifying gives

$$y = (2x + 1)\,e^{x}$$

Summary
The solution(s) found are the following

$$y = (2x + 1)\,e^{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = (2x + 1)\,e^{x}$$

Verified OK.

13.9.3 Maple step by step solution

Let's solve

$[y' - y = 2\,e^{x}, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y + 2\,e^{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = 2\,\mathrm{e}^{x}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - y) = 2\mu(x)\,\mathrm{e}^{x}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' - y) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\,\mathrm{e}^{x} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 2\mu(x)\,\mathrm{e}^{x} dx + c_1$
- Solve for y

  $y = \frac{\int 2\mu(x)\mathrm{e}^{x} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int 2\,\mathrm{e}^{-x}\mathrm{e}^{x} dx + c_1}{\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y = \frac{2x + c_1}{\mathrm{e}^{-x}}$
- Simplify

  $y = \mathrm{e}^{x}(2x + c_1)$
- Use initial condition $y(0) = 1$

  $1 = c_1$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = (2x + 1)\,\mathrm{e}^{x}$
- Solution to the IVP

$$y = (2x + 1)\,e^{x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 5.094 (sec). Leaf size: 12

```
dsolve([diff(y(x),x)-y(x)=2*exp(x),y(0) = 1],y(x), singsol=all)
```

$$y(x) = (2x + 1)\,e^{x}$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 14

```
DSolve[{y'[x]-y[x]==2*Exp[x],{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}(2x + 1)$$

13.10 problem 10

Internal problem ID [12794]
Internal file name [`OUTPUT/11446_Saturday_November_04_2023_08_47_21_AM_34637621/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 9y = x + 2}$$

With initial conditions

$$[y(0) = -1, y'(0) = 1]$$

13.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -9 \\ F &= x + 2 \end{aligned}$$

Hence the ode is

$$y'' - 9y = x + 2$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = x + 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 9Y(s) = \frac{2s+1}{s^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= -1\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + s - 9Y(s) = \frac{2s+1}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{s^3 - s^2 - 2s - 1}{s^2(s^2-9)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{9s^2} - \frac{11}{54(s-3)} - \frac{2}{9s} - \frac{31}{54(s+3)}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{9s^2}\right) = -\frac{x}{9}$$
$$\mathcal{L}^{-1}\left(-\frac{11}{54(s-3)}\right) = -\frac{11\,e^{3x}}{54}$$
$$\mathcal{L}^{-1}\left(-\frac{2}{9s}\right) = -\frac{2}{9}$$
$$\mathcal{L}^{-1}\left(-\frac{31}{54(s+3)}\right) = -\frac{31\,e^{-3x}}{54}$$

Adding the above results and simplifying gives

$$y = -\frac{7\cosh(3x)}{9} + \frac{10\sinh(3x)}{27} - \frac{x}{9} - \frac{2}{9}$$

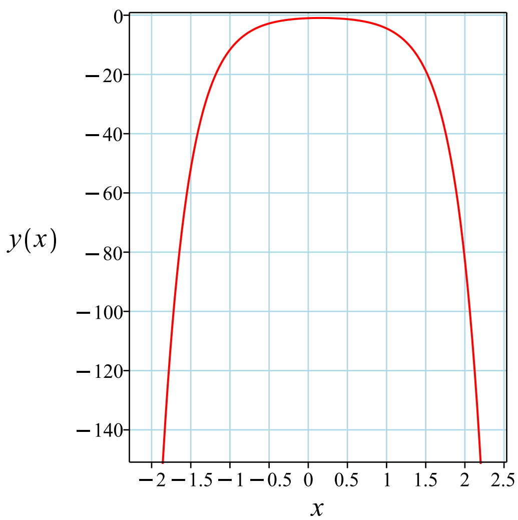
Simplifying the solution gives

$$y = -\frac{7\cosh(3x)}{9} + \frac{10\sinh(3x)}{27} - \frac{x}{9} - \frac{2}{9}$$

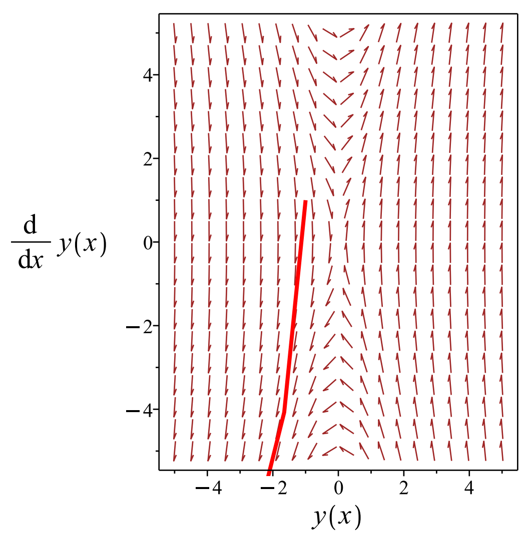
Summary

The solution(s) found are the following

$$y = -\frac{7\cosh(3x)}{9} + \frac{10\sinh(3x)}{27} - \frac{x}{9} - \frac{2}{9} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{7\cosh(3x)}{9} + \frac{10\sinh(3x)}{27} - \frac{x}{9} - \frac{2}{9}$$

Verified OK.

13.10.2 Maple step by step solution

Let's solve

$\left[y'' - 9y = x + 2, y(0) = -1, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 - 9 = 0$
- Factor the characteristic polynomial
  $(r-3)(r+3) = 0$
- Roots of the characteristic polynomial
  $r = (-3, 3)$
- 1st solution of the homogeneous ODE
  $y_1(x) = e^{-3x}$
- 2nd solution of the homogeneous ODE
  $y_2(x) = e^{3x}$
- General solution of the ODE
  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y = c_1 e^{-3x} + e^{3x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function
  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x + 2\right]$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x)\,,y_2(x)) = \begin{bmatrix} e^{-3x} & e^{3x} \\ -3\,e^{-3x} & 3\,e^{3x} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x)\,,y_2(x)) = 6$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{e^{-3x}\left(\int e^{3x}(x+2)dx\right)}{6} + \frac{e^{3x}\left(\int e^{-3x}(x+2)dx\right)}{6}$

- Compute integrals

  $y_p(x) = -\frac{2}{9} - \frac{x}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-3x} + e^{3x}c_2 - \frac{2}{9} - \frac{x}{9}$

☐ Check validity of solution $y = c_1 e^{-3x} + e^{3x}c_2 - \frac{2}{9} - \frac{x}{9}$

- Use initial condition $y(0) = -1$

  $-1 = c_1 + c_2 - \frac{2}{9}$

- Compute derivative of the solution

  $y' = -3c_1 e^{-3x} + 3\,e^{3x}c_2 - \frac{1}{9}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = -3c_1 + 3c_2 - \frac{1}{9}$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{31}{54}, c_2 = -\frac{11}{54}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{31\,e^{-3x}}{54} - \frac{11\,e^{3x}}{54} - \frac{2}{9} - \frac{x}{9}$

- Solution to the IVP

  $y = -\frac{31\,e^{-3x}}{54} - \frac{11\,e^{3x}}{54} - \frac{2}{9} - \frac{x}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.391 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$2)-9*y(x)=x+2,y(0) = -1, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{x}{9} - \frac{7\cosh(3x)}{9} + \frac{10\sinh(3x)}{27} - \frac{2}{9}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 33

```
DSolve[{y''[x]-9*y[x]==x+2,{y[0]==-1,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{54}e^{-3x}\left(-6e^{3x}(x+2) - 11e^{6x} - 31\right)$$

13.11 problem 11

Internal problem ID [12795]
Internal file name [`OUTPUT/11447_Saturday_November_04_2023_08_47_22_AM_80377253/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 9y = x + 2}$$

With initial conditions

$$[y(0) = -1, y'(0) = 1]$$

13.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 9 \\ F &= x + 2 \end{aligned}$$

Hence the ode is

$$y'' + 9y = x + 2$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = x + 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 9Y(s) = \frac{2s+1}{s^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= -1\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + s + 9Y(s) = \frac{2s+1}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{s^3 - s^2 - 2s - 1}{s^2\,(s^2 + 9)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{9s^2} + \frac{2}{9s} + \frac{-\frac{11}{18} - \frac{4i}{27}}{s - 3i} + \frac{-\frac{11}{18} + \frac{4i}{27}}{s + 3i}$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{9s^2}\right) &= \frac{x}{9} \\
\mathcal{L}^{-1}\left(\frac{2}{9s}\right) &= \frac{2}{9} \\
\mathcal{L}^{-1}\left(\frac{-\frac{11}{18}-\frac{4i}{27}}{s-3i}\right) &= \left(-\frac{11}{18}-\frac{4i}{27}\right)\mathrm{e}^{3ix} \\
\mathcal{L}^{-1}\left(\frac{-\frac{11}{18}+\frac{4i}{27}}{s+3i}\right) &= \left(-\frac{11}{18}+\frac{4i}{27}\right)\mathrm{e}^{-3ix}
\end{aligned}
$$

Adding the above results and simplifying gives

$$
y = -\frac{11\cos(3x)}{9} + \frac{8\sin(3x)}{27} + \frac{x}{9} + \frac{2}{9}
$$

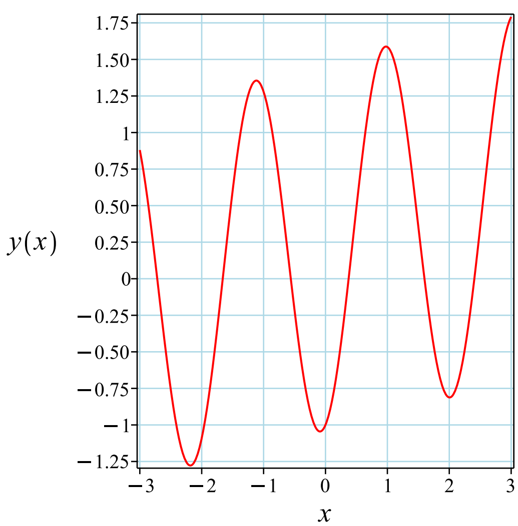
Simplifying the solution gives

$$
y = -\frac{11\cos(3x)}{9} + \frac{8\sin(3x)}{27} + \frac{x}{9} + \frac{2}{9}
$$

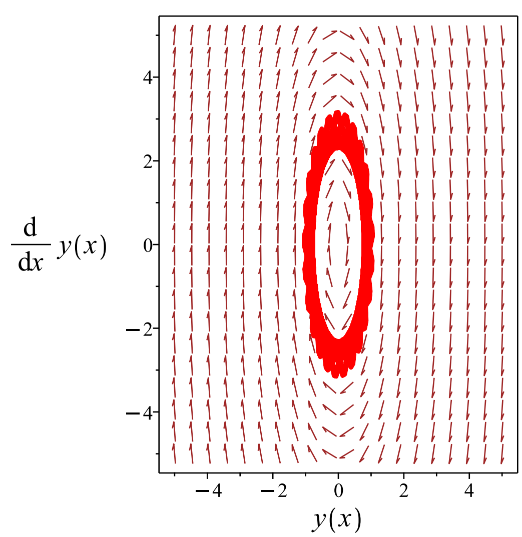
Summary

The solution(s) found are the following

$$
y = -\frac{11\cos(3x)}{9} + \frac{8\sin(3x)}{27} + \frac{x}{9} + \frac{2}{9} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{11\cos(3x)}{9} + \frac{8\sin(3x)}{27} + \frac{x}{9} + \frac{2}{9}$$

Verified OK.

13.11.2 Maple step by step solution

Let's solve

$\left[y'' + 9y = x + 2, y(0) = -1, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x + 2\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\cos(3x)\left(\int \sin(3x)(x+2)dx\right)}{3} + \frac{\sin(3x)\left(\int \cos(3x)(x+2)dx\right)}{3}$

- Compute integrals

  $y_p(x) = \frac{2}{9} + \frac{x}{9}$

• Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{2}{9} + \frac{x}{9}$

□ Check validity of solution $y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{2}{9} + \frac{x}{9}$

- Use initial condition $y(0) = -1$

  $-1 = c_1 + \frac{2}{9}$

- Compute derivative of the solution

  $y' = -3c_1 \sin(3x) + 3c_2 \cos(3x) + \frac{1}{9}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = \frac{1}{9} + 3c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{11}{9}, c_2 = \frac{8}{27}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{11\cos(3x)}{9} + \frac{8\sin(3x)}{27} + \frac{x}{9} + \frac{2}{9}$

• Solution to the IVP

  $y = -\frac{11\cos(3x)}{9} + \frac{8\sin(3x)}{27} + \frac{x}{9} + \frac{2}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.313 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$2)+9*y(x)=x+2,y(0) = -1, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{x}{9} - \frac{11\cos(3x)}{9} + \frac{8\sin(3x)}{27} + \frac{2}{9}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 26

```
DSolve[{y''[x]+9*y[x]==x+2,{y[0]==-1,y'[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{27}(3x + 8\sin(3x) - 33\cos(3x) + 6)$$

13.12 problem 12

Internal problem ID [12796]
Internal file name [`OUTPUT/11448_Saturday_November_04_2023_08_47_22_AM_84888398/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y' + 6y = -2\sin(3x)}$$

With initial conditions

$$[y(0) = 0, y'(0) = -1]$$

13.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= 6 \\ F &= -2\sin(3x) \end{aligned}$$

Hence the ode is

$$y'' - y' + 6y = -2\sin(3x)$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = -2\sin(3x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - sY(s) + y(0) + 6Y(s) = -\frac{6}{s^2+9} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= -1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 1 - sY(s) + 6Y(s) = -\frac{6}{s^2+9}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{s^2+15}{(s^2+9)(s^2-s+6)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{-\frac{1}{6}-\frac{i}{6}}{s-3i} + \frac{-\frac{1}{6}+\frac{i}{6}}{s+3i} + \frac{\frac{1}{6}+\frac{13i\sqrt{23}}{138}}{s-\frac{1}{2}-\frac{i\sqrt{23}}{2}} + \frac{\frac{1}{6}-\frac{13i\sqrt{23}}{138}}{s-\frac{1}{2}+\frac{i\sqrt{23}}{2}}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{6}-\frac{i}{6}}{s-3i}\right) = \left(-\frac{1}{6}-\frac{i}{6}\right) e^{3ix}$$
$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{6}+\frac{i}{6}}{s+3i}\right) = \left(-\frac{1}{6}+\frac{i}{6}\right) e^{-3ix}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{1}{6}+\frac{13i\sqrt{23}}{138}}{s-\frac{1}{2}-\frac{i\sqrt{23}}{2}}\right) = \frac{\left(13i\sqrt{23}+23\right) e^{\frac{\left(1+i\sqrt{23}\right)x}{2}}}{138}$$
$$\mathcal{L}^{-1}\left(\frac{\frac{1}{6}-\frac{13i\sqrt{23}}{138}}{s-\frac{1}{2}+\frac{i\sqrt{23}}{2}}\right) = \frac{\left(23-13i\sqrt{23}\right) e^{\frac{\left(1-i\sqrt{23}\right)x}{2}}}{138}$$

Adding the above results and simplifying gives

$$y = -\frac{\cos(3x)}{3} + \frac{\sin(3x)}{3} + \frac{\left(-13\sqrt{23}\,\sin\left(\frac{\sqrt{23}\,x}{2}\right) + 23\cos\left(\frac{\sqrt{23}\,x}{2}\right)\right) e^{\frac{x}{2}}}{69}$$

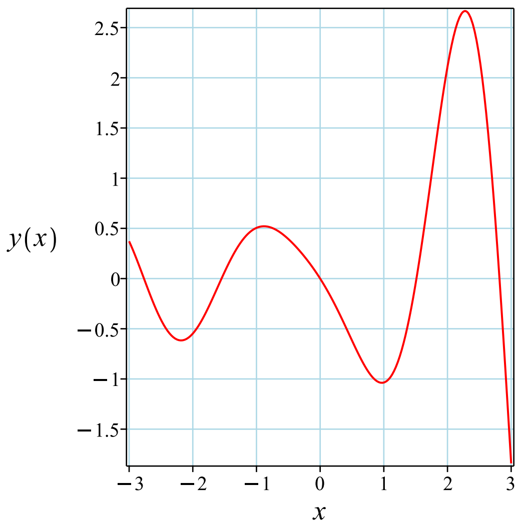
Simplifying the solution gives

$$y = -\frac{13\sqrt{23}\,e^{\frac{x}{2}}\sin\left(\frac{\sqrt{23}\,x}{2}\right)}{69} + \frac{e^{\frac{x}{2}}\cos\left(\frac{\sqrt{23}\,x}{2}\right)}{3} + \frac{\sin(3x)}{3} - \frac{\cos(3x)}{3}$$

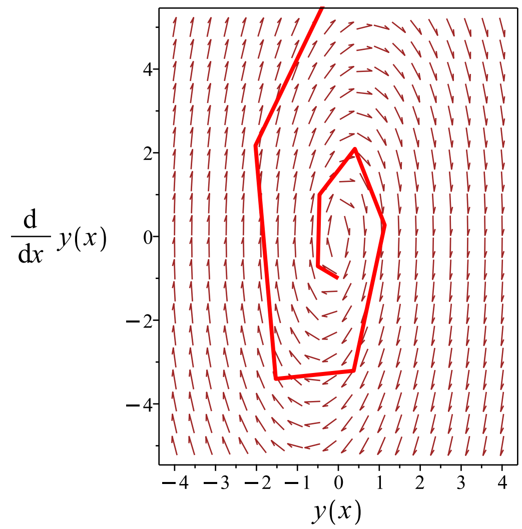
Summary

The solution(s) found are the following

$$y = -\frac{13\sqrt{23}\,e^{\frac{x}{2}}\sin\left(\frac{\sqrt{23}\,x}{2}\right)}{69} + \frac{e^{\frac{x}{2}}\cos\left(\frac{\sqrt{23}\,x}{2}\right)}{3} + \frac{\sin(3x)}{3} - \frac{\cos(3x)}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{13\sqrt{23}\, e^{\frac{x}{2}} \sin\left(\frac{\sqrt{23}\, x}{2}\right)}{69} + \frac{e^{\frac{x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right)}{3} + \frac{\sin(3x)}{3} - \frac{\cos(3x)}{3}$$

Verified OK.

13.12.2 Maple step by step solution

Let's solve

$$\left[y'' - y' + 6y = -2\sin(3x), y(0) = 0, y'\Big|_{\{x=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r + 6 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1 \pm (\sqrt{-23})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{I\sqrt{23}}{2}, \frac{1}{2} + \frac{I\sqrt{23}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{\frac{x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{\frac{x}{2}} \sin\left(\frac{\sqrt{23}\, x}{2}\right)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right) + c_2 e^{\frac{x}{2}} \sin\left(\frac{\sqrt{23}\, x}{2}\right) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -2\sin(3x)\right]$
- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{\frac{x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right) & e^{\frac{x}{2}} \sin\left(\frac{\sqrt{23}\, x}{2}\right) \\ \frac{e^{\frac{x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right)}{2} - \frac{\sqrt{23}\, e^{\frac{x}{2}} \sin\left(\frac{\sqrt{23}\, x}{2}\right)}{2} & \frac{e^{\frac{x}{2}} \sin\left(\frac{\sqrt{23}\, x}{2}\right)}{2} + \frac{e^{\frac{x}{2}} \sqrt{23}\, \cos\left(\frac{\sqrt{23}\, x}{2}\right)}{2} \end{bmatrix}$
- Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{23}\, e^{x}}{2}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{4\sqrt{23}\, e^{\frac{x}{2}} \left(\sin\left(\frac{\sqrt{23}\, x}{2}\right) \left(\int e^{-\frac{x}{2}} \sin(3x) \cos\left(\frac{\sqrt{23}\, x}{2}\right) dx\right) - \cos\left(\frac{\sqrt{23}\, x}{2}\right) \left(\int e^{-\frac{x}{2}} \sin(3x) \sin\left(\frac{\sqrt{23}\, x}{2}\right) dx\right)\right)}{23}$
- Compute integrals

  $y_p(x) = -\frac{\cos(3x)}{3} + \frac{\sin(3x)}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right) + c_2 e^{\frac{x}{2}} \sin\left(\frac{\sqrt{23}\, x}{2}\right) - \frac{\cos(3x)}{3} + \frac{\sin(3x)}{3}$

☐ Check validity of solution $y = c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right) + c_2 e^{\frac{x}{2}} \sin\left(\frac{\sqrt{23}\, x}{2}\right) - \frac{\cos(3x)}{3} + \frac{\sin(3x)}{3}$

- Use initial condition $y(0) = 0$

  $0 = -\frac{1}{3} + c_1$
- Compute derivative of the solution

$$y' = \frac{c_1 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right)}{2} - \frac{c_1 e^{\frac{x}{2}} \sqrt{23}\, \sin\left(\frac{\sqrt{23}\, x}{2}\right)}{2} + \frac{c_2 e^{\frac{x}{2}} \sin\left(\frac{\sqrt{23}\, x}{2}\right)}{2} + \frac{c_2 e^{\frac{x}{2}} \sqrt{23}\, \cos\left(\frac{\sqrt{23}\, x}{2}\right)}{2} + \sin(3x) + \cos(3x)$$

- Use the initial condition $y'\Big|_{\{x=0\}} = -1$

  $-1 = \frac{c_1}{2} + 1 + \frac{c_2\sqrt{23}}{2}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{3}, c_2 = -\frac{13\sqrt{23}}{69}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{13\sqrt{23}\, e^{\frac{x}{2}} \sin\left(\frac{\sqrt{23}\, x}{2}\right)}{69} + \frac{e^{\frac{x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right)}{3} + \frac{\sin(3x)}{3} - \frac{\cos(3x)}{3}$

- Solution to the IVP

  $y = -\frac{13\sqrt{23}\, e^{\frac{x}{2}} \sin\left(\frac{\sqrt{23}\, x}{2}\right)}{69} + \frac{e^{\frac{x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right)}{3} + \frac{\sin(3x)}{3} - \frac{\cos(3x)}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.438 (sec). Leaf size: 45

```
dsolve([diff(y(x),x$2)-diff(y(x),x)+6*y(x)=-2*sin(3*x),y(0) = 0, D(y)(0) = -1],y(x), singsol
```

$$y(x) = -\frac{13\, e^{\frac{x}{2}} \sqrt{23}\, \sin\left(\frac{\sqrt{23}\, x}{2}\right)}{69} + \frac{e^{\frac{x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right)}{3} + \frac{\sin(3x)}{3} - \frac{\cos(3x)}{3}$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 67

```
DSolve[{y''[x]-y'[x]+6*y[x]==-2*Sin[3*x],{y[0]==0,y'[0]==-1}},y[x],x,IncludeSingularSolution
```

$$y(x) \rightarrow \frac{1}{69}\left(23\sin(3x) - 13\sqrt{23}e^{x/2}\sin\left(\frac{\sqrt{23}x}{2}\right) - 23\cos(3x) + 23e^{x/2}\cos\left(\frac{\sqrt{23}x}{2}\right)\right)$$

13.13 problem 13

Internal problem ID [12797]
Internal file name [`OUTPUT/11449_Saturday_November_04_2023_08_47_22_AM_10331552/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y' + 2y = -x^2 + 1}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

13.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 2 \\ F &= -x^2 + 1 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 2y = -x^2 + 1$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = -x^2 + 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) + 2Y(s) = -\frac{2}{s^3} + \frac{1}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2 - s - 2sY(s) + 2Y(s) = -\frac{2}{s^3} + \frac{1}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^4 - 2s^3 + s^2 - 2}{s^3\left(s^2 - 2s + 2\right)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{2s - 2 - 2i} + \frac{1}{2s - 2 + 2i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{1}{s^2}\right) = -x$$

$$\mathcal{L}^{-1}\left(-\frac{1}{s^3}\right) = -\frac{x^2}{2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{2s-2-2i}\right) = \frac{\mathrm{e}^{(1+i)x}}{2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{2s-2+2i}\right) = \frac{\mathrm{e}^{(1-i)x}}{2}$$

Adding the above results and simplifying gives

$$y = \mathrm{e}^{x}\cos(x) - \frac{x^2}{2} - x$$

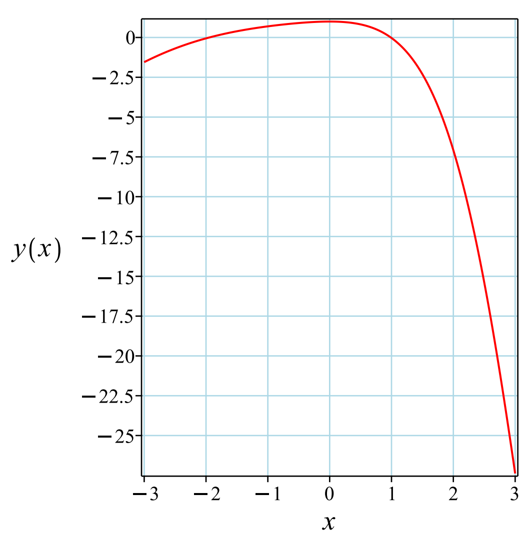
Simplifying the solution gives

$$y = \mathrm{e}^{x}\cos(x) - \frac{x^2}{2} - x$$

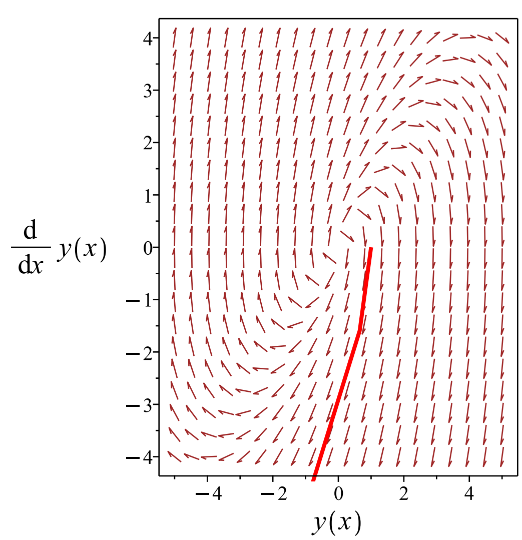
Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x}\cos(x) - \frac{x^2}{2} - x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^x \cos(x) - \frac{x^2}{2} - x$$

Verified OK.

13.13.2 Maple step by step solution

Let's solve

$\left[y'' - 2y' + 2y = -x^2 + 1, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - I, 1 + I)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^x \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^x \cos(x) + c_2 e^x \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = -x^2 + 1\right]$

  ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x \cos(x) & e^x \sin(x) \\ e^x \cos(x) - e^x \sin(x) & e^x \cos(x) + e^x \sin(x) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{2x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^x \left(\cos(x) \left(\int e^{-x} \sin(x) (x^2 - 1) \, dx\right) - \sin(x) \left(\int \cos(x) e^{-x}(x^2 - 1) \, dx\right)\right)$

- Compute integrals

  $y_p(x) = -\frac{x(x+2)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^x \cos(x) + c_2 e^x \sin(x) - \frac{x(x+2)}{2}$

☐ Check validity of solution $y = c_1 e^x \cos(x) + c_2 e^x \sin(x) - \frac{x(x+2)}{2}$

- Use initial condition $y(0) = 1$

  $1 = c_1$

- Compute derivative of the solution

  $y' = c_1 e^x \cos(x) - c_1 e^x \sin(x) + c_2 e^x \sin(x) + c_2 e^x \cos(x) - x - 1$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = c_1 + c_2 - 1$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = e^x \cos(x) - \frac{x^2}{2} - x$

- Solution to the IVP

  $y = e^x \cos(x) - \frac{x^2}{2} - x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.0 (sec). Leaf size: 18

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=1-x^2,y(0) = 1, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = -x - \frac{x^2}{2} + \cos(x)\,\mathrm{e}^x$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 20

```
DSolve[{y''[x]-2*y'[x]+2*y[x]==1-x^2,{y[0]==1,y'[0]==0}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to e^x \cos(x) - \frac{1}{2}x(x+2)$$

13.14 problem 14

Internal problem ID [12798]
Internal file name [`OUTPUT/11450_Saturday_November_04_2023_08_47_23_AM_43671262/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.2, page 248
Problem number: 14.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 3y'' + 2y' = x + \cos(x)}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1, y''(0) = 2]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}
\mathcal{L}(y') &= sY(s) - y(0) \\
\mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\
\mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0)
\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^3Y(s) - y''(0) - sy'(0) - s^2y(0) + 3s^2Y(s) - 3y'(0) - 3sy(0) + 2sY(s) - 2y(0) = \frac{1}{s^2} + \frac{s}{s^2+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= -1 \\
y''(0) &= 2
\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^3Y(s) - 1 - 2s - s^2 + 3s^2Y(s) + 2sY(s) = \frac{1}{s^2} + \frac{s}{s^2+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^6 + 2s^5 + 2s^4 + 3s^3 + 2s^2 + 1}{s^3(s^2+1)(s^2+3s+2)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{2(s+1)} + \frac{-\frac{3}{20} - \frac{i}{20}}{s-i} + \frac{-\frac{3}{20} + \frac{i}{20}}{s+i} + \frac{17}{40(s+2)} - \frac{3}{4s^2} + \frac{1}{2s^3} + \frac{11}{8s}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(-\frac{1}{2(s+1)}\right) &= -\frac{e^{-x}}{2} \\
\mathcal{L}^{-1}\left(\frac{-\frac{3}{20} - \frac{i}{20}}{s-i}\right) &= \left(-\frac{3}{20} - \frac{i}{20}\right) e^{ix} \\
\mathcal{L}^{-1}\left(\frac{-\frac{3}{20} + \frac{i}{20}}{s+i}\right) &= \left(-\frac{3}{20} + \frac{i}{20}\right) e^{-ix} \\
\mathcal{L}^{-1}\left(\frac{17}{40(s+2)}\right) &= \frac{17\,e^{-2x}}{40} \\
\mathcal{L}^{-1}\left(-\frac{3}{4s^2}\right) &= -\frac{3x}{4} \\
\mathcal{L}^{-1}\left(\frac{1}{2s^3}\right) &= \frac{x^2}{4} \\
\mathcal{L}^{-1}\left(\frac{11}{8s}\right) &= \frac{11}{8}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = \frac{17\,e^{-2x}}{40} - \frac{3\cos(x)}{10} + \frac{\sin(x)}{10} + \frac{x^2}{4} - \frac{e^{-x}}{2} - \frac{3x}{4} + \frac{11}{8}$$

Summary

The solution(s) found are the following

$$y = \frac{17\,e^{-2x}}{40} - \frac{3\cos(x)}{10} + \frac{\sin(x)}{10} + \frac{x^2}{4} - \frac{e^{-x}}{2} - \frac{3x}{4} + \frac{11}{8} \tag{1}$$

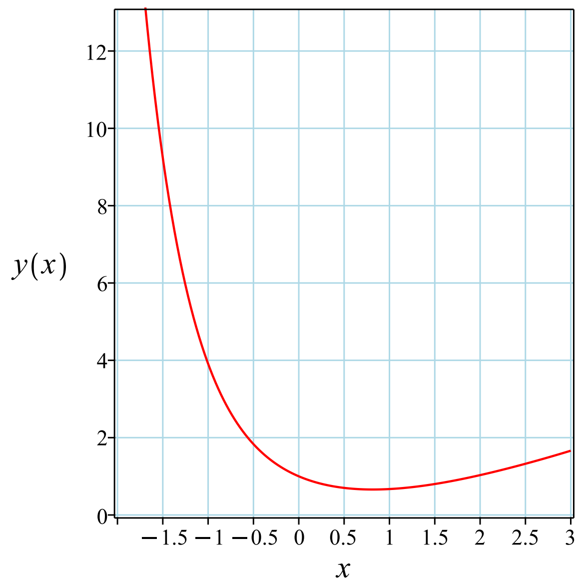


Figure 403: Solution plot

Verification of solutions

$$y = \frac{17\,\mathrm{e}^{-2x}}{40} - \frac{3\cos(x)}{10} + \frac{\sin(x)}{10} + \frac{x^2}{4} - \frac{\mathrm{e}^{-x}}{2} - \frac{3x}{4} + \frac{11}{8}$$

Verified OK.

13.14.1 Maple step by step solution

Let's solve

$$\left[y''' + 3y'' + 2y' = x + \cos(x), y(0) = 1, y'\Big|_{\{x=0\}} = -1, y''\Big|_{\{x=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

$y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = x + \cos(x) - 3y_3(x) - 2y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x + \cos(x) - 3y_3(x) - 2y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x + \cos(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x + \cos(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$$

□ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{-x} & 1 \\ -\frac{e^{-2x}}{2} & -e^{-x} & 0 \\ e^{-2x} & e^{-x} & 0 \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{-x} & 1 \\ -\frac{e^{-2x}}{2} & -e^{-x} & 0 \\ e^{-2x} & e^{-x} & 0 \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & 1 \\ -\frac{1}{2} & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & \frac{e^{-2x}}{2} - 2\,e^{-x} + \frac{3}{2} & \frac{e^{-2x}}{2} - e^{-x} + \frac{1}{2} \\ 0 & -e^{-2x} + 2\,e^{-x} & -e^{-2x} + e^{-x} \\ 0 & 2\,e^{-2x} - 2\,e^{-x} & 2\,e^{-2x} - e^{-x} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{\sin(x)}{10} - \frac{3\,\mathrm{e}^{-2x}}{40} - \frac{\mathrm{e}^{-x}}{2} - \frac{3x}{4} + \frac{7}{8} + \frac{x^2}{4} - \frac{3\cos(x)}{10} \\ \frac{3\,\mathrm{e}^{-2x}}{20} + \frac{\mathrm{e}^{-x}}{2} + \frac{x}{2} - \frac{3}{4} + \frac{\cos(x)}{10} + \frac{3\sin(x)}{10} \\ -\frac{3\,\mathrm{e}^{-2x}}{10} - \frac{\mathrm{e}^{-x}}{2} + \frac{1}{2} + \frac{3\cos(x)}{10} - \frac{\sin(x)}{10} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \begin{bmatrix} \frac{\sin(x)}{10} - \frac{3\,\mathrm{e}^{-2x}}{40} - \frac{\mathrm{e}^{-x}}{2} - \frac{3x}{4} + \frac{7}{8} + \frac{x^2}{4} - \frac{3\cos(x)}{10} \\ \frac{3\,\mathrm{e}^{-2x}}{20} + \frac{\mathrm{e}^{-x}}{2} + \frac{x}{2} - \frac{3}{4} + \frac{\cos(x)}{10} + \frac{3\sin(x)}{10} \\ -\frac{3\,\mathrm{e}^{-2x}}{10} - \frac{\mathrm{e}^{-x}}{2} + \frac{1}{2} + \frac{3\cos(x)}{10} - \frac{\sin(x)}{10} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = \frac{(10c_1-3)\mathrm{e}^{-2x}}{40} + \frac{(40c_2-20)\mathrm{e}^{-x}}{40} + \frac{x^2}{4} - \frac{3x}{4} + c_3 - \frac{3\cos(x)}{10} + \frac{\sin(x)}{10} + \frac{7}{8}$

- Use the initial condition $y(0) = 1$

  $1 = \frac{c_1}{4} + c_2 + c_3$

- Calculate the 1st derivative of the solution

  $y' = -\frac{(10c_1-3)\mathrm{e}^{-2x}}{20} - \frac{(40c_2-20)\mathrm{e}^{-x}}{40} + \frac{x}{2} - \frac{3}{4} + \frac{3\sin(x)}{10} + \frac{\cos(x)}{10}$

- Use the initial condition $y'\Big|_{\{x=0\}} = -1$

  $-1 = -\frac{c_1}{2} - c_2$

- Calculate the 2nd derivative of the solution

  $y'' = \frac{(10c_1-3)\mathrm{e}^{-2x}}{10} + \frac{(40c_2-20)\mathrm{e}^{-x}}{40} + \frac{1}{2} + \frac{3\cos(x)}{10} - \frac{\sin(x)}{10}$

- Use the initial condition $y''\Big|_{\{x=0\}} = 2$

  $2 = c_1 + c_2$

- Solve for the unknown coefficients

  $\left\{c_1 = 2, c_2 = 0, c_3 = \frac{1}{2}\right\}$

- Solution to the IVP

$$y = \frac{17\,e^{-2x}}{40} - \frac{3\cos(x)}{10} + \frac{\sin(x)}{10} + \frac{x^2}{4} - \frac{e^{-x}}{2} - \frac{3x}{4} + \frac{11}{8}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = -3*(diff(_b(_a), _a))-2*_b(_a
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   <- double symmetry of the form [xi=0, eta=F(x)] successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 5.36 (sec). Leaf size: 34

```
dsolve([diff(y(x),x$3)+3*diff(y(x),x$2)+2*diff(y(x),x)=x+cos(x),y(0) = 1, D(y)(0) = -1, (D@@
```

$$y(x) = -\frac{3\cos(x)}{10} + \frac{\sin(x)}{10} - \frac{e^{-x}}{2} - \frac{3x}{4} + \frac{x^2}{4} + \frac{17\,e^{-2x}}{40} + \frac{11}{8}$$

✓ Solution by Mathematica

Time used: 0.565 (sec). Leaf size: 41

```
DSolve[{y'''[x]+3*y''[x]+2*y'[x]==x+Cos[x],{y[0]==1,y'[0]==-1,y''[0]==2}},y[x],x,IncludeSing
```

$$y(x) \to \frac{1}{40}\left(10x^2 - 30x + 17e^{-2x} - 20e^{-x} + 4\sin(x) - 12\cos(x) + 55\right)$$

14 Chapter 5. The Laplace Transform Method. Exercises 5.3, page 255

14.1 problem 7

Internal problem ID [12799]
Internal file name [`OUTPUT/11451_Saturday_November_04_2023_08_47_23_AM_16935513/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.3, page 255
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' - 2y = 6}$$

With initial conditions

$$[y(0) = 2]$$

14.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -2$$
$$q(x) = 6$$

Hence the ode is

$$y' - 2y = 6$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

14.1.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - 2Y(s) = \frac{6}{s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 2 - 2Y(s) = \frac{6}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{6 + 2s}{s(s-2)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{5}{s-2} - \frac{3}{s}$$

The inverse Laplace of each term above is now found, which gives

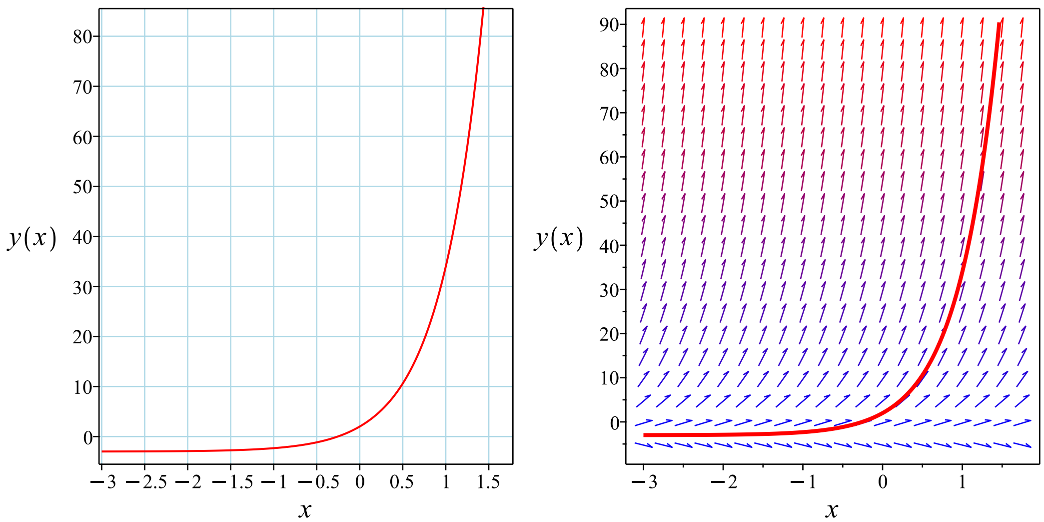
$$\mathcal{L}^{-1}\left(\frac{5}{s-2}\right) = 5\,e^{2x}$$
$$\mathcal{L}^{-1}\left(-\frac{3}{s}\right) = -3$$

Adding the above results and simplifying gives

$$y = 2\,e^{x}(\cosh(x) + 4\sinh(x))$$

Summary
The solution(s) found are the following

$$y = 2\,e^{x}(\cosh(x) + 4\sinh(x)) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 2\,e^{x}(\cosh(x) + 4\sinh(x))$$

Verified OK.

14.1.3 Maple step by step solution

Let's solve

$[y' - 2y = 6, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{2y+6} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{2y+6}dx = \int 1dx + c_1$

- Evaluate integral

  $\frac{\ln(y+3)}{2} = x + c_1$

- Solve for y

  $y = \mathrm{e}^{2x+2c_1} - 3$

- Use initial condition $y(0) = 2$

  $2 = \mathrm{e}^{2c_1} - 3$

- Solve for c_1

  $c_1 = \frac{\ln(5)}{2}$

- Substitute $c_1 = \frac{\ln(5)}{2}$ into general solution and simplify

  $y = 5\,\mathrm{e}^{2x} - 3$

- Solution to the IVP

  $y = 5\,\mathrm{e}^{2x} - 3$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 5.266 (sec). Leaf size: 15

```
dsolve([diff(y(x),x)-2*y(x)=6,y(0) = 2],y(x), singsol=all)
```

$$y(x) = 2\,\mathrm{e}^{x}(\cosh(x) + 4\sinh(x))$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 14

```
DSolve[{y'[x]-2*y[x]==6,{y[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 5e^{2x} - 3$$

14.2 problem 8

Internal problem ID [12800]
Internal file name [`OUTPUT/11452_Saturday_November_04_2023_08_47_23_AM_50152952/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.3, page 255
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = e^x}$$

With initial conditions

$$\left[y(0) = \frac{5}{2}\right]$$

14.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = e^x$$

Hence the ode is

$$y' + y = e^x$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

14.2.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + Y(s) = \frac{1}{s-1} \tag{1}$$

Replacing initial condition gives

$$sY(s) - \frac{5}{2} + Y(s) = \frac{1}{s-1}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{-3+5s}{2(s-1)(s+1)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{2s-2} + \frac{2}{s+1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{2s-2}\right) = \frac{e^x}{2}$$
$$\mathcal{L}^{-1}\left(\frac{2}{s+1}\right) = 2\,e^{-x}$$

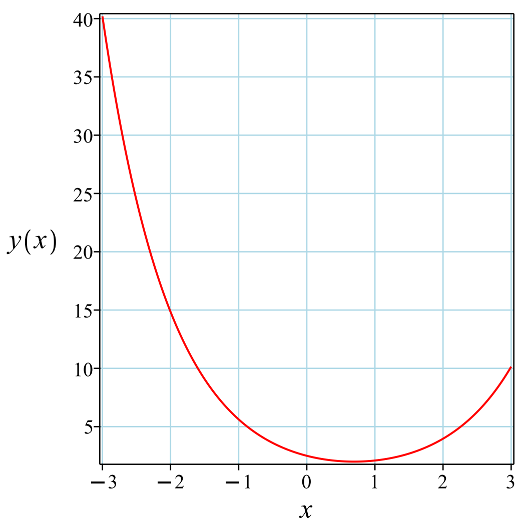
Adding the above results and simplifying gives

$$y = \frac{5\cosh(x)}{2} - \frac{3\sinh(x)}{2}$$

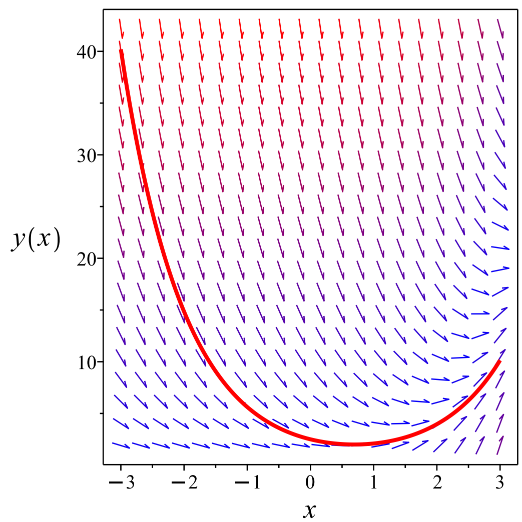
Summary

The solution(s) found are the following

$$y = \frac{5\cosh(x)}{2} - \frac{3\sinh(x)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{5\cosh(x)}{2} - \frac{3\sinh(x)}{2}$$

Verified OK.

14.2.3 Maple step by step solution

Let's solve

$\left[y' + y = e^x, y(0) = \frac{5}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = -y + e^x$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + y = e^x$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\,(y' + y) = \mu(x)\,e^x$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\,(y' + y) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \mu(x)$$

- Solve to find the integrating factor

$$\mu(x) = e^x$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,e^x dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \mu(x)\,e^x dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x) e^x dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = e^x$

$$y = \frac{\int (e^x)^2 dx + c_1}{e^x}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{(e^x)^2}{2} + c_1}{e^x}$$

- Simplify

$$y = c_1 e^{-x} + \frac{e^x}{2}$$

- Use initial condition $y(0) = \frac{5}{2}$

$$\frac{5}{2} = \frac{1}{2} + c_1$$

- Solve for c_1

$$c_1 = 2$$

- Substitute $c_1 = 2$ into general solution and simplify

$$y = 2\,e^{-x} + \frac{e^x}{2}$$

- Solution to the IVP

  $y = 2\,\mathrm{e}^{-x} + \frac{\mathrm{e}^{x}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 5.813 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)+y(x)=exp(x),y(0) = 5/2],y(x), singsol=all)
```

$$y(x) = \frac{5\cosh(x)}{2} - \frac{3\sinh(x)}{2}$$

✓ Solution by Mathematica

Time used: 0.066 (sec). Leaf size: 20

```
DSolve[{y'[x]+y[x]==Exp[x],{y[0]==5/2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2e^{-x} + \frac{e^{x}}{2}$$

14.3 problem 9

Internal problem ID [12801]
Internal file name [`OUTPUT/11453_Saturday_November_04_2023_08_47_23_AM_38420457/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.3, page 255
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 9y = 1}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

14.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 9 \\ F &= 1 \end{aligned}$$

Hence the ode is

$$y'' + 9y = 1$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 9Y(s) = \frac{1}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 9Y(s) = \frac{1}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{1}{s\,(s^2 + 9)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{9s} - \frac{1}{18\,(s - 3i)} - \frac{1}{18\,(s + 3i)}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{1}{9s}\right) = \frac{1}{9}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{18\left(s-3i\right)}\right) = -\frac{e^{3ix}}{18}$$
$$\mathcal{L}^{-1}\left(-\frac{1}{18\left(s+3i\right)}\right) = -\frac{e^{-3ix}}{18}$$

Adding the above results and simplifying gives

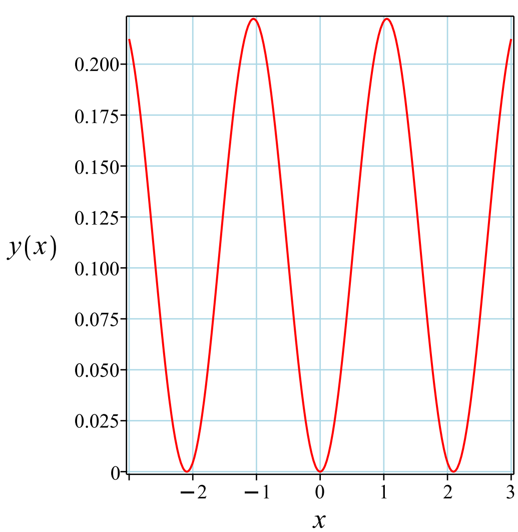
$$y = -\frac{\cos\left(3x\right)}{9}+\frac{1}{9}$$

Simplifying the solution gives

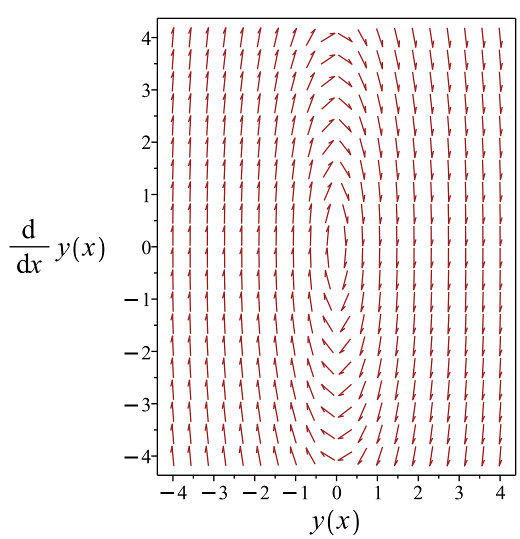
$$y = -\frac{\cos\left(3x\right)}{9}+\frac{1}{9}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{\cos\left(3x\right)}{9}+\frac{1}{9} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\cos(3x)}{9} + \frac{1}{9}$$

Verified OK.

14.3.2 Maple step by step solution

Let's solve

$\left[y'' + 9y = 1, y(0) = 0, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 1\right]$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\cos(3x)\left(\int \sin(3x)dx\right)}{3} + \frac{\sin(3x)\left(\int \cos(3x)dx\right)}{3}$

- Compute integrals

  $y_p(x) = \frac{1}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{1}{9}$

☐ Check validity of solution $y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{1}{9}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + \frac{1}{9}$

- Compute derivative of the solution

  $y' = -3c_1 \sin(3x) + 3c_2 \cos(3x)$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = 3c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{9}, c_2 = 0\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{\cos(3x)}{9} + \frac{1}{9}$

- Solution to the IVP

  $y = -\frac{\cos(3x)}{9} + \frac{1}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 4.921 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)+9*y(x)=1,y(0) = 0, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = -\frac{\cos(3x)}{9} + \frac{1}{9}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 17

```
DSolve[{y''[x]+9*y[x]==1,{y[0]==0,y'[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{9}\sin^2\left(\frac{3x}{2}\right)$$

14.4 problem 10

Internal problem ID [12802]
Internal file name [`OUTPUT/11454_Saturday_November_04_2023_08_47_24_AM_46660882/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.3, page 255
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 9y = 18\,\mathrm{e}^{3x}}$$

With initial conditions

$$[y(0) = -1, y'(0) = 6]$$

14.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 9 \\ F &= 18\,\mathrm{e}^{3x} \end{aligned}$$

Hence the ode is

$$y'' + 9y = 18\,\mathrm{e}^{3x}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 18\,\mathrm{e}^{3x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 9Y(s) = \frac{18}{s-3} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= -1\\ y'(0) &= 6\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 6 + s + 9Y(s) = \frac{18}{s-3}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{s(s-9)}{(s-3)(s^2+9)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{1}{s-3} + \frac{-1-\frac{i}{2}}{s-3i} + \frac{-1+\frac{i}{2}}{s+3i}$$

The inverse Laplace of each term above is now found, which gives

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) &= e^{3x} \\
\mathcal{L}^{-1}\left(\frac{-1-\frac{i}{2}}{s-3i}\right) &= \left(-1-\frac{i}{2}\right) e^{3ix} \\
\mathcal{L}^{-1}\left(\frac{-1+\frac{i}{2}}{s+3i}\right) &= \left(-1+\frac{i}{2}\right) e^{-3ix}
\end{aligned}
$$

Adding the above results and simplifying gives

$$y = -2\cos(3x) + \sin(3x) + e^{3x}$$

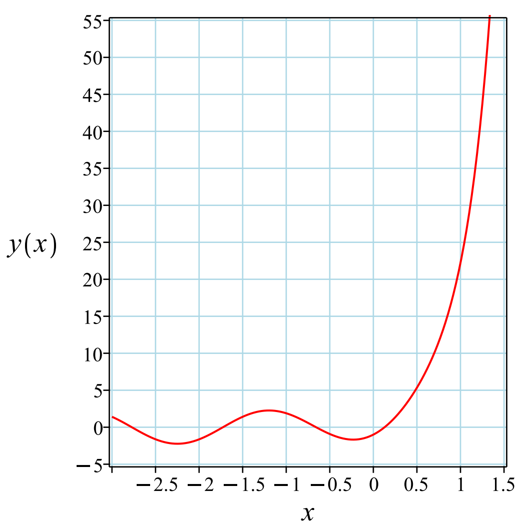
Simplifying the solution gives

$$y = -2\cos(3x) + \sin(3x) + e^{3x}$$

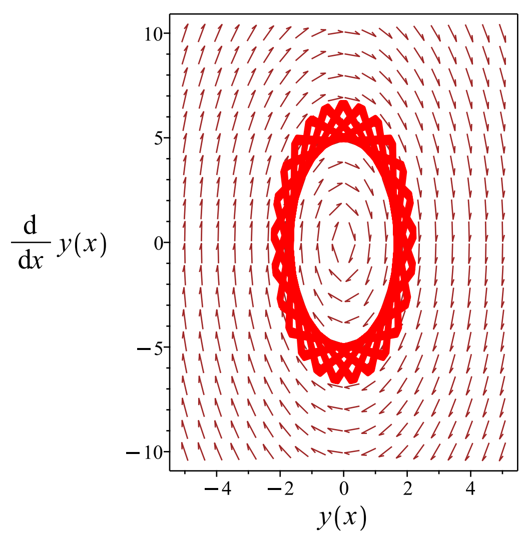
Summary

The solution(s) found are the following

$$y = -2\cos(3x) + \sin(3x) + e^{3x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2\cos(3x) + \sin(3x) + e^{3x}$$

Verified OK.

14.4.2 Maple step by step solution

Let's solve

$$\left[y'' + 9y = 18\,\mathrm{e}^{3x}, y(0) = -1, y'\Big|_{\{x=0\}} = 6\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 18\,\mathrm{e}^{3x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -6\cos(3x)\left(\int e^{3x}\sin(3x)\,dx\right) + 6\sin(3x)\left(\int e^{3x}\cos(3x)\,dx\right)$$

- Compute integrals

  $y_p(x) = e^{3x}$

- Substitute particular solution into general solution to ODE

  $y = c_1\cos(3x) + c_2\sin(3x) + e^{3x}$

□ Check validity of solution $y = c_1\cos(3x) + c_2\sin(3x) + e^{3x}$

- Use initial condition $y(0) = -1$

  $-1 = 1 + c_1$

- Compute derivative of the solution

  $y' = -3c_1\sin(3x) + 3c_2\cos(3x) + 3e^{3x}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 6$

  $6 = 3 + 3c_2$

- Solve for c_1 and c_2

  $\{c_1 = -2, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = -2\cos(3x) + \sin(3x) + e^{3x}$

- Solution to the IVP

  $y = -2\cos(3x) + \sin(3x) + e^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.922 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$2)+9*y(x)=18*exp(3*x),y(0) = -1, D(y)(0) = 6],y(x), singsol=all)
```

$$y(x) = -2\cos(3x) + \sin(3x) + e^{3x}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 21

```
DSolve[{y''[x]+9*y[x]==18*Exp[3*x],{y[0]==-1,y'[0]==6}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{3x} + \sin(3x) - 2\cos(3x)$$

14.5 problem 11

Internal problem ID [12803]
Internal file name [`OUTPUT/11455_Saturday_November_04_2023_08_47_24_AM_78086821/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.3, page 255
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' - 2y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 3]$$

14.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= -2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - y' - 2y = 0$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - sY(s) + y(0) - 2Y(s) = 0 \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 3\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 3 - sY(s) - 2Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{3}{s^2 - s - 2}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{1}{s+1} + \frac{1}{s-2}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}\mathcal{L}^{-1}\left(-\frac{1}{s+1}\right) &= -\mathrm{e}^{-x} \\ \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) &= \mathrm{e}^{2x}\end{aligned}$$

Adding the above results and simplifying gives

$$y = -e^{-x} + e^{2x}$$

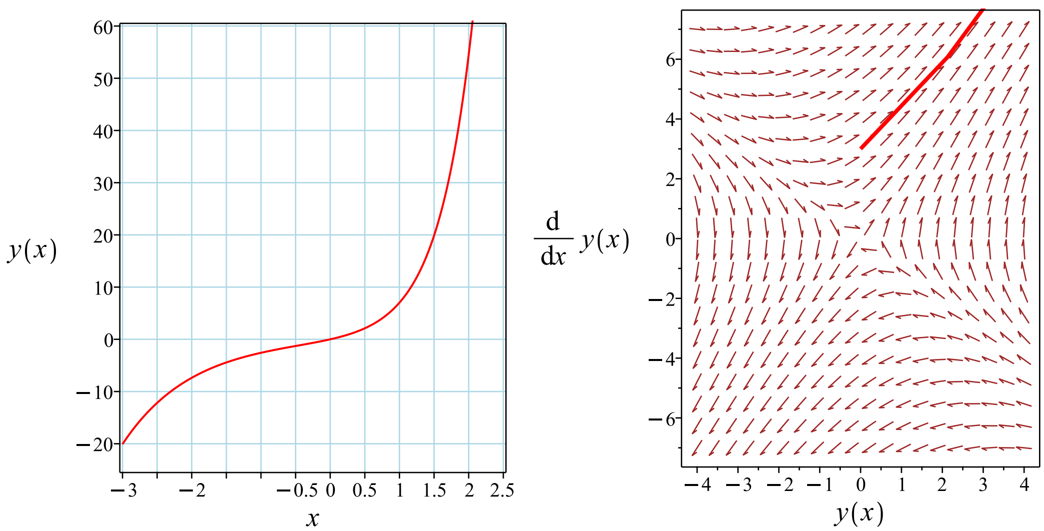
Simplifying the solution gives

$$y = -e^{-x} + e^{2x}$$

Summary

The solution(s) found are the following

$$y = -e^{-x} + e^{2x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -e^{-x} + e^{2x}$$

Verified OK.

14.5.2 Maple step by step solution

Let's solve

$$\left[y''-y'-2y=0, y(0)=0, y'\Big|_{\{x=0\}}=3\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of ODE

  $r^2-r-2=0$
- Factor the characteristic polynomial

  $(r+1)(r-2)=0$
- Roots of the characteristic polynomial

  $r=(-1,2)$
- 1st solution of the ODE

  $y_1(x)=\mathrm{e}^{-x}$
- 2nd solution of the ODE

  $y_2(x)=\mathrm{e}^{2x}$
- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)$
- Substitute in solutions

  $y=c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{2x}$

□ Check validity of solution $y=c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{2x}$

- Use initial condition $y(0)=0$

  $0=c_1+c_2$
- Compute derivative of the solution

  $y'=-c_1\mathrm{e}^{-x}+2c_2\mathrm{e}^{2x}$
- Use the initial condition $y'\Big|_{\{x=0\}}=3$

  $3=-c_1+2c_2$
- Solve for c_1 and c_2

  $\{c_1=-1, c_2=1\}$

- Substitute constant values into general solution and simplify

  $y = -e^{-x} + e^{2x}$

- Solution to the IVP

  $y = -e^{-x} + e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 5.203 (sec). Leaf size: 15

```
dsolve([diff(y(x),x$2)-diff(y(x),x)-2*y(x)=0,y(0) = 0, D(y)(0) = 3],y(x), singsol=all)
```

$$y(x) = -e^{-x} + e^{2x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 18

```
DSolve[{y''[x]-y'[x]-2*y[x]==0,{y[0]==0,y'[0]==3}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(e^{3x} - 1)$$

14.6 problem 12

Internal problem ID [12804]
Internal file name [`OUTPUT/11456_Saturday_November_04_2023_08_47_24_AM_94980606/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.3, page 255
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y' - 2y = x^2}$$

With initial conditions

$$\left[y(0) = \frac{11}{4}, y'(0) = \frac{1}{2}\right]$$

14.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= -2 \\ F &= x^2 \end{aligned}$$

Hence the ode is

$$y'' - y' - 2y = x^2$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - sY(s) + y(0) - 2Y(s) = \frac{2}{s^3} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= \frac{11}{4}\\ y'(0) &= \frac{1}{2}\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + \frac{9}{4} - \frac{11s}{4} - sY(s) - 2Y(s) = \frac{2}{s^3}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{11s^4 - 9s^3 + 8}{4s^3\,(s^2 - s - 2)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{3}{4s} - \frac{1}{s^3} + \frac{7}{3\,(s+1)} + \frac{7}{6\,(s-2)} + \frac{1}{2s^2}$$

The inverse Laplace of each term above is now found, which gives

$$\begin{aligned}
\mathcal{L}^{-1}\left(-\frac{3}{4s}\right) &= -\frac{3}{4} \\
\mathcal{L}^{-1}\left(-\frac{1}{s^3}\right) &= -\frac{x^2}{2} \\
\mathcal{L}^{-1}\left(\frac{7}{3\,(s+1)}\right) &= \frac{7\,\mathrm{e}^{-x}}{3} \\
\mathcal{L}^{-1}\left(\frac{7}{6\,(s-2)}\right) &= \frac{7\,\mathrm{e}^{2x}}{6} \\
\mathcal{L}^{-1}\left(\frac{1}{2s^2}\right) &= \frac{x}{2}
\end{aligned}$$

Adding the above results and simplifying gives

$$y = \frac{7\,\mathrm{e}^{2x}}{6} - \frac{x^2}{2} + \frac{7\,\mathrm{e}^{-x}}{3} + \frac{x}{2} - \frac{3}{4}$$

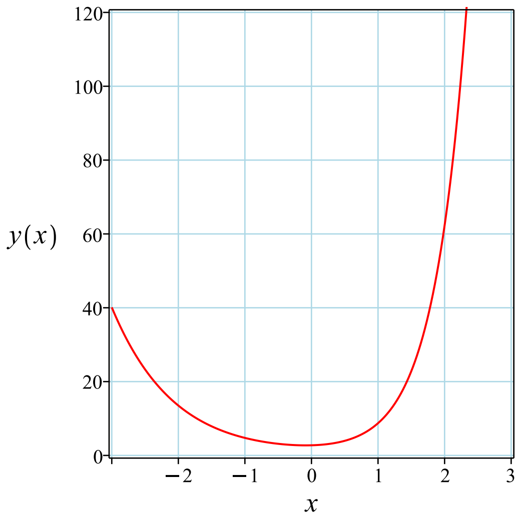
Simplifying the solution gives

$$y = \frac{7\,\mathrm{e}^{2x}}{6} - \frac{x^2}{2} + \frac{7\,\mathrm{e}^{-x}}{3} + \frac{x}{2} - \frac{3}{4}$$

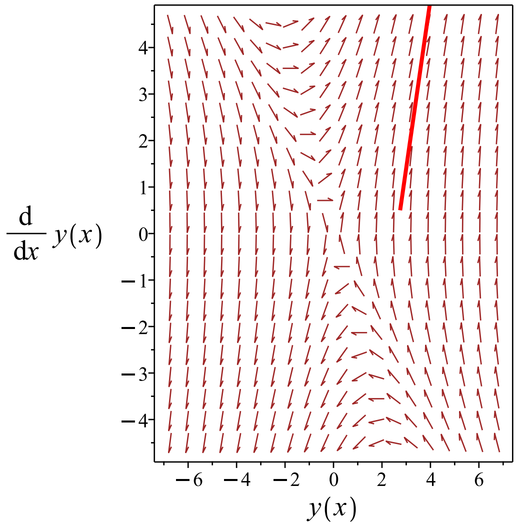
Summary

The solution(s) found are the following

$$y = \frac{7\,\mathrm{e}^{2x}}{6} - \frac{x^2}{2} + \frac{7\,\mathrm{e}^{-x}}{3} + \frac{x}{2} - \frac{3}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{7\,\mathrm{e}^{2x}}{6} - \frac{x^2}{2} + \frac{7\,\mathrm{e}^{-x}}{3} + \frac{x}{2} - \frac{3}{4}$$

Verified OK.

14.6.2 Maple step by step solution

Let's solve

$\left[y'' - y' - 2y = x^2, y(0) = \frac{11}{4}, y'\Big|_{\{x=0\}} = \frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 - r - 2 = 0$
- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 2)$
- 1st solution of the homogeneous ODE

$y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = x^2\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2\,e^{2x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,e^{x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{e^{-x}\left(\int x^2 e^{x} dx\right)}{3} + \frac{e^{2x}\left(\int x^2 e^{-2x} dx\right)}{3}$

- Compute integrals

  $y_p(x) = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 e^{2x} - \frac{x^2}{2} + \frac{x}{2} - \frac{3}{4}$

□ Check validity of solution $y = c_1 e^{-x} + c_2 e^{2x} - \frac{x^2}{2} + \frac{x}{2} - \frac{3}{4}$

- Use initial condition $y(0) = \frac{11}{4}$

  $\frac{11}{4} = c_1 + c_2 - \frac{3}{4}$

- Compute derivative of the solution

  $y' = -c_1 e^{-x} + 2c_2 e^{2x} - x + \frac{1}{2}$

- Use the initial condition $y'\Big|_{\{x=0\}} = \frac{1}{2}$

  $\frac{1}{2} = -c_1 + 2c_2 + \frac{1}{2}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{7}{3}, c_2 = \frac{7}{6}\right\}$
- Substitute constant values into general solution and simplify

  $y = \frac{7\,e^{2x}}{6} - \frac{x^2}{2} + \frac{7\,e^{-x}}{3} + \frac{x}{2} - \frac{3}{4}$

- Solution to the IVP

  $y = \frac{7\,e^{2x}}{6} - \frac{x^2}{2} + \frac{7\,e^{-x}}{3} + \frac{x}{2} - \frac{3}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.016 (sec). Leaf size: 26

```
dsolve([diff(y(x),x$2)-diff(y(x),x)-2*y(x)=x^2,y(0) = 11/4, D(y)(0) = 1/2],y(x), singsol=all
```

$$y(x) = \frac{7\,e^{-x}}{3} + \frac{x}{2} - \frac{x^2}{2} + \frac{7\,e^{2x}}{6} - \frac{3}{4}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 33

```
DSolve[{y''[x]-y'[x]-2*y[x]==x^2,{y[0]==11/4,y'[0]==1/2}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{12}\left(-6x^2 + 6x + 28e^{-x} + 14e^{2x} - 9\right)$$

14.7 problem 13

Internal problem ID [12805]
Internal file name [`OUTPUT/11457_Saturday_November_04_2023_08_47_24_AM_52105283/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.3, page 255
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = 2\sin(x)}$$

With initial conditions

$$[y(0) = -2, y'(0) = 0]$$

14.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 1 \\ F &= 2\sin(x) \end{aligned}$$

Hence the ode is

$$y'' - 2y' + y = 2\sin(x)$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 2\sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) + Y(s) = \frac{2}{s^2+1} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= -2 \\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 4 + 2s - 2sY(s) + Y(s) = \frac{2}{s^2+1}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{2(s^3 - 2s^2 + s - 3)}{(s^2+1)(s^2 - 2s + 1)}$$

Applying partial fractions decomposition results in

$$Y(s) = -\frac{3}{s-1} + \frac{3}{(s-1)^2} + \frac{1}{2s-2i} + \frac{1}{2s+2i}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(-\frac{3}{s-1}\right) = -3\,\mathrm{e}^{x}$$

$$\mathcal{L}^{-1}\left(\frac{3}{(s-1)^{2}}\right) = 3x\,\mathrm{e}^{x}$$

$$\mathcal{L}^{-1}\left(\frac{1}{2s-2i}\right) = \frac{\mathrm{e}^{ix}}{2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{2s+2i}\right) = \frac{\mathrm{e}^{-ix}}{2}$$

Adding the above results and simplifying gives

$$y = \cos(x) + 3\,\mathrm{e}^{x}(x-1)$$

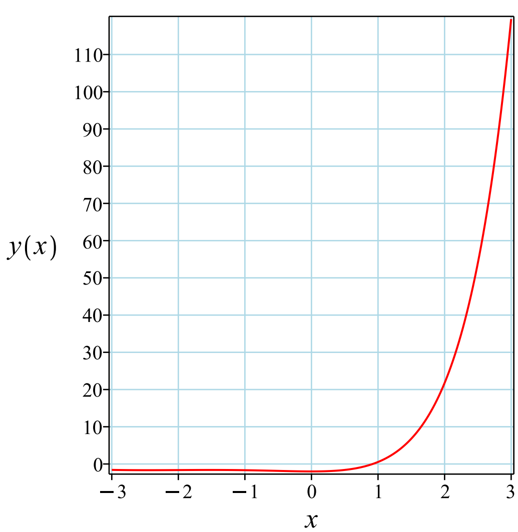
Simplifying the solution gives

$$y = (-3+3x)\,\mathrm{e}^{x} + \cos(x)$$

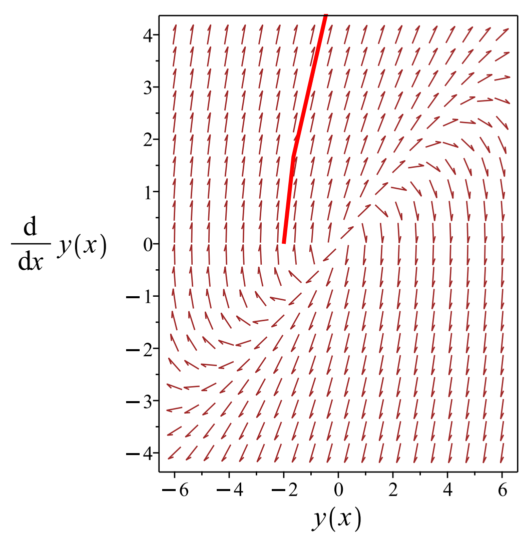
Summary

The solution(s) found are the following

$$y = (-3+3x)\,\mathrm{e}^{x} + \cos(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (-3+3x)\,\mathrm{e}^{x} + \cos(x)$$

Verified OK.

14.7.2 Maple step by step solution

Let's solve

$\left[y''-2y'+y=2\sin\left(x\right), y(0)=-2, y'\Big|_{\{x=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2-2r+1=0$

- Factor the characteristic polynomial

  $(r-1)^2=0$

- Root of the characteristic polynomial

  $r=1$

- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x)=x\,\mathrm{e}^x$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=\mathrm{e}^x c_1+c_2x\,\mathrm{e}^x+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=2\sin\left(x\right)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x))=\begin{bmatrix}\mathrm{e}^x & x\,\mathrm{e}^x\\ \mathrm{e}^x & x\,\mathrm{e}^x+\mathrm{e}^x\end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x))=\mathrm{e}^{2x}$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = 2\,\mathrm{e}^{x}\left(-\left(\int x\,\mathrm{e}^{-x}\sin(x)\,dx\right) + x\left(\int \mathrm{e}^{-x}\sin(x)\,dx\right)\right)$$

- Compute integrals

  $$y_p(x) = \cos(x)$$

- Substitute particular solution into general solution to ODE

  $$y = \mathrm{e}^{x}c_1 + c_2 x\,\mathrm{e}^{x} + \cos(x)$$

□ Check validity of solution $y = \mathrm{e}^{x}c_1 + c_2 x\mathrm{e}^{x} + \cos(x)$

- Use initial condition $y(0) = -2$

  $$-2 = 1 + c_1$$

- Compute derivative of the solution

  $$y' = \mathrm{e}^{x}c_1 + c_2\mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x} - \sin(x)$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $$0 = c_1 + c_2$$

- Solve for c_1 and c_2

  $$\{c_1 = -3, c_2 = 3\}$$

- Substitute constant values into general solution and simplify

  $$y = (-3 + 3x)\,\mathrm{e}^{x} + \cos(x)$$

- Solution to the IVP

  $$y = (-3 + 3x)\,\mathrm{e}^{x} + \cos(x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.109 (sec). Leaf size: 14

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+y(x)=2*sin(x),y(0) = -2, D(y)(0) = 0],y(x), singsol=al
```

$$y(x) = (3x - 3)\,\mathrm{e}^{x} + \cos(x)$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 16

```
DSolve[{y''[x]-2*y'[x]+y[x]==2*Sin[x],{y[0]==-2,y'[0]==0}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to 3e^{x}(x - 1) + \cos(x)$$

14.8 problem 14

Internal problem ID [12806]
Internal file name [`OUTPUT/11458_Saturday_November_04_2023_08_47_24_AM_22492681/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.3, page 255
Problem number: 14.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_laplace"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - y'' + 4y' - 4y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 5, y''(0) = 5]$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \\ \mathcal{L}(y''') &= s^3Y(s) - y''(0) - sy'(0) - s^2y(0)\end{aligned}$$

The given ode becomes an algebraic equation in the Laplace domain

$$s^3Y(s) - y''(0) - sy'(0) - s^2y(0) - s^2Y(s) + y'(0) + sy(0) + 4sY(s) - 4y(0) - 4Y(s) = 0 \quad (1)$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 5 \\ y''(0) &= 5\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^3Y(s) - 5s - s^2Y(s) + 4sY(s) - 4Y(s) = 0$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{5s}{s^3 - s^2 + 4s - 4}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{-\frac{1}{2} - i}{s - 2i} + \frac{-\frac{1}{2} + i}{s + 2i} + \frac{1}{s - 1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{2} - i}{s - 2i}\right) = \left(-\frac{1}{2} - i\right) e^{2ix}$$

$$\mathcal{L}^{-1}\left(\frac{-\frac{1}{2} + i}{s + 2i}\right) = \left(-\frac{1}{2} + i\right) e^{-2ix}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s - 1}\right) = e^{x}$$

Adding the above results and simplifying gives

$$y = e^{x} - \cos(2x) + 2\sin(2x)$$

Summary

The solution(s) found are the following

$$y = e^{x} - \cos(2x) + 2\sin(2x) \tag{1}$$

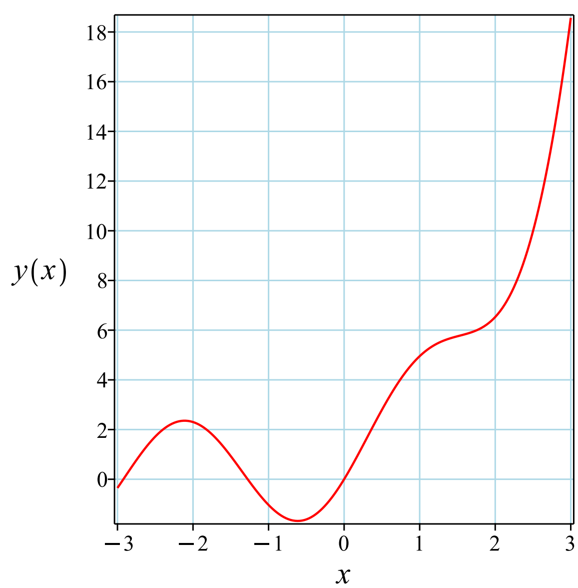


Figure 411: Solution plot

Verification of solutions

$$y = e^{x} - \cos(2x) + 2\sin(2x)$$

Verified OK.

