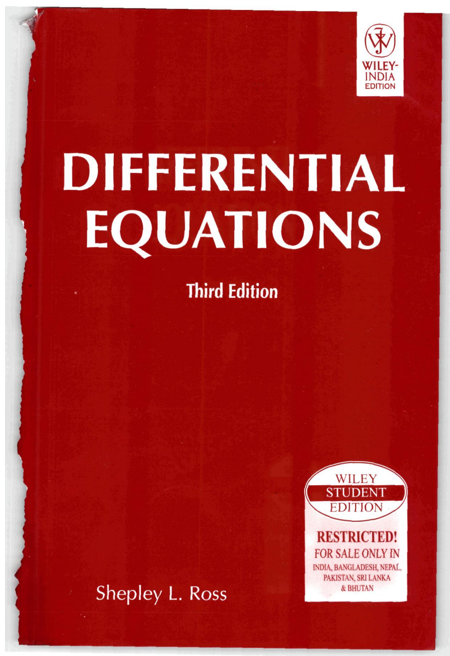
A Solution Manual For

Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.



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May 15, 2024

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1.1 problem 1(a)

Internal problem ID [11570]
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Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 1(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = 1 + x}$$

1.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 + x \end{aligned}$$

Hence the ode is

$$y' + y = 1 + x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(1+x) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} y) &= (\mathrm{e}^{x})\,(1+x) \\
\mathrm{d}(\mathrm{e}^{x} y) &= (\mathrm{e}^{x}(1+x))\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{x} y &= \int \mathrm{e}^{x}(1+x)\,\mathrm{d}x \\
\mathrm{e}^{x} y &= x\,\mathrm{e}^{x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$
y = \mathrm{e}^{-x} x\,\mathrm{e}^{x} + c_1 \mathrm{e}^{-x}
$$

which simplifies to

$$
y = x + c_1 \mathrm{e}^{-x}
$$

Summary

The solution(s) found are the following

$$
y = x + c_1 \mathrm{e}^{-x} \tag{1}
$$

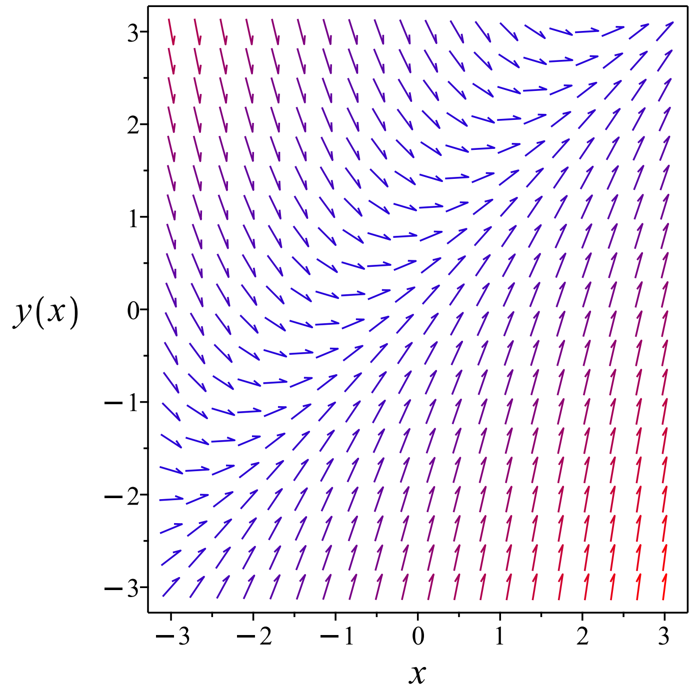


Figure 1: Slope field plot

Verification of solutions

$$y = x + c_1 e^{-x}$$

Verified OK.

1.1.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) + u(x)\,x = 1 + x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{(1+x)\,(-u+1)}{x} \end{aligned}$$

Where $f(x) = \frac{1+x}{x}$ and $g(u) = -u + 1$. Integrating both sides gives

$$\frac{1}{-u+1}\,du = \frac{1+x}{x}\,dx$$

$$
\int \frac{1}{-u+1}\, du = \int \frac{1+x}{x}\, dx
$$

$$
-\ln (u-1) = x + \ln (x) + c_2
$$

Raising both side to exponential gives

$$
\frac{1}{u-1} = \mathrm{e}^{x+\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{1}{u-1} = c_3 \mathrm{e}^{x+\ln(x)}
$$

Which simplifies to

$$
u(x) = \frac{\left(c_3 \mathrm{e}^{x} x\, \mathrm{e}^{c_2}+1\right) \mathrm{e}^{-x} \mathrm{e}^{-c_2}}{c_3 x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= \frac{\left(c_3 \mathrm{e}^{x} x\, \mathrm{e}^{c_2}+1\right) \mathrm{e}^{-x} \mathrm{e}^{-c_2}}{c_3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(c_3 \mathrm{e}^{x} x\, \mathrm{e}^{c_2}+1\right) \mathrm{e}^{-x} \mathrm{e}^{-c_2}}{c_3} \tag{1}
$$

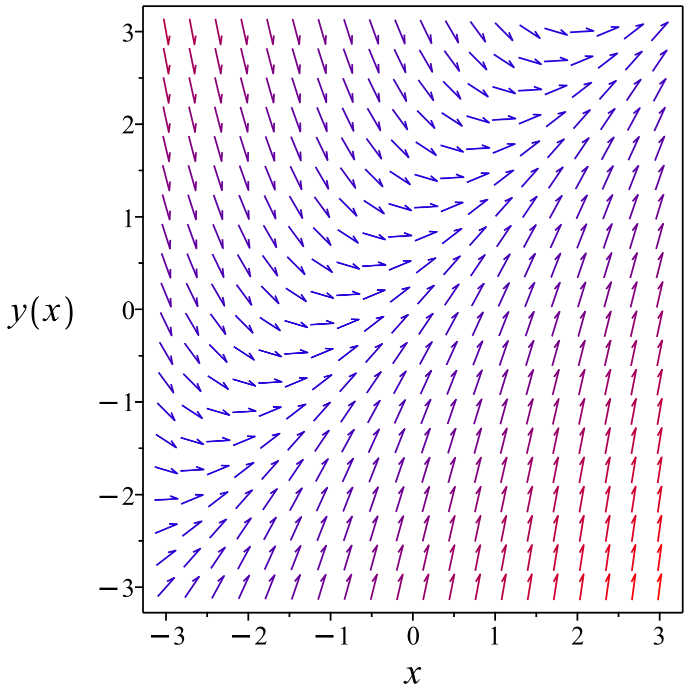


Figure 2: Slope field plot

Verification of solutions

$$y = \frac{(c_3 e^x x e^{c_2} + 1) e^{-x} e^{-c_2}}{c_3}$$

Verified OK.

1.1.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -y + 1 + x \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy \end{aligned}$$

Which results in

$$S = e^x y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y + 1 + x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^x y \\ S_y &= e^x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^x(1 + x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^R(1 + R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{R} R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,e^{x} = x\,e^{x} + c_1$$

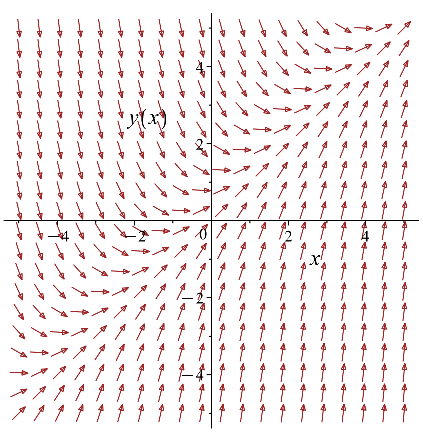
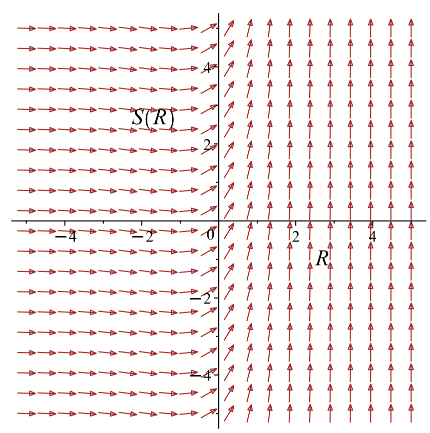
Which simplifies to

$$y\,e^{x} = x\,e^{x} + c_1$$

Which gives

$$y = (x\,e^{x} + c_1)\,e^{-x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y + 1 + x$ 	$R = x$ $S = e^{x} y$	$\frac{dS}{dR} = e^{R}(1 + R)$ 

Summary

The solution(s) found are the following

$$y = (x\,e^{x} + c_1)\,e^{-x} \tag{1}$$

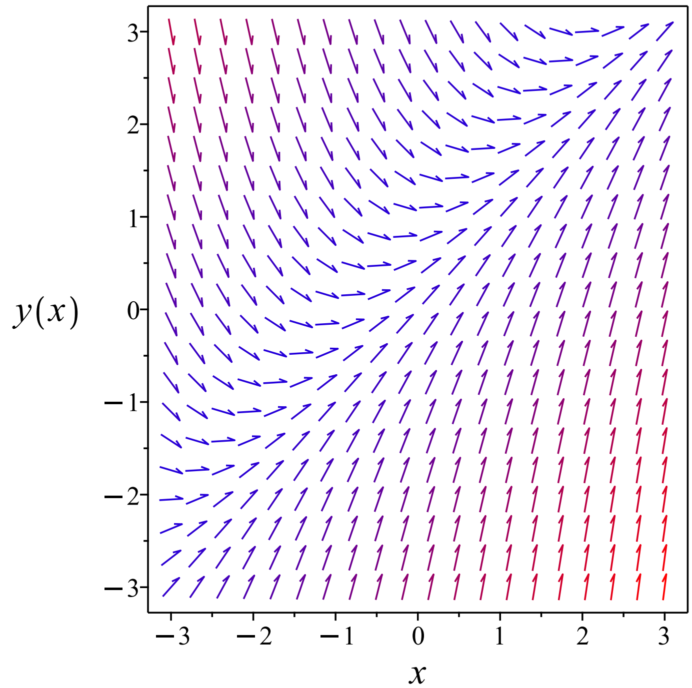


Figure 3: Slope field plot

Verification of solutions

$$y = (x\,e^{x} + c_1)\,e^{-x}$$

Verified OK.

1.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\mathrm{d}y &= (-y + 1 + x)\,\mathrm{d}x \\
(y - 1 - x)\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= y - 1 - x \\
N(x, y) &= 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - 1 - x) \\
&= 1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((1) - (0)) \\
&= 1
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x}(y - 1 - x) \\
&= -\mathrm{e}^{x}(-y + 1 + x)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(1) \\
&= \mathrm{e}^{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(-\mathrm{e}^{x}(-y + 1 + x)) + (\mathrm{e}^{x})\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\mathrm{e}^{x}(-y+1+x) \mathrm{d}x$$
$$\phi = -(-y+x)\,\mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -(-y+x)\,\mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -(-y+x)\,\mathrm{e}^{x}$$

The solution becomes

$$y = (x\,\mathrm{e}^{x} + c_1)\,\mathrm{e}^{-x}$$

Summary
The solution(s) found are the following

$$y = (x\,\mathrm{e}^{x} + c_1)\,\mathrm{e}^{-x} \tag{1}$$

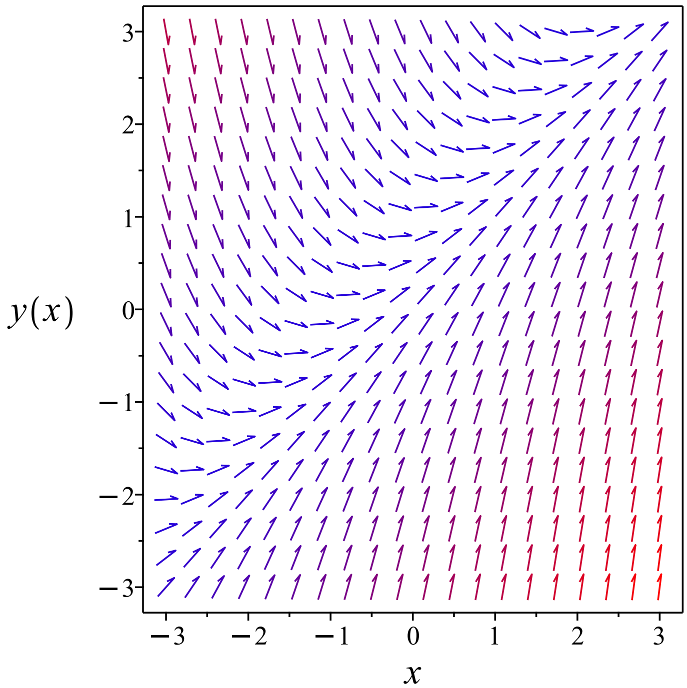


Figure 4: Slope field plot

Verification of solutions

$$y = (x\,e^{x} + c_1)\,e^{-x}$$

Verified OK.