14.8.1 Maple step by step solution

Let's solve

$$\left[y''' - y'' + 4y' - 4y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 5, y''\Big|_{\{x=0\}} = 5\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = y_3(x) - 4y_2(x) + 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = y_3(x) - 4y_2(x) + 4y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -4 & 1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -4 & 1 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-2\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(2x) - \mathrm{I}\sin(2x)) \cdot \left[\begin{array}{c} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\frac{\cos(2x)}{4} + \frac{\mathrm{I}\sin(2x)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \left[\begin{array}{c} -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{array}\right], \vec{y}_3(x) = \left[\begin{array}{c} \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = \mathrm{e}^x c_1 \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] + \left[\begin{array}{c} -\frac{c_2\cos(2x)}{4} + \frac{c_3\sin(2x)}{4} \\ \frac{c_2\sin(2x)}{2} + \frac{c_3\cos(2x)}{2} \\ c_2\cos(2x) - c_3\sin(2x) \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \mathrm{e}^x c_1 + \frac{c_3\sin(2x)}{4} - \frac{c_2\cos(2x)}{4}$$

- Use the initial condition $y(0) = 0$

$$0 = c_1 - \frac{c_2}{4}$$

- Calculate the 1st derivative of the solution

$$y' = \mathrm{e}^x c_1 + \frac{c_3\cos(2x)}{2} + \frac{c_2\sin(2x)}{2}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 5$

$$5 = c_1 + \frac{c_3}{2}$$

- Calculate the 2nd derivative of the solution

$$y'' = \mathrm{e}^x c_1 - c_3\sin(2x) + c_2\cos(2x)$$

- Use the initial condition $y''\Big|_{\{x=0\}} = 5$

  $5 = c_1 + c_2$
- Solve for the unknown coefficients

  $\{c_1 = 1, c_2 = 4, c_3 = 8\}$
- Solution to the IVP

  $y = e^x - \cos(2x) + 2\sin(2x)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 5.813 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$3)-diff(y(x),x$2)+4*diff(y(x),x)-4*y(x)=0,y(0) = 0, D(y)(0) = 5, (D@@2)(
```

$$y(x) = e^x - \cos(2x) + 2\sin(2x)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 21

```
DSolve[{y'''[x]-y''[x]+4*y'[x]-4*y[x]==0,{y[0]==0,y'[0]==5,y''[0]==5}},y[x],x,IncludeSingula
```

$$y(x) \to e^x + 2\sin(2x) - \cos(2x)$$

15 Chapter 5. The Laplace Transform Method. Exercises 5.4, page 265

15.1 problem 4 (a)

Internal problem ID [12807]
Internal file name [`OUTPUT/11459_Saturday_November_04_2023_08_47_25_AM_93984434/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.4, page 265
Problem number: 4 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$
\boxed{y' + 2y = \begin{cases} 2 & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}}
$$

With initial conditions

$$
[y(0) = 1]
$$

15.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 2 \\
q(x) &= \begin{cases} 0 & x < 0 \\ 2 & x < 1 \\ 1 & 1 \le x \end{cases}
\end{aligned}
$$

Hence the ode is

$$y' + 2y = \begin{cases} 0 & x < 0 \\ 2 & x < 1 \\ 1 & 1 \leq x \end{cases}$$

The domain of $p(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \begin{cases} 0 & x < 0 \\ 2 & x < 1 \\ 1 & 1 \leq x \end{cases}$ is

$$\{0 \leq x \leq 1, 1 \leq x \leq \infty, -\infty \leq x \leq 0\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

15.1.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + 2Y(s) = \frac{2 - \mathrm{e}^{-s}}{s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 1 + 2Y(s) = \frac{2 - \mathrm{e}^{-s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = -\frac{-2 + \mathrm{e}^{-s} - s}{s(s+2)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{-2+e^{-s}-s}{s(s+2)}\right) \\
&= 1-\frac{\text{Heaviside}(x-1)\left(1-e^{-2x+2}\right)}{2}
\end{aligned}
$$

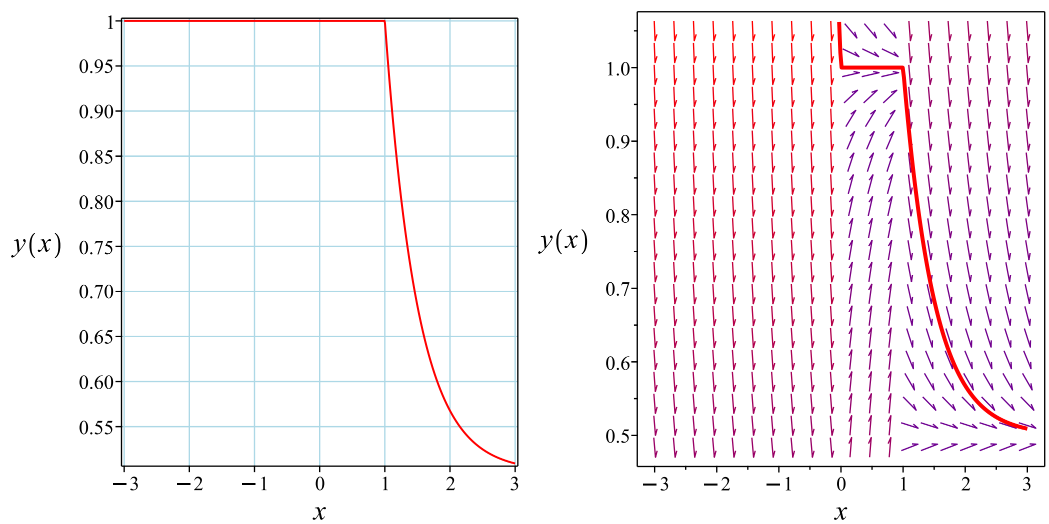
Hence the final solution is

$$
y = 1-\frac{\text{Heaviside}(x-1)\left(1-e^{-2x+2}\right)}{2}
$$

Summary

The solution(s) found are the following

$$
y = 1-\frac{\text{Heaviside}(x-1)\left(1-e^{-2x+2}\right)}{2} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = 1-\frac{\text{Heaviside}(x-1)\left(1-e^{-2x+2}\right)}{2}
$$

Verified OK.

15.1.3 Maple step by step solution

Let's solve

$$\left[y' + 2y = \left\{\begin{array}{cc} 2 & 0 \le x < 1 \\ 1 & 1 \le x \end{array}\right., y(0) = 1\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $$y' = -2y + \left\{\begin{array}{cc} 2 & 0 \le x < 1 \\ 1 & 1 \le x \end{array}\right.$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $$y' + 2y = \left\{\begin{array}{cc} 2 & 0 \le x < 1 \\ 1 & 1 \le x \end{array}\right.$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $$\mu(x)\left(y' + 2y\right) = \mu(x)\left(\left\{\begin{array}{cc} 2 & 0 \le x < 1 \\ 1 & 1 \le x \end{array}\right.\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + 2y\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = 2\mu(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{2x}$

- Integrate both sides with respect to x

  $$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\left(\left\{\begin{array}{cc} 2 & 0 \le x < 1 \\ 1 & 1 \le x \end{array}\right.\right) dx + c_1$$

- Evaluate the integral on the lhs

  $$\mu(x)\, y = \int \mu(x)\left(\left\{\begin{array}{cc} 2 & 0 \le x < 1 \\ 1 & 1 \le x \end{array}\right.\right) dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x) \left(\begin{cases} 2 & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}\right) dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = e^{2x}$

$$y = \frac{\int e^{2x} \left(\begin{cases} 2 & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}\right) dx + c_1}{e^{2x}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\begin{cases} 0 & x \leq 0 \\ e^{2x} - 1 & x \leq 1 \\ \frac{e^{2x}}{2} + \frac{e^2}{2} - 1 & 1 < x \end{cases} + c_1}{e^{2x}}$$

- Simplify

$$y = e^{-2x} \left(\begin{cases} 0 & x \leq 0 \\ e^{2x} - 1 & x \leq 1 \\ \frac{e^{2x}}{2} + \frac{e^2}{2} - 1 & 1 < x \end{cases} + c_1\right)$$

- Use initial condition $y(0) = 1$

$$1 = c_1$$

- Solve for c_1

$$c_1 = 1$$

- Substitute $c_1 = 1$ into general solution and simplify

$$y = \begin{cases} e^{-2x} & x \leq 0 \\ 1 & x \leq 1 \\ \frac{1}{2} + \frac{e^{-2x+2}}{2} & 1 < x \end{cases}$$

- Solution to the IVP

$$y = \begin{cases} e^{-2x} & x \leq 0 \\ 1 & x \leq 1 \\ \frac{1}{2} + \frac{e^{-2x+2}}{2} & 1 < x \end{cases}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 7.625 (sec). Leaf size: 22

```
dsolve([diff(y(x),x)+2*y(x)=piecewise(0<=x and x<1,2,1<=x,1),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \begin{cases} 1 & x < 1 \\ \frac{1}{2} + \frac{e^{2-2x}}{2} & 1 \leq x \end{cases}$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 37

```
DSolve[{y'[x]+2*y[x]==Piecewise[{ {2,0<=x<1},{1,1<=x}}],{y[0]==1}},y[x],x,IncludeSingularSol
```

$$y(x) \to \begin{cases} e^{-2x} & x \leq 0 \\ 1 & 0 < x \leq 1 \\ \frac{1}{2}(1 + e^{2-2x}) & \text{True} \end{cases}$$

15.2 problem 4 (b)

Internal problem ID [12808]
Internal file name [`OUTPUT/11460_Saturday_November_04_2023_08_47_25_AM_68786493/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.4, page 265
Problem number: 4 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' - y' - 2y = \begin{cases} 1 & 2 \leq x < 4 \\ 0 & \text{otherwise} \end{cases}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

15.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= -2 \\ F &= \begin{cases} 0 & x < 2 \\ 1 & x < 4 \\ 0 & 4 \leq x \end{cases} \end{aligned}$$

Hence the ode is

$$y'' - y' - 2y = \begin{cases} 0 & x < 2 \\ 1 & x < 4 \\ 0 & 4 \leq x \end{cases}$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & x < 2 \\ 1 & x < 4 \\ 0 & 4 \leq x \end{cases}$ is

$$\{2 \leq x \leq 4, 4 \leq x \leq \infty, -\infty \leq x \leq 2\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - sY(s) + y(0) - 2Y(s) = \frac{e^{-2s} - e^{-4s}}{s} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - sY(s) - 2Y(s) = \frac{\mathrm{e}^{-2s} - \mathrm{e}^{-4s}}{s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-2s} - \mathrm{e}^{-4s} + s}{s\,(s^2 - s - 2)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-2s} - \mathrm{e}^{-4s} + s}{s\,(s^2 - s - 2)}\right) \\
&= -\frac{\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}}{3} + \frac{\mathrm{e}^{2x-8}(-1 + \mathrm{Heaviside}\,(4 - x))}{6} + \frac{(1 - \mathrm{Heaviside}\,(-x + 2))\,\mathrm{e}^{2x-4}}{6} - \frac{\mathrm{Heaviside}\,(x - 2)\,(}{6}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y = &-\frac{\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}}{3} + \frac{\mathrm{e}^{2x-8}(-1 + \mathrm{Heaviside}\,(4 - x))}{6} + \frac{(1 - \mathrm{Heaviside}\,(-x + 2))\,\mathrm{e}^{2x-4}}{6} \\
&- \frac{\mathrm{Heaviside}\,(x - 2)\,(3 - 2\,\mathrm{e}^{-x+2})}{6} + \frac{\mathrm{Heaviside}\,(-4 + x)\,(3 - 2\,\mathrm{e}^{4-x})}{6}
\end{aligned}$$

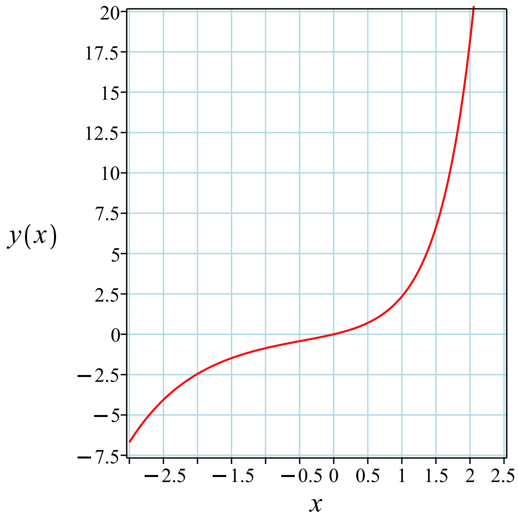
Simplifying the solution gives

$$\begin{aligned}
y = &-\frac{\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}}{3} - \frac{\mathrm{e}^{2x-8}\,\mathrm{Heaviside}\,(-4 + x)}{6} + \frac{\mathrm{e}^{2x-4}\,\mathrm{Heaviside}\,(x - 2)}{6} \\
&+ \frac{\mathrm{Heaviside}\,(x - 2)\,\mathrm{e}^{-x+2}}{3} - \frac{\mathrm{Heaviside}\,(x - 2)}{2} \\
&- \frac{\mathrm{Heaviside}\,(-4 + x)\,\mathrm{e}^{4-x}}{3} + \frac{\mathrm{Heaviside}\,(-4 + x)}{2}
\end{aligned}$$

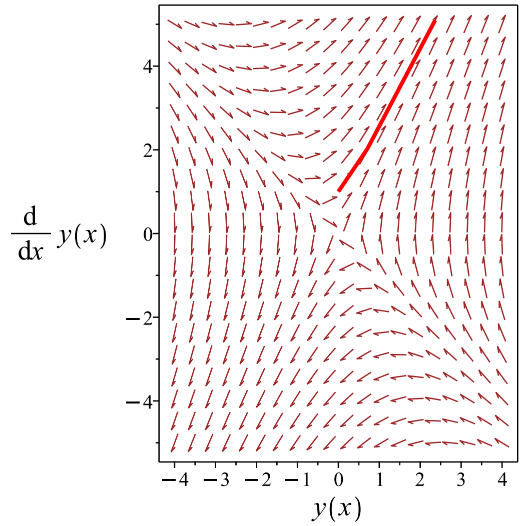
Summary

The solution(s) found are the following

$$\begin{aligned}
y = &-\frac{\mathrm{e}^{-x}}{3} + \frac{\mathrm{e}^{2x}}{3} - \frac{\mathrm{e}^{2x-8}\,\mathrm{Heaviside}\,(-4 + x)}{6} + \frac{\mathrm{e}^{2x-4}\,\mathrm{Heaviside}\,(x - 2)}{6} \\
&+ \frac{\mathrm{Heaviside}\,(x - 2)\,\mathrm{e}^{-x+2}}{3} - \frac{\mathrm{Heaviside}\,(x - 2)}{2} \\
&- \frac{\mathrm{Heaviside}\,(-4 + x)\,\mathrm{e}^{4-x}}{3} + \frac{\mathrm{Heaviside}\,(-4 + x)}{2}
\end{aligned} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-x}}{3} + \frac{e^{2x}}{3} - \frac{e^{2x-8}\,\text{Heaviside}\,(-4+x)}{6} + \frac{e^{2x-4}\,\text{Heaviside}\,(x-2)}{6} + \frac{\text{Heaviside}\,(x-2)\,e^{-x+2}}{3} - \frac{\text{Heaviside}\,(x-2)}{2} - \frac{\text{Heaviside}\,(-4+x)\,e^{4-x}}{3} + \frac{\text{Heaviside}\,(-4+x)}{2}$$

Verified OK.

15.2.2 Maple step by step solution

Let's solve

$$\left[y''-y'-2y = \begin{cases} 0 & x<2 \\ 1 & x<4 \\ 0 & 4\leq x \end{cases}, y(0)=0, y'\Big|_{\{x=0\}}=1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2-r-2=0$

- Factor the characteristic polynomial

$(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \left\{\begin{array}{cc} 0 & x < 2 \\ 1 & x < 4 \\ 0 & 4 \leq x \end{array}\right.\right]$$

- Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-x} & \mathrm{e}^{2x} \\ -\mathrm{e}^{-x} & 2\,\mathrm{e}^{2x} \end{array}\right]$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{x}$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\mathrm{e}^{-x}\left(\int \left(\left\{\begin{array}{cc} 0 & x < 2 \\ \frac{\mathrm{e}^{x}}{3} & x < 4 \\ 0 & 4 \leq x \end{array}\right.\right) dx\right) + \mathrm{e}^{2x}\left(\int \left(\left\{\begin{array}{cc} 0 & x < 2 \\ \frac{\mathrm{e}^{-2x}}{3} & x < 4 \\ 0 & 4 \leq x \end{array}\right.\right) dx\right)$$

- Compute integrals

$$y_p(x) = \frac{\left(\begin{cases} 0 & x \le 2 \\ e^{2x-4} - 3 + 2\,e^{-x+2} & x \le 4 \\ -e^{2x-8} + e^{2x-4} + 2\,e^{-x+2} - 2\,e^{4-x} & 4 < x \end{cases}\right)}{6}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-x} + c_2 e^{2x} + \frac{\left(\begin{cases} 0 & x \le 2 \\ e^{2x-4} - 3 + 2\,e^{-x+2} & x \le 4 \\ -e^{2x-8} + e^{2x-4} + 2\,e^{-x+2} - 2\,e^{4-x} & 4 < x \end{cases}\right)}{6}$$

□ Check validity of solution $y = c_1 e^{-x} + c_2 e^{2x} + \frac{\left(\begin{cases} 0 & x \le 2 \\ e^{2x-4} - 3 + 2e^{-x+2} & x \le 4 \\ -e^{2x-8} + e^{2x-4} + 2e^{-x+2} - 2e^{4-x} & 4 < x \end{cases}\right)}{6}$

- ○ Use initial condition $y(0) = 0$

  $0 = c_1 + c_2$

- ○ Compute derivative of the solution

$$y' = -c_1 e^{-x} + 2c_2 e^{2x} + \frac{\left(\begin{cases} 0 & x \le 2 \\ 2\,e^{2x-4} - 2\,e^{-x+2} & x \le 4 \\ -2\,e^{2x-8} + 2\,e^{2x-4} - 2\,e^{-x+2} + 2\,e^{4-x} & 4 < x \end{cases}\right)}{6}$$

- ○ Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = -c_1 + 2c_2$

- ○ Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{3}, c_2 = \frac{1}{3}\right\}$

- ○ Substitute constant values into general solution and simplify

$$y = -\frac{e^{-x}}{3} + \frac{e^{2x}}{3} - \frac{\left(\begin{cases} 0 & x \le 2 \\ -e^{2x-4} + 3 - 2\,e^{-x+2} & x \le 4 \\ e^{2x-8} - e^{2x-4} - 2\,e^{-x+2} + 2\,e^{4-x} & 4 < x \end{cases}\right)}{6}$$

- Solution to the IVP

$$y = -\frac{e^{-x}}{3} + \frac{e^{2x}}{3} - \frac{\left(\begin{cases} 0 & x \leq 2 \\ -e^{2x-4} + 3 - 2\,e^{-x+2} & x \leq 4 \\ e^{2x-8} - e^{2x-4} - 2\,e^{-x+2} + 2\,e^{4-x} & 4 < x \end{cases}\right)}{6}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 8.969 (sec). Leaf size: 136

```
dsolve([diff(y(x),x$2)-diff(y(x),x)-2*y(x)=piecewise(2<=x and x<4,1,true,0),y(0) = 0, D(y)(0
```

$$y(x) = \frac{\left(\begin{cases} -e^{-x} + e^{2x} & x < 2 \\ -\frac{1}{2} - e^{-2} + e^{4} & x = 2 \\ -e^{-x} + e^{2x} - \frac{3}{2} + e^{2-x} + \frac{e^{2x-4}}{2} & x < 4 \\ \frac{(2\,e^{12} + e^{8} - 2\,e^{4} + 2\,e^{2} - 2)\,e^{-4}}{2} & x = 4 \\ -e^{-x} + e^{2x} - e^{4-x} + e^{2-x} - \frac{e^{2x-8}}{2} + \frac{e^{2x-4}}{2} & 4 < x \end{cases}\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.068 (sec). Leaf size: 127

```
DSolve[{y''[x]-y'[x]-2*y[x]==Piecewise[{ {1,2<=x<4},{0,True}}],{y[0]==0,y'[0]==1}},y[x],x,In
```

$$y(x) \to \begin{cases} \frac{1}{3}e^{-x}(-1+e^{3x}) & x \leq 2 \\ \frac{1}{6}e^{-x-4}(-2e^4+2e^6+e^{3x}-3e^{x+4}+2e^{3x+4}) & 2 < x \leq 4 \\ \frac{1}{6}e^{-x-8}(-2e^8+2e^{10}-2e^{12}-e^{3x}+e^{3x+4}+2e^{3x+8}) & \text{True} \end{cases}$$

15.3 problem 4 (c)

Internal problem ID [12809]
Internal file name [`OUTPUT/11461_Saturday_November_04_2023_08_47_25_AM_41313843/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.4, page 265
Problem number: 4 (c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "exact linear second order ode", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - 2y' = \begin{cases} 0 & 0 \le x < 1 \\ (x-1)^2 & 1 \le x \end{cases}}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

15.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 0 \\ F &= \begin{cases} 0 & x < 1 \\ (x-1)^2 & 1 \le x \end{cases} \end{aligned}$$

Hence the ode is

$$y'' - 2y' = \begin{cases} 0 & x < 1 \\ (x-1)^2 & 1 \leq x \end{cases}$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $F = \begin{cases} 0 & x < 1 \\ (x-1)^2 & 1 \leq x \end{cases}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2 Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2 Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) = \frac{2\,\mathrm{e}^{-s}}{s^3} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2 Y(s) + 2 - s - 2sY(s) = \frac{2\,\mathrm{e}^{-s}}{s^3}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^4 - 2s^3 + 2\,\mathrm{e}^{-s}}{s^4 (s-2)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{s^4 - 2s^3 + 2\,\mathrm{e}^{-s}}{s^4\,(s-2)}\right) \\
&= 1 + \frac{(1 - \mathrm{Heaviside}\,(1-x))\,\mathrm{e}^{2x-2}}{8} - \frac{\mathrm{Heaviside}\,(x-1)\,(4x^3 - 6x^2 + 6x - 1)}{24}
\end{aligned}
$$

Hence the final solution is

$$
y = 1 + \frac{(1 - \mathrm{Heaviside}\,(1-x))\,\mathrm{e}^{2x-2}}{8} - \frac{\mathrm{Heaviside}\,(x-1)\,(4x^3 - 6x^2 + 6x - 1)}{24}
$$

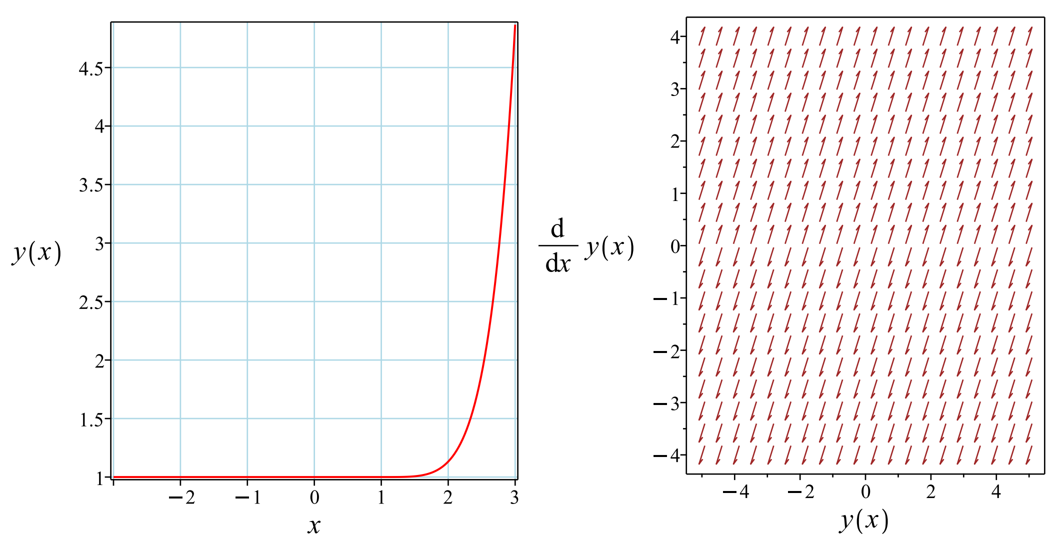
Simplifying the solution gives

$$
y = \frac{\mathrm{e}^{2x-2}\,\mathrm{Heaviside}\,(x-1)}{8} + 1 + \frac{(-4x^3 + 6x^2 - 6x + 1)\,\mathrm{Heaviside}\,(x-1)}{24}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\mathrm{e}^{2x-2}\,\mathrm{Heaviside}\,(x-1)}{8} + 1 + \frac{(-4x^3 + 6x^2 - 6x + 1)\,\mathrm{Heaviside}\,(x-1)}{24} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{e^{2x-2}\operatorname{Heaviside}(x-1)}{8} + 1 + \frac{(-4x^3+6x^2-6x+1)\operatorname{Heaviside}(x-1)}{24}$$

Verified OK.

15.3.2 Maple step by step solution

Let's solve

$$\left[y''-2y' = \begin{cases} 0 & x<1 \\ (x-1)^2 & 1\le x \end{cases}, y(0)=1, y'\Big|_{\{x=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2-2r=0$

- Factor the characteristic polynomial

  $r(r-2)=0$

- Roots of the characteristic polynomial

  $r=(0,2)$

- 1st solution of the homogeneous ODE

  $y_1(x)=1$

- 2nd solution of the homogeneous ODE

  $y_2(x)=e^{2x}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1+c_2e^{2x}+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x)=-y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=\begin{cases} 0 & x<1 \\ (x-1)^2 & 1\le x \end{cases}\right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2e^{2x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\left(\int \left(\left\{ \begin{array}{cc} 0 & x < 1 \\ \frac{(x-1)^2}{2} & 1 \leq x \end{array} \right.\right) dx\right) + e^{2x}\left(\int \left(\left\{ \begin{array}{cc} 0 & x < 1 \\ \frac{e^{-2x}(x-1)^2}{2} & 1 \leq x \end{array} \right.\right) dx\right)$$

- Compute integrals

$$y_p(x) = \left\{ \begin{array}{cc} 0 & x \leq 1 \\ \frac{e^{2x-2}}{8} - \frac{x^3}{6} + \frac{x^2}{4} - \frac{x}{4} + \frac{1}{24} & 1 < x \end{array} \right.$$

- Substitute particular solution into general solution to ODE

$$y = c_1 + c_2e^{2x} + \left\{ \begin{array}{cc} 0 & x \leq 1 \\ \frac{e^{2x-2}}{8} - \frac{x^3}{6} + \frac{x^2}{4} - \frac{x}{4} + \frac{1}{24} & 1 < x \end{array} \right.$$

☐ Check validity of solution $y = c_1 + c_2e^{2x} + \left\{ \begin{array}{cc} 0 & x \leq 1 \\ \frac{e^{2x-2}}{8} - \frac{x^3}{6} + \frac{x^2}{4} - \frac{x}{4} + \frac{1}{24} & 1 < x \end{array} \right.$

- Use initial condition $y(0) = 1$

$$1 = c_1 + c_2$$

- Compute derivative of the solution

$$y' = 2c_2e^{2x} + \left\{ \begin{array}{cc} 0 & x \leq 1 \\ \frac{e^{2x-2}}{4} - \frac{x^2}{2} + \frac{x}{2} - \frac{1}{4} & 1 < x \end{array} \right.$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

$$0 = 2c_2$$

- Solve for c_1 and c_2

$$\{c_1 = 1, c_2 = 0\}$$

- Substitute constant values into general solution and simplify

$$y = \begin{cases} 1 & x \leq 1 \\ \frac{25}{24}+\frac{e^{2x-2}}{8}-\frac{x^3}{6}+\frac{x^2}{4}-\frac{x}{4} & 1<x \end{cases}$$

- Solution to the IVP

$$y = \begin{cases} 1 & x \leq 1 \\ \frac{25}{24}+\frac{e^{2x-2}}{8}-\frac{x^3}{6}+\frac{x^2}{4}-\frac{x}{4} & 1<x \end{cases}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = Heaviside(_a-1)*_a^2-2*Heaviside(_a-1)*
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 9.25 (sec). Leaf size: 39

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)=piecewise(0<=x and x<1,0,1<=x,(x-1)^2),y(0) = 1, D(y)(
```

$$y(x) = \begin{cases} 1 & x<1 \\ \frac{7}{8} & x=1 \\ \frac{25}{24}+\frac{e^{2x-2}}{8}+\frac{x^2}{4}-\frac{x^3}{6}-\frac{x}{4} & 1<x \end{cases}$$

✓ Solution by Mathematica

Time used: 0.269 (sec). Leaf size: 40

```
DSolve[{y''[x]-2*y'[x]==Piecewise[{ {0,0<=x<1},{(x-1)^2,x>=1}}],{y[0]==1,y'[0]==0}},y[x],x,I
```

$$y(x) \to \begin{cases} 1 & x \leq 1 \\ \frac{1}{24}\left(-4x^3+6x^2-6x+3e^{2x-2}+25\right) & \text{True} \end{cases}$$

15.4 problem 4 (d)

Internal problem ID [12810]
Internal file name [`OUTPUT/11462_Saturday_November_04_2023_08_47_26_AM_49386920/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.4, page 265
Problem number: 4 (d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' - 2y' + y = \begin{cases} 0 & 0 \leq x < 1 \\ x^2 - 2x + 3 & 1 \leq x \end{cases}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

15.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = -2$$
$$q(x) = 1$$
$$F = \begin{cases} 0 & x < 1 \\ x^2 - 2x + 3 & 1 \leq x \end{cases}$$

Hence the ode is

$$y'' - 2y' + y = \begin{cases} 0 & x < 1 \\ x^2 - 2x + 3 & 1 \leq x \end{cases}$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & x < 1 \\ x^2 - 2x + 3 & 1 \leq x \end{cases}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) + Y(s) = 2\,\mathrm{e}^{-s}\left(\frac{1}{s} + \frac{1}{s^3}\right) \tag{1}$$

But the initial conditions are

$$y(0) = 0$$
$$y'(0) = 1$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - 2sY(s) + Y(s) = 2\,\mathrm{e}^{-s}\left(\frac{1}{s} + \frac{1}{s^3}\right)$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2\,\mathrm{e}^{-s} s^2 + s^3 + 2\,\mathrm{e}^{-s}}{s^3\,(s^2 - 2s + 1)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{2\,\mathrm{e}^{-s} s^2 + s^3 + 2\,\mathrm{e}^{-s}}{s^3\,(s^2 - 2s + 1)}\right) \\ &= x\,\mathrm{e}^{x} + 4(1 - \mathrm{Heaviside}\,(1 - x))\,\mathrm{e}^{x-1}(x - 3) + \left(x^2 + 2x + 5\right) \mathrm{Heaviside}\,(x - 1) \end{aligned}$$

Hence the final solution is

$$y = x\,\mathrm{e}^{x} + 4(1 - \mathrm{Heaviside}\,(1 - x))\,\mathrm{e}^{x-1}(x - 3) + \left(x^2 + 2x + 5\right) \mathrm{Heaviside}\,(x - 1)$$

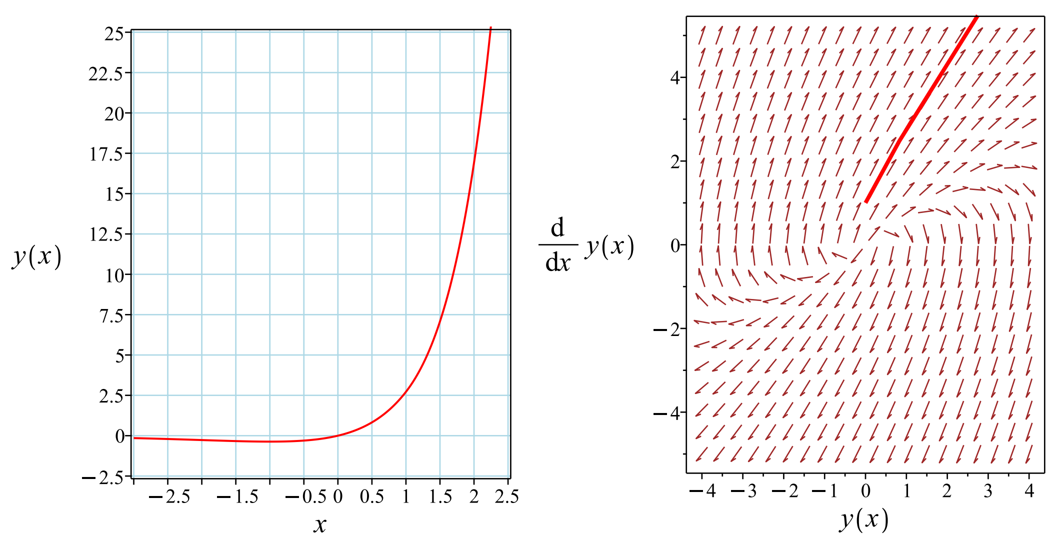
Simplifying the solution gives

$$y = \left(4\,\mathrm{e}^{x-1}(x - 3) + x^2 + 2x + 5\right) \mathrm{Heaviside}\,(x - 1) + x\,\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = \left(4\,\mathrm{e}^{x-1}(x - 3) + x^2 + 2x + 5\right) \mathrm{Heaviside}\,(x - 1) + x\,\mathrm{e}^{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \left(4\,\mathrm{e}^{x-1}(x-3) + x^2 + 2x + 5\right) \mathrm{Heaviside}\,(x-1) + x\,\mathrm{e}^{x}$$

Verified OK.

15.4.2 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + y = \left\{\begin{array}{cc} 0 & x < 1 \\ x^2 - 2x + 3 & 1 \leq x \end{array}\right., y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{x} c_1 + c_2 x\,\mathrm{e}^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \left\{\begin{array}{cc} 0 & x < 1 \\ x^2 - 2x + 3 & 1 \leq x \end{array}\right.\right.$$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & x e^x \\ e^x & x e^x + e^x \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{2x}$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = e^x \left(-\left(\int \left(\left(\begin{cases} 0 & x < 1 \\ x e^{-x}(x^2 - 2x + 3) & 1 \leq x \end{cases} \right) dx \right) + \left(\int \left(\left(\begin{cases} 0 & x < 1 \\ e^{-x}(x^2 - 2x + 3) & 1 \leq x \end{cases} \right.\right.\right.\right.$$

- Compute integrals

  $$y_p(x) = \begin{cases} 0 & x \leq 1 \\ 4 e^{x-1}(x-3) + x^2 + 2x + 5 & 1 < x \end{cases}$$

- Substitute particular solution into general solution to ODE

  $$y = e^x c_1 + c_2 x e^x + \begin{cases} 0 & x \leq 1 \\ 4 e^{x-1}(x-3) + x^2 + 2x + 5 & 1 < x \end{cases}$$

□ Check validity of solution $y = e^x c_1 + c_2 x e^x + \begin{cases} 0 & x \leq 1 \\ 4e^{x-1}(x-3) + x^2 + 2x + 5 & 1 < x \end{cases}$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $$y' = e^x c_1 + c_2 e^x + c_2 x e^x + \begin{cases} 0 & x \leq 1 \\ 4 e^{x-1}(x-3) + 4 e^{x-1} + 2x + 2 & 1 < x \end{cases}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 1\}$

- Substitute constant values into general solution and simplify

$$y = x\,e^{x} + \left\{\begin{array}{cc} 0 & x \le 1 \\ 4\,e^{x-1}(x-3) + x^{2} + 2x + 5 & 1 < x \end{array}\right.$$

- Solution to the IVP

$$y = x\,e^{x} + \left\{\begin{array}{cc} 0 & x \le 1 \\ 4\,e^{x-1}(x-3) + x^{2} + 2x + 5 & 1 < x \end{array}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 7.61 (sec). Leaf size: 43

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+y(x)=piecewise(0<=x and x<1,0,1<=x,x^2-2*x+3),y(0) = 0
```

$$y(x) = \left\{\begin{array}{cc} e^{x} x & x < 1 \\ e + 8 & x = 1 \\ e^{x} x + 5 + 4(-3+x)\,e^{-1+x} + x^{2} + 2x & 1 < x \end{array}\right.$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 39

```
DSolve[{y''[x]-2*y'[x]+y[x]==Piecewise[{ {0,0<=x<1},{x^2-2*x+3,x>=1}}],{y[0]==0,y'[0]==1}},y
```

$$y(x) \to \begin{cases} e^x x & x \leq 1 \\ x^2 + e^x x + 2x + 4e^{x-1}(x-3) + 5 & \text{True} \end{cases}$$

15.5 problem 4 (e)

Internal problem ID [12811]
Internal file name [`OUTPUT/11463_Saturday_November_04_2023_08_47_26_AM_12715766/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.4, page 265
Problem number: 4 (e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \left\{\begin{array}{cc} 0 & 0 \le x < \pi \\ -\sin(3x) & \pi \le x \end{array}\right.}$$

With initial conditions

$$[y(0) = 1, y'(0) = 1]$$

15.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 4 \\ F &= -\left(\left\{\begin{array}{cc} 0 & x < \pi \\ \sin(3x) & \pi \le x \end{array}\right.\right) \end{aligned}$$

Hence the ode is

$$y'' + 4y = -\left(\begin{cases} 0 & x < \pi \\ \sin(3x) & \pi \leq x \end{cases}\right)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = -\left(\begin{cases} 0 & x < \pi \\ \sin(3x) & \pi \leq x \end{cases}\right)$ is

$$\{x < \pi \vee \pi < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = \text{laplace}\left(\begin{cases} 0 & x < \pi \\ -\sin(3x) & \pi \leq x \end{cases}, x, s\right) \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 1 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - s + 4Y(s) = \text{laplace}\left(\begin{cases} 0 & x < \pi \\ -\sin(3x) & \pi \leq x \end{cases}, x, s\right)$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\operatorname{laplace}\left(\left\{\begin{array}{cc} 0 & x<\pi \\ -\sin(3x) & \pi \le x \end{array}\right., x, s\right) + s + 1}{s^2+4}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\operatorname{laplace}\left(\left\{\begin{array}{cc} 0 & x<\pi \\ -\sin(3x) & \pi \le x \end{array}\right., x, s\right) + s + 1}{s^2+4}\right) \\
&= \frac{\sin(3x)}{5} + \frac{4\sin(2x)}{5} + \cos(2x) - \frac{\left(\left\{\begin{array}{cc} 0 & \pi - x < 0 \\ 1 & \text{otherwise} \end{array}\right.\right)(3\sin(2x) + 2\sin(3x))}{10}
\end{aligned}$$

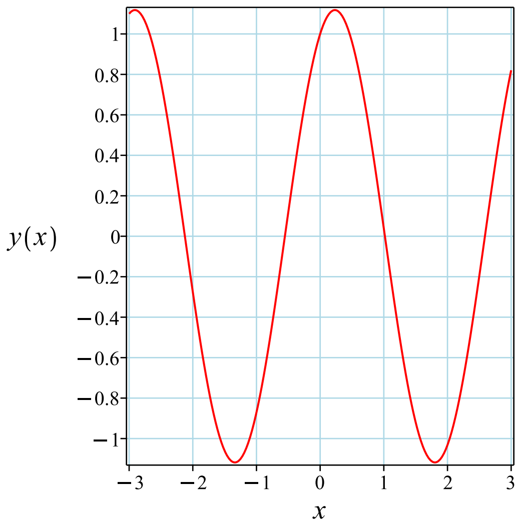
Simplifying the solution gives

$$y = \left\{\begin{array}{cc} \frac{\sin(2x)}{2} + \cos(2x) & x \le \pi \\ \frac{\left(4\cos(x)^2 + 8\cos(x) - 1\right)\sin(x)}{5} + 2\cos(x)^2 - 1 & \pi < x \end{array}\right.$$

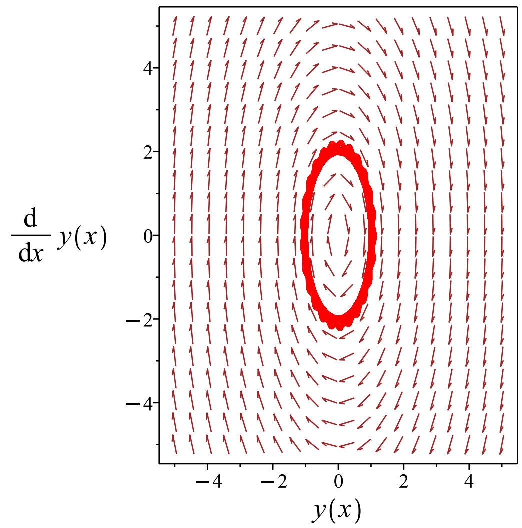
Summary

The solution(s) found are the following

$$y = \left\{\begin{array}{cc} \frac{\sin(2x)}{2} + \cos(2x) & x \le \pi \\ \frac{\left(4\cos(x)^2 + 8\cos(x) - 1\right)\sin(x)}{5} + 2\cos(x)^2 - 1 & \pi < x \end{array}\right. \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \begin{cases} \frac{\sin(2x)}{2} + \cos(2x) & x \leq \pi \\ \frac{\left(4\cos(x)^2+8\cos(x)-1\right)\sin(x)}{5} + 2\cos(x)^2 - 1 & \pi < x \end{cases}$$

Verified OK.

15.5.2 Maple step by step solution

Let's solve

$$\left[y'' + 4y = \begin{cases} 0 & x < \pi \\ -\sin(3x) & \pi \leq x \end{cases}, y(0) = 1, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 + 4 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{-16})}{2}$
- Roots of the characteristic polynomial

$r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \begin{cases} 0 & x < \pi \\ -\sin(3x) & \pi \leq x \end{cases}\right.$$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $$y_p(x) = -\cos(2x)\left(\int \left(\begin{cases} 0 & x < \pi \\ -\frac{\cos(x)}{4} + \frac{\cos(5x)}{4} & \pi \leq x \end{cases}\right) dx\right) + \sin(2x)\left(\int \left(\begin{cases} 0 \\ -\frac{\sin(5x)}{4} - \frac{\sin(x)}{4} \end{cases}\right.\right.$$

  - Compute integrals

    $$y_p(x) = \begin{cases} 0 & x \leq \pi \\ \frac{\left(4\cos(x)^2 + 3\cos(x) - 1\right)\sin(x)}{5} & \pi < x \end{cases}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 \cos(2x) + c_2 \sin(2x) + \begin{cases} 0 & x \leq \pi \\ \frac{\left(4\cos(x)^2 + 3\cos(x) - 1\right)\sin(x)}{5} & \pi < x \end{cases}$$

☐ Check validity of solution $y = c_1 \cos(2x) + c_2 \sin(2x) + \left\{ \begin{array}{cc} 0 & x \le \pi \\ \frac{\left(4\cos(x)^2+3\cos(x)-1\right)\sin(x)}{5} & \pi < x \end{array} \right.$

- Use initial condition $y(0) = 1$

  $1 = c_1$

- Compute derivative of the solution

  $$y' = -2c_1 \sin(2x) + 2c_2 \cos(2x) + \left\{ \begin{array}{cc} 0 & x \le \pi \\ \frac{(-8\cos(x)\sin(x)-3\sin(x))\sin(x)}{5} + \frac{\left(4\cos(x)^2+3\cos(x)-1\right)\cos(x)}{5} & \pi < x \end{array} \right.$$

- Use the initial condition $y'\big|_{\{x=0\}} = 1$

  $1 = 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = 1, c_2 = \frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $$y = \cos(2x) + \frac{\sin(2x)}{2} + \left\{ \begin{array}{cc} 0 & x \le \pi \\ \frac{\left(4\cos(x)^2+3\cos(x)-1\right)\sin(x)}{5} & \pi < x \end{array} \right.$$

- Solution to the IVP

  $$y = \cos(2x) + \frac{\sin(2x)}{2} + \left\{ \begin{array}{cc} 0 & x \le \pi \\ \frac{\left(4\cos(x)^2+3\cos(x)-1\right)\sin(x)}{5} & \pi < x \end{array} \right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 8.188 (sec). Leaf size: 39

```
dsolve([diff(y(x),x$2)+4*y(x)=piecewise(0<=x and x<Pi,0,Pi<=x,sin(3*(x-Pi))),y(0) = 1, D(y)(
```

$$y(x) = \cos(2x) + \left(\begin{cases} \frac{\sin(2x)}{2} & x < \pi \\ \frac{4\sin(2x)}{5} + \frac{\sin(3x)}{5} & \pi \leq x \end{cases}\right)$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 42

```
DSolve[{y''[x]+4*y[x]==Piecewise[{ {0,0<=x<Pi},{Sin[3*(x-Pi)],x>=Pi}}],{y[0]==1,y'[0]==1}},y
```

$$y(x) \to \begin{cases} \cos(2x) + \cos(x)\sin(x) & x \leq \pi \\ \frac{1}{5}(5\cos(2x) + 4\sin(2x) + \sin(3x)) & \text{True} \end{cases}$$

15.6 problem 4 (g)

Internal problem ID [12812]
Internal file name [`OUTPUT/11464_Saturday_November_04_2023_08_47_28_AM_1140586/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.4, page 265
Problem number: 4 (g).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' - 4y = \begin{cases} x & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

15.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = 0$$
$$q(x) = -4$$
$$F = \begin{cases} 0 & x < 0 \\ x & x < 1 \\ 1 & 1 \leq x \end{cases}$$

Hence the ode is

$$y'' - 4y = \begin{cases} 0 & x < 0 \\ x & x < 1 \\ 1 & 1 \leq x \end{cases}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & x < 0 \\ x & x < 1 \\ 1 & 1 \leq x \end{cases}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 4Y(s) = \frac{-e^{-s} + 1}{s^2} \tag{1}$$

But the initial conditions are

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 4Y(s) = \frac{-e^{-s} + 1}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{e^{-s} - 1}{s^2 (s^2 - 4)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{e^{-s} - 1}{s^2 (s^2 - 4)}\right) \\
&= -\frac{x \operatorname{Heaviside}(1 - x)}{4} + \frac{\sinh(2x)}{8} - \frac{\operatorname{Heaviside}(x - 1)(\sinh(2x - 2) + 2)}{8}
\end{aligned}$$

Hence the final solution is

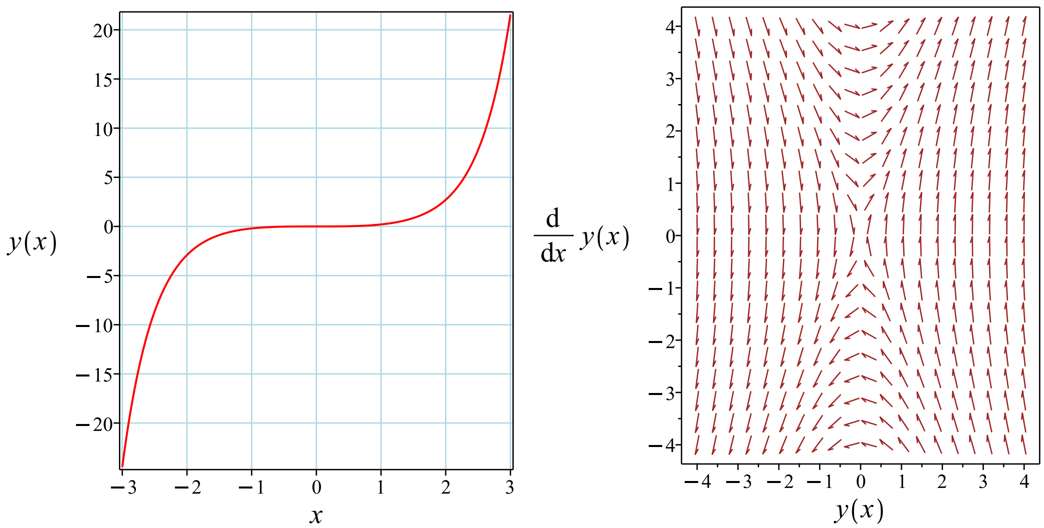
$$y = -\frac{x \operatorname{Heaviside}(1 - x)}{4} + \frac{\sinh(2x)}{8} - \frac{\operatorname{Heaviside}(x - 1)(\sinh(2x - 2) + 2)}{8}$$

Simplifying the solution gives

$$y = -\frac{\operatorname{Heaviside}(x - 1)\sinh(2x - 2)}{8} + \frac{(2x - 2)\operatorname{Heaviside}(x - 1)}{8} - \frac{x}{4} + \frac{\sinh(2x)}{8}$$

Summary
The solution(s) found are the following

$$y = -\frac{\operatorname{Heaviside}(x - 1)\sinh(2x - 2)}{8} + \frac{(2x - 2)\operatorname{Heaviside}(x - 1)}{8} - \frac{x}{4} + \frac{\sinh(2x)}{8} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{\text{Heaviside}\,(x-1)\sinh\,(2x-2)}{8} + \frac{(2x-2)\,\text{Heaviside}\,(x-1)}{8} - \frac{x}{4} + \frac{\sinh\,(2x)}{8}$$

Verified OK.

15.6.2 Maple step by step solution

Let's solve

$$\left[y'' - 4y = \begin{cases} 0 & x < 0 \\ x & x < 1 \\ 1 & 1 \leq x \end{cases}, y(0) = 0, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 - 4 = 0$
- Factor the characteristic polynomial
  $(r-2)\,(r+2) = 0$
- Roots of the characteristic polynomial

$r = (-2, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \begin{cases} 0 & x < 0 \\ x & x < 1 \\ 1 & 1 \leq x \end{cases}\right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{2x} \\ -2\,\mathrm{e}^{-2x} & 2\,\mathrm{e}^{2x} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 4$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\mathrm{e}^{-2x}\left(\int \mathrm{e}^{2x}\left(\begin{cases} 0 & x < 0 \\ x & x < 1 \\ 1 & 1 \leq x \end{cases}\right) dx\right)}{4} + \frac{\mathrm{e}^{2x}\left(\int \mathrm{e}^{-2x}\left(\begin{cases} 0 & x < 0 \\ x & x < 1 \\ 1 & 1 \leq x \end{cases}\right) dx\right)}{4}$$

- Compute integrals

$$y_p(x) = \frac{\left(\begin{cases} 0 & x \leq 0 \\ -4x - \mathrm{e}^{-2x} + \mathrm{e}^{2x} & x \leq 1 \\ \mathrm{e}^{-2x+2} - 4 - \mathrm{e}^{-2x} - \mathrm{e}^{2x-2} + \mathrm{e}^{2x} & 1 < x \end{cases}\right)}{16}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + \frac{\left(\begin{cases} 0 & x \leq 0 \\ -4x - \mathrm{e}^{-2x} + \mathrm{e}^{2x} & x \leq 1 \\ \mathrm{e}^{-2x+2} - 4 - \mathrm{e}^{-2x} - \mathrm{e}^{2x-2} + \mathrm{e}^{2x} & 1 < x \end{cases}\right)}{16}$$

- ☐ Check validity of solution $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + \frac{\left(\begin{cases} 0 & x \leq 0 \\ -4x - \mathrm{e}^{-2x} + \mathrm{e}^{2x} & x \leq 1 \\ \mathrm{e}^{-2x+2} - 4 - \mathrm{e}^{-2x} - \mathrm{e}^{2x-2} + \mathrm{e}^{2x} & 1 < x \end{cases}\right)}{16}$
  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2$
  - Compute derivative of the solution

    $$y' = -2c_1 \mathrm{e}^{-2x} + 2c_2 \mathrm{e}^{2x} + \frac{\left(\begin{cases} 0 & x \leq 0 \\ -4 + 2\,\mathrm{e}^{-2x} + 2\,\mathrm{e}^{2x} & x \leq 1 \\ -2\,\mathrm{e}^{-2x+2} + 2\,\mathrm{e}^{-2x} - 2\,\mathrm{e}^{2x-2} + 2\,\mathrm{e}^{2x} & 1 < x \end{cases}\right)}{16}$$
  - Use the initial condition $y'\Big|_{\{x=0\}} = 0$

    $0 = -2c_1 + 2c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 0\}$
  - Substitute constant values into general solution and simplify

    $$y = \frac{\left(\begin{cases} 0 & x \leq 0 \\ -4x - \mathrm{e}^{-2x} + \mathrm{e}^{2x} & x \leq 1 \\ \mathrm{e}^{-2x+2} - 4 - \mathrm{e}^{-2x} - \mathrm{e}^{2x-2} + \mathrm{e}^{2x} & 1 < x \end{cases}\right)}{16}$$
- Solution to the IVP

  $$y = \frac{\left(\begin{cases} 0 & x \leq 0 \\ -4x - \mathrm{e}^{-2x} + \mathrm{e}^{2x} & x \leq 1 \\ \mathrm{e}^{-2x+2} - 4 - \mathrm{e}^{-2x} - \mathrm{e}^{2x-2} + \mathrm{e}^{2x} & 1 < x \end{cases}\right)}{16}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 8.235 (sec). Leaf size: 46

```
dsolve([diff(y(x),x$2)-4*y(x)=piecewise(0<=x and x<1,x,1<=x,1),y(0) = 0, D(y)(0) = 0],y(x),
```

$$y(x) = \frac{\left(\begin{cases} \sinh(2x) - 2x & x < 1 \\ \sinh(2) - 4 & x = 1 \\ \sinh(2x) - \sinh(2x-2) - 2 & 1 < x \end{cases}\right)}{8}$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 36

```
DSolve[{y''[x]-4*y[x]==Piecewise[{ {x,0<=x<1},{x,x>=1}}],{y[0]==0,y'[0]==0}},y[x],x,IncludeS
```

$$y(x) \to \begin{cases} 0 & x \leq 0 \\ \frac{1}{16}e^{-2x}(-4e^{2x}x + e^{4x} - 1) & \text{True} \end{cases}$$

15.7 problem 4 (h)

Internal problem ID [12813]
Internal file name [`OUTPUT/11465_Saturday_November_04_2023_08_47_28_AM_77734816/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.4, page 265
Problem number: 4 (h).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' - 4y' + 5y = \begin{cases} x & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

15.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = -4$$
$$q(x) = 5$$
$$F = \begin{cases} 0 & x < 0 \\ x & x < 1 \\ 1 & 1 \leq x \end{cases}$$

Hence the ode is

$$y'' - 4y' + 5y = \begin{cases} 0 & x < 0 \\ x & x < 1 \\ 1 & 1 \leq x \end{cases}$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \begin{cases} 0 & x < 0 \\ x & x < 1 \\ 1 & 1 \leq x \end{cases}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$
$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 4sY(s) + 4y(0) + 5Y(s) = \frac{-e^{-s} + 1}{s^2} \tag{1}$$

But the initial conditions are

$$y(0) = 1$$
$$y'(0) = 0$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 4 - s - 4sY(s) + 5Y(s) = \frac{-e^{-s} + 1}{s^2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{-s^3 + 4s^2 + e^{-s} - 1}{s^2 (s^2 - 4s + 5)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(-\frac{-s^3 + 4s^2 + e^{-s} - 1}{s^2 (s^2 - 4s + 5)}\right) \\ &= \frac{4}{25} + \frac{x \,\text{Heaviside}(1 - x)}{5} + \frac{\text{Heaviside}(x - 1)}{25} + \frac{e^{2x}(21\cos(x) - 47\sin(x))}{25} + \frac{e^{2x-2}(\text{Heaviside}(1 - x}{} \end{aligned}$$

Hence the final solution is

$$y = \frac{4}{25} + \frac{x \,\text{Heaviside}(1 - x)}{5} + \frac{\text{Heaviside}(x - 1)}{25} + \frac{e^{2x}(21\cos(x) - 47\sin(x))}{25} + \frac{e^{2x-2}(\text{Heaviside}(1 - x) - 1)(-4\cos(x - 1) + 3\sin(x - 1))}{25}$$

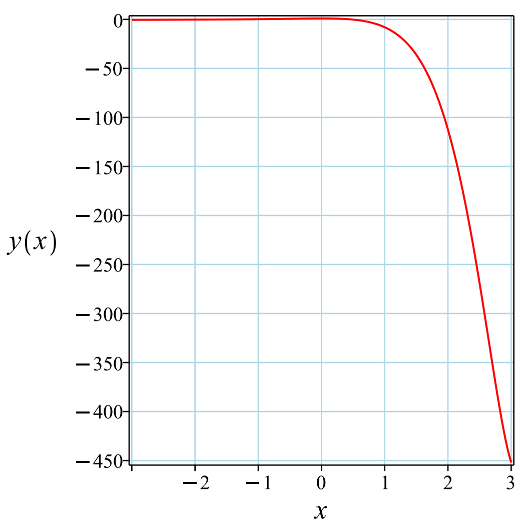
Simplifying the solution gives

$$y = \frac{4\left(\left(\cos(1) + \frac{3\sin(1)}{4}\right)\cos(x) - \frac{3\left(\cos(1) - \frac{4\sin(1)}{3}\right)\sin(x)}{4}\right)\text{Heaviside}(x - 1)\, e^{2x-2}}{25} + \frac{(-5x + 1)\,\text{Heaviside}(x - 1)}{25} + \frac{e^{2x}(21\cos(x) - 47\sin(x))}{25} + \frac{x}{5} + \frac{4}{25}$$

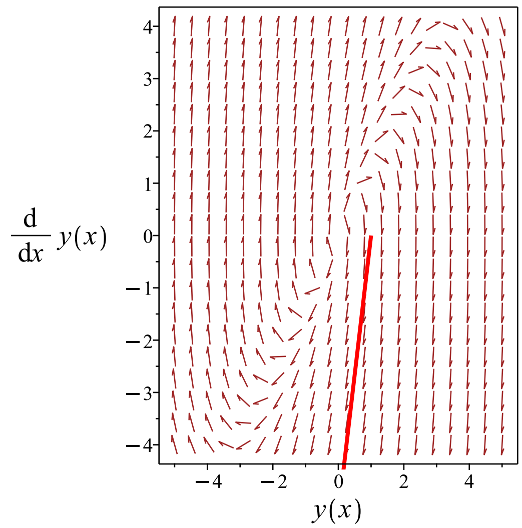
Summary

The solution(s) found are the following

$$y = \frac{4\left(\left(\cos(1) + \frac{3\sin(1)}{4}\right)\cos(x) - \frac{3\left(\cos(1) - \frac{4\sin(1)}{3}\right)\sin(x)}{4}\right)\text{Heaviside}(x - 1)\, e^{2x-2}}{25} + \frac{(-5x + 1)\,\text{Heaviside}(x - 1)}{25} + \frac{e^{2x}(21\cos(x) - 47\sin(x))}{25} + \frac{x}{5} + \frac{4}{25} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4\left(\left(\cos(1)+\frac{3\sin(1)}{4}\right)\cos(x)-\frac{3\left(\cos(1)-\frac{4\sin(1)}{3}\right)\sin(x)}{4}\right)\text{Heaviside}(x-1)\,\mathrm{e}^{2x-2}}{25} + \frac{(-5x+1)\,\text{Heaviside}(x-1)}{25} + \frac{\mathrm{e}^{2x}(21\cos(x)-47\sin(x))}{25} + \frac{x}{5} + \frac{4}{25}$$

Verified OK.

15.7.2 Maple step by step solution

Let's solve

$$\left[y''-4y'+5y = \begin{cases} 0 & x<0 \\ x & x<1 \\ 1 & 1\le x \end{cases}, y(0)=1, y'\Big|_{\{x=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2-4r+5=0$

- Use quadratic formula to solve for r

$r = \frac{4 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - \mathrm{I}, 2 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{2x} \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{2x} \cos(x) + c_2 \mathrm{e}^{2x} \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \left\{\begin{array}{cc} 0 & x < 0 \\ x & x < 1 \\ 1 & 1 \leq x \end{array}\right.\right]$$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{2x} \cos(x) & \mathrm{e}^{2x} \sin(x) \\ 2\,\mathrm{e}^{2x} \cos(x) - \mathrm{e}^{2x} \sin(x) & 2\,\mathrm{e}^{2x} \sin(x) + \mathrm{e}^{2x} \cos(x) \end{array}\right]$$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{4x}$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\mathrm{e}^{2x} \left(\cos(x) \left(\int \left(\left\{\begin{array}{cc} 0 & x < 0 \\ x & x < 1 \\ 1 & 1 \leq x \end{array}\right.\right) \sin(x)\,\mathrm{e}^{-2x} dx\right) - \sin(x) \left(\int \left(\left\{\begin{array}{cc} 0 & x < 0 \\ x & x < 1 \\ 1 & 1 \leq x \end{array}\right.\right) \cos($$

- ○ Compute integrals

$$y_p(x) = \frac{\left(\begin{cases} 0 \\ (-4\cos(x) + 3\sin(x))\,e^{2x} + 5x + 4 \\ \left((4\cos(1) + 3\sin(1))\cos(x) - 3\left(\cos(1) - \frac{4\sin(1)}{3}\right)\sin(x)\right)e^{2x-2} + 5 + (-4\cos(x) \end{cases}\right.}{25}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{2x}\cos(x) + c_2 e^{2x}\sin(x) + \frac{\left(\begin{cases} 0 \\ (-4\cos(x) + 3\sin(x))\,e^{2x} \\ \left((4\cos(1) + 3\sin(1))\cos(x) - 3\left(\cos(1) - \frac{4\sin(1)}{3}\right)\sin(x) \end{cases}\right.}{25}$$

☐ Check validity of solution $y = c_1 e^{2x}\cos(x) + c_2 e^{2x}\sin(x) + \frac{\left(\begin{cases} \\ \\ \left((4\cos(1) + 3\sin(1))\cos(x) - \end{cases}\right.}{}$

- Use initial condition $y(0) = 1$

  $1 = c_1$

- Compute derivative of the solution

  $$y' = 2c_1 e^{2x}\cos(x) - c_1 e^{2x}\sin(x) + 2c_2 e^{2x}\sin(x) + c_2 e^{2x}\cos(x) + \frac{\left(\begin{cases} \\ \\ \left(-(4\cos(1) + 3\sin(1))\,\mathrm{si} \end{cases}\right.}{}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = 2c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = -2\}$

- Substitute constant values into general solution and simplify

  $$y = e^{2x}(\cos(x) - 2\sin(x)) + \frac{\left(\begin{cases} 0 \\ (-4\cos(x) + 3\sin(x))\,e^{2x} + 5x \\ \left((4\cos(1) + 3\sin(1))\cos(x) - 3\left(\cos(1) - \frac{4\sin(1)}{3}\right)\sin(x)\right)e^{2x} \end{cases}\right.}{25}$$

- Solution to the IVP

$$y = e^{2x}(\cos(x) - 2\sin(x)) + \frac{\left(\begin{cases} 0 \\ (-4\cos(x) + 3\sin(x))\, e^{2x} + 5x \\ \left((4\cos(1) + 3\sin(1))\cos(x) - 3\left(\cos(1) - \frac{4\sin(1)}{3}\right)\sin(x)\right) e^{2x} \end{cases}\right)}{25}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 8.484 (sec). Leaf size: 87

```
dsolve([diff(y(x),x$2)-4*diff(y(x),x)+5*y(x)=piecewise(0<=x and x<1,x,1<=x,1),y(0) = 1, D(y)
```

$$y(x) = \frac{\left(\begin{cases} 4 + 5x + e^{2x}(21\cos(x) - 47\sin(x)) & x < 1 \\ 10 + e^{2}(21\cos(1) - 47\sin(1)) & x = 1 \\ (4\cos(-1+x) - 3\sin(-1+x))\, e^{2x-2} + 5 + e^{2x}(21\cos(x) - 47\sin(x)) & 1 < x \end{cases}\right)}{25}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 51

```
DSolve[{y''[x]-4*y'[x]+5*y[x]==Piecewise[{ {x,0<=x<1},{1,x>=1}}],{y[0]==1,y'[0]==0}},y[x],x,
```

$$y(x) \to \begin{cases} e^{2x}(\cos(x) - 2\sin(x)) & x \leq 0 \\ \frac{1}{25}(5x + 21e^{2x}\cos(x) - 47e^{2x}\sin(x) + 4) & \text{True} \end{cases}$$

16 Chapter 5. The Laplace Transform Method. Exercises 5.5, page 273

16.1 problem 1

Internal problem ID [12814]
Internal file name [`OUTPUT/11466_Saturday_November_04_2023_08_47_29_AM_38328001/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.5, page 273
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 3y = \delta(x - 2)}$$

With initial conditions

$$[y(0) = 1]$$

16.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 3 \\ q(x) &= \delta(x - 2) \end{aligned}$$

Hence the ode is

$$y' + 3y = \delta(x - 2)$$

The domain of $p(x) = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \delta(x-2)$ is

$$\{x < 2 \lor 2 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.1.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) + 3Y(s) = \mathrm{e}^{-2s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 1 + 3Y(s) = \mathrm{e}^{-2s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{\mathrm{e}^{-2s} + 1}{s + 3}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-2s} + 1}{s + 3}\right) \\ &= \text{Heaviside}\,(x-2)\,\mathrm{e}^{-3x+6} + \mathrm{e}^{-3x} \end{aligned}$$

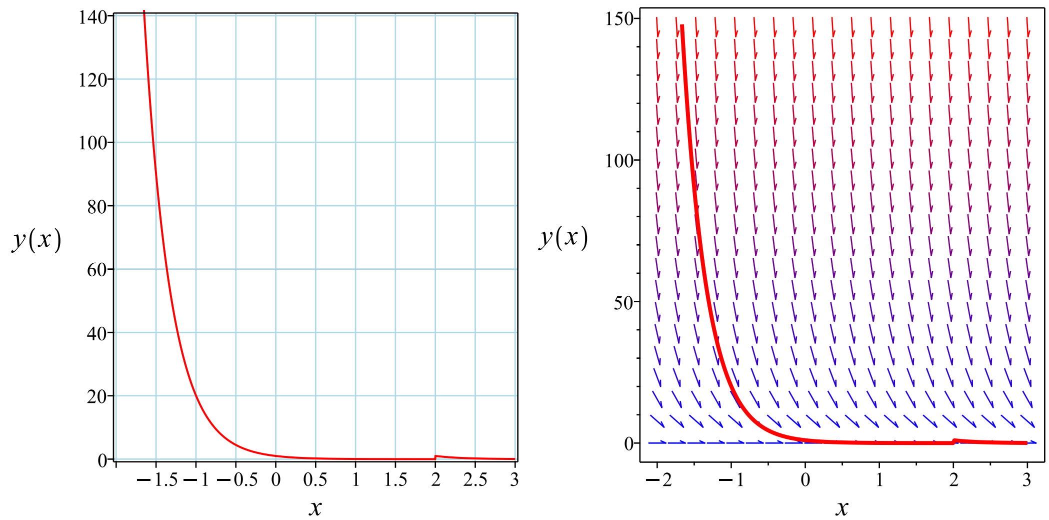
Hence the final solution is

$$y = \text{Heaviside}\,(x-2)\,\mathrm{e}^{-3x+6} + \mathrm{e}^{-3x}$$

Summary

The solution(s) found are the following

$$y = \text{Heaviside}\,(x-2)\,\mathrm{e}^{-3x+6} + \mathrm{e}^{-3x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \text{Heaviside}\,(x-2)\,\mathrm{e}^{-3x+6} + \mathrm{e}^{-3x}$$

Verified OK.

16.1.3 Maple step by step solution

Let's solve

$[y' + 3y = \mathit{Dirac}(x-2)\,, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -3y + \mathit{Dirac}(x-2)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 3y = \mathit{Dirac}(x-2)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)(y' + 3y) = \mu(x)\, Dirac(x-2)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$\mu(x)(y' + 3y) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

$\mu'(x) = 3\mu(x)$

- Solve to find the integrating factor

$\mu(x) = \mathrm{e}^{3x}$

- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, Dirac(x-2)\, dx + c_1$

- Evaluate the integral on the lhs

$\mu(x)\, y = \int \mu(x)\, Dirac(x-2)\, dx + c_1$

- Solve for y

$y = \frac{\int \mu(x) Dirac(x-2) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{3x}$

$y = \frac{\int \mathrm{e}^{3x} Dirac(x-2) dx + c_1}{\mathrm{e}^{3x}}$

- Evaluate the integrals on the rhs

$y = \frac{Heaviside(x-2)\mathrm{e}^{6} + c_1}{\mathrm{e}^{3x}}$

- Simplify

$y = \mathrm{e}^{-3x}(Heaviside(x-2)\, \mathrm{e}^{6} + c_1)$

- Use initial condition $y(0) = 1$

$1 = c_1$

- Solve for c_1

$c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

$y = \mathrm{e}^{-3x}(Heaviside(x-2)\, \mathrm{e}^{6} + 1)$

- Solution to the IVP

$y = \mathrm{e}^{-3x}(Heaviside(x-2)\, \mathrm{e}^{6} + 1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 5.921 (sec). Leaf size: 20

```
dsolve([diff(y(x),x)+3*y(x)=Dirac(x-2),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \text{Heaviside}(x-2)\,\mathrm{e}^{6-3x} + \mathrm{e}^{-3x}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 21

```
DSolve[{y'[x]+3*y[x]==DiracDelta[x-2],{y[0]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(e^6\theta(x-2)+1\right)$$

16.2 problem 2

Internal problem ID [12815]
Internal file name [`OUTPUT/11467_Saturday_November_04_2023_08_47_29_AM_86585364/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.5, page 273
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 3y = \delta(x - 1) + 2\,\text{Heaviside}\,(x - 2)}$$

With initial conditions

$$[y(0) = 0]$$

16.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= \delta(x - 1) + 2\,\text{Heaviside}\,(x - 2) \end{aligned}$$

Hence the ode is

$$y' - 3y = \delta(x - 1) + 2\,\text{Heaviside}\,(x - 2)$$

The domain of $p(x) = -3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \delta(x-1) + 2\,\text{Heaviside}\,(x-2)$ is

$$\{1 \leq x \leq 2, 2 \leq x \leq \infty, -\infty \leq x \leq 1\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

16.2.2 Solving as laplace ode

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\mathcal{L}(y') = sY(s) - y(0)$$

The given ode now becomes an algebraic equation in the Laplace domain

$$sY(s) - y(0) - 3Y(s) = \mathrm{e}^{-s} + \frac{2\,\mathrm{e}^{-2s}}{s} \tag{1}$$

Replacing initial condition gives

$$sY(s) - 3Y(s) = \mathrm{e}^{-s} + \frac{2\,\mathrm{e}^{-2s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{\mathrm{e}^{-s}s + 2\,\mathrm{e}^{-2s}}{s\,(s-3)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-s}s + 2\,\mathrm{e}^{-2s}}{s\,(s-3)}\right) \\
&= -\frac{2\,\text{Heaviside}\,(x-2)}{3} + \frac{2(1 - \text{Heaviside}\,(-x+2))\,\mathrm{e}^{3x-6}}{3} + \mathrm{e}^{-3+3x}(1 - \text{Heaviside}\,(1-x))
\end{aligned}$$

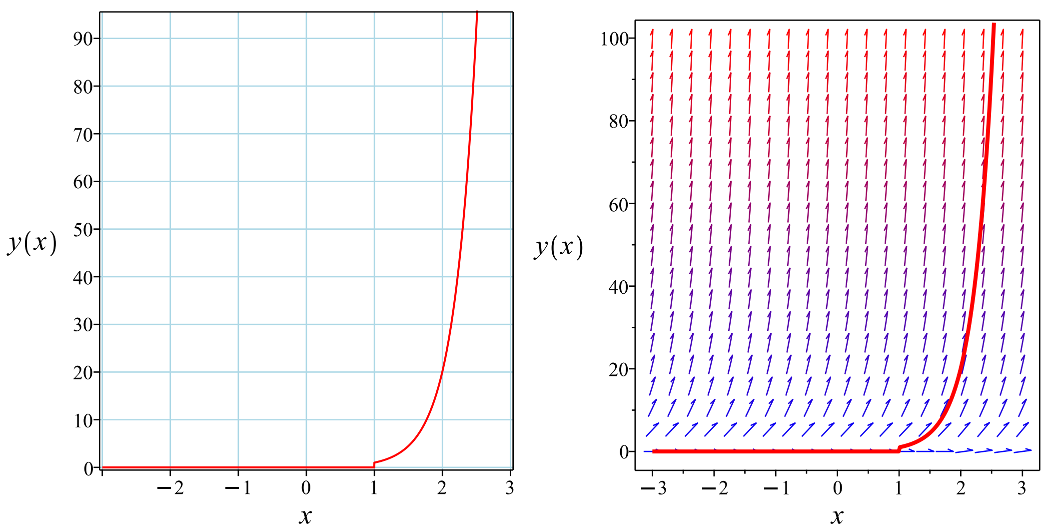
Hence the final solution is

$$y = -\frac{2\,\text{Heaviside}\,(x-2)}{3} + \frac{2(1-\text{Heaviside}\,(-x+2))\,\mathrm{e}^{3x-6}}{3} + \mathrm{e}^{-3+3x}(1-\text{Heaviside}\,(1-x))$$

Summary

The solution(s) found are the following

$$y = -\frac{2\,\text{Heaviside}\,(x-2)}{3} + \frac{2(1-\text{Heaviside}\,(-x+2))\,\mathrm{e}^{3x-6}}{3} + \mathrm{e}^{-3+3x}(1-\text{Heaviside}\,(1-x)) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{2\,\text{Heaviside}\,(x-2)}{3} + \frac{2(1-\text{Heaviside}\,(-x+2))\,\mathrm{e}^{3x-6}}{3} + \mathrm{e}^{-3+3x}(1-\text{Heaviside}\,(1-x))$$

Verified OK.

16.2.3 Maple step by step solution

Let's solve

$[y' - 3y = Dirac(x-1) + 2Heaviside(x-2), y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 3y + Dirac(x-1) + 2Heaviside(x-2)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 3y = Dirac(x-1) + 2Heaviside(x-2)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' - 3y) = \mu(x)(Dirac(x-1) + 2Heaviside(x-2))$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)y)$

  $\mu(x)(y' - 3y) = \mu'(x)y + \mu(x)y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -3\mu(x)$

- Solve to find the integrating factor

  $\mu(x) = e^{-3x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)y)\right) dx = \int \mu(x)(Dirac(x-1) + 2Heaviside(x-2)) dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)y = \int \mu(x)(Dirac(x-1) + 2Heaviside(x-2)) dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)(Dirac(x-1)+2Heaviside(x-2))dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{-3x}$

  $y = \frac{\int e^{-3x}(Dirac(x-1)+2Heaviside(x-2))dx + c_1}{e^{-3x}}$

- Evaluate the integrals on the rhs

  $y = \frac{e^{-3} Heaviside(x-1) - \frac{2e^{-3x} Heaviside(x-2)}{3} + \frac{2Heaviside(x-2)e^{-6}}{3} + c_1}{e^{-3x}}$

- Simplify

  $y = e^{-3+3x} Heaviside(x-1) - \frac{2Heaviside(x-2)}{3} + \frac{2e^{3x-6} Heaviside(x-2)}{3} + c_1 e^{3x}$

- Use initial condition $y(0) = 0$

  $0 = c_1$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = \mathrm{e}^{-3+3x} \mathit{Heaviside}(x-1) + \frac{2\mathit{Heaviside}(x-2)(\mathrm{e}^{3x-6}-1)}{3}$
- Solution to the IVP

  $y = \mathrm{e}^{-3+3x} \mathit{Heaviside}(x-1) + \frac{2\mathit{Heaviside}(x-2)(\mathrm{e}^{3x-6}-1)}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 6.234 (sec). Leaf size: 46

```
dsolve([diff(y(x),x)-3*y(x)=Dirac(x-1)+2*Heaviside(x-2),y(0) = 0],y(x), singsol=all)
```

$$y(x) = -\frac{2\,\mathrm{Heaviside}\,(x-2)}{3} + \frac{2\,\mathrm{Heaviside}\,(x-2)\,\mathrm{e}^{-6+3x}}{3} + \mathrm{Heaviside}\,(-1+x)\,\mathrm{e}^{3x-3}$$

✓ Solution by Mathematica

Time used: 0.212 (sec). Leaf size: 44

```
DSolve[{y'[x]-3*y[x]==DiracDelta[x-1]+2*UnitStep[x-2],{y[0]==0}},y[x],x,IncludeSingularSolut
```

$$y(x) \to e^{3x-3}\theta(x-1) + \frac{2(e^6 - e^{3x})\,(\theta(2-x)-1)}{3e^6}$$

16.3 problem 3

Internal problem ID [12816]
Internal file name [`OUTPUT/11468_Saturday_November_04_2023_08_47_29_AM_9423804/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.5, page 273
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 9y = \delta(x - \pi) + \delta(x - 3\pi)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 9 \\ F &= \delta(x - \pi) + \delta(x - 3\pi) \end{aligned}$$

Hence the ode is

$$y'' + 9y = \delta(x - \pi) + \delta(x - 3\pi)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \delta(x - \pi) + \delta(x - 3\pi)$ is

$$\{\pi \leq x \leq 3\pi, 3\pi \leq x \leq \infty, -\infty \leq x \leq \pi\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 9Y(s) = \mathrm{e}^{-s\pi} + \mathrm{e}^{-3s\pi} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0 \\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 9Y(s) = \mathrm{e}^{-s\pi} + \mathrm{e}^{-3s\pi}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{\mathrm{e}^{-s\pi} + \mathrm{e}^{-3s\pi}}{s^2 + 9}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{\mathrm{e}^{-s\pi}+\mathrm{e}^{-3s\pi}}{s^2+9}\right) \\
&= -\frac{\sin(3x)\left(\mathrm{Heaviside}(x-\pi)+\mathrm{Heaviside}(x-3\pi)\right)}{3}
\end{aligned}
$$

Hence the final solution is

$$
y = -\frac{\sin(3x)\left(\mathrm{Heaviside}(x-\pi)+\mathrm{Heaviside}(x-3\pi)\right)}{3}
$$

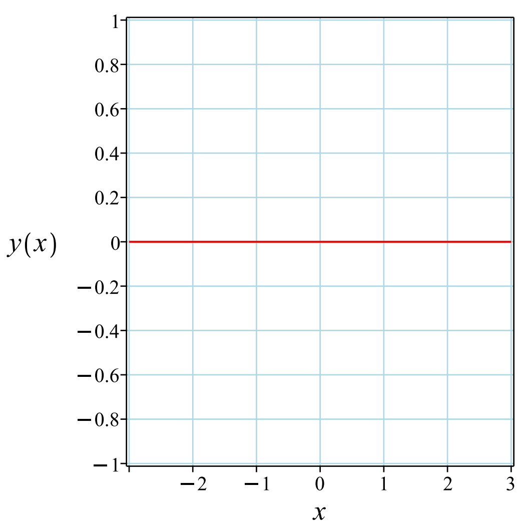
Simplifying the solution gives

$$
y = -\frac{\sin(3x)\left(\mathrm{Heaviside}(x-\pi)+\mathrm{Heaviside}(x-3\pi)\right)}{3}
$$

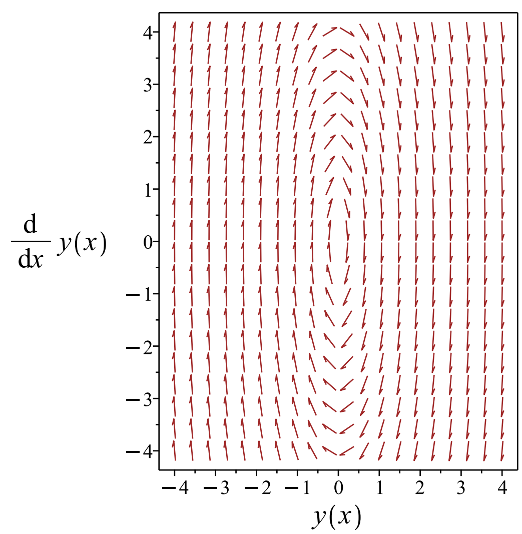
Summary

The solution(s) found are the following

$$
y = -\frac{\sin(3x)\left(\mathrm{Heaviside}(x-\pi)+\mathrm{Heaviside}(x-3\pi)\right)}{3} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = -\frac{\sin(3x)\left(\mathrm{Heaviside}(x-\pi)+\mathrm{Heaviside}(x-3\pi)\right)}{3}
$$

Verified OK.

16.3.2 Maple step by step solution

Let's solve

$$\left[y'' + 9y = Dirac(x - \pi) + Dirac(x - 3\pi), y(0) = 0, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = Dirac(x - \pi) + Dirac(\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3$

  - Substitute functions into equation for $y_p(x)$

$y_p(x) = \frac{\sin(3x)\left(\int(-\mathit{Dirac}(x-\pi)-\mathit{Dirac}(x-3\pi))dx\right)}{3}$

- Compute integrals

  $y_p(x) = -\frac{\sin(3x)(\mathit{Heaviside}(x-\pi)+\mathit{Heaviside}(x-3\pi))}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) - \frac{\sin(3x)(\mathit{Heaviside}(x-\pi)+\mathit{Heaviside}(x-3\pi))}{3}$

□ Check validity of solution $y = c_1 \cos(3x) + c_2 \sin(3x) - \frac{\sin(3x)(\mathit{Heaviside}(x-\pi)+\mathit{Heaviside}(x-3\pi))}{3}$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $y' = -3c_1 \sin(3x) + 3c_2 \cos(3x) - \cos(3x)(\mathit{Heaviside}(x-\pi) + \mathit{Heaviside}(x-3\pi)) - \frac{\sin(3x)(\mathit{Dira}}{}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = 3c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{\sin(3x)(\mathit{Heaviside}(x-\pi)+\mathit{Heaviside}(x-3\pi))}{3}$

- Solution to the IVP

  $y = -\frac{\sin(3x)(\mathit{Heaviside}(x-\pi)+\mathit{Heaviside}(x-3\pi))}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.469 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$2)+9*y(x)=Dirac(x-Pi)+Dirac(x-3*Pi),y(0) = 0, D(y)(0) = 0],y(x), singsol
```

$$y(x) = -\frac{(\text{Heaviside}\,(x-3\pi)+\text{Heaviside}\,(x-\pi))\sin\,(3x)}{3}$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 26

```
DSolve[{y''[x]+9*y[x]==DiracDelta[x-Pi]+DiracDelta[x-3*Pi],{y[0]==0,y'[0]==0}},y[x],x,Includ
```

$$y(x) \to -\frac{1}{3}(\theta(x-3\pi)+\theta(x-\pi))\sin(3x)$$

16.4 problem 4

Internal problem ID [12817]
Internal file name [`OUTPUT/11469_Saturday_November_04_2023_08_47_30_AM_42828604/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.5, page 273
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = 2\delta(x - 1)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

16.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 1 \\ F &= 2\delta(x - 1) \end{aligned}$$

Hence the ode is

$$y'' - 2y' + y = 2\delta(x - 1)$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 2\delta(x-1)$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) + Y(s) = 2\,\mathrm{e}^{-s} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 - 2sY(s) + Y(s) = 2\,\mathrm{e}^{-s}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2\,\mathrm{e}^{-s} + 1}{s^2 - 2s + 1}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}y &= \mathcal{L}^{-1}(Y(s))\\ &= \mathcal{L}^{-1}\left(\frac{2\,\mathrm{e}^{-s} + 1}{s^2 - 2s + 1}\right)\\ &= x\,\mathrm{e}^{x} + 2(1 - \text{Heaviside}\,(1-x))\,\mathrm{e}^{x-1}(x-1)\end{aligned}$$

Hence the final solution is

$$y = x\,e^{x} + 2(1 - \text{Heaviside}\,(1 - x))\,e^{x-1}(x - 1)$$

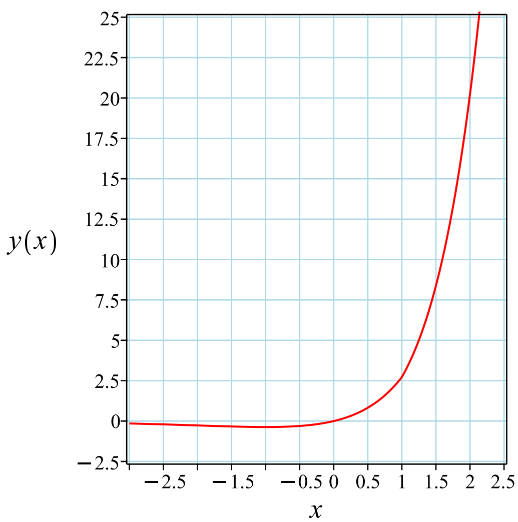
Simplifying the solution gives

$$y = 2\,e^{x-1}(x - 1)\,\text{Heaviside}\,(x - 1) + x\,e^{x}$$

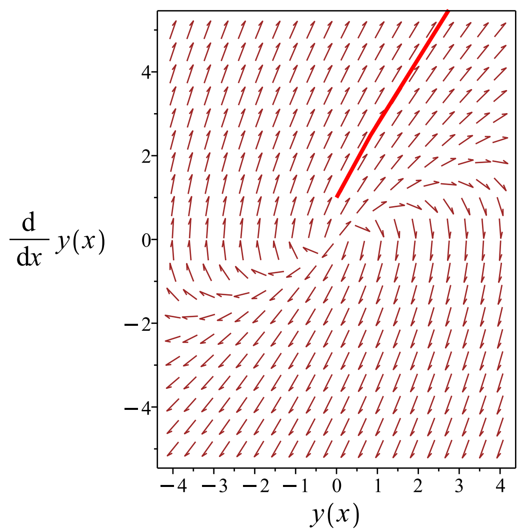
Summary

The solution(s) found are the following

$$y = 2\,e^{x-1}(x - 1)\,\text{Heaviside}\,(x - 1) + x\,e^{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,e^{x-1}(x - 1)\,\text{Heaviside}\,(x - 1) + x\,e^{x}$$

Verified OK.

16.4.2 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + y = 2Dirac(x-1), y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x c_1 + c_2 x\,e^x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2Dirac(x-1)\right]$$

- ○ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{2x}$

- ○ Substitute functions into equation for $y_p(x)$

$y_p(x) = 2\left(\int Dirac(x-1)\, dx\right) \mathrm{e}^{x-1}(x-1)$

- Compute integrals

  $y_p(x) = 2\,\mathrm{e}^{x-1}(x-1)\, Heaviside(x-1)$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{x} c_1 + c_2 x\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x-1}(x-1)\, Heaviside(x-1)$

□ Check validity of solution $y = \mathrm{e}^{x} c_1 + c_2 x\mathrm{e}^{x} + 2\mathrm{e}^{x-1}(x-1)\, Heaviside(x-1)$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $y' = \mathrm{e}^{x} c_1 + c_2\mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x-1}(x-1)\, Heaviside(x-1) + 2\,\mathrm{e}^{x-1} Heaviside(x-1) + 2\,\mathrm{e}^{x-1}(x -$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 1\}$

- Substitute constant values into general solution and simplify

  $y = 2\,\mathrm{e}^{x-1}(x-1)\, Heaviside(x-1) + x\,\mathrm{e}^{x}$

- Solution to the IVP

  $y = 2\,\mathrm{e}^{x-1}(x-1)\, Heaviside(x-1) + x\,\mathrm{e}^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.187 (sec). Leaf size: 28

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+y(x)=2*Dirac(x-1),y(0) = 0, D(y)(0) = 1],y(x), singsol
```

$$y(x) = 2\,\mathrm{Heaviside}\,(-1+x)\,\mathrm{e}^{-1+x}(-1+x)+\mathrm{e}^{x}x$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 24

```
DSolve[{y''[x]-2*y'[x]+y[x]==2*DiracDelta[x-1],{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSol
```

$$y(x) \to e^{x-1}(2(x-1)\theta(x-1)+ex)$$

16.5 problem 5

Internal problem ID [12818]
Internal file name [`OUTPUT/11470_Saturday_November_04_2023_08_47_30_AM_11197644/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.5, page 273
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + 5y = \cos(x) + 2\delta(x - \pi)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

16.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 5 \\ F &= \cos(x) + 2\delta(x - \pi) \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 5y = \cos(x) + 2\delta(x - \pi)$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \cos(x) + 2\delta(x - \pi)$ is

$$\{x < \pi \lor \pi < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) - 2sY(s) + 2y(0) + 5Y(s) = \frac{s}{s^2+1} + 2\,\mathrm{e}^{-s\pi} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 1\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + 2 - s - 2sY(s) + 5Y(s) = \frac{s}{s^2+1} + 2\,\mathrm{e}^{-s\pi}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{2\,\mathrm{e}^{-s\pi}s^2 + s^3 - 2s^2 + 2\,\mathrm{e}^{-s\pi} + 2s - 2}{(s^2+1)\,(s^2 - 2s + 5)}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(\frac{2\,\mathrm{e}^{-s\pi} s^2 + s^3 - 2s^2 + 2\,\mathrm{e}^{-s\pi} + 2s - 2}{(s^2+1)\,(s^2-2s+5)}\right) \\
&= \frac{\cos(x)}{5} - \frac{\sin(x)}{10} + \frac{4\,\mathrm{e}^{x}\cos(2x)}{5} + \frac{(-7\,\mathrm{e}^{x} + 20(1 - \text{Heaviside}(\pi - x))\,\mathrm{e}^{x-\pi})\sin(2x)}{20}
\end{aligned}
$$

Converting the above solution to piecewise it becomes

$$
y = \begin{cases} \frac{\cos(x)}{5} - \frac{\sin(x)}{10} + \frac{4\,\mathrm{e}^{x}\cos(2x)}{5} - \frac{7\,\mathrm{e}^{x}\sin(2x)}{20} & x \le \pi \\ \frac{\cos(x)}{5} - \frac{\sin(x)}{10} + \frac{4\,\mathrm{e}^{x}\cos(2x)}{5} + \frac{\sin(2x)(-7\,\mathrm{e}^{x}+20\,\mathrm{e}^{x-\pi})}{20} & \pi < x \end{cases}
$$

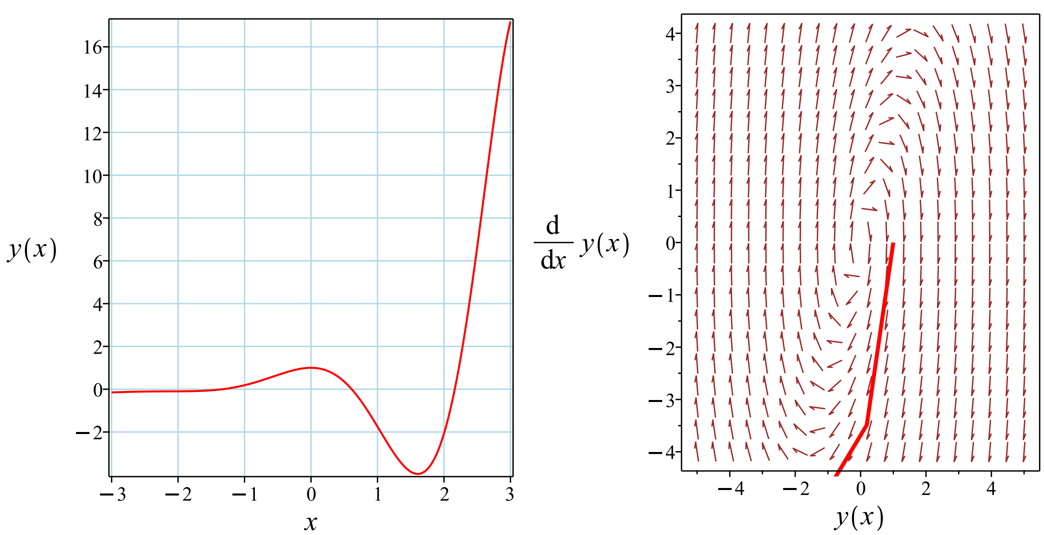
Simplifying the solution gives

$$
y = \frac{\cos(x)}{5} - \frac{\sin(x)}{10} + \frac{4\,\mathrm{e}^{x}\cos(2x)}{5} - \frac{7\,\mathrm{e}^{x}\sin(2x)}{20} + \left(\begin{cases} 0 & x \le \pi \\ \sin(2x)\,\mathrm{e}^{x-\pi} & \pi < x \end{cases}\right)
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{\cos(x)}{5} - \frac{\sin(x)}{10} + \frac{4\,\mathrm{e}^{x}\cos(2x)}{5} - \frac{7\,\mathrm{e}^{x}\sin(2x)}{20} + \left(\begin{cases} 0 & x \le \pi \\ \sin(2x)\,\mathrm{e}^{x-\pi} & \pi < x \end{cases}\right) \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\cos(x)}{5} - \frac{\sin(x)}{10} + \frac{4\,\mathrm{e}^{x}\cos(2x)}{5} - \frac{7\,\mathrm{e}^{x}\sin(2x)}{20} + \left(\left\{\begin{array}{cc} 0 & x \le \pi \\ \sin(2x)\,\mathrm{e}^{x-\pi} & \pi < x \end{array}\right.\right)$$

Verified OK.