1.1.5 Maple step by step solution

Let's solve

$y' + y = 1 + x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + 1 + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = 1 + x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + y) = \mu(x)\,(1 + x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\,(y'+y) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\,(1+x)\, dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\,(1+x)\, dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)(1+x)dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int \mathrm{e}^{x}(1+x)dx + c_1}{\mathrm{e}^{x}}$
- Evaluate the integrals on the rhs

  $y = \frac{x\,\mathrm{e}^{x} + c_1}{\mathrm{e}^{x}}$
- Simplify

  $y = x + c_1 \mathrm{e}^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x)+y(x)=1+x,y(x), singsol=all)
```

$$y(x) = x + c_1 \mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 15

```
DSolve[y'[x]+y[x]==1+x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_1 e^{-x}$$

1.2 problem 1(b)

Internal problem ID [11571]
Internal file name [`OUTPUT/10553_Thursday_May_18_2023_05_39_31_AM_64152595/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 1(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 7y' + 12y = 0}$$

1.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -7, C = 12$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 7\lambda e^{\lambda x} + 12 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 7\lambda + 12 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -7, C = 12$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{7}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-7^2 - (4)(1)(12)}\\ &= \frac{7}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{7}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{7}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 4$$
$$\lambda_2 = 3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(4)x} + c_2 e^{(3)x}$$

Or

$$y = c_1 \mathrm{e}^{4x} + \mathrm{e}^{3x} c_2$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{4x} + \mathrm{e}^{3x} c_2 \tag{1}$$

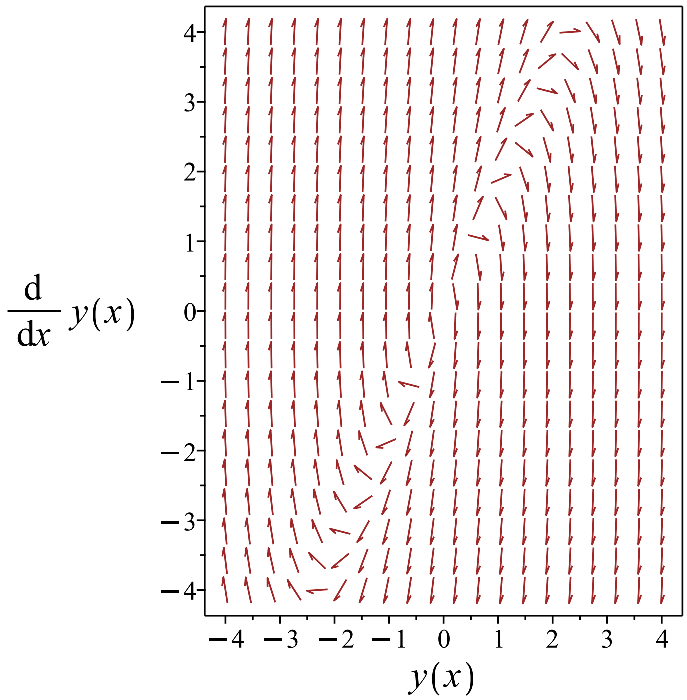


Figure 5: Slope field plot

Verification of solutions

$$y = c_1 e^{4x} + e^{3x} c_2$$

Verified OK.

1.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 7y' + 12y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -7 \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 4: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-7}{1}\,dx} \\ &= z_1 e^{\frac{7x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{7x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-7}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{7x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{3x}\right) + c_2 \left(e^{3x} \left(e^{x}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 e^{3x} + c_2 e^{4x} \tag{1}
$$

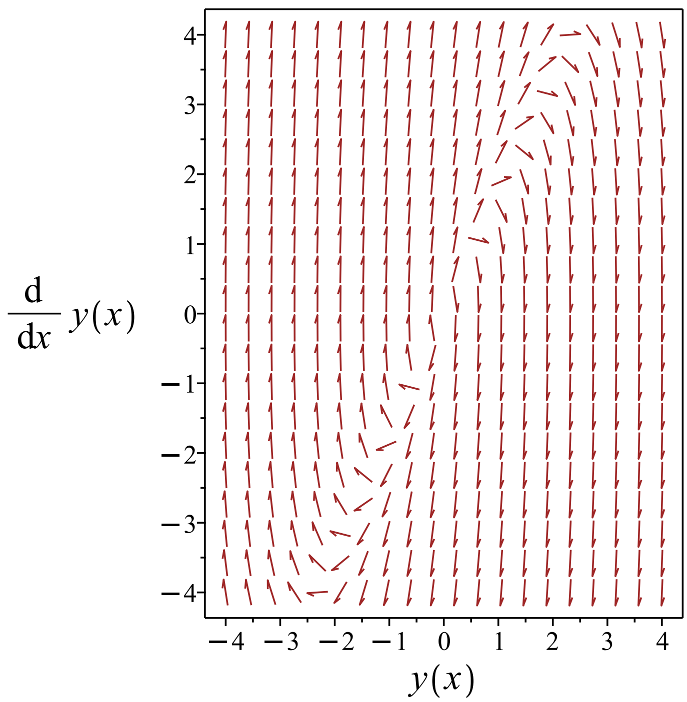


Figure 6: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 e^{3x} + c_2 e^{4x}
$$

Verified OK.

1.2.3 Maple step by step solution

Let's solve

$y'' - 7y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 7r + 12 = 0$

- Factor the characteristic polynomial

  $(r - 3)(r - 4) = 0$

- Roots of the characteristic polynomial

  $r = (3, 4)$

- 1st solution of the ODE

  $y_1(x) = e^{3x}$

- 2nd solution of the ODE

  $y_2(x) = e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{3x} + c_2 e^{4x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-7*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{4x} + c_2 e^{3x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 20

```
DSolve[y''[x]-7*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{3x}(c_2 e^x + c_1)$$

1.3 problem 1(c)

Internal problem ID [11572]
Internal file name [`OUTPUT/10554_Thursday_May_18_2023_05_39_32_AM_68847962/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 1(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 3y' + 2y = 4x^2}$$

1.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = 4x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=-3, C=2$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda\, \mathrm{e}^{\lambda x} + 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1, B=-3, C=2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)}\\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= \frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= 1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_3x^2 + 2A_2x - 6xA_3 + 2A_1 - 3A_2 + 2A_3 = 4x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 7, A_2 = 6, A_3 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x^2 + 6x + 7$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{2x} + c_2 e^x) + (2x^2 + 6x + 7) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^x + 2x^2 + 6x + 7 \tag{1}$$

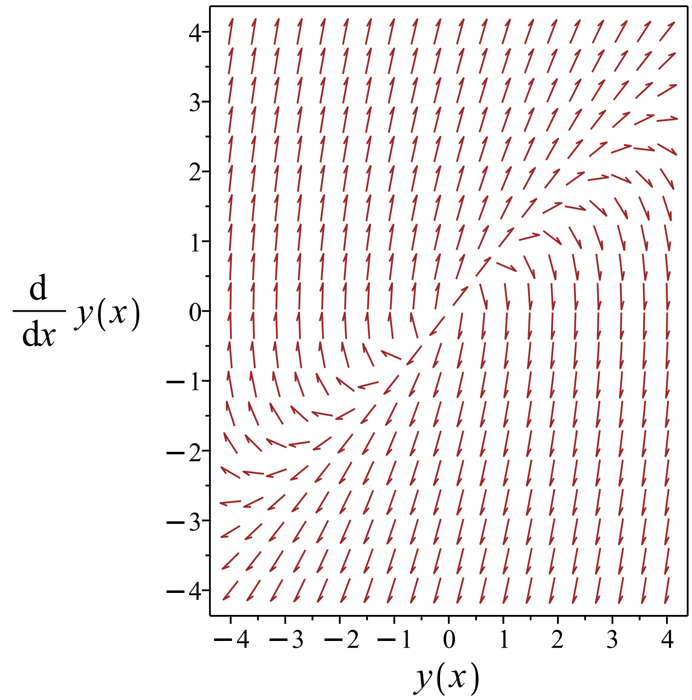


Figure 7: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{x} + 2x^2 + 6x + 7$$

Verified OK.

1.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 6: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\mathrm{e}^x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x c_1 + c_2 \mathrm{e}^{2x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^x, \mathrm{e}^{2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_3 x^2 + A_2 x + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_3 x^2 + 2A_2 x - 6x A_3 + 2A_1 - 3A_2 + 2A_3 = 4x^2
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 7, A_2 = 6, A_3 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x^2 + 6x + 7$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + c_2 e^{2x}\right) + \left(2x^2 + 6x + 7\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x c_1 + c_2 e^{2x} + 2x^2 + 6x + 7 \tag{1}$$

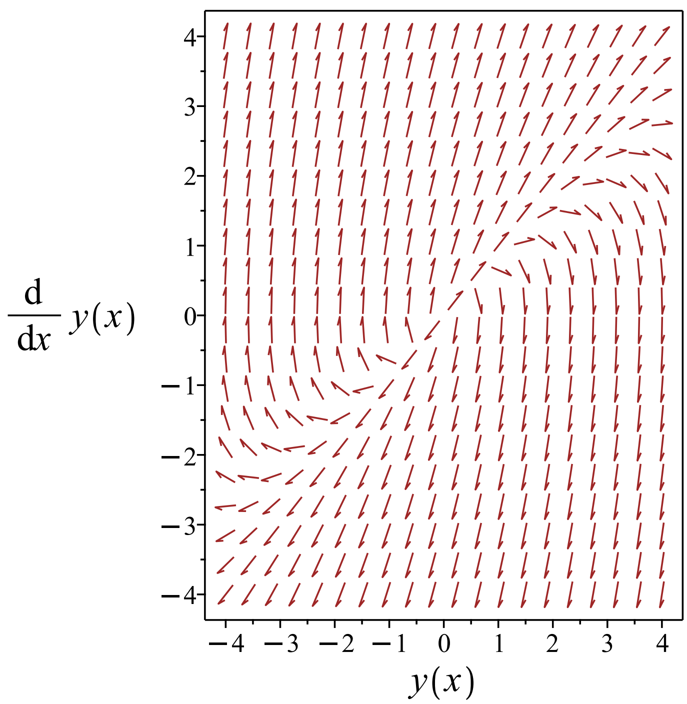


Figure 8: Slope field plot

Verification of solutions

$$y = e^x c_1 + c_2 e^{2x} + 2x^2 + 6x + 7$$

Verified OK.

1.3.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = 4x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x c_1 + c_2 e^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4x^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2\,e^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{3x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -4\,e^x\left(\int x^2 e^{-x} dx\right) + 4\,e^{2x}\left(\int x^2 e^{-2x} dx\right)$

- Compute integrals

  $y_p(x) = 2x^2 + 6x + 7$

- Substitute particular solution into general solution to ODE

  $y = e^x c_1 + c_2 e^{2x} + 2x^2 + 6x + 7$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=4*x^2,y(x), singsol=all)
```