16.5.2 Maple step by step solution

Let's solve

$\left[y'' - 2y' + 5y = \cos(x) + 2\mathit{Dirac}(x-\pi), y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (1-2\,\mathrm{I}, 1+2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{x}\cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}\sin(2x)$

- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\cos(2x)\,\mathrm{e}^{x} + c_2\mathrm{e}^{x}\sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \cos(x) + 2\mathit{Dirac}(x-\pi\right.$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x \cos(2x) & e^x \sin(2x) \\ e^x \cos(2x) - 2e^x \sin(2x) & e^x \sin(2x) + 2e^x \cos(2x) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 2e^{2x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{e^x\left(\cos(2x)\left(\int \sin(2x)\cos(x)e^{-x}dx\right) - \sin(2x)\left(\int \left(2e^{-\pi} Dirac(x-\pi) + e^{-x}\left(2\cos(x)^3 - \cos(x)\right)\right)dx\right)\right)}{2}$

- Compute integrals

  $y_p(x) = 2\cos(x)e^{x-\pi}\sin(x)\, Heaviside(x-\pi) - \frac{\sin(x)}{10} + \frac{\cos(x)}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2x)e^x + c_2 e^x \sin(2x) + 2\cos(x)e^{x-\pi}\sin(x)\, Heaviside(x-\pi) - \frac{\sin(x)}{10} + \frac{\cos(x)}{5}$

□ Check validity of solution $y = c_1 \cos(2x)e^x + c_2 e^x \sin(2x) + 2\cos(x)e^{x-\pi}\sin(x)\, Heaviside(x -$

- Use initial condition $y(0) = 1$

  $1 = c_1 + \frac{1}{5}$

- Compute derivative of the solution

  $y' = -2c_1 \sin(2x)e^x + c_1 \cos(2x)e^x + c_2 e^x \sin(2x) + 2c_2 e^x \cos(2x) - 2\sin(x)^2 e^{x-\pi} Heaviside(x$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = -\frac{1}{10} + c_1 + 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{4}{5}, c_2 = -\frac{7}{20}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{4e^x \cos(2x)}{5} - \frac{7e^x \sin(2x)}{20} + e^{x-\pi} Heaviside(x-\pi)\sin(2x) - \frac{\sin(x)}{10} + \frac{\cos(x)}{5}$

- Solution to the IVP

  $y = \frac{4e^x \cos(2x)}{5} - \frac{7e^x \sin(2x)}{20} + e^{x-\pi} Heaviside(x-\pi)\sin(2x) - \frac{\sin(x)}{10} + \frac{\cos(x)}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.625 (sec). Leaf size: 50

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+5*y(x)=cos(x)+2*Dirac(x-Pi),y(0) = 1, D(y)(0) = 0],y(x
```

$$y(x) = \sin(2x) \operatorname{Heaviside}(x-\pi) e^{x-\pi} + \frac{4 e^{x} \cos(2x)}{5} - \frac{7 e^{x} \sin(2x)}{20} - \frac{\sin(x)}{10} + \frac{\cos(x)}{5}$$

✓ Solution by Mathematica

Time used: 0.506 (sec). Leaf size: 54

```
DSolve[{y''[x]-2*y'[x]+5*y[x]==Cos[x]+2*DiracDelta[x-Pi],{y[0]==1,y'[0]==0}},y[x],x,IncludeS
```

$$y(x) \to \frac{1}{10}\left(10e^{x-\pi}\theta(x-\pi)\sin(2x) - \sin(x) + 8e^{x}\cos(2x) + (2-7e^{x}\sin(x))\cos(x)\right)$$

16.6 problem 6

Internal problem ID [12819]
Internal file name [`OUTPUT/11471_Saturday_November_04_2023_08_47_30_AM_8501762/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.5, page 273
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \delta(x - \pi)\cos(x)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

16.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 4 \\ F &= -\delta(x - \pi) \end{aligned}$$

Hence the ode is

$$y'' + 4y = -\delta(x - \pi)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = -\delta(x - \pi)$ is

$$\{x < \pi \vee \pi < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + 4Y(s) = -\mathrm{e}^{-s\pi} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 1\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) - 1 + 4Y(s) = -\mathrm{e}^{-s\pi}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = -\frac{\mathrm{e}^{-s\pi} - 1}{s^2 + 4}$$

Taking the inverse Laplace transform gives

$$
\begin{aligned}
y &= \mathcal{L}^{-1}(Y(s)) \\
&= \mathcal{L}^{-1}\left(-\frac{e^{-s\pi}-1}{s^2+4}\right) \\
&= \frac{\sin(2x)\,\text{Heaviside}(\pi - x)}{2}
\end{aligned}
$$

Hence the final solution is

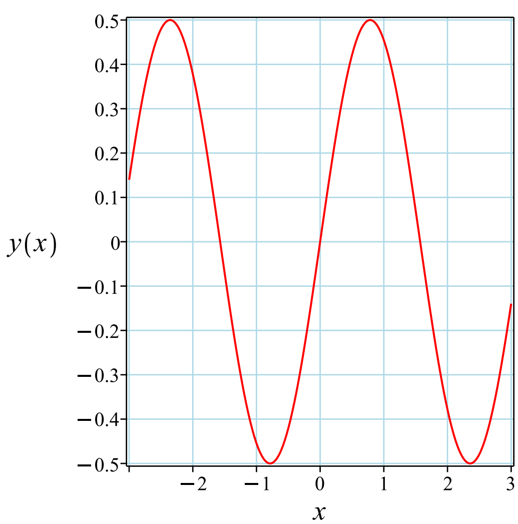
$$
y = \frac{\sin(2x)\,\text{Heaviside}(\pi - x)}{2}
$$

Simplifying the solution gives

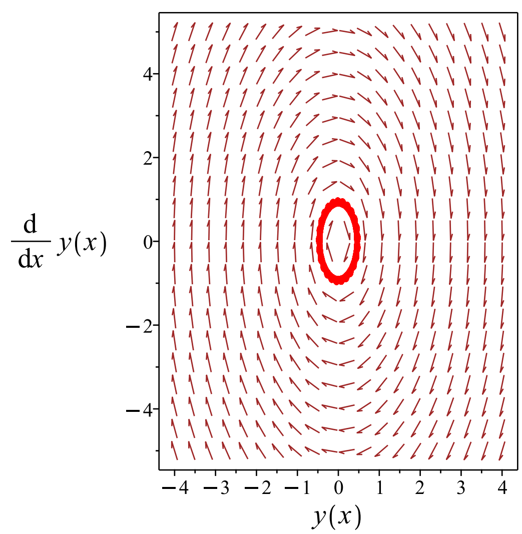
$$
y = -\frac{\sin(2x)\,(-1+\text{Heaviside}(x-\pi))}{2}
$$

Summary
The solution(s) found are the following

$$
y = -\frac{\sin(2x)\,(-1+\text{Heaviside}(x-\pi))}{2} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = -\frac{\sin(2x)\,(-1+\text{Heaviside}(x-\pi))}{2}
$$

Verified OK.

16.6.2 Maple step by step solution

Let's solve

$\left[y'' + 4y = -Dirac(x - \pi), y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -Dirac(x - \pi)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  ◦ Substitute functions into equation for $y_p(x)$

$y_p(x) = -\frac{\sin(2x)\left(\int Dirac(x-\pi)dx\right)}{2}$

- Compute integrals

  $y_p(x) = -\frac{\sin(2x) Heaviside(x-\pi)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) - \frac{\sin(2x) Heaviside(x-\pi)}{2}$

□ Check validity of solution $y = c_1 \cos(2x) + c_2 \sin(2x) - \frac{\sin(2x) Heaviside(x-\pi)}{2}$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Compute derivative of the solution

  $y' = -2c_1 \sin(2x) + 2c_2 \cos(2x) - \cos(2x) Heaviside(x-\pi) - \frac{\sin(2x) Dirac(x-\pi)}{2}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = 0, c_2 = \frac{1}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{\sin(2x)(-1+Heaviside(x-\pi))}{2}$

- Solution to the IVP

  $y = -\frac{\sin(2x)(-1+Heaviside(x-\pi))}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 5.968 (sec). Leaf size: 16

```
dsolve([diff(y(x),x$2)+4*y(x)=cos(x)*Dirac(x-Pi),y(0) = 0, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{\sin(2x)\,(-1 + \text{Heaviside}\,(x - \pi))}{2}$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 19

```
DSolve[{y''[x]+4*y[x]==Cos[x]*DiracDelta[x-Pi],{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSol
```

$$y(x) \to (\theta(x - \pi) - 1)\sin(x)(-\cos(x))$$

16.7 problem 7

Internal problem ID [12820]
Internal file name [`OUTPUT/11472_Saturday_November_04_2023_08_47_31_AM_24458766/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 5. The Laplace Transform Method. Exercises 5.5, page 273
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_laplace", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y a^2 = \delta(x-\pi)\, f(x)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

16.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= a^2 \\ F &= f(\pi)\, \delta(x-\pi) \end{aligned}$$

Hence the ode is

$$y'' + y a^2 = f(\pi)\, \delta(x-\pi)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = a^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = f(\pi)\,\delta(x - \pi)$ is

$$\{x < \pi \lor \pi < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned}\mathcal{L}(y') &= sY(s) - y(0)\\ \mathcal{L}(y'') &= s^2Y(s) - y'(0) - sy(0)\end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain

$$s^2Y(s) - y'(0) - sy(0) + a^2Y(s) = f(\pi)\,\mathrm{e}^{-s\pi} \tag{1}$$

But the initial conditions are

$$\begin{aligned}y(0) &= 0\\ y'(0) &= 0\end{aligned}$$

Substituting these initial conditions in above in Eq (1) gives

$$s^2Y(s) + a^2Y(s) = f(\pi)\,\mathrm{e}^{-s\pi}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{f(\pi)\,\mathrm{e}^{-s\pi}}{a^2 + s^2}$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y &= \mathcal{L}^{-1}(Y(s)) \\ &= \mathcal{L}^{-1}\left(\frac{f(\pi)\,\mathrm{e}^{-s\pi}}{a^2+s^2}\right) \\ &= \frac{\mathrm{Heaviside}\,(x-\pi)\, f(\pi) \sin\,(a(x-\pi))}{a} \end{aligned}$$

Hence the final solution is

$$y = \frac{\mathrm{Heaviside}\,(x-\pi)\, f(\pi) \sin\,(a(x-\pi))}{a}$$

Simplifying the solution gives

$$y = \frac{\mathrm{Heaviside}\,(x-\pi)\, f(\pi) \sin\,(a(x-\pi))}{a}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{Heaviside}\,(x-\pi)\, f(\pi) \sin\,(a(x-\pi))}{a} \tag{1}$$

Verification of solutions

$$y = \frac{\mathrm{Heaviside}\,(x-\pi)\, f(\pi) \sin\,(a(x-\pi))}{a}$$

Verified OK.

16.7.2 Maple step by step solution

Let's solve

$$\left[y'' + y a^2 = f(\pi)\, \mathit{Dirac}(x-\pi)\,, y(0) = 0, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $a^2 + r^2 = 0$
- Use quadratic formula to solve for r

$r=\frac{0\pm\left(\sqrt{-4a^2}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(\sqrt{-a^2}, -\sqrt{-a^2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{\sqrt{-a^2}\,x}$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\mathrm{e}^{-\sqrt{-a^2}\,x}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{\sqrt{-a^2}\,x}+c_2\mathrm{e}^{-\sqrt{-a^2}\,x}+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=f(\pi)\,Dirac(x-\pi)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x))=\begin{bmatrix}\mathrm{e}^{\sqrt{-a^2}\,x} & \mathrm{e}^{-\sqrt{-a^2}\,x}\\ \sqrt{-a^2}\,\mathrm{e}^{\sqrt{-a^2}\,x} & -\sqrt{-a^2}\,\mathrm{e}^{-\sqrt{-a^2}\,x}\end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x))=-2\sqrt{-a^2}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x)=\frac{f(\pi)\left(\int Dirac(x-\pi)dx\right)\left(-\mathrm{e}^{\sqrt{-a^2}\,(\pi-x)}+\mathrm{e}^{(x-\pi)\sqrt{-a^2}}\right)}{2\sqrt{-a^2}}$

  - Compute integrals

    $y_p(x)=\frac{f(\pi)Heaviside(x-\pi)\left(-\mathrm{e}^{\sqrt{-a^2}\,(\pi-x)}+\mathrm{e}^{(x-\pi)\sqrt{-a^2}}\right)}{2\sqrt{-a^2}}$

- Substitute particular solution into general solution to ODE

  $y=c_1\mathrm{e}^{\sqrt{-a^2}\,x}+c_2\mathrm{e}^{-\sqrt{-a^2}\,x}+\frac{f(\pi)Heaviside(x-\pi)\left(-\mathrm{e}^{\sqrt{-a^2}\,(\pi-x)}+\mathrm{e}^{(x-\pi)\sqrt{-a^2}}\right)}{2\sqrt{-a^2}}$

□ Check validity of solution $y=c_1\mathrm{e}^{\sqrt{-a^2}\,x}+c_2\mathrm{e}^{-\sqrt{-a^2}\,x}+\frac{f(\pi)Heaviside(x-\pi)\left(-\mathrm{e}^{\sqrt{-a^2}\,(\pi-x)}+\mathrm{e}^{(x-\pi)\sqrt{-a^2}}\right)}{2\sqrt{-a^2}}$

  - Use initial condition $y(0)=0$

$0 = c_1 + c_2$

- Compute derivative of the solution

$$y' = c_1\sqrt{-a^2}\, e^{\sqrt{-a^2}\, x} - c_2\sqrt{-a^2}\, e^{-\sqrt{-a^2}\, x} + \frac{f(\pi) Dirac(x-\pi)\left(-e^{\sqrt{-a^2}\,(\pi-x)}+e^{(x-\pi)\sqrt{-a^2}}\right)}{2\sqrt{-a^2}} + \frac{f(\pi) Heaviside(x-\pi)}{}$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

$0 = c_1\sqrt{-a^2} - c_2\sqrt{-a^2}$

- Solve for c_1 and c_2

$\{c_1 = 0, c_2 = 0\}$

- Substitute constant values into general solution and simplify

$$y = \frac{f(\pi) Heaviside(x-\pi)\left(-e^{\sqrt{-a^2}\,(\pi-x)}+e^{(x-\pi)\sqrt{-a^2}}\right)}{2\sqrt{-a^2}}$$

- Solution to the IVP

$$y = \frac{f(\pi) Heaviside(x-\pi)\left(-e^{\sqrt{-a^2}\,(\pi-x)}+e^{(x-\pi)\sqrt{-a^2}}\right)}{2\sqrt{-a^2}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 6.188 (sec). Leaf size: 24

```
dsolve([diff(y(x),x$2)+a^2*y(x)=Dirac(x-Pi)*f(x),y(0) = 0, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\text{Heaviside}\,(x-\pi)\sin\left(a(x-\pi)\right) f(\pi)}{a}$$

✓ Solution by Mathematica

Time used: 0.398 (sec). Leaf size: 26

```
DSolve[{y''[x]+a^2*y[x]==DiracDelta[x-Pi]*f[x],{y[0]==0,y'[0]==0}},y[x],x,IncludeSingularSol
```

$$y(x) \to -\frac{f(\pi)\theta(x-\pi)\sin(a(\pi-x))}{a}$$

17 Chapter 7. Systems of First-Order Differential Equations. Exercises page 329

17.1 problem 1

Internal problem ID [12821]
Internal file name [`OUTPUT/11473_Saturday_November_04_2023_08_47_31_AM_25985763/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 7. Systems of First-Order Differential Equations. Exercises page 329
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= 2y_1(x) - 3y_2(x)\\
y_2'(x) &= y_1(x) - 2y_2(x)
\end{aligned}
$$

17.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{-x}}{2} + \frac{3\,e^{x}}{2} & \frac{3\,e^{-x}}{2} - \frac{3\,e^{x}}{2} \\ -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{3\,e^{-x}}{2} - \frac{e^{x}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(x) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{e^{-x}}{2}+\frac{3e^{x}}{2} & \frac{3e^{-x}}{2}-\frac{3e^{x}}{2} \\ -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{3e^{-x}}{2}-\frac{e^{x}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\frac{e^{-x}}{2}+\frac{3e^{x}}{2}\right)c_1+\left(\frac{3e^{-x}}{2}-\frac{3e^{x}}{2}\right)c_2 \\ \left(-\frac{e^{-x}}{2}+\frac{e^{x}}{2}\right)c_1+\left(\frac{3e^{-x}}{2}-\frac{e^{x}}{2}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(3c_2-c_1)e^{-x}}{2}+\frac{3e^{x}(c_1-c_2)}{2} \\ \frac{(3c_2-c_1)e^{-x}}{2}+\frac{e^{x}(c_1-c_2)}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

17.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 1 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(x) = \vec{v}_1 e^{-x}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-x}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(x) = \vec{v}_2 e^{x}$$
$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{x}$$

Therefore the final solution is

$$\vec{x}_h(x) = c_1 \vec{x}_1(x) + c_2 \vec{x}_2(x)$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-x} \\ \mathrm{e}^{-x} \end{bmatrix} + c_2 \begin{bmatrix} 3\,\mathrm{e}^{x} \\ \mathrm{e}^{x} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{-x} + 3c_2 \mathrm{e}^{x} \\ c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} \end{bmatrix}$$

The following is the phase plot of the system.

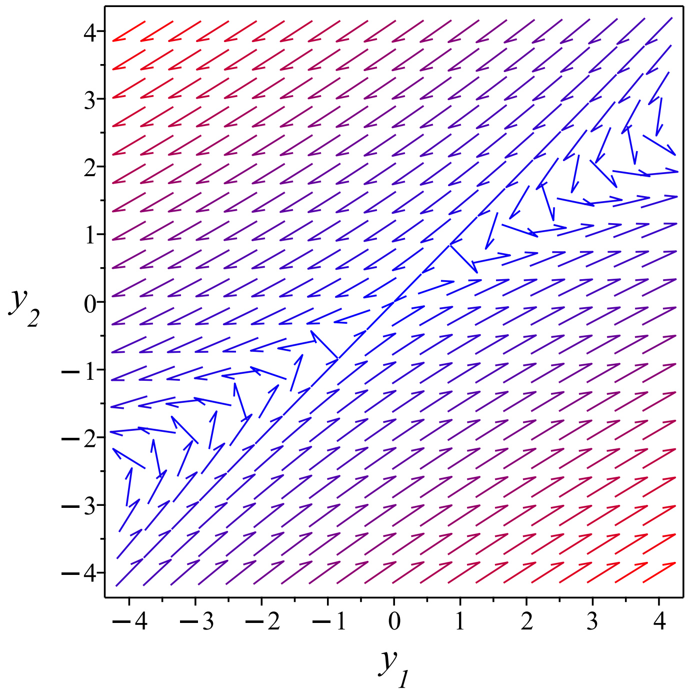


Figure 425: Phase plot

17.1.3 Maple step by step solution

Let's solve

$[y_1'(x) = 2y_1(x) - 3y_2(x), y_2'(x) = y_1(x) - 2y_2(x)]$

- Define vector

$$\underline{y}^{\rightarrow}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(x) = A \cdot \underline{\vec{y}}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_2 = \mathrm{e}^{x} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1 + c_2 \underline{\vec{y}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{x} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} c_1 e^{-x} + 3c_2 e^{x} \\ c_1 e^{-x} + c_2 e^{x} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\{y_1(x) = c_1 e^{-x} + 3c_2 e^{x}, y_2(x) = c_1 e^{-x} + c_2 e^{x}\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 31

```
dsolve([diff(y__1(x),x)=2*y__1(x)-3*y__2(x),diff(y__2(x),x)=y__1(x)-2*y__2(x)],singsol=all)
```

$$y_1(x) = c_1 e^{x} + c_2 e^{-x}$$
$$y_2(x) = \frac{c_1 e^{x}}{3} + c_2 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 72

```
DSolve[{y1'[x]==2*y1[x]-3*y2[x],y2'[x]==y1[x]-2*y2[x]},{y1[x],y2[x]},x,IncludeSingularSoluti
```

$$\text{y1}(x) \to \frac{1}{2} e^{-x} \left(c_1 \left(3e^{2x} - 1\right) - 3c_2 \left(e^{2x} - 1\right)\right)$$
$$\text{y2}(x) \to \frac{1}{2} e^{-x} \left(c_1 \left(e^{2x} - 1\right) - c_2 \left(e^{2x} - 3\right)\right)$$

17.2 problem 3

Internal problem ID [12822]
Internal file name [`OUTPUT/11474_Saturday_November_04_2023_08_47_31_AM_27779913/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 7. Systems of First-Order Differential Equations. Exercises page 329
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= y_1(x) - 2y_2(x)\\
y_2'(x) &= y_1(x) + 3y_2(x)
\end{aligned}
$$

17.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(x) = A\,\vec{x}(x)
$$

Or

$$
\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} \mathrm{e}^{2x}\cos(x) - \mathrm{e}^{2x}\sin(x) & -2\,\mathrm{e}^{2x}\sin(x) \\ \mathrm{e}^{2x}\sin(x) & \mathrm{e}^{2x}\cos(x) + \mathrm{e}^{2x}\sin(x) \end{bmatrix}\\
&= \begin{bmatrix} \mathrm{e}^{2x}(\cos(x) - \sin(x)) & -2\,\mathrm{e}^{2x}\sin(x) \\ \mathrm{e}^{2x}\sin(x) & \mathrm{e}^{2x}(\cos(x) + \sin(x)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(x) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2x}(\cos(x) - \sin(x)) & -2\,e^{2x}\sin(x) \\ e^{2x}\sin(x) & e^{2x}(\cos(x) + \sin(x)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2x}(\cos(x) - \sin(x))\,c_1 - 2\,e^{2x}\sin(x)\,c_2 \\ e^{2x}\sin(x)\,c_1 + e^{2x}(\cos(x) + \sin(x))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((-c_1 - 2c_2)\sin(x) + c_1\cos(x))\,e^{2x} \\ e^{2x}((c_1 + c_2)\sin(x) + c_2\cos(x)) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

17.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2 + i$$
$$\lambda_2 = 2 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2 + i$	1	complex eigenvalue
$2 - i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} - (2 - i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 + i & -2 \\ 1 & 1 + i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 + i & -2 & 0 \\ 1 & 1 + i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{2} + \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 + i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 + i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (-1 - i)\, t\}$

Hence the solution is

$$\begin{bmatrix} (-1 - \mathrm{I})\, t \\ t \end{bmatrix} = \begin{bmatrix} (-1 - i)\, t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (-1 - \mathrm{I})\, t \\ t \end{bmatrix} = t \begin{bmatrix} -1 - i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (-1 - \mathrm{I})\, t \\ t \end{bmatrix} = \begin{bmatrix} -1 - i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} - (2 + i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 - i & -2 \\ 1 & 1 - i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 - i & -2 & 0 \\ 1 & 1 - i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{2} - \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 - i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 - i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (-1+i)\,t\}$

Hence the solution is

$$\begin{bmatrix} (-1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} (-1+i)\,t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (-1+\mathrm{I})\,t \\ t \end{bmatrix} = t \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (-1+\mathrm{I})\,t \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2+i$	1	1	No	$\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$
$2-i$	1	1	No	$\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(x) = c_1\vec{x}_1(x) + c_2\vec{x}_2(x)$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = c_1 \begin{bmatrix} (-1+i)\, \mathrm{e}^{(2+i)x} \\ \mathrm{e}^{(2+i)x} \end{bmatrix} + c_2 \begin{bmatrix} (-1-i)\, \mathrm{e}^{(2-i)x} \\ \mathrm{e}^{(2-i)x} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} (-1+i)\, c_1 \mathrm{e}^{(2+i)x} + (-1-i)\, c_2 \mathrm{e}^{(2-i)x} \\ c_1 \mathrm{e}^{(2+i)x} + c_2 \mathrm{e}^{(2-i)x} \end{bmatrix}$$

The following is the phase plot of the system.

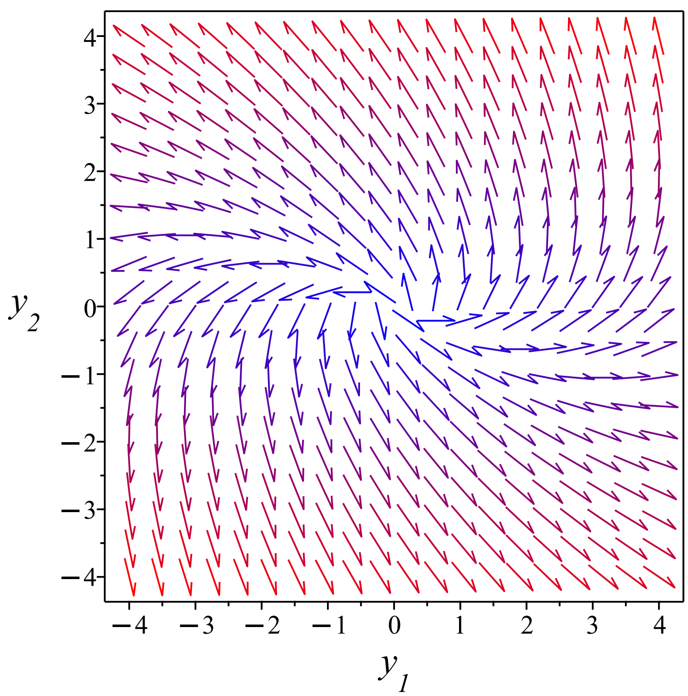


Figure 426: Phase plot

17.2.3 Maple step by step solution

Let's solve

$[y_1'(x) = y_1(x) - 2y_2(x)\, , y_2'(x) = y_1(x) + 3y_2(x)]$

- Define vector

$$\underline{\vec{y}}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{y}}'(x) = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \cdot \underline{\vec{y}}(x) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{y}}'(x) = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \cdot \underline{\vec{y}}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(x) = A \cdot \underline{\vec{y}}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2 - \mathrm{I}, \begin{bmatrix} -1 - \mathrm{I} \\ 1 \end{bmatrix}\right], \left[2 + \mathrm{I}, \begin{bmatrix} -1 + \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2 - \mathrm{I}, \begin{bmatrix} -1 - \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(2-\mathrm{I})x} \cdot \begin{bmatrix} -1 - \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{2x} \cdot (\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -1 - \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{2x} \cdot \begin{bmatrix} (-1-\mathrm{I})(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{y}}_1(x) = e^{2x} \cdot \begin{bmatrix} -\cos(x) - \sin(x) \\ \cos(x) \end{bmatrix}, \underline{\vec{y}}_2(x) = e^{2x} \cdot \begin{bmatrix} -\cos(x) + \sin(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1(x) + c_2 \underline{\vec{y}}_2(x)$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 e^{2x} \cdot \begin{bmatrix} -\cos(x) - \sin(x) \\ \cos(x) \end{bmatrix} + c_2 e^{2x} \cdot \begin{bmatrix} -\cos(x) + \sin(x) \\ -\sin(x) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -e^{2x}(\cos(x)(c_1+c_2) + \sin(x)(c_1-c_2)) \\ e^{2x}(c_1\cos(x) - c_2\sin(x)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{y_1(x) = -e^{2x}(\cos(x)(c_1+c_2) + \sin(x)(c_1-c_2)), y_2(x) = e^{2x}(c_1\cos(x) - c_2\sin(x))\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 46

```
dsolve([diff(y__1(x),x)=y__1(x)-2*y__2(x),diff(y__2(x),x)=y__1(x)+3*y__2(x)],singsol=all)
```

$$y_1(x) = e^{2x}(\sin(x)c_1 + \cos(x)c_2)$$
$$y_2(x) = -\frac{e^{2x}(\sin(x)c_1 - \sin(x)c_2 + \cos(x)c_1 + \cos(x)c_2)}{2}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 51

```
DSolve[{y1'[x]==y1[x]-2*y2[x],y2'[x]==y1[x]+3*y2[x]},{y1[x],y2[x]},x,IncludeSingularSolution
```

$$\mathrm{y1}(x) \to e^{2x}(c_1\cos(x) - (c_1+2c_2)\sin(x))$$
$$\mathrm{y2}(x) \to e^{2x}(c_2\cos(x) + (c_1+c_2)\sin(x))$$

17.3 problem 4

Internal problem ID [12823]
Internal file name [`OUTPUT/11475_Saturday_November_04_2023_08_47_32_AM_57192821/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 7. Systems of First-Order Differential Equations. Exercises page 329
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= y_1(x) + 2y_2(x) + x - 1 \\
y_2'(x) &= 3y_1(x) + 2y_2(x) - 5x - 2
\end{aligned}
$$

With initial conditions

$$
[y_1(0) = -2, y_2(0) = 3]
$$

17.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)
$$

Or

$$
\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} + \begin{bmatrix} x - 1 \\ -5x - 2 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(x) = \vec{x}_h(x) + \vec{x}_p(x)
$$

Where $\vec{x}_h(x)$ is the homogeneous solution to $\vec{x}'(x) = A\,\vec{x}(x)$ and $\vec{x}_p(x)$ is a particular solution to $\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{3\,e^{-x}}{5} + \frac{2\,e^{4x}}{5} & \frac{2\,e^{4x}}{5} - \frac{2\,e^{-x}}{5} \\ \frac{3\,e^{4x}}{5} - \frac{3\,e^{-x}}{5} & \frac{2\,e^{-x}}{5} + \frac{3\,e^{4x}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(x) &= e^{Ax}\vec{x}_0 \\ &= \begin{bmatrix} \frac{3\,e^{-x}}{5} + \frac{2\,e^{4x}}{5} & \frac{2\,e^{4x}}{5} - \frac{2\,e^{-x}}{5} \\ \frac{3\,e^{4x}}{5} - \frac{3\,e^{-x}}{5} & \frac{2\,e^{-x}}{5} + \frac{3\,e^{4x}}{5} \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{12\,e^{-x}}{5} + \frac{2\,e^{4x}}{5} \\ \frac{3\,e^{4x}}{5} + \frac{12\,e^{-x}}{5} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(x) = e^{Ax} \int e^{-Ax} \vec{G}(x)\,dx$$

But

$$\begin{aligned} e^{-Ax} &= (e^{Ax})^{-1} \\ &= \begin{bmatrix} \frac{(3\,e^{5x}+2)\,e^{-4x}}{5} & -\frac{2(e^{5x}-1)\,e^{-4x}}{5} \\ -\frac{3(e^{5x}-1)\,e^{-4x}}{5} & \frac{(2\,e^{5x}+3)\,e^{-4x}}{5} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(x) &= \begin{bmatrix} \frac{3\,e^{-x}}{5} + \frac{2\,e^{4x}}{5} & \frac{2\,e^{4x}}{5} - \frac{2\,e^{-x}}{5} \\ \frac{3\,e^{4x}}{5} - \frac{3\,e^{-x}}{5} & \frac{2\,e^{-x}}{5} + \frac{3\,e^{4x}}{5} \end{bmatrix} \int \begin{bmatrix} \frac{(3\,e^{5x}+2)\,e^{-4x}}{5} & -\frac{2(e^{5x}-1)\,e^{-4x}}{5} \\ -\frac{3(e^{5x}-1)\,e^{-4x}}{5} & \frac{(2\,e^{5x}+3)\,e^{-4x}}{5} \end{bmatrix} \begin{bmatrix} x-1 \\ -5x-2 \end{bmatrix} dx \\ &= \begin{bmatrix} \frac{3\,e^{-x}}{5} + \frac{2\,e^{4x}}{5} & \frac{2\,e^{4x}}{5} - \frac{2\,e^{-x}}{5} \\ \frac{3\,e^{4x}}{5} - \frac{3\,e^{-x}}{5} & \frac{2\,e^{-x}}{5} + \frac{3\,e^{4x}}{5} \end{bmatrix} \begin{bmatrix} \frac{(13x\,e^{5x}-12\,e^{5x}+2x+2)\,e^{-4x}}{5} \\ -\frac{(13x\,e^{5x}-12\,e^{5x}-3x-3)\,e^{-4x}}{5} \end{bmatrix} \\ &= \begin{bmatrix} 3x-2 \\ -2x+3 \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(x) &= \vec{x}_h(x) + \vec{x}_p(x) \\ &= \begin{bmatrix} -\frac{12\,e^{-x}}{5} + \frac{2\,e^{4x}}{5} + 3x - 2 \\ \frac{3\,e^{4x}}{5} + \frac{12\,e^{-x}}{5} - 2x + 3 \end{bmatrix} \end{aligned}$$

17.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} + \begin{bmatrix} x-1 \\ -5x-2 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(x) = \vec{x}_h(x) + \vec{x}_p(x)$$

Where $\vec{x}_h(x)$ is the homogeneous solution to $\vec{x}'(x) = A\,\vec{x}(x)$ and $\vec{x}_p(x)$ is a particular solution to $\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3\lambda - 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = t\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
4	1	1	No	$\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(x) &= \vec{v}_1 e^{4x} \\ &= \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} e^{4x} \end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(x) &= \vec{v}_2 e^{-x} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-x} \end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(x) = c_1 \vec{x}_1(x) + c_2 \vec{x}_2(x)$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = c_1 \begin{bmatrix} \frac{2\,\mathrm{e}^{4x}}{3} \\ \mathrm{e}^{4x} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-x} \\ \mathrm{e}^{-x} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(x)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(x) = \begin{bmatrix} \frac{2\,\mathrm{e}^{4x}}{3} & -\mathrm{e}^{-x} \\ \mathrm{e}^{4x} & \mathrm{e}^{-x} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(x) = \Phi \int \Phi^{-1} \vec{G}(x)\, dx$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{3\,\mathrm{e}^{-4x}}{5} & \frac{3\,\mathrm{e}^{-4x}}{5} \\ -\frac{3\,\mathrm{e}^{x}}{5} & \frac{2\,\mathrm{e}^{x}}{5} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(x) &= \begin{bmatrix} \frac{2\,\mathrm{e}^{4x}}{3} & -\mathrm{e}^{-x} \\ \mathrm{e}^{4x} & \mathrm{e}^{-x} \end{bmatrix} \int \begin{bmatrix} \frac{3\,\mathrm{e}^{-4x}}{5} & \frac{3\,\mathrm{e}^{-4x}}{5} \\ -\frac{3\,\mathrm{e}^{x}}{5} & \frac{2\,\mathrm{e}^{x}}{5} \end{bmatrix} \begin{bmatrix} x-1 \\ -5x-2 \end{bmatrix} dx \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{4x}}{3} & -\mathrm{e}^{-x} \\ \mathrm{e}^{4x} & \mathrm{e}^{-x} \end{bmatrix} \int \begin{bmatrix} -\frac{3\,\mathrm{e}^{-4x}(4x+3)}{5} \\ -\frac{\mathrm{e}^{x}(13x+1)}{5} \end{bmatrix} dx \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{4x}}{3} & -\mathrm{e}^{-x} \\ \mathrm{e}^{4x} & \mathrm{e}^{-x} \end{bmatrix} \begin{bmatrix} \frac{3\,\mathrm{e}^{-4x}(x+1)}{5} \\ -\frac{\mathrm{e}^{x}(13x-12)}{5} \end{bmatrix} \\
&= \begin{bmatrix} 3x-2 \\ -2x+3 \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(x) = \vec{x}_h(x) + \vec{x}_p(x)$$
$$\left[\begin{array}{c} y_1(x) \\ y_2(x) \end{array}\right] = \left[\begin{array}{c} \frac{2c_1 e^{4x}}{3} \\ c_1 e^{4x} \end{array}\right] + \left[\begin{array}{c} -c_2 e^{-x} \\ c_2 e^{-x} \end{array}\right] + \left[\begin{array}{c} 3x-2 \\ -2x+3 \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} y_1(x) \\ y_2(x) \end{array}\right] = \left[\begin{array}{c} \frac{2c_1 e^{4x}}{3} - c_2 e^{-x} + 3x - 2 \\ c_1 e^{4x} + c_2 e^{-x} - 2x + 3 \end{array}\right]$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\left[\begin{array}{c} y_1(0) = -2 \\ y_2(0) = 3 \end{array}\right] \tag{1}$$

Substituting initial conditions into the above solution at $x = 0$ gives

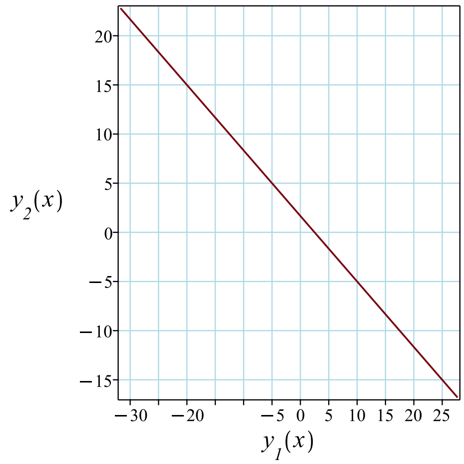
$$\left[\begin{array}{c} -2 \\ 3 \end{array}\right] = \left[\begin{array}{c} \frac{2c_1}{3} - c_2 - 2 \\ c_1 + c_2 + 3 \end{array}\right]$$

Solving for the constants of integrations gives

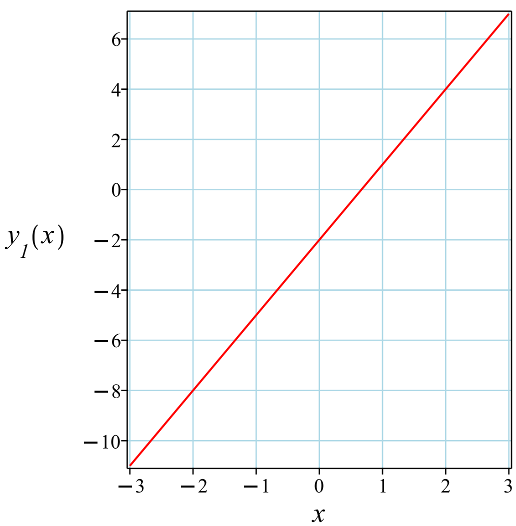
$$\left[\begin{array}{c} c_1 = 0 \\ c_2 = 0 \end{array}\right]$$

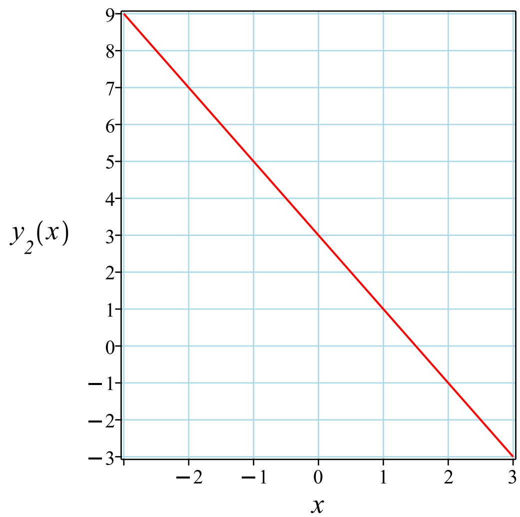
Substituting these constants back in original solution in Eq. (1) gives

$$\left[\begin{array}{c} y_1(x) \\ y_2(x) \end{array}\right] = \left[\begin{array}{c} 3x-2 \\ -2x+3 \end{array}\right]$$



The following are plots of each solution.





✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve([diff(y__1(x),x) = y__1(x)+2*y__2(x)+x-1, diff(y__2(x),x) = 3*y__1(x)+2*y__2(x)-5*x-2
```

$$y_1(x) = -2 + 3x$$
$$y_2(x) = 3 - 2x$$

✓ Solution by Mathematica

Time used: 0.316 (sec). Leaf size: 18

```
DSolve[{y1'[x]==y1[x]+2*y2[x]+x-1,y2'[x]==3*y1[x]+2*y2[x]-5*x-2},{y1[0]==-2,y2[0]==3},{y1[x]
```

$$\begin{aligned} \text{y1}(x) &\to 3x-2 \\ \text{y2}(x) &\to 3-2x \end{aligned}$$

17.4 problem 5

Internal problem ID [12824]
Internal file name [`OUTPUT/11476_Saturday_November_04_2023_08_47_32_AM_68493840/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 7. Systems of First-Order Differential Equations. Exercises page 329
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$y_1'(x) = \frac{2y_1(x)}{x} - \frac{y_2(x)}{x^2} - 3 + \frac{1}{x} - \frac{1}{x^2}$$
$$y_2'(x) = 2y_1(x) + 1 - 6x$$

With initial conditions

$$[y_1(1) = -2, y_2(1) = -5]$$

Does not currently support non autonomous system of first order linear differential equations. The following is the phase plot

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 20

```
dsolve([diff(y__1(x),x) = 2*y__1(x)/x-y__2(x)/x^2-3+1/x-1/x^2, diff(y__2(x),x) = 2*y__1(x)+1
```

$$y_1(x) = -2x$$
$$y_2(x) = -1 + x(-5x + 1)$$

✓ Solution by Mathematica
Time used: 0.012 (sec). Leaf size: 19

```
DSolve[{y1'[x]==2*y1[x]/x-y2[x]/x^2-3+1/x-1/x^2,y2'[x]==2*y1[x]+1-6*x},{y1[1]==-2,y2[1]==-5}
```

$$\text{y1}(x) \to -2x$$
$$\text{y2}(x) \to -5x^2 + x - 1$$

17.5 problem 6

Internal problem ID [12825]
Internal file name [`OUTPUT/11477_Saturday_November_04_2023_08_47_32_AM_78012659/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 7. Systems of First-Order Differential Equations. Exercises page 329
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$
\begin{aligned}
y_1'(x) &= \frac{5y_1(x)}{x} + \frac{4y_2(x)}{x} - 2x \\
y_2'(x) &= -\frac{6y_1(x)}{x} - \frac{5y_2(x)}{x} + 5x
\end{aligned}
$$

With initial conditions

$$[y_1(-1) = 3, y_2(-1) = -3]$$

Does not currently support non autonomous system of first order linear differential equations. The following is the phase plot

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
dsolve([diff(y__1(x),x) = 5*y__1(x)/x+4*y__2(x)/x-2*x, diff(y__2(x),x) = -6*y__1(x)/x-5*y__2
```

$$
\begin{aligned}
y_1(x) &= \frac{2x^3 + x^2 - 2}{x} \\
y_2(x) &= -\frac{2x^3 + 2x^2 - 6}{2x}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 33

```
DSolve[{y1'[x]==5*y1[x]/x+4*y2[x]/x-2*x,y2'[x]==-6*y1[x]/x-5*y2[x]/x+5*x},{y1[-1]==3,y2[-1]=
```

$$\mathrm{y1}(x)\to 2x^2+x-\frac{2}{x}$$
$$\mathrm{y2}(x)\to -\frac{x^3+x^2-3}{x}$$

17.6 problem 13 (a)

Internal problem ID [12826]
Internal file name [OUTPUT/11478_Saturday_November_04_2023_08_47_33_AM_4687413/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 7. Systems of First-Order Differential Equations. Exercises page 329
Problem number: 13 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$y_1'(x) = 3y_1(x) - 2y_2(x)$$
$$y_2'(x) = -y_1(x) + y_2(x)$$

With initial conditions

$$[y_1(0) = 1, y_2(0) = -1]$$

17.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(-\sqrt{3}+3\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)x}}{6} + \frac{\mathrm{e}^{\left(2+\sqrt{3}\right)x}\left(\sqrt{3}+3\right)}{6} & \frac{\left(-\mathrm{e}^{\left(2+\sqrt{3}\right)x}+\mathrm{e}^{-\left(\sqrt{3}-2\right)x}\right)\sqrt{3}}{3} \\ \frac{\left(-\mathrm{e}^{\left(2+\sqrt{3}\right)x}+\mathrm{e}^{-\left(\sqrt{3}-2\right)x}\right)\sqrt{3}}{6} & \frac{\left(\sqrt{3}+3\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)x}}{6} - \frac{\mathrm{e}^{\left(2+\sqrt{3}\right)x}\left(\sqrt{3}-3\right)}{6} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(x) = e^{Ax}\vec{x}_0$$

$$= \begin{bmatrix} \frac{\left(-\sqrt{3}+3\right)e^{-\left(\sqrt{3}-2\right)x}}{6} + \frac{e^{\left(2+\sqrt{3}\right)x}\left(\sqrt{3}+3\right)}{6} & \frac{\left(-e^{\left(2+\sqrt{3}\right)x}+e^{-\left(\sqrt{3}-2\right)x}\right)\sqrt{3}}{3} \\ \frac{\left(-e^{\left(2+\sqrt{3}\right)x}+e^{-\left(\sqrt{3}-2\right)x}\right)\sqrt{3}}{6} & \frac{\left(\sqrt{3}+3\right)e^{-\left(\sqrt{3}-2\right)x}}{6} - \frac{e^{\left(2+\sqrt{3}\right)x}\left(\sqrt{3}-3\right)}{6} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left(-\sqrt{3}+3\right)e^{-\left(\sqrt{3}-2\right)x}}{6} + \frac{e^{\left(2+\sqrt{3}\right)x}\left(\sqrt{3}+3\right)}{6} - \frac{\left(-e^{\left(2+\sqrt{3}\right)x}+e^{-\left(\sqrt{3}-2\right)x}\right)\sqrt{3}}{3} \\ \frac{\left(-e^{\left(2+\sqrt{3}\right)x}+e^{-\left(\sqrt{3}-2\right)x}\right)\sqrt{3}}{6} - \frac{\left(\sqrt{3}+3\right)e^{-\left(\sqrt{3}-2\right)x}}{6} + \frac{e^{\left(2+\sqrt{3}\right)x}\left(\sqrt{3}-3\right)}{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\left(1-\sqrt{3}\right)e^{-\left(\sqrt{3}-2\right)x}}{2} + \frac{e^{\left(2+\sqrt{3}\right)x}\left(1+\sqrt{3}\right)}{2} \\ -\frac{e^{-\left(\sqrt{3}-2\right)x}}{2} - \frac{e^{\left(2+\sqrt{3}\right)x}}{2} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

17.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -2 \\ -1 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2 + \sqrt{3}$$
$$\lambda_2 = 2 - \sqrt{3}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2 - \sqrt{3}$	1	real eigenvalue
$2 + \sqrt{3}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2 - \sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} - \left(2 - \sqrt{3}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 + \sqrt{3} & -2 \\ -1 & \sqrt{3} - 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 + \sqrt{3} & -2 & 0 \\ -1 & \sqrt{3} - 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{1 + \sqrt{3}} \Longrightarrow \left[\begin{array}{cc|c} 1 + \sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 + \sqrt{3} & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{1+\sqrt{3}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{1+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{1+\sqrt{3}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{1+\sqrt{3}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{1+\sqrt{3}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{1+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{1+\sqrt{3}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{1+\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{1+\sqrt{3}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 + \sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} - \left(2 + \sqrt{3}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 - \sqrt{3} & -2 \\ -1 & -1 - \sqrt{3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 - \sqrt{3} & -2 & 0 \\ -1 & -1 - \sqrt{3} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{1 - \sqrt{3}} \Longrightarrow \left[\begin{array}{cc|c} 1 - \sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-\sqrt{3} & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{\sqrt{3}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{\sqrt{3}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{\sqrt{3}-1} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{\sqrt{3}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{3}-1} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2+\sqrt{3}$	1	1	No	$\begin{bmatrix} -\frac{2}{\sqrt{3}-1} \\ 1 \end{bmatrix}$
$2-\sqrt{3}$	1	1	No	$\begin{bmatrix} -\frac{2}{-1-\sqrt{3}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $2+\sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(x) &= \vec{v}_1 e^{\left(2+\sqrt{3}\right)x} \\
&= \begin{bmatrix} -\frac{2}{\sqrt{3}-1} \\ 1 \end{bmatrix} e^{\left(2+\sqrt{3}\right)x}
\end{aligned}
$$

Since eigenvalue $2-\sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(x) &= \vec{v}_2 e^{\left(2-\sqrt{3}\right)x} \\
&= \begin{bmatrix} -\frac{2}{-1-\sqrt{3}} \\ 1 \end{bmatrix} e^{\left(2-\sqrt{3}\right)x}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(x) = c_1\vec{x}_1(x) + c_2\vec{x}_2(x)
$$

Which is written as

$$
\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{2\,\mathrm{e}^{\left(2+\sqrt{3}\right)x}}{\sqrt{3}-1} \\ \mathrm{e}^{\left(2+\sqrt{3}\right)x} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{2\,\mathrm{e}^{\left(2-\sqrt{3}\right)x}}{-1-\sqrt{3}} \\ \mathrm{e}^{\left(2-\sqrt{3}\right)x} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} c_2\left(\sqrt{3}-1\right)\mathrm{e}^{-\left(\sqrt{3}-2\right)x} - c_1\mathrm{e}^{\left(2+\sqrt{3}\right)x}\left(1+\sqrt{3}\right) \\ c_1\mathrm{e}^{\left(2+\sqrt{3}\right)x} + c_2\mathrm{e}^{-\left(\sqrt{3}-2\right)x} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} y_1(0) = 1 \\ y_2(0) = -1 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $x=0$ gives

$$
\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \left(-c_1+c_2\right)\sqrt{3} - c_1 - c_2 \\ c_1 + c_2 \end{bmatrix}
$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = -\frac{1}{2} \\ c_2 = -\frac{1}{2} \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} -\frac{\left(\sqrt{3}-1\right)e^{-\left(\sqrt{3}-2\right)x}}{2} + \frac{e^{\left(2+\sqrt{3}\right)x}\left(1+\sqrt{3}\right)}{2} \\ -\frac{e^{-\left(\sqrt{3}-2\right)x}}{2} - \frac{e^{\left(2+\sqrt{3}\right)x}}{2} \end{bmatrix}$$

The following is the phase plot of the system.

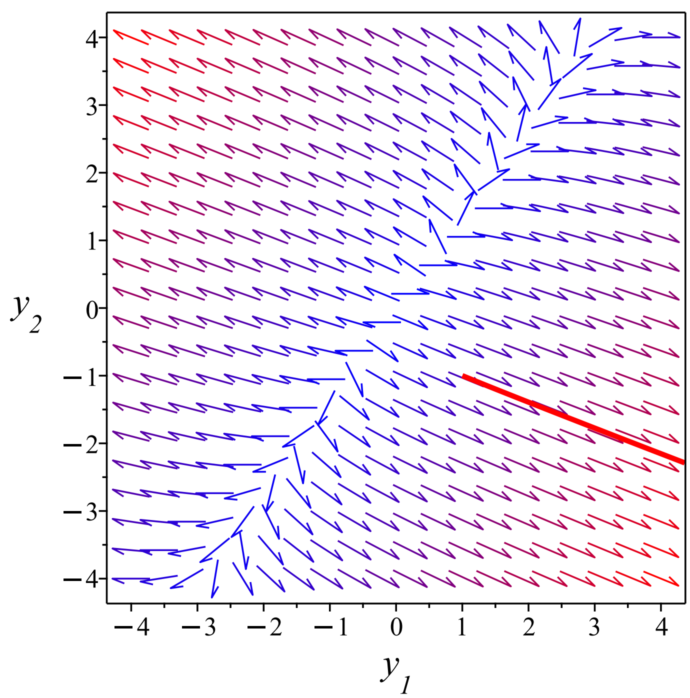
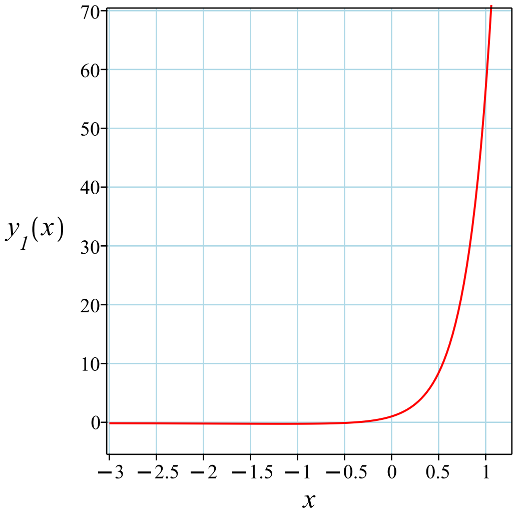
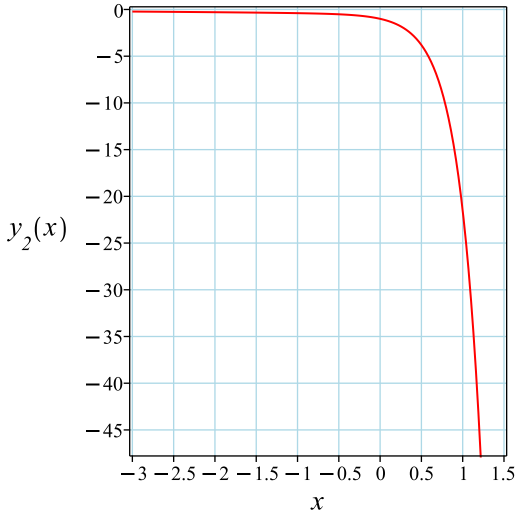


Figure 427: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 119

```
dsolve([diff(y__1(x),x) = 3*y__1(x)-2*y__2(x), diff(y__2(x),x) = -y__1(x)+y__2(x), y__1(0) =
```

$$y_1(x) = \left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right) e^{\left(2+\sqrt{3}\right)x} + \left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right) e^{-\left(-2+\sqrt{3}\right)x}$$

$$y_2(x) = -\frac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right) e^{\left(2+\sqrt{3}\right)x}\sqrt{3}}{2} + \frac{\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right) e^{-\left(-2+\sqrt{3}\right)x}\sqrt{3}}{2} + \frac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right) e^{\left(2+\sqrt{3}\right)x}}{2} + \frac{\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right) e^{-\left(-2+\sqrt{3}\right)x}}{2}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 79

```
DSolve[{y1'[x]==3*y1[x]-2*y2[x],y2'[x]==-y1[x]+y2[x]},{y1[0]==1,y2[0]==-1},{y1[x],y2[x]},x,I
```

$$\text{y1}(x) \to \frac{1}{2}e^{-\left(\left(\sqrt{3}-2\right)x\right)}\left(\left(1+\sqrt{3}\right)e^{2\sqrt{3}x}+1-\sqrt{3}\right)$$

$$\text{y2}(x) \to -\frac{1}{2}e^{-\left(\left(\sqrt{3}-2\right)x\right)}\left(e^{2\sqrt{3}x}+1\right)$$

17.7 problem 13 (b(i))

Internal problem ID [12827]
Internal file name [`OUTPUT/11479_Saturday_November_04_2023_08_47_33_AM_58002003/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 7. Systems of First-Order Differential Equations. Exercises page 329
Problem number: 13 (b(i)).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$y_1'(x) = \sin(x) y_1(x) + \sqrt{x}\, y_2(x) + \ln(x)$$
$$y_2'(x) = \tan(x) y_1(x) - e^x y_2(x) + 1$$

With initial conditions

$$[y_1(1) = 1, y_2(1) = -1]$$

Does not currently support non autonomous system of first order linear differential equations. The following is the phase plot

✗ Solution by Maple

```
dsolve([diff(y__1(x),x) = sin(x)*y__1(x)+x^(1/2)*y__2(x)+ln(x), diff(y__2(x),x) = tan(x)*y__
```

No solution found

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y1'[x]==Sin[x]*y1[x]+Sqrt[x]*y2[x]+Log[x],y2'[x]==Tan[x]*y1[x]-Exp[x]*y2[x]+1},{y1[1
```

Not solved

17.8 problem 13 (b(ii))

Internal problem ID [12828]
Internal file name [`OUTPUT/11480_Saturday_November_04_2023_08_47_33_AM_12439869/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 7. Systems of First-Order Differential Equations. Exercises page 329
Problem number: 13 (b(ii)).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$y_1'(x) = \sin(x)\, y_1(x) + \sqrt{x}\, y_2(x) + \ln(x)$$
$$y_2'(x) = \tan(x)\, y_1(x) - e^x y_2(x) + 1$$

With initial conditions

$$[y_1(2) = 1, y_2(2) = -1]$$

Does not currently support non autonomous system of first order linear differential equations. The following is the phase plot

✗ Solution by Maple

```
dsolve([diff(y__1(x),x) = sin(x)*y__1(x)+x^(1/2)*y__2(x)+ln(x), diff(y__2(x),x) = tan(x)*y__
```

No solution found

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y1'[x]==Sin[x]*y1[x]+Sqrt[x]*y2[x]+Log[x],y2'[x]==Tan[x]*y1[x]-Exp[x]*y2[x]+1},{y1[2
```

Not solved

17.9 problem 13 (c(i))

Internal problem ID [12829]
Internal file name [`OUTPUT/11481_Saturday_November_04_2023_08_47_33_AM_96590617/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 7. Systems of First-Order Differential Equations. Exercises page 329
Problem number: 13 (c(i)).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$y_1'(x) = e^{-x} y_1(x) - \sqrt{x+1}\, y_2(x) + x^2$$
$$y_2'(x) = \frac{y_1(x)}{x^2 - 4x + 4}$$

With initial conditions

$$[y_1(0) = 0, y_2(0) = 1]$$

Does not currently support non autonomous system of first order linear differential equations. The following is the phase plot

✗ Solution by Maple

```
dsolve([diff(y__1(x),x) = exp(-x)*y__1(x)-(1+x)^(1/2)*y__2(x)+x^2, diff(y__2(x),x) = y__1(x)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y1'[x]==Exp[-x]*y1[x]-Sqrt[x+1]*y2[x]+x^2,y2'[x]==y1[x]/(x-2)^2},{y1[0]==0,y2[0]==1}
```

Not solved

17.10 problem 13 (c(ii))

Internal problem ID [12830]
Internal file name [OUTPUT/11482_Saturday_November_04_2023_08_47_33_AM_94405046/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 7. Systems of First-Order Differential Equations. Exercises page 329
Problem number: 13 (c(ii)).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$y_1'(x) = e^{-x} y_1(x) - \sqrt{x+1}\, y_2(x) + x^2$$
$$y_2'(x) = \frac{y_1(x)}{x^2 - 4x + 4}$$

With initial conditions

$$[y_1(3) = 1, y_2(3) = 0]$$

Does not currently support non autonomous system of first order linear differential equations. The following is the phase plot

✗ Solution by Maple

```
dsolve([diff(y__1(x),x) = exp(-x)*y__1(x)-(1+x)^(1/2)*y__2(x)+x^2, diff(y__2(x),x) = y__1(x)
```

No solution found

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y1'[x]==Exp[-x]*y1[x]-Sqrt[x+1]*y2[x]+x^2,y2'[x]==y1[x]/(x-2)^2},{y1[3]==1,y2[3]==0}
```

Not solved

18 Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.2 page 362

18.1 problem 1

Internal problem ID [12831]
Internal file name [`OUTPUT/11483_Saturday_November_04_2023_08_47_34_AM_14623495/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.2 page 362
Problem number: 1.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} -2 & -4 \\ 1 & 3 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} -2 & -4 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} -2-\lambda & -4 \\ 1 & 3-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - \lambda - 2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} -2 & -4 \\ 1 & 3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} -2 & -4 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -1 & -4 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & -4 & 0 \\ 1 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -4t\}$

Hence the solution is

$$\begin{bmatrix} -4t \\ t \end{bmatrix} = \begin{bmatrix} -4t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -4t \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -4t \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} -2 & -4 \\ 1 & 3 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} -2 & -4 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -4 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & -4 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} -4 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-1	1	2	No	$\begin{bmatrix} -4 \\ 1 \end{bmatrix}$
2	1	2	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} -4 & -1 \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} -2 & -4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -4 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

18.2 problem 2

Internal problem ID [12832]
Internal file name [`OUTPUT/11484_Saturday_November_04_2023_08_47_34_AM_17140583/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.2 page 362
Problem number: 2.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} -3-\lambda & -1 \\ 2 & -1-\lambda \end{bmatrix} = 0$$

$$\lambda^2 + 4\lambda + 5 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = -2 + i$$
$$\lambda_2 = -2 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-2-i$	1	complex eigenvalue
$-2+i$	1	complex eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -2 - i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} - (-2 - i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} -2 - i & 0 \\ 0 & -2 - i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -1 + i & -1 \\ 2 & 1 + i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 + i & -1 & 0 \\ 2 & 1 + i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1 + i) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 + i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 + i & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} - \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2}-\frac{I}{2}\right)t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t=1$ then the eigenvector is

$$\begin{bmatrix} \left(-\frac{1}{2}-\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \left(-\frac{1}{2}-\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -1-i \\ 2 \end{bmatrix}$$

Considering $\lambda = -2+i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v}-\lambda\boldsymbol{v} &= \mathbf{0} \\ (A-\lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} - (-2+i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} -2+i & 0 \\ 0 & -2+i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -1-i & -1 \\ 2 & 1-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -1-i & -1 & 0 \\ 2 & 1-i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1-i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} -1-i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
$-2-i$	1	2	No	$\begin{bmatrix} -1-i \\ 2 \end{bmatrix}$
$-2+i$	1	2	No	$\begin{bmatrix} -1+i \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -2-i & 0 \\ 0 & -2+i \end{bmatrix}$$

$$P = \begin{bmatrix} -1-i & -1+i \\ 2 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1-i & -1+i \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -2-i & 0 \\ 0 & -2+i \end{bmatrix} \begin{bmatrix} -1-i & -1+i \\ 2 & 2 \end{bmatrix}^{-1}$$

18.3 problem 3

Internal problem ID [12833]
Internal file name [`OUTPUT/11485_Saturday_November_04_2023_08_47_34_AM_93570398/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.2 page 362
Problem number: 3.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & -1 \\ -2 & 0 & -1-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + \lambda^2 - \lambda + 1 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0}
\end{aligned}$$

$$\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -2 & 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 3 gives

$$\left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} - (i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} - \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1-i & 0 & 1 \\ 0 & 1-i & -1 \\ -2 & 0 & -1-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1-i & 0 & 1 & 0 \\ 0 & 1-i & -1 & 0 \\ -2 & 0 & -1-i & 0 \end{array}\right]$$

$$R_3 = R_3 + (1+i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1-i & 0 & 1 & 0 \\ 0 & 1-i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i & 0 & 1 \\ 0 & 1-i & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} - \frac{i}{2}\right)t, v_2 = \left(\frac{1}{2} + \frac{i}{2}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} - \frac{i}{2}\right) t \\ \left(\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) t \\ \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1 - i \\ 1 + i \\ 2 \end{bmatrix}$$

Considering $\lambda = -i$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} - \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1+i & 0 & 1 \\ 0 & 1+i & -1 \\ -2 & 0 & -1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 1+i & 0 & 1 & 0 \\ 0 & 1+i & -1 & 0 \\ -2 & 0 & -1+i & 0 \end{array}\right]
$$

$$
R_3 = R_3 + (1-i)\,R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1+i & 0 & 1 & 0 \\ 0 & 1+i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 1+i & 0 & 1 \\ 0 & 1+i & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} + \frac{i}{2}\right)t, v_2 = \left(\frac{1}{2} - \frac{i}{2}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ \left(\frac{1}{2}-\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right)t \\ \left(\frac{1}{2}-\frac{i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ \left(\frac{1}{2}-\frac{I}{2}\right)t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ \frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t=1$ then the eigenvector is

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ \left(\frac{1}{2}-\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ \frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \left(-\frac{1}{2}+\frac{I}{2}\right)t \\ \left(\frac{1}{2}-\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ 1-i \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	3	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
i	1	3	No	$\begin{bmatrix} -1-i \\ 1+i \\ 2 \end{bmatrix}$
$-i$	1	3	No	$\begin{bmatrix} -1+i \\ 1-i \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & -1-i & -1+i \\ 1 & 1+i & 1-i \\ 0 & 2 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1-i & -1+i \\ 1 & 1+i & 1-i \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix} \begin{bmatrix} 0 & -1-i & -1+i \\ 1 & 1+i & 1-i \\ 0 & 2 & 2 \end{bmatrix}^{-1}$$

18.4 problem 4

Internal problem ID [12834]
Internal file name [`OUTPUT/11486_Saturday_November_04_2023_08_47_35_AM_37651106/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.2 page 362
Problem number: 4.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det \begin{bmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3 & 3 & -1-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	2	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 3 & 3 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 2 & -1 & 0 \\ 3 & 3 & -2 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 3 & 3 & -2 & 0 \end{array}\right]
$$

$$R_3 = R_3 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}, v_2 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 3 & 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t + s\}$

Hence the solution is

$$\begin{bmatrix} -t+s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t+s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -t+s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -t+s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$
2	2	3	No	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}^{-1}$$

18.5 problem 5

Internal problem ID [12835]
Internal file name [OUTPUT/11487_Saturday_November_04_2023_08_47_35_AM_83219047/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.2 page 362
Problem number: 5.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$
$$\det \begin{bmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{bmatrix} = 0$$
$$-\lambda^3 + 10\lambda^2 - 31\lambda + 30 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 5$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue
5	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 5 & -1 & 6 \\ -10 & 2 & -12 \\ -2 & 1 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 5 & -1 & 6 & 0 \\ -10 & 2 & -12 & 0 \\ -2 & 1 & -3 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 5 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 1 & -3 & 0 \end{array}\right]
$$

$$R_3 = R_3 + \frac{2R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{3}{5} & -\frac{3}{5} & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 5 & -1 & 6 & 0 \\ 0 & \frac{3}{5} & -\frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & -1 & 6 \\ 0 & \frac{3}{5} & -\frac{3}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 4 & -1 & 6 \\ -10 & 1 & -12 \\ -2 & 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 4 & -1 & 6 & 0 \\ -10 & 1 & -12 & 0 \\ -2 & 1 & -4 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + \frac{5R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & -1 & 6 & 0 \\ 0 & -\frac{3}{2} & 3 & 0 \\ -2 & 1 & -4 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & -1 & 6 & 0 \\ 0 & -\frac{3}{2} & 3 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} 4 & -1 & 6 & 0 \\ 0 & -\frac{3}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -1 & 6 \\ 0 & -\frac{3}{2} & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Considering $\lambda = 5$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & -1 & 6 \\ -10 & -1 & -12 \\ -2 & 1 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ -10 & -1 & -12 & 0 \\ -2 & 1 & -6 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + 5R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & -6 & 18 & 0 \\ -2 & 1 & -6 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & -6 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 2 & -1 & 6 \\ 0 & -6 & 18 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3t}{2}, v_2 = 3t\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{3t}{2} \\ 3t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3t}{2} \\ 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{3t}{2} \\ 3t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{2} \\ 3 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{3t}{2} \\ 3t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ 3 \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{3t}{2} \\ 3t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ 2 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
2	1	3	No	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$
3	1	3	No	$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$
5	1	3	No	$\begin{bmatrix} -3 \\ 6 \\ 2 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & -3 \\ 1 & 2 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ 1 & 2 & 6 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & -1 & -3 \\ 1 & 2 & 6 \\ 1 & 1 & 2 \end{bmatrix}^{-1}$$

18.6 problem 6

Internal problem ID [12836]
Internal file name [`OUTPUT/11488_Saturday_November_04_2023_08_47_35_AM_69163266/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.2 page 362
Problem number: 6.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{bmatrix} = 0$$

$$\lambda^4 - 4\lambda^3 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$
\begin{aligned}
\lambda_1 &= 4 \\
\lambda_2 &= 0 \\
\lambda_3 &= 0 \\
\lambda_4 &= 0
\end{aligned}
$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	3	real eigenvalue
4	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 0$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_4 = R_4 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3, v_4\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Let $v_4 = r$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t - s - r\}$

Hence the solution is

$$\begin{bmatrix} -t-s-r \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} -t-s-r \\ t \\ s \\ r \end{bmatrix}$$

Since there are three free Variable, we have found three eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t-s-r \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \\ 0 \end{bmatrix}$$

$$= t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ and $r = 1$ then the above becomes

$$\begin{bmatrix} -t-s-r \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the three eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Considering $\lambda = 4$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cccc|c} -3 & 1 & 1 & 1 & 0 \\ 1 & -3 & 1 & 1 & 0 \\ 1 & 1 & -3 & 1 & 0 \\ 1 & 1 & 1 & -3 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + \frac{R_1}{3} \implies \left[\begin{array}{cccc|c} -3 & 1 & 1 & 1 & 0 \\ 0 & -\frac{8}{3} & \frac{4}{3} & \frac{4}{3} & 0 \\ 1 & 1 & -3 & 1 & 0 \\ 1 & 1 & 1 & -3 & 0 \end{array}\right]
$$

$$R_3 = R_3 + \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} -3 & 1 & 1 & 1 & 0 \\ 0 & -\frac{8}{3} & \frac{4}{3} & \frac{4}{3} & 0 \\ 0 & \frac{4}{3} & -\frac{8}{3} & \frac{4}{3} & 0 \\ 1 & 1 & 1 & -3 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} -3 & 1 & 1 & 1 & 0 \\ 0 & -\frac{8}{3} & \frac{4}{3} & \frac{4}{3} & 0 \\ 0 & \frac{4}{3} & -\frac{8}{3} & \frac{4}{3} & 0 \\ 0 & \frac{4}{3} & \frac{4}{3} & -\frac{8}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{2} \Longrightarrow \left[\begin{array}{cccc|c} -3 & 1 & 1 & 1 & 0 \\ 0 & -\frac{8}{3} & \frac{4}{3} & \frac{4}{3} & 0 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & \frac{4}{3} & \frac{4}{3} & -\frac{8}{3} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{R_2}{2} \Longrightarrow \left[\begin{array}{cccc|c} -3 & 1 & 1 & 1 & 0 \\ 0 & -\frac{8}{3} & \frac{4}{3} & \frac{4}{3} & 0 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & 2 & -2 & 0 \end{array}\right]$$

$$R_4 = R_4 + R_3 \Longrightarrow \left[\begin{array}{cccc|c} -3 & 1 & 1 & 1 & 0 \\ 0 & -\frac{8}{3} & \frac{4}{3} & \frac{4}{3} & 0 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -\frac{8}{3} & \frac{4}{3} & \frac{4}{3} \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t, v_3 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
0	3	4	No	$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
4	1	4	No	$\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}^{-1}$$

18.7 problem 6

Internal problem ID [12837]
Internal file name [`OUTPUT/11489_Saturday_November_04_2023_08_47_35_AM_65202513/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.2 page 362
Problem number: 6.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 6 & 10 & 14 \\ 3 & 9 & 15 & 21 \\ 6 & 18 & 30 & 42 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 6 & 10 & 14 \\ 3 & 9 & 15 & 21 \\ 6 & 18 & 30 & 42 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 3 & 5 & 7 \\ 2 & 6-\lambda & 10 & 14 \\ 3 & 9 & 15-\lambda & 21 \\ 6 & 18 & 30 & 42-\lambda \end{bmatrix} = 0$$

$$\lambda^4 - 64\lambda^3 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$
\begin{aligned}
\lambda_1 &= 64 \\
\lambda_2 &= 0 \\
\lambda_3 &= 0 \\
\lambda_4 &= 0
\end{aligned}
$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	3	real eigenvalue
64	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 0$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0}
\end{aligned}
$$

$$
\left(\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 6 & 10 & 14 \\ 3 & 9 & 15 & 21 \\ 6 & 18 & 30 & 42 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\left(\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 6 & 10 & 14 \\ 3 & 9 & 15 & 21 \\ 6 & 18 & 30 & 42 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 6 & 10 & 14 \\ 3 & 9 & 15 & 21 \\ 6 & 18 & 30 & 42 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & 7 & 0 \\ 2 & 6 & 10 & 14 & 0 \\ 3 & 9 & 15 & 21 & 0 \\ 6 & 18 & 30 & 42 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 9 & 15 & 21 & 0 \\ 6 & 18 & 30 & 42 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_1 \Longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 18 & 30 & 42 & 0 \end{array}\right]$$

$$R_4 = R_4 - 6R_1 \Longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 5 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3, v_4\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Let $v_4 = r$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -3t - 5s - 7r\}$

Hence the solution is

$$\begin{bmatrix} -3t-5s-7r \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} -3t-5s-7r \\ t \\ s \\ r \end{bmatrix}$$

Since there are three free Variable, we have found three eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -3t-5s-7r \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} -3t \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -5s \\ 0 \\ s \\ 0 \end{bmatrix}$$

$$= t\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -7 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

By letting $t = 1$ and $s = 1$ and $r = 1$ then the above becomes

$$\begin{bmatrix} -3t-5s-7r \\ t \\ s \\ r \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -7 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the three eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Considering $\lambda = 64$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0}
\end{aligned}$$

$$\left(\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 6 & 10 & 14 \\ 3 & 9 & 15 & 21 \\ 6 & 18 & 30 & 42 \end{bmatrix} - (64)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 6 & 10 & 14 \\ 3 & 9 & 15 & 21 \\ 6 & 18 & 30 & 42 \end{bmatrix} - \begin{bmatrix} 64 & 0 & 0 & 0 \\ 0 & 64 & 0 & 0 \\ 0 & 0 & 64 & 0 \\ 0 & 0 & 0 & 64 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -63 & 3 & 5 & 7 \\ 2 & -58 & 10 & 14 \\ 3 & 9 & -49 & 21 \\ 6 & 18 & 30 & -22 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cccc|c} -63 & 3 & 5 & 7 & 0 \\ 2 & -58 & 10 & 14 & 0 \\ 3 & 9 & -49 & 21 & 0 \\ 6 & 18 & 30 & -22 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{63} \implies \left[\begin{array}{cccc|c} -63 & 3 & 5 & 7 & 0 \\ 0 & -\frac{1216}{21} & \frac{640}{63} & \frac{128}{9} & 0 \\ 3 & 9 & -49 & 21 & 0 \\ 6 & 18 & 30 & -22 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{21} \implies \left[\begin{array}{cccc|c} -63 & 3 & 5 & 7 & 0 \\ 0 & -\frac{1216}{21} & \frac{640}{63} & \frac{128}{9} & 0 \\ 0 & \frac{64}{7} & -\frac{1024}{21} & \frac{64}{3} & 0 \\ 6 & 18 & 30 & -22 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{2R_1}{21} \implies \left[\begin{array}{cccc|c} -63 & 3 & 5 & 7 & 0 \\ 0 & -\frac{1216}{21} & \frac{640}{63} & \frac{128}{9} & 0 \\ 0 & \frac{64}{7} & -\frac{1024}{21} & \frac{64}{3} & 0 \\ 0 & \frac{128}{7} & \frac{640}{21} & -\frac{64}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_2}{19} \implies \left[\begin{array}{cccc|c} -63 & 3 & 5 & 7 & 0 \\ 0 & -\frac{1216}{21} & \frac{640}{63} & \frac{128}{9} & 0 \\ 0 & 0 & -\frac{896}{19} & \frac{448}{19} & 0 \\ 0 & \frac{128}{7} & \frac{640}{21} & -\frac{64}{3} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{6R_2}{19} \implies \left[\begin{array}{cccc|c} -63 & 3 & 5 & 7 & 0 \\ 0 & -\frac{1216}{21} & \frac{640}{63} & \frac{128}{9} & 0 \\ 0 & 0 & -\frac{896}{19} & \frac{448}{19} & 0 \\ 0 & 0 & \frac{640}{19} & -\frac{320}{19} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{5R_3}{7} \implies \left[\begin{array}{cccc|c} -63 & 3 & 5 & 7 & 0 \\ 0 & -\frac{1216}{21} & \frac{640}{63} & \frac{128}{9} & 0 \\ 0 & 0 & -\frac{896}{19} & \frac{448}{19} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -63 & 3 & 5 & 7 \\ 0 & -\frac{1216}{21} & \frac{640}{63} & \frac{128}{9} \\ 0 & 0 & -\frac{896}{19} & \frac{448}{19} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{6}, v_2 = \frac{t}{3}, v_3 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{6} \\ \frac{t}{3} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{6} \\ \frac{t}{3} \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{6} \\ \frac{t}{3} \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{t}{6} \\ \frac{t}{3} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{t}{6} \\ \frac{t}{3} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
0	3	4	No	$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
64	1	4	No	$\begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 64 \end{bmatrix}$$

$$P = \begin{bmatrix} -3 & -5 & -7 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 6 & 10 & 14 \\ 3 & 9 & 15 & 21 \\ 6 & 18 & 30 & 42 \end{bmatrix} = \begin{bmatrix} -3 & -5 & -7 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 64 \end{bmatrix} \begin{bmatrix} -3 & -5 & -7 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{bmatrix}^{-1}$$

18.8 problem 7

Internal problem ID [12838]
Internal file name [`OUTPUT/11490_Saturday_November_04_2023_08_47_36_AM_4836427/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.2 page 362
Problem number: 7.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 3 & 5 & 2 & 4 \\ 5 & 2-\lambda & 4 & 1 & 3 \\ 4 & 1 & 3-\lambda & 5 & 2 \\ 3 & 5 & 2 & 4-\lambda & 1 \\ 2 & 4 & 1 & 3 & 5-\lambda \end{bmatrix} = 0$$

$$-\lambda^5 + 15\lambda^4 + 125\lambda - 1875 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 15$$
$$\lambda_2 = 5^{\frac{3}{4}}$$
$$\lambda_3 = i5^{\frac{3}{4}}$$
$$\lambda_4 = -5^{\frac{3}{4}}$$
$$\lambda_5 = -i5^{\frac{3}{4}}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
15	1	real eigenvalue
$-5^{\frac{3}{4}}$	1	real eigenvalue
$5^{\frac{3}{4}}$	1	real eigenvalue
$-i5^{\frac{3}{4}}$	1	complex eigenvalue
$i5^{\frac{3}{4}}$	1	complex eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 15$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix} - (15)\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 15 & 0 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 & 0 \\ 0 & 0 & 15 & 0 & 0 \\ 0 & 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 & 15 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -14 & 3 & 5 & 2 & 4 \\ 5 & -13 & 4 & 1 & 3 \\ 4 & 1 & -12 & 5 & 2 \\ 3 & 5 & 2 & -11 & 1 \\ 2 & 4 & 1 & 3 & -10 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccccc|c} -14 & 3 & 5 & 2 & 4 & 0 \\ 5 & -13 & 4 & 1 & 3 & 0 \\ 4 & 1 & -12 & 5 & 2 & 0 \\ 3 & 5 & 2 & -11 & 1 & 0 \\ 2 & 4 & 1 & 3 & -10 & 0 \end{array}\right]
$$

$$R_2 = R_2 + \frac{5R_1}{14} \Longrightarrow \left[\begin{array}{ccccc|c} -14 & 3 & 5 & 2 & 4 & 0 \\ 0 & -\frac{167}{14} & \frac{81}{14} & \frac{12}{7} & \frac{31}{7} & 0 \\ 4 & 1 & -12 & 5 & 2 & 0 \\ 3 & 5 & 2 & -11 & 1 & 0 \\ 2 & 4 & 1 & 3 & -10 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{2R_1}{7} \Longrightarrow \left[\begin{array}{ccccc|c} -14 & 3 & 5 & 2 & 4 & 0 \\ 0 & -\frac{167}{14} & \frac{81}{14} & \frac{12}{7} & \frac{31}{7} & 0 \\ 0 & \frac{13}{7} & -\frac{74}{7} & \frac{39}{7} & \frac{22}{7} & 0 \\ 3 & 5 & 2 & -11 & 1 & 0 \\ 2 & 4 & 1 & 3 & -10 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{3R_1}{14} \Longrightarrow \left[\begin{array}{ccccc|c} -14 & 3 & 5 & 2 & 4 & 0 \\ 0 & -\frac{167}{14} & \frac{81}{14} & \frac{12}{7} & \frac{31}{7} & 0 \\ 0 & \frac{13}{7} & -\frac{74}{7} & \frac{39}{7} & \frac{22}{7} & 0 \\ 0 & \frac{79}{14} & \frac{43}{14} & -\frac{74}{7} & \frac{13}{7} & 0 \\ 2 & 4 & 1 & 3 & -10 & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{R_1}{7} \Longrightarrow \left[\begin{array}{ccccc|c} -14 & 3 & 5 & 2 & 4 & 0 \\ 0 & -\frac{167}{14} & \frac{81}{14} & \frac{12}{7} & \frac{31}{7} & 0 \\ 0 & \frac{13}{7} & -\frac{74}{7} & \frac{39}{7} & \frac{22}{7} & 0 \\ 0 & \frac{79}{14} & \frac{43}{14} & -\frac{74}{7} & \frac{13}{7} & 0 \\ 0 & \frac{31}{7} & \frac{12}{7} & \frac{23}{7} & -\frac{66}{7} & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{26R_2}{167} \Longrightarrow \left[\begin{array}{ccccc|c} -14 & 3 & 5 & 2 & 4 & 0 \\ 0 & -\frac{167}{14} & \frac{81}{14} & \frac{12}{7} & \frac{31}{7} & 0 \\ 0 & 0 & -\frac{1615}{167} & \frac{975}{167} & \frac{640}{167} & 0 \\ 0 & \frac{79}{14} & \frac{43}{14} & -\frac{74}{7} & \frac{13}{7} & 0 \\ 0 & \frac{31}{7} & \frac{12}{7} & \frac{23}{7} & -\frac{66}{7} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{79R_2}{167} \Longrightarrow \left[\begin{array}{ccccc|c} -14 & 3 & 5 & 2 & 4 & 0 \\ 0 & -\frac{167}{14} & \frac{81}{14} & \frac{12}{7} & \frac{31}{7} & 0 \\ 0 & 0 & -\frac{1615}{167} & \frac{975}{167} & \frac{640}{167} & 0 \\ 0 & 0 & \frac{970}{167} & -\frac{1630}{167} & \frac{660}{167} & 0 \\ 0 & \frac{31}{7} & \frac{12}{7} & \frac{23}{7} & -\frac{66}{7} & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{62R_2}{167} \Longrightarrow \left[\begin{array}{ccccc|c} -14 & 3 & 5 & 2 & 4 & 0 \\ 0 & -\frac{167}{14} & \frac{81}{14} & \frac{12}{7} & \frac{31}{7} & 0 \\ 0 & 0 & -\frac{1615}{167} & \frac{975}{167} & \frac{640}{167} & 0 \\ 0 & 0 & \frac{970}{167} & -\frac{1630}{167} & \frac{660}{167} & 0 \\ 0 & 0 & \frac{645}{167} & \frac{655}{167} & -\frac{1300}{167} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{194R_3}{323} \Longrightarrow \left[\begin{array}{ccccc|c} -14 & 3 & 5 & 2 & 4 & 0 \\ 0 & -\frac{167}{14} & \frac{81}{14} & \frac{12}{7} & \frac{31}{7} & 0 \\ 0 & 0 & -\frac{1615}{167} & \frac{975}{167} & \frac{640}{167} & 0 \\ 0 & 0 & 0 & -\frac{2020}{323} & \frac{2020}{323} & 0 \\ 0 & 0 & \frac{645}{167} & \frac{655}{167} & -\frac{1300}{167} & 0 \end{array}\right]$$

$$R_5 = R_5 + \frac{129R_3}{323} \Longrightarrow \left[\begin{array}{ccccc|c} -14 & 3 & 5 & 2 & 4 & 0 \\ 0 & -\frac{167}{14} & \frac{81}{14} & \frac{12}{7} & \frac{31}{7} & 0 \\ 0 & 0 & -\frac{1615}{167} & \frac{975}{167} & \frac{640}{167} & 0 \\ 0 & 0 & 0 & -\frac{2020}{323} & \frac{2020}{323} & 0 \\ 0 & 0 & 0 & \frac{2020}{323} & -\frac{2020}{323} & 0 \end{array}\right]$$

$$R_5 = R_5 + R_4 \Longrightarrow \left[\begin{array}{ccccc|c} -14 & 3 & 5 & 2 & 4 & 0 \\ 0 & -\frac{167}{14} & \frac{81}{14} & \frac{12}{7} & \frac{31}{7} & 0 \\ 0 & 0 & -\frac{1615}{167} & \frac{975}{167} & \frac{640}{167} & 0 \\ 0 & 0 & 0 & -\frac{2020}{323} & \frac{2020}{323} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -14 & 3 & 5 & 2 & 4 \\ 0 & -\frac{167}{14} & \frac{81}{14} & \frac{12}{7} & \frac{31}{7} \\ 0 & 0 & -\frac{1615}{167} & \frac{975}{167} & \frac{640}{167} \\ 0 & 0 & 0 & -\frac{2020}{323} & \frac{2020}{323} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_5\}$ and the leading variables are $\{v_1, v_2, v_3, v_4\}$. Let $v_5 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t, v_3 = t, v_4 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = -5^{\frac{3}{4}}$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0}
\end{aligned}
$$

$$
\left(\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix} - \left(-5^{\frac{3}{4}}\right) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\left(\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix} - \begin{bmatrix} -5^{\frac{3}{4}} & 0 & 0 & 0 & 0 \\ 0 & -5^{\frac{3}{4}} & 0 & 0 & 0 \\ 0 & 0 & -5^{\frac{3}{4}} & 0 & 0 \\ 0 & 0 & 0 & -5^{\frac{3}{4}} & 0 \\ 0 & 0 & 0 & 0 & -5^{\frac{3}{4}} \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 1+5^{\frac{3}{4}} & 3 & 5 & 2 & 4 \\ 5 & 2+5^{\frac{3}{4}} & 4 & 1 & 3 \\ 4 & 1 & 3+5^{\frac{3}{4}} & 5 & 2 \\ 3 & 5 & 2 & 4+5^{\frac{3}{4}} & 1 \\ 2 & 4 & 1 & 3 & 5+5^{\frac{3}{4}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccccc|c} 1+5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 5 & 2+5^{\frac{3}{4}} & 4 & 1 & 3 & 0 \\ 4 & 1 & 3+5^{\frac{3}{4}} & 5 & 2 & 0 \\ 3 & 5 & 2 & 4+5^{\frac{3}{4}} & 1 & 0 \\ 2 & 4 & 1 & 3 & 5+5^{\frac{3}{4}} & 0 \end{array}\right]
$$

$$R_2 = R_2 - \frac{5R_1}{1+5^{\frac{3}{4}}} \Longrightarrow \left[\begin{array}{ccccc|c} 1+5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{5\sqrt{5}+3\,5^{\frac{3}{4}}-13}{1+5^{\frac{3}{4}}} & \frac{4\,5^{\frac{3}{4}}-21}{1+5^{\frac{3}{4}}} & \frac{5^{\frac{3}{4}}-9}{1+5^{\frac{3}{4}}} & \frac{3\,5^{\frac{3}{4}}-17}{1+5^{\frac{3}{4}}} & 0 \\ 4 & 1 & 3+5^{\frac{3}{4}} & 5 & 2 & 0 \\ 3 & 5 & 2 & 4+5^{\frac{3}{4}} & 1 & 0 \\ 2 & 4 & 1 & 3 & 5+5^{\frac{3}{4}} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{4R_1}{1+5^{\frac{3}{4}}} \Longrightarrow \left[\begin{array}{ccccc|c} 1+5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{5\sqrt{5}+3\,5^{\frac{3}{4}}-13}{1+5^{\frac{3}{4}}} & \frac{4\,5^{\frac{3}{4}}-21}{1+5^{\frac{3}{4}}} & \frac{5^{\frac{3}{4}}-9}{1+5^{\frac{3}{4}}} & \frac{3\,5^{\frac{3}{4}}-17}{1+5^{\frac{3}{4}}} & 0 \\ 0 & \frac{5^{\frac{3}{4}}-11}{1+5^{\frac{3}{4}}} & \frac{5\sqrt{5}+4\,5^{\frac{3}{4}}-17}{1+5^{\frac{3}{4}}} & \frac{5\,5^{\frac{3}{4}}-3}{1+5^{\frac{3}{4}}} & \frac{2\,5^{\frac{3}{4}}-14}{1+5^{\frac{3}{4}}} & 0 \\ 3 & 5 & 2 & 4+5^{\frac{3}{4}} & 1 & 0 \\ 2 & 4 & 1 & 3 & 5+5^{\frac{3}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{3R_1}{1+5^{\frac{3}{4}}} \Longrightarrow \left[\begin{array}{ccccc|c} 1+5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{5\sqrt{5}+3\,5^{\frac{3}{4}}-13}{1+5^{\frac{3}{4}}} & \frac{4\,5^{\frac{3}{4}}-21}{1+5^{\frac{3}{4}}} & \frac{5^{\frac{3}{4}}-9}{1+5^{\frac{3}{4}}} & \frac{3\,5^{\frac{3}{4}}-17}{1+5^{\frac{3}{4}}} & 0 \\ 0 & \frac{5^{\frac{3}{4}}-11}{1+5^{\frac{3}{4}}} & \frac{5\sqrt{5}+4\,5^{\frac{3}{4}}-17}{1+5^{\frac{3}{4}}} & \frac{5\,5^{\frac{3}{4}}-3}{1+5^{\frac{3}{4}}} & \frac{2\,5^{\frac{3}{4}}-14}{1+5^{\frac{3}{4}}} & 0 \\ 0 & \frac{5\,5^{\frac{3}{4}}-4}{1+5^{\frac{3}{4}}} & \frac{2\,5^{\frac{3}{4}}-13}{1+5^{\frac{3}{4}}} & \frac{5\sqrt{5}+5\,5^{\frac{3}{4}}-2}{1+5^{\frac{3}{4}}} & \frac{5^{\frac{3}{4}}-11}{1+5^{\frac{3}{4}}} & 0 \\ 2 & 4 & 1 & 3 & 5+5^{\frac{3}{4}} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{2R_1}{1+5^{\frac{3}{4}}} \Longrightarrow \left[\begin{array}{ccccc|c} 1+5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{5\sqrt{5}+3\,5^{\frac{3}{4}}-13}{1+5^{\frac{3}{4}}} & \frac{4\,5^{\frac{3}{4}}-21}{1+5^{\frac{3}{4}}} & \frac{5^{\frac{3}{4}}-9}{1+5^{\frac{3}{4}}} & \frac{3\,5^{\frac{3}{4}}-17}{1+5^{\frac{3}{4}}} & 0 \\ 0 & \frac{5^{\frac{3}{4}}-11}{1+5^{\frac{3}{4}}} & \frac{5\sqrt{5}+4\,5^{\frac{3}{4}}-17}{1+5^{\frac{3}{4}}} & \frac{5\,5^{\frac{3}{4}}-3}{1+5^{\frac{3}{4}}} & \frac{2\,5^{\frac{3}{4}}-14}{1+5^{\frac{3}{4}}} & 0 \\ 0 & \frac{5\,5^{\frac{3}{4}}-4}{1+5^{\frac{3}{4}}} & \frac{2\,5^{\frac{3}{4}}-13}{1+5^{\frac{3}{4}}} & \frac{5\sqrt{5}+5\,5^{\frac{3}{4}}-2}{1+5^{\frac{3}{4}}} & \frac{5^{\frac{3}{4}}-11}{1+5^{\frac{3}{4}}} & 0 \\ 0 & \frac{4\,5^{\frac{3}{4}}-2}{1+5^{\frac{3}{4}}} & \frac{5^{\frac{3}{4}}-9}{1+5^{\frac{3}{4}}} & \frac{3\,5^{\frac{3}{4}}-1}{1+5^{\frac{3}{4}}} & \frac{5\sqrt{5}+6\,5^{\frac{3}{4}}-3}{1+5^{\frac{3}{4}}} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{\left(5^{\frac{3}{4}} - 11\right) R_2}{5\sqrt{5} + 3\,5^{\frac{3}{4}} - 13} \Longrightarrow \begin{bmatrix} 1+5^{\frac{3}{4}} & 3 & 5 & 2 \\ 0 & \frac{5\sqrt{5}+3\,5^{\frac{3}{4}}-13}{1+5^{\frac{3}{4}}} & \frac{4\,5^{\frac{3}{4}}-21}{1+5^{\frac{3}{4}}} & \frac{5^{\frac{3}{4}}-9}{1+5^{\frac{3}{4}}} \\ 0 & 0 & \frac{-110\sqrt{5}+175\,5^{\frac{1}{4}}-38\,5^{\frac{3}{4}}+115}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} & \frac{55\sqrt{5}+125\,5^{\frac{1}{4}}-54\,5^{\frac{3}{4}}-60}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} \\ 0 & \frac{5\,5^{\frac{3}{4}}-4}{1+5^{\frac{3}{4}}} & \frac{2\,5^{\frac{3}{4}}-13}{1+5^{\frac{3}{4}}} & \frac{5\sqrt{5}+5\,5^{\frac{3}{4}}-2}{1+5^{\frac{3}{4}}} \\ 0 & \frac{4\,5^{\frac{3}{4}}-2}{1+5^{\frac{3}{4}}} & \frac{5^{\frac{3}{4}}-9}{1+5^{\frac{3}{4}}} & \frac{3\,5^{\frac{3}{4}}-1}{1+5^{\frac{3}{4}}} \end{bmatrix}$$

$$R_4 = R_4 - \frac{\left(5\,5^{\frac{3}{4}} - 4\right) R_2}{5\sqrt{5} + 3\,5^{\frac{3}{4}} - 13} \Longrightarrow \begin{bmatrix} 1+5^{\frac{3}{4}} & 3 & 5 & 2 \\ 0 & \frac{5\sqrt{5}+3\,5^{\frac{3}{4}}-13}{1+5^{\frac{3}{4}}} & \frac{4\,5^{\frac{3}{4}}-21}{1+5^{\frac{3}{4}}} & \frac{5^{\frac{3}{4}}-9}{1+5^{\frac{3}{4}}} \\ 0 & 0 & \frac{-110\sqrt{5}+175\,5^{\frac{1}{4}}-38\,5^{\frac{3}{4}}+115}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} & \frac{55\sqrt{5}+125\,5^{\frac{1}{4}}-54\,5^{\frac{3}{4}}-60}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} \\ 0 & 0 & \frac{-135\sqrt{5}+50\,5^{\frac{1}{4}}+56\,5^{\frac{3}{4}}+85}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} & \frac{-25\sqrt{5}+200\,5^{\frac{1}{4}}-22\,5^{\frac{3}{4}}+115}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} \\ 0 & \frac{4\,5^{\frac{3}{4}}-2}{1+5^{\frac{3}{4}}} & \frac{5^{\frac{3}{4}}-9}{1+5^{\frac{3}{4}}} & \frac{3\,5^{\frac{3}{4}}-1}{1+5^{\frac{3}{4}}} \end{bmatrix}$$

$$R_5 = R_5 - \frac{\left(4\,5^{\frac{3}{4}} - 2\right) R_2}{5\sqrt{5} + 3\,5^{\frac{3}{4}} - 13} \Longrightarrow \begin{bmatrix} 1+5^{\frac{3}{4}} & 3 & 5 & 2 \\ 0 & \frac{5\sqrt{5}+3\,5^{\frac{3}{4}}-13}{1+5^{\frac{3}{4}}} & \frac{4\,5^{\frac{3}{4}}-21}{1+5^{\frac{3}{4}}} & \frac{5^{\frac{3}{4}}-9}{1+5^{\frac{3}{4}}} \\ 0 & 0 & \frac{-110\sqrt{5}+175\,5^{\frac{1}{4}}-38\,5^{\frac{3}{4}}+115}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} & \frac{55\sqrt{5}+125\,5^{\frac{1}{4}}-54\,5^{\frac{3}{4}}-60}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} \\ 0 & 0 & \frac{-135\sqrt{5}+50\,5^{\frac{1}{4}}+56\,5^{\frac{3}{4}}+85}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} & \frac{-25\sqrt{5}+200\,5^{\frac{1}{4}}-22\,5^{\frac{3}{4}}+115}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} \\ 0 & 0 & \frac{-110\sqrt{5}+25\,5^{\frac{1}{4}}+52\,5^{\frac{3}{4}}+75}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} & \frac{20\sqrt{5}+75\,5^{\frac{1}{4}}-4\,5^{\frac{3}{4}}-5}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} \end{bmatrix}$$

$$R_4 = R_4 - \frac{\left(-135\sqrt{5} + 50\,5^{\frac{1}{4}} + 56\,5^{\frac{3}{4}} + 85\right) R_3}{-110\sqrt{5} + 175\,5^{\frac{1}{4}} - 38\,5^{\frac{3}{4}} + 115} \Longrightarrow \begin{bmatrix} 1+5^{\frac{3}{4}} & 3 & 5 \\ 0 & \frac{5\sqrt{5}+3\,5^{\frac{3}{4}}-13}{1+5^{\frac{3}{4}}} & \frac{4\,5^{\frac{3}{4}}-21}{1+5^{\frac{3}{4}}} \\ 0 & 0 & \frac{-110\sqrt{5}+175\,5^{\frac{1}{4}}-38\,5^{\frac{3}{4}}+115}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-110\sqrt{5}+25\,5^{\frac{1}{4}}+52\,5^{\frac{3}{4}}+75}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} \end{bmatrix}$$

$$R_5 = R_5 - \frac{\left(-110\sqrt{5} + 25\,5^{\frac{1}{4}} + 52\,5^{\frac{3}{4}} + 75\right) R_3}{-110\sqrt{5} + 175\,5^{\frac{1}{4}} - 38\,5^{\frac{3}{4}} + 115} \Longrightarrow \begin{bmatrix} 1+5^{\frac{3}{4}} & 3 & 5 & \cdots \\ 0 & \frac{5\sqrt{5}+3\,5^{\frac{3}{4}}-13}{1+5^{\frac{3}{4}}} & \frac{4\,5^{\frac{3}{4}}-21}{1+5^{\frac{3}{4}}} & \cdots \\ 0 & 0 & \frac{-110\sqrt{5}+175\,5^{\frac{1}{4}}-38\,5^{\frac{3}{4}}+115}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \end{bmatrix}$$

$$R_5 = R_5 + \frac{\left(581\,5^{\frac{3}{4}} - 1829\,5^{\frac{1}{4}} + 677\sqrt{5} - 813\right) R_4}{448\,5^{\frac{3}{4}} - 20\,5^{\frac{1}{4}} - 790\sqrt{5} + 382} \Longrightarrow \begin{bmatrix} 1+5^{\frac{3}{4}} & 3 & 5 & \cdots \\ 0 & \frac{5\sqrt{5}+3\,5^{\frac{3}{4}}-13}{1+5^{\frac{3}{4}}} & \frac{4\,5^{\frac{3}{4}}-21}{1+5^{\frac{3}{4}}} & \cdots \\ 0 & 0 & \frac{-110\sqrt{5}+175\,5^{\frac{1}{4}}-38\,5^{\frac{3}{4}}+115}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \end{bmatrix}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+5^{\frac{3}{4}} & 3 & 5 & 2 & \cdots \\ 0 & \frac{5\sqrt{5}+3\,5^{\frac{3}{4}}-13}{1+5^{\frac{3}{4}}} & \frac{4\,5^{\frac{3}{4}}-21}{1+5^{\frac{3}{4}}} & \frac{5^{\frac{3}{4}}-9}{1+5^{\frac{3}{4}}} & \cdots \\ 0 & 0 & \frac{-110\sqrt{5}+175\,5^{\frac{1}{4}}-38\,5^{\frac{3}{4}}+115}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} & \frac{55\sqrt{5}+125\,5^{\frac{1}{4}}-54\,5^{\frac{3}{4}}-60}{\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} & \cdots \\ 0 & 0 & 0 & \frac{-19750\sqrt{5}+9550-500\,5^{\frac{1}{4}}+11200\,5^{\frac{3}{4}}}{\left(38\,5^{\frac{3}{4}}-175\,5^{\frac{1}{4}}+110\sqrt{5}-115\right)\left(5\sqrt{5}+3\,5^{\frac{3}{4}}-13\right)\left(1+5^{\frac{3}{4}}\right)} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

The free variables are $\{v_5\}$ and the leading variables are $\{v_1, v_2, v_3, v_4\}$. Let $v_5 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(2\,5^{\frac{3}{4}} + 4\,5^{\frac{1}{4}} + 3\sqrt{5} + 6\right)t, v_2 = -\left(5^{\frac{3}{4}} + 2\,5^{\frac{1}{4}} + \sqrt{5} + 4\right)t, v_3 = -\left(\cdots\right.\right.$

Hence the solution is

$$\begin{bmatrix} \left(2\,5^{\frac{3}{4}}+4\,5^{\frac{1}{4}}+3\sqrt{5}+6\right)t \\ -\left(5^{\frac{3}{4}}+2\,5^{\frac{1}{4}}+\sqrt{5}+4\right)t \\ -\left(5^{\frac{3}{4}}+3\,5^{\frac{1}{4}}+2\sqrt{5}+4\right)t \\ \left(5^{\frac{1}{4}}+1\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(2\,5^{\frac{3}{4}}+4\,5^{\frac{1}{4}}+3\sqrt{5}+6\right)t \\ -\left(5^{\frac{3}{4}}+2\,5^{\frac{1}{4}}+\sqrt{5}+4\right)t \\ -\left(5^{\frac{3}{4}}+3\,5^{\frac{1}{4}}+2\sqrt{5}+4\right)t \\ \left(5^{\frac{1}{4}}+1\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(2\,5^{\frac{3}{4}}+4\,5^{\frac{1}{4}}+3\sqrt{5}+6\right)t \\ -\left(5^{\frac{3}{4}}+2\,5^{\frac{1}{4}}+\sqrt{5}+4\right)t \\ -\left(5^{\frac{3}{4}}+3\,5^{\frac{1}{4}}+2\sqrt{5}+4\right)t \\ \left(5^{\frac{1}{4}}+1\right)t \\ t \end{bmatrix} = t\begin{bmatrix} 2\,5^{\frac{3}{4}}+4\,5^{\frac{1}{4}}+3\sqrt{5}+6 \\ -5^{\frac{3}{4}}-2\,5^{\frac{1}{4}}-\sqrt{5}-4 \\ -5^{\frac{3}{4}}-3\,5^{\frac{1}{4}}-2\sqrt{5}-4 \\ 5^{\frac{1}{4}}+1 \\ 1 \end{bmatrix}$$

Or, by letting $t=1$ then the eigenvector is

$$\begin{bmatrix} \left(2\,5^{\frac{3}{4}}+4\,5^{\frac{1}{4}}+3\sqrt{5}+6\right)t \\ -\left(5^{\frac{3}{4}}+2\,5^{\frac{1}{4}}+\sqrt{5}+4\right)t \\ -\left(5^{\frac{3}{4}}+3\,5^{\frac{1}{4}}+2\sqrt{5}+4\right)t \\ \left(5^{\frac{1}{4}}+1\right)t \\ t \end{bmatrix} = \begin{bmatrix} 2\,5^{\frac{3}{4}}+4\,5^{\frac{1}{4}}+3\sqrt{5}+6 \\ -5^{\frac{3}{4}}-2\,5^{\frac{1}{4}}-\sqrt{5}-4 \\ -5^{\frac{3}{4}}-3\,5^{\frac{1}{4}}-2\sqrt{5}-4 \\ 5^{\frac{1}{4}}+1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 5^{\frac{3}{4}}$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0}
\end{aligned}
$$

$$
\left(\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix} - \left(5^{\frac{3}{4}}\right) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\left(\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 5^{\frac{3}{4}} & 0 & 0 & 0 & 0 \\ 0 & 5^{\frac{3}{4}} & 0 & 0 & 0 \\ 0 & 0 & 5^{\frac{3}{4}} & 0 & 0 \\ 0 & 0 & 0 & 5^{\frac{3}{4}} & 0 \\ 0 & 0 & 0 & 0 & 5^{\frac{3}{4}} \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 1 - 5^{\frac{3}{4}} & 3 & 5 & 2 & 4 \\ 5 & 2 - 5^{\frac{3}{4}} & 4 & 1 & 3 \\ 4 & 1 & 3 - 5^{\frac{3}{4}} & 5 & 2 \\ 3 & 5 & 2 & 4 - 5^{\frac{3}{4}} & 1 \\ 2 & 4 & 1 & 3 & 5 - 5^{\frac{3}{4}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccccc|c} 1 - 5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 5 & 2 - 5^{\frac{3}{4}} & 4 & 1 & 3 & 0 \\ 4 & 1 & 3 - 5^{\frac{3}{4}} & 5 & 2 & 0 \\ 3 & 5 & 2 & 4 - 5^{\frac{3}{4}} & 1 & 0 \\ 2 & 4 & 1 & 3 & 5 - 5^{\frac{3}{4}} & 0 \end{array} \right]
$$

$$R_2 = R_2 - \frac{5R_1}{1-5^{\frac{3}{4}}} \implies \left[\begin{array}{ccccc|c} 1-5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{-5\sqrt{5}+3\,5^{\frac{3}{4}}+13}{5^{\frac{3}{4}}-1} & \frac{4\,5^{\frac{3}{4}}+21}{5^{\frac{3}{4}}-1} & \frac{5^{\frac{3}{4}}+9}{5^{\frac{3}{4}}-1} & \frac{3\,5^{\frac{3}{4}}+17}{5^{\frac{3}{4}}-1} & 0 \\ 4 & 1 & 3-5^{\frac{3}{4}} & 5 & 2 & 0 \\ 3 & 5 & 2 & 4-5^{\frac{3}{4}} & 1 & 0 \\ 2 & 4 & 1 & 3 & 5-5^{\frac{3}{4}} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{4R_1}{1-5^{\frac{3}{4}}} \implies \left[\begin{array}{ccccc|c} 1-5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{-5\sqrt{5}+3\,5^{\frac{3}{4}}+13}{5^{\frac{3}{4}}-1} & \frac{4\,5^{\frac{3}{4}}+21}{5^{\frac{3}{4}}-1} & \frac{5^{\frac{3}{4}}+9}{5^{\frac{3}{4}}-1} & \frac{3\,5^{\frac{3}{4}}+17}{5^{\frac{3}{4}}-1} & 0 \\ 0 & \frac{5^{\frac{3}{4}}+11}{5^{\frac{3}{4}}-1} & \frac{-5\sqrt{5}+4\,5^{\frac{3}{4}}+17}{5^{\frac{3}{4}}-1} & \frac{5\,5^{\frac{3}{4}}+3}{5^{\frac{3}{4}}-1} & \frac{2\,5^{\frac{3}{4}}+14}{5^{\frac{3}{4}}-1} & 0 \\ 3 & 5 & 2 & 4-5^{\frac{3}{4}} & 1 & 0 \\ 2 & 4 & 1 & 3 & 5-5^{\frac{3}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{3R_1}{1-5^{\frac{3}{4}}} \implies \left[\begin{array}{ccccc|c} 1-5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{-5\sqrt{5}+3\,5^{\frac{3}{4}}+13}{5^{\frac{3}{4}}-1} & \frac{4\,5^{\frac{3}{4}}+21}{5^{\frac{3}{4}}-1} & \frac{5^{\frac{3}{4}}+9}{5^{\frac{3}{4}}-1} & \frac{3\,5^{\frac{3}{4}}+17}{5^{\frac{3}{4}}-1} & 0 \\ 0 & \frac{5^{\frac{3}{4}}+11}{5^{\frac{3}{4}}-1} & \frac{-5\sqrt{5}+4\,5^{\frac{3}{4}}+17}{5^{\frac{3}{4}}-1} & \frac{5\,5^{\frac{3}{4}}+3}{5^{\frac{3}{4}}-1} & \frac{2\,5^{\frac{3}{4}}+14}{5^{\frac{3}{4}}-1} & 0 \\ 0 & \frac{5\,5^{\frac{3}{4}}+4}{5^{\frac{3}{4}}-1} & \frac{2\,5^{\frac{3}{4}}+13}{5^{\frac{3}{4}}-1} & \frac{-5\sqrt{5}+5\,5^{\frac{3}{4}}+2}{5^{\frac{3}{4}}-1} & \frac{5^{\frac{3}{4}}+11}{5^{\frac{3}{4}}-1} & 0 \\ 2 & 4 & 1 & 3 & 5-5^{\frac{3}{4}} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{2R_1}{1-5^{\frac{3}{4}}} \implies \left[\begin{array}{ccccc|c} 1-5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{-5\sqrt{5}+3\,5^{\frac{3}{4}}+13}{5^{\frac{3}{4}}-1} & \frac{4\,5^{\frac{3}{4}}+21}{5^{\frac{3}{4}}-1} & \frac{5^{\frac{3}{4}}+9}{5^{\frac{3}{4}}-1} & \frac{3\,5^{\frac{3}{4}}+17}{5^{\frac{3}{4}}-1} & 0 \\ 0 & \frac{5^{\frac{3}{4}}+11}{5^{\frac{3}{4}}-1} & \frac{-5\sqrt{5}+4\,5^{\frac{3}{4}}+17}{5^{\frac{3}{4}}-1} & \frac{5\,5^{\frac{3}{4}}+3}{5^{\frac{3}{4}}-1} & \frac{2\,5^{\frac{3}{4}}+14}{5^{\frac{3}{4}}-1} & 0 \\ 0 & \frac{5\,5^{\frac{3}{4}}+4}{5^{\frac{3}{4}}-1} & \frac{2\,5^{\frac{3}{4}}+13}{5^{\frac{3}{4}}-1} & \frac{-5\sqrt{5}+5\,5^{\frac{3}{4}}+2}{5^{\frac{3}{4}}-1} & \frac{5^{\frac{3}{4}}+11}{5^{\frac{3}{4}}-1} & 0 \\ 0 & \frac{4\,5^{\frac{3}{4}}+2}{5^{\frac{3}{4}}-1} & \frac{5^{\frac{3}{4}}+9}{5^{\frac{3}{4}}-1} & \frac{3\,5^{\frac{3}{4}}+1}{5^{\frac{3}{4}}-1} & \frac{-5\sqrt{5}+6\,5^{\frac{3}{4}}+3}{5^{\frac{3}{4}}-1} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{\left(5^{\frac{3}{4}}+11\right)R_2}{-5\sqrt{5}+3\,5^{\frac{3}{4}}+13} \Longrightarrow \begin{bmatrix} 1-5^{\frac{3}{4}} & 3 & 5 & 2 \\ 0 & \frac{-5\sqrt{5}+3\,5^{\frac{3}{4}}+13}{5^{\frac{3}{4}}-1} & \frac{4\,5^{\frac{3}{4}}+21}{5^{\frac{3}{4}}-1} & \frac{5^{\frac{3}{4}}+9}{5^{\frac{3}{4}}-1} \\ 0 & 0 & \frac{-110\sqrt{5}-175\,5^{\frac{1}{4}}+38\,5^{\frac{3}{4}}+115}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right)} & \frac{55\sqrt{5}-125\,5^{\frac{1}{4}}+54\,5^{\frac{3}{4}}-60}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right.} \\ 0 & \frac{5\,5^{\frac{3}{4}}+4}{5^{\frac{3}{4}}-1} & \frac{2\,5^{\frac{3}{4}}+13}{5^{\frac{3}{4}}-1} & \frac{-5\sqrt{5}+5\,5^{\frac{3}{4}}+2}{5^{\frac{3}{4}}-1} \\ 0 & \frac{4\,5^{\frac{3}{4}}+2}{5^{\frac{3}{4}}-1} & \frac{5^{\frac{3}{4}}+9}{5^{\frac{3}{4}}-1} & \frac{3\,5^{\frac{3}{4}}+1}{5^{\frac{3}{4}}-1} \end{bmatrix}$$

$$R_4 = R_4 - \frac{\left(5\,5^{\frac{3}{4}}+4\right)R_2}{-5\sqrt{5}+3\,5^{\frac{3}{4}}+13} \Longrightarrow \begin{bmatrix} 1-5^{\frac{3}{4}} & 3 & 5 & 2 \\ 0 & \frac{-5\sqrt{5}+3\,5^{\frac{3}{4}}+13}{5^{\frac{3}{4}}-1} & \frac{4\,5^{\frac{3}{4}}+21}{5^{\frac{3}{4}}-1} & \frac{5^{\frac{3}{4}}+9}{5^{\frac{3}{4}}-1} \\ 0 & 0 & \frac{-110\sqrt{5}-175\,5^{\frac{1}{4}}+38\,5^{\frac{3}{4}}+115}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right)} & \frac{55\sqrt{5}-125\,5^{\frac{1}{4}}+54\,5^{\frac{3}{4}}-60}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right.} \\ 0 & 0 & \frac{-135\sqrt{5}-50\,5^{\frac{1}{4}}-56\,5^{\frac{3}{4}}+85}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right)} & \frac{-25\sqrt{5}-200\,5^{\frac{1}{4}}+22\,5^{\frac{3}{4}}+1}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right.} \\ 0 & \frac{4\,5^{\frac{3}{4}}+2}{5^{\frac{3}{4}}-1} & \frac{5^{\frac{3}{4}}+9}{5^{\frac{3}{4}}-1} & \frac{3\,5^{\frac{3}{4}}+1}{5^{\frac{3}{4}}-1} \end{bmatrix}$$

$$R_5 = R_5 - \frac{\left(4\,5^{\frac{3}{4}}+2\right)R_2}{-5\sqrt{5}+3\,5^{\frac{3}{4}}+13} \Longrightarrow \begin{bmatrix} 1-5^{\frac{3}{4}} & 3 & 5 & 2 \\ 0 & \frac{-5\sqrt{5}+3\,5^{\frac{3}{4}}+13}{5^{\frac{3}{4}}-1} & \frac{4\,5^{\frac{3}{4}}+21}{5^{\frac{3}{4}}-1} & \frac{5^{\frac{3}{4}}+9}{5^{\frac{3}{4}}-1} \\ 0 & 0 & \frac{-110\sqrt{5}-175\,5^{\frac{1}{4}}+38\,5^{\frac{3}{4}}+115}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right)} & \frac{55\sqrt{5}-125\,5^{\frac{1}{4}}+54\,5^{\frac{3}{4}}-60}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right.} \\ 0 & 0 & \frac{-135\sqrt{5}-50\,5^{\frac{1}{4}}-56\,5^{\frac{3}{4}}+85}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right)} & \frac{-25\sqrt{5}-200\,5^{\frac{1}{4}}+22\,5^{\frac{3}{4}}+1}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right.} \\ 0 & 0 & \frac{-110\sqrt{5}-25\,5^{\frac{1}{4}}-52\,5^{\frac{3}{4}}+75}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right)} & \frac{20\sqrt{5}-75\,5^{\frac{1}{4}}+4\,5^{\frac{3}{4}}-5}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right.} \end{bmatrix}$$

$$R_4 = R_4 - \frac{\left(-135\sqrt{5}-50\,5^{\frac{1}{4}}-56\,5^{\frac{3}{4}}+85\right)R_3}{-110\sqrt{5}-175\,5^{\frac{1}{4}}+38\,5^{\frac{3}{4}}+115} \Longrightarrow \begin{bmatrix} 1-5^{\frac{3}{4}} & 3 & 5 \\ 0 & \frac{-5\sqrt{5}+3\,5^{\frac{3}{4}}+13}{5^{\frac{3}{4}}-1} & \frac{4\,5^{\frac{3}{4}}+21}{5^{\frac{3}{4}}-1} \\ 0 & 0 & \frac{-110\sqrt{5}-175\,5^{\frac{1}{4}}+38\,5^{\frac{3}{4}}+115}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right)} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-110\sqrt{5}-25\,5^{\frac{1}{4}}-52\,5^{\frac{3}{4}}+75}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right)} \end{bmatrix}$$

$$R_5 = R_5 - \frac{\left(-110\sqrt{5} - 25\,5^{\frac{1}{4}} - 52\,5^{\frac{3}{4}} + 75\right) R_3}{-110\sqrt{5} - 175\,5^{\frac{1}{4}} + 38\,5^{\frac{3}{4}} + 115} \Longrightarrow \begin{bmatrix} 1-5^{\frac{3}{4}} & 3 & 5 \\ 0 & \frac{-5\sqrt{5}+3\,5^{\frac{3}{4}}+13}{5^{\frac{3}{4}}-1} & \frac{4\,5^{\frac{3}{4}}+21}{5^{\frac{3}{4}}-1} \\ 0 & 0 & \frac{-110\sqrt{5}-175\,5^{\frac{1}{4}}+38\,5^{\frac{3}{4}}+115}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_5 = R_5 + \frac{\left(581\,5^{\frac{3}{4}} - 1829\,5^{\frac{1}{4}} - 677\sqrt{5} + 813\right) R_4}{448\,5^{\frac{3}{4}} - 20\,5^{\frac{1}{4}} + 790\sqrt{5} - 382} \Longrightarrow \begin{bmatrix} 1-5^{\frac{3}{4}} & 3 & 5 \\ 0 & \frac{-5\sqrt{5}+3\,5^{\frac{3}{4}}+13}{5^{\frac{3}{4}}-1} & \frac{4\,5^{\frac{3}{4}}+21}{5^{\frac{3}{4}}-1} \\ 0 & 0 & \frac{-110\sqrt{5}-175\,5^{\frac{1}{4}}+38\,5^{\frac{3}{4}}+115}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-5^{\frac{3}{4}} & 3 & 5 & 2 \\ 0 & \frac{-5\sqrt{5}+3\,5^{\frac{3}{4}}+13}{5^{\frac{3}{4}}-1} & \frac{4\,5^{\frac{3}{4}}+21}{5^{\frac{3}{4}}-1} & \frac{5^{\frac{3}{4}}+9}{5^{\frac{3}{4}}-1} \\ 0 & 0 & \frac{-110\sqrt{5}-175\,5^{\frac{1}{4}}+38\,5^{\frac{3}{4}}+115}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right)} & \frac{55\sqrt{5}-125\,5^{\frac{1}{4}}+54\,5^{\frac{3}{4}}-60}{\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right)} \\ 0 & 0 & 0 & \frac{19750\sqrt{5}-9550-500\,5^{\frac{1}{4}}+11200\,5^{\frac{3}{4}}}{\left(-110\sqrt{5}-175\,5^{\frac{1}{4}}+38\,5^{\frac{3}{4}}+115\right)\left(-5\sqrt{5}+3\,5^{\frac{3}{4}}+13\right)\left(5^{\frac{3}{4}}-1\right)} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The free variables are $\{v_5\}$ and the leading variables are $\{v_1, v_2, v_3, v_4\}$. Let $v_5 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\left(2\,5^{\frac{3}{4}} + 4\,5^{\frac{1}{4}} - 3\sqrt{5} - 6\right) t, v_2 = \left(5^{\frac{3}{4}} + 2\,5^{\frac{1}{4}} - \sqrt{5} - 4\right) t, v_3 = \left(5^{\frac{3}{4}}\right.\right.$

Hence the solution is

$$\begin{bmatrix} -\left(2\,5^{\frac{3}{4}}+4\,5^{\frac{1}{4}}-3\sqrt{5}-6\right)t \\ \left(5^{\frac{3}{4}}+2\,5^{\frac{1}{4}}-\sqrt{5}-4\right)t \\ \left(5^{\frac{3}{4}}+3\,5^{\frac{1}{4}}-2\sqrt{5}-4\right)t \\ -\left(5^{\frac{1}{4}}-1\right)t \\ t \end{bmatrix} = \begin{bmatrix} -\left(2\,5^{\frac{3}{4}}+4\,5^{\frac{1}{4}}-3\sqrt{5}-6\right)t \\ \left(5^{\frac{3}{4}}+2\,5^{\frac{1}{4}}-\sqrt{5}-4\right)t \\ \left(5^{\frac{3}{4}}+3\,5^{\frac{1}{4}}-2\sqrt{5}-4\right)t \\ -\left(5^{\frac{1}{4}}-1\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\left(2\,5^{\frac{3}{4}}+4\,5^{\frac{1}{4}}-3\sqrt{5}-6\right)t \\ \left(5^{\frac{3}{4}}+2\,5^{\frac{1}{4}}-\sqrt{5}-4\right)t \\ \left(5^{\frac{3}{4}}+3\,5^{\frac{1}{4}}-2\sqrt{5}-4\right)t \\ -\left(5^{\frac{1}{4}}-1\right)t \\ t \end{bmatrix} = t\begin{bmatrix} -2\,5^{\frac{3}{4}}-4\,5^{\frac{1}{4}}+3\sqrt{5}+6 \\ 5^{\frac{3}{4}}+2\,5^{\frac{1}{4}}-\sqrt{5}-4 \\ 5^{\frac{3}{4}}+3\,5^{\frac{1}{4}}-2\sqrt{5}-4 \\ -5^{\frac{1}{4}}+1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\left(2\,5^{\frac{3}{4}}+4\,5^{\frac{1}{4}}-3\sqrt{5}-6\right)t \\ \left(5^{\frac{3}{4}}+2\,5^{\frac{1}{4}}-\sqrt{5}-4\right)t \\ \left(5^{\frac{3}{4}}+3\,5^{\frac{1}{4}}-2\sqrt{5}-4\right)t \\ -\left(5^{\frac{1}{4}}-1\right)t \\ t \end{bmatrix} = \begin{bmatrix} -2\,5^{\frac{3}{4}}-4\,5^{\frac{1}{4}}+3\sqrt{5}+6 \\ 5^{\frac{3}{4}}+2\,5^{\frac{1}{4}}-\sqrt{5}-4 \\ 5^{\frac{3}{4}}+3\,5^{\frac{1}{4}}-2\sqrt{5}-4 \\ -5^{\frac{1}{4}}+1 \\ 1 \end{bmatrix}$$

Considering $\lambda = -i5^{\frac{3}{4}}$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0}
\end{aligned}
$$

$$
\left(\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix} - \left(-i5^{\frac{3}{4}}\right)\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\left(\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix} - \begin{bmatrix} -i5^{\frac{3}{4}} & 0 & 0 & 0 & 0 \\ 0 & -i5^{\frac{3}{4}} & 0 & 0 & 0 \\ 0 & 0 & -i5^{\frac{3}{4}} & 0 & 0 \\ 0 & 0 & 0 & -i5^{\frac{3}{4}} & 0 \\ 0 & 0 & 0 & 0 & -i5^{\frac{3}{4}} \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 1+i5^{\frac{3}{4}} & 3 & 5 & 2 & 4 \\ 5 & 2+i5^{\frac{3}{4}} & 4 & 1 & 3 \\ 4 & 1 & 3+i5^{\frac{3}{4}} & 5 & 2 \\ 3 & 5 & 2 & 4+i5^{\frac{3}{4}} & 1 \\ 2 & 4 & 1 & 3 & 5+i5^{\frac{3}{4}} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccccc|c} 1+i5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 5 & 2+i5^{\frac{3}{4}} & 4 & 1 & 3 & 0 \\ 4 & 1 & 3+i5^{\frac{3}{4}} & 5 & 2 & 0 \\ 3 & 5 & 2 & 4+i5^{\frac{3}{4}} & 1 & 0 \\ 2 & 4 & 1 & 3 & 5+i5^{\frac{3}{4}} & 0 \end{array}\right]
$$

$$R_2 = R_2 - \frac{5R_1}{1+i5^{\frac{3}{4}}} \Longrightarrow \left[\begin{array}{ccccc|c} 1+i5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{5i\sqrt{5}+3\,5^{\frac{3}{4}}+13i}{-i+5^{\frac{3}{4}}} & \frac{-4\,5^{\frac{3}{4}}-21i}{i-5^{\frac{3}{4}}} & \frac{-5^{\frac{3}{4}}-9i}{i-5^{\frac{3}{4}}} & \frac{-3\,5^{\frac{3}{4}}-17i}{i-5^{\frac{3}{4}}} & 0 \\ 4 & 1 & 3+i5^{\frac{3}{4}} & 5 & 2 & 0 \\ 3 & 5 & 2 & 4+i5^{\frac{3}{4}} & 1 & 0 \\ 2 & 4 & 1 & 3 & 5+i5^{\frac{3}{4}} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{4R_1}{1+i5^{\frac{3}{4}}} \Longrightarrow \left[\begin{array}{ccccc|c} 1+i5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{5i\sqrt{5}+3\,5^{\frac{3}{4}}+13i}{-i+5^{\frac{3}{4}}} & \frac{-4\,5^{\frac{3}{4}}-21i}{i-5^{\frac{3}{4}}} & \frac{-5^{\frac{3}{4}}-9i}{i-5^{\frac{3}{4}}} & \frac{-3\,5^{\frac{3}{4}}-17i}{i-5^{\frac{3}{4}}} & 0 \\ 0 & \frac{-5^{\frac{3}{4}}-11i}{i-5^{\frac{3}{4}}} & \frac{5i\sqrt{5}+4\,5^{\frac{3}{4}}+17i}{-i+5^{\frac{3}{4}}} & \frac{-5\,5^{\frac{3}{4}}-3i}{i-5^{\frac{3}{4}}} & \frac{-2\,5^{\frac{3}{4}}-14i}{i-5^{\frac{3}{4}}} & 0 \\ 3 & 5 & 2 & 4+i5^{\frac{3}{4}} & 1 & 0 \\ 2 & 4 & 1 & 3 & 5+i5^{\frac{3}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{3R_1}{1+i5^{\frac{3}{4}}} \Longrightarrow \left[\begin{array}{ccccc|c} 1+i5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{5i\sqrt{5}+3\,5^{\frac{3}{4}}+13i}{-i+5^{\frac{3}{4}}} & \frac{-4\,5^{\frac{3}{4}}-21i}{i-5^{\frac{3}{4}}} & \frac{-5^{\frac{3}{4}}-9i}{i-5^{\frac{3}{4}}} & \frac{-3\,5^{\frac{3}{4}}-17i}{i-5^{\frac{3}{4}}} & 0 \\ 0 & \frac{-5^{\frac{3}{4}}-11i}{i-5^{\frac{3}{4}}} & \frac{5i\sqrt{5}+4\,5^{\frac{3}{4}}+17i}{-i+5^{\frac{3}{4}}} & \frac{-5\,5^{\frac{3}{4}}-3i}{i-5^{\frac{3}{4}}} & \frac{-2\,5^{\frac{3}{4}}-14i}{i-5^{\frac{3}{4}}} & 0 \\ 0 & \frac{-5\,5^{\frac{3}{4}}-4i}{i-5^{\frac{3}{4}}} & \frac{-2\,5^{\frac{3}{4}}-13i}{i-5^{\frac{3}{4}}} & \frac{5i\sqrt{5}+5\,5^{\frac{3}{4}}+2i}{-i+5^{\frac{3}{4}}} & \frac{-5^{\frac{3}{4}}-11i}{i-5^{\frac{3}{4}}} & 0 \\ 2 & 4 & 1 & 3 & 5+i5^{\frac{3}{4}} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{2R_1}{1+i5^{\frac{3}{4}}} \Longrightarrow \left[\begin{array}{ccccc|c} 1+i5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{5i\sqrt{5}+3\,5^{\frac{3}{4}}+13i}{-i+5^{\frac{3}{4}}} & \frac{-4\,5^{\frac{3}{4}}-21i}{i-5^{\frac{3}{4}}} & \frac{-5^{\frac{3}{4}}-9i}{i-5^{\frac{3}{4}}} & \frac{-3\,5^{\frac{3}{4}}-17i}{i-5^{\frac{3}{4}}} & 0 \\ 0 & \frac{-5^{\frac{3}{4}}-11i}{i-5^{\frac{3}{4}}} & \frac{5i\sqrt{5}+4\,5^{\frac{3}{4}}+17i}{-i+5^{\frac{3}{4}}} & \frac{-5\,5^{\frac{3}{4}}-3i}{i-5^{\frac{3}{4}}} & \frac{-2\,5^{\frac{3}{4}}-14i}{i-5^{\frac{3}{4}}} & 0 \\ 0 & \frac{-5\,5^{\frac{3}{4}}-4i}{i-5^{\frac{3}{4}}} & \frac{-2\,5^{\frac{3}{4}}-13i}{i-5^{\frac{3}{4}}} & \frac{5i\sqrt{5}+5\,5^{\frac{3}{4}}+2i}{-i+5^{\frac{3}{4}}} & \frac{-5^{\frac{3}{4}}-11i}{i-5^{\frac{3}{4}}} & 0 \\ 0 & \frac{-4\,5^{\frac{3}{4}}-2i}{i-5^{\frac{3}{4}}} & \frac{-5^{\frac{3}{4}}-9i}{i-5^{\frac{3}{4}}} & \frac{-3\,5^{\frac{3}{4}}-i}{i-5^{\frac{3}{4}}} & \frac{5i\sqrt{5}+6\,5^{\frac{3}{4}}+3i}{-i+5^{\frac{3}{4}}} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{\left(-5^{\frac{3}{4}} - 11i\right)\left(-i + 5^{\frac{3}{4}}\right)R_2}{\left(i - 5^{\frac{3}{4}}\right)\left(5i\sqrt{5} + 3\,5^{\frac{3}{4}} + 13i\right)} \Longrightarrow \begin{bmatrix} 1 + i5^{\frac{3}{4}} & 3 & 5 & \\ 0 & \frac{5i\sqrt{5}+3\,5^{\frac{3}{4}}+13i}{-i+5^{\frac{3}{4}}} & \frac{-4\,5^{\frac{3}{4}}-21i}{i-5^{\frac{3}{4}}} & \\ 0 & 0 & \frac{110i\sqrt{5}+175\,5^{\frac{1}{4}}+115i+38\,5^{\frac{3}{4}}}{\left(-5\sqrt{5}+3i5^{\frac{3}{4}}-13\right)\left(i-5^{\frac{3}{4}}\right)} & \frac{-55\sqrt{5}-}{\left(1+i5^{\frac{3}{4}}\right)} \\ 0 & \frac{-5\,5^{\frac{3}{4}}-4i}{i-5^{\frac{3}{4}}} & \frac{-2\,5^{\frac{3}{4}}-13i}{i-5^{\frac{3}{4}}} & 5i\sqrt{} \\ 0 & \frac{-4\,5^{\frac{3}{4}}-2i}{i-5^{\frac{3}{4}}} & \frac{-5^{\frac{3}{4}}-9i}{i-5^{\frac{3}{4}}} & \end{bmatrix}$$

$$R_4 = R_4 - \frac{\left(-5\,5^{\frac{3}{4}} - 4i\right)\left(-i + 5^{\frac{3}{4}}\right)R_2}{\left(i - 5^{\frac{3}{4}}\right)\left(5i\sqrt{5} + 3\,5^{\frac{3}{4}} + 13i\right)} \Longrightarrow \begin{bmatrix} 1 + i5^{\frac{3}{4}} & 3 & 5 & \\ 0 & \frac{5i\sqrt{5}+3\,5^{\frac{3}{4}}+13i}{-i+5^{\frac{3}{4}}} & \frac{-4\,5^{\frac{3}{4}}-21i}{i-5^{\frac{3}{4}}} & \\ 0 & 0 & \frac{110i\sqrt{5}+175\,5^{\frac{1}{4}}+115i+38\,5^{\frac{3}{4}}}{\left(-5\sqrt{5}+3i5^{\frac{3}{4}}-13\right)\left(i-5^{\frac{3}{4}}\right)} & \frac{-55\sqrt{5}-}{\left(1+i5^{\frac{3}{4}}\right)} \\ 0 & 0 & \frac{-56\,5^{\frac{3}{4}}+50\,5^{\frac{1}{4}}+135i\sqrt{5}+85i}{\left(-5\sqrt{5}+3i5^{\frac{3}{4}}-13\right)\left(i-5^{\frac{3}{4}}\right)} & \frac{25i\sqrt{5}+2}{\left(-5\sqrt{5}+\right.} \\ 0 & \frac{-4\,5^{\frac{3}{4}}-2i}{i-5^{\frac{3}{4}}} & \frac{-5^{\frac{3}{4}}-9i}{i-5^{\frac{3}{4}}} & \end{bmatrix}$$

$$R_5 = R_5 - \frac{\left(-4\,5^{\frac{3}{4}} - 2i\right)\left(-i + 5^{\frac{3}{4}}\right)R_2}{\left(i - 5^{\frac{3}{4}}\right)\left(5i\sqrt{5} + 3\,5^{\frac{3}{4}} + 13i\right)} \Longrightarrow \begin{bmatrix} 1 + i5^{\frac{3}{4}} & 3 & 5 & \\ 0 & \frac{5i\sqrt{5}+3\,5^{\frac{3}{4}}+13i}{-i+5^{\frac{3}{4}}} & \frac{-4\,5^{\frac{3}{4}}-21i}{i-5^{\frac{3}{4}}} & \\ 0 & 0 & \frac{110i\sqrt{5}+175\,5^{\frac{1}{4}}+115i+38\,5^{\frac{3}{4}}}{\left(-5\sqrt{5}+3i5^{\frac{3}{4}}-13\right)\left(i-5^{\frac{3}{4}}\right)} & \frac{-55\sqrt{5}-}{\left(1+i5^{\frac{3}{4}}\right)} \\ 0 & 0 & \frac{-56\,5^{\frac{3}{4}}+50\,5^{\frac{1}{4}}+135i\sqrt{5}+85i}{\left(-5\sqrt{5}+3i5^{\frac{3}{4}}-13\right)\left(i-5^{\frac{3}{4}}\right)} & \frac{25i\sqrt{5}+2}{\left(-5\sqrt{5}+\right.} \\ 0 & 0 & \frac{-52\,5^{\frac{3}{4}}+25\,5^{\frac{1}{4}}+110i\sqrt{5}+75i}{\left(-5\sqrt{5}+3i5^{\frac{3}{4}}-13\right)\left(i-5^{\frac{3}{4}}\right)} & \frac{-20\sqrt{5}}{\left(1+i5^{\frac{3}{4}}\right)} \end{bmatrix}$$

$$R_4 = R_4 - \frac{\left(-56\,5^{\frac{3}{4}} + 50\,5^{\frac{1}{4}} + 135i\sqrt{5} + 85i\right)R_3}{110i\sqrt{5} + 175\,5^{\frac{1}{4}} + 115i + 38\,5^{\frac{3}{4}}} \Longrightarrow \begin{bmatrix} 1 + i5^{\frac{3}{4}} & 3 & 5 \\ 0 & \frac{5i\sqrt{5}+3\,5^{\frac{3}{4}}+13i}{-i+5^{\frac{3}{4}}} & \frac{-4\,5^{\frac{3}{4}}-21i}{i-5^{\frac{3}{4}}} \\ 0 & 0 & \frac{110i\sqrt{5}+175\,5^{\frac{1}{4}}+115i+38\,5^{\frac{3}{4}}}{\left(-5\sqrt{5}+3i5^{\frac{3}{4}}-13\right)\left(i-5^{\frac{3}{4}}\right)} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-52\,5^{\frac{3}{4}}+25\,5^{\frac{1}{4}}+110i\sqrt{5}+75i}{\left(-5\sqrt{5}+3i5^{\frac{3}{4}}-13\right)\left(i-5^{\frac{3}{4}}\right)} \end{bmatrix}$$

$$R_5 = R_5 - \frac{\left(-52\,5^{\frac{3}{4}} + 25\,5^{\frac{1}{4}} + 110i\sqrt{5} + 75i\right) R_3}{110i\sqrt{5} + 175\,5^{\frac{1}{4}} + 115i + 38\,5^{\frac{3}{4}}} \Longrightarrow \begin{bmatrix} 1+i5^{\frac{3}{4}} & 3 & 5 & \cdots \\ 0 & \frac{5i\sqrt{5}+3\,5^{\frac{3}{4}}+13i}{-i+5^{\frac{3}{4}}} & \frac{-4\,5^{\frac{3}{4}}-21i}{i-5^{\frac{3}{4}}} & \cdots \\ 0 & 0 & \frac{110i\sqrt{5}+175\,5^{\frac{1}{4}}+115i+38\,5^{\frac{3}{4}}}{\left(-5\sqrt{5}+3i5^{\frac{3}{4}}-13\right)\left(i-5^{\frac{3}{4}}\right)} & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \end{bmatrix}$$

$$R_5 = R_5 + \frac{\left(581\,5^{\frac{3}{4}} + 677i\sqrt{5} + 813i + 1829\,5^{\frac{1}{4}}\right)\left(5\sqrt{5} - 3i5^{\frac{3}{4}} + 13\right)\left(-i + 5^{\frac{3}{4}}\right)^2 R_4}{2\left(1 + i5^{\frac{3}{4}}\right)\left(-5\sqrt{5} + 3i5^{\frac{3}{4}} - 13\right)\left(415\,5^{\frac{3}{4}} + 1985\,5^{\frac{1}{4}} + 725i\sqrt{5} - 141i\right)} \Longrightarrow \begin{bmatrix} 1+i5^{\frac{3}{4}} & \cdots \\ 0 & \cdots \\ 0 & \cdots \\ 0 & \cdots \\ 0 & \cdots \end{bmatrix}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i5^{\frac{3}{4}} & 3 & 5 & 2 & \cdots \\ 0 & \frac{5i\sqrt{5}+3\,5^{\frac{3}{4}}+13i}{-i+5^{\frac{3}{4}}} & \frac{-4\,5^{\frac{3}{4}}-21i}{i-5^{\frac{3}{4}}} & \frac{-5^{\frac{3}{4}}-9i}{i-5^{\frac{3}{4}}} & \cdots \\ 0 & 0 & \frac{110i\sqrt{5}+175\,5^{\frac{1}{4}}+115i+38\,5^{\frac{3}{4}}}{\left(-5\sqrt{5}+3i5^{\frac{3}{4}}-13\right)\left(i-5^{\frac{3}{4}}\right)} & \frac{-55\sqrt{5}-125i5^{\frac{1}{4}}-54i5^{\frac{3}{4}}-60}{\left(1+i5^{\frac{3}{4}}\right)\left(-5\sqrt{5}+3i5^{\frac{3}{4}}-13\right)} & \cdots \\ 0 & 0 & 0 & \frac{20750\,5^{\frac{3}{4}}+99250\,5^{\frac{1}{4}}+36250i\sqrt{5}-7050i}{\left(5\sqrt{5}-3i5^{\frac{3}{4}}+13\right)\left(-i+5^{\frac{3}{4}}\right)^2\left(110i\sqrt{5}+175\,5^{\frac{1}{4}}+115i+38\,5^{\frac{3}{4}}\right)} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

The free variables are $\{v_5\}$ and the leading variables are $\{v_1, v_2, v_3, v_4\}$. Let $v_5 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t\left(1120939140i-227340276\,5^{\frac{3}{4}}-513924580\,5^{\frac{1}{4}}+502641300i\sqrt{5}\right)}{-221544380i+409045164\,5^{\frac{3}{4}}+869213100\,5^{\frac{1}{4}}-129265180i\sqrt{5}}, v_2 = \frac{t\left(-28\cdots\right.}{\left(3\,5^{\frac{3}{4}}+25i\sqrt{5}+2i\right)\cdots}\right.$

Hence the solution is

$$\left[\begin{array}{c} \frac{t\left(1120939140 \,\mathrm{I}-227340276\, 5^{\frac{3}{4}}-513924580\, 5^{\frac{1}{4}}+502641300 \,\mathrm{I} \sqrt{5}\right)}{-221544380 \,\mathrm{I}+4090451645^{\frac{3}{4}}+869213100\, 5^{\frac{1}{4}}-129265180 \,\mathrm{I} \sqrt{5}} \\ \frac{t\left(-289010568883680 \,\mathrm{I}+684198522768928\, 5^{\frac{3}{4}}+1528289069064160\, 5^{\frac{1}{4}}-130209759077280 \,\mathrm{I} \sqrt{5}\right)}{\left(3\, 5^{\frac{3}{4}}+25 \,\mathrm{I} \sqrt{5}+2 \,\mathrm{I}\right)\left(-221544380 \,\mathrm{I}+409045164\, 5^{\frac{3}{4}}+869213100\, 5^{\frac{1}{4}}-129265180 \,\mathrm{I} \sqrt{5}\right)\left(2800 \,\mathrm{I} \sqrt{5}+9784 \,\mathrm{I}-1640\, 5^{\frac{1}{4}}+556\, 5^{\frac{3}{4}}\right)}-\frac{t\left(1202774792474240 \,\mathrm{I}+312962603795392\, 5^{\frac{3}{4}}+689928752306240\, 5^{\frac{1}{4}}+531127942499840 \,\mathrm{I} \sqrt{5}\right)}{\left(3\, 5^{\frac{3}{4}}+25 \,\mathrm{I} \sqrt{5}+2 \,\mathrm{I}\right)\left(-221544380 \,\mathrm{I}+409045164\, 5^{\frac{3}{4}}+869213100\, 5^{\frac{1}{4}}-129265180 \,\mathrm{I} \sqrt{5}\right)\left(2800 \,\mathrm{I} \sqrt{5}+9784 \,\mathrm{I}-1640\, 5^{\frac{1}{4}}+556\, 5^{\frac{3}{4}}\right)} \\ \frac{t\left(8144\, 5^{\frac{1}{4}}+7004 \,\mathrm{I}+4440 \,\mathrm{I} \sqrt{5}+3356\, 5^{\frac{3}{4}}\right)}{2800 \,\mathrm{I} \sqrt{5}+9784 \,\mathrm{I}-1640\, 5^{\frac{1}{4}}+556\, 5^{\frac{3}{4}}} \\ t \end{array}\right]=\left[\right.$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} \frac{t\left(1120939140 \,\mathrm{I}-227340276\, 5^{\frac{3}{4}}-513924580\, 5^{\frac{1}{4}}+502641300 \,\mathrm{I} \sqrt{5}\right)}{-221544380 \,\mathrm{I}+409045164\, 5^{\frac{3}{4}}+869213100\, 5^{\frac{1}{4}}-129265180 \,\mathrm{I} \sqrt{5}} \\ \frac{t\left(-289010568883680 \,\mathrm{I}+684198522768928\, 5^{\frac{3}{4}}+1528289069064160\, 5^{\frac{1}{4}}-130209759077280 \,\mathrm{I} \sqrt{5}\right)}{\left(3\, 5^{\frac{3}{4}}+25 \,\mathrm{I} \sqrt{5}+2 \,\mathrm{I}\right)\left(-221544380 \,\mathrm{I}+409045164\, 5^{\frac{3}{4}}+869213100\, 5^{\frac{1}{4}}-129265180 \,\mathrm{I} \sqrt{5}\right)\left(2800 \,\mathrm{I} \sqrt{5}+9784 \,\mathrm{I}-1640\, 5^{\frac{1}{4}}+556\, 5^{\frac{3}{4}}\right)}-\frac{t\left(1202774792474240 \,\mathrm{I}+312962603795392\, 5^{\frac{3}{4}}+689928752306240\, 5^{\frac{1}{4}}+531127942499840 \,\mathrm{I} \sqrt{5}\right)}{\left(3\, 5^{\frac{3}{4}}+25 \,\mathrm{I} \sqrt{5}+2 \,\mathrm{I}\right)\left(-221544380 \,\mathrm{I}+409045164\, 5^{\frac{3}{4}}+869213100\, 5^{\frac{1}{4}}-129265180 \,\mathrm{I} \sqrt{5}\right)\left(2800 \,\mathrm{I} \sqrt{5}+9784 \,\mathrm{I}-1640\, 5^{\frac{1}{4}}+556\, 5^{\frac{3}{4}}\right)} \\ \frac{t\left(8144\, 5^{\frac{1}{4}}+7004 \,\mathrm{I}+4440 \,\mathrm{I} \sqrt{5}+3356\, 5^{\frac{3}{4}}\right)}{2800 \,\mathrm{I} \sqrt{5}+9784 \,\mathrm{I}-1640\, 5^{\frac{1}{4}}+556\, 5^{\frac{3}{4}}} \\ t \end{array}\right]=t\left[\right.$$

Or, by letting $t=1$ then the eigenvector is

$$\left[\begin{array}{c} \frac{t\left(1120939140 \,\mathrm{I}-227340276\, 5^{\frac{3}{4}}-513924580\, 5^{\frac{1}{4}}+502641300 \,\mathrm{I} \sqrt{5}\right)}{-221544380 \,\mathrm{I}+409045164\, 5^{\frac{3}{4}}+869213100\, 5^{\frac{1}{4}}-129265180 \,\mathrm{I} \sqrt{5}} \\ \frac{t\left(-289010568883680 \,\mathrm{I}+684198522768928\, 5^{\frac{3}{4}}+1528289069064160\, 5^{\frac{1}{4}}-130209759077280 \,\mathrm{I} \sqrt{5}\right)}{\left(3\, 5^{\frac{3}{4}}+25 \,\mathrm{I} \sqrt{5}+2 \,\mathrm{I}\right)\left(-221544380 \,\mathrm{I}+409045164\, 5^{\frac{3}{4}}+869213100\, 5^{\frac{1}{4}}-129265180 \,\mathrm{I} \sqrt{5}\right)\left(2800 \,\mathrm{I} \sqrt{5}+9784 \,\mathrm{I}-1640\, 5^{\frac{1}{4}}+556\, 5^{\frac{3}{4}}\right)}-\frac{t\left(1202774792474240 \,\mathrm{I}+312962603795392\, 5^{\frac{3}{4}}+689928752306240\, 5^{\frac{1}{4}}+531127942499840 \,\mathrm{I} \sqrt{5}\right)}{\left(3\, 5^{\frac{3}{4}}+25 \,\mathrm{I} \sqrt{5}+2 \,\mathrm{I}\right)\left(-221544380 \,\mathrm{I}+409045164\, 5^{\frac{3}{4}}+869213100\, 5^{\frac{1}{4}}-129265180 \,\mathrm{I} \sqrt{5}\right)\left(2800 \,\mathrm{I} \sqrt{5}+9784 \,\mathrm{I}-1640\, 5^{\frac{1}{4}}+556\, 5^{\frac{3}{4}}\right)} \\ \frac{t\left(8144\, 5^{\frac{1}{4}}+7004 \,\mathrm{I}+4440 \,\mathrm{I} \sqrt{5}+3356\, 5^{\frac{3}{4}}\right)}{2800 \,\mathrm{I} \sqrt{5}+9784 \,\mathrm{I}-1640\, 5^{\frac{1}{4}}+556\, 5^{\frac{3}{4}}} \\ t \end{array}\right]=\left[\right.$$

Which can be normalized to

$$\left[\begin{array}{c} \frac{t \left(1120939140\,\mathrm{I}-227340276\,5^{\frac{3}{4}}-513924580\,5^{\frac{1}{4}}+502641300\,\mathrm{I}\,\sqrt{5}\right)}{-221544380\,\mathrm{I}+409045164\,5^{\frac{3}{4}}+869213100\,5^{\frac{1}{4}}-129265180\,\mathrm{I}\,\sqrt{5}} \\ \frac{t \left(-289010568883680\,\mathrm{I}+684198522768928\,5^{\frac{3}{4}}+1528289069064160\,5^{\frac{1}{4}}-130209759077280\,\mathrm{I}\,\sqrt{5}\right)}{\left(3\,5^{\frac{3}{4}}+25\,\mathrm{I}\,\sqrt{5}+2\,\mathrm{I}\right) \left(-221544380\,\mathrm{I}+409045164\,5^{\frac{3}{4}}+869213100\,5^{\frac{1}{4}}-129265180\,\mathrm{I}\,\sqrt{5}\right) \left(2800\,\mathrm{I}\,\sqrt{5}+9784\,\mathrm{I}-1640\,5^{\frac{1}{4}}+556\,5^{\frac{3}{4}}\right)} \\ -\frac{t \left(1202774792474240\,\mathrm{I}+312962603795392\,5^{\frac{3}{4}}+689928752306240\,5^{\frac{1}{4}}+531127942499840\,\mathrm{I}\,\sqrt{5}\right)}{\left(3\,5^{\frac{3}{4}}+25\,\mathrm{I}\,\sqrt{5}+2\,\mathrm{I}\right) \left(-221544380\,\mathrm{I}+409045164\,5^{\frac{3}{4}}+869213100\,5^{\frac{1}{4}}-129265180\,\mathrm{I}\,\sqrt{5}\right) \left(2800\,\mathrm{I}\,\sqrt{5}+9784\,\mathrm{I}-1640\,5^{\frac{1}{4}}+556\,5^{\frac{3}{4}}\right)} \\ \frac{t \left(8144\,5^{\frac{1}{4}}+7004\,\mathrm{I}+4440\,\mathrm{I}\,\sqrt{5}+3356\,5^{\frac{3}{4}}\right)}{2800\,\mathrm{I}\,\sqrt{5}+9784\,\mathrm{I}-1640\,5^{\frac{1}{4}}+556\,5^{\frac{3}{4}}} \\ t \end{array}\right] = \left[\right.$$