$$y(x) = e^{2x} c_1 + c_2 e^x + 2x^2 + 6x + 7$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 29

```
DSolve[y''[x]-3*y'[x]+2*y[x]==4*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2x^2 + 6x + c_1 e^x + c_2 e^{2x} + 7$$

1.4 problem 1(d)

Internal problem ID [11573]
Internal file name [`OUTPUT/10555_Thursday_May_18_2023_05_39_32_AM_64086885/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 1(d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x^2+1)\,y''+4y'x+2y=0}$$

1.4.1 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y''+p(x)\,y'+\frac{\left(p(x)^2+p'(x)\right)y}{2}=f(x)$$

Where $p(x) = \frac{4x}{x^2+1}$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int \frac{4x}{x^2+1}\,dx} \\
&= x^2 + 1
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\left(x^2+1\right)y\right)'' &= 0
\end{aligned}
$$

Integrating once gives

$$
\left(\left(x^2+1\right)y\right)' = c_1
$$

Integrating again gives

$$
\left(\left(x^2+1\right)y\right) = c_1 x + c_2
$$

Hence the solution is

$$
y = \frac{c_1 x + c_2}{x^2+1}
$$

Or

$$
y = \frac{c_1 x}{x^2+1} + \frac{c_2}{x^2+1}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x}{x^2+1} + \frac{c_2}{x^2+1} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 x}{x^2+1} + \frac{c_2}{x^2+1}
$$

Verified OK.

1.4.2 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{4x}{x^2+1}\\
q(x) &= \frac{2}{x^2+1}
\end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4}\\
&= \frac{2}{x^2+1} - \frac{\left(\frac{4x}{x^2+1}\right)'}{2} - \frac{\left(\frac{4x}{x^2+1}\right)^2}{4}\\
&= \frac{2}{x^2+1} - \frac{\left(\frac{4}{x^2+1} - \frac{8x^2}{(x^2+1)^2}\right)}{2} - \frac{\left(\frac{16x^2}{(x^2+1)^2}\right)}{4}\\
&= \frac{2}{x^2+1} - \left(\frac{2}{x^2+1} - \frac{4x^2}{(x^2+1)^2}\right) - \frac{4x^2}{(x^2+1)^2}\\
&= 0
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)}\\
&= e^{-\int \frac{\frac{4x}{x^2+1}}{2}}\\
&= \frac{1}{x^2+1}
\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{x^2+1} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) = 0$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$v(x) = c_1x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1x + c_2)\, (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x^2 + 1}$$

Hence (7) becomes

$$y = \frac{c_1x + c_2}{x^2 + 1}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1x + c_2}{x^2 + 1} \tag{1}$$

Verification of solutions

$$y = \frac{c_1x + c_2}{x^2 + 1}$$

Verified OK.

1.4.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned} \int \left(\left(x^2 + 1\right) y'' + 4y'x + 2y\right) dx &= 0 \\ 2yx + \left(x^2 + 1\right) y' &= c_1 \end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2x}{x^2+1}\\
q(x) &= \frac{c_1}{x^2+1}
\end{aligned}
$$

Hence the ode is

$$
y' + \frac{2xy}{x^2+1} = \frac{c_1}{x^2+1}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{2x}{x^2+1}dx}\\
&= x^2+1
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2+1}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\big((x^2+1)\,y\big) &= (x^2+1)\left(\frac{c_1}{x^2+1}\right)\\
\mathrm{d}\big((x^2+1)\,y\big) &= c_1\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
(x^2+1)\,y &= \int c_1\,\mathrm{d}x\\
(x^2+1)\,y &= c_1x + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2+1$ results in

$$
y = \frac{c_1x}{x^2+1} + \frac{c_2}{x^2+1}
$$

which simplifies to

$$
y = \frac{c_1x+c_2}{x^2+1}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1x+c_2}{x^2+1} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1x+c_2}{x^2+1}
$$

Verified OK.

1.4.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$\left(x^{2}+1\right) y''+4y'x+2y=0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(x^{2}+1\right) y''+4y'x+2y\right) dx=0$$
$$2yx+\left(x^{2}+1\right) y'=c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{2x}{x^{2}+1}$$
$$q(x)=\frac{c_1}{x^{2}+1}$$

Hence the ode is

$$y'+\frac{2xy}{x^{2}+1}=\frac{c_1}{x^{2}+1}$$

The integrating factor μ is

$$\mu=\mathrm{e}^{\int \frac{2x}{x^{2}+1}dx}$$
$$=x^{2}+1$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)=(\mu)\left(\frac{c_1}{x^{2}+1}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^{2}+1\right) y\right)=\left(x^{2}+1\right)\left(\frac{c_1}{x^{2}+1}\right)$$
$$\mathrm{d}\left(\left(x^{2}+1\right) y\right)=c_1\,\mathrm{d}x$$

Integrating gives

$$\left(x^{2}+1\right) y=\int c_1\,\mathrm{d}x$$
$$\left(x^{2}+1\right) y=c_1x+c_2$$

Dividing both sides by the integrating factor $\mu = x^2 + 1$ results in

$$y = \frac{c_1 x}{x^2 + 1} + \frac{c_2}{x^2 + 1}$$

which simplifies to

$$y = \frac{c_1 x + c_2}{x^2 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x + c_2}{x^2 + 1} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x + c_2}{x^2 + 1}$$

Verified OK.

1.4.5 Solving using Kovacic algorithm

Writing the ode as

$$\left(x^2 + 1\right) y'' + 4y'x + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 + 1 \\ B &= 4x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 8: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2+1}\,dx} \\
&= z_1 e^{-\ln(x^2+1)} \\
&= z_1 \left(\frac{1}{x^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^2+1}\right) + c_2\left(\frac{1}{x^2+1}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^2+1} + \frac{c_2 x}{x^2+1} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x^2+1} + \frac{c_2 x}{x^2+1}
$$

Verified OK.

1.4.6 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x^2 + 1 \\
q(x) &= 4x \\
r(x) &= 2 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 4
\end{aligned}
$$

Therefore (1) becomes

$$
2 - (4) + (2) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
\left(p(x)\, y' + (q(x) - p'(x))\, y\right)' = s(x)
$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$2yx + \left(x^2+1\right)y' = c_1$$

We now have a first order ode to solve which is

$$2yx + \left(x^2+1\right)y' = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= \frac{2x}{x^2+1}\\
q(x) &= \frac{c_1}{x^2+1}
\end{aligned}$$

Hence the ode is

$$y' + \frac{2xy}{x^2+1} = \frac{c_1}{x^2+1}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{2x}{x^2+1}dx}\\
&= x^2+1
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2+1}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2+1\right)y\right) &= \left(x^2+1\right)\left(\frac{c_1}{x^2+1}\right)\\
\mathrm{d}\left(\left(x^2+1\right)y\right) &= c_1\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\left(x^2+1\right)y &= \int c_1\,\mathrm{d}x\\
\left(x^2+1\right)y &= c_1x + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2 + 1$ results in

$$y = \frac{c_1 x}{x^2 + 1} + \frac{c_2}{x^2 + 1}$$

which simplifies to

$$y = \frac{c_1 x + c_2}{x^2 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x + c_2}{x^2 + 1} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x + c_2}{x^2 + 1}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve((1+x^2)*diff(y(x),x$2)+4*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x + c_2}{x^2 + 1}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 20

```
DSolve[(1+x^2)*y''[x]+4*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x + c_1}{x^2 + 1}$$

1.5 problem 2(a)

Internal problem ID [11574]
Internal file name [`OUTPUT/10556_Thursday_May_18_2023_05_39_34_AM_10656506/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 2(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _Bernoulli]
```

$$\boxed{2xyy' + y^2 = -x^2}$$

1.5.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2x^2u(x)\,(u'(x)\,x + u(x)) + u(x)^2\,x^2 = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u^2+1}{2ux} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{3u^2+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{3u^2+1}{u}}\,du = -\frac{1}{2x}\,dx$$

$$\int \frac{1}{\frac{3u^2+1}{u}}\,du = \int -\frac{1}{2x}\,dx$$

$$\frac{\ln(3u^2+1)}{6} = -\frac{\ln(x)}{2} + c_2$$

Raising both side to exponential gives

$$\left(3u^2+1\right)^{\frac{1}{6}} = \mathrm{e}^{-\frac{\ln(x)}{2}+c_2}$$

Which simplifies to

$$\left(3u^2+1\right)^{\frac{1}{6}} = \frac{c_3}{\sqrt{x}}$$

Which simplifies to

$$\left(3u(x)^2+1\right)^{\frac{1}{6}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}$$

The solution is

$$\left(3u(x)^2+1\right)^{\frac{1}{6}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{3y^2}{x^2}+1\right)^{\frac{1}{6}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}$$

$$\left(\frac{3y^2+x^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$\left(\frac{3y^2+x^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}} \tag{1}$$

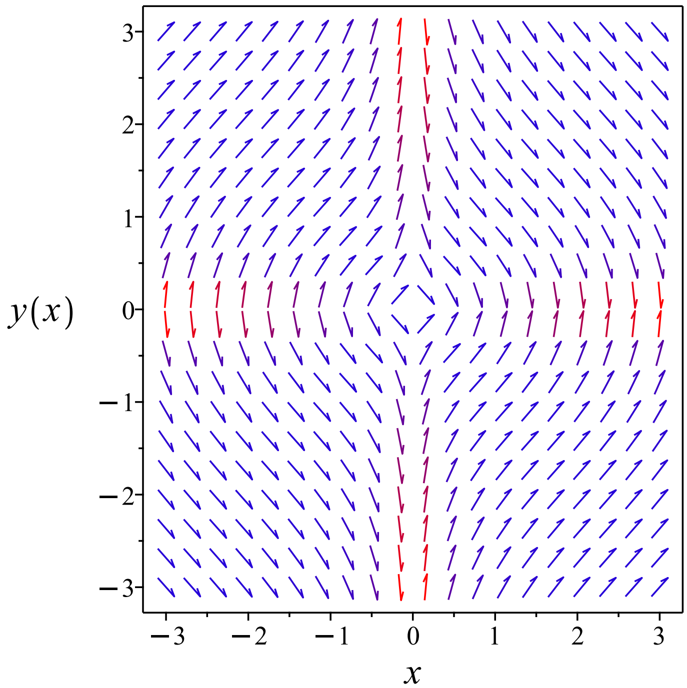


Figure 9: Slope field plot

Verification of solutions

$$\left(\frac{3y^2+x^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$

Verified OK.

1.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2+y^2}{2yx}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 9: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{xy}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{xy}} dy \end{aligned}$$

Which results in

$$S = \frac{x\,y^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x^2 + y^2}{2yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^2}{2} \\ S_y &= xy \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{x^2}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R^2}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{R^3}{6} + c_1 \tag{4}$$

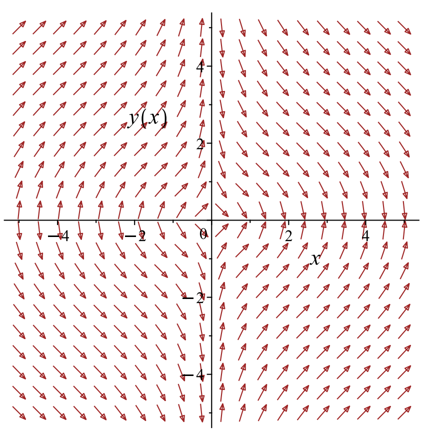
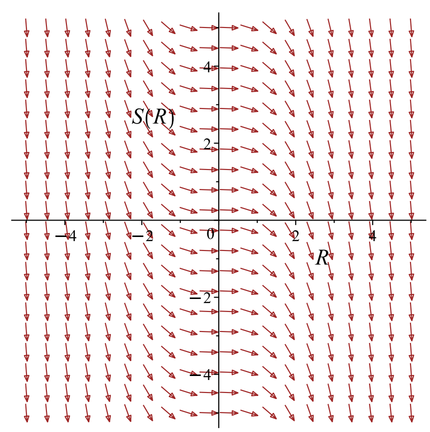
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2 x}{2} = -\frac{x^3}{6} + c_1$$

Which simplifies to

$$\frac{y^2 x}{2} = -\frac{x^3}{6} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{x^2+y^2}{2yx}$

</td><td>$R = x$
$S = \frac{x\,y^2}{2}$</td><td>$\frac{dS}{dR} = -\frac{R^2}{2}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{y^2 x}{2} = -\frac{x^3}{6} + c_1 \tag{1}$$

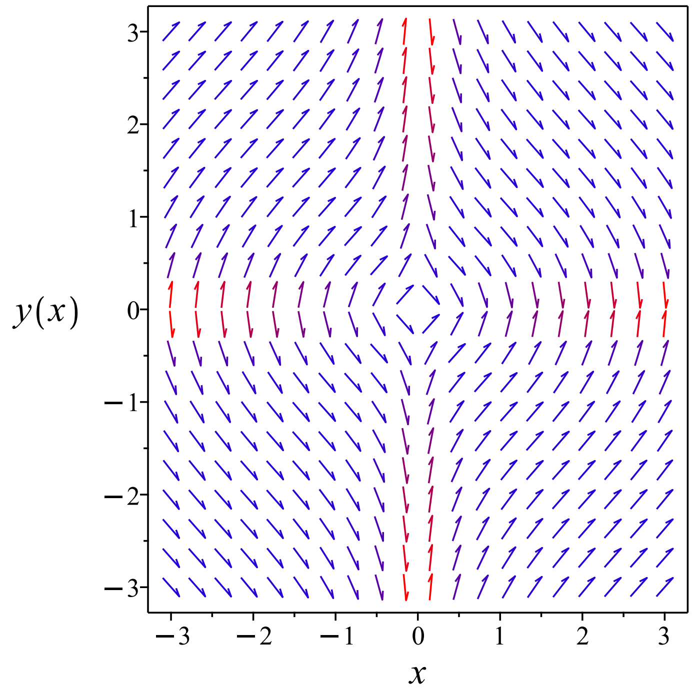


Figure 10: Slope field plot

Verification of solutions

$$\frac{y^2 x}{2} = -\frac{x^3}{6} + c_1$$

Verified OK.

1.5.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{x^2 + y^2}{2yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{2x} y - \frac{x}{2} \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x) y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{2x} \\ f_1(x) &= -\frac{x}{2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{2x} - \frac{x}{2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -\frac{w(x)}{2x} - \frac{x}{2} \\ w' &= -\frac{w}{x} - x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -x \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = -x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-x) \\ \frac{d}{dx}(xw) &= (x)(-x) \\ d(xw) &= (-x^2)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xw &= \int -x^2\, dx \\ xw &= -\frac{x^3}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = -\frac{x^2}{3} + \frac{c_1}{x}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -\frac{x^2}{3} + \frac{c_1}{x}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x} \\ y(x) &= -\frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x} \tag{1}$$

$$y = -\frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x} \tag{2}$$

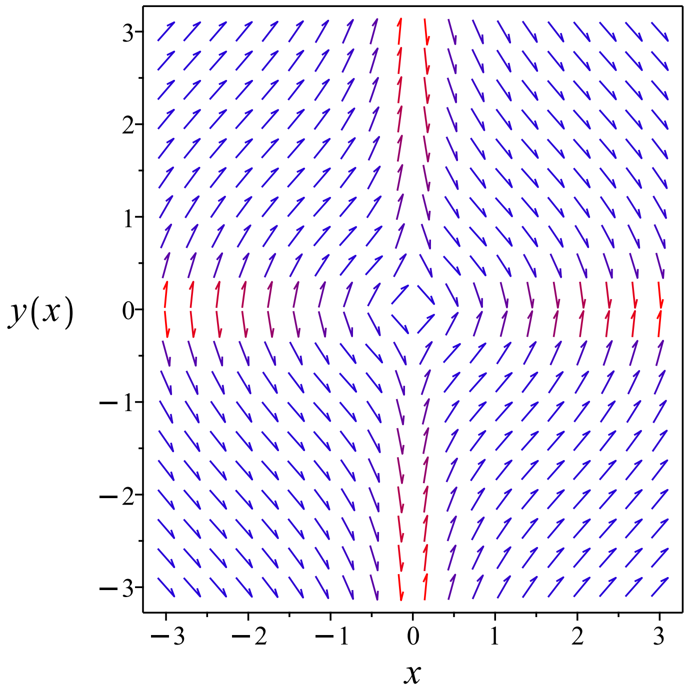


Figure 11: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x}$$

Verified OK.

$$y = -\frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x}$$

Verified OK.

1.5.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy)\,\mathrm{d}y &= \left(-x^2 - y^2\right)\mathrm{d}x \\ \left(x^2 + y^2\right)\mathrm{d}x + (2xy)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x^2 + y^2 \\ N(x,y) &= 2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + y^2\right) \\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy)\\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int x^2 + y^2\, \mathrm{d}x\end{aligned}$$

$$\phi = \frac{1}{3}x^3 + x\,y^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2xy + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2xy$. Therefore equation (4) becomes

$$2xy = 2xy + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{3}x^3 + x\,y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3}x^3 + x\,y^2$$

Summary

The solution(s) found are the following

$$\frac{x^3}{3} + y^2x = c_1 \tag{1}$$

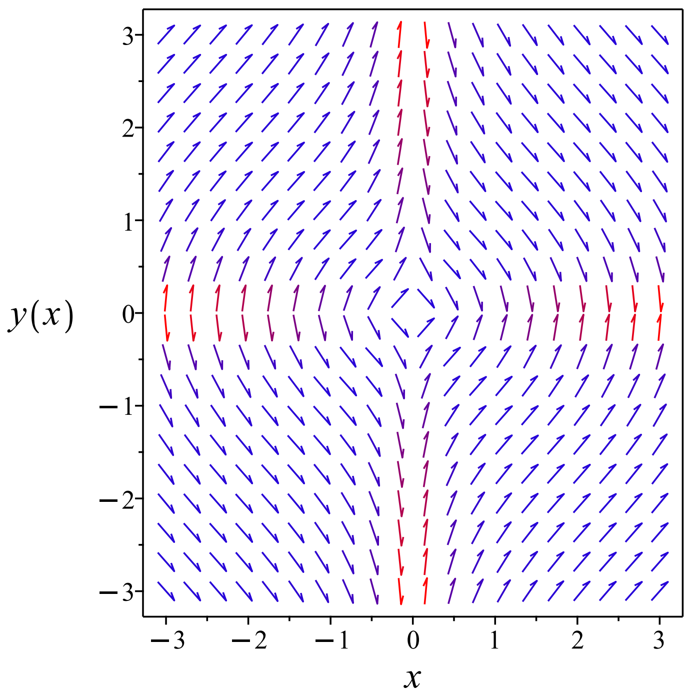


Figure 12: Slope field plot

Verification of solutions

$$\frac{x^3}{3} + y^2x = c_1$$

Verified OK.

1.5.5 Maple step by step solution

Let's solve

$2xyy' + y^2 = -x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  - Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right) y' = 0$

  - Evaluate derivatives

    $2y = 2y$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (x^2 + y^2)\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \frac{x^3}{3} + x\,y^2 + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $2xy = 2xy + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$F(x, y) = \frac{1}{3}x^3 + x\,y^2$

- Substitute $F(x, y)$ into the solution of the ODE

  $\frac{1}{3}x^3 + x\,y^2 = c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{3}\,\sqrt{x(-x^3+3c_1)}}{3x}, y = \frac{\sqrt{3}\,\sqrt{x(-x^3+3c_1)}}{3x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(2*x*y(x)*diff(y(x),x)+x^2+y(x)^2=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x}$$
$$y(x) = \frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x}$$

✓ Solution by Mathematica

Time used: 0.377 (sec). Leaf size: 60