Considering $\lambda = i5^{\frac{3}{4}}$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix} - \left(i5^{\frac{3}{4}}\right) \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix} - \begin{bmatrix} i5^{\frac{3}{4}} & 0 & 0 & 0 & 0 \\ 0 & i5^{\frac{3}{4}} & 0 & 0 & 0 \\ 0 & 0 & i5^{\frac{3}{4}} & 0 & 0 \\ 0 & 0 & 0 & i5^{\frac{3}{4}} & 0 \\ 0 & 0 & 0 & 0 & i5^{\frac{3}{4}} \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 - i5^{\frac{3}{4}} & 3 & 5 & 2 & 4 \\ 5 & 2 - i5^{\frac{3}{4}} & 4 & 1 & 3 \\ 4 & 1 & 3 - i5^{\frac{3}{4}} & 5 & 2 \\ 3 & 5 & 2 & 4 - i5^{\frac{3}{4}} & 1 \\ 2 & 4 & 1 & 3 & 5 - i5^{\frac{3}{4}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccccc|c} 1-i5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 5 & 2-i5^{\frac{3}{4}} & 4 & 1 & 3 & 0 \\ 4 & 1 & 3-i5^{\frac{3}{4}} & 5 & 2 & 0 \\ 3 & 5 & 2 & 4-i5^{\frac{3}{4}} & 1 & 0 \\ 2 & 4 & 1 & 3 & 5-i5^{\frac{3}{4}} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{5R_1}{1-i5^{\frac{3}{4}}} \Longrightarrow \left[\begin{array}{ccccc|c} 1-i5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{3\,5^{\frac{3}{4}}-5i\sqrt{5}-13i}{5^{\frac{3}{4}}+i} & \frac{4\,5^{\frac{3}{4}}-21i}{5^{\frac{3}{4}}+i} & \frac{5^{\frac{3}{4}}-9i}{5^{\frac{3}{4}}+i} & \frac{3\,5^{\frac{3}{4}}-17i}{5^{\frac{3}{4}}+i} & 0 \\ 4 & 1 & 3-i5^{\frac{3}{4}} & 5 & 2 & 0 \\ 3 & 5 & 2 & 4-i5^{\frac{3}{4}} & 1 & 0 \\ 2 & 4 & 1 & 3 & 5-i5^{\frac{3}{4}} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{4R_1}{1-i5^{\frac{3}{4}}} \Longrightarrow \left[\begin{array}{ccccc|c} 1-i5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{3\,5^{\frac{3}{4}}-5i\sqrt{5}-13i}{5^{\frac{3}{4}}+i} & \frac{4\,5^{\frac{3}{4}}-21i}{5^{\frac{3}{4}}+i} & \frac{5^{\frac{3}{4}}-9i}{5^{\frac{3}{4}}+i} & \frac{3\,5^{\frac{3}{4}}-17i}{5^{\frac{3}{4}}+i} & 0 \\ 0 & \frac{5^{\frac{3}{4}}-11i}{5^{\frac{3}{4}}+i} & \frac{4\,5^{\frac{3}{4}}-5i\sqrt{5}-17i}{5^{\frac{3}{4}}+i} & \frac{5\,5^{\frac{3}{4}}-3i}{5^{\frac{3}{4}}+i} & \frac{2\,5^{\frac{3}{4}}-14i}{5^{\frac{3}{4}}+i} & 0 \\ 3 & 5 & 2 & 4-i5^{\frac{3}{4}} & 1 & 0 \\ 2 & 4 & 1 & 3 & 5-i5^{\frac{3}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{3R_1}{1-i5^{\frac{3}{4}}} \Longrightarrow \left[\begin{array}{ccccc|c} 1-i5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{3\,5^{\frac{3}{4}}-5i\sqrt{5}-13i}{5^{\frac{3}{4}}+i} & \frac{4\,5^{\frac{3}{4}}-21i}{5^{\frac{3}{4}}+i} & \frac{5^{\frac{3}{4}}-9i}{5^{\frac{3}{4}}+i} & \frac{3\,5^{\frac{3}{4}}-17i}{5^{\frac{3}{4}}+i} & 0 \\ 0 & \frac{5^{\frac{3}{4}}-11i}{5^{\frac{3}{4}}+i} & \frac{4\,5^{\frac{3}{4}}-5i\sqrt{5}-17i}{5^{\frac{3}{4}}+i} & \frac{5\,5^{\frac{3}{4}}-3i}{5^{\frac{3}{4}}+i} & \frac{2\,5^{\frac{3}{4}}-14i}{5^{\frac{3}{4}}+i} & 0 \\ 0 & \frac{5\,5^{\frac{3}{4}}-4i}{5^{\frac{3}{4}}+i} & \frac{2\,5^{\frac{3}{4}}-13i}{5^{\frac{3}{4}}+i} & \frac{5\,5^{\frac{3}{4}}-5i\sqrt{5}-2i}{5^{\frac{3}{4}}+i} & \frac{5^{\frac{3}{4}}-11i}{5^{\frac{3}{4}}+i} & 0 \\ 2 & 4 & 1 & 3 & 5-i5^{\frac{3}{4}} & 0 \end{array}\right]$$

$$R_5 = R_5 - \frac{2R_1}{1 - i5^{\frac{3}{4}}} \Longrightarrow \left[\begin{array}{ccccc|c} 1 - i5^{\frac{3}{4}} & 3 & 5 & 2 & 4 & 0 \\ 0 & \frac{3\,5^{\frac{3}{4}}-5i\sqrt{5}-13i}{5^{\frac{3}{4}}+i} & \frac{4\,5^{\frac{3}{4}}-21i}{5^{\frac{3}{4}}+i} & \frac{5^{\frac{3}{4}}-9i}{5^{\frac{3}{4}}+i} & \frac{3\,5^{\frac{3}{4}}-17i}{5^{\frac{3}{4}}+i} & 0 \\ 0 & \frac{5^{\frac{3}{4}}-11i}{5^{\frac{3}{4}}+i} & \frac{4\,5^{\frac{3}{4}}-5i\sqrt{5}-17i}{5^{\frac{3}{4}}+i} & \frac{5\,5^{\frac{3}{4}}-3i}{5^{\frac{3}{4}}+i} & \frac{2\,5^{\frac{3}{4}}-14i}{5^{\frac{3}{4}}+i} & 0 \\ 0 & \frac{5\,5^{\frac{3}{4}}-4i}{5^{\frac{3}{4}}+i} & \frac{2\,5^{\frac{3}{4}}-13i}{5^{\frac{3}{4}}+i} & \frac{5\,5^{\frac{3}{4}}-5i\sqrt{5}-2i}{5^{\frac{3}{4}}+i} & \frac{5^{\frac{3}{4}}-11i}{5^{\frac{3}{4}}+i} & 0 \\ 0 & \frac{4\,5^{\frac{3}{4}}-2i}{5^{\frac{3}{4}}+i} & \frac{5^{\frac{3}{4}}-9i}{5^{\frac{3}{4}}+i} & \frac{3\,5^{\frac{3}{4}}-i}{5^{\frac{3}{4}}+i} & \frac{6\,5^{\frac{3}{4}}-5i\sqrt{5}-3i}{5^{\frac{3}{4}}+i} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{\left(5^{\frac{3}{4}} - 11i\right) R_2}{3\,5^{\frac{3}{4}} - 5i\sqrt{5} - 13i} \Longrightarrow \left[\begin{array}{cccc} 1 - i5^{\frac{3}{4}} & 3 & 5 & 2 \\ 0 & \frac{3\,5^{\frac{3}{4}}-5i\sqrt{5}-13i}{5^{\frac{3}{4}}+i} & \frac{4\,5^{\frac{3}{4}}-21i}{5^{\frac{3}{4}}+i} & \frac{5^{\frac{3}{4}}-9i}{5^{\frac{3}{4}}+i} \\ 0 & 0 & \frac{38\,5^{\frac{3}{4}}+175\,5^{\frac{1}{4}}-110i\sqrt{5}-115i}{25\,5^{\frac{1}{4}}+10\,5^{\frac{3}{4}}+20i\sqrt{5}+13i} & \frac{54\,5^{\frac{3}{4}}+125\,5^{\frac{1}{4}}+55i\sqrt{5}+60}{\left(3i5^{\frac{3}{4}}+5\sqrt{5}+13\right)\left(5^{\frac{3}{4}}+i\right)} \\ 0 & \frac{5\,5^{\frac{3}{4}}-4i}{5^{\frac{3}{4}}+i} & \frac{2\,5^{\frac{3}{4}}-13i}{5^{\frac{3}{4}}+i} & \frac{5\,5^{\frac{3}{4}}-5i\sqrt{5}-2i}{5^{\frac{3}{4}}+i} \\ 0 & \frac{4\,5^{\frac{3}{4}}-2i}{5^{\frac{3}{4}}+i} & \frac{5^{\frac{3}{4}}-9i}{5^{\frac{3}{4}}+i} & \frac{3\,5^{\frac{3}{4}}-i}{5^{\frac{3}{4}}+i} \end{array}\right.$$

$$R_4 = R_4 - \frac{\left(5\,5^{\frac{3}{4}} - 4i\right) R_2}{3\,5^{\frac{3}{4}} - 5i\sqrt{5} - 13i} \Longrightarrow \left[\begin{array}{cccc} 1 - i5^{\frac{3}{4}} & 3 & 5 & 2 \\ 0 & \frac{3\,5^{\frac{3}{4}}-5i\sqrt{5}-13i}{5^{\frac{3}{4}}+i} & \frac{4\,5^{\frac{3}{4}}-21i}{5^{\frac{3}{4}}+i} & \frac{5^{\frac{3}{4}}-9i}{5^{\frac{3}{4}}+i} \\ 0 & 0 & \frac{38\,5^{\frac{3}{4}}+175\,5^{\frac{1}{4}}-110i\sqrt{5}-115i}{25\,5^{\frac{1}{4}}+10\,5^{\frac{3}{4}}+20i\sqrt{5}+13i} & \frac{54\,5^{\frac{3}{4}}+125\,5^{\frac{1}{4}}+55i\sqrt{5}+60}{\left(3i5^{\frac{3}{4}}+5\sqrt{5}+13\right)\left(5^{\frac{3}{4}}+i\right)} \\ 0 & 0 & \frac{-135i\sqrt{5}+50\,5^{\frac{1}{4}}-56\,5^{\frac{3}{4}}-85i}{\left(3i5^{\frac{3}{4}}+5\sqrt{5}+13\right)\left(5^{\frac{3}{4}}+i\right)} & \frac{22\,5^{\frac{3}{4}}+200\,5^{\frac{1}{4}}-25i\sqrt{5}-11}{25\,5^{\frac{1}{4}}+10\,5^{\frac{3}{4}}+20i\sqrt{5}+13} \\ 0 & \frac{4\,5^{\frac{3}{4}}-2i}{5^{\frac{3}{4}}+i} & \frac{5^{\frac{3}{4}}-9i}{5^{\frac{3}{4}}+i} & \frac{3\,5^{\frac{3}{4}}-i}{5^{\frac{3}{4}}+i} \end{array}\right.$$

$$R_5 = R_5 - \frac{\left(4\,5^{\frac{3}{4}} - 2i\right) R_2}{3\,5^{\frac{3}{4}} - 5i\sqrt{5} - 13i} \Longrightarrow \left[\begin{array}{cccc} 1 - i5^{\frac{3}{4}} & 3 & 5 & 2 \\ 0 & \frac{3\,5^{\frac{3}{4}}-5i\sqrt{5}-13i}{5^{\frac{3}{4}}+i} & \frac{4\,5^{\frac{3}{4}}-21i}{5^{\frac{3}{4}}+i} & \frac{5^{\frac{3}{4}}-9i}{5^{\frac{3}{4}}+i} \\ 0 & 0 & \frac{38\,5^{\frac{3}{4}}+175\,5^{\frac{1}{4}}-110i\sqrt{5}-115i}{25\,5^{\frac{1}{4}}+10\,5^{\frac{3}{4}}+20i\sqrt{5}+13i} & \frac{54\,5^{\frac{3}{4}}+125\,5^{\frac{1}{4}}+55i\sqrt{5}+60}{\left(3i5^{\frac{3}{4}}+5\sqrt{5}+13\right)\left(5^{\frac{3}{4}}+i\right)} \\ 0 & 0 & \frac{-135i\sqrt{5}+50\,5^{\frac{1}{4}}-56\,5^{\frac{3}{4}}-85i}{\left(3i5^{\frac{3}{4}}+5\sqrt{5}+13\right)\left(5^{\frac{3}{4}}+i\right)} & \frac{22\,5^{\frac{3}{4}}+200\,5^{\frac{1}{4}}-25i\sqrt{5}-11}{25\,5^{\frac{1}{4}}+10\,5^{\frac{3}{4}}+20i\sqrt{5}+13} \\ 0 & 0 & \frac{-110i\sqrt{5}+25\,5^{\frac{1}{4}}-52\,5^{\frac{3}{4}}-75i}{\left(3i5^{\frac{3}{4}}+5\sqrt{5}+13\right)\left(5^{\frac{3}{4}}+i\right)} & \frac{4\,5^{\frac{3}{4}}+75\,5^{\frac{1}{4}}+20i\sqrt{5}+5i}{\left(3i5^{\frac{3}{4}}+5\sqrt{5}+13\right)\left(5^{\frac{3}{4}}+i\right)} \end{array}\right.$$

$$R_4 = R_4 - \frac{\left(-135i\sqrt{5} + 50\,5^{\frac{1}{4}} - 56\,5^{\frac{3}{4}} - 85i\right)\left(25\,5^{\frac{1}{4}} + 10\,5^{\frac{3}{4}} + 20i\sqrt{5} + 13i\right)R_3}{\left(3i5^{\frac{3}{4}} + 5\sqrt{5} + 13\right)\left(5^{\frac{3}{4}} + i\right)\left(38\,5^{\frac{3}{4}} + 175\,5^{\frac{1}{4}} - 110i\sqrt{5} - 115i\right)} \Longrightarrow \begin{bmatrix} 1 - i5^{\frac{3}{4}} & 3 \\ 0 & \frac{3\,5^{\frac{3}{4}} - 5i}{5^{\frac{3}{4}}} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$R_5 = R_5 - \frac{\left(-110i\sqrt{5} + 25\,5^{\frac{1}{4}} - 52\,5^{\frac{3}{4}} - 75i\right)\left(25\,5^{\frac{1}{4}} + 10\,5^{\frac{3}{4}} + 20i\sqrt{5} + 13i\right)R_3}{\left(3i5^{\frac{3}{4}} + 5\sqrt{5} + 13\right)\left(5^{\frac{3}{4}} + i\right)\left(38\,5^{\frac{3}{4}} + 175\,5^{\frac{1}{4}} - 110i\sqrt{5} - 115i\right)} \Longrightarrow \begin{bmatrix} 1 - i5^{\frac{3}{4}} & 3 \\ 0 & \frac{3\,5^{\frac{3}{4}} - 5i}{5^{\frac{3}{4}}} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$R_5 = R_5 - \frac{\left(9539i5^{\frac{3}{4}} + 49918\sqrt{5} + 121172 + 13851i5^{\frac{1}{4}}\right)\left(38\,5^{\frac{3}{4}} + 175\,5^{\frac{1}{4}} - 110i\sqrt{5} - 115i\right)\left(25\,5^{\frac{1}{4}} + 10\,5\right.}{4\left(i5^{\frac{3}{4}} - 1\right)^2\left(38i5^{\frac{3}{4}} + 175i5^{\frac{1}{4}} + 110\sqrt{5} + 115\right)\left(301804i5^{\frac{3}{4}} + 578435i5^{\frac{1}{4}} + 4210\right.}$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i5^{\frac{3}{4}} & 3 & 5 & 2 \\ 0 & \frac{3\,5^{\frac{3}{4}}-5i\sqrt{5}-13i}{5^{\frac{3}{4}}+i} & \frac{4\,5^{\frac{3}{4}}-21i}{5^{\frac{3}{4}}+i} & \frac{5^{\frac{3}{4}}-9i}{5^{\frac{3}{4}}+i} \\ 0 & 0 & \frac{38\,5^{\frac{3}{4}}+175\,5^{\frac{1}{4}}-110i\sqrt{5}-115i}{25\,5^{\frac{1}{4}}+10\,5^{\frac{3}{4}}+20i\sqrt{5}+13i} & \frac{54\,5^{\frac{3}{4}}+125\,5^{\frac{1}{4}}+55i\sqrt{5}+60i}{\left(3i5^{\frac{3}{4}}+5\sqrt{5}+13\right)\left(5^{\frac{3}{4}}+i\right)} \\ 0 & 0 & 0 & \frac{60360800i5^{\frac{3}{4}}+115687000i5^{\frac{1}{4}}+84200000\sqrt{5}+214454800}{\left(38\,5^{\frac{3}{4}}+175\,5^{\frac{1}{4}}-110i\sqrt{5}-115i\right)\left(25\,5^{\frac{1}{4}}+10\,5^{\frac{3}{4}}+20i\sqrt{5}+13i\right)\left(3i5^{\frac{3}{4}}+5\sqrt{5}+\right.} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The free variables are $\{v_5\}$ and the leading variables are $\{v_1, v_2, v_3, v_4\}$. Let $v_5 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t\left(-535254249891900i+952896311573490\,5^{\frac{3}{4}}-239503585554020i\sqrt{5}+2130940072176550\,5\right.}{-2632502774254600i+562948189520990\,5^{\frac{3}{4}}-1173944063380280i\sqrt{5}+1253785384570450\,5}\right.$

Hence the solution is

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Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t\left(-535254249891900\,\mathrm{I}+952896311573490\,5^{\frac{3}{4}}-239503585554020\,\mathrm{I}\sqrt{5}+2130940072176550\,5^{\frac{1}{4}}\right)}{-2632502774254600\,\mathrm{I}+562948189520990\,5^{\frac{3}{4}}-1173944063380280\,\mathrm{I}\sqrt{5}+1253785384570450\,5^{\frac{1}{4}}} \\ -\frac{t\left(-2796476061925888744241118000-1250611566079379186356530000\sqrt{5}+2759160749959802888034944000\,\mathrm{I}\,5^{\frac{1}{4}}+1233927714\right.}{\left(-3\,5^{\frac{3}{4}}+25\,\mathrm{I}\sqrt{5}+2\,\mathrm{I}\right)\left(-2632502774254600\,\mathrm{I}+562948189520990\,5^{\frac{3}{4}}-1173944063380280\,\mathrm{I}\sqrt{5}+1253785384570450\,5^{\frac{1}{4}}\right)\left(7796255\sqrt{5}+500\right.} \\ \frac{t\left(1288366538676959815237630000+576243943027962529373066000\sqrt{5}+2122108438874472087257716000\,\mathrm{I}\,5^{\frac{1}{4}}+948989661429\right.}{\left(-3\,5^{\frac{3}{4}}+25\,\mathrm{I}\sqrt{5}+2\,\mathrm{I}\right)\left(-2632502774254600\,\mathrm{I}+562948189520990\,5^{\frac{3}{4}}-1173944063380280\,\mathrm{I}\sqrt{5}+1253785384570450\,5^{\frac{1}{4}}\right)\left(7796255\sqrt{5}+5001\right.} \\ -\frac{t\left(-220552977\,\mathrm{I}\,5^{\frac{3}{4}}-481030535\,\mathrm{I}\,5^{\frac{1}{4}}+1082903250+492353920\sqrt{5}\right)}{7796255\sqrt{5}+500150175\,\mathrm{I}\,5^{\frac{1}{4}}+212756722\,\mathrm{I}\,5^{\frac{3}{4}}-19119640} \\ t \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

Expression too large to display

Which can be normalized to

Expression too large to display

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
15	1	5	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$
$-5^{\frac{3}{4}}$	1	5	No	$\begin{bmatrix} 2\,5^{\frac{3}{4}}+4\,5^{\frac{1}{4}}+3\sqrt{5}+6 \\ -5^{\frac{3}{4}}-2\,5^{\frac{1}{4}}-\sqrt{5}-4 \\ -5^{\frac{3}{4}}-3\,5^{\frac{1}{4}}-2\sqrt{5}-4 \\ 5^{\frac{1}{4}}+1 \\ 1 \end{bmatrix}$
$5^{\frac{3}{4}}$	1	5	No	$\begin{bmatrix} -2\,5^{\frac{3}{4}}-4\,5^{\frac{1}{4}}+3\sqrt{5}+6 \\ 5^{\frac{3}{4}}+2\,5^{\frac{1}{4}}-\sqrt{5}-4 \\ 5^{\frac{3}{4}}+3\,5^{\frac{1}{4}}-2\sqrt{5}-4 \\ -5^{\frac{1}{4}}+1 \\ 1 \end{bmatrix}$
$-i5^{\frac{3}{4}}$	1	5	No	$\begin{bmatrix} \frac{1120939140i-227340276\,5^{\frac{3}{4}}-51392458}{-221544380i+409045164\,5^{\frac{3}{4}}+86921310} \\ \frac{-289010568883680i+684198522768928\,5^{\frac{3}{4}}+15282890}{\left(3\,5^{\frac{3}{4}}+25i\sqrt{5}+2i\right)\left(-221544380i+409045164\,5^{\frac{3}{4}}+869213100\,5^{\frac{1}{4}}-12\right.} \\ -\frac{1202774792474240i+312962603795392\,5^{\frac{3}{4}}+6899287}{\left(3\,5^{\frac{3}{4}}+25i\sqrt{5}+2i\right)\left(-221544380i+409045164\,5^{\frac{3}{4}}+869213100\,5^{\frac{1}{4}}-1\right.} \\ \frac{8144\,5^{\frac{1}{4}}+7004i+4440i\sqrt{5}+}{2800i\sqrt{5}+9784i-1640\,5^{\frac{1}{4}}-} \\ 1 \end{bmatrix}$
$i5^{\frac{3}{4}}$	1	5	No	$\begin{bmatrix} -\frac{-535254249891900i+9528}{-2632502774254600i+5629} \\ -\frac{-27964760619258887442411800 0-125061156607937 9}{\left(-3\,5^{\frac{3}{4}}+25i\sqrt{5}+2i\right)\left(-2632502774254600i+562948189520990\,5^{\frac{3}{4}}-\right.} \\ \frac{1288366538676959815237630 00+5762439430279625}{\left(-3\,5^{\frac{3}{4}}+25i\sqrt{5}+2i\right)\left(-2632502774254600i+562948189520990\,5^{\frac{3}{4}}-1\right.} \\ -\frac{-22055297}{7796255} \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 15 & 0 & 0 & 0 & 0 \\ 0 & -5^{\frac{3}{4}} & 0 & 0 & 0 \\ 0 & 0 & 5^{\frac{3}{4}} & 0 & 0 \\ 0 & 0 & 0 & -i5^{\frac{3}{4}} & 0 \\ 0 & 0 & 0 & 0 & i5^{\frac{3}{4}} \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2\,5^{\frac{3}{4}} + 4\,5^{\frac{1}{4}} + 3\sqrt{5} + 6 & -2\,5^{\frac{3}{4}} - 4\,5^{\frac{1}{4}} + 3\sqrt{5} + 6 & \frac{1120939140i - 227340276\,5^{\frac{3}{4}} \cdots}{-221544380i + 409045164\,5^{\frac{3}{4}} \cdots} & \cdots \\ 1 & -5^{\frac{3}{4}} - 2\,5^{\frac{1}{4}} - \sqrt{5} - 4 & 5^{\frac{3}{4}} + 2\,5^{\frac{1}{4}} - \sqrt{5} - 4 & \frac{-289010568883680i + 684198522768928\,5^{\frac{3}{4}} \cdots}{\left(3\,5^{\frac{3}{4}} + 25i\sqrt{5} + 2i\right)\left(-221544380i + 409045164\,5^{\frac{3}{4}} + 869213 \cdots\right.} & \cdots \\ 1 & -5^{\frac{3}{4}} - 3\,5^{\frac{1}{4}} - 2\sqrt{5} - 4 & 5^{\frac{3}{4}} + 3\,5^{\frac{1}{4}} - 2\sqrt{5} - 4 & -\frac{1202774792474240i + 312962603795392\,5^{\frac{3}{4}} \cdots}{\left(3\,5^{\frac{3}{4}} + 25i\sqrt{5} + 2i\right)\left(-221544380i + 409045164\,5^{\frac{3}{4}} + 86921 \cdots\right.} & \cdots \\ 1 & 5^{\frac{1}{4}} + 1 & -5^{\frac{1}{4}} + 1 & \frac{8144\,5^{\frac{1}{4}} + 7004i \cdots}{2800i\sqrt{5} + 9784i \cdots} & \cdots \\ 1 & 1 & 1 & \cdots & \cdots \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2\,5^{\frac{3}{4}} + 4\,5^{\frac{1}{4}} + 3\sqrt{5} + 6 & -2\,5^{\frac{3}{4}} - 4\,5^{\frac{1}{4}} + 3\sqrt{5} + 6 & \frac{1120 \cdots}{-221 \cdots} & \cdots \\ 1 & -5^{\frac{3}{4}} - 2\,5^{\frac{1}{4}} - \sqrt{5} - 4 & 5^{\frac{3}{4}} + 2\,5^{\frac{1}{4}} - \sqrt{5} - 4 & \frac{-289010568883680i \cdots}{\left(3\,5^{\frac{3}{4}} + 25i\sqrt{5} + 2i\right)\left(-221544380i + \cdots\right.} & \cdots \\ 1 & -5^{\frac{3}{4}} - 3\,5^{\frac{1}{4}} - 2\sqrt{5} - 4 & 5^{\frac{3}{4}} + 3\,5^{\frac{1}{4}} - 2\sqrt{5} - 4 & -\frac{120277479247424 \cdots}{\left(3\,5^{\frac{3}{4}} + 25i\sqrt{5} + 2i\right)\left(-221544380i \cdots\right.} & \cdots \\ 1 & 5^{\frac{1}{4}} + 1 & -5^{\frac{1}{4}} + 1 & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & \cdots \end{bmatrix}$$

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19.1 problem 1

Internal problem ID [12839]
Internal file name [`OUTPUT/11491_Saturday_November_04_2023_09_17_37_AM_72152883/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= 2y_1(x) - 3y_2(x) + 5\,\mathrm{e}^{x} \\
y_2'(x) &= y_1(x) + 4y_2(x) - 2\,\mathrm{e}^{-x}
\end{aligned}
$$

19.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} + \begin{bmatrix} 5\,\mathrm{e}^{x} \\ -2\,\mathrm{e}^{-x} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(x) = \vec{x}_h(x) + \vec{x}_p(x)$$

Where $\vec{x}_h(x)$ is the homogeneous solution to $\vec{x}'(x) = A\,\vec{x}(x)$ and $\vec{x}_p(x)$ is a particular solution to $\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{3x}\cos\left(\sqrt{2}\,x\right) - \frac{e^{3x}\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{2} & -\frac{3\,e^{3x}\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{2} \\ \frac{e^{3x}\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{2} & e^{3x}\cos\left(\sqrt{2}\,x\right) + \frac{e^{3x}\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{e^{3x}\left(\sin\left(\sqrt{2}\,x\right)\sqrt{2}-2\cos\left(\sqrt{2}\,x\right)\right)}{2} & -\frac{3\,e^{3x}\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{2} \\ \frac{e^{3x}\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{2} & \frac{e^{3x}\left(\sin\left(\sqrt{2}\,x\right)\sqrt{2}+2\cos\left(\sqrt{2}\,x\right)\right)}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(x) = e^{At}\vec{c}$$

$$= \begin{bmatrix} -\frac{e^{3x}\left(\sin\left(\sqrt{2}\,x\right)\sqrt{2}-2\cos\left(\sqrt{2}\,x\right)\right)}{2} & -\frac{3\,e^{3x}\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{2} \\ \frac{e^{3x}\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{2} & \frac{e^{3x}\left(\sin\left(\sqrt{2}\,x\right)\sqrt{2}+2\cos\left(\sqrt{2}\,x\right)\right)}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{e^{3x}\left(\sin\left(\sqrt{2}\,x\right)\sqrt{2}-2\cos\left(\sqrt{2}\,x\right)\right)c_1}{2} - \frac{3\,e^{3x}\sin\left(\sqrt{2}\,x\right)\sqrt{2}\,c_2}{2} \\ \frac{e^{3x}\sin\left(\sqrt{2}\,x\right)\sqrt{2}\,c_1}{2} + \frac{e^{3x}\left(\sin\left(\sqrt{2}\,x\right)\sqrt{2}+2\cos\left(\sqrt{2}\,x\right)\right)c_2}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\left(\sqrt{2}\,(c_1+3c_2)\sin\left(\sqrt{2}\,x\right)-2\cos\left(\sqrt{2}\,x\right)c_1\right)e^{3x}}{2} \\ \frac{e^{3x}\left(\sqrt{2}\,(c_1+c_2)\sin\left(\sqrt{2}\,x\right)+2\cos\left(\sqrt{2}\,x\right)c_2\right)}{2} \end{bmatrix}$$

The particular solution given by

$$\vec{x}_p(x) = e^{Ax}\int e^{-Ax}\vec{G}(x)\,dx$$

But

$$e^{-Ax} = \left(e^{Ax}\right)^{-1}$$

$$= \begin{bmatrix} \frac{\left(\sin\left(\sqrt{2}\,x\right)\sqrt{2}+2\cos\left(\sqrt{2}\,x\right)\right)e^{-3x}}{2} & \frac{3\sin\left(\sqrt{2}\,x\right)\sqrt{2}\,e^{-3x}}{2} \\ -\frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}\,e^{-3x}}{2} & -\frac{\left(\sin\left(\sqrt{2}\,x\right)\sqrt{2}-2\cos\left(\sqrt{2}\,x\right)\right)e^{-3x}}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(x) = \begin{bmatrix} -\frac{e^{3x}\left(\sin\left(\sqrt{2}\, x\right)\sqrt{2}-2\cos\left(\sqrt{2}\, x\right)\right)}{2} & -\frac{3\, e^{3x}\sin\left(\sqrt{2}\, x\right)\sqrt{2}}{2} \\ \frac{e^{3x}\sin\left(\sqrt{2}\, x\right)\sqrt{2}}{2} & \frac{e^{3x}\left(\sin\left(\sqrt{2}\, x\right)\sqrt{2}+2\cos\left(\sqrt{2}\, x\right)\right)}{2} \end{bmatrix} \int \begin{bmatrix} \frac{\left(\sin\left(\sqrt{2}\, x\right)\sqrt{2}+2\cos\left(\sqrt{2}\, x\right)\right)e^{-3x}}{2} & \\ -\frac{\sin\left(\sqrt{2}\, x\right)\sqrt{2}\, e^{-3x}}{2} & -\frac{(s}{} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{e^{3x}\left(\sin\left(\sqrt{2}\, x\right)\sqrt{2}-2\cos\left(\sqrt{2}\, x\right)\right)}{2} & -\frac{3\, e^{3x}\sin\left(\sqrt{2}\, x\right)\sqrt{2}}{2} \\ \frac{e^{3x}\sin\left(\sqrt{2}\, x\right)\sqrt{2}}{2} & \frac{e^{3x}\left(\sin\left(\sqrt{2}\, x\right)\sqrt{2}+2\cos\left(\sqrt{2}\, x\right)\right)}{2} \end{bmatrix} \begin{bmatrix} \frac{\left(2\, e^{-4x}-15\, e^{-2x}\right)\cos\left(\sqrt{2}\, x\right)}{6} + \frac{2\sqrt{2}\, e^{-4x}}{} \\ \frac{\left(2\, e^{-4x}+5\, e^{-2x}\right)\cos\left(\sqrt{2}\, x\right)}{6} - \frac{\sin\left(\sqrt{2}\, x\right)\left(e^{-}\right.}{} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5\, e^{x}}{2} + \frac{e^{-x}}{3} \\ \frac{5\, e^{x}}{6} + \frac{e^{-x}}{3} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(x) &= \vec{x}_h(x) + \vec{x}_p(x) \\ &= \begin{bmatrix} -\frac{\sqrt{2}\,(c_1+3c_2)e^{3x}\sin\left(\sqrt{2}\, x\right)}{2} + e^{3x}\cos\left(\sqrt{2}\, x\right)c_1 - \frac{5\, e^{x}}{2} + \frac{e^{-x}}{3} \\ \frac{\sqrt{2}\,(c_1+c_2)e^{3x}\sin\left(\sqrt{2}\, x\right)}{2} + e^{3x}\cos\left(\sqrt{2}\, x\right)c_2 + \frac{5\, e^{x}}{6} + \frac{e^{-x}}{3} \end{bmatrix} \end{aligned}$$

19.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} + \begin{bmatrix} 5\, e^{x} \\ -2\, e^{-x} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(x) = \vec{x}_h(x) + \vec{x}_p(x)$$

Where $\vec{x}_h(x)$ is the homogeneous solution to $\vec{x}'(x) = A\,\vec{x}(x)$ and $\vec{x}_p(x)$ is a particular solution to $\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}2 & -3\\1 & 4\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}2-\lambda & -3\\1 & 4-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 11 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3 + i\sqrt{2}$$
$$\lambda_2 = 3 - i\sqrt{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$3 - i\sqrt{2}$	1	complex eigenvalue
$3 + i\sqrt{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3 - i\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}2 & -3\\1 & 4\end{bmatrix} - \left(3 - i\sqrt{2}\right)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$
$$\begin{bmatrix}i\sqrt{2}-1 & -3\\1 & 1+i\sqrt{2}\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}i\sqrt{2}-1 & -3 & 0\\1 & 1+i\sqrt{2} & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{i\sqrt{2}-1} \implies \left[\begin{array}{cc|c} i\sqrt{2}-1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i\sqrt{2}-1 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{i\sqrt{2}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{\mathrm{I}\sqrt{2}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{i\sqrt{2}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{\mathrm{I}\sqrt{2}-1} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{i\sqrt{2}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{\mathrm{I}\sqrt{2}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{i\sqrt{2}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{\mathrm{I}\sqrt{2}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{i\sqrt{2}-1} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3 + i\sqrt{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} - \left(3 + i\sqrt{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 - i\sqrt{2} & -3 \\ 1 & 1 - i\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1-i\sqrt{2} & -3 & 0 \\ 1 & 1-i\sqrt{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{-1-i\sqrt{2}} \Longrightarrow \left[\begin{array}{cc|c} -1-i\sqrt{2} & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-i\sqrt{2} & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3t}{1+i\sqrt{2}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{3t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3t}{1+i\sqrt{2}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{3t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{3t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{3t}{1+\mathrm{I}\sqrt{2}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{1+i\sqrt{2}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$3+i\sqrt{2}$	1	1	No	$\left[\begin{array}{c} -\frac{3}{1+i\sqrt{2}} \\ 1 \end{array}\right]$
$3-i\sqrt{2}$	1	1	No	$\left[\begin{array}{c} -\frac{3}{1-i\sqrt{2}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(x) = c_1\vec{x}_1(x) + c_2\vec{x}_2(x)$$

Which is written as

$$\left[\begin{array}{c} y_1(x) \\ y_2(x) \end{array}\right] = c_1 \left[\begin{array}{c} -\frac{3\,\mathrm{e}^{\left(3+i\sqrt{2}\right)x}}{1+i\sqrt{2}} \\ \mathrm{e}^{\left(3+i\sqrt{2}\right)x} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{3\,\mathrm{e}^{\left(3-i\sqrt{2}\right)x}}{1-i\sqrt{2}} \\ \mathrm{e}^{\left(3-i\sqrt{2}\right)x} \end{array}\right]$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(x)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(x) = \left[\begin{array}{cc} -\frac{3\,\mathrm{e}^{\left(3+i\sqrt{2}\right)x}}{1+i\sqrt{2}} & -\frac{3\,\mathrm{e}^{\left(3-i\sqrt{2}\right)x}}{1-i\sqrt{2}} \\ \mathrm{e}^{\left(3+i\sqrt{2}\right)x} & \mathrm{e}^{\left(3-i\sqrt{2}\right)x} \end{array}\right]$$

The particular solution is then given by

$$\vec{x}_p(x) = \Phi \int \Phi^{-1}\vec{G}(x)\,dx$$

But

$$\Phi^{-1} = \left[\begin{array}{cc} -\frac{i\sqrt{2}\,\mathrm{e}^{-\left(3+i\sqrt{2}\right)x}}{4} & -\frac{\sqrt{2}\,\mathrm{e}^{-\left(3+i\sqrt{2}\right)x}\left(i-\sqrt{2}\right)}{4} \\ \frac{i\sqrt{2}\,\mathrm{e}^{\left(i\sqrt{2}-3\right)x}}{4} & \frac{\mathrm{e}^{\left(i\sqrt{2}-3\right)x}\sqrt{2}\left(i+\sqrt{2}\right)}{4} \end{array}\right]$$

Hence

$$
\vec{x}_p(x) = \left[\begin{array}{cc} -\frac{3\,\mathrm{e}^{\left(3+i\sqrt{2}\right)x}}{1+i\sqrt{2}} & -\frac{3\,\mathrm{e}^{\left(3-i\sqrt{2}\right)x}}{1-i\sqrt{2}} \\ \mathrm{e}^{\left(3+i\sqrt{2}\right)x} & \mathrm{e}^{\left(3-i\sqrt{2}\right)x} \end{array}\right] \int \left[\begin{array}{cc} -\frac{i\sqrt{2}\,\mathrm{e}^{-\left(3+i\sqrt{2}\right)x}}{4} & -\frac{\sqrt{2}\,\mathrm{e}^{-\left(3+i\sqrt{2}\right)x}\left(i-\sqrt{2}\right)}{4} \\ \frac{i\sqrt{2}\,\mathrm{e}^{\left(i\sqrt{2}-3\right)x}}{4} & \frac{\mathrm{e}^{\left(i\sqrt{2}-3\right)x}\sqrt{2}\left(i+\sqrt{2}\right)}{4} \end{array}\right] \left[\begin{array}{c} 5\,\mathrm{e}^{x} \\ -2\,\mathrm{e}^{-x} \end{array}\right] dx
$$

$$
= \left[\begin{array}{cc} -\frac{3\,\mathrm{e}^{\left(3+i\sqrt{2}\right)x}}{1+i\sqrt{2}} & -\frac{3\,\mathrm{e}^{\left(3-i\sqrt{2}\right)x}}{1-i\sqrt{2}} \\ \mathrm{e}^{\left(3+i\sqrt{2}\right)x} & \mathrm{e}^{\left(3-i\sqrt{2}\right)x} \end{array}\right] \int \left[\begin{array}{c} -\frac{5\sqrt{2}\left(\frac{2\,\mathrm{e}^{-\left(i\sqrt{2}+4\right)x}\left(-i+\sqrt{2}\right)}{5}+i\mathrm{e}^{-\left(2+i\sqrt{2}\right)x}\right)}{4} \\ -\frac{\left(\left(i+\sqrt{2}\right)\mathrm{e}^{\left(i\sqrt{2}-4\right)x}-\frac{5i\mathrm{e}^{\left(i\sqrt{2}-2\right)x}}{2}\right)\sqrt{2}}{2} \end{array}\right] dx
$$

$$
= \left[\begin{array}{cc} -\frac{3\,\mathrm{e}^{\left(3+i\sqrt{2}\right)x}}{1+i\sqrt{2}} & -\frac{3\,\mathrm{e}^{\left(3-i\sqrt{2}\right)x}}{1-i\sqrt{2}} \\ \mathrm{e}^{\left(3+i\sqrt{2}\right)x} & \mathrm{e}^{\left(3-i\sqrt{2}\right)x} \end{array}\right] \left[\begin{array}{c} \frac{\sqrt{2}\left(-5\sqrt{2}+20i\right)\mathrm{e}^{-\left(2+i\sqrt{2}\right)x}+12\,\mathrm{e}^{-\left(i\sqrt{2}+4\right)x}}{4\left(2+i\sqrt{2}\right)\left(i\sqrt{2}+4\right)} \\ \frac{\sqrt{2}\left(20i+5\sqrt{2}\right)\mathrm{e}^{\left(i\sqrt{2}-2\right)x}-12\,\mathrm{e}^{\left(i\sqrt{2}-4\right)x}}{-24+24i\sqrt{2}} \end{array}\right]
$$

$$
= \left[\begin{array}{c} -\frac{5\,\mathrm{e}^{x}}{2}+\frac{\mathrm{e}^{-x}}{3} \\ \frac{5\,\mathrm{e}^{x}}{6}+\frac{\mathrm{e}^{-x}}{3} \end{array}\right]
$$

Now that we found particular solution, the final solution is

$$
\vec{x}(x) = \vec{x}_h(x) + \vec{x}_p(x)
$$

$$
\left[\begin{array}{c} y_1(x) \\ y_2(x) \end{array}\right] = \left[\begin{array}{c} -\frac{3c_1\mathrm{e}^{\left(3+i\sqrt{2}\right)x}}{1+i\sqrt{2}} \\ c_1\mathrm{e}^{\left(3+i\sqrt{2}\right)x} \end{array}\right] + \left[\begin{array}{c} -\frac{3c_2\mathrm{e}^{\left(3-i\sqrt{2}\right)x}}{1-i\sqrt{2}} \\ c_2\mathrm{e}^{\left(3-i\sqrt{2}\right)x} \end{array}\right] + \left[\begin{array}{c} -\frac{5\,\mathrm{e}^{x}}{2}+\frac{\mathrm{e}^{-x}}{3} \\ \frac{5\,\mathrm{e}^{x}}{6}+\frac{\mathrm{e}^{-x}}{3} \end{array}\right]
$$

Which becomes

$$
\left[\begin{array}{c} y_1(x) \\ y_2(x) \end{array}\right] = \left[\begin{array}{c} c_2\left(-1-i\sqrt{2}\right)\mathrm{e}^{-\left(i\sqrt{2}-3\right)x}+\left(i\sqrt{2}-1\right)c_1\mathrm{e}^{\left(3+i\sqrt{2}\right)x}-\frac{5\,\mathrm{e}^{x}}{2}+\frac{\mathrm{e}^{-x}}{3} \\ c_1\mathrm{e}^{\left(3+i\sqrt{2}\right)x}+c_2\mathrm{e}^{-\left(i\sqrt{2}-3\right)x}+\frac{5\,\mathrm{e}^{x}}{6}+\frac{\mathrm{e}^{-x}}{3} \end{array}\right]
$$

19.1.3 Maple step by step solution

Let's solve

$\left[y_1'(x) = 2y_1(x) - 3y_2(x) + 5\,\mathrm{e}^{x}, y_2'(x) = y_1(x) + 4y_2(x) - \frac{2}{\mathrm{e}^{x}}\right]$

- Define vector

$$\underline{\vec{y}}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{y}}'(x) = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \cdot \underline{\vec{y}}(x) + \begin{bmatrix} 5\,\mathrm{e}^{x} \\ \frac{4\,\mathrm{e}^{x} y_2(x) + y_1(x)\mathrm{e}^{x} - 2}{\mathrm{e}^{x}} - y_1(x) - 4 y_2(x) \end{bmatrix}$$

- System to solve

$$\underline{\vec{y}}'(x) = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \cdot \underline{\vec{y}}(x) + \begin{bmatrix} 5\,\mathrm{e}^{x} \\ 0 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 5\,\mathrm{e}^{x} \\ 0 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(x) = A \cdot \underline{\vec{y}}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[3 - \mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{3}{1-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}\right], \left[3 + \mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{3}{1+\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[3 - \mathrm{I}\sqrt{2}, \begin{bmatrix} -\frac{3}{1-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(3-\mathrm{I}\sqrt{2}\right)x} \cdot \begin{bmatrix} -\frac{3}{1-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{3x} \cdot \left(\cos\left(\sqrt{2}\, x\right) - \mathrm{I} \sin\left(\sqrt{2}\, x\right)\right) \cdot \left[\begin{array}{c} -\frac{3}{1-\mathrm{I}\sqrt{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{3x} \cdot \left[\begin{array}{c} -\frac{3\left(\cos\left(\sqrt{2}\, x\right)-\mathrm{I}\sin\left(\sqrt{2}\, x\right)\right)}{1-\mathrm{I}\sqrt{2}} \\ \cos\left(\sqrt{2}\, x\right) - \mathrm{I}\sin\left(\sqrt{2}\, x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{y}^{\rightarrow}{}_{1}(x) = e^{3x} \cdot \left[\begin{array}{c} -\cos\left(\sqrt{2}\, x\right) - \sin\left(\sqrt{2}\, x\right)\sqrt{2} \\ \cos\left(\sqrt{2}\, x\right) \end{array}\right], \underline{y}^{\rightarrow}{}_{2}(x) = e^{3x} \cdot \left[\begin{array}{c} -\cos\left(\sqrt{2}\, x\right)\sqrt{2} + \sin \\ -\sin\left(\sqrt{2}\, x\right) \end{array}\right.\right.$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{y}^{\rightarrow}$

$$\underline{y}^{\rightarrow}(x) = c_1 \underline{y}^{\rightarrow}{}_{1}(x) + c_2 \underline{y}^{\rightarrow}{}_{2}(x) + \underline{y}^{\rightarrow}{}_{p}(x)$$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \left[\begin{array}{cc} e^{3x}\left(-\cos\left(\sqrt{2}\, x\right) - \sin\left(\sqrt{2}\, x\right)\sqrt{2}\right) & e^{3x}\left(-\cos\left(\sqrt{2}\, x\right)\sqrt{2} + \sin\left(\sqrt{2}\, x\right)\right) \\ e^{3x}\cos\left(\sqrt{2}\, x\right) & -e^{3x}\sin\left(\sqrt{2}\, x\right) \end{array}\right]$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \left[\begin{array}{cc} e^{3x}\left(-\cos\left(\sqrt{2}\, x\right) - \sin\left(\sqrt{2}\, x\right)\sqrt{2}\right) & e^{3x}\left(-\cos\left(\sqrt{2}\, x\right)\sqrt{2} + \sin\left(\sqrt{2}\, x\right)\right) \\ e^{3x}\cos\left(\sqrt{2}\, x\right) & -e^{3x}\sin\left(\sqrt{2}\, x\right) \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} -1 & - \\ 1 & \end{array}\right.}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \left[\begin{array}{cc} \frac{\left(\cos\left(\sqrt{2}\, x\right)\sqrt{2}-\sin\left(\sqrt{2}\, x\right)\right)\sqrt{2}\, e^{3x}}{2} & -\frac{3\, e^{3x}\sin\left(\sqrt{2}\, x\right)\sqrt{2}}{2} \\ \frac{e^{3x}\sin\left(\sqrt{2}\, x\right)\sqrt{2}}{2} & \frac{e^{3x}\left(\sin\left(\sqrt{2}\, x\right)\sqrt{2}+2\cos\left(\sqrt{2}\, x\right)\right)}{2} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\underline{\overrightarrow{y}}_{p}(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\underline{\overrightarrow{y}}_{p}'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\underline{\overrightarrow{y}}_{p}(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\underline{\overrightarrow{y}}_{p}(x) = \left[\begin{array}{c} \frac{5\,\mathrm{e}^{x}\left(\mathrm{e}^{2x}\cos\left(\sqrt{2}\,x\right)-1\right)}{2} \\ \frac{5\,\mathrm{e}^{x}\left(\mathrm{e}^{2x}\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)-\mathrm{e}^{2x}\cos\left(\sqrt{2}\,x\right)+1\right)}{6} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\underline{\overrightarrow{y}}(x) = c_1\underline{\overrightarrow{y}}_{1}(x) + c_2\underline{\overrightarrow{y}}_{2}(x) + \left[\begin{array}{c} \frac{5\,\mathrm{e}^{x}\left(\mathrm{e}^{2x}\cos\left(\sqrt{2}\,x\right)-1\right)}{2} \\ \frac{5\,\mathrm{e}^{x}\left(\mathrm{e}^{2x}\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)-\mathrm{e}^{2x}\cos\left(\sqrt{2}\,x\right)+1\right)}{6} \end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} y_1(x) \\ y_2(x) \end{array}\right] = \left[\begin{array}{c} -\mathrm{e}^{3x}\left(\sqrt{2}\,c_2 + c_1 - \frac{5}{2}\right)\cos\left(\sqrt{2}\,x\right) - \mathrm{e}^{3x}\left(\sqrt{2}\,c_1 - c_2\right)\sin\left(\sqrt{2}\,x\right) - \frac{5\,\mathrm{e}^{x}}{2} \\ \frac{\mathrm{e}^{3x}(6c_1-5)\cos\left(\sqrt{2}\,x\right)}{6} - \mathrm{e}^{3x}\left(c_2 - \frac{5\sqrt{2}}{6}\right)\sin\left(\sqrt{2}\,x\right) + \frac{5\,\mathrm{e}^{x}}{6} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{y_1(x) = -\mathrm{e}^{3x}\left(\sqrt{2}\,c_2 + c_1 - \frac{5}{2}\right)\cos\left(\sqrt{2}\,x\right) - \mathrm{e}^{3x}\left(\sqrt{2}\,c_1 - c_2\right)\sin\left(\sqrt{2}\,x\right) - \frac{5\,\mathrm{e}^{x}}{2}, y_2(x) = \frac{\mathrm{e}^{3x}(6c_1-5)}{6}\right.$$

✓ Solution by Maple

Time used: 0.157 (sec). Leaf size: 112

```
dsolve([diff(y__1(x),x)=2*y__1(x)-3*y__2(x)+5*exp(x),diff(y__2(x),x)=y__1(x)+4*y__2(x)-2*exp
```

$$y_1(x) = \mathrm{e}^{3x}\cos\left(\sqrt{2}\,x\right)c_2 + \mathrm{e}^{3x}\sin\left(\sqrt{2}\,x\right)c_1 + \frac{\mathrm{e}^{-x}}{3} - \frac{5\,\mathrm{e}^{x}}{2}$$

$$y_2(x) = -\frac{\mathrm{e}^{3x}\cos\left(\sqrt{2}\,x\right)c_2}{3} + \frac{\mathrm{e}^{3x}\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)c_2}{3} - \frac{\mathrm{e}^{3x}\sin\left(\sqrt{2}\,x\right)c_1}{3} - \frac{\mathrm{e}^{3x}\sqrt{2}\,\cos\left(\sqrt{2}\,x\right)c_1}{3} + \frac{\mathrm{e}^{-x}}{3} + \frac{5\,\mathrm{e}^{x}}{6}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 108

```
DSolve[{y1'[x]==2*y1[x]-3*y2[x]+5*Exp[x],y2'[x]==y1[x]+4*y2[x]-2*Exp[-x]},{y1[x],y2[x]},x,In
```

$$\mathrm{y1}(x) \to -\frac{1}{2}e^{x}\left(-2c_1e^{2x}\cos\left(\sqrt{2}x\right) + \sqrt{2}(c_1+3c_2)e^{2x}\sin\left(\sqrt{2}x\right) + 5\right)$$

$$\mathrm{y2}(x) \to \frac{5e^{x}}{6} + c_2e^{3x}\cos\left(\sqrt{2}x\right) + \frac{(c_1+c_2)e^{3x}\sin\left(\sqrt{2}x\right)}{\sqrt{2}}$$

19.2 problem 2

Internal problem ID [12840]
Internal file name [`OUTPUT/11492_Monday_November_06_2023_01_31_00_PM_64086885/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= y_2(x) - 2y_1(x) + 2\cos(x)\sin(x) \\
y_2'(x) &= -3y_1(x) + y_2(x) - 8\cos(x)^3 + 6\cos(x)
\end{aligned}
$$

19.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} + \begin{bmatrix} 2\cos(x)\sin(x) \\ -8\cos(x)^3 + 6\cos(x) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(x) = \vec{x}_h(x) + \vec{x}_p(x)$$

Where $\vec{x}_h(x)$ is the homogeneous solution to $\vec{x}'(x) = A\,\vec{x}(x)$ and $\vec{x}_p(x)$ is a particular solution to $\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) - e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3} & \frac{2\,e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} \\ -2\,e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3} & e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\right) e^{-\frac{x}{2}} & \frac{2\,e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} \\ -2\,e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3} & \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\right) e^{-\frac{x}{2}} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(x) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\right) e^{-\frac{x}{2}} & \frac{2\,e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} \\ -2\,e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3} & \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\right) e^{-\frac{x}{2}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\right) e^{-\frac{x}{2}} c_1 + \frac{2\,e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\,c_2}{3} \\ -2\,e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\,c_1 + \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\right) e^{-\frac{x}{2}} c_2 \end{bmatrix} \\ &= \begin{bmatrix} -e^{-\frac{x}{2}}\left(\sqrt{3}\left(c_1 - \frac{2c_2}{3}\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right) - \cos\left(\frac{\sqrt{3}\,x}{2}\right)c_1\right) \\ -2\,e^{-\frac{x}{2}}\left(\sqrt{3}\left(c_1 - \frac{c_2}{2}\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right) - \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)c_2}{2}\right) \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(x) = e^{Ax}\int e^{-Ax}\vec{G}(x)\,dx$$

But

$$\begin{aligned} e^{-Ax} &= \left(e^{Ax}\right)^{-1} \\ &= \begin{bmatrix} \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\right) e^{\frac{x}{2}} & -\frac{2\sqrt{3}\,e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3} \\ 2\sqrt{3}\,e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) & \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\right) e^{\frac{x}{2}} \end{bmatrix} \end{aligned}$$

Hence

$$\vec{x}_p(x) = \begin{bmatrix} \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\right)\mathrm{e}^{-\frac{x}{2}} & \frac{2\,\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} \\ -2\,\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3} & \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\right)\mathrm{e}^{-\frac{x}{2}} \end{bmatrix} \int \begin{bmatrix} \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \mathrm{s}\cdots \\ 2\sqrt{3}\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{s}\cdots \end{bmatrix}$$

$$= \begin{bmatrix} \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\right)\mathrm{e}^{-\frac{x}{2}} & \frac{2\,\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} \\ -2\,\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3} & \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\right)\mathrm{e}^{-\frac{x}{2}} \end{bmatrix} \begin{bmatrix} \frac{80\,\mathrm{e}^{\frac{x}{2}}\left(\left(\frac{12\cos(x)^3}{5}+\left(-\frac{9}{\cdots}\right.\right.\right.}{\cdots} \\ \frac{72\,\mathrm{e}^{\frac{x}{2}}\left(\left(\frac{7\cos(x)^3}{9}+\left(-\cdots\right.\right.\right.}{\cdots} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7\sin(2x)}{13} - \frac{6\sin(3x)}{73} + \frac{16\cos(3x)}{73} - \frac{4\cos(2x)}{13} \\ \frac{9\sin(2x)}{13} - \frac{60\sin(3x)}{73} + \frac{14\cos(3x)}{73} + \frac{6\cos(2x)}{13} \end{bmatrix}$$

Hence the complete solution is

$$\vec{x}(x) = \vec{x}_h(x) + \vec{x}_p(x)$$

$$= \begin{bmatrix} -\sqrt{3}\left(c_1 - \frac{2c_2}{3}\right)\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) + \cos\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}c_1 - \frac{4\cos(2x)}{13} + \frac{16\cos(3x)}{73} + \frac{7\sin(2x)}{13} - \frac{6\sin(3x)}{73} \\ -2\sqrt{3}\left(c_1 - \frac{c_2}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) + \cos\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}c_2 + \frac{6\cos(2x)}{13} + \frac{14\cos(3x)}{73} + \frac{9\sin(2x)}{13} - \frac{60\sin(3x)}{73} \end{bmatrix}$$

19.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} + \begin{bmatrix} 2\cos(x)\sin(x) \\ -8\cos(x)^3 + 6\cos(x) \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(x) = \vec{x}_h(x) + \vec{x}_p(x)$$

Where $\vec{x}_h(x)$ is the homogeneous solution to $\vec{x}'(x) = A\,\vec{x}(x)$ and $\vec{x}_p(x)$ is a particular solution to $\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}-2 & 1\\ -3 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-2-\lambda & 1\\ -3 & 1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\frac{1}{2} - \frac{i\sqrt{3}}{2}$	1	complex eigenvalue
$-\frac{1}{2} + \frac{i\sqrt{3}}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}-2 & 1\\ -3 & 1\end{bmatrix} - \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}-\frac{3}{2} + \frac{i\sqrt{3}}{2} & 1\\ -3 & \frac{3}{2} + \frac{i\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}-\frac{3}{2} + \frac{i\sqrt{3}}{2} & 1 & 0\\ -3 & \frac{3}{2} + \frac{i\sqrt{3}}{2} & 0\end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{-\frac{3}{2} + \frac{i\sqrt{3}}{2}} \implies \left[\begin{array}{cc|c} -\frac{3}{2} + \frac{i\sqrt{3}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{3}{2} + \frac{i\sqrt{3}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{i\sqrt{3}-3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{i\sqrt{3}-3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{i\sqrt{3}-3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{i\sqrt{3}-3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{i\sqrt{3}-3} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix} - \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3}{2} - \frac{i\sqrt{3}}{2} & 1 \\ -3 & \frac{3}{2} - \frac{i\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{3}{2}-\frac{i\sqrt{3}}{2} & 1 & 0 \\ -3 & \frac{3}{2}-\frac{i\sqrt{3}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{-\frac{3}{2}-\frac{i\sqrt{3}}{2}} \Longrightarrow \left[\begin{array}{cc|c} -\frac{3}{2}-\frac{i\sqrt{3}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{3}{2}-\frac{i\sqrt{3}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{i\sqrt{3}+3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{\mathrm{I}\sqrt{3}+3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{i\sqrt{3}+3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{\mathrm{I}\sqrt{3}+3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{i\sqrt{3}+3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{\mathrm{I}\sqrt{3}+3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{i\sqrt{3}+3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{\mathrm{I}\sqrt{3}+3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{i\sqrt{3}+3} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	1	1	No	$\left[\begin{array}{c} \frac{1}{\frac{3}{2}+\frac{i\sqrt{3}}{2}} \\ 1 \end{array}\right]$
$-\frac{1}{2}-\frac{i\sqrt{3}}{2}$	1	1	No	$\left[\begin{array}{c} \frac{1}{\frac{3}{2}-\frac{i\sqrt{3}}{2}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(x) = c_1\vec{x}_1(x) + c_2\vec{x}_2(x)$$

Which is written as

$$\left[\begin{array}{c} y_1(x) \\ y_2(x) \end{array}\right] = c_1 \left[\begin{array}{c} \frac{e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}}{\frac{3}{2}+\frac{i\sqrt{3}}{2}} \\ e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} \end{array}\right] + c_2 \left[\begin{array}{c} \frac{e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}}{\frac{3}{2}-\frac{i\sqrt{3}}{2}} \\ e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} \end{array}\right]$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(x)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(x) = \left[\begin{array}{cc} \frac{e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}}{\frac{3}{2}+\frac{i\sqrt{3}}{2}} & \frac{e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}}{\frac{3}{2}-\frac{i\sqrt{3}}{2}} \\ e^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} & e^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} \end{array}\right]$$

The particular solution is then given by

$$\vec{x}_p(x) = \Phi \int \Phi^{-1}\vec{G}(x)\, dx$$

But

$$\Phi^{-1} = \left[\begin{array}{cc} i\sqrt{3}\,{\mathrm e}^{-\frac{\left(i \sqrt{3}-1\right) x}{2}} & -\frac{\left(3 i-\sqrt{3}\right) \sqrt{3}\,{\mathrm e}^{-\frac{\left(i \sqrt{3}-1\right) x}{2}}}{6} \\ -i\sqrt{3}\,{\mathrm e}^{\frac{\left(1+i \sqrt{3}\right) x}{2}} & \frac{\sqrt{3}\,{\mathrm e}^{\frac{\left(1+i \sqrt{3}\right) x}{2}} \left(\sqrt{3}+3 i\right)}{6} \end{array}\right]$$

Hence

$$\vec{x}_p(x) = \left[\begin{array}{cc} \frac{{\mathrm e}^{\left(-\frac{1}{2}+\frac{i \sqrt{3}}{2}\right) x}}{\frac{3}{2}+\frac{i \sqrt{3}}{2}} & \frac{{\mathrm e}^{\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right) x}}{\frac{3}{2}-\frac{i \sqrt{3}}{2}} \\ {\mathrm e}^{\left(-\frac{1}{2}+\frac{i \sqrt{3}}{2}\right) x} & {\mathrm e}^{\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right) x} \end{array}\right] \int \left[\begin{array}{cc} i\sqrt{3}\,{\mathrm e}^{-\frac{\left(i \sqrt{3}-1\right) x}{2}} & -\frac{\left(3 i-\sqrt{3}\right) \sqrt{3}\,{\mathrm e}^{-\frac{\left(i \sqrt{3}-1\right) x}{2}}}{6} \\ -i\sqrt{3}\,{\mathrm e}^{\frac{\left(1+i \sqrt{3}\right) x}{2}} & \frac{\sqrt{3}\,{\mathrm e}^{\frac{\left(1+i \sqrt{3}\right) x}{2}} \left(\sqrt{3}+3 i\right)}{6} \end{array}\right] \left[\begin{array}{c} 2 \cos\left(x\right) \sin\left(x\right) \\ -8 \cos\left(x\right)^{3}+6 \cos \end{array}\right]$$

$$= \left[\begin{array}{cc} \frac{{\mathrm e}^{\left(-\frac{1}{2}+\frac{i \sqrt{3}}{2}\right) x}}{\frac{3}{2}+\frac{i \sqrt{3}}{2}} & \frac{{\mathrm e}^{\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right) x}}{\frac{3}{2}-\frac{i \sqrt{3}}{2}} \\ {\mathrm e}^{\left(-\frac{1}{2}+\frac{i \sqrt{3}}{2}\right) x} & {\mathrm e}^{\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right) x} \end{array}\right] \int \left[\begin{array}{c} 4 \cos\left(x\right) {\mathrm e}^{-\frac{\left(i \sqrt{3}-1\right) x}{2}} \left(\frac{3}{4}+i \left(\cos\left(x\right)^{2}+\frac{\sin\left(x\right)}{2}-\frac{3}{4}\right) \sqrt{3}-\cos\left(x\right)^{2} \right. \\ 4 \left(\frac{3}{4}-i \left(\cos\left(x\right)^{2}+\frac{\sin\left(x\right)}{2}-\frac{3}{4}\right) \sqrt{3}-\cos\left(x\right)^{2}\right) \cos\left(x\right) {\mathrm e}^{\frac{\left(1+i \sqrt{3}\right) x}{2}} \end{array}\right]$$

$$= \left[\begin{array}{cc} \frac{{\mathrm e}^{\left(-\frac{1}{2}+\frac{i \sqrt{3}}{2}\right) x}}{\frac{3}{2}+\frac{i \sqrt{3}}{2}} & \frac{{\mathrm e}^{\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right) x}}{\frac{3}{2}-\frac{i \sqrt{3}}{2}} \\ {\mathrm e}^{\left(-\frac{1}{2}+\frac{i \sqrt{3}}{2}\right) x} & {\mathrm e}^{\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right) x} \end{array}\right] \left[\begin{array}{c} \frac{\left(\left(68 i \sqrt{3}+12\right) \cos\left(2 x\right)+\left(-12 i \sqrt{3}-20\right) \cos\left(3 x\right)-14 i \sin\left(2 x\right) \sqrt{3}-48 i \sqrt{3}\, \sin\left(3 x\right)-54 \sin(2x \right.}{-116+24 i \sqrt{3}} \\ \frac{\left(\left(68 i \sqrt{3}-12\right) \cos\left(2 x\right)+\left(-12 i \sqrt{3}+20\right) \cos\left(3 x\right)-14 i \sin\left(2 x\right) \sqrt{3}-48 i \sqrt{3}\, \sin\left(3 x\right)+54 \sin(2 \right.}{\left(i \sqrt{3}+17\right) \left(i \sqrt{3}+7\right)} \end{array}\right]$$

$$= \left[\begin{array}{c} \frac{4}{13}+\frac{64 \cos\left(x\right)^{3}}{73}+\frac{8 \left(-73-39 \sin\left(x\right)\right) \cos\left(x\right)^{2}}{949}+\frac{2 \left(-312+511 \sin\left(x\right)\right) \cos\left(x\right)}{949}+\frac{6 \sin\left(x\right)}{73} \\ -\frac{6}{13}+\frac{56 \cos\left(x\right)^{3}}{73}+\frac{12 \left(73-260 \sin\left(x\right)\right) \cos\left(x\right)^{2}}{949}+\frac{6 \left(-91+219 \sin\left(x\right)\right) \cos\left(x\right)}{949}+\frac{60 \sin\left(x\right)}{73} \end{array}\right]$$

Now that we found particular solution, the final solution is

$$\vec{x}(x) = \vec{x}_h(x) + \vec{x}_p(x)$$

$$\left[\begin{array}{c} y_{1}\left(x\right) \\ y_{2}\left(x\right) \end{array}\right] = \left[\begin{array}{c} \frac{c_{1} {\mathrm e}^{\left(-\frac{1}{2}+\frac{i \sqrt{3}}{2}\right) x}}{\frac{3}{2}+\frac{i \sqrt{3}}{2}} \\ c_{1} {\mathrm e}^{\left(-\frac{1}{2}+\frac{i \sqrt{3}}{2}\right) x} \end{array}\right] + \left[\begin{array}{c} \frac{c_{2} {\mathrm e}^{\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right) x}}{\frac{3}{2}-\frac{i \sqrt{3}}{2}} \\ c_{2} {\mathrm e}^{\left(-\frac{1}{2}-\frac{i \sqrt{3}}{2}\right) x} \end{array}\right] + \left[\begin{array}{c} \frac{4}{13}+\frac{64 \cos\left(x\right)^{3}}{73}+\frac{8 \left(-73-39 \sin\left(x\right)\right) \cos\left(x\right)^{2}}{949}+2(-312+51 \\ -\frac{6}{13}+\frac{56 \cos\left(x\right)^{3}}{73}+\frac{12 \left(73-260 \sin\left(x\right)\right) \cos\left(x\right)^{2}}{949}+6(-91+21 \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} y_{1}\left(x\right) \\ y_{2}\left(x\right) \end{array}\right] = \left[\begin{array}{c} \frac{c_{2} \left(i \sqrt{3}+3\right) {\mathrm e}^{-\frac{\left(1+i \sqrt{3}\right) x}{2}}}{6}-\frac{\left(i \sqrt{3}-3\right) c_{1} {\mathrm e}^{\frac{\left(i \sqrt{3}-1\right) x}{2}}}{6}+\frac{64 \cos\left(x\right)^{3}}{73}+\frac{\left(-1872 \sin\left(x\right)-3504\right) \cos\left(x\right)^{2}}{5694}+\frac{(6132 \sin\left(x\right)-374}{5694} \\ c_{1} {\mathrm e}^{\frac{\left(i \sqrt{3}-1\right) x}{2}}+c_{2} {\mathrm e}^{-\frac{\left(1+i \sqrt{3}\right) x}{2}}-\frac{6}{13}+\frac{56 \cos\left(x\right)^{3}}{73}+\frac{12 \left(73-260 \sin\left(x\right)\right) \cos\left(x\right)^{2}}{949}+\frac{6(-91+219 \sin\left(x\right)) \operatorname{co}}{949} \end{array}\right]$$

19.2.3 Maple step by step solution

Let's solve

$[y_1'(x) = y_2(x) - 2y_1(x) + 2\cos(x)\sin(x), y_2'(x) = -3y_1(x) + y_2(x) - 8\cos(x)^3 + 6\cos(x)]$

- Define vector

$$\underline{\vec{y}}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{y}}'(x) = \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix} \cdot \underline{\vec{y}}(x) + \begin{bmatrix} 2\cos(x)\sin(x) \\ -8\cos(x)^3 + 6\cos(x) \end{bmatrix}$$

- System to solve

$$\underline{\vec{y}}'(x) = \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix} \cdot \underline{\vec{y}}(x) + \begin{bmatrix} 2\cos(x)\sin(x) \\ -8\cos(x)^3 + 6\cos(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 2\cos(x)\sin(x) \\ -8\cos(x)^3 + 6\cos(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(x) = A \cdot \underline{\vec{y}}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{1}{2} - \frac{I\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\frac{3}{2} - \frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{I\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\frac{3}{2} + \frac{I\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2}-\frac{I\sqrt{3}}{2}, \left[\begin{array}{c}\frac{1}{\frac{3}{2}-\frac{I\sqrt{3}}{2}}\\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)x}\cdot\left[\begin{array}{c}\frac{1}{\frac{3}{2}-\frac{I\sqrt{3}}{2}}\\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-\frac{x}{2}}\cdot\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\cdot\left[\begin{array}{c}\frac{1}{\frac{3}{2}-\frac{I\sqrt{3}}{2}}\\ 1\end{array}\right]$$

- Simplify expression

$$e^{-\frac{x}{2}}\cdot\left[\begin{array}{c}\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\frac{3}{2}-\frac{I\sqrt{3}}{2}}\\ \cos\left(\frac{\sqrt{3}\,x}{2}\right)-I\sin\left(\frac{\sqrt{3}\,x}{2}\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{y}^{\rightarrow}{}_{1}(x)=e^{-\frac{x}{2}}\cdot\left[\begin{array}{c}\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{6}\\ \cos\left(\frac{\sqrt{3}\,x}{2}\right)\end{array}\right], \underline{y}^{\rightarrow}{}_{2}(x)=e^{-\frac{x}{2}}\cdot\left[\begin{array}{c}\frac{\sqrt{3}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{6}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right)\end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{y}^{\rightarrow}$

$$\underline{y}^{\rightarrow}(x)=c_{1}\underline{y}^{\rightarrow}{}_{1}(x)+c_{2}\underline{y}^{\rightarrow}{}_{2}(x)+\underline{y}^{\rightarrow}{}_{p}(x)$$

☐ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\left[\begin{array}{cc}e^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{6}\right) & e^{-\frac{x}{2}}\left(\frac{\sqrt{3}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{6}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right)\\ e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\end{array}\right]$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

$$\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{6}\right) & e^{-\frac{x}{2}}\left(\frac{\sqrt{3}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{6} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} \\ 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{\left(\sqrt{3}\cos\left(\frac{\sqrt{3}\,x}{2}\right) - 3\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\sqrt{3}\,e^{-\frac{x}{2}}}{3} & \frac{2\,e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} \\ -2\,e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3} & \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\right)e^{-\frac{x}{2}} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\underset{\_\_\_\_}{\vec{y}}_p(x) = \left[\begin{array}{c} \frac{4}{13} + \frac{84\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{949} - \frac{1492\,e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2847} + \frac{64\cos(x)^3}{73} + \frac{8(-73-39\sin(x))\cos(x)^2}{949} + \frac{2(-312 \\ -\frac{6}{13} - \frac{620\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{949} - \frac{788\,e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{949} + \frac{56\cos(x)^3}{73} + \frac{12(73-260\sin(x))\cos(x)^2}{949} + \frac{6(-91 \end{array}\right.$$

- Plug particular solution back into general solution

$$\underset{\_\_\_\_}{\vec{y}}(x) = c_1\underset{\_\_\_\_}{\vec{y}}_1(x) + c_2\underset{\_\_\_\_}{\vec{y}}_2(x) + \left[\begin{array}{c} \frac{4}{13} + \frac{84\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{949} - \frac{1492\,e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2847} + \frac{64\cos(x)^3}{73} + 8(\\ -\frac{6}{13} - \frac{620\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{949} - \frac{788\,e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{949} + \frac{56\cos(x)^3}{73} + 1 \end{array}\right.$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} y_1(x) \\ y_2(x) \end{array}\right] = \left[\begin{array}{c} \frac{4}{13} + \frac{\left(\frac{c_2\sqrt{3}}{3}+c_1+\frac{168}{949}\right)e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}}\left(\left(c_1-\frac{2984}{949}\right)\sqrt{3}-3c_2\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{6} + \frac{64\cos(x)^3}{73} + \frac{8(-73- \\ -\frac{6}{13} + \frac{e^{-\frac{x}{2}}(949c_1-620)\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{949} - e^{-\frac{x}{2}}\left(c_2+\frac{788\sqrt{3}}{949}\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{56\cos(x)^3}{73} + \frac{12(73-26 \end{array}\right.$$

- Solution to the system of ODEs

$$\left\{y_1(x) = \frac{4}{13} + \frac{\left(\frac{c_2\sqrt{3}}{3}+c_1+\frac{168}{949}\right)e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}}\left(\left(c_1-\frac{2984}{949}\right)\sqrt{3}-3c_2\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{6} + \frac{64\cos(x)^3}{73} + \frac{8(-73-39\sin(x)}{949}\right.$$

✓ Solution by Maple

Time used: 1.578 (sec). Leaf size: 146

```
dsolve([diff(y__1(x),x)=y__2(x)-2*y__1(x)+sin(2*x),diff(y__2(x),x)=-3*y__1(x)+y__2(x)-2*cos(
```

$$y_1(x) = c_2 e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) + e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)c_1 + \frac{16\cos(3x)}{73} - \frac{4\cos(2x)}{13} - \frac{6\sin(3x)}{73} + \frac{7\sin(2x)}{13}$$

$$y_2(x) = \frac{3c_2 e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{c_2 e^{-\frac{x}{2}}\sqrt{3}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{3\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)c_1}{2} - \frac{e^{-\frac{x}{2}}\sqrt{3}\sin\left(\frac{\sqrt{3}\,x}{2}\right)c_1}{2} - \frac{60\sin(3x)}{73} + \frac{9\sin(2x)}{13} + \frac{14\cos(3x)}{73} + \frac{6\cos(2x)}{13}$$

✓ Solution by Mathematica

Time used: 4.455 (sec). Leaf size: 223

```
DSolve[{y1'[x]==y2[x]-2*y1[x]+Sin[2*x],y2'[x]==-3*y1[x]+y2[x]-2*Cos[3*x]},{y1[x],y2[x]},x,In
```

$$
\begin{aligned}
\text{y1}(x) \to{} & \frac{7}{13}\sin(2x) - \frac{6}{73}\sin(3x) - \frac{4}{13}\cos(2x) + \frac{16}{73}\cos(3x) \\
& + c_1 e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right) - \sqrt{3}c_1 e^{-x/2}\sin\left(\frac{\sqrt{3}x}{2}\right) + \frac{2c_2 e^{-x/2}\sin\left(\frac{\sqrt{3}x}{2}\right)}{\sqrt{3}} \\
\text{y2}(x) \to{} & \frac{9}{13}\sin(2x) - \frac{60}{73}\sin(3x) + \frac{6}{13}\cos(2x) + \frac{14}{73}\cos(3x) \\
& + c_2 e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right) - 2\sqrt{3}c_1 e^{-x/2}\sin\left(\frac{\sqrt{3}x}{2}\right) + \sqrt{3}c_2 e^{-x/2}\sin\left(\frac{\sqrt{3}x}{2}\right)
\end{aligned}
$$

19.3 problem 3

Internal problem ID [12841]
Internal file name [`OUTPUT/11493_Monday_November_06_2023_01_31_03_PM_19236192/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= 2y_2(x) \\
y_2'(x) &= 3y_1(x) \\
y_3'(x) &= 2y_3(x) - y_1(x)
\end{aligned}
$$

19.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(x) = A\,\vec{x}(x)
$$

Or

$$
\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{\sqrt{6}\,x}}{2}+\frac{e^{-\sqrt{6}\,x}}{2} & \frac{\left(-e^{-\sqrt{6}\,x}+e^{\sqrt{6}\,x}\right)\sqrt{6}}{6} & 0 \\ \frac{\left(-e^{-\sqrt{6}\,x}+e^{\sqrt{6}\,x}\right)\sqrt{6}}{4} & \frac{e^{\sqrt{6}\,x}}{2}+\frac{e^{-\sqrt{6}\,x}}{2} & 0 \\ \frac{\left(\sqrt{6}-2\right)e^{-\sqrt{6}\,x}}{4}+\frac{\left(-\sqrt{6}-2\right)e^{\sqrt{6}\,x}}{4}+e^{2x} & \frac{\left(\sqrt{6}-3\right)e^{-\sqrt{6}\,x}}{6}+\frac{\left(-\sqrt{6}-3\right)e^{\sqrt{6}\,x}}{6}+e^{2x} & e^{2x} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(x) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{\sqrt{6}\,x}}{2}+\frac{e^{-\sqrt{6}\,x}}{2} & \frac{\left(-e^{-\sqrt{6}\,x}+e^{\sqrt{6}\,x}\right)\sqrt{6}}{6} & 0 \\ \frac{\left(-e^{-\sqrt{6}\,x}+e^{\sqrt{6}\,x}\right)\sqrt{6}}{4} & \frac{e^{\sqrt{6}\,x}}{2}+\frac{e^{-\sqrt{6}\,x}}{2} & 0 \\ \frac{\left(\sqrt{6}-2\right)e^{-\sqrt{6}\,x}}{4}+\frac{\left(-\sqrt{6}-2\right)e^{\sqrt{6}\,x}}{4}+e^{2x} & \frac{\left(\sqrt{6}-3\right)e^{-\sqrt{6}\,x}}{6}+\frac{\left(-\sqrt{6}-3\right)e^{\sqrt{6}\,x}}{6}+e^{2x} & e^{2x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{\sqrt{6}\,x}}{2}+\frac{e^{-\sqrt{6}\,x}}{2}\right)c_1+\frac{\left(-e^{-\sqrt{6}\,x}+e^{\sqrt{6}\,x}\right)\sqrt{6}\,c_2}{6} \\ \frac{\left(-e^{-\sqrt{6}\,x}+e^{\sqrt{6}\,x}\right)\sqrt{6}\,c_1}{4}+\left(\frac{e^{\sqrt{6}\,x}}{2}+\frac{e^{-\sqrt{6}\,x}}{2}\right)c_2 \\ \left(\frac{\left(\sqrt{6}-2\right)e^{-\sqrt{6}\,x}}{4}+\frac{\left(-\sqrt{6}-2\right)e^{\sqrt{6}\,x}}{4}+e^{2x}\right)c_1+\left(\frac{\left(\sqrt{6}-3\right)e^{-\sqrt{6}\,x}}{6}+\frac{\left(-\sqrt{6}-3\right)e^{\sqrt{6}\,x}}{6}+e^{2x}\right)c_2+e^{2x}c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\left(-c_2\sqrt{6}+3c_1\right)e^{-\sqrt{6}\,x}}{6}+\frac{e^{\sqrt{6}\,x}\left(\frac{c_2\sqrt{6}}{3}+c_1\right)}{2} \\ \frac{\left(-c_1\sqrt{6}+2c_2\right)e^{-\sqrt{6}\,x}}{4}+\frac{e^{\sqrt{6}\,x}\left(c_1\sqrt{6}+2c_2\right)}{4} \\ \frac{\left((3c_1+2c_2)\sqrt{6}-6c_1-6c_2\right)e^{-\sqrt{6}\,x}}{12}+\frac{\left((-3c_1-2c_2)\sqrt{6}-6c_1-6c_2\right)e^{\sqrt{6}\,x}}{12}+e^{2x}(c_1+c_2+c_3) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

19.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 2 & 0 \\ 3 & -\lambda & 0 \\ -1 & 0 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 2\lambda^2 - 6\lambda + 12 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \sqrt{6}$$
$$\lambda_2 = -\sqrt{6}$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
$\sqrt{6}$	1	real eigenvalue
$-\sqrt{6}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & 2 & 0 \\ 3 & -2 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \sqrt{6}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix} - \left(\sqrt{6}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\sqrt{6} & 2 & 0 \\ 3 & -\sqrt{6} & 0 \\ -1 & 0 & -\sqrt{6}+2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -\sqrt{6} & 2 & 0 & 0 \\ 3 & -\sqrt{6} & 0 & 0 \\ -1 & 0 & -\sqrt{6}+2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{\sqrt{6}\, R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -\sqrt{6} & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -\sqrt{6}+2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{\sqrt{6}\, R_1}{6} \Longrightarrow \left[\begin{array}{ccc|c} -\sqrt{6} & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{6}}{3} & -\sqrt{6}+2 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -\sqrt{6} & 2 & 0 & 0 \\ 0 & -\frac{\sqrt{6}}{3} & -\sqrt{6}+2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\sqrt{6} & 2 & 0 \\ 0 & -\frac{\sqrt{6}}{3} & -\sqrt{6}+2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\left(\sqrt{6}-2\right) t, v_2 = \left(\sqrt{6}-3\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} -\left(\sqrt{6}-2\right) t \\ \left(\sqrt{6}-3\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\left(\sqrt{6}-2\right) t \\ \left(\sqrt{6}-3\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\left(\sqrt{6}-2\right) t \\ \left(\sqrt{6}-3\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\sqrt{6}+2 \\ \sqrt{6}-3 \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} -(\sqrt{6}-2)\,t \\ (\sqrt{6}-3)\,t \\ t \end{bmatrix} = \begin{bmatrix} -\sqrt{6}+2 \\ \sqrt{6}-3 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -\sqrt{6}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix} - \left(-\sqrt{6}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{6} & 2 & 0 \\ 3 & \sqrt{6} & 0 \\ -1 & 0 & \sqrt{6}+2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} \sqrt{6} & 2 & 0 & 0 \\ 3 & \sqrt{6} & 0 & 0 \\ -1 & 0 & \sqrt{6}+2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{\sqrt{6}\,R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} \sqrt{6} & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & \sqrt{6}+2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{\sqrt{6}\,R_1}{6} \Longrightarrow \left[\begin{array}{ccc|c} \sqrt{6} & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{6}}{3} & \sqrt{6}+2 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} \sqrt{6} & 2 & 0 & 0 \\ 0 & \frac{\sqrt{6}}{3} & \sqrt{6}+2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \sqrt{6} & 2 & 0 \\ 0 & \frac{\sqrt{6}}{3} & \sqrt{6}+2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\sqrt{6}+2\right)t, v_2 = -\left(\sqrt{6}+3\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\sqrt{6}+2\right)t \\ -\left(\sqrt{6}+3\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(\sqrt{6}+2\right)t \\ -\left(\sqrt{6}+3\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\sqrt{6}+2\right)t \\ -\left(\sqrt{6}+3\right)t \\ t \end{bmatrix} = t \begin{bmatrix} \sqrt{6}+2 \\ -\sqrt{6}-3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\sqrt{6}+2\right)t \\ -\left(\sqrt{6}+3\right)t \\ t \end{bmatrix} = \begin{bmatrix} \sqrt{6}+2 \\ -\sqrt{6}-3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$\sqrt{6}$	1	1	No	$\left[\begin{array}{c} -\sqrt{6}+2 \\ -\frac{\left(\sqrt{6}-2\right)\sqrt{6}}{2} \\ 1 \end{array}\right]$
$-\sqrt{6}$	1	1	No	$\left[\begin{array}{c} \sqrt{6}+2 \\ \frac{\left(-\sqrt{6}-2\right)\sqrt{6}}{2} \\ 1 \end{array}\right]$
2	1	1	No	$\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\sqrt{6}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}
\vec{x}_1(x) &= \vec{v}_1 e^{\sqrt{6}\, x} \\
&= \left[\begin{array}{c} -\sqrt{6}+2 \\ -\frac{\left(\sqrt{6}-2\right)\sqrt{6}}{2} \\ 1 \end{array}\right] e^{\sqrt{6}\, x}
\end{aligned}$$

Since eigenvalue $-\sqrt{6}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}
\vec{x}_2(x) &= \vec{v}_2 e^{-\sqrt{6}\, x} \\
&= \left[\begin{array}{c} \sqrt{6}+2 \\ \frac{\left(-\sqrt{6}-2\right)\sqrt{6}}{2} \\ 1 \end{array}\right] e^{-\sqrt{6}\, x}
\end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(x) = \vec{v}_3 e^{2x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2x}$$

Therefore the final solution is

$$\vec{x}_h(x) = c_1\vec{x}_1(x) + c_2\vec{x}_2(x) + c_3\vec{x}_3(x)$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{\sqrt{6}\,x}\left(-\sqrt{6}+2\right) \\ -\frac{\mathrm{e}^{\sqrt{6}\,x}\left(\sqrt{6}-2\right)\sqrt{6}}{2} \\ \mathrm{e}^{\sqrt{6}\,x} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-\sqrt{6}\,x}\left(\sqrt{6}+2\right) \\ \frac{\mathrm{e}^{-\sqrt{6}\,x}\left(-\sqrt{6}-2\right)\sqrt{6}}{2} \\ \mathrm{e}^{-\sqrt{6}\,x} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{2x} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} c_2\mathrm{e}^{-\sqrt{6}\,x}\left(\sqrt{6}+2\right) - \mathrm{e}^{\sqrt{6}\,x}c_1\left(\sqrt{6}-2\right) \\ -c_2\left(\sqrt{6}+3\right)\mathrm{e}^{-\sqrt{6}\,x} + \mathrm{e}^{\sqrt{6}\,x}c_1\left(\sqrt{6}-3\right) \\ c_1\mathrm{e}^{\sqrt{6}\,x} + c_2\mathrm{e}^{-\sqrt{6}\,x} + c_3\mathrm{e}^{2x} \end{bmatrix}$$

19.3.3 Maple step by step solution

Let's solve

$[y_1'(x) = 2y_2(x), y_2'(x) = 3y_1(x), y_3'(x) = 2y_3(x) - y_1(x)]$

- Define vector

$$\underline{y}^{\rightarrow}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow}{}'(x) = \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow}{}'(x) = \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{y}^{\rightarrow}{}'(x) = A \cdot \underline{y}^{\rightarrow}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[\sqrt{6}, \begin{bmatrix} -\sqrt{6}+2 \\ -\frac{(\sqrt{6}-2)\sqrt{6}}{2} \\ 1 \end{bmatrix}\right], \left[-\sqrt{6}, \begin{bmatrix} \sqrt{6}+2 \\ \frac{(-\sqrt{6}-2)\sqrt{6}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{y}^{\rightarrow}{}_1 = \mathrm{e}^{2x} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[\sqrt{6}, \left[\begin{array}{c} -\sqrt{6}+2 \\ -\frac{\left(\sqrt{6}-2\right)\sqrt{6}}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{y}^{\rightarrow}{}_{2} = \mathrm{e}^{\sqrt{6}\, x} \cdot \left[\begin{array}{c} -\sqrt{6}+2 \\ -\frac{\left(\sqrt{6}-2\right)\sqrt{6}}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-\sqrt{6}, \left[\begin{array}{c} \sqrt{6}+2 \\ \frac{\left(-\sqrt{6}-2\right)\sqrt{6}}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{y}^{\rightarrow}{}_{3} = \mathrm{e}^{-\sqrt{6}\, x} \cdot \left[\begin{array}{c} \sqrt{6}+2 \\ \frac{\left(-\sqrt{6}-2\right)\sqrt{6}}{2} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\underline{y}^{\rightarrow} = c_1 \underline{y}^{\rightarrow}{}_{1} + c_2 \underline{y}^{\rightarrow}{}_{2} + c_3 \underline{y}^{\rightarrow}{}_{3}$$

- Substitute solutions into the general solution

$$\underline{y}^{\rightarrow} = c_1 \mathrm{e}^{2x} \cdot \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] + c_2 \mathrm{e}^{\sqrt{6}\, x} \cdot \left[\begin{array}{c} -\sqrt{6}+2 \\ -\frac{\left(\sqrt{6}-2\right)\sqrt{6}}{2} \\ 1 \end{array}\right] + c_3 \mathrm{e}^{-\sqrt{6}\, x} \cdot \left[\begin{array}{c} \sqrt{6}+2 \\ \frac{\left(-\sqrt{6}-2\right)\sqrt{6}}{2} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right] = \left[\begin{array}{c} c_3 \mathrm{e}^{-\sqrt{6}\, x}\left(\sqrt{6}+2\right) - \mathrm{e}^{\sqrt{6}\, x} c_2 \left(\sqrt{6}-2\right) \\ -c_3\left(\sqrt{6}+3\right)\mathrm{e}^{-\sqrt{6}\, x} + \mathrm{e}^{\sqrt{6}\, x} c_2 \left(\sqrt{6}-3\right) \\ c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{\sqrt{6}\, x} + c_3 \mathrm{e}^{-\sqrt{6}\, x} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{y_1(x) = c_3 \mathrm{e}^{-\sqrt{6}\,x}\left(\sqrt{6}+2\right) - \mathrm{e}^{\sqrt{6}\,x} c_2\left(\sqrt{6}-2\right), y_2(x) = -c_3\left(\sqrt{6}+3\right)\mathrm{e}^{-\sqrt{6}\,x} + \mathrm{e}^{\sqrt{6}\,x} c_2\left(\sqrt{6}-3\right), y_3\right.$$