```
DSolve[2*x*y[x]*y'[x]+x^2+y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-x^3 + 3c_1}}{\sqrt{3}\sqrt{x}}$$
$$y(x) \to \frac{\sqrt{-x^3 + 3c_1}}{\sqrt{3}\sqrt{x}}$$

1.6 problem 2(b)

Internal problem ID [11575]
Internal file name [`OUTPUT/10557_Thursday_May_18_2023_05_39_35_AM_47002682/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 2(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y'x + y - y^3x^3 = 0}$$

1.6.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(y^2x^3 - 1)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 12: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= y^3 x^2 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^3 x^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{2x^2 y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(y^2 x^3 - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x^3 y^2} \\ S_y &= \frac{1}{y^3 x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

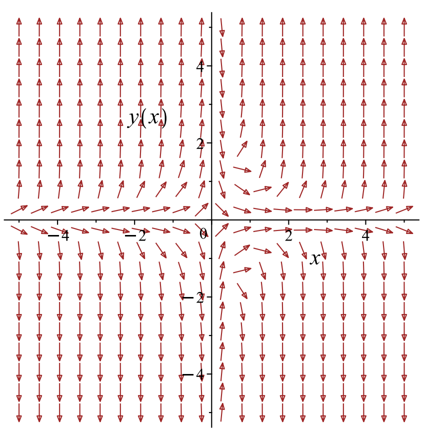
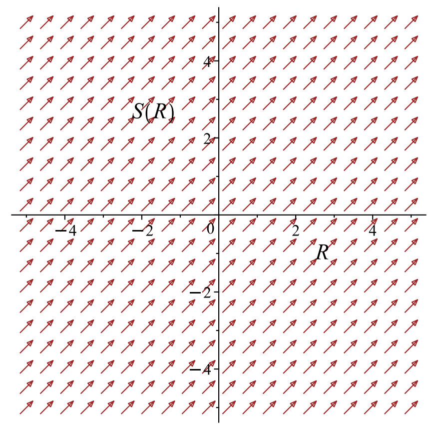
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{2x^2y^2} = x + c_1$$

Which simplifies to

$$-\frac{1}{2x^2y^2} = x + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(y^2x^3-1)}{x}$ 	$R = x$ $S = -\frac{1}{2x^2y^2}$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$-\frac{1}{2x^2y^2} = x + c_1 \tag{1}$$

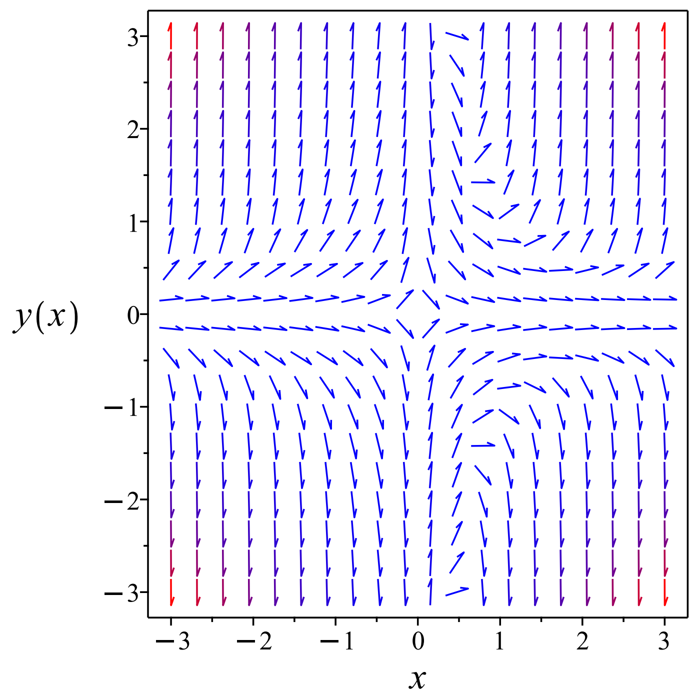


Figure 13: Slope field plot

<u>Verification of solutions</u>

$$-\frac{1}{2x^2y^2} = x + c_1$$

Verified OK.

1.6.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(y^2x^3 - 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + x^2y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= x^2 \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y' \frac{1}{y^3} = -\frac{1}{x\, y^2} + x^2 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= -\frac{w(x)}{x} + x^2 \\
w' &= \frac{2w}{x} - 2x^2
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= -2x^2
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = -2x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-2x^2\right) \\ \frac{d}{dx}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(-2x^2\right) \\ d\left(\frac{w}{x^2}\right) &= -2\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^2} &= \int -2\, dx \\ \frac{w}{x^2} &= -2x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = c_1 x^2 - 2x^3$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = c_1 x^2 - 2x^3$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{1}{\sqrt{c_1 - 2x}\, x} \\ y(x) &= -\frac{1}{\sqrt{c_1 - 2x}\, x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{c_1 - 2x}\, x} \tag{1}$$

$$y = -\frac{1}{\sqrt{c_1 - 2x}\, x} \tag{2}$$

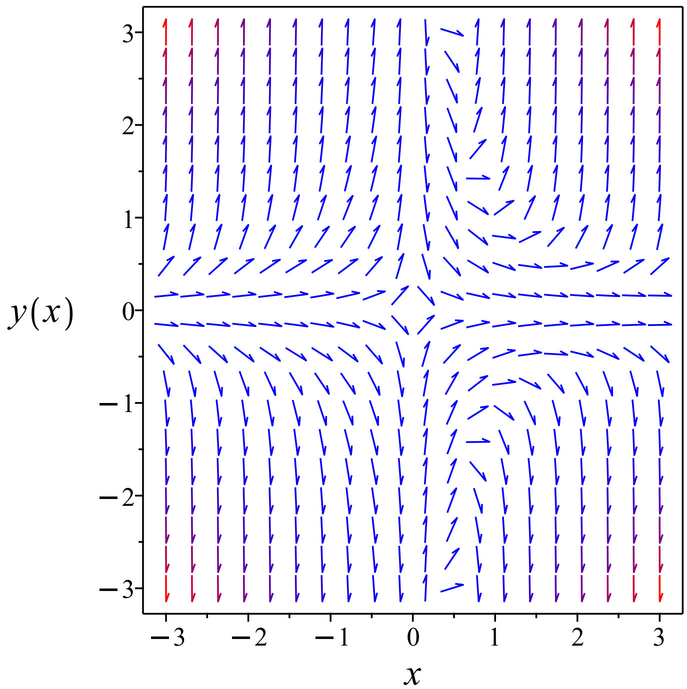


Figure 14: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{c_1 - 2x}\, x}$$

Verified OK.

$$y = -\frac{1}{\sqrt{c_1 - 2x}\, x}$$

Verified OK.