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 107

```
dsolve([diff(y__1(x),x)=2*y__2(x),diff(y__2(x),x)=3*y__1(x),diff(y__3(x),x)=2*y__3(x)-y__1(x
```

$$y_1(x) = \mathrm{e}^{\sqrt{6}\,x} c_2 + c_3 \mathrm{e}^{-\sqrt{6}\,x}$$

$$y_2(x) = \frac{\sqrt{6}\left(\mathrm{e}^{\sqrt{6}\,x} c_2 - c_3 \mathrm{e}^{-\sqrt{6}\,x}\right)}{2}$$

$$y_3(x) = \frac{2\,\mathrm{e}^{2x} c_1}{\left(2+\sqrt{6}\right)\left(-2+\sqrt{6}\right)} + \frac{\mathrm{e}^{-\sqrt{6}\,x} c_3}{2+\sqrt{6}} - \frac{\mathrm{e}^{\sqrt{6}\,x} c_2}{-2+\sqrt{6}}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 232

```
DSolve[{y1'[x]==2*y2[x],y2'[x]==3*y1[x],y3'[x]==2*y3[x]-y1[x]},{y1[x],y2[x],y3[x]},x,Include
```

$$\mathrm{y1}(x) \to \frac{1}{6} e^{-\sqrt{6}x}\left(3c_1\left(e^{2\sqrt{6}x}+1\right) + \sqrt{6}c_2\left(e^{2\sqrt{6}x}-1\right)\right)$$

$$\mathrm{y2}(x) \to \frac{1}{4} e^{-\sqrt{6}x}\left(\sqrt{6}c_1\left(e^{2\sqrt{6}x}-1\right) + 2c_2\left(e^{2\sqrt{6}x}+1\right)\right)$$

$$\mathrm{y3}(x) \to \frac{1}{12} e^{-\sqrt{6}x}\left(2\left(c_2\left(-\left(3+\sqrt{6}\right)e^{2\sqrt{6}x} + 6e^{\left(2+\sqrt{6}\right)x} - 3 + \sqrt{6}\right) + 6c_3 e^{\left(2+\sqrt{6}\right)x}\right)\right.$$
$$\left. - 3c_1\left(\left(2+\sqrt{6}\right)e^{2\sqrt{6}x} - 4e^{\left(2+\sqrt{6}\right)x} + 2 - \sqrt{6}\right)\right)$$

19.4 problem 4

Internal problem ID [12842]
Internal file name [`OUTPUT/11494_Monday_November_06_2023_01_31_03_PM_39604673/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$y_1'(x) = 2y_1(x)\, x - x^2 y_2(x) + 4x$$
$$y_2'(x) = y_1(x)\, e^x + 3\, e^{-x} y_2(x) - 4\cos(x)^3 + 3\cos(x)$$

Does not currently support non autonomous system of first order linear differential equations. The following is the phase plot

✗ Solution by Maple

```
dsolve([diff(y__1(x),x)=2*x*y__1(x)-x^2*y__2(x)+4*x,diff(y__2(x),x)=exp(x)*y__1(x)+3*exp(-x)
```

No solution found

✗ Solution by Mathematica
Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y1'[x]==2*x*y1[x]-x^2*y2[x]+4*x,y2'[x]==Exp[x]*y1[x]+3*Exp[-x]*y2[x]-Cos[3*x]},{y1[x
```

Not solved

19.5 problem 5 a

Internal problem ID [12843]
Internal file name [OUTPUT/11495_Monday_November_06_2023_01_31_03_PM_20314798/index.tex]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 5 a.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= 2y_1(x) - 3y_2(x) \\
y_2'(x) &= y_1(x) - 2y_2(x)
\end{aligned}
$$

19.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{-x}}{2} + \frac{3\,e^{x}}{2} & \frac{3\,e^{-x}}{2} - \frac{3\,e^{x}}{2} \\ -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{3\,e^{-x}}{2} - \frac{e^{x}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(x) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{e^{-x}}{2} + \frac{3e^{x}}{2} & \frac{3e^{-x}}{2} - \frac{3e^{x}}{2} \\ -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{3e^{-x}}{2} - \frac{e^{x}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\frac{e^{-x}}{2} + \frac{3e^{x}}{2}\right) c_1 + \left(\frac{3e^{-x}}{2} - \frac{3e^{x}}{2}\right) c_2 \\ \left(-\frac{e^{-x}}{2} + \frac{e^{x}}{2}\right) c_1 + \left(\frac{3e^{-x}}{2} - \frac{e^{x}}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(-c_1+3c_2)e^{-x}}{2} + \frac{3e^{x}(c_1-c_2)}{2} \\ \frac{(-c_1+3c_2)e^{-x}}{2} + \frac{e^{x}(c_1-c_2)}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

19.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(x) = A\,\vec{x}(x)
$$

Or

$$
\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 1 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(x) = \vec{v}_1 e^{x}$$
$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{x}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(x) = \vec{v}_2 e^{-x}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-x}$$

Therefore the final solution is

$$\vec{x}_h(x) = c_1 \vec{x}_1(x) + c_2 \vec{x}_2(x)$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = c_1 \begin{bmatrix} 3\,\mathrm{e}^{x} \\ \mathrm{e}^{x} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{-x} \\ \mathrm{e}^{-x} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 3c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} \\ c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} \end{bmatrix}$$

The following is the phase plot of the system.

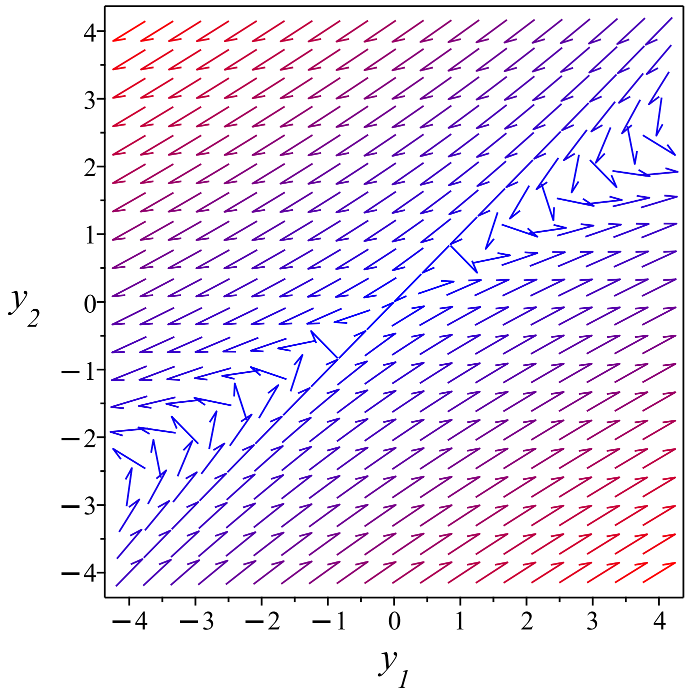

Figure 428: Phase plot

19.5.3 Maple step by step solution

Let's solve

$[y_1'(x) = 2y_1(x) - 3y_2(x), y_2'(x) = y_1(x) - 2y_2(x)]$

- Define vector

$$\underline{y}^{\rightarrow}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow}{}'(x) = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow}{}'(x) = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(x) = A \cdot \underline{\vec{y}}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_2 = \mathrm{e}^{x} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1 + c_2 \underline{\vec{y}}_2$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{x} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^{-x} + 3c_2 \mathrm{e}^{x} \\ c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} \end{bmatrix}$$

- Solution to the system of ODEs
  $\{y_1(x) = c_1 \mathrm{e}^{-x} + 3c_2 \mathrm{e}^{x}, y_2(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x}\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve([diff(y__1(x),x)=2*y__1(x)-3*y__2(x),diff(y__2(x),x)=y__1(x)-2*y__2(x)],singsol=all)
```

$$y_1(x) = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$
$$y_2(x) = \frac{c_1 \mathrm{e}^{x}}{3} + c_2 \mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 81

```
DSolve[{y1'[x]==-2*y1[x]-3*y2[x],y2'[x]==y1[x]-2*y2[x]},{y1[x],y2[x]},x,IncludeSingularSolut
```

$$\text{y1}(x) \to e^{-2x}\left(c_1 \cos\left(\sqrt{3}x\right) - \sqrt{3}c_2 \sin\left(\sqrt{3}x\right)\right)$$
$$\text{y2}(x) \to \frac{1}{3}e^{-2x}\left(3c_2 \cos\left(\sqrt{3}x\right) + \sqrt{3}c_1 \sin\left(\sqrt{3}x\right)\right)$$

19.6 problem 5 c

Internal problem ID [12844]
Internal file name [`OUTPUT/11496_Monday_November_06_2023_01_31_04_PM_69505427/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 5 c.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= 2y_1(x) - 3y_2(x) + 4x - 2 \\
y_2'(x) &= y_1(x) - 2y_2(x) + 3x
\end{aligned}
$$

19.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)
$$

Or

$$
\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} + \begin{bmatrix} 4x-2 \\ 3x \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(x) = \vec{x}_h(x) + \vec{x}_p(x)
$$

Where $\vec{x}_h(x)$ is the homogeneous solution to $\vec{x}'(x) = A\,\vec{x}(x)$ and $\vec{x}_p(x)$ is a particular solution to $\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{-x}}{2}+\frac{3\,e^{x}}{2} & \frac{3\,e^{-x}}{2}-\frac{3\,e^{x}}{2} \\ -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{3\,e^{-x}}{2}-\frac{e^{x}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(x) &= e^{At}\vec{c} \\ &= \begin{bmatrix} -\frac{e^{-x}}{2}+\frac{3\,e^{x}}{2} & \frac{3\,e^{-x}}{2}-\frac{3\,e^{x}}{2} \\ -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{3\,e^{-x}}{2}-\frac{e^{x}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(-\frac{e^{-x}}{2}+\frac{3\,e^{x}}{2}\right)c_1+\left(\frac{3\,e^{-x}}{2}-\frac{3\,e^{x}}{2}\right)c_2 \\ \left(-\frac{e^{-x}}{2}+\frac{e^{x}}{2}\right)c_1+\left(\frac{3\,e^{-x}}{2}-\frac{e^{x}}{2}\right)c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(-c_1+3c_2)e^{-x}}{2}+\frac{3\,e^{x}(c_1-c_2)}{2} \\ \frac{(-c_1+3c_2)e^{-x}}{2}+\frac{e^{x}(c_1-c_2)}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(x) = e^{Ax}\int e^{-Ax}\vec{G}(x)\,dx$$

But

$$\begin{aligned} e^{-Ax} &= (e^{Ax})^{-1} \\ &= \begin{bmatrix} \frac{3\,e^{-x}}{2}-\frac{e^{x}}{2} & -\frac{3\,e^{-x}}{2}+\frac{3\,e^{x}}{2} \\ \frac{e^{-x}}{2}-\frac{e^{x}}{2} & -\frac{e^{-x}}{2}+\frac{3\,e^{x}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(x) &= \begin{bmatrix} -\frac{e^{-x}}{2}+\frac{3\,e^{x}}{2} & \frac{3\,e^{-x}}{2}-\frac{3\,e^{x}}{2} \\ -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{3\,e^{-x}}{2}-\frac{e^{x}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{3\,e^{-x}}{2}-\frac{e^{x}}{2} & -\frac{3\,e^{-x}}{2}+\frac{3\,e^{x}}{2} \\ \frac{e^{-x}}{2}-\frac{e^{x}}{2} & -\frac{e^{-x}}{2}+\frac{3\,e^{x}}{2} \end{bmatrix} \begin{bmatrix} 4x-2 \\ 3x \end{bmatrix} dx \\ &= \begin{bmatrix} -\frac{e^{-x}}{2}+\frac{3\,e^{x}}{2} & \frac{3\,e^{-x}}{2}-\frac{3\,e^{x}}{2} \\ -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{3\,e^{-x}}{2}-\frac{e^{x}}{2} \end{bmatrix} \begin{bmatrix} \frac{(3-3x)e^{-x}}{2}+\frac{(5x-3)e^{x}}{2} \\ \frac{e^{-x}(1-x)}{2}+\frac{(5x-3)e^{x}}{2} \end{bmatrix} \\ &= \begin{bmatrix} x \\ 2x-1 \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(x) = \vec{x}_h(x) + \vec{x}_p(x)$$
$$= \begin{bmatrix} \frac{(-c_1+3c_2)\mathrm{e}^{-x}}{2} + \frac{(3c_1-3c_2)\mathrm{e}^{x}}{2} + x \\ \frac{(-c_1+3c_2)\mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^{x}(c_1-c_2)}{2} + 2x - 1 \end{bmatrix}$$

19.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} + \begin{bmatrix} 4x-2 \\ 3x \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(x) = \vec{x}_h(x) + \vec{x}_p(x)$$

Where $\vec{x}_h(x)$ is the homogeneous solution to $\vec{x}'(x) = A\,\vec{x}(x)$ and $\vec{x}_p(x)$ is a particular solution to $\vec{x}'(x) = A\,\vec{x}(x) + \vec{G}(x)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 1 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(x) &= \vec{v}_1 e^{-x} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-x} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(x) = \vec{v}_2 e^x$$
$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^x$$

Therefore the homogeneous solution is

$$\vec{x}_h(x) = c_1 \vec{x}_1(x) + c_2 \vec{x}_2(x)$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{-x} \\ \mathrm{e}^{-x} \end{bmatrix} + c_2 \begin{bmatrix} 3\,\mathrm{e}^{x} \\ \mathrm{e}^{x} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(x)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(x) = \begin{bmatrix} \mathrm{e}^{-x} & 3\,\mathrm{e}^{x} \\ \mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(x) = \Phi \int \Phi^{-1} \vec{G}(x)\, dx$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{\mathrm{e}^{x}}{2} & \frac{3\,\mathrm{e}^{x}}{2} \\ \frac{\mathrm{e}^{-x}}{2} & -\frac{\mathrm{e}^{-x}}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(x) = \begin{bmatrix} e^{-x} & 3\,e^{x} \\ e^{-x} & e^{x} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{x}}{2} & \frac{3\,e^{x}}{2} \\ \frac{e^{-x}}{2} & -\frac{e^{-x}}{2} \end{bmatrix} \begin{bmatrix} 4x-2 \\ 3x \end{bmatrix} dx$$

$$= \begin{bmatrix} e^{-x} & 3\,e^{x} \\ e^{-x} & e^{x} \end{bmatrix} \int \begin{bmatrix} e^{x}\left(\frac{5x}{2}+1\right) \\ e^{-x}\left(\frac{x}{2}-1\right) \end{bmatrix} dx$$

$$= \begin{bmatrix} e^{-x} & 3\,e^{x} \\ e^{-x} & e^{x} \end{bmatrix} \begin{bmatrix} \frac{(5x-3)e^{x}}{2} \\ -\frac{(x-1)e^{-x}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} x \\ 2x-1 \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(x) = \vec{x}_h(x) + \vec{x}_p(x)$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} c_1 e^{-x} \\ c_1 e^{-x} \end{bmatrix} + \begin{bmatrix} 3c_2 e^{x} \\ c_2 e^{x} \end{bmatrix} + \begin{bmatrix} x \\ 2x-1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} c_1 e^{-x} + 3c_2 e^{x} + x \\ c_1 e^{-x} + c_2 e^{x} + 2x - 1 \end{bmatrix}$$

19.6.3 Maple step by step solution

Let's solve

$[y_1'(x) = 2y_1(x) - 3y_2(x) + 4x - 2, y_2'(x) = y_1(x) - 2y_2(x) + 3x]$

- Define vector

$$\underline{y}^{\rightarrow}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x) + \begin{bmatrix} 4x-2 \\ 3x \end{bmatrix}$$

- System to solve

$$\underline{\vec{y}}'(x) = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \cdot \underline{\vec{y}}(x) + \begin{bmatrix} 4x-2 \\ 3x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 4x-2 \\ 3x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(x) = A \cdot \underline{\vec{y}}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_2 = \mathrm{e}^{x} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{\vec{y}}$

  $\underline{\vec{y}}(x) = c_1 \underline{\vec{y}}_1 + c_2 \underline{\vec{y}}_2 + \underline{\vec{y}}_p(x)$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $\phi(x) = \begin{bmatrix} e^{-x} & 3\,e^{x} \\ e^{-x} & e^{x} \end{bmatrix}$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

  $\Phi(x) = \begin{bmatrix} e^{-x} & 3\,e^{x} \\ e^{-x} & e^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}}$

- Evaluate and simplify to get the fundamental matrix

  $\Phi(x) = \begin{bmatrix} -\frac{e^{-x}}{2} + \frac{3\,e^{x}}{2} & \frac{3\,e^{-x}}{2} - \frac{3\,e^{x}}{2} \\ -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{3\,e^{-x}}{2} - \frac{e^{x}}{2} \end{bmatrix}$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\underline{\vec{y}}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\underline{\vec{y}}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\underline{\vec{y}}_{p}(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $\underline{\vec{y}}_{p}(x) = \begin{bmatrix} x + \frac{3\,e^{-x}}{2} - \frac{3\,e^{x}}{2} \\ 2x - 1 + \frac{3\,e^{-x}}{2} - \frac{e^{x}}{2} \end{bmatrix}$

- Plug particular solution back into general solution

  $\underline{\vec{y}}(x) = c_1 \underline{\vec{y}}_{1} + c_2 \underline{\vec{y}}_{2} + \begin{bmatrix} x + \frac{3\,e^{-x}}{2} - \frac{3\,e^{x}}{2} \\ 2x - 1 + \frac{3\,e^{-x}}{2} - \frac{e^{x}}{2} \end{bmatrix}$

- Substitute in vector of dependent variables

  $\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} c_1 e^{-x} + 3c_2 e^{x} + x + \frac{3\,e^{-x}}{2} - \frac{3\,e^{x}}{2} \\ c_1 e^{-x} + c_2 e^{x} + 2x - 1 + \frac{3\,e^{-x}}{2} - \frac{e^{x}}{2} \end{bmatrix}$

- Solution to the system of ODEs

  $\left\{y_1(x) = c_1 e^{-x} + 3c_2 e^{x} + x + \frac{3\,e^{-x}}{2} - \frac{3\,e^{x}}{2}, y_2(x) = c_1 e^{-x} + c_2 e^{x} + 2x - 1 + \frac{3\,e^{-x}}{2} - \frac{e^{x}}{2}\right\}$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 36

```
dsolve([diff(y__1(x),x)=2*y__1(x)-3*y__2(x)+4*x-2,diff(y__2(x),x)=y__1(x)-2*y__2(x)+3*x],sin
```

$$y_1(x) = c_2 e^{x} + e^{-x} c_1 + x$$
$$y_2(x) = \frac{c_2 e^{x}}{3} + e^{-x} c_1 - 1 + 2x$$

✓ Solution by Mathematica

Time used: 3.724 (sec). Leaf size: 101

```
DSolve[{y1'[x]==-2*y1[x]-3*y2[x]+4*x-2,y2'[x]==y1[x]-2*y2[x]+3*x},{y1[x],y2[x]},x,IncludeSin
```

$$\text{y1}(x)\to -\frac{x}{7}+c_1e^{-2x}\cos\left(\sqrt{3}x\right)-\sqrt{3}c_2e^{-2x}\sin\left(\sqrt{3}x\right)+\frac{4}{49}$$
$$\text{y2}(x)\to \frac{10x}{7}+c_2e^{-2x}\cos\left(\sqrt{3}x\right)+\frac{c_1e^{-2x}\sin\left(\sqrt{3}x\right)}{\sqrt{3}}-\frac{33}{49}$$

19.7 problem 6 a

Internal problem ID [12845]
Internal file name [`OUTPUT/11497_Monday_November_06_2023_01_31_04_PM_8086785/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 6 a.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$
\begin{aligned}
y_1'(x) &= \frac{5y_1(x)}{x} + \frac{4y_2(x)}{x} \\
y_2'(x) &= -\frac{6y_1(x)}{x} - \frac{5y_2(x)}{x}
\end{aligned}
$$

Does not currently support non autonomous system of first order linear differential equations. The following is the phase plot

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 34

```
dsolve([diff(y__1(x),x)=5/x*y__1(x)+4/x*y__2(x),diff(y__2(x),x)=-6/x*y__1(x)-5/x*y__2(x)],si
```

$$
\begin{aligned}
y_1(x) &= \frac{c_1x^2 + c_2}{x} \\
y_2(x) &= -\frac{2c_1x^2 + 3c_2}{2x}
\end{aligned}
$$

✓ Solution by Mathematica
Time used: 0.017 (sec). Leaf size: 34

```
DSolve[{y1'[x]==5/x*y1[x]+4/x*y2[x],y2'[x]==-6/x*y1[x]-5/x*y2[x]},{y1[x],y2[x]},x,IncludeSin
```

$$
\begin{aligned}
\text{y1}(x) &\to \frac{c_1}{x} + c_2x \\
\text{y2}(x) &\to -\frac{3c_1}{2x} - c_2x
\end{aligned}
$$

19.8 problem 6 c

Internal problem ID [12846]
Internal file name [`OUTPUT/11498_Monday_November_06_2023_01_31_04_PM_4917633/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 6 c.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"** Unable to solve or complete t

Solve

$$
\begin{aligned}
y_1'(x) &= \frac{5y_1(x)}{x} + \frac{4y_2(x)}{x} - 2x \\
y_2'(x) &= -\frac{6y_1(x)}{x} - \frac{5y_2(x)}{x} + 5x
\end{aligned}
$$

Does not currently support non autonomous system of first order linear differential equations. The following is the phase plot

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 44

```
dsolve([diff(y__1(x),x)=5/x*y__1(x)+4/x*y__2(x)-2*x,diff(y__2(x),x)=-6/x*y__1(x)-5/x*y__2(x)
```

$$
\begin{aligned}
y_1(x) &= \frac{c_1x^2 + 2x^3 + c_2}{x} \\
y_2(x) &= -\frac{2c_1x^2 + 2x^3 + 3c_2}{2x}
\end{aligned}
$$

✓ Solution by Mathematica
Time used: 0.013 (sec). Leaf size: 44

```
DSolve[{y1'[x]==5/x*y1[x]+4/x*y2[x]-2*x,y2'[x]==-6/x*y1[x]-5/x*y2[x]+5*x},{y1[x],y2[x]},x,In
```

$$
\begin{aligned}
\text{y1}(x) &\to 2x^2 + c_2x + \frac{c_1}{x} \\
\text{y2}(x) &\to -x^2 - c_2x - \frac{3c_1}{2x}
\end{aligned}
$$

19.9 problem 7

Internal problem ID [12847]
Internal file name [`OUTPUT/11499_Monday_November_06_2023_01_31_04_PM_90196884/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= 2y_1(x) + y_2(x) - 2y_3(x) \\
y_2'(x) &= 3y_2(x) - 2y_3(x) \\
y_3'(x) &= 3y_1(x) + y_2(x) - 3y_3(x)
\end{aligned}
$$

19.9.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & -2 \\ 3 & 1 & -3 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -e^{2x} + 3\,e^{x} - e^{-x} & -e^{x} + e^{2x} & -e^{x} + e^{-x} \\ 3\,e^{x} - 2\,e^{2x} - e^{-x} & -e^{x} + 2\,e^{2x} & -e^{x} + e^{-x} \\ -2\,e^{-x} - e^{2x} + 3\,e^{x} & -e^{x} + e^{2x} & -e^{x} + 2\,e^{-x} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(x) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\mathrm{e}^{2x}+3\,\mathrm{e}^{x}-\mathrm{e}^{-x} & -\mathrm{e}^{x}+\mathrm{e}^{2x} & -\mathrm{e}^{x}+\mathrm{e}^{-x} \\ 3\,\mathrm{e}^{x}-2\,\mathrm{e}^{2x}-\mathrm{e}^{-x} & -\mathrm{e}^{x}+2\,\mathrm{e}^{2x} & -\mathrm{e}^{x}+\mathrm{e}^{-x} \\ -2\,\mathrm{e}^{-x}-\mathrm{e}^{2x}+3\,\mathrm{e}^{x} & -\mathrm{e}^{x}+\mathrm{e}^{2x} & -\mathrm{e}^{x}+2\,\mathrm{e}^{-x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (-\mathrm{e}^{2x}+3\,\mathrm{e}^{x}-\mathrm{e}^{-x})\,c_1+(-\mathrm{e}^{x}+\mathrm{e}^{2x})\,c_2+(-\mathrm{e}^{x}+\mathrm{e}^{-x})\,c_3 \\ (3\,\mathrm{e}^{x}-2\,\mathrm{e}^{2x}-\mathrm{e}^{-x})\,c_1+(-\mathrm{e}^{x}+2\,\mathrm{e}^{2x})\,c_2+(-\mathrm{e}^{x}+\mathrm{e}^{-x})\,c_3 \\ (-2\,\mathrm{e}^{-x}-\mathrm{e}^{2x}+3\,\mathrm{e}^{x})\,c_1+(-\mathrm{e}^{x}+\mathrm{e}^{2x})\,c_2+(-\mathrm{e}^{x}+2\,\mathrm{e}^{-x})\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} (-c_1+c_3)\,\mathrm{e}^{-x}+(-c_1+c_2)\,\mathrm{e}^{2x}+3\,\mathrm{e}^{x}\left(-\frac{c_3}{3}+c_1-\frac{c_2}{3}\right) \\ (-c_1+c_3)\,\mathrm{e}^{-x}+(-2c_1+2c_2)\,\mathrm{e}^{2x}+3\,\mathrm{e}^{x}\left(-\frac{c_3}{3}+c_1-\frac{c_2}{3}\right) \\ (-2c_1+2c_3)\,\mathrm{e}^{-x}+(-c_1+c_2)\,\mathrm{e}^{2x}+3\,\mathrm{e}^{x}\left(-\frac{c_3}{3}+c_1-\frac{c_2}{3}\right) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

19.9.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(x) = A\,\vec{x}(x)
$$

Or

$$
\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & -2 \\ 3 & 1 & -3 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A-\lambda I\right)=0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & -2 \\ 3 & 1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & -2 \\ 0 & 3-\lambda & -2 \\ 3 & 1 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & -2 \\ 3 & 1 & -3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -2 \\ 0 & 4 & -2 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & 0 \\ 0 & 4 & -2 & 0 \\ 3 & 1 & -2 & 0 \end{array}\right]$$

$$
R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3 & 1 & -2 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 3 & 1 & -2 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$
\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}
$$

Which is normalized to

$$
\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}
$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & -2 \\ 3 & 1 & -3 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & -2 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 3 & 1 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & -2 \\ 3 & 1 & -3 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -2 \\ 0 & 1 & -2 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 3 & 1 & -5 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 3 gives

$$\left[\begin{array}{ccc|c} 3 & 1 & -5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 3 & 1 & -5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 1 & -5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(x) &= \vec{v}_1 e^{x} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{x}
\end{aligned}
$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(x) &= \vec{v}_2 e^{-x} \\
&= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} e^{-x}
\end{aligned}
$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(x) &= \vec{v}_3 e^{2x} \\
&= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{2x}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(x) = c_1 \vec{x}_1(x) + c_2 \vec{x}_2(x) + c_3 \vec{x}_3(x)
$$

Which is written as

$$
\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = c_1 \begin{bmatrix} e^{x} \\ e^{x} \\ e^{x} \end{bmatrix} + c_2 \begin{bmatrix} \frac{e^{-x}}{2} \\ \frac{e^{-x}}{2} \\ e^{-x} \end{bmatrix} + c_3 \begin{bmatrix} e^{2x} \\ 2\, e^{2x} \\ e^{2x} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} c_1 e^{x} + \frac{c_2 e^{-x}}{2} + c_3 e^{2x} \\ c_1 e^{x} + \frac{c_2 e^{-x}}{2} + 2c_3 e^{2x} \\ c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} \end{bmatrix}
$$

19.9.3 Maple step by step solution

Let's solve

$[y_1'(x) = 2y_1(x) + y_2(x) - 2y_3(x), y_2'(x) = 3y_2(x) - 2y_3(x), y_3'(x) = 3y_1(x) + y_2(x) - 3y_3(x)]$

- Define vector

$$
\underline{y}^{\rightarrow}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}
$$

- Convert system into a vector equation

$$\underline{\vec{y}}'(x) = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & -2 \\ 3 & 1 & -3 \end{bmatrix} \cdot \underline{\vec{y}}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{y}}'(x) = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & -2 \\ 3 & 1 & -3 \end{bmatrix} \cdot \underline{\vec{y}}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & -2 \\ 3 & 1 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(x) = A \cdot \underline{\vec{y}}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = e^{-x} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_{2} = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_{3} = e^{2x} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_{1} + c_2 \underline{\vec{y}}_{2} + c_3 \underline{\vec{y}}_{3}$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 e^{-x} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + e^{2x} c_3 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{-x}}{2} + c_2 e^{x} + e^{2x} c_3 \\ \frac{c_1 e^{-x}}{2} + c_2 e^{x} + 2 e^{2x} c_3 \\ c_1 e^{-x} + c_2 e^{x} + e^{2x} c_3 \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{y_1(x) = \frac{c_1 e^{-x}}{2} + c_2 e^{x} + e^{2x} c_3, y_2(x) = \frac{c_1 e^{-x}}{2} + c_2 e^{x} + 2 e^{2x} c_3, y_3(x) = c_1 e^{-x} + c_2 e^{x} + e^{2x} c_3\right\}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 65

```
dsolve([diff(y__1(x),x)=2*y__1(x)+y__2(x)-2*y__3(x),diff(y__2(x),x)=3*y__2(x)-2*y__3(x),diff
```

$$y_1(x) = c_1 e^x + \frac{c_2 e^{2x}}{2} + c_3 e^{-x}$$
$$y_2(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{-x}$$
$$y_3(x) = c_1 e^x + \frac{c_2 e^{2x}}{2} + 2c_3 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 159

```
DSolve[{y1'[x]==2*y1[x]+y2[x]-2*y3[x],y2'[x]==3*y2[x]-2*y3[x],y3'[x]==3*y1[x]+y2[x]-3*y3[x]}
```

$$\text{y1}(x) \to e^{-x}\left((e^x-1)\left(c_2e^{2x}-c_3e^x-c_3\right)-c_1\left(-3e^{2x}+e^{3x}+1\right)\right)$$
$$\text{y2}(x) \to e^{-x}\left(-\left(c_1(2e^x+1)\left(e^x-1\right)^2\right)+2c_2e^{3x}-(c_2+c_3)e^{2x}+c_3\right)$$
$$\text{y3}(x) \to e^{-x}\left(-\left(c_1\left(-3e^{2x}+e^{3x}+2\right)\right)+c_2e^{3x}-(c_2+c_3)e^{2x}+2c_3\right)$$

19.10 problem 8

Internal problem ID [12848]
Internal file name [`OUTPUT/11500_Monday_November_06_2023_01_31_05_PM_5352351/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= 5y_1(x) - 5y_2(x) - 5y_3(x) \\
y_2'(x) &= -y_1(x) + 4y_2(x) + 2y_3(x) \\
y_3'(x) &= 3y_1(x) - 5y_2(x) - 3y_3(x)
\end{aligned}
$$

19.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(x) = A\,\vec{x}(x)
$$

Or

$$
\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \left[\begin{array}{ccc} e^{2x}\cos(x)+3\,e^{2x}\sin(x) & -5\,e^{2x}\sin(x) & -5\,e^{2x}\sin(x) \\ -e^{2x}\cos(x)-e^{2x}\sin(x)+e^{2x} & e^{2x}\cos(x)+2\,e^{2x}\sin(x) & e^{2x}\cos(x)+2\,e^{2x}\sin(x)-e^{2x} \\ e^{2x}\cos(x)+3\,e^{2x}\sin(x)-e^{2x} & -5\,e^{2x}\sin(x) & e^{2x}-5\,e^{2x}\sin(x) \end{array}\right]$$

$$= \left[\begin{array}{ccc} e^{2x}(\cos(x)+3\sin(x)) & -5\,e^{2x}\sin(x) & -5\,e^{2x}\sin(x) \\ -e^{2x}(-1+\cos(x)+\sin(x)) & e^{2x}(\cos(x)+2\sin(x)) & e^{2x}(-1+\cos(x)+2\sin(x)) \\ e^{2x}(-1+\cos(x)+3\sin(x)) & -5\,e^{2x}\sin(x) & e^{2x}(1-5\sin(x)) \end{array}\right]$$

Therefore the homogeneous solution is

$$\vec{x}_h(x) = e^{At}\vec{c}$$

$$= \left[\begin{array}{ccc} e^{2x}(\cos(x)+3\sin(x)) & -5\,e^{2x}\sin(x) & -5\,e^{2x}\sin(x) \\ -e^{2x}(-1+\cos(x)+\sin(x)) & e^{2x}(\cos(x)+2\sin(x)) & e^{2x}(-1+\cos(x)+2\sin(x)) \\ e^{2x}(-1+\cos(x)+3\sin(x)) & -5\,e^{2x}\sin(x) & e^{2x}(1-5\sin(x)) \end{array}\right]\left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array}\right]$$

$$= \left[\begin{array}{c} e^{2x}(\cos(x)+3\sin(x))\,c_1-5\,e^{2x}\sin(x)\,c_2-5\,e^{2x}\sin(x)\,c_3 \\ -e^{2x}(-1+\cos(x)+\sin(x))\,c_1+e^{2x}(\cos(x)+2\sin(x))\,c_2+e^{2x}(-1+\cos(x)+2\sin(x))\,c_3 \\ e^{2x}(-1+\cos(x)+3\sin(x))\,c_1-5\,e^{2x}\sin(x)\,c_2+e^{2x}(1-5\sin(x))\,c_3 \end{array}\right]$$

$$= \left[\begin{array}{c} ((3c_1-5c_2-5c_3)\sin(x)+c_1\cos(x))\,e^{2x} \\ -e^{2x}((c_1-c_2-c_3)\cos(x)+(c_1-2c_2-2c_3)\sin(x)-c_1+c_3) \\ e^{2x}((3c_1-5c_2-5c_3)\sin(x)+c_1\cos(x)-c_1+c_3) \end{array}\right]$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

19.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\left[\begin{array}{c} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{array}\right] = \left[\begin{array}{ccc} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{array}\right]\left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & -5 & -5 \\ -1 & 4-\lambda & 2 \\ 3 & -5 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 + 13\lambda - 10 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2 + i$$
$$\lambda_2 = 2 - i$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
$2+i$	1	complex eigenvalue
$2-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -5 & -5 \\ -1 & 2 & 2 \\ 3 & -5 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & -5 & -5 & 0 \\ -1 & 2 & 2 & 0 \\ 3 & -5 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 3 & -5 & -5 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 3 & -5 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3 & -5 & -5 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -5 & -5 \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix} - (2-i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3+i & -5 & -5 \\ -1 & 2+i & 2 \\ 3 & -5 & -5+i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3+i & -5 & -5 & 0 \\ -1 & 2+i & 2 & 0 \\ 3 & -5 & -5+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{3}{10} - \frac{i}{10}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3+i & -5 & -5 & 0 \\ 0 & \frac{1}{2} + \frac{3i}{2} & \frac{1}{2} + \frac{i}{2} & 0 \\ 3 & -5 & -5+i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{9}{10} + \frac{3i}{10}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} 3+i & -5 & -5 & 0 \\ 0 & \frac{1}{2} + \frac{3i}{2} & \frac{1}{2} + \frac{i}{2} & 0 \\ 0 & -\frac{1}{2} - \frac{3i}{2} & -\frac{1}{2} - \frac{i}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 3+i & -5 & -5 & 0 \\ 0 & \frac{1}{2}+\frac{3i}{2} & \frac{1}{2}+\frac{i}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3+i & -5 & -5 \\ 0 & \frac{1}{2}+\frac{3i}{2} & \frac{1}{2}+\frac{i}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t, v_2 = -\frac{2}{5}t + \frac{1}{5}it\right\}$

Hence the solution is

$$\begin{bmatrix} t \\ -\frac{2t}{5} + \frac{\mathrm{I}t}{5} \\ t \end{bmatrix} = \begin{bmatrix} t \\ -\frac{2}{5}t + \frac{1}{5}it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -\frac{2t}{5} + \frac{\mathrm{I}t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ \frac{-\frac{2}{5}t+\frac{1}{5}it}{t} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -\frac{2t}{5} + \frac{\mathrm{I}t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} t \\ -\frac{2t}{5} + \frac{\mathrm{I}t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ -2+i \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 2 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix} - (2+i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-i & -5 & -5 \\ -1 & 2-i & 2 \\ 3 & -5 & -5-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3-i & -5 & -5 & 0 \\ -1 & 2-i & 2 & 0 \\ 3 & -5 & -5-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{3}{10} + \frac{i}{10}\right) R_1 \implies \left[\begin{array}{ccc|c} 3-i & -5 & -5 & 0 \\ 0 & \frac{1}{2} - \frac{3i}{2} & \frac{1}{2} - \frac{i}{2} & 0 \\ 3 & -5 & -5-i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{9}{10} - \frac{3i}{10}\right) R_1 \implies \left[\begin{array}{ccc|c} 3-i & -5 & -5 & 0 \\ 0 & \frac{1}{2} - \frac{3i}{2} & \frac{1}{2} - \frac{i}{2} & 0 \\ 0 & -\frac{1}{2} + \frac{3i}{2} & -\frac{1}{2} + \frac{i}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \implies \left[\begin{array}{ccc|c} 3-i & -5 & -5 & 0 \\ 0 & \frac{1}{2} - \frac{3i}{2} & \frac{1}{2} - \frac{i}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3-i & -5 & -5 \\ 0 & \frac{1}{2} - \frac{3i}{2} & \frac{1}{2} - \frac{i}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = t, v_2 = -\frac{2}{5}t - \frac{1}{5}it\right\}$

Hence the solution is

$$\begin{bmatrix} t \\ -\frac{2t}{5} - \frac{\mathrm{I}t}{5} \\ t \end{bmatrix} = \begin{bmatrix} t \\ -\frac{2}{5}t - \frac{1}{5}it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -\frac{2t}{5} - \frac{\mathrm{I}t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ \frac{-\frac{2}{5}t - \frac{1}{5}it}{t} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -\frac{2t}{5} - \frac{\mathrm{I}t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} t \\ -\frac{2t}{5} - \frac{\mathrm{I}t}{5} \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ -2 - i \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
$2+i$	1	1	No	$\begin{bmatrix} 1 \\ -\frac{2}{5}-\frac{i}{5} \\ 1 \end{bmatrix}$
$2-i$	1	1	No	$\begin{bmatrix} 1 \\ -\frac{2}{5}+\frac{i}{5} \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(x) &= \vec{v}_1 e^{2x} \\ &= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{2x}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(x) = c_1\vec{x}_1(x) + c_2\vec{x}_2(x) + c_3\vec{x}_3(x)$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{(2+i)x} \\ \left(-\frac{2}{5}-\frac{i}{5}\right)\mathrm{e}^{(2+i)x} \\ \mathrm{e}^{(2+i)x} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{(2-i)x} \\ \left(-\frac{2}{5}+\frac{i}{5}\right)\mathrm{e}^{(2-i)x} \\ \mathrm{e}^{(2-i)x} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -\mathrm{e}^{2x} \\ \mathrm{e}^{2x} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} c_1 e^{(2+i)x} + c_2 e^{(2-i)x} \\ \left(-\frac{2}{5} - \frac{i}{5}\right) c_1 e^{(2+i)x} + \left(-\frac{2}{5} + \frac{i}{5}\right) c_2 e^{(2-i)x} - c_3 e^{2x} \\ c_1 e^{(2+i)x} + c_2 e^{(2-i)x} + c_3 e^{2x} \end{bmatrix}$$

19.10.3 Maple step by step solution

Let's solve

$[y_1'(x) = 5y_1(x) - 5y_2(x) - 5y_3(x), y_2'(x) = -y_1(x) + 4y_2(x) + 2y_3(x), y_3'(x) = 3y_1(x) - 5y_2(x) -$

- Define vector

$$\underline{y}^{\rightarrow}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix}$$

- Rewrite the system as

$\underline{y}^{\rightarrow\prime}(x) = A \cdot \underline{y}^{\rightarrow}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right]\right], \left[2-\mathrm{I}, \left[\begin{array}{c} 1 \\ -\frac{2}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]\right], \left[2+\mathrm{I}, \left[\begin{array}{c} 1 \\ -\frac{2}{5}-\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\overrightarrow{y}}_{1} = \mathrm{e}^{2x} \cdot \left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2-\mathrm{I}, \left[\begin{array}{c} 1 \\ -\frac{2}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(2-\mathrm{I})x} \cdot \left[\begin{array}{c} 1 \\ -\frac{2}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{2x} \cdot (\cos(x) - \mathrm{I}\sin(x)) \cdot \left[\begin{array}{c} 1 \\ -\frac{2}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{2x} \cdot \left[\begin{array}{c} \cos(x) - \mathrm{I}\sin(x) \\ \left(-\frac{2}{5}+\frac{\mathrm{I}}{5}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{y}^{\rightarrow}{}_{2}(x)=\mathrm{e}^{2x}\cdot\left[\begin{array}{c}\cos(x)\\ -\frac{2\cos(x)}{5}+\frac{\sin(x)}{5}\\ \cos(x)\end{array}\right],\underline{y}^{\rightarrow}{}_{3}(x)=\mathrm{e}^{2x}\cdot\left[\begin{array}{c}-\sin(x)\\ \frac{2\sin(x)}{5}+\frac{\cos(x)}{5}\\ -\sin(x)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\underline{y}^{\rightarrow}=c_1\underline{y}^{\rightarrow}{}_{1}+c_2\underline{y}^{\rightarrow}{}_{2}(x)+c_3\underline{y}^{\rightarrow}{}_{3}(x)$$

- Substitute solutions into the general solution

$$\underline{y}^{\rightarrow}=c_1\mathrm{e}^{2x}\cdot\left[\begin{array}{c}0\\ -1\\ 1\end{array}\right]+c_2\mathrm{e}^{2x}\cdot\left[\begin{array}{c}\cos(x)\\ -\frac{2\cos(x)}{5}+\frac{\sin(x)}{5}\\ \cos(x)\end{array}\right]+\mathrm{e}^{2x}c_3\cdot\left[\begin{array}{c}-\sin(x)\\ \frac{2\sin(x)}{5}+\frac{\cos(x)}{5}\\ -\sin(x)\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}y_1(x)\\ y_2(x)\\ y_3(x)\end{array}\right]=\left[\begin{array}{c}\mathrm{e}^{2x}(c_2\cos(x)-c_3\sin(x))\\ -\left(\frac{(2c_2-c_3)\cos(x)}{5}+\frac{(-c_2-2c_3)\sin(x)}{5}+c_1\right)\mathrm{e}^{2x}\\ \mathrm{e}^{2x}(c_1+c_2\cos(x)-c_3\sin(x))\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{y_1(x)=\mathrm{e}^{2x}(c_2\cos(x)-c_3\sin(x)),y_2(x)=-\left(\frac{(2c_2-c_3)\cos(x)}{5}+\frac{(-c_2-2c_3)\sin(x)}{5}+c_1\right)\mathrm{e}^{2x},y_3(x)=\mathrm{e}^{2x}\right.$$

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 71

```
dsolve([diff(y__1(x),x)=5*y__1(x)-5*y__2(x)-5*y__3(x),diff(y__2(x),x)=-1*y__1(x)+4*y__2(x)+2
```

$$y_1(x)=\mathrm{e}^{2x}(\sin(x)c_2+\cos(x)c_3)$$
$$y_2(x)=-\frac{(2\sin(x)c_2-\sin(x)c_3+\cos(x)c_2+2\cos(x)c_3-5c_1)\mathrm{e}^{2x}}{5}$$
$$y_3(x)=\mathrm{e}^{2x}(\sin(x)c_2+\cos(x)c_3-c_1)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 109

```
DSolve[{y1'[x]==5*y1[x]-5*y2[x]-5*y3[x],y2'[x]==-1*y1[x]+4*y2[x]+2*y3[x],y3'[x]==3*y1[x]-5*y
```

$$\text{y1}(x)\to e^{2x}(c_1\cos(x)+(3c_1-5(c_2+c_3))\sin(x))$$
$$\text{y2}(x)\to e^{2x}(-c_1(\sin(x)+\cos(x)-1)+c_3(2\sin(x)+\cos(x)-1)+c_2(2\sin(x)+\cos(x)))$$
$$\text{y3}(x)\to e^{2x}(c_1\cos(x)+(3c_1-5(c_2+c_3))\sin(x)-c_1+c_3)$$

19.11 problem 9

Internal problem ID [12849]
Internal file name [`OUTPUT/11501_Monday_November_06_2023_01_31_05_PM_39861667/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= 4y_1(x) + 6y_2(x) + 6y_3(x)\\
y_2'(x) &= y_1(x) + 3y_2(x) + 2y_3(x)\\
y_3'(x) &= -y_1(x) - 4y_2(x) - 3y_3(x)
\end{aligned}
$$

19.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$
\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{4x} & \frac{6\,e^{4x}}{5} - \frac{6\,e^{-x}}{5} & \frac{6\,e^{4x}}{5} - \frac{6\,e^{-x}}{5} \\ \frac{e^{4x}}{3} - \frac{e^{x}}{3} & -\frac{2\,e^{-x}}{5} + \frac{2\,e^{4x}}{5} + e^{x} & \frac{2\,e^{4x}}{5} - \frac{2\,e^{-x}}{5} \\ -\frac{e^{4x}}{3} + \frac{e^{x}}{3} & -\frac{2\,e^{4x}}{5} - e^{x} + \frac{7\,e^{-x}}{5} & \frac{7\,e^{-x}}{5} - \frac{2\,e^{4x}}{5} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(x) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{4x} & \frac{6\,e^{4x}}{5}-\frac{6\,e^{-x}}{5} & \frac{6\,e^{4x}}{5}-\frac{6\,e^{-x}}{5} \\ \frac{e^{4x}}{3}-\frac{e^{x}}{3} & -\frac{2\,e^{-x}}{5}+\frac{2\,e^{4x}}{5}+e^{x} & \frac{2\,e^{4x}}{5}-\frac{2\,e^{-x}}{5} \\ -\frac{e^{4x}}{3}+\frac{e^{x}}{3} & -\frac{2\,e^{4x}}{5}-e^{x}+\frac{7\,e^{-x}}{5} & \frac{7\,e^{-x}}{5}-\frac{2\,e^{4x}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{4x}c_1+\left(\frac{6\,e^{4x}}{5}-\frac{6\,e^{-x}}{5}\right)c_2+\left(\frac{6\,e^{4x}}{5}-\frac{6\,e^{-x}}{5}\right)c_3 \\ \left(\frac{e^{4x}}{3}-\frac{e^{x}}{3}\right)c_1+\left(-\frac{2\,e^{-x}}{5}+\frac{2\,e^{4x}}{5}+e^{x}\right)c_2+\left(\frac{2\,e^{4x}}{5}-\frac{2\,e^{-x}}{5}\right)c_3 \\ \left(-\frac{e^{4x}}{3}+\frac{e^{x}}{3}\right)c_1+\left(-\frac{2\,e^{4x}}{5}-e^{x}+\frac{7\,e^{-x}}{5}\right)c_2+\left(\frac{7\,e^{-x}}{5}-\frac{2\,e^{4x}}{5}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(5c_1+6c_2+6c_3)e^{4x}}{5}-\frac{6\,e^{-x}(c_2+c_3)}{5} \\ \frac{(5c_1+6c_2+6c_3)e^{4x}}{15}+\frac{2(-c_2-c_3)e^{-x}}{5}-\frac{e^{x}(c_1-3c_2)}{3} \\ \frac{(-5c_1-6c_2-6c_3)e^{4x}}{15}+\frac{7\,e^{-x}(c_2+c_3)}{5}+\frac{e^{x}(c_1-3c_2)}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

19.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4$$
$$\lambda_2 = -1$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 5 & 6 & 6 & 0 \\ 1 & 4 & 2 & 0 \\ -1 & -4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & 6 & 6 & 0 \\ 0 & \frac{14}{5} & \frac{4}{5} & 0 \\ -1 & -4 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} 5 & 6 & 6 & 0 \\ 0 & \frac{14}{5} & \frac{4}{5} & 0 \\ 0 & -\frac{14}{5} & -\frac{4}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} 5 & 6 & 6 & 0 \\ 0 & \frac{14}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 6 & 6 \\ 0 & \frac{14}{5} & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{6t}{7}, v_2 = -\frac{2t}{7}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{6t}{7} \\ -\frac{2t}{7} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{6t}{7} \\ -\frac{2t}{7} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{6t}{7} \\ -\frac{2t}{7} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{6}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{6t}{7} \\ -\frac{2t}{7} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{6}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{6t}{7} \\ -\frac{2t}{7} \\ t \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 6 & 6 & 0 \\ 1 & 2 & 2 & 0 \\ -1 & -4 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \implies \left[\begin{array}{ccc|c} 3 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -4 & -4 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{3} \implies \left[\begin{array}{ccc|c} 3 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 3 & 6 & 6 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 6 & 6 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 6 & 6 & 0 \\ 1 & -1 & 2 & 0 \\ -1 & -4 & -7 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 6 & 6 & 0 \\ -1 & -4 & -7 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 6 & 6 & 0 \\ 0 & -5 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{5R_2}{6} \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 6 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -3t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} -3t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -3t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -3t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -3t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
4	1	1	No	$\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -\frac{6}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(x) &= \vec{v}_1 e^{4x} \\ &= \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} e^{4x}\end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(x) &= \vec{v}_2 e^{-x} \\ &= \begin{bmatrix} -\frac{6}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix} e^{-x}\end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(x) = \vec{v}_3 e^{x}$$
$$= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{x}$$

Therefore the final solution is

$$\vec{x}_h(x) = c_1\vec{x}_1(x) + c_2\vec{x}_2(x) + c_3\vec{x}_3(x)$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = c_1 \begin{bmatrix} -3\,\mathrm{e}^{4x} \\ -\mathrm{e}^{4x} \\ \mathrm{e}^{4x} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{6\,\mathrm{e}^{-x}}{7} \\ -\frac{2\,\mathrm{e}^{-x}}{7} \\ \mathrm{e}^{-x} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -\mathrm{e}^{x} \\ \mathrm{e}^{x} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} -3c_1\mathrm{e}^{4x} - \frac{6c_2\mathrm{e}^{-x}}{7} \\ -c_1\mathrm{e}^{4x} - \frac{2c_2\mathrm{e}^{-x}}{7} - c_3\mathrm{e}^{x} \\ c_1\mathrm{e}^{4x} + c_2\mathrm{e}^{-x} + c_3\mathrm{e}^{x} \end{bmatrix}$$

19.11.3 Maple step by step solution

Let's solve

$[y_1'(x) = 4y_1(x) + 6y_2(x) + 6y_3(x)\,, y_2'(x) = y_1(x) + 3y_2(x) + 2y_3(x)\,, y_3'(x) = -y_1(x) - 4y_2(x) - 3$

- Define vector

$$\underline{y}^{\rightarrow}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{\vec{y}}'(x) = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} \cdot \underline{\vec{y}}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{\vec{y}}'(x) = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} \cdot \underline{\vec{y}}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{y}}'(x) = A \cdot \underline{\vec{y}}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -\frac{6}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -\frac{6}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = e^{-x} \cdot \begin{bmatrix} -\frac{6}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_{2} = e^{x} \cdot \left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[4, \left[\begin{array}{c} -3 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_{3} = e^{4x} \cdot \left[\begin{array}{c} -3 \\ -1 \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_{1} + c_2 \underline{\vec{y}}_{2} + c_3 \underline{\vec{y}}_{3}$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 e^{-x} \cdot \left[\begin{array}{c} -\frac{6}{7} \\ -\frac{2}{7} \\ 1 \end{array}\right] + c_2 e^{x} \cdot \left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right] + e^{4x} c_3 \cdot \left[\begin{array}{c} -3 \\ -1 \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right] = \left[\begin{array}{c} -\frac{6 c_1 e^{-x}}{7} - 3\, e^{4x} c_3 \\ -\frac{2 c_1 e^{-x}}{7} - c_2 e^{x} - e^{4x} c_3 \\ c_1 e^{-x} + c_2 e^{x} + e^{4x} c_3 \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{y_1(x) = -\frac{6 c_1 e^{-x}}{7} - 3\, e^{4x} c_3, y_2(x) = -\frac{2 c_1 e^{-x}}{7} - c_2 e^{x} - e^{4x} c_3, y_3(x) = c_1 e^{-x} + c_2 e^{x} + e^{4x} c_3\right\}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 63

```
dsolve([diff(y__1(x),x)=4*y__1(x)+6*y__2(x)+6*y__3(x),diff(y__2(x),x)=1*y__1(x)+3*y__2(x)+2*
```

$$y_1(x) = c_2 e^{4x} + c_3 e^{-x}$$
$$y_2(x) = \frac{c_2 e^{4x}}{3} + \frac{c_3 e^{-x}}{3} + c_1 e^{x}$$
$$y_3(x) = -\frac{7c_3 e^{-x}}{6} - \frac{c_2 e^{4x}}{3} - c_1 e^{x}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 145

```
DSolve[{y1'[x]==4*y1[x]+6*y2[x]+6*y3[x],y2'[x]==1*y1[x]+3*y2[x]+2*y3[x],y3'[x]==-1*y1[x]-4*y
```

$$\text{y1}(x) \to \frac{1}{5}e^{-x}\left((5c_1 + 6(c_2 + c_3))e^{5x} - 6(c_2 + c_3)\right)$$
$$\text{y2}(x) \to \frac{1}{15}e^{-x}\left(-5(c_1 - 3c_2)e^{2x} + (5c_1 + 6(c_2 + c_3))e^{5x} - 6(c_2 + c_3)\right)$$
$$\text{y3}(x) \to \frac{1}{3}(c_1 - 3c_2)e^{x} + \frac{7}{5}(c_2 + c_3)e^{-x} - \frac{1}{15}(5c_1 + 6(c_2 + c_3))e^{4x}$$

19.12 problem 10

Internal problem ID [12850]
Internal file name [`OUTPUT/11502_Monday_November_06_2023_01_31_06_PM_61324329/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= y_1(x) + 2y_2(x) - 3y_3(x) \\
y_2'(x) &= -3y_1(x) + 4y_2(x) - 2y_3(x) \\
y_3'(x) &= 2y_1(x) + y_3(x)
\end{aligned}
$$

19.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$
\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 4 & -2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{11\,e^{2x}\cos(3x)}{9} - \frac{e^{2x}\sin(3x)}{3} - \frac{2\,e^{2x}}{9} & -\frac{2\,e^{2x}\cos(3x)}{9} + \frac{2\,e^{2x}\sin(3x)}{3} + \frac{2\,e^{2x}}{9} & -\frac{2\,e^{2x}\cos(3x)}{9} - e^{2x}\sin(3x) + \frac{2\,e^{2x}}{9} \\ \frac{7\,e^{2x}\cos(3x)}{9} - e^{2x}\sin(3x) - \frac{7\,e^{2x}}{9} & \frac{2\,e^{2x}\cos(3x)}{9} + \frac{2\,e^{2x}\sin(3x)}{3} + \frac{7\,e^{2x}}{9} & -\frac{7\,e^{2x}\cos(3x)}{9} - \frac{2\,e^{2x}\sin(3x)}{3} + \frac{7\,e^{2x}}{9} \\ \frac{4\,e^{2x}\cos(3x)}{9} + \frac{2\,e^{2x}\sin(3x)}{3} - \frac{4\,e^{2x}}{9} & -\frac{4\,e^{2x}\cos(3x)}{9} + \frac{4\,e^{2x}}{9} & \frac{5\,e^{2x}\cos(3x)}{9} - \frac{e^{2x}\sin(3x)}{3} + \frac{4\,e^{2x}}{9} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{2x}(-2+11\cos(3x)-3\sin(3x))}{9} & -\frac{2\,e^{2x}(-1+\cos(3x)-3\sin(3x))}{9} & -\frac{e^{2x}(-2+2\cos(3x)+9\sin(3x))}{9} \\ \frac{e^{2x}(-7+7\cos(3x)-9\sin(3x))}{9} & \frac{e^{2x}(7+2\cos(3x)+6\sin(3x))}{9} & -\frac{e^{2x}(-7+7\cos(3x)+6\sin(3x))}{9} \\ \frac{2\,e^{2x}(-2+2\cos(3x)+3\sin(3x))}{9} & -\frac{4\,e^{2x}(\cos(3x)-1)}{9} & \frac{e^{2x}(4+5\cos(3x)-3\sin(3x))}{9} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(x) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{e^{2x}(-2+11\cos(3x)-3\sin(3x))}{9} & -\frac{2\,e^{2x}(-1+\cos(3x)-3\sin(3x))}{9} & -\frac{e^{2x}(-2+2\cos(3x)+9\sin(3x))}{9} \\ \frac{e^{2x}(-7+7\cos(3x)-9\sin(3x))}{9} & \frac{e^{2x}(7+2\cos(3x)+6\sin(3x))}{9} & -\frac{e^{2x}(-7+7\cos(3x)+6\sin(3x))}{9} \\ \frac{2\,e^{2x}(-2+2\cos(3x)+3\sin(3x))}{9} & -\frac{4\,e^{2x}(\cos(3x)-1)}{9} & \frac{e^{2x}(4+5\cos(3x)-3\sin(3x))}{9} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{2x}(-2+11\cos(3x)-3\sin(3x))c_1}{9} - \frac{2\,e^{2x}(-1+\cos(3x)-3\sin(3x))c_2}{9} - \frac{e^{2x}(-2+2\cos(3x)+9\sin(3x))c_3}{9} \\ \frac{e^{2x}(-7+7\cos(3x)-9\sin(3x))c_1}{9} + \frac{e^{2x}(7+2\cos(3x)+6\sin(3x))c_2}{9} - \frac{e^{2x}(-7+7\cos(3x)+6\sin(3x))c_3}{9} \\ \frac{2\,e^{2x}(-2+2\cos(3x)+3\sin(3x))c_1}{9} - \frac{4\,e^{2x}(\cos(3x)-1)c_2}{9} + \frac{e^{2x}(4+5\cos(3x)-3\sin(3x))c_3}{9} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11\,e^{2x}\left(\left(c_1-\frac{2c_2}{11}-\frac{2c_3}{11}\right)\cos(3x)+\frac{3(-c_1+2c_2-3c_3)\sin(3x)}{11}-\frac{2c_1}{11}+\frac{2c_2}{11}+\frac{2c_3}{11}\right)}{9} \\ \frac{7\,e^{2x}\left(\left(c_1+\frac{2c_2}{7}-c_3\right)\cos(3x)+\frac{3(-3c_1+2c_2-2c_3)\sin(3x)}{7}-c_1+c_2+c_3\right)}{9} \\ \frac{4\,e^{2x}\left(\left(c_1-c_2+\frac{5c_3}{4}\right)\cos(3x)+\left(\frac{3c_1}{2}-\frac{3c_3}{4}\right)\sin(3x)-c_1+c_2+c_3\right)}{9} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

19.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 4 & -2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 & -3 \\ -3 & 4 & -2 \\ 2 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 2 & -3 \\ -3 & 4-\lambda & -2 \\ 2 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 + 21\lambda - 26 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 2+3i \\ \lambda_2 &= 2-3i \\ \lambda_3 &= 2 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
$2-3i$	1	complex eigenvalue
$2+3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & -3 \\ -3 & 4 & -2 \\ 2 & 0 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -3 \\ -3 & 2 & -2 \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ -3 & 2 & -2 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 0 & -4 & 7 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 0 & -4 & 7 & 0 \\ 0 & 4 & -7 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 0 & -4 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 2 & -3 \\ 0 & -4 & 7 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = \frac{t}{2}, v_2 = \frac{7t}{4}\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ \frac{7t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ \frac{7t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ \frac{7t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ \frac{7}{4} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ \frac{7t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{7}{4} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ \frac{7t}{4} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 - 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & -3 \\ -3 & 4 & -2 \\ 2 & 0 & 1 \end{bmatrix} - (2 - 3i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 + 3i & 2 & -3 \\ -3 & 2 + 3i & -2 \\ 2 & 0 & -1 + 3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1+3i & 2 & -3 & 0 \\ -3 & 2+3i & -2 & 0 \\ 2 & 0 & -1+3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{10} - \frac{9i}{10}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1+3i & 2 & -3 & 0 \\ 0 & \frac{7}{5}+\frac{6i}{5} & -\frac{11}{10}+\frac{27i}{10} & 0 \\ 2 & 0 & -1+3i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(\frac{1}{5} + \frac{3i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1+3i & 2 & -3 & 0 \\ 0 & \frac{7}{5}+\frac{6i}{5} & -\frac{11}{10}+\frac{27i}{10} & 0 \\ 0 & \frac{2}{5}+\frac{6i}{5} & -\frac{8}{5}+\frac{6i}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{10}{17} - \frac{6i}{17}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1+3i & 2 & -3 & 0 \\ 0 & \frac{7}{5}+\frac{6i}{5} & -\frac{11}{10}+\frac{27i}{10} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+3i & 2 & -3 \\ 0 & \frac{7}{5}+\frac{6i}{5} & -\frac{11}{10}+\frac{27i}{10} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{3i}{2}\right)t, v_2 = -\frac{1}{2}t - \frac{3}{2}it\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{3\,I}{2}\right)t \\ -\frac{t}{2} - \frac{3\,I\,t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{3i}{2}\right)t \\ -\frac{1}{2}t - \frac{3}{2}it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2}-\frac{3\,\mathrm{I}}{2}\right)t \\ -\frac{t}{2}-\frac{3\,\mathrm{I}\,t}{2} \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2}-\frac{3i}{2} \\ \frac{-\frac{1}{2}t-\frac{3}{2}it}{t} \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2}-\frac{3\,\mathrm{I}}{2}\right)t \\ -\frac{t}{2}-\frac{3\,\mathrm{I}\,t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2}-\frac{3i}{2} \\ -\frac{1}{2}-\frac{3i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2}-\frac{3\,\mathrm{I}}{2}\right)t \\ -\frac{t}{2}-\frac{3\,\mathrm{I}\,t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1-3i \\ -1-3i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 2+3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 & -3 \\ -3 & 4 & -2 \\ 2 & 0 & 1 \end{bmatrix} - (2+3i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1-3i & 2 & -3 \\ -3 & 2-3i & -2 \\ 2 & 0 & -1-3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1-3i & 2 & -3 & 0 \\ -3 & 2-3i & -2 & 0 \\ 2 & 0 & -1-3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{3}{10}+\frac{9i}{10}\right)R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1-3i & 2 & -3 & 0 \\ 0 & \frac{7}{5}-\frac{6i}{5} & -\frac{11}{10}-\frac{27i}{10} & 0 \\ 2 & 0 & -1-3i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(\frac{1}{5} - \frac{3i}{5}\right) R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1-3i & 2 & -3 & 0 \\ 0 & \frac{7}{5}-\frac{6i}{5} & -\frac{11}{10}-\frac{27i}{10} & 0 \\ 0 & \frac{2}{5}-\frac{6i}{5} & -\frac{8}{5}-\frac{6i}{5} & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{10}{17} + \frac{6i}{17}\right) R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1-3i & 2 & -3 & 0 \\ 0 & \frac{7}{5}-\frac{6i}{5} & -\frac{11}{10}-\frac{27i}{10} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-3i & 2 & -3 \\ 0 & \frac{7}{5}-\frac{6i}{5} & -\frac{11}{10}-\frac{27i}{10} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{3i}{2}\right) t, v_2 = -\frac{1}{2}t + \frac{3}{2}it\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{3\,\mathrm{I}}{2}\right) t \\ -\frac{t}{2} + \frac{3\,\mathrm{I}\,t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{3i}{2}\right) t \\ -\frac{1}{2}t + \frac{3}{2}it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{3\,\mathrm{I}}{2}\right) t \\ -\frac{t}{2} + \frac{3\,\mathrm{I}\,t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{3i}{2} \\ \frac{-\frac{1}{2}t + \frac{3}{2}it}{t} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{3\,\mathrm{I}}{2}\right) t \\ -\frac{t}{2} + \frac{3\,\mathrm{I}\,t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{3i}{2} \\ -\frac{1}{2} + \frac{3i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2}+\frac{3\,\mathrm{I}}{2}\right)t \\ -\frac{t}{2}+\frac{3\,\mathrm{I}\,t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1+3i \\ -1+3i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$2+3i$	1	1	No	$\begin{bmatrix} \frac{1}{2}+\frac{3i}{2} \\ -\frac{1}{2}+\frac{3i}{2} \\ 1 \end{bmatrix}$
$2-3i$	1	1	No	$\begin{bmatrix} \frac{1}{2}-\frac{3i}{2} \\ -\frac{1}{2}-\frac{3i}{2} \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ \frac{7}{4} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(x) = \vec{v}_1 e^{2x}$$
$$= \begin{bmatrix} \frac{1}{2} \\ \frac{7}{4} \\ 1 \end{bmatrix} e^{2x}$$

Therefore the final solution is

$$\vec{x}_h(x) = c_1\vec{x}_1(x) + c_2\vec{x}_2(x) + c_3\vec{x}_3(x)$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{2}+\frac{3i}{2}\right) \mathrm{e}^{(2+3i)x} \\ \left(-\frac{1}{2}+\frac{3i}{2}\right) \mathrm{e}^{(2+3i)x} \\ \mathrm{e}^{(2+3i)x} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}-\frac{3i}{2}\right) \mathrm{e}^{(2-3i)x} \\ \left(-\frac{1}{2}-\frac{3i}{2}\right) \mathrm{e}^{(2-3i)x} \\ \mathrm{e}^{(2-3i)x} \end{bmatrix} + c_3 \begin{bmatrix} \frac{\mathrm{e}^{2x}}{2} \\ \frac{7\,\mathrm{e}^{2x}}{4} \\ \mathrm{e}^{2x} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+\frac{3i}{2}\right) c_1\mathrm{e}^{(2+3i)x} + \left(\frac{1}{2}-\frac{3i}{2}\right) c_2\mathrm{e}^{(2-3i)x} + \frac{c_3\mathrm{e}^{2x}}{2} \\ \left(-\frac{1}{2}+\frac{3i}{2}\right) c_1\mathrm{e}^{(2+3i)x} + \left(-\frac{1}{2}-\frac{3i}{2}\right) c_2\mathrm{e}^{(2-3i)x} + \frac{7c_3\mathrm{e}^{2x}}{4} \\ c_1\mathrm{e}^{(2+3i)x} + c_2\mathrm{e}^{(2-3i)x} + c_3\mathrm{e}^{2x} \end{bmatrix}$$

19.12.3 Maple step by step solution

Let's solve

$[y_1'(x) = y_1(x) + 2y_2(x) - 3y_3(x), y_2'(x) = -3y_1(x) + 4y_2(x) - 2y_3(x), y_3'(x) = 2y_1(x) + y_3(x)]$

- Define vector

$$\underline{y}^{\rightarrow}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 4 & -2 \\ 2 & 0 & 1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 4 & -2 \\ 2 & 0 & 1 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 4 & -2 \\ 2 & 0 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\underline{y}^{\rightarrow\prime}(x) = A \cdot \underline{y}^{\rightarrow}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{2} \\ \frac{7}{4} \\ 1 \end{bmatrix}\right], \left[2-3\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{3\,\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{3\,\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2+3\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}+\frac{3\,\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{3\,\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{2} \\ \frac{7}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{y}^{\rightarrow}{}_{1} = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{7}{4} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2-3\,\mathrm{I}, \begin{bmatrix} \frac{1}{2}-\frac{3\,\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{3\,\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(2-3\,\mathrm{I})x} \cdot \begin{bmatrix} \frac{1}{2}-\frac{3\,\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{3\,\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{2x} \cdot (\cos(3x) - \mathrm{I}\sin(3x)) \cdot \begin{bmatrix} \frac{1}{2}-\frac{3\,\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{3\,\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{2x} \cdot \begin{bmatrix} \left(\frac{1}{2}-\frac{3\,\mathrm{I}}{2}\right)(\cos(3x) - \mathrm{I}\sin(3x)) \\ \left(-\frac{1}{2}-\frac{3\,\mathrm{I}}{2}\right)(\cos(3x) - \mathrm{I}\sin(3x)) \\ \cos(3x) - \mathrm{I}\sin(3x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{y}}_2(x) = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{\cos(3x)}{2} - \frac{3\sin(3x)}{2} \\ -\frac{\cos(3x)}{2} - \frac{3\sin(3x)}{2} \\ \cos(3x) \end{bmatrix}, \underline{\vec{y}}_3(x) = \mathrm{e}^{2x} \cdot \begin{bmatrix} -\frac{\sin(3x)}{2} - \frac{3\cos(3x)}{2} \\ \frac{\sin(3x)}{2} - \frac{3\cos(3x)}{2} \\ -\sin(3x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1\underline{\vec{y}}_1 + c_2\underline{\vec{y}}_2(x) + c_3\underline{\vec{y}}_3(x)$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1\mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{7}{4} \\ 1 \end{bmatrix} + c_2\mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{\cos(3x)}{2} - \frac{3\sin(3x)}{2} \\ -\frac{\cos(3x)}{2} - \frac{3\sin(3x)}{2} \\ \cos(3x) \end{bmatrix} + \mathrm{e}^{2x}c_3 \cdot \begin{bmatrix} -\frac{\sin(3x)}{2} - \frac{3\cos(3x)}{2} \\ \frac{\sin(3x)}{2} - \frac{3\cos(3x)}{2} \\ -\sin(3x) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} \frac{e^{2x}((c_2-3c_3)\cos(3x)+(-3c_2-c_3)\sin(3x)+c_1)}{2} \\ \frac{7\,e^{2x}\left(\frac{2(-c_2-3c_3)\cos(3x)}{7}+\frac{2(-3c_2+c_3)\sin(3x)}{7}+c_1\right)}{4} \\ e^{2x}(c_1 + c_2\cos(3x) - c_3\sin(3x)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{y_1(x) = \frac{e^{2x}((c_2-3c_3)\cos(3x)+(-3c_2-c_3)\sin(3x)+c_1)}{2}, y_2(x) = \frac{7\,e^{2x}\left(\frac{2(-c_2-3c_3)\cos(3x)}{7}+\frac{2(-3c_2+c_3)\sin(3x)}{7}+c_1\right)}{4}, y_3(x) =\right.$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 102

```
dsolve([diff(y__1(x),x)=1*y__1(x)+2*y__2(x)-3*y__3(x),diff(y__2(x),x)=-3*y__1(x)+4*y__2(x)-2
```

$$y_1(x) = \frac{e^{2x}(3\cos(3x)\,c_2 + \cos(3x)\,c_3 + \sin(3x)\,c_2 - 3\sin(3x)\,c_3 + c_1)}{2}$$
$$y_2(x) = \frac{e^{2x}(6\cos(3x)\,c_2 - 2\cos(3x)\,c_3 - 2\sin(3x)\,c_2 - 6\sin(3x)\,c_3 + 7c_1)}{4}$$
$$y_3(x) = e^{2x}(c_1 + \sin(3x)\,c_2 + \cos(3x)\,c_3)$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 176

```
DSolve[{y1'[x]==1*y1[x]+2*y2[x]-3*y3[x],y2'[x]==-3*y1[x]+4*y2[x]-2*y3[x],y3'[x]==2*y1[x]+0*y
```

$$\text{y1}(x) \to \frac{1}{9}e^{2x}((11c_1 - 2(c_2+c_3))\cos(3x) - 3(c_1 - 2c_2 + 3c_3)\sin(3x) + 2(-c_1 + c_2 + c_3))$$
$$\text{y2}(x) \to \frac{1}{9}e^{2x}((7c_1 + 2c_2 - 7c_3)\cos(3x) + (-9c_1 + 6c_2 - 6c_3)\sin(3x) + 7(-c_1 + c_2 + c_3))$$
$$\text{y3}(x) \to \frac{1}{9}e^{2x}((4c_1 - 4c_2 + 5c_3)\cos(3x) + (6c_1 - 3c_3)\sin(3x) + 4(-c_1 + c_2 + c_3))$$

19.13 problem 11

Internal problem ID [12851]
Internal file name [`OUTPUT/11503_Monday_November_06_2023_01_31_06_PM_56194562/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= -2y_1(x) - y_2(x) + y_3(x)\\
y_2'(x) &= -y_1(x) - 2y_2(x) - y_3(x)\\
y_3'(x) &= y_1(x) - y_2(x) - 2y_3(x)
\end{aligned}
$$

19.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$
\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{2\,\mathrm{e}^{-3x}}{3}+\frac{1}{3} & -\frac{1}{3}+\frac{\mathrm{e}^{-3x}}{3} & \frac{1}{3}-\frac{\mathrm{e}^{-3x}}{3} \\ -\frac{1}{3}+\frac{\mathrm{e}^{-3x}}{3} & \frac{2\,\mathrm{e}^{-3x}}{3}+\frac{1}{3} & -\frac{1}{3}+\frac{\mathrm{e}^{-3x}}{3} \\ \frac{1}{3}-\frac{\mathrm{e}^{-3x}}{3} & -\frac{1}{3}+\frac{\mathrm{e}^{-3x}}{3} & \frac{2\,\mathrm{e}^{-3x}}{3}+\frac{1}{3} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(x) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{-3x}}{3}+\frac{1}{3} & -\frac{1}{3}+\frac{\mathrm{e}^{-3x}}{3} & \frac{1}{3}-\frac{\mathrm{e}^{-3x}}{3} \\ -\frac{1}{3}+\frac{\mathrm{e}^{-3x}}{3} & \frac{2\,\mathrm{e}^{-3x}}{3}+\frac{1}{3} & -\frac{1}{3}+\frac{\mathrm{e}^{-3x}}{3} \\ \frac{1}{3}-\frac{\mathrm{e}^{-3x}}{3} & -\frac{1}{3}+\frac{\mathrm{e}^{-3x}}{3} & \frac{2\,\mathrm{e}^{-3x}}{3}+\frac{1}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{2\,\mathrm{e}^{-3x}}{3}+\frac{1}{3}\right)c_1+\left(-\frac{1}{3}+\frac{\mathrm{e}^{-3x}}{3}\right)c_2+\left(\frac{1}{3}-\frac{\mathrm{e}^{-3x}}{3}\right)c_3 \\ \left(-\frac{1}{3}+\frac{\mathrm{e}^{-3x}}{3}\right)c_1+\left(\frac{2\,\mathrm{e}^{-3x}}{3}+\frac{1}{3}\right)c_2+\left(-\frac{1}{3}+\frac{\mathrm{e}^{-3x}}{3}\right)c_3 \\ \left(\frac{1}{3}-\frac{\mathrm{e}^{-3x}}{3}\right)c_1+\left(-\frac{1}{3}+\frac{\mathrm{e}^{-3x}}{3}\right)c_2+\left(\frac{2\,\mathrm{e}^{-3x}}{3}+\frac{1}{3}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1+c_2-c_3)\mathrm{e}^{-3x}}{3}+\frac{c_1}{3}-\frac{c_2}{3}+\frac{c_3}{3} \\ \frac{(c_1+2c_2+c_3)\mathrm{e}^{-3x}}{3}-\frac{c_1}{3}+\frac{c_2}{3}-\frac{c_3}{3} \\ \frac{(-c_1+c_2+2c_3)\mathrm{e}^{-3x}}{3}+\frac{c_1}{3}-\frac{c_2}{3}+\frac{c_3}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

19.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-2-\lambda & -1 & 1\\ -1 & -2-\lambda & -1\\ 1 & -1 & -2-\lambda\end{bmatrix}\right)=0$$

Which gives the characteristic equation

$$\lambda^3+6\lambda^2+9\lambda=0$$

The roots of the above are the eigenvalues.

$$\lambda_1=0$$
$$\lambda_2=-3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
−3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1=-3$

We need to solve $A\vec{v}=\lambda\vec{v}$ or $(A-\lambda I)\vec{v}=\vec{0}$ which becomes

$$\left(\begin{bmatrix}-2 & -1 & 1\\ -1 & -2 & -1\\ 1 & -1 & -2\end{bmatrix}-(-3)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}1 & -1 & 1 & 0\\ -1 & 1 & -1 & 0\\ 1 & -1 & 1 & 0\end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t - s\}$

Hence the solution is

$$\begin{bmatrix} t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} t - s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} t - s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right)$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}-2 & -1 & 1\\-1 & -2 & -1\\1 & -1 & -2\end{bmatrix} - (0)\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-2 & -1 & 1\\-1 & -2 & -1\\1 & -1 & -2\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}-2 & -1 & 1 & 0\\-1 & -2 & -1 & 0\\1 & -1 & -2 & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c}-2 & -1 & 1 & 0\\0 & -\frac{3}{2} & -\frac{3}{2} & 0\\1 & -1 & -2 & 0\end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c}-2 & -1 & 1 & 0\\0 & -\frac{3}{2} & -\frac{3}{2} & 0\\0 & -\frac{3}{2} & -\frac{3}{2} & 0\end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c}-2 & -1 & 1 & 0\\0 & -\frac{3}{2} & -\frac{3}{2} & 0\\0 & 0 & 0 & 0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -1 & 1 \\ 0 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
0	1	1	No	$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
-3	2	2	No	$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(x) &= \vec{v}_1 e^{0} \\
&= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{0}
\end{aligned}
$$

eigenvalue -3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

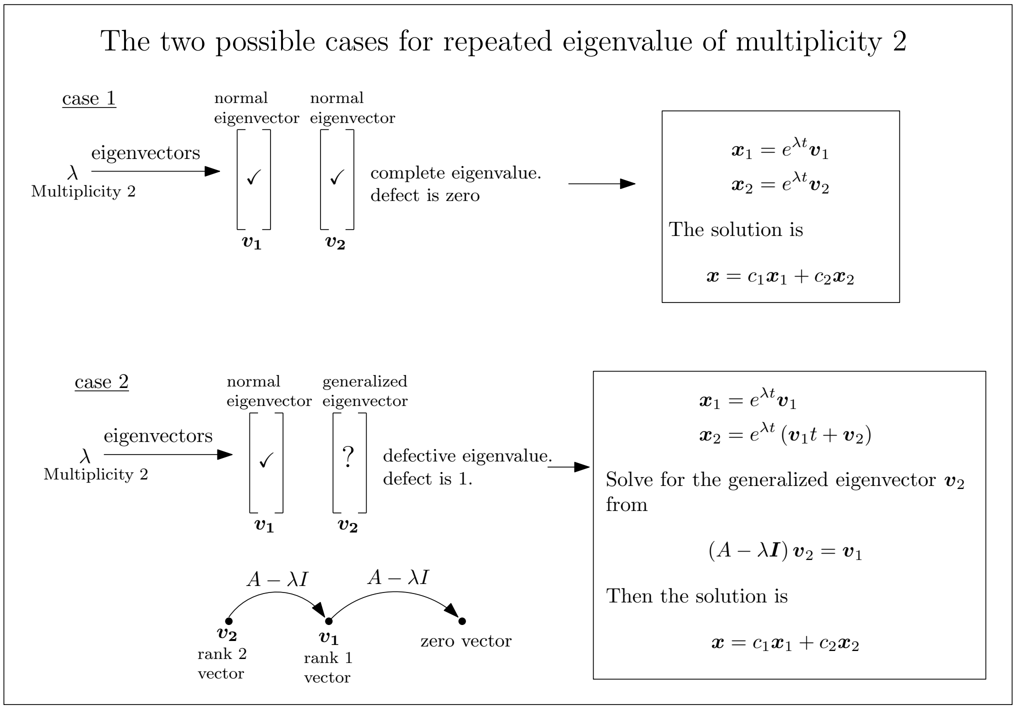

Figure 429: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_2(x) &= \vec{v}_2 e^{-3x} \\
&= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-3x}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_3(x) &= \vec{v}_3 e^{-3x} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-3x}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(x) = c_1\vec{x}_1(x) + c_2\vec{x}_2(x) + c_3\vec{x}_3(x)
$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -e^{-3x} \\ 0 \\ e^{-3x} \end{bmatrix} + c_3 \begin{bmatrix} e^{-3x} \\ e^{-3x} \\ 0 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} (-c_2 + c_3) e^{-3x} + c_1 \\ -c_1 + c_3 e^{-3x} \\ c_1 + c_2 e^{-3x} \end{bmatrix}$$

19.13.3 Maple step by step solution

Let's solve

$[y_1'(x) = -2y_1(x) - y_2(x) + y_3(x), y_2'(x) = -y_1(x) - 2y_2(x) - y_3(x), y_3'(x) = y_1(x) - y_2(x) - 2y_3($

- Define vector

$$\underline{y}^{\rightarrow}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$$

- Rewrite the system as

  $\underline{\vec{y}}'(x) = A \cdot \underline{\vec{y}}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-3, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[-3, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[-3, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -3

  $$\underline{\vec{y}}_1(x) = e^{-3x} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -3$ is the eigenvalue, an

  $\underline{\vec{y}}_2(x) = e^{\lambda x}\left(x\vec{v} + \vec{p}\right)$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{y}}_2(x)$ into the homogeneous system

  $\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = \left(e^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = e^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\underline{\vec{y}}_2(x)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -3

$$\left(\left[\begin{array}{ccc} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{array}\right] - (-3) \cdot \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right) \cdot \vec{p} = \left[\begin{array}{c} -1 \\ 0 \\ 1 \end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p} = \left[\begin{array}{c} -1 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue -3

$$\underline{\vec{y}}_2(x) = \mathrm{e}^{-3x} \cdot \left(x \cdot \left[\begin{array}{c} -1 \\ 0 \\ 1 \end{array}\right] + \left[\begin{array}{c} -1 \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_3 = \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