1.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(x^3y^3-y\right)\mathrm{d}x\\ \left(-x^3y^3+y\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^3y^3+y\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^3y^3 + y) \\ &= -3y^2x^3 + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}\left((-3y^2x^3 + 1) - (1)\right) \\ &= -3x^2y^2\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{-x^3y^3 + y}\left((1) - (-3y^2x^3 + 1)\right) \\ &= -\frac{3y\,x^3}{y^2x^3 - 1}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - (-3y^2x^3 + 1)}{x\,(-x^3y^3 + y) - y\,(x)} \\
&= -\frac{3}{xy}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{3}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \, \mathrm{d}t} \\
&= e^{\int \left(-\frac{3}{t}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-3\ln(t)} \\
&= \frac{1}{t^3}
\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^3y^3}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^3y^3}\left(-x^3y^3 + y\right) \\
&= \frac{-y^2x^3 + 1}{y^2x^3}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^3y^3}(x) \\
&= \frac{1}{y^3x^2}
\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-y^2x^3+1}{y^2x^3}\right) + \left(\frac{1}{y^3x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-y^2x^3+1}{y^2x^3}\,\mathrm{d}x$$
$$\phi = \frac{-x\,y^2 - \frac{1}{2x^2}}{y^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{2\left(-x\,y^2 - \frac{1}{2x^2}\right)}{y^3} - \frac{2x}{y} + f'(y) \tag{4}$$
$$= \frac{1}{y^3x^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^3x^2}$. Therefore equation (4) becomes

$$\frac{1}{y^3x^2} = \frac{1}{y^3x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x\,y^2 - \frac{1}{2x^2}}{y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x\,y^2 - \frac{1}{2x^2}}{y^2}$$

Summary

The solution(s) found are the following

$$\frac{-y^2 x - \frac{1}{2x^2}}{y^2} = c_1 \tag{1}$$

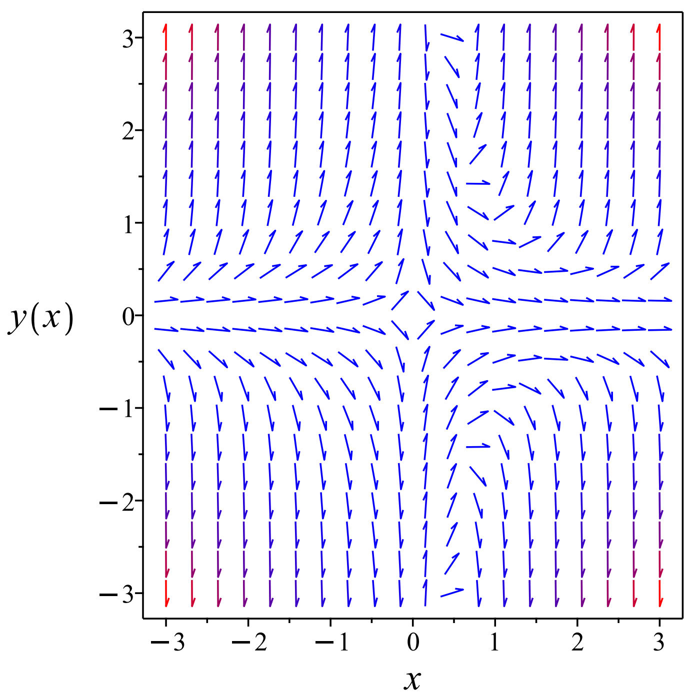


Figure 15: Slope field plot

Verification of solutions

$$\frac{-y^2x - \frac{1}{2x^2}}{y^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(x*diff(y(x),x)+y(x)=x^3*y(x)^3,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{-2x + c_1}\, x}$$
$$y(x) = -\frac{1}{\sqrt{-2x + c_1}\, x}$$

✓ Solution by Mathematica

Time used: 0.6 (sec). Leaf size: 44

```
DSolve[x*y'[x]+y[x]==x^3*y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{x^2(-2x + c_1)}}$$
$$y(x) \to \frac{1}{\sqrt{x^2(-2x + c_1)}}$$
$$y(x) \to 0$$

1.7 problem 3(a)

Internal problem ID [11576]
Internal file name [`OUTPUT/10558_Thursday_May_18_2023_05_39_37_AM_88378866/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 3(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 3y = 3x^2 e^{-3x}}$$

1.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 3$$
$$q(x) = 3x^2 e^{-3x}$$

Hence the ode is

$$y' + 3y = 3x^2 e^{-3x}$$

The integrating factor μ is

$$\mu = e^{\int 3dx}$$
$$= e^{3x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(3x^2 e^{-3x}\right)$$
$$\frac{d}{dx}\left(e^{3x} y\right) = \left(e^{3x}\right)\left(3x^2 e^{-3x}\right)$$
$$d\left(e^{3x} y\right) = \left(3x^2\right) dx$$

Integrating gives

$$e^{3x} y = \int 3x^2 \, dx$$
$$e^{3x} y = x^3 + c_1$$

Dividing both sides by the integrating factor $\mu = e^{3x}$ results in

$$y = e^{-3x} x^3 + c_1 e^{-3x}$$

which simplifies to

$$y = e^{-3x}\left(x^3 + c_1\right)$$

Summary
The solution(s) found are the following

$$y = e^{-3x}\left(x^3 + c_1\right) \tag{1}$$

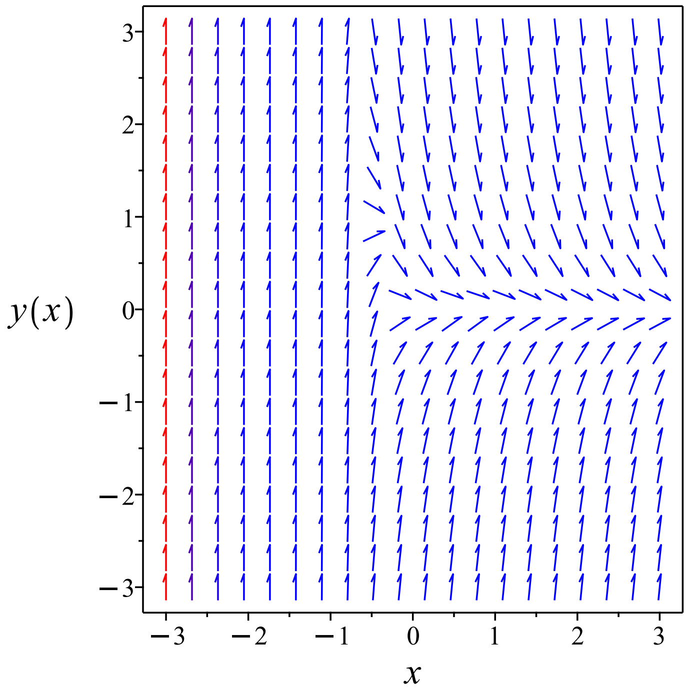


Figure 16: Slope field plot

Verification of solutions

$$y = e^{-3x}(x^3 + c_1)$$

Verified OK.

1.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -3y + 3x^2 e^{-3x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 14: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-3x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-3x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{3x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -3y + 3x^2 \mathrm{e}^{-3x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3\,\mathrm{e}^{3x} y \\ S_y &= \mathrm{e}^{3x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^3 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{3x}y = x^3 + c_1$$

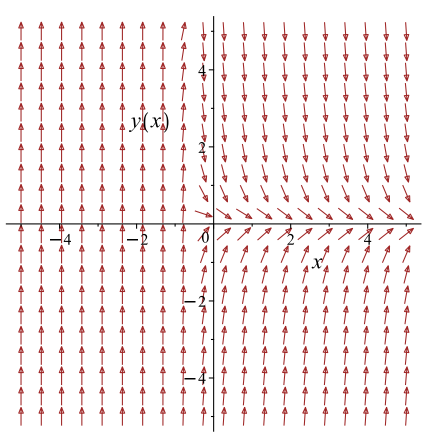
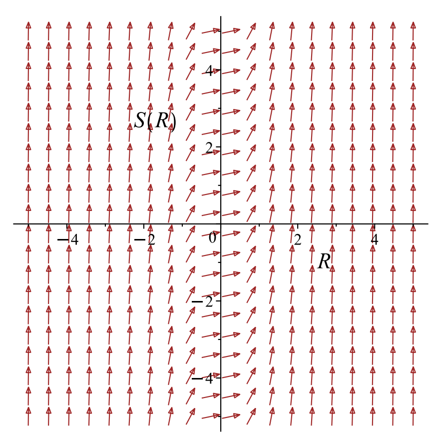
Which simplifies to

$$e^{3x}y = x^3 + c_1$$

Which gives

$$y = e^{-3x}(x^3 + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -3y + 3x^2 e^{-3x}$ 	$R = x$ $S = e^{3x}y$	$\frac{dS}{dR} = 3R^2$ 

Summary

The solution(s) found are the following

$$y = e^{-3x}(x^3 + c_1) \tag{1}$$

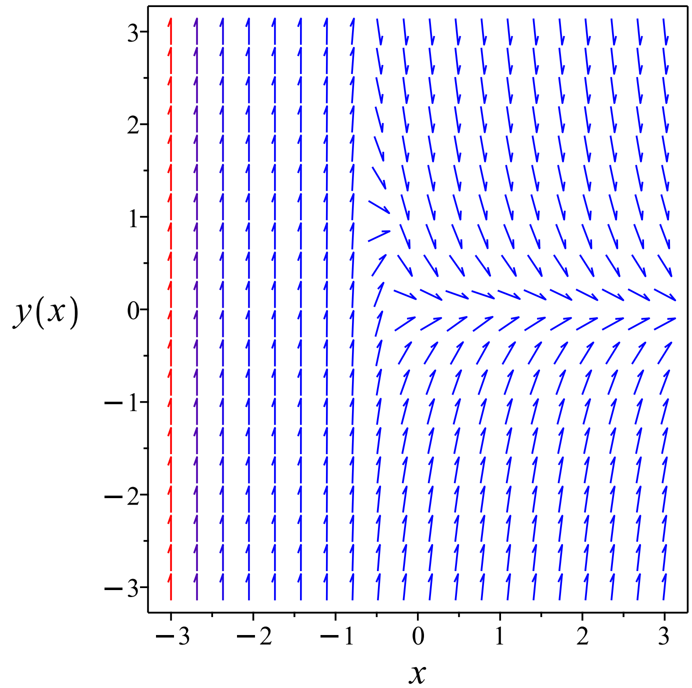


Figure 17: Slope field plot

Verification of solutions

$$y = e^{-3x}\left(x^3 + c_1\right)$$

Verified OK.

1.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(-3y + 3x^2 \mathrm{e}^{-3x}\right) \mathrm{d}x \\
\left(3y - 3x^2 \mathrm{e}^{-3x}\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= 3y - 3x^2 \mathrm{e}^{-3x} \\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y - 3x^2 \mathrm{e}^{-3x}\right) \\
&= 3
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((3) - (0)) \\
&= 3
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 3 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{3x} \\
&= \mathrm{e}^{3x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{3x}\left(3y - 3x^2\mathrm{e}^{-3x}\right) \\
&= 3\,\mathrm{e}^{3x}y - 3x^2
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{3x}(1) \\
&= \mathrm{e}^{3x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(3\,\mathrm{e}^{3x}y - 3x^2\right) + \left(\mathrm{e}^{3x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \overline{M} \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int 3 \, e^{3x} y - 3x^2 \, dx$$

$$\phi = e^{3x} y - x^3 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{3x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{3x}$. Therefore equation (4) becomes

$$e^{3x} = e^{3x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = e^{3x} y - x^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{3x} y - x^3$$

The solution becomes

$$y = e^{-3x} (x^3 + c_1)$$

Summary
The solution(s) found are the following

$$y = e^{-3x} (x^3 + c_1) \tag{1}$$

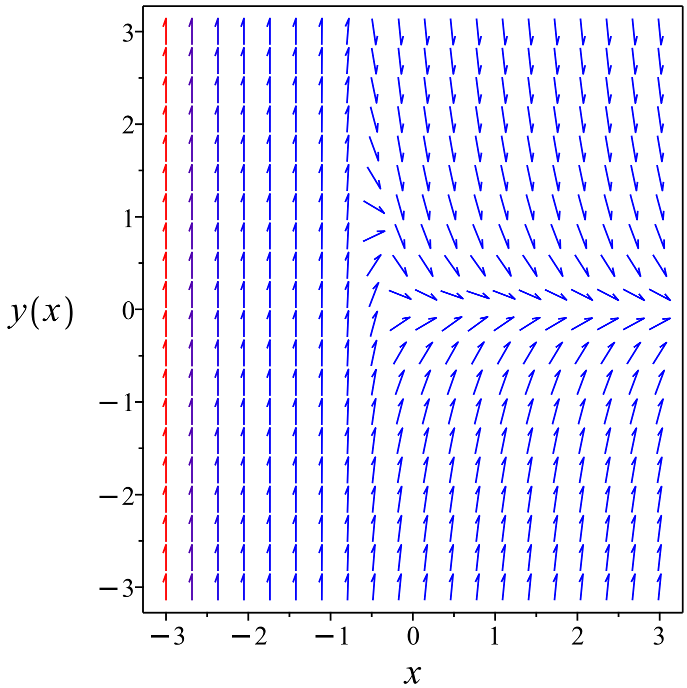


Figure 18: Slope field plot

Verification of solutions

$$y = e^{-3x}(x^3 + c_1)$$

Verified OK.

1.7.4 Maple step by step solution

Let's solve

$y' + 3y = 3x^2 e^{-3x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -3y + 3x^2 e^{-3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 3y = 3x^2 e^{-3x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' + 3y) = 3\mu(x)\, x^2 e^{-3x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\,(y' + 3y) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = 3\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{3x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 3\mu(x)\, x^2 \mathrm{e}^{-3x} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 3\mu(x)\, x^2 \mathrm{e}^{-3x} dx + c_1$
- Solve for y

  $y = \frac{\int 3\mu(x) x^2 \mathrm{e}^{-3x} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{3x}$

  $y = \frac{\int 3x^2 \mathrm{e}^{-3x} \mathrm{e}^{3x} dx + c_1}{\mathrm{e}^{3x}}$
- Evaluate the integrals on the rhs

  $y = \frac{x^3 + c_1}{\mathrm{e}^{3x}}$
- Simplify

  $y = \mathrm{e}^{-3x}(x^3 + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+3*y(x)=3*x^2*exp(-3*x),y(x), singsol=all)
```