  $\underline{\vec{y}} = c_1 \underline{\vec{y}}_1(x) + c_2 \underline{\vec{y}}_2(x) + c_3 \underline{\vec{y}}_3$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 e^{-3x} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-3x} \cdot \left(x \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} c_3 \\ -c_3 \\ c_3 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} ((-x-1)\,c_2 - c_1)\,e^{-3x} + c_3 \\ -c_3 \\ e^{-3x}(c_2 x + c_1) + c_3 \end{bmatrix}$$

- Solution to the system of ODEs

$$\{y_1(x) = ((-x-1)\,c_2 - c_1)\,e^{-3x} + c_3, y_2(x) = -c_3, y_3(x) = e^{-3x}(c_2 x + c_1) + c_3\}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 51

```
dsolve([diff(y__1(x),x)=-2*y__1(x)-1*y__2(x)+1*y__3(x),diff(y__2(x),x)=-1*y__1(x)-2*y__2(x)-
```

$$\begin{aligned} y_1(x) &= c_2 + c_3 e^{-3x} \\ y_2(x) &= -c_2 - c_3 e^{-3x} + c_1 e^{-3x} \\ y_3(x) &= -2c_3 e^{-3x} + c_2 + c_1 e^{-3x} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 130

```
DSolve[{y1'[x]==-2*y1[x]-1*y2[x]+1*y3[x],y2'[x]==-1*y1[x]-2*y2[x]-1*y3[x],y3'[x]==1*y1[x]-1*
```

$$\begin{aligned} \text{y1}(x) &\to \frac{1}{3} e^{-3x} \left(c_1 \left(e^{3x}+2\right) - (c_2 - c_3)\left(e^{3x}-1\right)\right) \\ \text{y2}(x) &\to \frac{1}{3} e^{-3x} \left(-\left(c_1\left(e^{3x}-1\right)\right) + c_2\left(e^{3x}+2\right) - c_3\left(e^{3x}-1\right)\right) \\ \text{y3}(x) &\to \frac{1}{3} e^{-3x} \left(c_1\left(e^{3x}-1\right) - c_2\left(e^{3x}-1\right) + c_3\left(e^{3x}+2\right)\right) \end{aligned}$$

19.14 problem 12

Internal problem ID [12852]
Internal file name [`OUTPUT/11504_Monday_November_06_2023_01_31_07_PM_37422183/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= y_1(x) + y_2(x) + 2y_3(x) \\
y_2'(x) &= y_1(x) + y_2(x) + 2y_3(x) \\
y_3'(x) &= 2y_1(x) + 2y_2(x) + 4y_3(x)
\end{aligned}
$$

19.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(x) = A\,\vec{x}(x)
$$

Or

$$
\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{5}{6} + \frac{e^{6x}}{6} & \frac{e^{6x}}{6} - \frac{1}{6} & \frac{e^{6x}}{3} - \frac{1}{3} \\ \frac{e^{6x}}{6} - \frac{1}{6} & \frac{5}{6} + \frac{e^{6x}}{6} & \frac{e^{6x}}{3} - \frac{1}{3} \\ \frac{e^{6x}}{3} - \frac{1}{3} & \frac{e^{6x}}{3} - \frac{1}{3} & \frac{1}{3} + \frac{2\,e^{6x}}{3} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(x) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{5}{6}+\frac{e^{6x}}{6} & \frac{e^{6x}}{6}-\frac{1}{6} & \frac{e^{6x}}{3}-\frac{1}{3} \\ \frac{e^{6x}}{6}-\frac{1}{6} & \frac{5}{6}+\frac{e^{6x}}{6} & \frac{e^{6x}}{3}-\frac{1}{3} \\ \frac{e^{6x}}{3}-\frac{1}{3} & \frac{e^{6x}}{3}-\frac{1}{3} & \frac{1}{3}+\frac{2e^{6x}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{5}{6}+\frac{e^{6x}}{6}\right)c_1 + \left(\frac{e^{6x}}{6}-\frac{1}{6}\right)c_2 + \left(\frac{e^{6x}}{3}-\frac{1}{3}\right)c_3 \\ \left(\frac{e^{6x}}{6}-\frac{1}{6}\right)c_1 + \left(\frac{5}{6}+\frac{e^{6x}}{6}\right)c_2 + \left(\frac{e^{6x}}{3}-\frac{1}{3}\right)c_3 \\ \left(\frac{e^{6x}}{3}-\frac{1}{3}\right)c_1 + \left(\frac{e^{6x}}{3}-\frac{1}{3}\right)c_2 + \left(\frac{1}{3}+\frac{2e^{6x}}{3}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1+c_2+2c_3)e^{6x}}{6} + \frac{5c_1}{6} - \frac{c_2}{6} - \frac{c_3}{3} \\ \frac{(c_1+c_2+2c_3)e^{6x}}{6} - \frac{c_1}{6} + \frac{5c_2}{6} - \frac{c_3}{3} \\ \frac{(c_1+c_2+2c_3)e^{6x}}{3} - \frac{c_1}{3} - \frac{c_2}{3} + \frac{c_3}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

19.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 & 2 \\ 1 & 1-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 0$$
$$\lambda_2 = 6$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t - 2s\}$

Hence the solution is

$$\begin{bmatrix} -t - 2s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t - 2s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -t - 2s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -2s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -t - 2s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

Considering the eigenvalue $\lambda_2 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 1 & 2 \\ 1 & -5 & 2 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & 1 & 2 & 0 \\ 1 & -5 & 2 & 0 \\ 2 & 2 & -2 & 0 \end{array} \right]$$

$$R_2 = R_2 + \frac{R_1}{5} \implies \left[\begin{array}{ccc|c} -5 & 1 & 2 & 0 \\ 0 & -\frac{24}{5} & \frac{12}{5} & 0 \\ 2 & 2 & -2 & 0 \end{array} \right]$$

$$R_3 = R_3 + \frac{2R_1}{5} \implies \left[\begin{array}{ccc|c} -5 & 1 & 2 & 0 \\ 0 & -\frac{24}{5} & \frac{12}{5} & 0 \\ 0 & \frac{12}{5} & -\frac{6}{5} & 0 \end{array} \right]$$

$$R_3 = R_3 + \frac{R_2}{2} \implies \left[\begin{array}{ccc|c} -5 & 1 & 2 & 0 \\ 0 & -\frac{24}{5} & \frac{12}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & 1 & 2 \\ 0 & -\frac{24}{5} & \frac{12}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
0	2	2	No	$\begin{bmatrix} -2 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
6	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 0 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

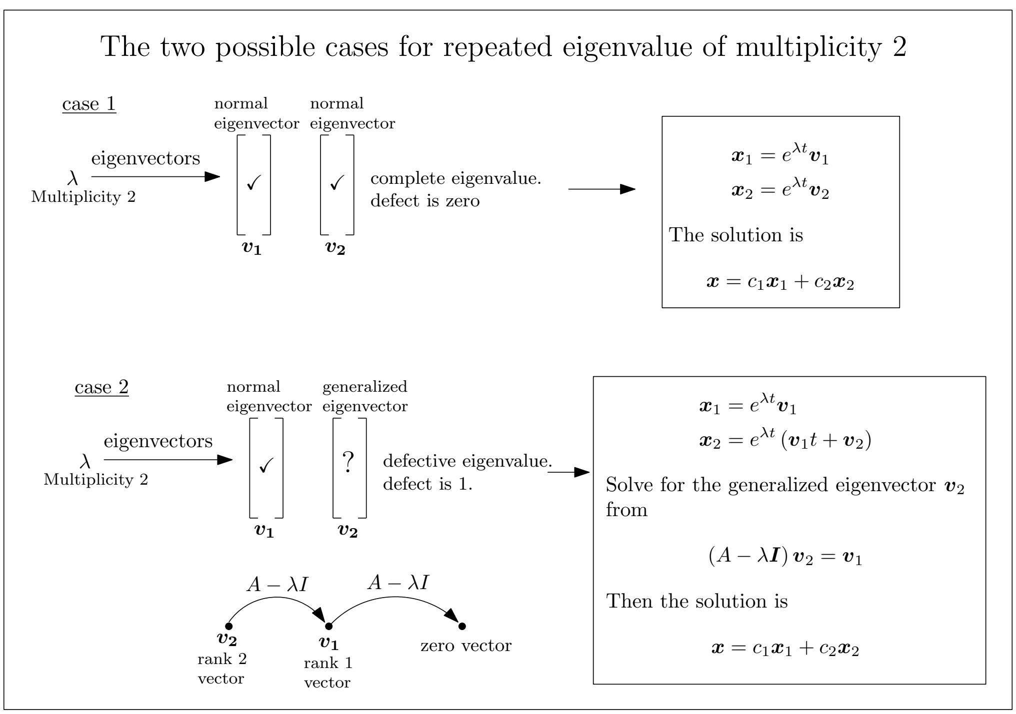

Figure 430: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\vec{x}_1(x) = \vec{v}_1 e^0$$
$$= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^0$$

$$\vec{x}_2(x) = \vec{v}_2 e^0$$
$$= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^0$$

Since eigenvalue 6 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(x) = \vec{v}_3 e^{6x}$$
$$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} e^{6x}$$

Therefore the final solution is

$$\vec{x}_h(x) = c_1\vec{x}_1(x) + c_2\vec{x}_2(x) + c_3\vec{x}_3(x)$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = c_1 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \frac{e^{6x}}{2} \\ \frac{e^{6x}}{2} \\ e^{6x} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} -2c_1 - c_2 + \frac{c_3 e^{6x}}{2} \\ c_2 + \frac{c_3 e^{6x}}{2} \\ c_1 + c_3 e^{6x} \end{bmatrix}$$

19.14.3 Maple step by step solution

Let's solve

$[y_1'(x) = y_1(x) + y_2(x) + 2y_3(x), y_2'(x) = y_1(x) + y_2(x) + 2y_3(x), y_3'(x) = 2y_1(x) + 2y_2(x) + 4y_3(x$

- Define vector

$$\underline{y}^{\rightarrow}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\underline{y}^{\rightarrow\prime}(x) = A \cdot \underline{y}^{\rightarrow}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[6, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[6, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{y}}_3 = \mathrm{e}^{6x} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_1 + c_2 \underline{\vec{y}}_2 + c_3 \underline{\vec{y}}_3$$

- Substitute solutions into the general solution

$$\underline{\overrightarrow{y}} = c_3 e^{6x} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -c_2 - 2c_1 \\ c_2 \\ c_1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} -2c_1 - c_2 + \frac{c_3 e^{6x}}{2} \\ c_2 + \frac{c_3 e^{6x}}{2} \\ c_1 + c_3 e^{6x} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{y_1(x) = -2c_1 - c_2 + \frac{c_3 e^{6x}}{2}, y_2(x) = c_2 + \frac{c_3 e^{6x}}{2}, y_3(x) = c_1 + c_3 e^{6x}\right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 42

```
dsolve([diff(y__1(x),x)=1*y__1(x)+1*y__2(x)+2*y__3(x),diff(y__2(x),x)=1*y__1(x)+1*y__2(x)+2*
```

$$\begin{aligned} y_1(x) &= c_2 + c_3 e^{6x} \\ y_2(x) &= c_2 + c_3 e^{6x} + c_1 \\ y_3(x) &= 2c_3 e^{6x} - c_2 - \frac{c_1}{2} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 114

```
DSolve[{y1'[x]==1*y1[x]+1*y2[x]+2*y3[x],y2'[x]==1*y1[x]+1*y2[x]+2*y3[x],y3'[x]==2*y1[x]+2*y2
```

$$\begin{aligned} \text{y1}(x) &\to \frac{1}{6}\left(c_1\left(e^{6x}+5\right)+\left(c_2+2c_3\right)\left(e^{6x}-1\right)\right) \\ \text{y2}(x) &\to \frac{1}{6}\left(c_1\left(e^{6x}-1\right)+c_2\left(e^{6x}+5\right)+2c_3\left(e^{6x}-1\right)\right) \\ \text{y3}(x) &\to \frac{1}{3}\left(c_1\left(e^{6x}-1\right)+c_2\left(e^{6x}-1\right)+c_3\left(2e^{6x}+1\right)\right) \end{aligned}$$

19.15 problem 13

Internal problem ID [12853]
Internal file name [`OUTPUT/11505_Monday_November_06_2023_01_31_07_PM_88694947/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} y_1'(x) &= 2y_1(x) + y_2(x) \\ y_2'(x) &= -y_1(x) + 2y_2(x) \\ y_3'(x) &= 3y_3(x) - 4y_4(x) \\ y_4'(x) &= 4y_3(x) + 3y_4(x) \end{aligned}$$

19.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2x}\cos(x) & e^{2x}\sin(x) & 0 & 0 \\ -e^{2x}\sin(x) & e^{2x}\cos(x) & 0 & 0 \\ 0 & 0 & e^{3x}\cos(4x) & -e^{3x}\sin(4x) \\ 0 & 0 & e^{3x}\sin(4x) & e^{3x}\cos(4x) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(x) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2x}\cos(x) & e^{2x}\sin(x) & 0 & 0 \\ -e^{2x}\sin(x) & e^{2x}\cos(x) & 0 & 0 \\ 0 & 0 & e^{3x}\cos(4x) & -e^{3x}\sin(4x) \\ 0 & 0 & e^{3x}\sin(4x) & e^{3x}\cos(4x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{2x}\cos(x)c_1 + e^{2x}\sin(x)c_2 \\ -e^{2x}\sin(x)c_1 + e^{2x}\cos(x)c_2 \\ e^{3x}\cos(4x)c_3 - e^{3x}\sin(4x)c_4 \\ e^{3x}\sin(4x)c_3 + e^{3x}\cos(4x)c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{2x}(\cos(x)c_1 + \sin(x)c_2) \\ -e^{2x}(\sin(x)c_1 - \cos(x)c_2) \\ e^{3x}(\cos(4x)c_3 - \sin(4x)c_4) \\ e^{3x}(\sin(4x)c_3 + \cos(4x)c_4) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

19.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & 0 & 0 \\ -1 & 2-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & -4 \\ 0 & 0 & 4 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 10\lambda^3 + 54\lambda^2 - 130\lambda + 125 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 2+i \\ \lambda_2 &= 2-i \\ \lambda_3 &= 3+4i \\ \lambda_4 &= 3-4i \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$3+4i$	1	complex eigenvalue
$3-4i$	1	complex eigenvalue
$2+i$	1	complex eigenvalue
$2-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} - (2-i)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i & 1 & 0 & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & 1+i & -4 \\ 0 & 0 & 4 & 1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} i & 1 & 0 & 0 & 0 \\ -1 & i & 0 & 0 & 0 \\ 0 & 0 & 1+i & -4 & 0 \\ 0 & 0 & 4 & 1+i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+i & -4 & 0 \\ 0 & 0 & 4 & 1+i & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} i & 1 & 0 & 0 & 0 \\ 0 & 0 & 1+i & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1+i & 0 \end{array}\right]$$

$$R_4 = R_4 + (-2+2i)\,R_2 \Longrightarrow \left[\begin{array}{cccc|c} i & 1 & 0 & 0 & 0 \\ 0 & 0 & 1+i & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9-7i & 0 \end{array}\right]$$

Since the current pivot $A(3,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} i & 1 & 0 & 0 & 0 \\ 0 & 0 & 1+i & -4 & 0 \\ 0 & 0 & 0 & 9-7i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i & 1 & 0 & 0 \\ 0 & 0 & 1+i & -4 \\ 0 & 0 & 0 & 9-7i \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} it \\ t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{I}\,t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} - (2+i) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 1-i & -4 \\ 0 & 0 & 4 & 1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -i & 1 & 0 & 0 & 0 \\ -1 & -i & 0 & 0 & 0 \\ 0 & 0 & 1-i & -4 & 0 \\ 0 & 0 & 4 & 1-i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} -i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-i & -4 & 0 \\ 0 & 0 & 4 & 1-i & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} -i & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-i & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1-i & 0 \end{array}\right]$$

$$R_4 = R_4 + (-2-2i)\, R_2 \Longrightarrow \left[\begin{array}{cccc|c} -i & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-i & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9+7i & 0 \end{array}\right]$$

Since the current pivot $A(3,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} -i & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-i & -4 & 0 \\ 0 & 0 & 0 & 9+7i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i & 1 & 0 & 0 \\ 0 & 0 & 1-i & -4 \\ 0 & 0 & 0 & 9+7i \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -it \\ t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3 - 4i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} - (3 - 4i) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1+4i & 1 & 0 & 0 \\ -1 & -1+4i & 0 & 0 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -1+4i & 1 & 0 & 0 & 0 \\ -1 & -1+4i & 0 & 0 & 0 \\ 0 & 0 & 4i & -4 & 0 \\ 0 & 0 & 4 & 4i & 0 \end{array} \right]$$

$$R_2 = R_2 + \left(-\frac{1}{17} - \frac{4i}{17}\right) R_1 \Longrightarrow \left[\begin{array}{cccc|c} -1+4i & 1 & 0 & 0 & 0 \\ 0 & -\frac{18}{17}+\frac{64i}{17} & 0 & 0 & 0 \\ 0 & 0 & 4i & -4 & 0 \\ 0 & 0 & 4 & 4i & 0 \end{array}\right]$$

$$R_4 = iR_3 + R_4 \Longrightarrow \left[\begin{array}{cccc|c} -1+4i & 1 & 0 & 0 & 0 \\ 0 & -\frac{18}{17}+\frac{64i}{17} & 0 & 0 & 0 \\ 0 & 0 & 4i & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+4i & 1 & 0 & 0 \\ 0 & -\frac{18}{17}+\frac{64i}{17} & 0 & 0 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0, v_3 = -it\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ \text{-I}\,t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ \text{-I}\,t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ -i \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = 3+4i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} - (3+4i) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1-4i & 1 & 0 & 0 \\ -1 & -1-4i & 0 & 0 \\ 0 & 0 & -4i & -4 \\ 0 & 0 & 4 & -4i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -1-4i & 1 & 0 & 0 & 0 \\ -1 & -1-4i & 0 & 0 & 0 \\ 0 & 0 & -4i & -4 & 0 \\ 0 & 0 & 4 & -4i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(-\frac{1}{17} + \frac{4i}{17}\right) R_1 \Longrightarrow \left[\begin{array}{cccc|c} -1-4i & 1 & 0 & 0 & 0 \\ 0 & -\frac{18}{17} - \frac{64i}{17} & 0 & 0 & 0 \\ 0 & 0 & -4i & -4 & 0 \\ 0 & 0 & 4 & -4i & 0 \end{array}\right]$$

$$R_4 = -iR_3 + R_4 \Longrightarrow \left[\begin{array}{cccc|c} -1-4i & 1 & 0 & 0 & 0 \\ 0 & -\frac{18}{17}-\frac{64i}{17} & 0 & 0 & 0 \\ 0 & 0 & -4i & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-4i & 1 & 0 & 0 \\ 0 & -\frac{18}{17}-\frac{64i}{17} & 0 & 0 \\ 0 & 0 & -4i & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0, v_3 = it\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$2+i$	1	1	No	$\begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}$
$2-i$	1	1	No	$\begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}$
$3+4i$	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}$
$3-4i$	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(x) = c_1\vec{x}_1(x) + c_2\vec{x}_2(x) + c_3\vec{x}_3(x) + c_4\vec{x}_4(x)$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = c_1 \begin{bmatrix} -i\mathrm{e}^{(2+i)x} \\ \mathrm{e}^{(2+i)x} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} i\mathrm{e}^{(2-i)x} \\ \mathrm{e}^{(2-i)x} \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ i\mathrm{e}^{(3+4i)x} \\ \mathrm{e}^{(3+4i)x} \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ -i\mathrm{e}^{(3-4i)x} \\ \mathrm{e}^{(3-4i)x} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} -i\left(c_1\mathrm{e}^{(2+i)x} - c_2\mathrm{e}^{(2-i)x}\right) \\ c_1\mathrm{e}^{(2+i)x} + c_2\mathrm{e}^{(2-i)x} \\ -i\left(c_4\mathrm{e}^{(3-4i)x} - c_3\mathrm{e}^{(3+4i)x}\right) \\ c_3\mathrm{e}^{(3+4i)x} + c_4\mathrm{e}^{(3-4i)x} \end{bmatrix}$$

19.15.3 Maple step by step solution

Let's solve

$[y_1'(x) = 2y_1(x) + y_2(x), y_2'(x) = -y_1(x) + 2y_2(x), y_3'(x) = 3y_3(x) - 4y_4(x), y_4'(x) = 4y_3(x) + 3y_4$

- Define vector

$$\underline{y}^{\rightarrow}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y^{\rightarrow}}'(x) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \cdot \underline{y^{\rightarrow}}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\underline{y^{\rightarrow}}'(x) = A \cdot \underline{y^{\rightarrow}}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2 - \mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[2 + \mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[3 - 4\,\mathrm{I}, \begin{bmatrix} 0 \\ 0 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[3 + 4\,\mathrm{I}, \begin{bmatrix} 0 \\ 0 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2 - \mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(2-\mathrm{I})x} \cdot \begin{bmatrix} \mathrm{I} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{2x} \cdot (\cos(x) - I\sin(x)) \cdot \begin{bmatrix} I \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- Simplify expression

$$e^{2x} \cdot \begin{bmatrix} I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \\ 0 \\ 0 \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = e^{2x} \cdot \begin{bmatrix} \sin(x) \\ \cos(x) \\ 0 \\ 0 \end{bmatrix}, \vec{y}_2(x) = e^{2x} \cdot \begin{bmatrix} \cos(x) \\ -\sin(x) \\ 0 \\ 0 \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[3 - 4I, \begin{bmatrix} 0 \\ 0 \\ -I \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(3-4I)x} \cdot \begin{bmatrix} 0 \\ 0 \\ -I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{3x} \cdot (\cos(4x) - I\sin(4x)) \cdot \begin{bmatrix} 0 \\ 0 \\ -I \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{3x} \cdot \begin{bmatrix} 0 \\ 0 \\ -I(\cos(4x) - I\sin(4x)) \\ \cos(4x) - I\sin(4x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{y}}_{3}(x) = e^{3x} \cdot \begin{bmatrix} 0 \\ 0 \\ -\sin(4x) \\ \cos(4x) \end{bmatrix}, \underline{\vec{y}}_{4}(x) = e^{3x} \cdot \begin{bmatrix} 0 \\ 0 \\ -\cos(4x) \\ -\sin(4x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\underline{\vec{y}} = c_1 \underline{\vec{y}}_{1}(x) + c_2 \underline{\vec{y}}_{2}(x) + c_3 \underline{\vec{y}}_{3}(x) + c_4 \underline{\vec{y}}_{4}(x)$$

- Substitute solutions into the general solution

$$\underline{\vec{y}} = c_1 e^{2x} \cdot \begin{bmatrix} \sin(x) \\ \cos(x) \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2x} \cdot \begin{bmatrix} \cos(x) \\ -\sin(x) \\ 0 \\ 0 \end{bmatrix} + c_3 e^{3x} \cdot \begin{bmatrix} 0 \\ 0 \\ -\sin(4x) \\ \cos(4x) \end{bmatrix} + c_4 e^{3x} \cdot \begin{bmatrix} 0 \\ 0 \\ -\cos(4x) \\ -\sin(4x) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} e^{2x}(\sin(x) c_1 + c_2 \cos(x)) \\ e^{2x}(c_1 \cos(x) - c_2 \sin(x)) \\ -e^{3x}(c_4 \cos(4x) + c_3 \sin(4x)) \\ e^{3x}(c_3 \cos(4x) - c_4 \sin(4x)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{y_1(x) = e^{2x}(\sin(x) c_1 + c_2 \cos(x)), y_2(x) = e^{2x}(c_1 \cos(x) - c_2 \sin(x)), y_3(x) = -e^{3x}(c_4 \cos(4x) +$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 82

```
dsolve([diff(y__1(x),x)=2*y__1(x)+1*y__2(x)+0*y__3(x)+0*y__4(x),diff(y__2(x),x)=-1*y__1(x)+2
```

$$y_1(x) = e^{2x}(\sin(x) c_3 + c_4 \cos(x))$$
$$y_2(x) = -e^{2x}(\sin(x) c_4 - \cos(x) c_3)$$
$$y_3(x) = e^{3x}(\cos(4x) c_2 + \sin(4x) c_1)$$
$$y_4(x) = -e^{3x}(\cos(4x) c_1 - \sin(4x) c_2)$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 92

```
DSolve[{y1'[x]==2*y1[x]+1*y2[x]+0*y3[x]+0*y4[x],y2'[x]==-1*y1[x]+2*y2[x]+0*y3[x]+0*y4[x],y3'
```

$$\text{y1}(x) \to e^{2x}(c_1 \cos(x) + c_2 \sin(x))$$
$$\text{y2}(x) \to e^{2x}(c_2 \cos(x) - c_1 \sin(x))$$
$$\text{y3}(x) \to e^{3x}(c_3 \cos(4x) - c_4 \sin(4x))$$
$$\text{y4}(x) \to e^{3x}(c_4 \cos(4x) + c_3 \sin(4x))$$

19.16 problem 14

Internal problem ID [12854]
Internal file name [`OUTPUT/11506_Monday_November_06_2023_01_31_08_PM_2243585/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= y_2(x)\\
y_2'(x) &= -3y_1(x) + 2y_3(x)\\
y_3'(x) &= y_4(x)\\
y_4'(x) &= 2y_1(x) - 5y_3(x)
\end{aligned}
$$

19.16.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$
\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -5 & 0 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \text{Expression too large to display}\\
&= \text{Expression too large to display}
\end{aligned}
$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(x) &= e^{At}\vec{c} \\
&= \text{Expression too large to display} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \text{Expression too large to display} \\
&= \text{Expression too large to display}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

19.16.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -5 & 0 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -5 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 & 0 & 0 \\ -3 & -\lambda & 2 & 0 \\ 0 & 0 & -\lambda & 1 \\ 2 & 0 & -5 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 + 8\lambda^2 + 11 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i\sqrt{4+\sqrt{5}}$$
$$\lambda_2 = -i\sqrt{4+\sqrt{5}}$$
$$\lambda_3 = i\sqrt{4-\sqrt{5}}$$
$$\lambda_4 = -i\sqrt{4-\sqrt{5}}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i\sqrt{4-\sqrt{5}}$	1	complex eigenvalue
$i\sqrt{4+\sqrt{5}}$	1	complex eigenvalue
$i\sqrt{4-\sqrt{5}}$	1	complex eigenvalue
$-i\sqrt{4+\sqrt{5}}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i\sqrt{4-\sqrt{5}}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -5 & 0 \end{bmatrix} - \left(-i\sqrt{4-\sqrt{5}}\right)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i\sqrt{4-\sqrt{5}} & 1 & 0 & 0 \\ -3 & i\sqrt{4-\sqrt{5}} & 2 & 0 \\ 0 & 0 & i\sqrt{4-\sqrt{5}} & 1 \\ 2 & 0 & -5 & i\sqrt{4-\sqrt{5}} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} i\sqrt{4-\sqrt{5}} & 1 & 0 & 0 & 0 \\ -3 & i\sqrt{4-\sqrt{5}} & 2 & 0 & 0 \\ 0 & 0 & i\sqrt{4-\sqrt{5}} & 1 & 0 \\ 2 & 0 & -5 & i\sqrt{4-\sqrt{5}} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3iR_1}{\sqrt{4-\sqrt{5}}} \Longrightarrow \left[\begin{array}{cccc|c} i\sqrt{4-\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & -\frac{i\left(\sqrt{5}-1\right)}{\sqrt{4-\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & i\sqrt{4-\sqrt{5}} & 1 & 0 \\ 2 & 0 & -5 & i\sqrt{4-\sqrt{5}} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{2iR_1}{\sqrt{4-\sqrt{5}}} \Longrightarrow \left[\begin{array}{cccc|c} i\sqrt{4-\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & -\frac{i\left(\sqrt{5}-1\right)}{\sqrt{4-\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & i\sqrt{4-\sqrt{5}} & 1 & 0 \\ 0 & \frac{2i}{\sqrt{4-\sqrt{5}}} & -5 & i\sqrt{4-\sqrt{5}} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{2R_2}{\sqrt{5}-1} \Longrightarrow \left[\begin{array}{cccc|c} i\sqrt{4-\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & -\frac{i\left(\sqrt{5}-1\right)}{\sqrt{4-\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & i\sqrt{4-\sqrt{5}} & 1 & 0 \\ 0 & 0 & \frac{-5\sqrt{5}+9}{\sqrt{5}-1} & i\sqrt{4-\sqrt{5}} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{i\left(-5\sqrt{5}+9\right) R_3}{\left(\sqrt{5}-1\right)\sqrt{4-\sqrt{5}}} \Longrightarrow \left[\begin{array}{cccc|c} i\sqrt{4-\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & -\frac{i\left(\sqrt{5}-1\right)}{\sqrt{4-\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & i\sqrt{4-\sqrt{5}} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i\sqrt{4-\sqrt{5}} & 1 & 0 & 0 \\ 0 & -\frac{i\left(\sqrt{5}-1\right)}{\sqrt{4-\sqrt{5}}} & 2 & 0 \\ 0 & 0 & i\sqrt{4-\sqrt{5}} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2it}{\sqrt{4-\sqrt{5}}\left(\sqrt{5}-1\right)}, v_2 = \frac{2t}{\sqrt{5}-1}, v_3 = \frac{it}{\sqrt{4-\sqrt{5}}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2\,\mathrm{I}t}{\sqrt{4-\sqrt{5}}\left(\sqrt{5}-1\right)} \\ \frac{2t}{\sqrt{5}-1} \\ \frac{\mathrm{I}t}{\sqrt{4-\sqrt{5}}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2it}{\sqrt{4-\sqrt{5}}\left(\sqrt{5}-1\right)} \\ \frac{2t}{\sqrt{5}-1} \\ \frac{it}{\sqrt{4-\sqrt{5}}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this

eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2\,\mathrm{I}\,t}{\sqrt{4-\sqrt{5}}\left(\sqrt{5}-1\right)} \\ \frac{2t}{\sqrt{5}-1} \\ \frac{\mathrm{I}\,t}{\sqrt{4-\sqrt{5}}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2i}{\sqrt{4-\sqrt{5}}\left(\sqrt{5}-1\right)} \\ \frac{2}{\sqrt{5}-1} \\ \frac{i}{\sqrt{4-\sqrt{5}}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2\,\mathrm{I}\,t}{\sqrt{4-\sqrt{5}}\left(\sqrt{5}-1\right)} \\ \frac{2t}{\sqrt{5}-1} \\ \frac{\mathrm{I}\,t}{\sqrt{4-\sqrt{5}}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2i}{\sqrt{4-\sqrt{5}}\left(\sqrt{5}-1\right)} \\ \frac{2}{\sqrt{5}-1} \\ \frac{i}{\sqrt{4-\sqrt{5}}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2\,\mathrm{I}\,t}{\sqrt{4-\sqrt{5}}\left(\sqrt{5}-1\right)} \\ \frac{2t}{\sqrt{5}-1} \\ \frac{\mathrm{I}\,t}{\sqrt{4-\sqrt{5}}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2i}{\sqrt{4-\sqrt{5}}\left(\sqrt{5}-1\right)} \\ \frac{2}{\sqrt{5}-1} \\ \frac{i}{\sqrt{4-\sqrt{5}}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -i\sqrt{4+\sqrt{5}}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -5 & 0 \end{bmatrix} - \left(-i\sqrt{4+\sqrt{5}}\right) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i\sqrt{4+\sqrt{5}} & 1 & 0 & 0 \\ -3 & i\sqrt{4+\sqrt{5}} & 2 & 0 \\ 0 & 0 & i\sqrt{4+\sqrt{5}} & 1 \\ 2 & 0 & -5 & i\sqrt{4+\sqrt{5}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} i\sqrt{4+\sqrt{5}} & 1 & 0 & 0 & 0 \\ -3 & i\sqrt{4+\sqrt{5}} & 2 & 0 & 0 \\ 0 & 0 & i\sqrt{4+\sqrt{5}} & 1 & 0 \\ 2 & 0 & -5 & i\sqrt{4+\sqrt{5}} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3iR_1}{\sqrt{4+\sqrt{5}}} \Longrightarrow \left[\begin{array}{cccc|c} i\sqrt{4+\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & \frac{i\left(\sqrt{5}+1\right)}{\sqrt{4+\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & i\sqrt{4+\sqrt{5}} & 1 & 0 \\ 2 & 0 & -5 & i\sqrt{4+\sqrt{5}} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{2iR_1}{\sqrt{4+\sqrt{5}}} \Longrightarrow \left[\begin{array}{cccc|c} i\sqrt{4+\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & \frac{i\left(\sqrt{5}+1\right)}{\sqrt{4+\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & i\sqrt{4+\sqrt{5}} & 1 & 0 \\ 0 & \frac{2i}{\sqrt{4+\sqrt{5}}} & -5 & i\sqrt{4+\sqrt{5}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{2R_2}{\sqrt{5}+1} \Longrightarrow \left[\begin{array}{cccc|c} i\sqrt{4+\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & \frac{i\left(\sqrt{5}+1\right)}{\sqrt{4+\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & i\sqrt{4+\sqrt{5}} & 1 & 0 \\ 0 & 0 & \frac{-5\sqrt{5}-9}{\sqrt{5}+1} & i\sqrt{4+\sqrt{5}} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{i\left(-5\sqrt{5}-9\right)R_3}{\left(\sqrt{5}+1\right)\sqrt{4+\sqrt{5}}} \Longrightarrow \left[\begin{array}{cccc|c} i\sqrt{4+\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & \frac{i\left(\sqrt{5}+1\right)}{\sqrt{4+\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & i\sqrt{4+\sqrt{5}} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i\sqrt{4+\sqrt{5}} & 1 & 0 & 0 \\ 0 & \frac{i(\sqrt{5}+1)}{\sqrt{4+\sqrt{5}}} & 2 & 0 \\ 0 & 0 & i\sqrt{4+\sqrt{5}} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2it}{(\sqrt{5}+1)\sqrt{4+\sqrt{5}}}, v_2 = -\frac{2t}{\sqrt{5}+1}, v_3 = \frac{it}{\sqrt{4+\sqrt{5}}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{-2\,\mathrm{I}\,t}{(\sqrt{5}+1)\sqrt{4+\sqrt{5}}} \\ -\frac{2t}{\sqrt{5}+1} \\ \frac{\mathrm{I}\,t}{\sqrt{4+\sqrt{5}}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2it}{(\sqrt{5}+1)\sqrt{4+\sqrt{5}}} \\ -\frac{2t}{\sqrt{5}+1} \\ \frac{it}{\sqrt{4+\sqrt{5}}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{-2\,\mathrm{I}\,t}{(\sqrt{5}+1)\sqrt{4+\sqrt{5}}} \\ -\frac{2t}{\sqrt{5}+1} \\ \frac{\mathrm{I}\,t}{\sqrt{4+\sqrt{5}}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2i}{(\sqrt{5}+1)\sqrt{4+\sqrt{5}}} \\ -\frac{2}{\sqrt{5}+1} \\ \frac{i}{\sqrt{4+\sqrt{5}}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{-2\,\mathrm{I}\,t}{(\sqrt{5}+1)\sqrt{4+\sqrt{5}}} \\ -\frac{2t}{\sqrt{5}+1} \\ \frac{\mathrm{I}\,t}{\sqrt{4+\sqrt{5}}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2i}{(\sqrt{5}+1)\sqrt{4+\sqrt{5}}} \\ -\frac{2}{\sqrt{5}+1} \\ \frac{i}{\sqrt{4+\sqrt{5}}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{-2\,\mathrm{I}\,t}{\left(\sqrt{5}+1\right)\sqrt{4+\sqrt{5}}} \\ -\frac{2t}{\sqrt{5}+1} \\ \frac{\mathrm{I}\,t}{\sqrt{4+\sqrt{5}}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2i}{\left(\sqrt{5}+1\right)\sqrt{4+\sqrt{5}}} \\ -\frac{2}{\sqrt{5}+1} \\ \frac{i}{\sqrt{4+\sqrt{5}}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = i\sqrt{4-\sqrt{5}}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -5 & 0 \end{bmatrix} - \left(i\sqrt{4-\sqrt{5}}\right)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i\sqrt{4-\sqrt{5}} & 1 & 0 & 0 \\ -3 & -i\sqrt{4-\sqrt{5}} & 2 & 0 \\ 0 & 0 & -i\sqrt{4-\sqrt{5}} & 1 \\ 2 & 0 & -5 & -i\sqrt{4-\sqrt{5}} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -i\sqrt{4-\sqrt{5}} & 1 & 0 & 0 & 0 \\ -3 & -i\sqrt{4-\sqrt{5}} & 2 & 0 & 0 \\ 0 & 0 & -i\sqrt{4-\sqrt{5}} & 1 & 0 \\ 2 & 0 & -5 & -i\sqrt{4-\sqrt{5}} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3iR_1}{\sqrt{4-\sqrt{5}}} \Longrightarrow \left[\begin{array}{cccc|c} -i\sqrt{4-\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & \frac{i\left(\sqrt{5}-1\right)}{\sqrt{4-\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & -i\sqrt{4-\sqrt{5}} & 1 & 0 \\ 2 & 0 & -5 & -i\sqrt{4-\sqrt{5}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{2iR_1}{\sqrt{4-\sqrt{5}}} \Longrightarrow \left[\begin{array}{cccc|c} -i\sqrt{4-\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & \frac{i\left(\sqrt{5}-1\right)}{\sqrt{4-\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & -i\sqrt{4-\sqrt{5}} & 1 & 0 \\ 0 & -\frac{2i}{\sqrt{4-\sqrt{5}}} & -5 & -i\sqrt{4-\sqrt{5}} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{2R_2}{\sqrt{5}-1} \Longrightarrow \left[\begin{array}{cccc|c} -i\sqrt{4-\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & \frac{i\left(\sqrt{5}-1\right)}{\sqrt{4-\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & -i\sqrt{4-\sqrt{5}} & 1 & 0 \\ 0 & 0 & \frac{-5\sqrt{5}+9}{\sqrt{5}-1} & -i\sqrt{4-\sqrt{5}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{i\left(-5\sqrt{5}+9\right)R_3}{\left(\sqrt{5}-1\right)\sqrt{4-\sqrt{5}}} \Longrightarrow \left[\begin{array}{cccc|c} -i\sqrt{4-\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & \frac{i\left(\sqrt{5}-1\right)}{\sqrt{4-\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & -i\sqrt{4-\sqrt{5}} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i\sqrt{4-\sqrt{5}} & 1 & 0 & 0 \\ 0 & \frac{i\left(\sqrt{5}-1\right)}{\sqrt{4-\sqrt{5}}} & 2 & 0 \\ 0 & 0 & -i\sqrt{4-\sqrt{5}} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2it}{\sqrt{4-\sqrt{5}}\left(\sqrt{5}-1\right)}, v_2 = \frac{2t}{\sqrt{5}-1}, v_3 = -\frac{it}{\sqrt{4-\sqrt{5}}}\right\}$

Hence the solution is

$$\left[\begin{array}{c} \frac{-2\,\mathrm{I}\,t}{\sqrt{4-\sqrt{5}}\,\left(\sqrt{5}-1\right)} \\ \frac{2t}{\sqrt{5}-1} \\ \frac{\text{-}\mathrm{I}\,t}{\sqrt{4-\sqrt{5}}} \\ t \end{array}\right] = \left[\begin{array}{c} -\frac{2it}{\sqrt{4-\sqrt{5}}\,\left(\sqrt{5}-1\right)} \\ \frac{2t}{\sqrt{5}-1} \\ -\frac{it}{\sqrt{4-\sqrt{5}}} \\ t \end{array}\right]$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} \frac{-2\,\mathrm{I}\,t}{\sqrt{4-\sqrt{5}}\,\left(\sqrt{5}-1\right)} \\ \frac{2t}{\sqrt{5}-1} \\ \frac{\text{-}\mathrm{I}\,t}{\sqrt{4-\sqrt{5}}} \\ t \end{array}\right] = t\left[\begin{array}{c} -\frac{2i}{\sqrt{4-\sqrt{5}}\,\left(\sqrt{5}-1\right)} \\ \frac{2}{\sqrt{5}-1} \\ -\frac{i}{\sqrt{4-\sqrt{5}}} \\ 1 \end{array}\right]$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} \frac{-2\,\mathrm{I}\,t}{\sqrt{4-\sqrt{5}}\,\left(\sqrt{5}-1\right)} \\ \frac{2t}{\sqrt{5}-1} \\ \frac{\text{-}\mathrm{I}\,t}{\sqrt{4-\sqrt{5}}} \\ t \end{array}\right] = \left[\begin{array}{c} -\frac{2i}{\sqrt{4-\sqrt{5}}\,\left(\sqrt{5}-1\right)} \\ \frac{2}{\sqrt{5}-1} \\ -\frac{i}{\sqrt{4-\sqrt{5}}} \\ 1 \end{array}\right]$$

Which is normalized to

$$\left[\begin{array}{c} \frac{-2\,\mathrm{I}\,t}{\sqrt{4-\sqrt{5}}\,\left(\sqrt{5}-1\right)} \\ \frac{2t}{\sqrt{5}-1} \\ \frac{\text{-}\mathrm{I}\,t}{\sqrt{4-\sqrt{5}}} \\ t \end{array}\right] = \left[\begin{array}{c} -\frac{2i}{\sqrt{4-\sqrt{5}}\,\left(\sqrt{5}-1\right)} \\ \frac{2}{\sqrt{5}-1} \\ -\frac{i}{\sqrt{4-\sqrt{5}}} \\ 1 \end{array}\right]$$

Considering the eigenvalue $\lambda_4 = i\sqrt{4+\sqrt{5}}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -5 & 0 \end{bmatrix} - \left(i\sqrt{4+\sqrt{5}}\right)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i\sqrt{4+\sqrt{5}} & 1 & 0 & 0 \\ -3 & -i\sqrt{4+\sqrt{5}} & 2 & 0 \\ 0 & 0 & -i\sqrt{4+\sqrt{5}} & 1 \\ 2 & 0 & -5 & -i\sqrt{4+\sqrt{5}} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -i\sqrt{4+\sqrt{5}} & 1 & 0 & 0 & 0 \\ -3 & -i\sqrt{4+\sqrt{5}} & 2 & 0 & 0 \\ 0 & 0 & -i\sqrt{4+\sqrt{5}} & 1 & 0 \\ 2 & 0 & -5 & -i\sqrt{4+\sqrt{5}} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3iR_1}{\sqrt{4+\sqrt{5}}} \Longrightarrow \left[\begin{array}{cccc|c} -i\sqrt{4+\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & -\frac{i\left(\sqrt{5}+1\right)}{\sqrt{4+\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & -i\sqrt{4+\sqrt{5}} & 1 & 0 \\ 2 & 0 & -5 & -i\sqrt{4+\sqrt{5}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{2iR_1}{\sqrt{4+\sqrt{5}}} \Longrightarrow \left[\begin{array}{cccc|c} -i\sqrt{4+\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & -\frac{i\left(\sqrt{5}+1\right)}{\sqrt{4+\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & -i\sqrt{4+\sqrt{5}} & 1 & 0 \\ 0 & -\frac{2i}{\sqrt{4+\sqrt{5}}} & -5 & -i\sqrt{4+\sqrt{5}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{2R_2}{\sqrt{5}+1} \Longrightarrow \left[\begin{array}{cccc|c} -i\sqrt{4+\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & -\frac{i(\sqrt{5}+1)}{\sqrt{4+\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & -i\sqrt{4+\sqrt{5}} & 1 & 0 \\ 0 & 0 & \frac{-5\sqrt{5}-9}{\sqrt{5}+1} & -i\sqrt{4+\sqrt{5}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{i(-5\sqrt{5}-9)\,R_3}{(\sqrt{5}+1)\,\sqrt{4+\sqrt{5}}} \Longrightarrow \left[\begin{array}{cccc|c} -i\sqrt{4+\sqrt{5}} & 1 & 0 & 0 & 0 \\ 0 & -\frac{i(\sqrt{5}+1)}{\sqrt{4+\sqrt{5}}} & 2 & 0 & 0 \\ 0 & 0 & -i\sqrt{4+\sqrt{5}} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i\sqrt{4+\sqrt{5}} & 1 & 0 & 0 \\ 0 & -\frac{i(\sqrt{5}+1)}{\sqrt{4+\sqrt{5}}} & 2 & 0 \\ 0 & 0 & -i\sqrt{4+\sqrt{5}} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2it}{(\sqrt{5}+1)\sqrt{4+\sqrt{5}}}, v_2 = -\frac{2t}{\sqrt{5}+1}, v_3 = -\frac{it}{\sqrt{4+\sqrt{5}}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2\,\mathrm{I}\,t}{(\sqrt{5}+1)\sqrt{4+\sqrt{5}}} \\ -\frac{2t}{\sqrt{5}+1} \\ \frac{-\mathrm{I}\,t}{\sqrt{4+\sqrt{5}}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2it}{(\sqrt{5}+1)\sqrt{4+\sqrt{5}}} \\ -\frac{2t}{\sqrt{5}+1} \\ -\frac{it}{\sqrt{4+\sqrt{5}}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this

eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2\,\mathrm{I}\,t}{\left(\sqrt{5}+1\right)\sqrt{4+\sqrt{5}}} \\ -\frac{2t}{\sqrt{5}+1} \\ \frac{\text{-I}\,t}{\sqrt{4+\sqrt{5}}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2i}{\left(\sqrt{5}+1\right)\sqrt{4+\sqrt{5}}} \\ -\frac{2}{\sqrt{5}+1} \\ -\frac{i}{\sqrt{4+\sqrt{5}}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2\,\mathrm{I}\,t}{\left(\sqrt{5}+1\right)\sqrt{4+\sqrt{5}}} \\ -\frac{2t}{\sqrt{5}+1} \\ \frac{\text{-I}\,t}{\sqrt{4+\sqrt{5}}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2i}{\left(\sqrt{5}+1\right)\sqrt{4+\sqrt{5}}} \\ -\frac{2}{\sqrt{5}+1} \\ -\frac{i}{\sqrt{4+\sqrt{5}}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2\,\mathrm{I}\,t}{\left(\sqrt{5}+1\right)\sqrt{4+\sqrt{5}}} \\ -\frac{2t}{\sqrt{5}+1} \\ \frac{\text{-I}\,t}{\sqrt{4+\sqrt{5}}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2i}{\left(\sqrt{5}+1\right)\sqrt{4+\sqrt{5}}} \\ -\frac{2}{\sqrt{5}+1} \\ -\frac{i}{\sqrt{4+\sqrt{5}}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
$i\sqrt{4+\sqrt{5}}$	1	1	No	$\left[\begin{array}{c} -\frac{22i}{\left(4+\sqrt{5}\right)^{\frac{3}{2}}\left(1-3\sqrt{5}\right)} \\ \frac{22}{\left(4+\sqrt{5}\right)\left(1-3\sqrt{5}\right)} \\ -\frac{i}{\sqrt{4+\sqrt{5}}} \\ 1 \end{array}\right]$
$-i\sqrt{4+\sqrt{5}}$	1	1	No	$\left[\begin{array}{c} \frac{22i}{\left(4+\sqrt{5}\right)^{\frac{3}{2}}\left(1-3\sqrt{5}\right)} \\ \frac{22}{\left(4+\sqrt{5}\right)\left(1-3\sqrt{5}\right)} \\ \frac{i}{\sqrt{4+\sqrt{5}}} \\ 1 \end{array}\right]$
$i\sqrt{4-\sqrt{5}}$	1	1	No	$\left[\begin{array}{c} -\frac{22i}{\left(4-\sqrt{5}\right)^{\frac{3}{2}}\left(1+3\sqrt{5}\right)} \\ \frac{22}{\left(4-\sqrt{5}\right)\left(1+3\sqrt{5}\right)} \\ -\frac{i}{\sqrt{4-\sqrt{5}}} \\ 1 \end{array}\right]$
$-i\sqrt{4-\sqrt{5}}$	1	1	No	$\left[\begin{array}{c} \frac{22i}{\left(4-\sqrt{5}\right)^{\frac{3}{2}}\left(1+3\sqrt{5}\right)} \\ \frac{22}{\left(4-\sqrt{5}\right)\left(1+3\sqrt{5}\right)} \\ \frac{i}{\sqrt{4-\sqrt{5}}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(x) = c_1\vec{x}_1(x) + c_2\vec{x}_2(x) + c_3\vec{x}_3(x) + c_4\vec{x}_4(x)$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{22i\mathrm{e}^{i\sqrt{4+\sqrt{5}}\,x}}{\left(4+\sqrt{5}\right)^{\frac{3}{2}}\left(1-3\sqrt{5}\right)} \\ \frac{22\,\mathrm{e}^{i\sqrt{4+\sqrt{5}}\,x}}{\left(4+\sqrt{5}\right)\left(1-3\sqrt{5}\right)} \\ -\frac{i\mathrm{e}^{i\sqrt{4+\sqrt{5}}\,x}}{\sqrt{4+\sqrt{5}}} \\ \mathrm{e}^{i\sqrt{4+\sqrt{5}}\,x} \end{bmatrix} + c_2 \begin{bmatrix} \frac{22i\mathrm{e}^{-i\sqrt{4+\sqrt{5}}\,x}}{\left(4+\sqrt{5}\right)^{\frac{3}{2}}\left(1-3\sqrt{5}\right)} \\ \frac{22\,\mathrm{e}^{-i\sqrt{4+\sqrt{5}}\,x}}{\left(4+\sqrt{5}\right)\left(1-3\sqrt{5}\right)} \\ \frac{i\mathrm{e}^{-i\sqrt{4+\sqrt{5}}\,x}}{\sqrt{4+\sqrt{5}}} \\ \mathrm{e}^{-i\sqrt{4+\sqrt{5}}\,x} \end{bmatrix} + c_3 \begin{bmatrix} -\frac{22i\mathrm{e}^{i\sqrt{4-\sqrt{5}}\,x}}{\left(4-\sqrt{5}\right)^{\frac{3}{2}}\left(1+3\sqrt{5}\right)} \\ \frac{22\,\mathrm{e}^{i\sqrt{4-\sqrt{5}}\,x}}{\left(4-\sqrt{5}\right)\left(1+3\sqrt{5}\right)} \\ -\frac{i\mathrm{e}^{i\sqrt{4-\sqrt{5}}\,x}}{\sqrt{4-\sqrt{5}}} \\ \mathrm{e}^{i\sqrt{4-\sqrt{5}}\,x} \end{bmatrix} + c_4 \begin{bmatrix} \frac{22i\mathrm{e}^{-i\cdots}}{\left(4-\sqrt{5}\right)\cdots} \\ \frac{22\,\mathrm{e}^{-i\cdots}}{\left(4-\sqrt{5}\right)\cdots} \\ \frac{i\mathrm{e}^{-i\cdots}}{\sqrt{\cdots}} \\ \mathrm{e}^{-i\sqrt{\cdots}} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} \frac{i\left(\left(-c_2\mathrm{e}^{-i\sqrt{4+\sqrt{5}}\,x}+c_1\mathrm{e}^{i\sqrt{4+\sqrt{5}}\,x}\right)\left(\sqrt{5}-1\right)\sqrt{4-\sqrt{5}}-\left(c_3\mathrm{e}^{i\sqrt{4-\sqrt{5}}\,x}-c_4\mathrm{e}^{-i\sqrt{4-\sqrt{5}}\,x}\right)\sqrt{4+\sqrt{5}}\left(\sqrt{5}+1\right)\right)\sqrt{4-\sqrt{5}}\,\sqrt{4+\sqrt{5}\cdots}}{22} \\ \frac{c_4\left(\sqrt{5}+1\right)\mathrm{e}^{-i\sqrt{4-\sqrt{5}}\,x}}{2} - \frac{c_2\left(\sqrt{5}-1\right)\mathrm{e}^{-i\sqrt{4+\sqrt{5}}\,x}}{2} + \frac{c_3\left(\sqrt{5}+1\right)\mathrm{e}^{i\sqrt{4-\sqrt{5}}\,x}}{2} - \frac{c_1\mathrm{e}^{i\sqrt{4+\sqrt{5}}\,x}\left(\sqrt{5}-1\right)}{2} \\ \frac{i\left(-\mathrm{e}^{i\sqrt{4-\sqrt{5}}\,x}\sqrt{4+\sqrt{5}}\,c_3+\mathrm{e}^{-i\sqrt{4-\sqrt{5}}\,x}\sqrt{4+\sqrt{5}}\,c_4+\mathrm{e}^{-i\sqrt{4+\sqrt{5}}\,x}\sqrt{4-\sqrt{5}}\,c_2-\mathrm{e}^{i\sqrt{4+\sqrt{5}}\,x}\sqrt{4-\sqrt{5}}\,c_1\right)}{\sqrt{4+\sqrt{5}}\,\sqrt{4-\sqrt{5}}} \\ c_1\mathrm{e}^{i\sqrt{4+\sqrt{5}}\,x} + c_2\mathrm{e}^{-i\sqrt{4+\sqrt{5}}\,x} + c_3\mathrm{e}^{i\sqrt{4-\sqrt{5}}\,x} + c_4\mathrm{e}^{-i\sqrt{4-\sqrt{5}}\,x} \end{bmatrix}$$

19.16.3 Maple step by step solution

Let's solve

$[y_1'(x) = y_2(x)\,, y_2'(x) = -3y_1(x) + 2y_3(x)\,, y_3'(x) = y_4(x)\,, y_4'(x) = 2y_1(x) - 5y_3(x)]$

- Define vector

$$y\underline{\overset{\rightarrow}{\quad}}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -5 & 0 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -5 & 0 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -5 & 0 \end{bmatrix}$$

- Rewrite the system as

$\underline{y}^{\rightarrow\prime}(x) = A \cdot \underline{y}^{\rightarrow}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}\sqrt{4-\sqrt{5}}, \begin{bmatrix} \frac{22\,\mathrm{I}}{\left(4-\sqrt{5}\right)^{\frac{3}{2}}\left(1+3\sqrt{5}\right)} \\ \frac{22}{\left(4-\sqrt{5}\right)\left(1+3\sqrt{5}\right)} \\ \frac{\mathrm{I}}{\sqrt{4-\sqrt{5}}} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}\sqrt{4+\sqrt{5}}, \begin{bmatrix} \frac{22\,\mathrm{I}}{\left(4+\sqrt{5}\right)^{\frac{3}{2}}\left(1-3\sqrt{5}\right)} \\ \frac{22}{\left(4+\sqrt{5}\right)\left(1-3\sqrt{5}\right)} \\ \frac{\mathrm{I}}{\sqrt{4+\sqrt{5}}} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{4-\sqrt{5}}, \begin{bmatrix} \frac{}{(4-} \\ \frac{}{(4-} \\ \\ \end{bmatrix}\right.\right.$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I\sqrt{4-\sqrt{5}}, \left[\begin{array}{c} \frac{22\,I}{\left(4-\sqrt{5}\right)^{\frac{3}{2}}\left(1+3\sqrt{5}\right)} \\ \frac{22}{\left(4-\sqrt{5}\right)\left(1+3\sqrt{5}\right)} \\ \frac{I}{\sqrt{4-\sqrt{5}}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-I\sqrt{4-\sqrt{5}}\,x} \cdot \left[\begin{array}{c} \frac{22\,I}{\left(4-\sqrt{5}\right)^{\frac{3}{2}}\left(1+3\sqrt{5}\right)} \\ \frac{22}{\left(4-\sqrt{5}\right)\left(1+3\sqrt{5}\right)} \\ \frac{I}{\sqrt{4-\sqrt{5}}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos\left(\sqrt{4-\sqrt{5}}\,x\right) - I\sin\left(\sqrt{4-\sqrt{5}}\,x\right)\right) \cdot \left[\begin{array}{c} \frac{22\,I}{\left(4-\sqrt{5}\right)^{\frac{3}{2}}\left(1+3\sqrt{5}\right)} \\ \frac{22}{\left(4-\sqrt{5}\right)\left(1+3\sqrt{5}\right)} \\ \frac{I}{\sqrt{4-\sqrt{5}}} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} \frac{22\,I\left(\cos\left(\sqrt{4-\sqrt{5}}\,x\right)-I\sin\left(\sqrt{4-\sqrt{5}}\,x\right)\right)}{\left(4-\sqrt{5}\right)^{\frac{3}{2}}\left(1+3\sqrt{5}\right)} \\ \frac{22\left(\cos\left(\sqrt{4-\sqrt{5}}\,x\right)-I\sin\left(\sqrt{4-\sqrt{5}}\,x\right)\right)}{\left(4-\sqrt{5}\right)\left(1+3\sqrt{5}\right)} \\ \frac{I\left(\cos\left(\sqrt{4-\sqrt{5}}\,x\right)-I\sin\left(\sqrt{4-\sqrt{5}}\,x\right)\right)}{\sqrt{4-\sqrt{5}}} \\ \cos\left(\sqrt{4-\sqrt{5}}\,x\right) - I\sin\left(\sqrt{4-\sqrt{5}}\,x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = \left[\begin{array}{c} \frac{22\sin\left(\sqrt{4-\sqrt{5}}\,x\right)}{\left(4-\sqrt{5}\right)^{\frac{3}{2}}\left(1+3\sqrt{5}\right)} \\ \frac{22\cos\left(\sqrt{4-\sqrt{5}}\,x\right)}{\left(4-\sqrt{5}\right)\left(1+3\sqrt{5}\right)} \\ \frac{\sin\left(\sqrt{4-\sqrt{5}}\,x\right)}{\sqrt{4-\sqrt{5}}} \\ \cos\left(\sqrt{4-\sqrt{5}}\,x\right) \end{array}\right], \vec{y}_2(x) = \left[\begin{array}{c} \frac{22\cos\left(\sqrt{4-\sqrt{5}}\,x\right)}{\left(4-\sqrt{5}\right)^{\frac{3}{2}}\left(1+3\sqrt{5}\right)} \\ -\frac{22\sin\left(\sqrt{4-\sqrt{5}}\,x\right)}{\left(4-\sqrt{5}\right)\left(1+3\sqrt{5}\right)} \\ \frac{\cos\left(\sqrt{4-\sqrt{5}}\,x\right)}{\sqrt{4-\sqrt{5}}} \\ -\sin\left(\sqrt{4-\sqrt{5}}\,x\right) \end{array}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}\sqrt{4+\sqrt{5}}, \left[\begin{array}{c} \frac{22\,\mathrm{I}}{\left(4+\sqrt{5}\right)^{\frac{3}{2}}\left(1-3\sqrt{5}\right)} \\ \frac{22}{\left(4+\sqrt{5}\right)\left(1-3\sqrt{5}\right)} \\ \frac{\mathrm{I}}{\sqrt{4+\sqrt{5}}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}\sqrt{4+\sqrt{5}}\,x} \cdot \left[\begin{array}{c} \frac{22\,\mathrm{I}}{\left(4+\sqrt{5}\right)^{\frac{3}{2}}\left(1-3\sqrt{5}\right)} \\ \frac{22}{\left(4+\sqrt{5}\right)\left(1-3\sqrt{5}\right)} \\ \frac{\mathrm{I}}{\sqrt{4+\sqrt{5}}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos\left(\sqrt{4+\sqrt{5}}\,x\right) - \mathrm{I}\sin\left(\sqrt{4+\sqrt{5}}\,x\right)\right) \cdot \left[\begin{array}{c} \frac{22\,\mathrm{I}}{\left(4+\sqrt{5}\right)^{\frac{3}{2}}\left(1-3\sqrt{5}\right)} \\ \frac{22}{\left(4+\sqrt{5}\right)\left(1-3\sqrt{5}\right)} \\ \frac{\mathrm{I}}{\sqrt{4+\sqrt{5}}} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} \frac{22 \mathrm{I}\left(\cos\left(\sqrt{4+\sqrt{5}}\, x\right)-\mathrm{I} \sin\left(\sqrt{4+\sqrt{5}}\, x\right)\right)}{\left(4+\sqrt{5}\right)^{\frac{3}{2}} \left(1-3\sqrt{5}\right)} \\ \frac{22\left(\cos\left(\sqrt{4+\sqrt{5}}\, x\right)-\mathrm{I} \sin\left(\sqrt{4+\sqrt{5}}\, x\right)\right)}{\left(4+\sqrt{5}\right) \left(1-3\sqrt{5}\right)} \\ \frac{\mathrm{I}\left(\cos\left(\sqrt{4+\sqrt{5}}\, x\right)-\mathrm{I} \sin\left(\sqrt{4+\sqrt{5}}\, x\right)\right)}{\sqrt{4+\sqrt{5}}} \\ \cos\left(\sqrt{4+\sqrt{5}}\, x\right)-\mathrm{I} \sin\left(\sqrt{4+\sqrt{5}}\, x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{3}(x) = \left[\begin{array}{c} \frac{22 \sin\left(\sqrt{4+\sqrt{5}}\, x\right)}{\left(4+\sqrt{5}\right)^{\frac{3}{2}} \left(1-3\sqrt{5}\right)} \\ \frac{22 \cos\left(\sqrt{4+\sqrt{5}}\, x\right)}{\left(4+\sqrt{5}\right) \left(1-3\sqrt{5}\right)} \\ \frac{\sin\left(\sqrt{4+\sqrt{5}}\, x\right)}{\sqrt{4+\sqrt{5}}} \\ \cos\left(\sqrt{4+\sqrt{5}}\, x\right) \end{array}\right], \vec{y}_{4}(x) = \left[\begin{array}{c} \frac{22 \cos\left(\sqrt{4+\sqrt{5}}\, x\right)}{\left(4+\sqrt{5}\right)^{\frac{3}{2}} \left(1-3\sqrt{5}\right)} \\ -\frac{22 \sin\left(\sqrt{4+\sqrt{5}}\, x\right)}{\left(4+\sqrt{5}\right) \left(1-3\sqrt{5}\right)} \\ \frac{\cos\left(\sqrt{4+\sqrt{5}}\, x\right)}{\sqrt{4+\sqrt{5}}} \\ -\sin\left(\sqrt{4+\sqrt{5}}\, x\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_{1}(x) + c_2 \vec{y}_{2}(x) + c_3 \vec{y}_{3}(x) + c_4 \vec{y}_{4}(x)$$

- Substitute solutions into the general solution

$$\vec{y} = \left[\begin{array}{c} \frac{22 c_4 \cos\left(\sqrt{4+\sqrt{5}}\, x\right)}{\left(4+\sqrt{5}\right)^{\frac{3}{2}} \left(1-3\sqrt{5}\right)} + \frac{22 c_3 \sin\left(\sqrt{4+\sqrt{5}}\, x\right)}{\left(4+\sqrt{5}\right)^{\frac{3}{2}} \left(1-3\sqrt{5}\right)} + \frac{22 c_2 \cos\left(\sqrt{4-\sqrt{5}}\, x\right)}{\left(4-\sqrt{5}\right)^{\frac{3}{2}} \left(1+3\sqrt{5}\right)} + \frac{22 c_1 \sin\left(\sqrt{4-\sqrt{5}}\, x\right)}{\left(4-\sqrt{5}\right)^{\frac{3}{2}} \left(1+3\sqrt{5}\right)} \\ -\frac{22 c_4 \sin\left(\sqrt{4+\sqrt{5}}\, x\right)}{\left(4+\sqrt{5}\right) \left(1-3\sqrt{5}\right)} + \frac{22 c_3 \cos\left(\sqrt{4+\sqrt{5}}\, x\right)}{\left(4+\sqrt{5}\right) \left(1-3\sqrt{5}\right)} - \frac{22 c_2 \sin\left(\sqrt{4-\sqrt{5}}\, x\right)}{\left(4-\sqrt{5}\right) \left(1+3\sqrt{5}\right)} + \frac{22 c_1 \cos\left(\sqrt{4-\sqrt{5}}\, x\right)}{\left(4-\sqrt{5}\right) \left(1+3\sqrt{5}\right)} \\ \frac{c_4 \cos\left(\sqrt{4+\sqrt{5}}\, x\right)}{\sqrt{4+\sqrt{5}}} + \frac{c_3 \sin\left(\sqrt{4+\sqrt{5}}\, x\right)}{\sqrt{4+\sqrt{5}}} + \frac{c_2 \cos\left(\sqrt{4-\sqrt{5}}\, x\right)}{\sqrt{4-\sqrt{5}}} + \frac{c_1 \sin\left(\sqrt{4-\sqrt{5}}\, x\right)}{\sqrt{4-\sqrt{5}}} \\ -c_4 \sin\left(\sqrt{4+\sqrt{5}}\, x\right) + c_3 \cos\left(\sqrt{4+\sqrt{5}}\, x\right) - c_2 \sin\left(\sqrt{4-\sqrt{5}}\, x\right) + c_1 \cos\left(\sqrt{4-\sqrt{5} \end{array}\right.$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} -\frac{11\left(\left(\sqrt{5}-1\right)\sqrt{4-\sqrt{5}}\left(c_3\sin\left(\sqrt{4+\sqrt{5}}\,x\right)+c_4\cos\left(\sqrt{4+\sqrt{5}}\,x\right)\right)-\left(\sqrt{5}+1\right)\sqrt{4+\sqrt{5}}\left(\cos\left(\sqrt{4-\sqrt{5}}\,x\right)c_2+\sin\left(\sqrt{\right.\right.\right.}{2\left(4+\sqrt{5}\right)^{\frac{3}{2}}\left(4-\sqrt{5}\right)^{\frac{3}{2}}} \\ \frac{\left(\sqrt{5}+1\right)\cos\left(\sqrt{4-\sqrt{5}}\,x\right)c_1}{2} - \frac{\left(\sqrt{5}+1\right)\sin\left(\sqrt{4-\sqrt{5}}\,x\right)c_2}{2} - \frac{\left(\sqrt{5}-1\right)\left(-c_4\sin\left(\sqrt{4+\sqrt{5}}\,x\right)+c_3\cos\left(\sqrt{4\right.\right.}{2} \\ \frac{\sqrt{4-\sqrt{5}}\left(c_3\sin\left(\sqrt{4+\sqrt{5}}\,x\right)+c_4\cos\left(\sqrt{4+\sqrt{5}}\,x\right)\right)+\sqrt{4+\sqrt{5}}\left(\cos\left(\sqrt{4-\sqrt{5}}\,x\right)c_2+\sin\left(\sqrt{4-\sqrt{5}}\,x\right)c_{\right.}}{\sqrt{4+\sqrt{5}}\,\sqrt{4-\sqrt{5}}} \\ -c_4\sin\left(\sqrt{4+\sqrt{5}}\,x\right)+c_3\cos\left(\sqrt{4+\sqrt{5}}\,x\right)-c_2\sin\left(\sqrt{4-\sqrt{5}}\,x\right)+c_1\cos\left(\sqrt{4\right. \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{y_1(x) = -\frac{11\left(\left(\sqrt{5}-1\right)\sqrt{4-\sqrt{5}}\left(c_3\sin\left(\sqrt{4+\sqrt{5}}\,x\right)+c_4\cos\left(\sqrt{4+\sqrt{5}}\,x\right)\right)-\left(\sqrt{5}+1\right)\sqrt{4+\sqrt{5}}\left(\cos\left(\sqrt{4-\sqrt{5}}\,x\right)c_2+\sin\left(\sqrt{4-\sqrt{5}}\,x\right.\right.\right.}{2\left(4+\sqrt{5}\right)^{\frac{3}{2}}\left(4-\sqrt{5}\right)^{\frac{3}{2}}}\right.$$

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 548

```
dsolve([diff(y__1(x),x)=0*y__1(x)+1*y__2(x)+0*y__3(x)+0*y__4(x),diff(y__2(x),x)=-3*y__1(x)+0
```

$$
\begin{aligned}
y_1(x) &= -\frac{c_1\left(4+\sqrt{5}\right)^{\frac{3}{2}} \cos\left(\sqrt{4+\sqrt{5}}\,x\right)}{11} - \frac{c_2\left(4-\sqrt{5}\right)^{\frac{3}{2}} \cos\left(\sqrt{4-\sqrt{5}}\,x\right)}{11} \\
&\quad - \frac{c_3\left(4+\sqrt{5}\right)^{\frac{3}{2}} \sin\left(\sqrt{4+\sqrt{5}}\,x\right)}{11} - \frac{c_4\left(4-\sqrt{5}\right)^{\frac{3}{2}} \sin\left(\sqrt{4-\sqrt{5}}\,x\right)}{11} \\
&\quad + \frac{8c_1\sqrt{4+\sqrt{5}}\,\cos\left(\sqrt{4+\sqrt{5}}\,x\right)}{11} + \frac{8c_2\sqrt{4-\sqrt{5}}\,\cos\left(\sqrt{4-\sqrt{5}}\,x\right)}{11} \\
&\quad + \frac{8c_3\sqrt{4+\sqrt{5}}\,\sin\left(\sqrt{4+\sqrt{5}}\,x\right)}{11} + \frac{8c_4\sqrt{4-\sqrt{5}}\,\sin\left(\sqrt{4-\sqrt{5}}\,x\right)}{11} \\
y_2(x) &= -c_1\sin\left(\sqrt{4+\sqrt{5}}\,x\right) - c_2\sin\left(\sqrt{4-\sqrt{5}}\,x\right) \\
&\quad + c_3\cos\left(\sqrt{4+\sqrt{5}}\,x\right) + c_4\cos\left(\sqrt{4-\sqrt{5}}\,x\right) \\
y_3(x) &= \frac{13c_1\sqrt{4+\sqrt{5}}\,\cos\left(\sqrt{4+\sqrt{5}}\,x\right)}{22} + \frac{13c_2\sqrt{4-\sqrt{5}}\,\cos\left(\sqrt{4-\sqrt{5}}\,x\right)}{22} \\
&\quad + \frac{13c_3\sqrt{4+\sqrt{5}}\,\sin\left(\sqrt{4+\sqrt{5}}\,x\right)}{22} + \frac{13c_4\sqrt{4-\sqrt{5}}\,\sin\left(\sqrt{4-\sqrt{5}}\,x\right)}{22} \\
&\quad - \frac{3c_1\left(4+\sqrt{5}\right)^{\frac{3}{2}} \cos\left(\sqrt{4+\sqrt{5}}\,x\right)}{22} - \frac{3c_2\left(4-\sqrt{5}\right)^{\frac{3}{2}} \cos\left(\sqrt{4-\sqrt{5}}\,x\right)}{22} \\
&\quad - \frac{3c_3\left(4+\sqrt{5}\right)^{\frac{3}{2}} \sin\left(\sqrt{4+\sqrt{5}}\,x\right)}{22} - \frac{3c_4\left(4-\sqrt{5}\right)^{\frac{3}{2}} \sin\left(\sqrt{4-\sqrt{5}}\,x\right)}{22} \\
y_4(x) &= \frac{c_1\sin\left(\sqrt{4+\sqrt{5}}\,x\right)\sqrt{5}}{2} - \frac{c_2\sin\left(\sqrt{4-\sqrt{5}}\,x\right)\sqrt{5}}{2} \\
&\quad - \frac{c_3\cos\left(\sqrt{4+\sqrt{5}}\,x\right)\sqrt{5}}{2} + \frac{c_4\cos\left(\sqrt{4-\sqrt{5}}\,x\right)\sqrt{5}}{2} + \frac{c_1\sin\left(\sqrt{4+\sqrt{5}}\,x\right)}{2} \\
&\quad + \frac{c_2\sin\left(\sqrt{4-\sqrt{5}}\,x\right)}{2} - \frac{c_3\cos\left(\sqrt{4+\sqrt{5}}\,x\right)}{2} - \frac{c_4\cos\left(\sqrt{4-\sqrt{5}}\,x\right)}{2}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.099 (sec). Leaf size: 730

```
DSolve[{y1'[x]==0*y1[x]+1*y2[x]+0*y3[x]+0*y4[x],y2'[x]==-3*y1[x]+0*y2[x]+2*y3[x]+0*y4[x],y3'
```

$$
\begin{aligned}
\mathrm{y1}(x) \rightarrow & \frac{1}{2} c_3 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{e^{\#1x}}{\#1^2+4}\&\right] \\
& +\frac{1}{4} c_1 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{\#1^2 e^{\#1x}+5e^{\#1x}}{\#1^2+4}\&\right] \\
& +\frac{1}{2} c_4 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{e^{\#1x}}{\#1^3+4\#1}\&\right] \\
& +\frac{1}{4} c_2 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{\#1^2 e^{\#1x}+5e^{\#1x}}{\#1^3+4\#1}\&\right] \\
\mathrm{y2}(x) \rightarrow & \frac{1}{2} c_4 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{e^{\#1x}}{\#1^2+4}\&\right] \\
& +\frac{1}{2} c_3 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{\#1e^{\#1x}}{\#1^2+4}\&\right] \\
& +\frac{1}{4} c_2 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{\#1^2 e^{\#1x}+5e^{\#1x}}{\#1^2+4}\&\right] \\
& -\frac{1}{4} c_1 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{3\#1^2 e^{\#1x}+11e^{\#1x}}{\#1^3+4\#1}\&\right] \\
\mathrm{y3}(x) \rightarrow & \frac{1}{2} c_1 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{e^{\#1x}}{\#1^2+4}\&\right] \\
& +\frac{1}{4} c_3 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{\#1^2 e^{\#1x}+3e^{\#1x}}{\#1^2+4}\&\right] \\
& +\frac{1}{2} c_2 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{e^{\#1x}}{\#1^3+4\#1}\&\right] \\
& +\frac{1}{4} c_4 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{\#1^2 e^{\#1x}+3e^{\#1x}}{\#1^3+4\#1}\&\right] \\
\mathrm{y4}(x) \rightarrow & \frac{1}{2} c_2 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{e^{\#1x}}{\#1^2+4}\&\right] \\
& +\frac{1}{2} c_1 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{\#1e^{\#1x}}{\#1^2+4}\&\right] \\
& +\frac{1}{4} c_4 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{\#1^2 e^{\#1x}+3e^{\#1x}}{\#1^2+4}\&\right] \\
& -\frac{1}{4} c_3 \mathrm{RootSum}\left[\#1^4+8\#1^2+11\&, \frac{5\#1^2 e^{\#1x}+11e^{\#1x}}{\#1^3+4\#1}\&\right]
\end{aligned}
$$

19.17 problem 15

Internal problem ID [12855]
Internal file name [`OUTPUT/11507_Monday_November_06_2023_01_31_10_PM_44261061/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= 3y_1(x) + 2y_2(x) \\
y_2'(x) &= -2y_1(x) + 3y_2(x) \\
y_3'(x) &= y_3(x) \\
y_4'(x) &= 2y_4(x)
\end{aligned}
$$

19.17.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$
\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{3x}\cos(2x) & \sin(2x)\,e^{3x} & 0 & 0 \\ -\sin(2x)\,e^{3x} & e^{3x}\cos(2x) & 0 & 0 \\ 0 & 0 & e^{x} & 0 \\ 0 & 0 & 0 & e^{2x} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(x) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{3x}\cos(2x) & \sin(2x)\,e^{3x} & 0 & 0 \\ -\sin(2x)\,e^{3x} & e^{3x}\cos(2x) & 0 & 0 \\ 0 & 0 & e^{x} & 0 \\ 0 & 0 & 0 & e^{2x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{3x}\cos(2x)\,c_1 + \sin(2x)\,e^{3x}c_2 \\ -\sin(2x)\,e^{3x}c_1 + e^{3x}\cos(2x)\,c_2 \\ e^{x}c_3 \\ e^{2x}c_4 \end{bmatrix} \\
&= \begin{bmatrix} e^{3x}(\cos(2x)\,c_1 + \sin(2x)\,c_2) \\ -e^{3x}(\sin(2x)\,c_1 - \cos(2x)\,c_2) \\ e^{x}c_3 \\ e^{2x}c_4 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

19.17.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 2 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 2 & 0 & 0 \\ -2 & 3-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 9\lambda^3 + 33\lambda^2 - 51\lambda + 26 = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 3 + 2i \\ \lambda_3 &= 3 - 2i \\ \lambda_4 &= 2 \end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue
$3 - 2i$	1	complex eigenvalue
$3 + 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cccc|c} 2 & 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(3,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2, v_4\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \implies \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix}0\\0\\0\\t\end{bmatrix} = \begin{bmatrix}0\\0\\0\\t\end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix}0\\0\\0\\t\end{bmatrix} = t\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix}0\\0\\0\\t\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 3 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}3 & 2 & 0 & 0\\-2 & 3 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 2\end{bmatrix} - (3-2i)\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}2i & 2 & 0 & 0\\-2 & 2i & 0 & 0\\0 & 0 & -2+2i & 0\\0 & 0 & 0 & -1+2i\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\\v_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 2i & 2 & 0 & 0 & 0 \\ -2 & 2i & 0 & 0 & 0 \\ 0 & 0 & -2+2i & 0 & 0 \\ 0 & 0 & 0 & -1+2i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} 2i & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2+2i & 0 & 0 \\ 0 & 0 & 0 & -1+2i & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} 2i & 2 & 0 & 0 & 0 \\ 0 & 0 & -2+2i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1+2i & 0 \end{array}\right]$$

Since the current pivot $A(3,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} 2i & 2 & 0 & 0 & 0 \\ 0 & 0 & -2+2i & 0 & 0 \\ 0 & 0 & 0 & -1+2i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\left[\begin{array}{cccc} 2i & 2 & 0 & 0 \\ 0 & 0 & -2+2i & 0 \\ 0 & 0 & 0 & -1+2i \\ 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} it \\ t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = 3 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - (3 + 2i) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & 2 & 0 & 0 \\ -2 & -2i & 0 & 0 \\ 0 & 0 & -2 - 2i & 0 \\ 0 & 0 & 0 & -1 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -2i & 2 & 0 & 0 & 0 \\ -2 & -2i & 0 & 0 & 0 \\ 0 & 0 & -2-2i & 0 & 0 \\ 0 & 0 & 0 & -1-2i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} -2i & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2-2i & 0 & 0 \\ 0 & 0 & 0 & -1-2i & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} -2i & 2 & 0 & 0 & 0 \\ 0 & 0 & -2-2i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-2i & 0 \end{array}\right]$$

Since the current pivot $A(3,4)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 3 and row 4 gives

$$\left[\begin{array}{cccc|c} -2i & 2 & 0 & 0 & 0 \\ 0 & 0 & -2-2i & 0 & 0 \\ 0 & 0 & 0 & -1-2i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & 2 & 0 & 0 \\ 0 & 0 & -2-2i & 0 \\ 0 & 0 & 0 & -1-2i \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3, v_4\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it, v_3 = 0, v_4 = 0\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -it \\ t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}\,t \\ t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$
$3+2i$	1	1	No	$\begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}$
$3-2i$	1	1	No	$\begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(x) &= \vec{v}_1 e^{x} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{x} \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(x) = \vec{v}_2 e^{2x}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{2x}$$

Therefore the final solution is

$$\vec{x}_h(x) = c_1\vec{x}_1(x) + c_2\vec{x}_2(x) + c_3\vec{x}_3(x) + c_4\vec{x}_4(x)$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^x \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -i\mathrm{e}^{(3+2i)x} \\ \mathrm{e}^{(3+2i)x} \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} i\mathrm{e}^{(3-2i)x} \\ \mathrm{e}^{(3-2i)x} \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathrm{e}^{2x} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} i\left(c_3\mathrm{e}^{(3-2i)x} - c_2\mathrm{e}^{(3+2i)x}\right) \\ c_2\mathrm{e}^{(3+2i)x} + c_3\mathrm{e}^{(3-2i)x} \\ c_1\mathrm{e}^x \\ c_4\mathrm{e}^{2x} \end{bmatrix}$$

19.17.3 Maple step by step solution

Let's solve

$[y_1'(x) = 3y_1(x) + 2y_2(x), y_2'(x) = -2y_1(x) + 3y_2(x), y_3'(x) = y_3(x), y_4'(x) = 2y_4(x)]$

- Define vector

$$\underline{y}^{\rightarrow}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(x)=\left[\begin{array}{cccc}3 & 2 & 0 & 0\\ -2 & 3 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\end{array}\right]\cdot\underline{y}^{\rightarrow}(x)+\left[\begin{array}{c}0\\ 0\\ 0\\ 0\end{array}\right]$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(x)=\left[\begin{array}{cccc}3 & 2 & 0 & 0\\ -2 & 3 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\end{array}\right]\cdot\underline{y}^{\rightarrow}(x)$$

- Define the coefficient matrix

$$A=\left[\begin{array}{cccc}3 & 2 & 0 & 0\\ -2 & 3 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\end{array}\right]$$

- Rewrite the system as
$\underline{y}^{\rightarrow\prime}(x)=A\cdot\underline{y}^{\rightarrow}(x)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1,\left[\begin{array}{c}0\\ 0\\ 1\\ 0\end{array}\right]\right],\left[2,\left[\begin{array}{c}0\\ 0\\ 0\\ 1\end{array}\right]\right],\left[3-2\,\mathrm{I},\left[\begin{array}{c}\mathrm{I}\\ 1\\ 0\\ 0\end{array}\right]\right],\left[3+2\,\mathrm{I},\left[\begin{array}{c}-\mathrm{I}\\ 1\\ 0\\ 0\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1,\left[\begin{array}{c}0\\ 0\\ 1\\ 0\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{x} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2x} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[3 - 2\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(3-2\mathrm{I})x} \cdot \begin{bmatrix} \mathrm{I} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{3x} \cdot (\cos(2x) - I\sin(2x)) \cdot \begin{bmatrix} I \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- Simplify expression

$$e^{3x} \cdot \begin{bmatrix} I(\cos(2x) - I\sin(2x)) \\ \cos(2x) - I\sin(2x) \\ 0 \\ 0 \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{3x} \cdot \begin{bmatrix} \sin(2x) \\ \cos(2x) \\ 0 \\ 0 \end{bmatrix}, \vec{y}_4(x) = e^{3x} \cdot \begin{bmatrix} \cos(2x) \\ -\sin(2x) \\ 0 \\ 0 \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = e^{x}c_1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2x} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{3x} \cdot \begin{bmatrix} \sin(2x) \\ \cos(2x) \\ 0 \\ 0 \end{bmatrix} + c_4 e^{3x} \cdot \begin{bmatrix} \cos(2x) \\ -\sin(2x) \\ 0 \\ 0 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} e^{3x}(c_3\sin(2x) + c_4\cos(2x)) \\ e^{3x}(c_3\cos(2x) - c_4\sin(2x)) \\ e^{x}c_1 \\ c_2 e^{2x} \end{bmatrix}$$

- Solution to the system of ODEs

$$\{y_1(x) = e^{3x}(c_3\sin(2x) + c_4\cos(2x)), y_2(x) = e^{3x}(c_3\cos(2x) - c_4\sin(2x)), y_3(x) = e^{x}c_1, y_4(x) =$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 62

```
dsolve([diff(y__1(x),x)=3*y__1(x)+2*y__2(x)+0*y__3(x)+0*y__4(x),diff(y__2(x),x)=-2*y__1(x)+3
```

$$
\begin{aligned}
y_1(x) &= e^{3x}(\sin(2x)\, c_1 + \cos(2x)\, c_2) \\
y_2(x) &= -e^{3x}(\sin(2x)\, c_2 - \cos(2x)\, c_1) \\
y_3(x) &= c_4 e^{x} \\
y_4(x) &= c_3 e^{2x}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.067 (sec). Leaf size: 255

```
DSolve[{y1'[x]==3*y1[x]+2*y2[x]+0*y3[x]+0*y4[x],y2'[x]==-2*y1[x]+3*y2[x]+0*y3[x]+0*y4[x],y3'
```

$$
\begin{aligned}
&\text{y1}(x) \to e^{3x}(c_1 \cos(2x) + c_2 \sin(2x)) \\
&\text{y2}(x) \to e^{3x}(c_2 \cos(2x) - c_1 \sin(2x)) \\
&\text{y3}(x) \to c_3 e^{x} \\
&\text{y4}(x) \to c_4 e^{2x} \\
&\text{y1}(x) \to e^{3x}(c_1 \cos(2x) + c_2 \sin(2x)) \\
&\text{y2}(x) \to e^{3x}(c_2 \cos(2x) - c_1 \sin(2x)) \\
&\text{y3}(x) \to c_3 e^{x} \\
&\text{y4}(x) \to 0 \\
&\text{y1}(x) \to e^{3x}(c_1 \cos(2x) + c_2 \sin(2x)) \\
&\text{y2}(x) \to e^{3x}(c_2 \cos(2x) - c_1 \sin(2x)) \\
&\text{y3}(x) \to 0 \\
&\text{y4}(x) \to c_4 e^{2x} \\
&\text{y1}(x) \to e^{3x}(c_1 \cos(2x) + c_2 \sin(2x)) \\
&\text{y2}(x) \to e^{3x}(c_2 \cos(2x) - c_1 \sin(2x)) \\
&\text{y3}(x) \to 0 \\
&\text{y4}(x) \to 0
\end{aligned}
$$

19.18 problem 16

Internal problem ID [12856]
Internal file name [`OUTPUT/11508_Monday_November_06_2023_01_31_11_PM_97635255/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 8. Linear Systems of First-Order Differential Equations. Exercises 8.3 page 379
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
y_1'(x) &= y_2(x) + y_4(x) \\
y_2'(x) &= y_1(x) - y_3(x) \\
y_3'(x) &= y_4(x) \\
y_4'(x) &= y_3(x)
\end{aligned}
$$

19.18.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$
\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-x}}{2}+\frac{e^{x}}{2} & -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & 0 & -\frac{e^{-x}}{2}+\frac{e^{x}}{2} \\ -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{e^{-x}}{2}-\frac{e^{x}}{2} & 0 \\ 0 & 0 & \frac{e^{-x}}{2}+\frac{e^{x}}{2} & -\frac{e^{-x}}{2}+\frac{e^{x}}{2} \\ 0 & 0 & -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{e^{-x}}{2}+\frac{e^{x}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(x) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-x}}{2}+\frac{e^{x}}{2} & -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & 0 & -\frac{e^{-x}}{2}+\frac{e^{x}}{2} \\ -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{e^{-x}}{2}-\frac{e^{x}}{2} & 0 \\ 0 & 0 & \frac{e^{-x}}{2}+\frac{e^{x}}{2} & -\frac{e^{-x}}{2}+\frac{e^{x}}{2} \\ 0 & 0 & -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{e^{-x}}{2}+\frac{e^{x}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-x}}{2}+\frac{e^{x}}{2}\right)c_1 + \left(-\frac{e^{-x}}{2}+\frac{e^{x}}{2}\right)c_2 + \left(-\frac{e^{-x}}{2}+\frac{e^{x}}{2}\right)c_4 \\ \left(-\frac{e^{-x}}{2}+\frac{e^{x}}{2}\right)c_1 + \left(\frac{e^{-x}}{2}+\frac{e^{x}}{2}\right)c_2 + \left(\frac{e^{-x}}{2}-\frac{e^{x}}{2}\right)c_3 \\ \left(\frac{e^{-x}}{2}+\frac{e^{x}}{2}\right)c_3 + \left(-\frac{e^{-x}}{2}+\frac{e^{x}}{2}\right)c_4 \\ \left(-\frac{e^{-x}}{2}+\frac{e^{x}}{2}\right)c_3 + \left(\frac{e^{-x}}{2}+\frac{e^{x}}{2}\right)c_4 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-c_2-c_4)e^{-x}}{2} + \frac{e^{x}(c_1+c_2+c_4)}{2} \\ \frac{(-c_1+c_2+c_3)e^{-x}}{2} + \frac{e^{x}(c_1+c_2-c_3)}{2} \\ \frac{(c_3-c_4)e^{-x}}{2} + \frac{e^{x}(c_3+c_4)}{2} \\ \frac{(-c_3+c_4)e^{-x}}{2} + \frac{e^{x}(c_3+c_4)}{2} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(x)$ above.

19.18.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(x) = A\,\vec{x}(x)$$

Or

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 & 0 & 1 \\ 1 & -\lambda & -1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 2\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$R_4 = R_4 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_4\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Let $v_4 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t - s, v_3 = -s\}$

Hence the solution is

$$\begin{bmatrix} -t - s \\ t \\ -s \\ s \end{bmatrix} = \begin{bmatrix} -t - s \\ t \\ -s \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -t - s \\ t \\ -s \\ s \end{bmatrix} &= \begin{bmatrix} -t \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ -s \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -t-s \\ t \\ -s \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right)$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cccc|c} -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{cccc|c} -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_4 = R_4 + R_2 \Longrightarrow \left[\begin{array}{cccc|c} -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_4\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Let $v_4 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t + s, v_3 = s\}$

Hence the solution is

$$\begin{bmatrix} t+s \\ t \\ s \\ s \end{bmatrix} = \begin{bmatrix} t+s \\ t \\ s \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this

eigenvalue. The above can be written as

$$
\begin{aligned}
\begin{bmatrix} t+s \\ t \\ s \\ s \end{bmatrix} &= \begin{bmatrix} t \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \\ s \\ s \end{bmatrix} \\
&= t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$
\begin{bmatrix} t+s \\ t \\ s \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}
$$

Hence the two eigenvectors associated with this eigenvalue are

$$
\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)
$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-1	2	2	No	$\begin{bmatrix} -1 & -1 \\ 0 & 1 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}$
1	2	2	No	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

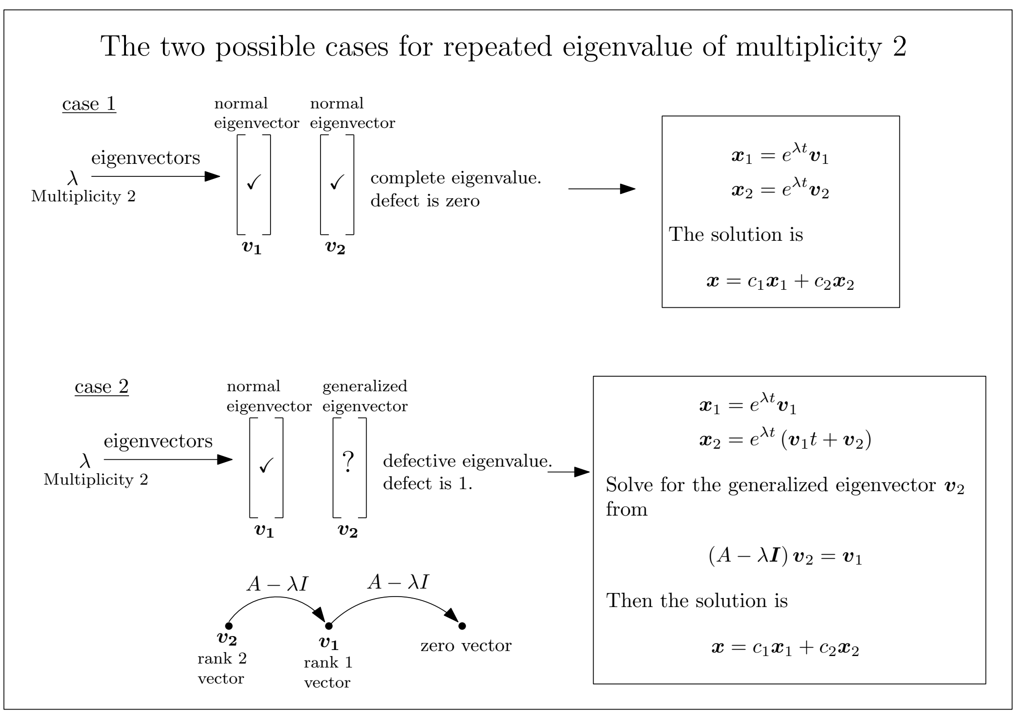

Figure 431: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_1(x) &= \vec{v}_1 e^{-x} \\
&= \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} e^{-x}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_2(x) &= \vec{v}_2 e^{-x} \\
&= \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-x}
\end{aligned}
$$

eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

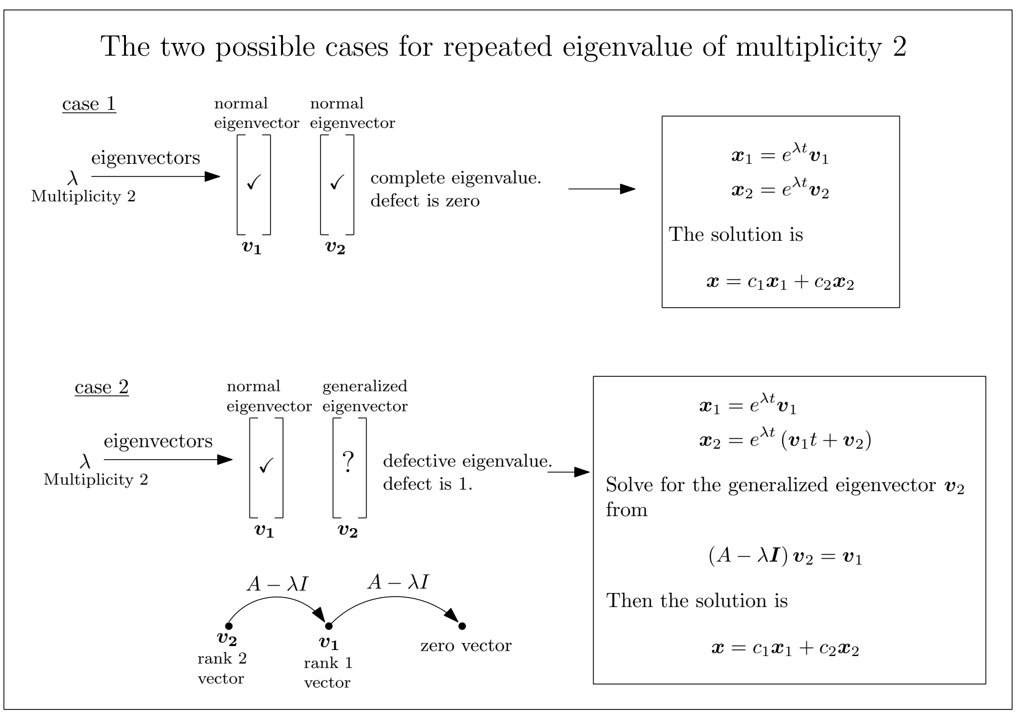

Figure 432: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\begin{aligned}\vec{x}_3(x) &= \vec{v}_3 e^{x} \\ &= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{x}\end{aligned}$$

$$\begin{aligned}\vec{x}_4(x) &= \vec{v}_4 e^{x} \\ &= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} e^{x}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(x) = c_1 \vec{x}_1(x) + c_2 \vec{x}_2(x) + c_3 \vec{x}_3(x) + c_4 \vec{x}_4(x)$$

Which is written as

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-x} \\ 0 \\ -\mathrm{e}^{-x} \\ \mathrm{e}^{-x} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-x} \\ \mathrm{e}^{-x} \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{x} \\ \mathrm{e}^{x} \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} \mathrm{e}^{x} \\ 0 \\ \mathrm{e}^{x} \\ \mathrm{e}^{x} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} (-c_1 - c_2)\, \mathrm{e}^{-x} + \mathrm{e}^{x}(c_3 + c_4) \\ c_2 \mathrm{e}^{-x} + c_3 \mathrm{e}^{x} \\ -c_1 \mathrm{e}^{-x} + c_4 \mathrm{e}^{x} \\ c_1 \mathrm{e}^{-x} + c_4 \mathrm{e}^{x} \end{bmatrix}$$

19.18.3 Maple step by step solution

Let's solve

$[y_1'(x) = y_2(x) + y_4(x), y_2'(x) = y_1(x) - y_3(x), y_3'(x) = y_4(x), y_4'(x) = y_3(x)]$

- Define vector

$$\underline{y}^{\rightarrow}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{y}^{\rightarrow\prime}(x) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \underline{y}^{\rightarrow}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$\underline{y}^{\rightarrow\prime}(x) = A \cdot \underline{y}^{\rightarrow}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}\right], \left[-1, \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}\right], \left[1, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}\right], \left[1, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\underline{\vec{y}}_1(x) = \mathrm{e}^{-x} \cdot \begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, an

$\underline{\vec{y}}_2(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\underline{\vec{y}}_2(x)$ into the homogeneous system

$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$

- Use the fact that $\vec{v}$ is an eigenvector of A

$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \mathrm{e}^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$

- Simplify equation

$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\underline{\vec{y}}_2(x)$ to be a solution to the homogeneous system

$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} - (-1) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

$$\vec{y}_2(x) = e^{-x} \cdot \left(x \cdot \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

$$\vec{y}_3(x) = e^{x} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, and

$$\vec{y}_4(x) = e^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{y}_4(x)$ into the homogeneous system

$$\lambda \mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v}=\left(\mathrm{e}^{\lambda x}A\right)\cdot\left(x\vec{v}+\vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda \mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v}=\mathrm{e}^{\lambda x}\left(\lambda x\vec{v}+A\cdot\vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_4(x)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix}0&1&0&1\\1&0&-1&0\\0&0&0&1\\0&0&1&0\end{bmatrix}-1\cdot\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}1\\0\\1\\1\end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p}=\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{y}_4(x)=\mathrm{e}^{x}\cdot\left(x\cdot\begin{bmatrix}1\\0\\1\\1\end{bmatrix}+\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1(x)+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-x} \cdot \left(x \cdot \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + e^{x} c_4 \cdot \left(x \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \right.$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix} = \begin{bmatrix} ((-x-1) c_2 - c_1) e^{-x} + e^{x}((x-1) c_4 + c_3) \\ 0 \\ (-c_2 x - c_1) e^{-x} + e^{x}(c_4 x + c_3) \\ e^{-x}(c_2 x + c_1) + e^{x}(c_4 x + c_3) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{y_1(x) = ((-x-1) c_2 - c_1) e^{-x} + e^{x}((x-1) c_4 + c_3), y_2(x) = 0, y_3(x) = (-c_2 x - c_1) e^{-x} + e^{x}(c_4 x$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 71

```
dsolve([diff(y__1(x),x)=0*y__1(x)+1*y__2(x)+0*y__3(x)+1*y__4(x),diff(y__2(x),x)=1*y__1(x)+0*
```

$$y_1(x) = c_1 e^{x} + c_2 e^{-x}$$
$$y_2(x) = c_1 e^{x} - c_2 e^{-x} - c_3 e^{x} + c_4 e^{-x}$$
$$y_3(x) = c_3 e^{x} + c_4 e^{-x}$$
$$y_4(x) = c_3 e^{x} - c_4 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 148

```
DSolve[{y1'[x]==0*y1[x]+1*y2[x]+0*y3[x]+1*y4[x],y2'[x]==1*y1[x]+0*y2[x]-1*y3[x]+0*y4[x],y3'[
```

$$\text{y1}(x) \to \frac{1}{2} e^{-x} \left(c_1 \left(e^{2x} + 1\right) + (c_2 + c_4) \left(e^{2x} - 1\right)\right)$$
$$\text{y2}(x) \to \frac{1}{2} e^{-x} \left(c_1 \left(e^{2x} - 1\right) + c_2 e^{2x} - c_3 e^{2x} + c_2 + c_3\right)$$
$$\text{y3}(x) \to \frac{1}{2} e^{-x} \left(c_3 \left(e^{2x} + 1\right) + c_4 \left(e^{2x} - 1\right)\right)$$
$$\text{y4}(x) \to \frac{1}{2} e^{-x} \left(c_3 \left(e^{2x} - 1\right) + c_4 \left(e^{2x} + 1\right)\right)$$

20 Chapter 10. Applications of Systems of Equations. Exercises 10.2 page 432

20.1 problem 1

Internal problem ID [12857]
Internal file name [`OUTPUT/11509_Monday_November_06_2023_01_31_11_PM_97262255/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 10. Applications of Systems of Equations. Exercises 10.2 page 432
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -2x(t) + 3y(t) \\
y'(t) &= -x(t) + 2y(t)
\end{aligned}
$$

20.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{3\,\mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{t}}{2} & \frac{3\,\mathrm{e}^{t}}{2} - \frac{3\,\mathrm{e}^{-t}}{2} \\ -\frac{\mathrm{e}^{t}}{2} + \frac{\mathrm{e}^{-t}}{2} & -\frac{\mathrm{e}^{-t}}{2} + \frac{3\,\mathrm{e}^{t}}{2} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{3\,\mathrm{e}^{-t}}{2}-\frac{\mathrm{e}^{t}}{2} & \frac{3\,\mathrm{e}^{t}}{2}-\frac{3\,\mathrm{e}^{-t}}{2} \\ -\frac{\mathrm{e}^{t}}{2}+\frac{\mathrm{e}^{-t}}{2} & -\frac{\mathrm{e}^{-t}}{2}+\frac{3\,\mathrm{e}^{t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{3\,\mathrm{e}^{-t}}{2}-\frac{\mathrm{e}^{t}}{2}\right)c_1+\left(\frac{3\,\mathrm{e}^{t}}{2}-\frac{3\,\mathrm{e}^{-t}}{2}\right)c_2 \\ \left(-\frac{\mathrm{e}^{t}}{2}+\frac{\mathrm{e}^{-t}}{2}\right)c_1+\left(-\frac{\mathrm{e}^{-t}}{2}+\frac{3\,\mathrm{e}^{t}}{2}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(3c_1-3c_2)\mathrm{e}^{-t}}{2}-\frac{\mathrm{e}^{t}(-3c_2+c_1)}{2} \\ \frac{(c_1-c_2)\mathrm{e}^{-t}}{2}-\frac{\mathrm{e}^{t}(-3c_2+c_1)}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 3 \\ -1 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
−1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ -1 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^t$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-t}$$
$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^t \\ \mathrm{e}^t \end{bmatrix} + c_2 \begin{bmatrix} 3\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 \mathrm{e}^t + 3c_2 \mathrm{e}^{-t} \\ c_1 \mathrm{e}^t + c_2 \mathrm{e}^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

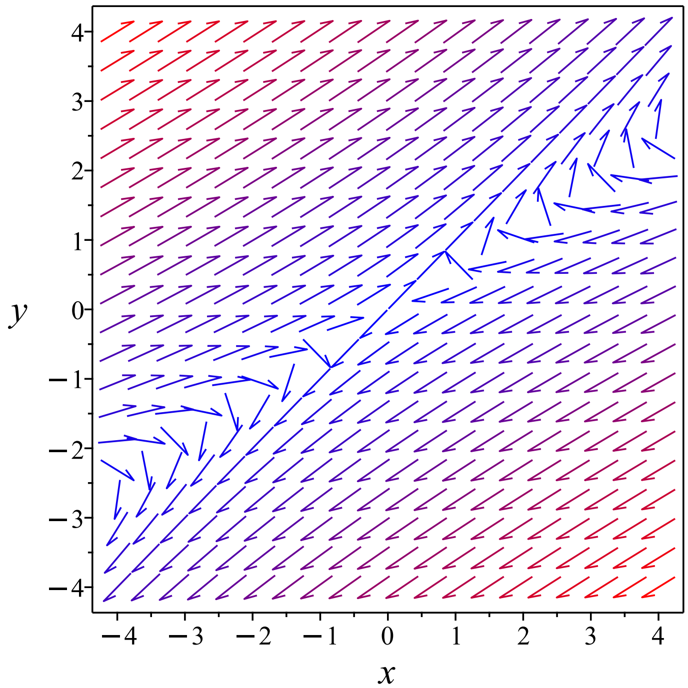

Figure 433: Phase plot

20.1.3 Maple step by step solution

Let's solve

$[x'(t) = -2x(t) + 3y(t), y'(t) = -x(t) + 2y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{-t} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x(t) = 3c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t}, y(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t}\}$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve([diff(x(t),t)=-2*x(t)+3*y(t),diff(y(t),t)=-x(t)+2*y(t)],singsol=all)
```

$$x(t) = c_1 \mathrm{e}^{t} + c_2 \mathrm{e}^{-t}$$
$$y(t) = c_1 \mathrm{e}^{t} + \frac{c_2 \mathrm{e}^{-t}}{3}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 72

```
DSolve[{x'[t]==-2*x[t]+3*y[t],y'[t]==-x[t]+2*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{1}{2} e^{-t} \left(3c_2 \left(e^{2t} - 1\right) - c_1 \left(e^{2t} - 3\right)\right)$$
$$y(t) \to -\frac{1}{2} e^{-t} \left(c_1 \left(e^{2t} - 1\right) + c_2 \left(1 - 3e^{2t}\right)\right)$$

20.2 problem 2

Internal problem ID [12858]
Internal file name [`OUTPUT/11510_Monday_November_06_2023_01_31_12_PM_48820822/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 10. Applications of Systems of Equations. Exercises 10.2 page 432
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -x(t) + 2y(t) \\
y'(t) &= -2x(t) + 3y(t)
\end{aligned}
$$

20.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{t}(1-2t) & 2t\,\mathrm{e}^{t} \\ -2t\,\mathrm{e}^{t} & \mathrm{e}^{t}(2t+1) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{c}\\ &= \begin{bmatrix} e^{t}(1-2t) & 2t\,e^{t} \\ -2t\,e^{t} & e^{t}(2t+1) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\\ &= \begin{bmatrix} e^{t}(1-2t)\,c_1 + 2t\,e^{t}c_2 \\ -2t\,e^{t}c_1 + e^{t}(2t+1)\,c_2 \end{bmatrix}\\ &= \begin{bmatrix} (c_1(1-2t) + 2c_2t)\,e^{t} \\ (c_2(2t+1) - 2tc_1)\,e^{t} \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 2 \\ -2 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	2	1	Yes	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

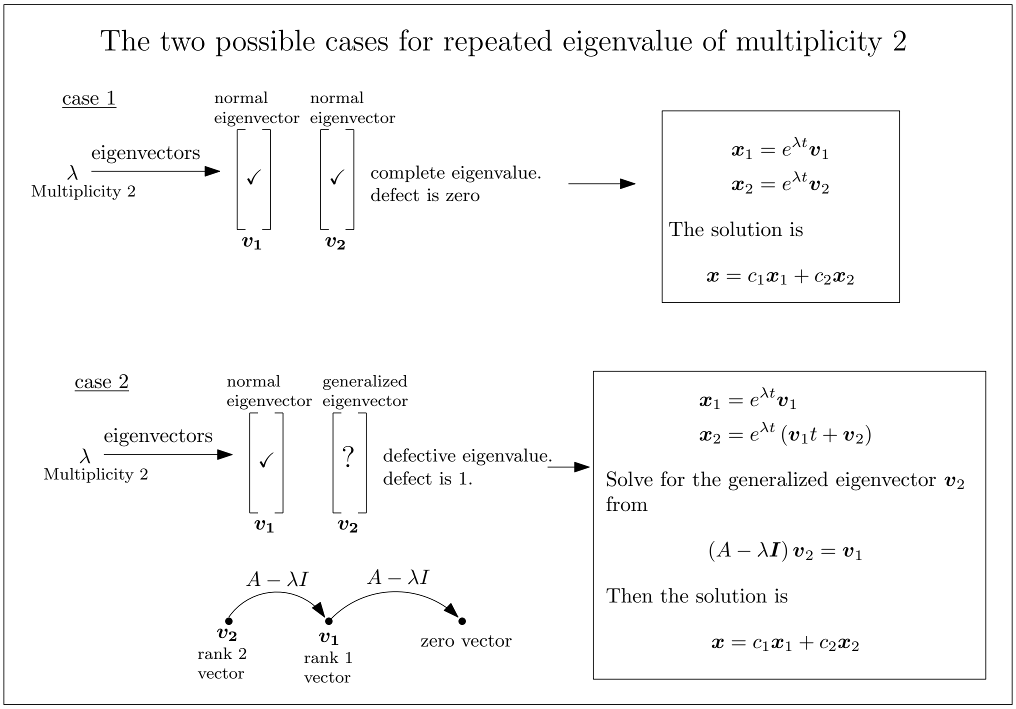

Figure 434: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 1. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathrm{e}^{t} \\ &= \begin{bmatrix} \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right) \mathrm{e}^{t} \\ &= \begin{bmatrix} \frac{\mathrm{e}^{t}(2t+1)}{2} \\ \mathrm{e}^{t}(1+t) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t}\left(t + \frac{1}{2}\right) \\ \mathrm{e}^{t}(1+t) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{t}\left(c_1 + c_2 t + \frac{1}{2} c_2\right) \\ \mathrm{e}^{t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

The following is the phase plot of the system.

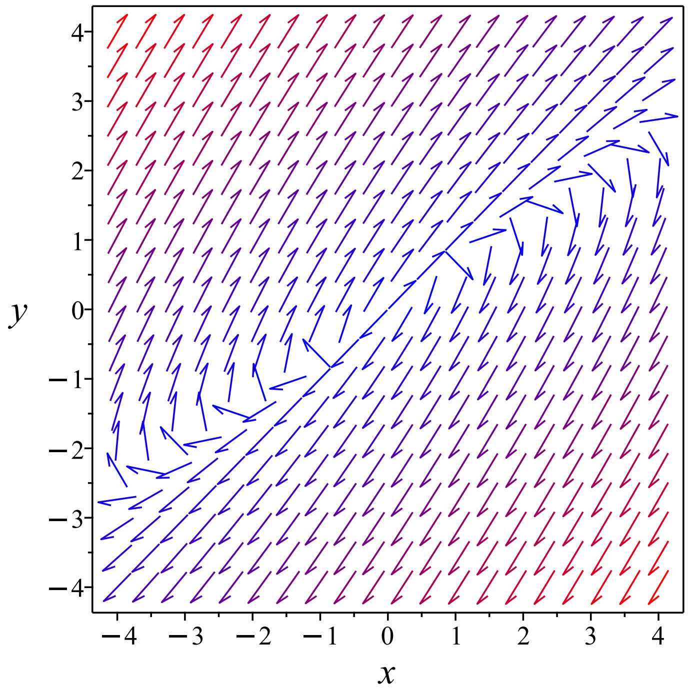

Figure 435: Phase plot

20.2.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) + 2y(t), y'(t) = -2x(t) + 3y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{x}'(t) = A \cdot \vec{x}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$$
- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[1, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$
- First solution from eigenvalue 1

  $$\vec{x}_1(t) = e^t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, and

  $\vec{x}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$
- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_2(t)$ into the homogeneous system

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$
- Use the fact that $\vec{v}$ is an eigenvector of A

  $\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$
- Simplify equation

  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$
- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$
- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix}-1 & 2 \\ -2 & 3\end{bmatrix} - 1 \cdot \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix}1 \\ 1\end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix}-\frac{1}{2} \\ 0\end{bmatrix}$$

- Second solution from eigenvalue 1

$$\overrightarrow{x}_2(t) = e^t \cdot \left(t \cdot \begin{bmatrix}1 \\ 1\end{bmatrix} + \begin{bmatrix}-\frac{1}{2} \\ 0\end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1(t) + c_2 \overrightarrow{x}_2(t)$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 e^t \cdot \begin{bmatrix}1 \\ 1\end{bmatrix} + c_2 e^t \cdot \left(t \cdot \begin{bmatrix}1 \\ 1\end{bmatrix} + \begin{bmatrix}-\frac{1}{2} \\ 0\end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix}x(t) \\ y(t)\end{bmatrix} = \begin{bmatrix}e^t\left(c_1 + c_2 t - \frac{1}{2}c_2\right) \\ e^t(c_2 t + c_1)\end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = e^t\left(c_1 + c_2 t - \frac{1}{2}c_2\right), y(t) = e^t(c_2 t + c_1)\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 29

```
dsolve([diff(x(t),t)=-x(t)+2*y(t),diff(y(t),t)=-2*x(t)+3*y(t)],singsol=all)
```

$$x(t) = e^t(c_2 t + c_1)$$

$$y(t) = \frac{e^t(2c_2 t + 2c_1 + c_2)}{2}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 42

```
DSolve[{x'[t]==-x[t]+2*y[t],y'[t]==-2*x[t]+3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to e^t(-2c_1t + 2c_2t + c_1)$$
$$y(t) \to e^t(-2c_1t + 2c_2t + c_2)$$

20.3 problem 3

Internal problem ID [12859]
Internal file name [`OUTPUT/11511_Monday_November_06_2023_01_31_12_PM_491032/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 10. Applications of Systems of Equations. Exercises 10.2 page 432
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -x(t) - 2y(t)$$
$$y'(t) = 2x(t) - 3y(t)$$

20.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-2t}\cos\left(\sqrt{3}\,t\right) + \frac{\mathrm{e}^{-2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} & -\frac{2\,\mathrm{e}^{-2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} \\ \frac{2\,\mathrm{e}^{-2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} & \mathrm{e}^{-2t}\cos\left(\sqrt{3}\,t\right) - \frac{\mathrm{e}^{-2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\mathrm{e}^{-2t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}+3\cos\left(\sqrt{3}\,t\right)\right)}{3} & -\frac{2\,\mathrm{e}^{-2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} \\ \frac{2\,\mathrm{e}^{-2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} & -\frac{\mathrm{e}^{-2t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}-3\cos\left(\sqrt{3}\,t\right)\right)}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-2t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}+3\cos\left(\sqrt{3}\,t\right)\right)}{3} & -\frac{2\,e^{-2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} \\ \frac{2\,e^{-2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} & -\frac{e^{-2t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}-3\cos\left(\sqrt{3}\,t\right)\right)}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-2t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}+3\cos\left(\sqrt{3}\,t\right)\right)c_1}{3} - \frac{2\,e^{-2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}\,c_2}{3} \\ \frac{2\,e^{-2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}\,c_1}{3} - \frac{e^{-2t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}-3\cos\left(\sqrt{3}\,t\right)\right)c_2}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\left(\sqrt{3}\,(c_1-2c_2)\sin\left(\sqrt{3}\,t\right)+3\cos\left(\sqrt{3}\,t\right)c_1\right)e^{-2t}}{3} \\ \frac{2\,e^{-2t}\left(\sqrt{3}\,\left(c_1-\frac{c_2}{2}\right)\sin\left(\sqrt{3}\,t\right)+\frac{3\cos\left(\sqrt{3}\,t\right)c_2}{2}\right)}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -2 \\ 2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & -2 \\ 2 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 4\lambda + 7 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i\sqrt{3} - 2$$
$$\lambda_2 = -2 - i\sqrt{3}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-2 - i\sqrt{3}$	1	complex eigenvalue
$i\sqrt{3} - 2$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2 - i\sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -2 \\ 2 & -3 \end{bmatrix} - \left(-2 - i\sqrt{3}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 + i\sqrt{3} & -2 \\ 2 & i\sqrt{3} - 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 + i\sqrt{3} & -2 & 0 \\ 2 & i\sqrt{3} - 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{1 + i\sqrt{3}} \Longrightarrow \left[\begin{array}{cc|c} 1 + i\sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 + i\sqrt{3} & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{1+i\sqrt{3}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{1+i\sqrt{3}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i\sqrt{3} - 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -2 \\ 2 & -3 \end{bmatrix} - \left(i\sqrt{3} - 2\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 - i\sqrt{3} & -2 \\ 2 & -i\sqrt{3} - 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 - i\sqrt{3} & -2 & 0 \\ 2 & -i\sqrt{3} - 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{1 - i\sqrt{3}} \Longrightarrow \left[\begin{array}{cc|c} 1 - i\sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 - i\sqrt{3} & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{i\sqrt{3}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{i\sqrt{3}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$i\sqrt{3}-2$	1	1	No	$\left[\begin{array}{c} -\frac{2}{i\sqrt{3}-1} \\ 1 \end{array}\right]$
$-2-i\sqrt{3}$	1	1	No	$\left[\begin{array}{c} -\frac{2}{-i\sqrt{3}-1} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = c_1 \left[\begin{array}{c} -\frac{2\,\mathrm{e}^{\left(i\sqrt{3}-2\right)t}}{i\sqrt{3}-1} \\ \mathrm{e}^{\left(i\sqrt{3}-2\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{2\,\mathrm{e}^{\left(-2-i\sqrt{3}\right)t}}{-i\sqrt{3}-1} \\ \mathrm{e}^{\left(-2-i\sqrt{3}\right)t} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} -\frac{ic_2\left(\sqrt{3}+i\right)\mathrm{e}^{-\left(2+i\sqrt{3}\right)t}}{2} - \frac{i\mathrm{e}^{\left(i\sqrt{3}-2\right)t}c_1\left(i-\sqrt{3}\right)}{2} \\ c_1\mathrm{e}^{\left(i\sqrt{3}-2\right)t} + c_2\mathrm{e}^{-\left(2+i\sqrt{3}\right)t} \end{array}\right]$$

The following is the phase plot of the system.

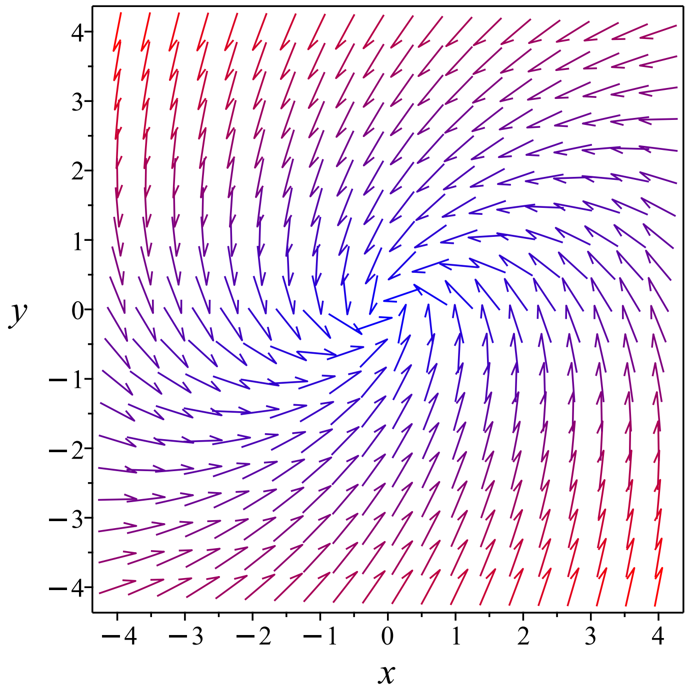

Figure 436: Phase plot

20.3.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) - 2y(t), y'(t) = 2x(t) - 3y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & -2 \\ 2 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & -2 \\ 2 & -3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & -2 \\ 2 & -3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

  $$\left[\left[-2 - \mathrm{I}\sqrt{3}, \begin{bmatrix} -\frac{2}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{3} - 2, \begin{bmatrix} -\frac{2}{\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[-2 - \mathrm{I}\sqrt{3}, \begin{bmatrix} -\frac{2}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

  $$\mathrm{e}^{\left(-2-\mathrm{I}\sqrt{3}\right)t} \cdot \begin{bmatrix} -\frac{2}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

  $$\mathrm{e}^{-2t} \cdot \left(\cos\left(\sqrt{3}\,t\right) - \mathrm{I}\sin\left(\sqrt{3}\,t\right)\right) \cdot \begin{bmatrix} -\frac{2}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}$$

- Simplify expression

  $$\mathrm{e}^{-2t} \cdot \begin{bmatrix} -\frac{2\left(\cos\left(\sqrt{3}\,t\right) - \mathrm{I}\sin\left(\sqrt{3}\,t\right)\right)}{-\mathrm{I}\sqrt{3}-1} \\ \cos\left(\sqrt{3}\,t\right) - \mathrm{I}\sin\left(\sqrt{3}\,t\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\vec{x}_1(t) = \mathrm{e}^{-2t} \cdot \begin{bmatrix} \frac{\cos\left(\sqrt{3}\,t\right)}{2} - \frac{\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{2} \\ \cos\left(\sqrt{3}\,t\right) \end{bmatrix}, \vec{x}_2(t) = \mathrm{e}^{-2t} \cdot \begin{bmatrix} -\frac{\cos\left(\sqrt{3}\,t\right)\sqrt{3}}{2} - \frac{\sin\left(\sqrt{3}\,t\right)}{2} \\ -\sin\left(\sqrt{3}\,t\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{-2t} \cdot \left[\begin{array}{c} \frac{\cos\left(\sqrt{3}\,t\right)}{2} - \frac{\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{2} \\ \cos\left(\sqrt{3}\,t\right) \end{array}\right] + c_2 \mathrm{e}^{-2t} \cdot \left[\begin{array}{c} -\frac{\cos\left(\sqrt{3}\,t\right)\sqrt{3}}{2} - \frac{\sin\left(\sqrt{3}\,t\right)}{2} \\ -\sin\left(\sqrt{3}\,t\right) \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} -\frac{\mathrm{e}^{-2t}\left(\left(c_2\sqrt{3}-c_1\right)\cos\left(\sqrt{3}\,t\right)+\sin\left(\sqrt{3}\,t\right)\left(\sqrt{3}\,c_1+c_2\right)\right)}{2} \\ \mathrm{e}^{-2t}\left(\cos\left(\sqrt{3}\,t\right)c_1 - \sin\left(\sqrt{3}\,t\right)c_2\right) \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{\mathrm{e}^{-2t}\left(\left(c_2\sqrt{3}-c_1\right)\cos\left(\sqrt{3}\,t\right)+\sin\left(\sqrt{3}\,t\right)\left(\sqrt{3}\,c_1+c_2\right)\right)}{2}, y(t) = \mathrm{e}^{-2t}\left(\cos\left(\sqrt{3}\,t\right)c_1 - \sin\left(\sqrt{3}\,t\right)c_2\right)\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 76

```
dsolve([diff(x(t),t)=-x(t)-2*y(t),diff(y(t),t)=2*x(t)-3*y(t)],singsol=all)
```

$$x(t) = \mathrm{e}^{-2t}\left(\sin\left(\sqrt{3}\,t\right)c_1 + \cos\left(\sqrt{3}\,t\right)c_2\right)$$

$$y(t) = \frac{\mathrm{e}^{-2t}\left(\sqrt{3}\,\sin\left(\sqrt{3}\,t\right)c_2 - \sqrt{3}\,\cos\left(\sqrt{3}\,t\right)c_1 + \sin\left(\sqrt{3}\,t\right)c_1 + \cos\left(\sqrt{3}\,t\right)c_2\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 96

```
DSolve[{x'[t]==-x[t]-2*y[t],y'[t]==2*x[t]-3*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{1}{3}e^{-2t}\left(3c_1\cos\left(\sqrt{3}t\right) + \sqrt{3}(c_1 - 2c_2)\sin\left(\sqrt{3}t\right)\right)$$

$$y(t) \to \frac{1}{3}e^{-2t}\left(3c_2\cos\left(\sqrt{3}t\right) + \sqrt{3}(2c_1 - c_2)\sin\left(\sqrt{3}t\right)\right)$$

20.4 problem 4

Internal problem ID [12860]
Internal file name [`OUTPUT/11512_Monday_November_06_2023_01_31_12_PM_39576688/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 10. Applications of Systems of Equations. Exercises 10.2 page 432
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -x(t) - 2y(t)$$
$$y'(t) = 5x(t) + y(t)$$

20.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(3t) - \frac{\sin(3t)}{3} & -\frac{2\sin(3t)}{3} \\ \frac{5\sin(3t)}{3} & \cos(3t) + \frac{\sin(3t)}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(3t) - \frac{\sin(3t)}{3} & -\frac{2\sin(3t)}{3} \\ \frac{5\sin(3t)}{3} & \cos(3t) + \frac{\sin(3t)}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\cos(3t) - \frac{\sin(3t)}{3}\right) c_1 - \frac{2\sin(3t)c_2}{3} \\ \frac{5\sin(3t)c_1}{3} + \left(\cos(3t) + \frac{\sin(3t)}{3}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(-c_1-2c_2)\sin(3t)}{3} + c_1\cos(3t) \\ \frac{(5c_1+c_2)\sin(3t)}{3} + c_2\cos(3t) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -2 \\ 5 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & -2 \\ 5 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$3i$	1	complex eigenvalue
$-3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -2 \\ 5 & 1 \end{bmatrix} - (-3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1+3i & -2 \\ 5 & 1+3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1+3i & -2 & 0 \\ 5 & 1+3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{2} + \frac{3i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1+3i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+3i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{5} - \frac{3i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{5} - \frac{3i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{5} - \frac{3i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} - \frac{3i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{5} - \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1 - 3i \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -2 \\ 5 & 1 \end{bmatrix} - (3i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 - 3i & -2 \\ 5 & 1 - 3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 - 3i & -2 & 0 \\ 5 & 1 - 3i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{1}{2} - \frac{3i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} -1-3i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-3i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{5} + \frac{3i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{5} + \frac{3i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{5} + \frac{3i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} + \frac{3i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{5} + \frac{3\,\mathrm{I}}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1+3i \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
$3i$	1	1	No	$\left[\begin{array}{c} -\frac{1}{5}+\frac{3i}{5} \\ 1 \end{array}\right]$
$-3i$	1	1	No	$\left[\begin{array}{c} -\frac{1}{5}-\frac{3i}{5} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = c_1 \left[\begin{array}{c} \left(-\frac{1}{5}+\frac{3i}{5}\right) \mathrm{e}^{3it} \\ \mathrm{e}^{3it} \end{array}\right] + c_2 \left[\begin{array}{c} \left(-\frac{1}{5}-\frac{3i}{5}\right) \mathrm{e}^{-3it} \\ \mathrm{e}^{-3it} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} \left(-\frac{1}{5}+\frac{3i}{5}\right) c_1 \mathrm{e}^{3it} + \left(-\frac{1}{5}-\frac{3i}{5}\right) c_2 \mathrm{e}^{-3it} \\ c_1 \mathrm{e}^{3it} + c_2 \mathrm{e}^{-3it} \end{array}\right]$$

The following is the phase plot of the system.

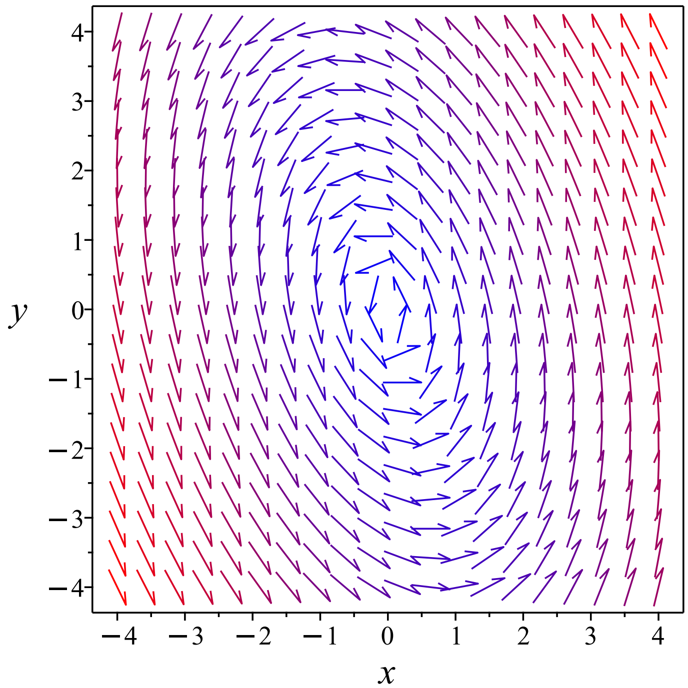

Figure 437: Phase plot

20.4.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) - 2y(t), y'(t) = 5x(t) + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & -2 \\ 5 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & -2 \\ 5 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & -2 \\ 5 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{5} - \frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right], \left[3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{5} + \frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{5} - \frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-3\,\mathrm{I}t} \cdot \begin{bmatrix} -\frac{1}{5} - \frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos(3t) - \mathrm{I}\sin(3t)\right) \cdot \begin{bmatrix} -\frac{1}{5} - \frac{3\,\mathrm{I}}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} \left(-\frac{1}{5} - \frac{3\,\mathrm{I}}{5}\right)\left(\cos(3t) - \mathrm{I}\sin(3t)\right) \\ \cos(3t) - \mathrm{I}\sin(3t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \begin{bmatrix} -\frac{\cos(3t)}{5} - \frac{3\sin(3t)}{5} \\ \cos(3t) \end{bmatrix}, \vec{x}_2(t) = \begin{bmatrix} \frac{\sin(3t)}{5} - \frac{3\cos(3t)}{5} \\ -\sin(3t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = \begin{bmatrix} c_2\left(\frac{\sin(3t)}{5} - \frac{3\cos(3t)}{5}\right) + c_1\left(-\frac{\cos(3t)}{5} - \frac{3\sin(3t)}{5}\right) \\ c_1 \cos(3t) - c_2 \sin(3t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{(-c_1-3c_2)\cos(3t)}{5} - \frac{3\left(c_1-\frac{c_2}{3}\right)\sin(3t)}{5} \\ c_1 \cos(3t) - c_2 \sin(3t) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{(-c_1-3c_2)\cos(3t)}{5} - \frac{3\left(c_1-\frac{c_2}{3}\right)\sin(3t)}{5}, y(t) = c_1 \cos(3t) - c_2 \sin(3t)\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 50

```
dsolve([diff(x(t),t)=-x(t)-2*y(t),diff(y(t),t)=5*x(t)+1*y(t)],singsol=all)
```

$$x(t) = c_1 \sin(3t) + c_2 \cos(3t)$$
$$y(t) = -\frac{3c_1 \cos(3t)}{2} + \frac{3c_2 \sin(3t)}{2} - \frac{c_1 \sin(3t)}{2} - \frac{c_2 \cos(3t)}{2}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 54

```
DSolve[{x'[t]==-x[t]-2*y[t],y'[t]==5*x[t]+1*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to c_1 \cos(3t) - \frac{1}{3}(c_1 + 2c_2)\sin(3t)$$
$$y(t) \to c_2 \cos(3t) + \frac{1}{3}(5c_1 + c_2)\sin(3t)$$

20.5 problem 5

Internal problem ID [12861]
Internal file name [`OUTPUT/11513_Monday_November_06_2023_01_31_13_PM_44578035/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 10. Applications of Systems of Equations. Exercises 10.2 page 432
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -x(t) + 2y(t) \\
y'(t) &= -2x(t) - y(t)
\end{aligned}
$$

20.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\left[\begin{array}{c} x'(t) \\ y'(t) \end{array}\right] = \left[\begin{array}{cc} -1 & 2 \\ -2 & -1 \end{array}\right] \left[\begin{array}{c} x(t) \\ y(t) \end{array}\right]
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \left[\begin{array}{cc} \mathrm{e}^{-t}\cos(2t) & \mathrm{e}^{-t}\sin(2t) \\ -\mathrm{e}^{-t}\sin(2t) & \mathrm{e}^{-t}\cos(2t) \end{array}\right]
$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t}\cos(2t) & e^{-t}\sin(2t) \\ -e^{-t}\sin(2t) & e^{-t}\cos(2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}\cos(2t)\, c_1 + e^{-t}\sin(2t)\, c_2 \\ -e^{-t}\sin(2t)\, c_1 + e^{-t}\cos(2t)\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}(\cos(2t)\, c_1 + \sin(2t)\, c_2) \\ e^{-t}(-\sin(2t)\, c_1 + \cos(2t)\, c_2) \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{bmatrix} \right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1 + 2i$$
$$\lambda_2 = -1 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-1 - 2i$	1	complex eigenvalue
$-1 + 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} - (-1 - 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2i & 2 & 0 \\ -2 & 2i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} - (-1 + 2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$-1+2i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$-1-2i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -i\mathrm{e}^{(-1+2i)t} \\ \mathrm{e}^{(-1+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} i\mathrm{e}^{(-1-2i)t} \\ \mathrm{e}^{(-1-2i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -i\left(c_1\mathrm{e}^{(-1+2i)t} - c_2\mathrm{e}^{(-1-2i)t}\right) \\ c_1\mathrm{e}^{(-1+2i)t} + c_2\mathrm{e}^{(-1-2i)t} \end{bmatrix}$$

The following is the phase plot of the system.

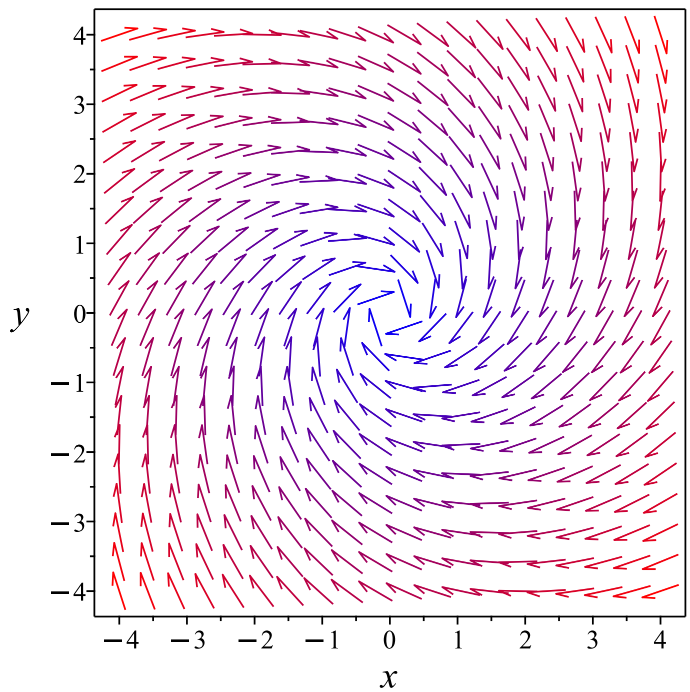

Figure 438: Phase plot

20.5.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t) + 2y(t), y'(t) = -2x(t) - y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1-2\,\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right], \left[-1+2\,\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-2\,\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-2\,\mathrm{I})t} \cdot \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-t} \cdot \begin{bmatrix} \mathrm{I}(\cos(2t) - \mathrm{I}\sin(2t)) \\ \cos(2t) - \mathrm{I}\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = e^{-t} \cdot \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}, \vec{x}_2(t) = e^{-t} \cdot \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-t} \cdot \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 e^{-t} \cdot \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{-t}(c_1\sin(2t) + c_2\cos(2t)) \\ e^{-t}(c_1\cos(2t) - c_2\sin(2t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = e^{-t}(c_1\sin(2t) + c_2\cos(2t)), y(t) = e^{-t}(c_1\cos(2t) - c_2\sin(2t))\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
dsolve([diff(x(t),t)=-x(t)+2*y(t),diff(y(t),t)=-2*x(t)-1*y(t)],singsol=all)
```

$$x(t) = e^{-t}(c_1\sin(2t) + c_2\cos(2t))$$
$$y(t) = e^{-t}(c_1\cos(2t) - c_2\sin(2t))$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 51

```
DSolve[{x'[t]==-x[t]+2*y[t],y'[t]==-2*x[t]-1*y[t]},{x[t],y[t]},t,IncludeSingularSolutions ->
```

$$x(t) \to e^{-t}(c_1\cos(2t) + c_2\sin(2t))$$
$$y(t) \to e^{-t}(c_2\cos(2t) - c_1\sin(2t))$$

20.6 problem 6

Internal problem ID [12862]
Internal file name [`OUTPUT/11514_Monday_November_06_2023_01_31_13_PM_87283299/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 10. Applications of Systems of Equations. Exercises 10.2 page 432
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) - 2y(t) \\
y'(t) &= 2x(t) + y(t)
\end{aligned}
$$

20.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{t} \cos(2t) & -\mathrm{e}^{t} \sin(2t) \\ \mathrm{e}^{t} \sin(2t) & \mathrm{e}^{t} \cos(2t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t \cos(2t) & -e^t \sin(2t) \\ e^t \sin(2t) & e^t \cos(2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t \cos(2t) c_1 - e^t \sin(2t) c_2 \\ e^t \sin(2t) c_1 + e^t \cos(2t) c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t(\cos(2t) c_1 - \sin(2t) c_2) \\ e^t(\sin(2t) c_1 + \cos(2t) c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1 + 2i$	1	complex eigenvalue
$1 - 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} - (1 - 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2i & -2 \\ 2 & 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2i & -2 & 0 \\ 2 & 2i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} - (1 + 2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2i & -2 & 0 \\ 2 & -2i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$1 + 2i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
$1 - 2i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} i\mathrm{e}^{(1+2i)t} \\ \mathrm{e}^{(1+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} -i\mathrm{e}^{(1-2i)t} \\ \mathrm{e}^{(1-2i)t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} i\left(c_1\mathrm{e}^{(1+2i)t} - c_2\mathrm{e}^{(1-2i)t}\right) \\ c_1\mathrm{e}^{(1+2i)t} + c_2\mathrm{e}^{(1-2i)t} \end{bmatrix}$$

The following is the phase plot of the system.

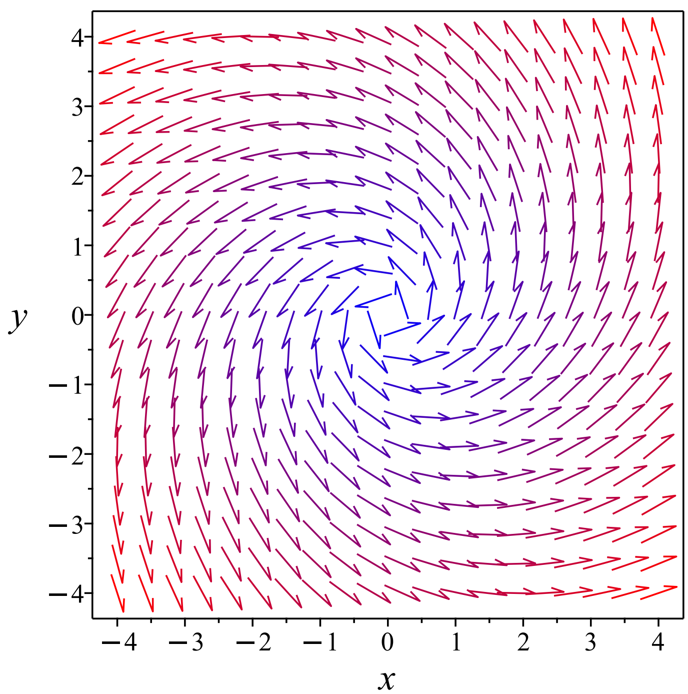

Figure 439: Phase plot

20.6.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) - 2y(t), y'(t) = 2x(t) + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1-2\,\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[1+2\,\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-2\,\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(1-2\,\mathrm{I})t} \cdot \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{t} \cdot (\cos(2t) - \mathrm{I}\sin(2t)) \cdot \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^t \cdot \begin{bmatrix} -I(\cos(2t) - I\sin(2t)) \\ \cos(2t) - I\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = e^t \cdot \begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix}, \vec{x}_2(t) = e^t \cdot \begin{bmatrix} -\cos(2t) \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^t \cdot \begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix} + c_2 e^t \cdot \begin{bmatrix} -\cos(2t) \\ -\sin(2t) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -e^t(c_1\sin(2t) + c_2\cos(2t)) \\ e^t(c_1\cos(2t) - c_2\sin(2t)) \end{bmatrix}$$

- Solution to the system of ODEs

$$\{x(t) = -e^t(c_1\sin(2t) + c_2\cos(2t)), y(t) = e^t(c_1\cos(2t) - c_2\sin(2t))\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 42

```
dsolve([diff(x(t),t)=x(t)-2*y(t),diff(y(t),t)=2*x(t)+1*y(t)],singsol=all)
```

$$x(t) = e^t(c_1\sin(2t) + c_2\cos(2t))$$
$$y(t) = -e^t(c_1\cos(2t) - c_2\sin(2t))$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 47

```
DSolve[{x'[t]==x[t]-2*y[t],y'[t]==2*x[t]+1*y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to e^t(c_1\cos(2t) - c_2\sin(2t))$$
$$y(t) \to e^t(c_2\cos(2t) + c_1\sin(2t))$$

20.7 problem 7

Internal problem ID [12863]
Internal file name [`OUTPUT/11515_Monday_November_06_2023_01_31_13_PM_3281850/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 10. Applications of Systems of Equations. Exercises 10.2 page 432
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -5x(t) - y(t) + 2 \\
y'(t) &= 3x(t) - y(t) - 3
\end{aligned}
$$

20.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{3\,e^{-4t}}{2} - \frac{e^{-2t}}{2} & -\frac{e^{-2t}}{2} + \frac{e^{-4t}}{2} \\ \frac{3\,e^{-2t}}{2} - \frac{3\,e^{-4t}}{2} & -\frac{e^{-4t}}{2} + \frac{3\,e^{-2t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{3\,e^{-4t}}{2} - \frac{e^{-2t}}{2} & -\frac{e^{-2t}}{2} + \frac{e^{-4t}}{2} \\ \frac{3\,e^{-2t}}{2} - \frac{3\,e^{-4t}}{2} & -\frac{e^{-4t}}{2} + \frac{3\,e^{-2t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{3\,e^{-4t}}{2} - \frac{e^{-2t}}{2}\right) c_1 + \left(-\frac{e^{-2t}}{2} + \frac{e^{-4t}}{2}\right) c_2 \\ \left(\frac{3\,e^{-2t}}{2} - \frac{3\,e^{-4t}}{2}\right) c_1 + \left(-\frac{e^{-4t}}{2} + \frac{3\,e^{-2t}}{2}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(3c_1+c_2)e^{-4t}}{2} - \frac{e^{-2t}(c_1+c_2)}{2} \\ \frac{(-3c_1-c_2)e^{-4t}}{2} + \frac{3\,e^{-2t}(c_1+c_2)}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} -\frac{e^{2t}}{2} + \frac{3\,e^{4t}}{2} & \frac{(e^{2t}-1)e^{2t}}{2} \\ -\frac{3(e^{2t}-1)e^{2t}}{2} & -\frac{(e^{2t}-3)e^{2t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{3\,e^{-4t}}{2} - \frac{e^{-2t}}{2} & -\frac{e^{-2t}}{2} + \frac{e^{-4t}}{2} \\ \frac{3\,e^{-2t}}{2} - \frac{3\,e^{-4t}}{2} & -\frac{e^{-4t}}{2} + \frac{3\,e^{-2t}}{2} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{2t}}{2} + \frac{3\,e^{4t}}{2} & \frac{(e^{2t}-1)e^{2t}}{2} \\ -\frac{3(e^{2t}-1)e^{2t}}{2} & -\frac{(e^{2t}-3)e^{2t}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{3\,e^{-4t}}{2} - \frac{e^{-2t}}{2} & -\frac{e^{-2t}}{2} + \frac{e^{-4t}}{2} \\ \frac{3\,e^{-2t}}{2} - \frac{3\,e^{-4t}}{2} & -\frac{e^{-4t}}{2} + \frac{3\,e^{-2t}}{2} \end{bmatrix} \begin{bmatrix} \frac{3\,e^{4t}}{8} + \frac{e^{2t}}{4} \\ -\frac{3\,e^{4t}}{8} - \frac{3\,e^{2t}}{4} \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{8} \\ -\frac{9}{8} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{5}{8} + \frac{(3c_1+c_2)\mathrm{e}^{-4t}}{2} + \frac{(-c_1-c_2)\mathrm{e}^{-2t}}{2} \\ \frac{(-3c_1-c_2)\mathrm{e}^{-4t}}{2} + \frac{3\,\mathrm{e}^{-2t}(c_1+c_2)}{2} - \frac{9}{8} \end{bmatrix}$$

20.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -5-\lambda & -1 \\ 3 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 6\lambda + 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -4$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
-4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} - (-4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + 3R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & -1 & 0 \\ 3 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \implies \left[\begin{array}{cc|c} -3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-4	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -4 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-4t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-4t}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{-2t}$$
$$= \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} e^{-2t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-4t} \\ \mathrm{e}^{-4t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{\mathrm{e}^{-2t}}{3} \\ \mathrm{e}^{-2t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\mathrm{e}^{-4t} & -\frac{\mathrm{e}^{-2t}}{3} \\ \mathrm{e}^{-4t} & \mathrm{e}^{-2t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{3\,\mathrm{e}^{4t}}{2} & -\frac{\mathrm{e}^{4t}}{2} \\ \frac{3\,\mathrm{e}^{2t}}{2} & \frac{3\,\mathrm{e}^{2t}}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -\mathrm{e}^{-4t} & -\frac{\mathrm{e}^{-2t}}{3} \\ \mathrm{e}^{-4t} & \mathrm{e}^{-2t} \end{bmatrix} \int \begin{bmatrix} -\frac{3\,\mathrm{e}^{4t}}{2} & -\frac{\mathrm{e}^{4t}}{2} \\ \frac{3\,\mathrm{e}^{2t}}{2} & \frac{3\,\mathrm{e}^{2t}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} dt$$

$$= \begin{bmatrix} -\mathrm{e}^{-4t} & -\frac{\mathrm{e}^{-2t}}{3} \\ \mathrm{e}^{-4t} & \mathrm{e}^{-2t} \end{bmatrix} \int \begin{bmatrix} -\frac{3\,\mathrm{e}^{4t}}{2} \\ -\frac{3\,\mathrm{e}^{2t}}{2} \end{bmatrix} dt$$

$$= \begin{bmatrix} -\mathrm{e}^{-4t} & -\frac{\mathrm{e}^{-2t}}{3} \\ \mathrm{e}^{-4t} & \mathrm{e}^{-2t} \end{bmatrix} \begin{bmatrix} -\frac{3\,\mathrm{e}^{4t}}{8} \\ -\frac{3\,\mathrm{e}^{2t}}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{8} \\ -\frac{9}{8} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1\mathrm{e}^{-4t} \\ c_1\mathrm{e}^{-4t} \end{bmatrix} + \begin{bmatrix} -\frac{c_2\mathrm{e}^{-2t}}{3} \\ c_2\mathrm{e}^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{5}{8} \\ -\frac{9}{8} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1\mathrm{e}^{-4t} - \frac{c_2\mathrm{e}^{-2t}}{3} + \frac{5}{8} \\ c_1\mathrm{e}^{-4t} + c_2\mathrm{e}^{-2t} - \frac{9}{8} \end{bmatrix}$$

The following is the phase plot of the system.

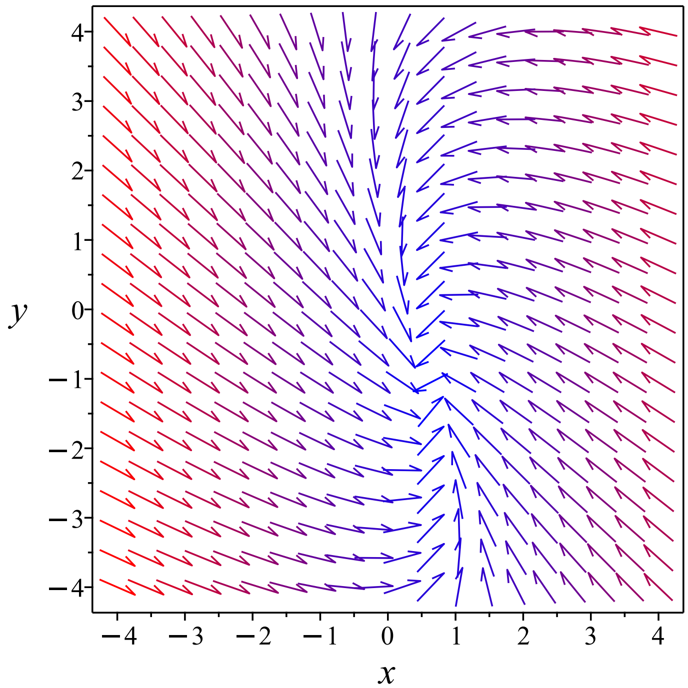

Figure 440: Phase plot

20.7.3 Maple step by step solution

Let's solve

$[x'(t) = -5x(t) - y(t) + 2, y'(t) = 3x(t) - y(t) - 3]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-4, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-4t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{-2t} \cdot \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \vec{x}_p(t)$$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -\mathrm{e}^{-4t} & -\frac{\mathrm{e}^{-2t}}{3} \\ \mathrm{e}^{-4t} & \mathrm{e}^{-2t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $$\Phi(t) = \begin{bmatrix} -\mathrm{e}^{-4t} & -\frac{\mathrm{e}^{-2t}}{3} \\ \mathrm{e}^{-4t} & \mathrm{e}^{-2t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & -\frac{1}{3} \\ 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(t) = \begin{bmatrix} \frac{3\,\mathrm{e}^{-4t}}{2} - \frac{\mathrm{e}^{-2t}}{2} & -\frac{\mathrm{e}^{-2t}}{2} + \frac{\mathrm{e}^{-4t}}{2} \\ \frac{3\,\mathrm{e}^{-2t}}{2} - \frac{3\,\mathrm{e}^{-4t}}{2} & -\frac{\mathrm{e}^{-4t}}{2} + \frac{3\,\mathrm{e}^{-2t}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} -\frac{3\,\mathrm{e}^{-4t}}{8} + \frac{5}{8} - \frac{\mathrm{e}^{-2t}}{4} \\ \frac{3\,\mathrm{e}^{-2t}}{4} - \frac{9}{8} + \frac{3\,\mathrm{e}^{-4t}}{8} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \begin{bmatrix} -\frac{3\,\mathrm{e}^{-4t}}{8} + \frac{5}{8} - \frac{\mathrm{e}^{-2t}}{4} \\ \frac{3\,\mathrm{e}^{-2t}}{4} - \frac{9}{8} + \frac{3\,\mathrm{e}^{-4t}}{8} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{(-24c_1-9)\mathrm{e}^{-4t}}{24} + \frac{5}{8} + \frac{(-8c_2-6)\mathrm{e}^{-2t}}{24} \\ \frac{(3+8c_1)\mathrm{e}^{-4t}}{8} - \frac{9}{8} + \frac{(8c_2+6)\mathrm{e}^{-2t}}{8} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{(-24c_1-9)\mathrm{e}^{-4t}}{24} + \frac{5}{8} + \frac{(-8c_2-6)\mathrm{e}^{-2t}}{24}, y(t) = \frac{(3+8c_1)\mathrm{e}^{-4t}}{8} - \frac{9}{8} + \frac{(8c_2+6)\mathrm{e}^{-2t}}{8}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 39

```
dsolve([diff(x(t),t)=-5*x(t)-y(t)+2,diff(y(t),t)=3*x(t)-1*y(t)-3],singsol=all)
```

$$x(t) = \frac{5}{8} - \frac{\mathrm{e}^{-4t} c_1}{2} + c_2 \mathrm{e}^{-2t}$$
$$y(t) = \frac{\mathrm{e}^{-4t} c_1}{2} - 3c_2 \mathrm{e}^{-2t} - \frac{9}{8}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 93

```
DSolve[{x'[t]==-5*x[t]-y[t]+2,y'[t]==3*x[t]-1*y[t]-3},{x[t],y[t]},t,IncludeSingularSolutions
```

$$x(t) \to \frac{1}{48} e^{-4t} \left(30e^{4t} - (1 + 24c_1 + 24c_2)e^{2t} + 3 + 72c_1 + 24c_2\right)$$
$$y(t) \to \frac{1}{16} e^{-4t} \left(-18e^{4t} + (1 + 24c_1 + 24c_2)e^{2t} - 1 - 24c_1 - 8c_2\right)$$

20.8 problem 8

Internal problem ID [12864]
Internal file name [`OUTPUT/11516_Monday_November_06_2023_01_31_14_PM_70028064/index.tex`]

Book: Ordinary Differential Equations by Charles E. Roberts, Jr. CRC Press. 2010
Section: Chapter 10. Applications of Systems of Equations. Exercises 10.2 page 432
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 3x(t) - 2y(t) - 6 \\
y'(t) &= 4x(t) - y(t) + 2
\end{aligned}
$$

20.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^t \cos(2t) + e^t \sin(2t) & -e^t \sin(2t) \\ 2\,e^t \sin(2t) & e^t \cos(2t) - e^t \sin(2t) \end{bmatrix} \\
&= \begin{bmatrix} e^t(\sin(2t) + \cos(2t)) & -e^t \sin(2t) \\ 2\,e^t \sin(2t) & e^t(\cos(2t) - \sin(2t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t(\sin(2t) + \cos(2t)) & -e^t \sin(2t) \\ 2\,e^t \sin(2t) & e^t(\cos(2t) - \sin(2t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t(\sin(2t) + \cos(2t))\,c_1 - e^t \sin(2t)\,c_2 \\ 2\,e^t \sin(2t)\,c_1 + e^t(\cos(2t) - \sin(2t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^t((c_1 - c_2)\sin(2t) + c_1 \cos(2t)) \\ e^t(2c_1 - c_2)\sin(2t) + e^t \cos(2t)\,c_2 \end{bmatrix}
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\,dt
$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} e^{-t}(\cos(2t) - \sin(2t)) & e^{-t}\sin(2t) \\ -2\,e^{-t}\sin(2t) & e^{-t}(\sin(2t) + \cos(2t)) \end{bmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^t(\sin(2t) + \cos(2t)) & -e^t \sin(2t) \\ 2\,e^t \sin(2t) & e^t(\cos(2t) - \sin(2t)) \end{bmatrix} \int \begin{bmatrix} e^{-t}(\cos(2t) - \sin(2t)) & e^{-t}\sin(2t) \\ -2\,e^{-t}\sin(2t) & e^{-t}(\sin(2t) + \cos(\end{bmatrix} \\
&= \begin{bmatrix} e^t(\sin(2t) + \cos(2t)) & -e^t \sin(2t) \\ 2\,e^t \sin(2t) & e^t(\cos(2t) - \sin(2t)) \end{bmatrix} \begin{bmatrix} -2(2\sin(2t) + \cos(2t))\,e^{-t} \\ -2(3\cos(2t) + \sin(2t))\,e^{-t} \end{bmatrix} \\
&= \begin{bmatrix} -2 \\ -6 \end{bmatrix}
\end{aligned}
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} e^t(c_1 - c_2)\sin(2t) + e^t\cos(2t)\,c_1 - 2 \\ 2\left(c_1 - \frac{c_2}{2}\right)e^t\sin(2t) + e^t\cos(2t)\,c_2 - 6 \end{bmatrix}
\end{aligned}
$$

20.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda + 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1 + 2i$	1	complex eigenvalue
$1 - 2i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1 - 2i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 + 2i & -2 \\ 4 & -2 + 2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 + 2i & -2 & 0 \\ 4 & -2 + 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 + i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 + 2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 + 2i & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} - \frac{i}{2}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 - i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + 2i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1 + 2i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - 2i & -2 \\ 4 & -2 - 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 - 2i & -2 & 0 \\ 4 & -2 - 2i & 0 \end{array}\right]$$

$$R_2 = R_2 + (-1 - i)\, R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 - 2i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 - 2i & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} 1 + i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$1 + 2i$	1	1	No	$\begin{bmatrix} \frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$
$1 - 2i$	1	1	No	$\begin{bmatrix} \frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) e^{(1+2i)t} \\ e^{(1+2i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) e^{(1-2i)t} \\ e^{(1-2i)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) e^{(1+2i)t} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{(1-2i)t} \\ e^{(1+2i)t} & e^{(1-2i)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -i e^{(-1-2i)t} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{(-1-2i)t} \\ i e^{(-1+2i)t} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{(-1+2i)t} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) e^{(1+2i)t} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{(1-2i)t} \\ e^{(1+2i)t} & e^{(1-2i)t} \end{bmatrix} \int \begin{bmatrix} -i e^{(-1-2i)t} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{(-1-2i)t} \\ i e^{(-1+2i)t} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{(-1+2i)t} \end{bmatrix} \begin{bmatrix} -6 \\ 2 \end{bmatrix} dt \\
&= \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) e^{(1+2i)t} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{(1-2i)t} \\ e^{(1+2i)t} & e^{(1-2i)t} \end{bmatrix} \int \begin{bmatrix} (1+7i) e^{(-1-2i)t} \\ (1-7i) e^{(-1+2i)t} \end{bmatrix} dt \\
&= \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) e^{(1+2i)t} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{(1-2i)t} \\ e^{(1+2i)t} & e^{(1-2i)t} \end{bmatrix} \begin{bmatrix} (-3-i) e^{(-1-2i)t} \\ (-3+i) e^{(-1+2i)t} \end{bmatrix} \\
&= \begin{bmatrix} -2 \\ -6 \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) c_1 \mathrm{e}^{(1+2i)t} \\ c_1 \mathrm{e}^{(1+2i)t} \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{2}-\frac{i}{2}\right) c_2 \mathrm{e}^{(1-2i)t} \\ c_2 \mathrm{e}^{(1-2i)t} \end{bmatrix} + \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}+\frac{i}{2}\right) c_1 \mathrm{e}^{(1+2i)t} + \left(\frac{1}{2}-\frac{i}{2}\right) c_2 \mathrm{e}^{(1-2i)t} - 2 \\ c_1 \mathrm{e}^{(1+2i)t} + c_2 \mathrm{e}^{(1-2i)t} - 6 \end{bmatrix}$$

The following is the phase plot of the system.

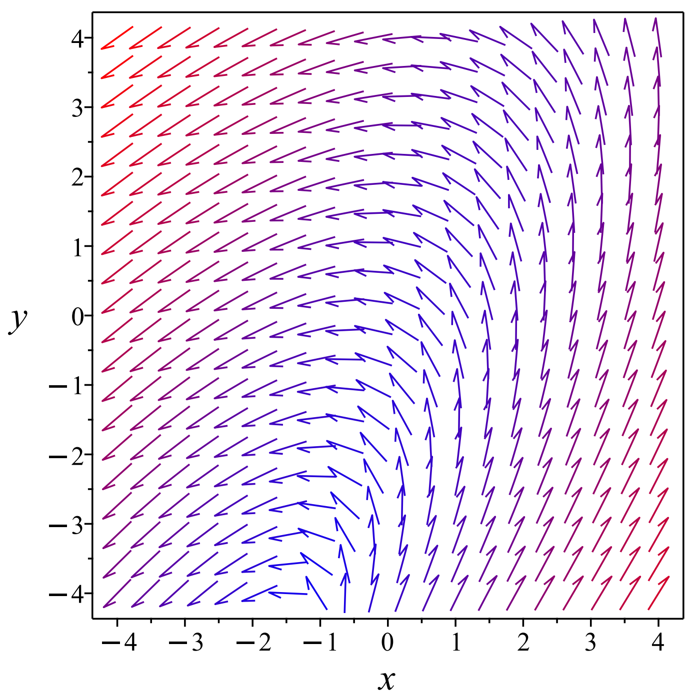

Figure 441: Phase plot

20.8.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t) - 2y(t) - 6, y'(t) = 4x(t) - y(t) + 2]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1 - 2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1 + 2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - 2\,\mathrm{I}, \begin{bmatrix} \frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(1-2\,\mathrm{I})t} \cdot \begin{bmatrix} \frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^t \cdot (\cos(2t) - I\sin(2t)) \cdot \begin{bmatrix} \frac{1}{2} - \frac{I}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^t \cdot \begin{bmatrix} \left(\frac{1}{2} - \frac{I}{2}\right)(\cos(2t) - I\sin(2t)) \\ \cos(2t) - I\sin(2t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = e^t \cdot \begin{bmatrix} \frac{\cos(2t)}{2} - \frac{\sin(2t)}{2} \\ \cos(2t) \end{bmatrix}, \vec{x}_2(t) = e^t \cdot \begin{bmatrix} -\frac{\sin(2t)}{2} - \frac{\cos(2t)}{2} \\ -\sin(2t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_p(t)$$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} e^t\left(\frac{\cos(2t)}{2} - \frac{\sin(2t)}{2}\right) & e^t\left(-\frac{\sin(2t)}{2} - \frac{\cos(2t)}{2}\right) \\ e^t\cos(2t) & -e^t\sin(2t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} e^t\left(\frac{\cos(2t)}{2} - \frac{\sin(2t)}{2}\right) & e^t\left(-\frac{\sin(2t)}{2} - \frac{\cos(2t)}{2}\right) \\ e^t\cos(2t) & -e^t\sin(2t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^t(\sin(2t) + \cos(2t)) & -e^t\sin(2t) \\ 2\,e^t\sin(2t) & e^t(\cos(2t) - \sin(2t)) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t) = \left[\begin{array}{c} -4\,\mathrm{e}^t \sin(2t) + 2\,\mathrm{e}^t \cos(2t) - 2 \\ -2\,\mathrm{e}^t \sin(2t) + 6\,\mathrm{e}^t \cos(2t) - 6 \end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \left[\begin{array}{c} -4\,\mathrm{e}^t \sin(2t) + 2\,\mathrm{e}^t \cos(2t) - 2 \\ -2\,\mathrm{e}^t \sin(2t) + 6\,\mathrm{e}^t \cos(2t) - 6 \end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{\mathrm{e}^t (c_1 - c_2 + 4) \cos(2t)}{2} - 2 - \frac{\mathrm{e}^t (c_1 + c_2 + 8) \sin(2t)}{2} \\ \mathrm{e}^t (c_1 + 6) \cos(2t) - 6 - \mathrm{e}^t (c_2 + 2) \sin(2t) \end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x(t) = \frac{\mathrm{e}^t (c_1 - c_2 + 4) \cos(2t)}{2} - 2 - \frac{\mathrm{e}^t (c_1 + c_2 + 8) \sin(2t)}{2}, y(t) = \mathrm{e}^t (c_1 + 6) \cos(2t) - 6 - \mathrm{e}^t (c_2 + 2) \sin(2t)\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 57

```
dsolve([diff(x(t),t)=3*x(t)-2*y(t)-6,diff(y(t),t)=4*x(t)-1*y(t)+2],singsol=all)
```

$$x(t) = -2 + e^t(c_1 \cos(2t) + c_2 \sin(2t))$$
$$y(t) = -6 + e^t(c_1 \cos(2t) - c_2 \cos(2t) + c_1 \sin(2t) + c_2 \sin(2t))$$

✓ Solution by Mathematica

Time used: 0.358 (sec). Leaf size: 64

```
DSolve[{x'[t]==3*x[t]-2*y[t]-6,y'[t]==4*x[t]-1*y[t]+2},{x[t],y[t]},t,IncludeSingularSolution
```

$$x(t) \to c_1 e^t \cos(2t) + (c_1 - c_2) e^t \sin(2t) - 2$$
$$y(t) \to c_2 e^t \cos(2t) + (2c_1 - c_2) e^t \sin(2t) - 6$$