$$y(x) = (x^3 + c_1) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 17

```
DSolve[y'[x]+3*y[x]==3*x^2*Exp[-3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}(x^3 + c_1)$$

1.8 problem 3(b)

Internal problem ID [11577]
Internal file name [`OUTPUT/10559_Thursday_May_18_2023_05_56_26_PM_75656588/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 3(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + 4yx = 8x}$$

1.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= x(-4y + 8) \end{aligned}$$

Where $f(x) = x$ and $g(y) = -4y + 8$. Integrating both sides gives

$$\frac{1}{-4y + 8}\, dy = x\, dx$$

$$\int \frac{1}{-4y + 8}\, dy = \int x\, dx$$

$$-\frac{\ln(y-2)}{4} = \frac{x^2}{2} + c_1$$

Raising both side to exponential gives

$$\frac{1}{(y-2)^{\frac{1}{4}}} = e^{\frac{x^2}{2}+c_1}$$

Which simplifies to

$$\frac{1}{(y-2)^{\frac{1}{4}}} = c_2 e^{\frac{x^2}{2}}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(2c_2^4 e^{2x^2+4c_1} + 1\right) e^{-2x^2-4c_1}}{c_2^4} \tag{1}$$

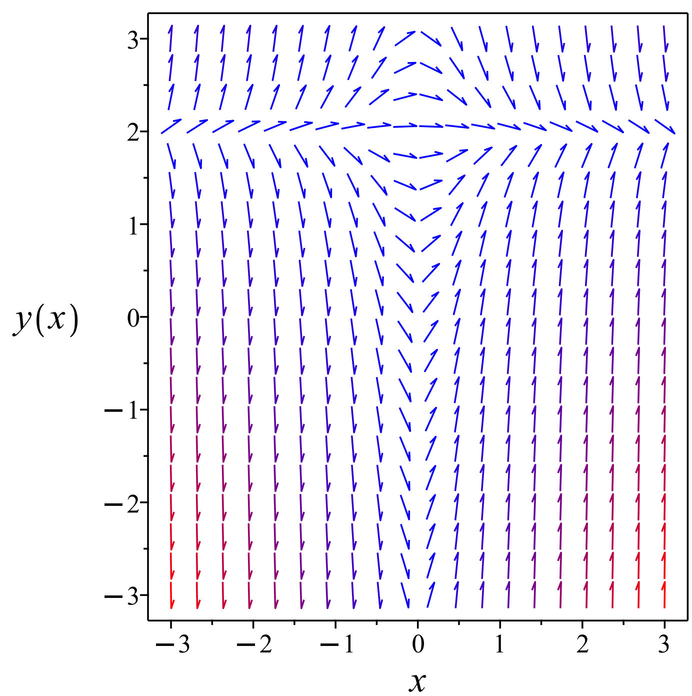


Figure 19: Slope field plot

Verification of solutions

$$y = \frac{\left(2c_2^4 e^{2x^2+4c_1} + 1\right) e^{-2x^2-4c_1}}{c_2^4}$$

Verified OK.

1.8.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 4x \\ q(x) &= 8x \end{aligned}$$

Hence the ode is

$$y' + 4yx = 8x$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 4x dx} \\ &= \mathrm{e}^{2x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(8x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x^2} y\right) &= \left(\mathrm{e}^{2x^2}\right)(8x) \\ \mathrm{d}\left(\mathrm{e}^{2x^2} y\right) &= \left(8x\,\mathrm{e}^{2x^2}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{2x^2} y &= \int 8x\,\mathrm{e}^{2x^2}\, \mathrm{d}x \\ \mathrm{e}^{2x^2} y &= 2\,\mathrm{e}^{2x^2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2x^2}$ results in

$$y = 2\,\mathrm{e}^{-2x^2}\mathrm{e}^{2x^2} + c_1\mathrm{e}^{-2x^2}$$

which simplifies to

$$y = 2 + c_1\mathrm{e}^{-2x^2}$$

Summary

The solution(s) found are the following

$$y = 2 + c_1\mathrm{e}^{-2x^2} \tag{1}$$

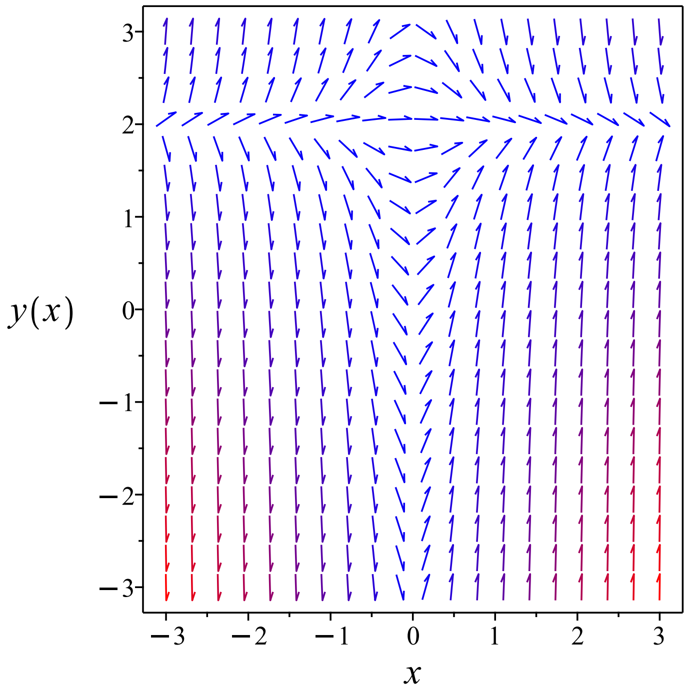


Figure 20: Slope field plot

Verification of solutions

$$y = 2 + c_1 e^{-2x^2}$$

Verified OK.

1.8.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -4xy + 8x \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 17: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{-2x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-2x^2}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{2x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -4xy + 8x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 4x\,\mathrm{e}^{2x^2} y \\ S_y &= \mathrm{e}^{2x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 8x\,\mathrm{e}^{2x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 8R\,\mathrm{e}^{2R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\,\mathrm{e}^{2R^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{2x^2} y = 2\,\mathrm{e}^{2x^2} + c_1$$

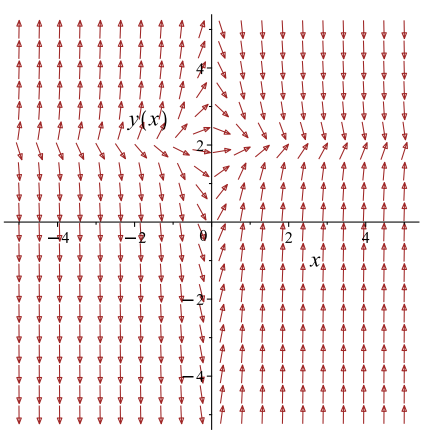
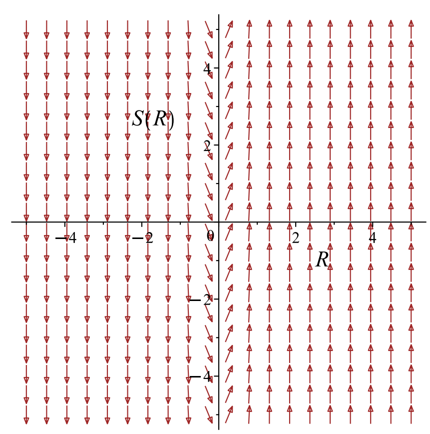
Which simplifies to

$$(y-2)\,\mathrm{e}^{2x^2} - c_1 = 0$$

Which gives

$$y = \left(2\,\mathrm{e}^{2x^2} + c_1\right)\mathrm{e}^{-2x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -4xy + 8x$  	$R = x$ $S = \mathrm{e}^{2x^2} y$	$\frac{dS}{dR} = 8R\,\mathrm{e}^{2R^2}$ 

Summary
The solution(s) found are the following

$$y = \left(2\,\mathrm{e}^{2x^2} + c_1\right)\mathrm{e}^{-2x^2} \tag{1}$$

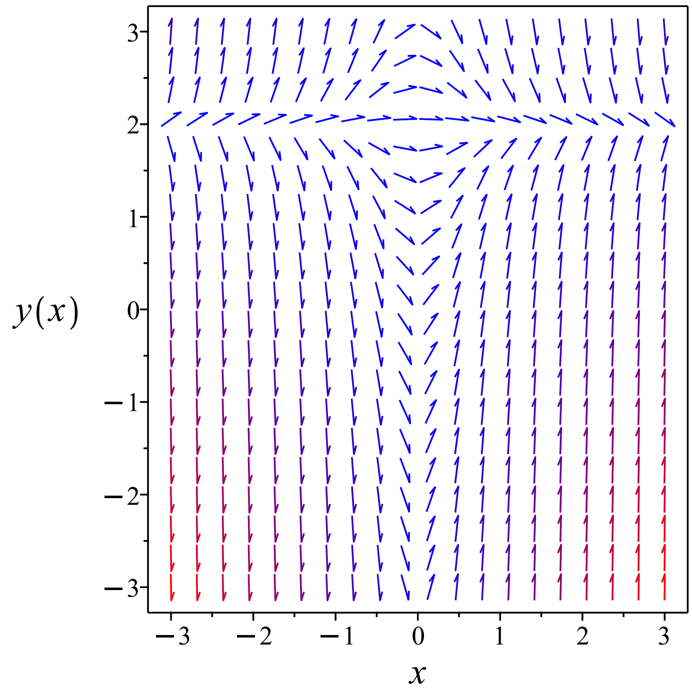


Figure 21: Slope field plot

Verification of solutions

$$y = \left(2\,e^{2x^2} + c_1\right) e^{-2x^2}$$

Verified OK.

1.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{-4y+8}\right)\mathrm{d}y = (x)\,\mathrm{d}x$$
$$(-x)\,\mathrm{d}x + \left(\frac{1}{-4y+8}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -x$$
$$N(x,y) = \frac{1}{-4y+8}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x)$$
$$= 0$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-4y+8}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-4y+8}$. Therefore equation (4) becomes

$$\frac{1}{-4y+8} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{4(y-2)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{4y-8}\right)\mathrm{d}y$$

$$f(y) = -\frac{\ln(y-2)}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{\ln(y-2)}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{\ln(y-2)}{4}$$

The solution becomes

$$y = \mathrm{e}^{-2x^2-4c_1} + 2$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x^2-4c_1} + 2 \tag{1}$$

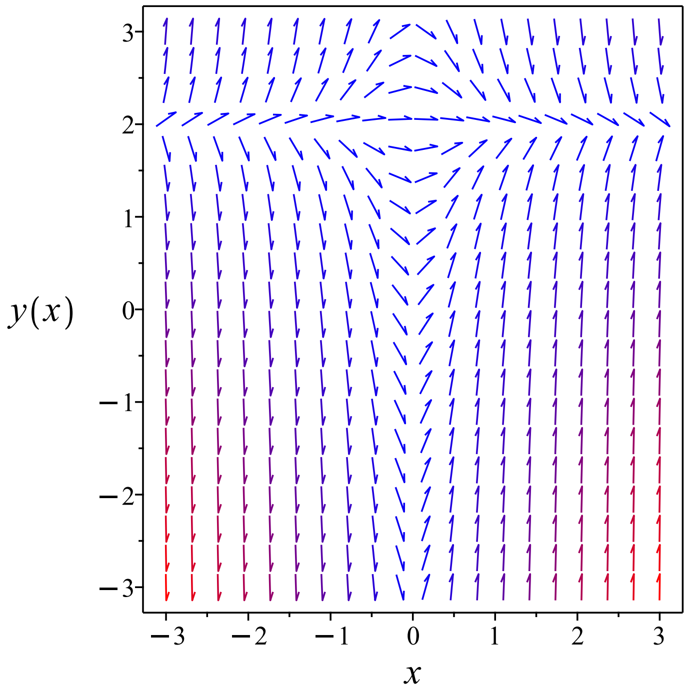


Figure 22: Slope field plot

Verification of solutions

$$y = e^{-2x^2-4c_1} + 2$$

Verified OK.

1.8.5 Maple step by step solution

Let's solve

$y' + 4yx = 8x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y-2} = -4x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y-2}dx = \int -4xdx + c_1$

- Evaluate integral

$$\ln(y-2) = -2x^2 + c_1$$

- Solve for y

$$y = e^{-2x^2+c_1} + 2$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+4*x*y(x)=8*x,y(x), singsol=all)
```

$$y(x) = 2 + e^{-2x^2} c_1$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 22

```
DSolve[y'[x]+4*x*y[x]==8*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2 + c_1 e^{-2x^2}$$
$$y(x) \to 2$$

1.9 problem 4(a)

Internal problem ID [11578]
Internal file name [`OUTPUT/10560_Thursday_May_18_2023_05_56_27_PM_64152595/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 4(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' - 8y = 0}$$

1.9.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -8$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} - 8\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -8$ into the above gives

$$\lambda_{1,2} = \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(-8)}$$
$$= 1 \pm 3$$

Hence

$$\lambda_1 = 1 + 3$$
$$\lambda_2 = 1 - 3$$

Which simplifies to

$$\lambda_1 = 4$$
$$\lambda_2 = -2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(4)x} + c_2 e^{(-2)x}$$

Or

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-2x} \tag{1}$$

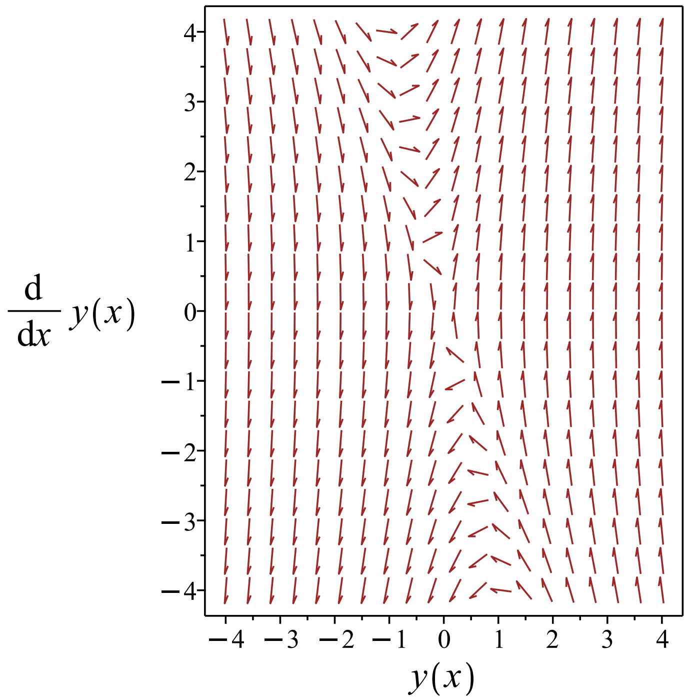


Figure 23: Slope field plot

Verification of solutions

$$y = c_1 e^{4x} + c_2 e^{-2x}$$

Verified OK.

1.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 20: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-3x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1(\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{6x}}{6}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{6x}}{6}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{4x}}{6} \tag{1}
$$

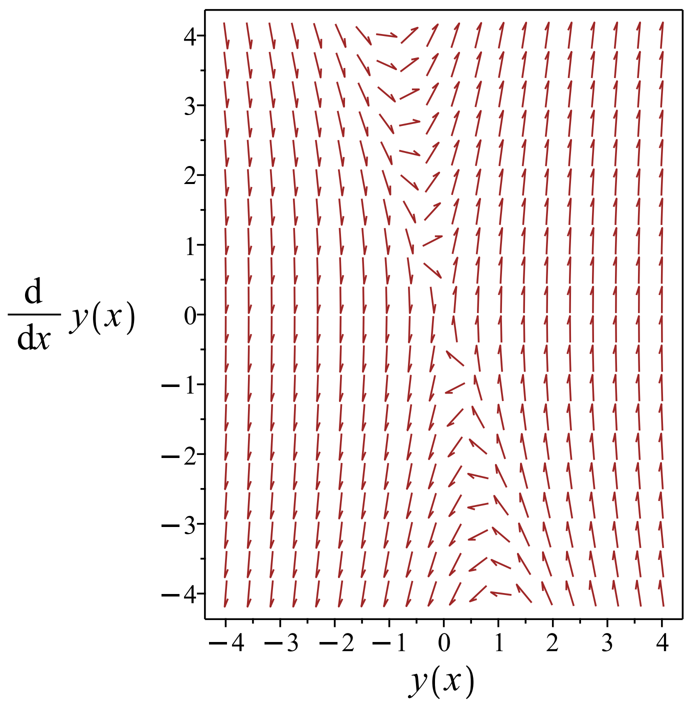


Figure 24: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{4x}}{6}
$$

Verified OK.

1.9.3 Maple step by step solution

Let's solve

$y'' - 2y' - 8y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r - 8 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-4) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 4)$

- 1st solution of the ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} + c_2 e^{4x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{4x} + e^{-2x} c_2$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 22

```
DSolve[y''[x]-2*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x} \left(c_2 e^{6x} + c_1 \right)$$

1.10 problem 4(b)

Internal problem ID [11579]
Internal file name [`OUTPUT/10561_Thursday_May_18_2023_05_56_28_PM_68847962/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 4(b).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 2y'' - 4y' + 8y = 0}$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$$

The roots of the above equation are

$$\lambda_1 = -2$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{2x} + x\, e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = e^{2x}$$
$$y_3 = e^{2x} x$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{2x} + x e^{2x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{2x} + x e^{2x} c_3$$

Verified OK.

1.10.1 Maple step by step solution

Let's solve

$y''' - 2y'' - 4y' + 8y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 2y_3(x) + 4y_2(x) - 8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 2y_3(x) + 4y_2(x) - 8y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 4 & 2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 4 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

$$\vec{y}_2(x) = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, an

$$\vec{y}_3(x) = e^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_3(x)$ into the homogeneous system

$$\lambda\, e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = \left(e^{\lambda x} A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\, e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = e^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_3(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 4 & 2 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

$$\vec{y}_3(x) = e^{2x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

  $\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + e^{2x} c_3 \cdot \left(x \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}\right)$$

- First component of the vector is the solution to the ODE

  $y = \frac{((2x-1)c_3+2c_2)e^{2x}}{8} + \frac{c_1 e^{-2x}}{4}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-2*diff(y(x),x$2)-4*diff(y(x),x)+8*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_3x + c_2)\, e^{2x} + e^{-2x} c_1$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 27

```
DSolve[y'''[x]-2*y''[x]-4*y'[x]+8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(e^{4x}(c_3x + c_2) + c_1\right)$$

1.11 problem 5(a)

Internal problem ID [11580]
Internal file name [`OUTPUT/10562_Thursday_May_18_2023_05_56_29_PM_64086885/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 5(a).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 3y'' - 4y' + 12y = 0}$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = e^{2x}$$
$$y_3 = e^{3x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^{3x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^{3x}$$

Verified OK.

1.11.1 Maple step by step solution

Let's solve

$y''' - 3y'' - 4y' + 12y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 3y_3(x) + 4y_2(x) - 12y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3y_3(x) + 4y_2(x) - 12y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(4c_3 e^{5x} + 9c_2 e^{4x} + 9c_1) e^{-2x}}{36}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)-4*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 \mathrm{e}^{5x} + c_2 \mathrm{e}^{4x} + c_3\right) \mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 29

```
DSolve[y'''[x]-3*y''[x]-4*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(e^{4x}(c_3 e^x + c_2) + c_1\right)$$

1.12 problem 5(b)

Internal problem ID [11581]
Internal file name [`OUTPUT/10563_Thursday_May_18_2023_05_56_30_PM_47063498/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 5(b).
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _fully, _exact, _linear]]
```

$$\boxed{x^3y''' + 2x^2y'' - 10y'x - 8y = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' + 2x^2y'' - 10y'x - 8y = 0$$

gives

$$-10x\lambda\, x^{\lambda-1} + 2x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} - 8x^\lambda = 0$$

Which simplifies to

$$-10\lambda\, x^\lambda + 2\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda - 8x^\lambda = 0$$

And since $x^{\lambda} \neq 0$ then dividing through by x^{λ}, the above becomes

$$-10\lambda + 2\lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) - 8 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - \lambda^2 - 10\lambda - 8 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 4$$
$$\lambda_2 = -2$$
$$\lambda_3 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
-2	1	real root
4	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^{\lambda}$ basis solution. Each real root of multiplicty two, generates $c_1 x^{\lambda}$ and $c_2 x^{\lambda} \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^{\lambda}$ and $c_2 x^{\lambda} \ln(x)$ and $c_3 x^{\lambda} \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x) x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2 x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} + c_3 x^4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{1}{x^2}$$
$$y_3 = x^4$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} + c_3 x^4 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} + c_3 x^4$$

Verified OK.

1.12.1 Maple step by step solution

Let's solve

$x^3 y''' + 2y'' x^2 - 10y' x - 8y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{8y}{x^3} - \frac{2(y'' x - 5y')}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{2y''}{x} - \frac{10y'}{x^2} - \frac{8y}{x^3} = 0$

- Multiply by denominators of the ODE

  $x^3 y''' + 2y'' x^2 - 10y' x - 8y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

$y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

- Calculate the 3rd derivative of y with respect to x , using the chain rule

  $y''' = \left(\frac{d^3}{dt^3}y(t)\right)t'(x)^3 + 3t'(x)\,t''(x)\left(\frac{d^2}{dt^2}y(t)\right) + t'''(x)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) + 2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - 10\frac{d}{dt}y(t) - 8y(t) = 0$

- Simplify

  $\frac{d^3}{dt^3}y(t) - \frac{d^2}{dt^2}y(t) - 10\frac{d}{dt}y(t) - 8y(t) = 0$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = y_3(t) + 10y_2(t) + 8y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = y_3(t) + 10y_2(t) + 8y_1(t)\right]$

- Define vector

  $$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 10 & 1 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 10 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{4t} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2t} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{4t} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y(t) = \frac{(c_3 e^{6t} + 16c_2 e^{t} + 4c_1)e^{-2t}}{16}$$

- Change variables back using $t = \ln(x)$

$$y = \frac{c_3 x^6 + 16c_2 x + 4c_1}{16x^2}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve(x^3*diff(y(x),x$3)+2*x^2*diff(y(x),x$2)-10*x*diff(y(x),x)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^6 + c_2 x + c_3}{x^2}$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 22

```
DSolve[x^3*y'''[x]+2*x^2*y''[x]-10*x*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{c_3 x^6 + c_2 x + c_1}{x^2}$$

1.13 problem 6(a)

Internal problem ID [11582]
Internal file name [`OUTPUT/10564_Thursday_May_18_2023_05_56_31_PM_784968/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 6(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 2y = 6\,\mathrm{e}^{x} + 4x\,\mathrm{e}^{-2x}}$$

1.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= \left(6\,\mathrm{e}^{3x} + 4x\right)\mathrm{e}^{-2x} \end{aligned}$$

Hence the ode is

$$y' + 2y = \left(6\,\mathrm{e}^{3x} + 4x\right)\mathrm{e}^{-2x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 2dx} \\
&= \mathrm{e}^{2x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\left(6\,\mathrm{e}^{3x}+4x\right)\mathrm{e}^{-2x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{2x}\right) &= \left(\mathrm{e}^{2x}\right)\left(\left(6\,\mathrm{e}^{3x}+4x\right)\mathrm{e}^{-2x}\right) \\
\mathrm{d}\left(y\,\mathrm{e}^{2x}\right) &= \left(6\,\mathrm{e}^{3x}+4x\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,\mathrm{e}^{2x} &= \int 6\,\mathrm{e}^{3x}+4x\,\mathrm{d}x \\
y\,\mathrm{e}^{2x} &= 2x^2+2\,\mathrm{e}^{3x}+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2x}$ results in

$$
y = \mathrm{e}^{-2x}\left(2x^2+2\,\mathrm{e}^{3x}\right)+c_1\mathrm{e}^{-2x}
$$

which simplifies to

$$
y = \left(2\,\mathrm{e}^{3x}+2x^2+c_1\right)\mathrm{e}^{-2x}
$$

Summary

The solution(s) found are the following

$$
y = \left(2\,\mathrm{e}^{3x}+2x^2+c_1\right)\mathrm{e}^{-2x} \tag{1}
$$

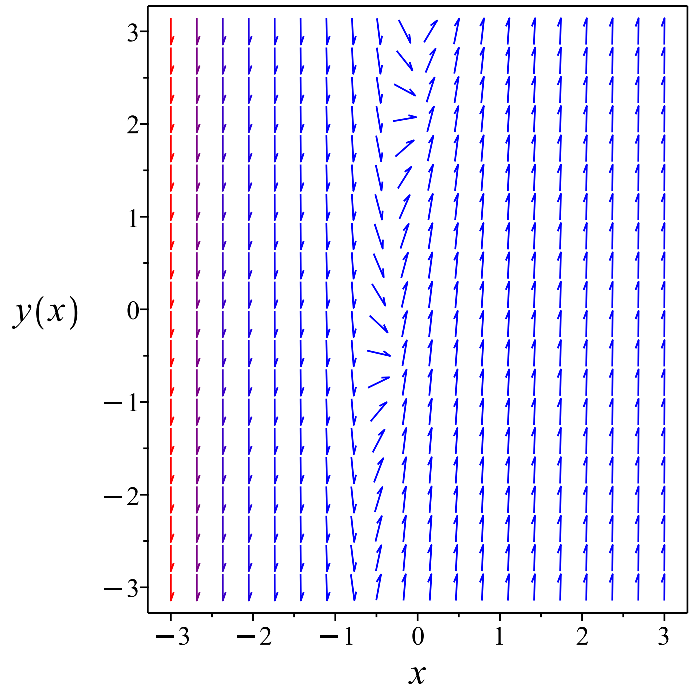


Figure 25: Slope field plot

Verification of solutions

$$y = \left(2\,\mathrm{e}^{3x} + 2x^2 + c_1\right)\mathrm{e}^{-2x}$$

Verified OK.

1.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -2y + 6\,\mathrm{e}^{x} + 4x\,\mathrm{e}^{-2x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 25: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-2x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-2x}} dy \end{aligned}$$

Which results in

$$S = y\,e^{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2y + 6\,e^{x} + 4x\,e^{-2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2y\,e^{2x} \\ S_y &= e^{2x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 6\,e^{3x} + 4x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 6\,e^{3R} + 4R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2R^2 + 2\,e^{3R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,e^{2x} = 2\,e^{3x} + 2x^2 + c_1$$

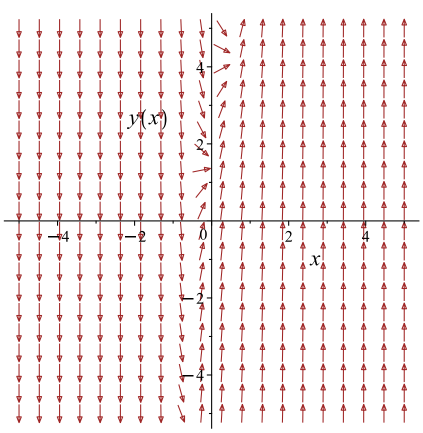
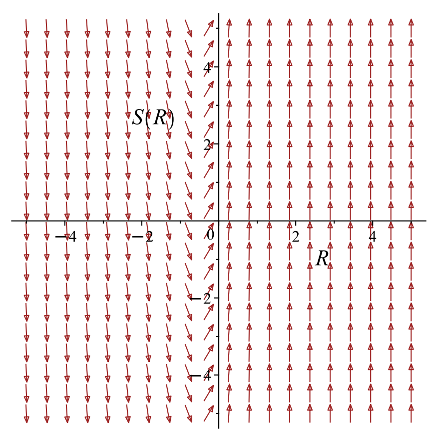
Which simplifies to

$$y\,e^{2x} = 2\,e^{3x} + 2x^2 + c_1$$

Which gives

$$y = \left(2\,e^{3x} + 2x^2 + c_1\right) e^{-2x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2y + 6\,e^{x} + 4x\,e^{-2x}$ 	$R = x$ $S = y\,e^{2x}$	$\frac{dS}{dR} = 6\,e^{3R} + 4R$ 

Summary
The solution(s) found are the following

$$y = \left(2\,e^{3x} + 2x^2 + c_1\right) e^{-2x} \tag{1}$$

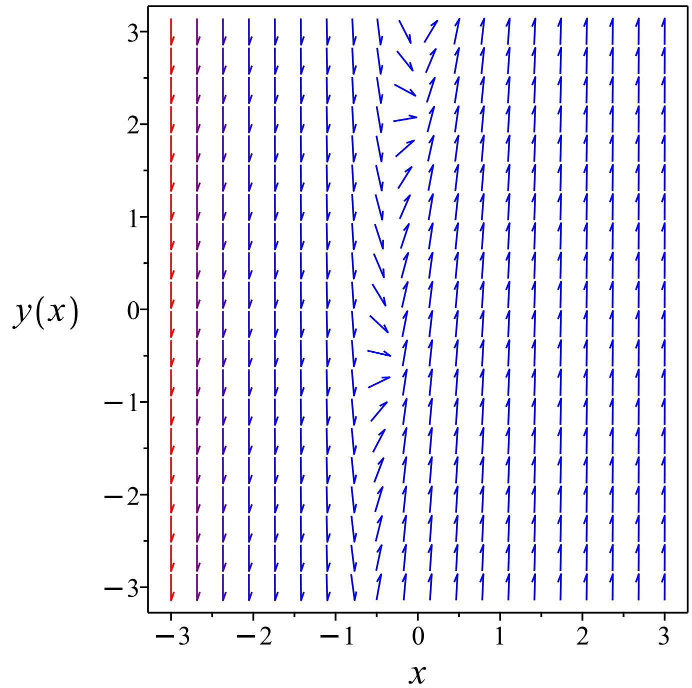


Figure 26: Slope field plot

Verification of solutions

$$y = \left(2\,\mathrm{e}^{3x} + 2x^2 + c_1\right)\mathrm{e}^{-2x}$$

Verified OK.

1.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(-2y + 6\,\mathrm{e}^{x} + 4x\,\mathrm{e}^{-2x}\right)\mathrm{d}x \\
\left(2y - 6\,\mathrm{e}^{x} - 4x\,\mathrm{e}^{-2x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 2y - 6\,\mathrm{e}^{x} - 4x\,\mathrm{e}^{-2x} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y - 6\,\mathrm{e}^{x} - 4x\,\mathrm{e}^{-2x}\right) \\
&= 2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((2) - (0)) \\ &= 2 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 2 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{2x} \\ &= \mathrm{e}^{2x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{2x}\left(2y - 6\,\mathrm{e}^{x} - 4x\,\mathrm{e}^{-2x}\right) \\ &= -6\,\mathrm{e}^{3x} + 2y\,\mathrm{e}^{2x} - 4x \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{2x}(1) \\ &= \mathrm{e}^{2x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-6\,\mathrm{e}^{3x} + 2y\,\mathrm{e}^{2x} - 4x\right) + \left(\mathrm{e}^{2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -6\,\mathrm{e}^{3x} + 2y\,\mathrm{e}^{2x} - 4x \,\mathrm{d}x \\
\phi &= y\,\mathrm{e}^{2x} - 2x^2 - 2\,\mathrm{e}^{3x} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x} + f'(y) \qquad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x}$. Therefore equation (4) becomes

$$\mathrm{e}^{2x} = \mathrm{e}^{2x} + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,\mathrm{e}^{2x} - 2x^2 - 2\,\mathrm{e}^{3x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,\mathrm{e}^{2x} - 2x^2 - 2\,\mathrm{e}^{3x}$$

The solution becomes

$$y = \left(2\,\mathrm{e}^{3x} + 2x^2 + c_1\right) \mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = \left(2\,\mathrm{e}^{3x} + 2x^2 + c_1\right) \mathrm{e}^{-2x} \qquad (1)$$

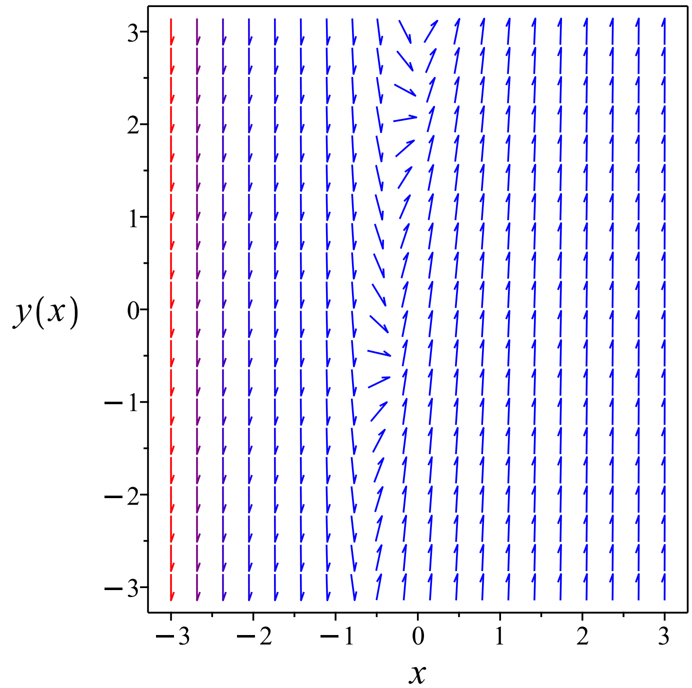


Figure 27: Slope field plot

Verification of solutions

$$y = (2\,e^{3x} + 2x^2 + c_1)\,e^{-2x}$$

Verified OK.

1.13.4 Maple step by step solution

Let's solve

$y' + 2y = 6\,e^{x} + 4x\,e^{-2x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2y + 6\,e^{x} + 4x\,e^{-2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2y = 6\,e^{x} + 4x\,e^{-2x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + 2y) = \mu(x)\,(6\,e^{x} + 4x\,e^{-2x})$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'+2y) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = 2\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = e^{2x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,(6\,e^{x} + 4x\,e^{-2x})\,dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,(6\,e^{x} + 4x\,e^{-2x})\,dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)(6\,e^{x}+4x\,e^{-2x})dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = e^{2x}$

  $y = \frac{\int e^{2x}(6\,e^{x}+4x\,e^{-2x})dx + c_1}{e^{2x}}$
- Evaluate the integrals on the rhs

  $y = \frac{2(e^{x})^{3}+2x^{2}+c_1}{e^{2x}}$
- Simplify

  $y = (2\,e^{3x} + 2x^{2} + c_1)\,e^{-2x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x)+2*y(x)=6*exp(x)+4*x*exp(-2*x),y(x), singsol=all)
```

$$y(x) = \left(2x^2 + 2\,e^{3x} + c_1\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.129 (sec). Leaf size: 26

```
DSolve[y'[x]+2*y[x]==6*Exp[x]+4*x*Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(2x^2 + 2e^{3x} + c_1\right)$$

1.14 problem 6(b)

Internal problem ID [11583]
Internal file name [`OUTPUT/10565_Thursday_May_18_2023_05_56_32_PM_79597876/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 6(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y' + 4y = -8\sin(2x)}$$

1.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = -8\sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-4)^2 - (4)(1)(4)}\\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 e^{2x} + c_2 e^{2x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{2x} x$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-8\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x} x, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 \sin(2x) - 8A_2 \cos(2x) = -8 \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{2x} x\right) + \left(-\cos(2x)\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) - \cos(2x)$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) - \cos(2x) \tag{1}$$

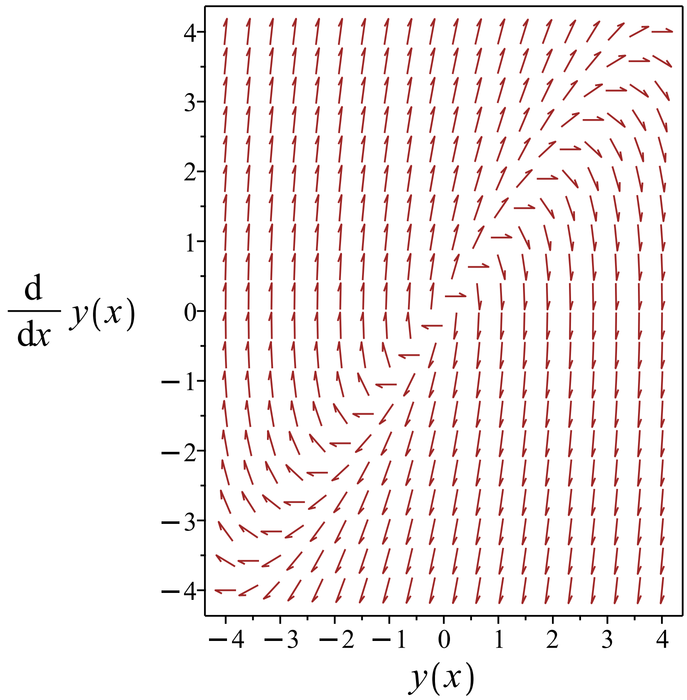


Figure 28: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2 x + c_1) - \cos(2x)$$

Verified OK.

1.14.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -4\,dx} \\ &= e^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= -8\,\mathrm{e}^{-2x} \sin(2x) \\
\left(\mathrm{e}^{-2x} y\right)'' &= -8\,\mathrm{e}^{-2x} \sin(2x)
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{-2x} y\right)' = 2\,\mathrm{e}^{-2x}(\cos(2x) + \sin(2x)) + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{-2x} y\right) = c_1 x - \mathrm{e}^{-2x} \cos(2x) + c_2
$$

Hence the solution is

$$
y = \frac{c_1 x - \mathrm{e}^{-2x} \cos(2x) + c_2}{\mathrm{e}^{-2x}}
$$

Or

$$
y = \mathrm{e}^{2x} c_1 x + c_2 \mathrm{e}^{2x} - 2\cos(x)^2 + 1
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \mathrm{e}^{2x} c_1 x + c_2 \mathrm{e}^{2x} - 2\cos(x)^2 + 1 \tag{1}
$$

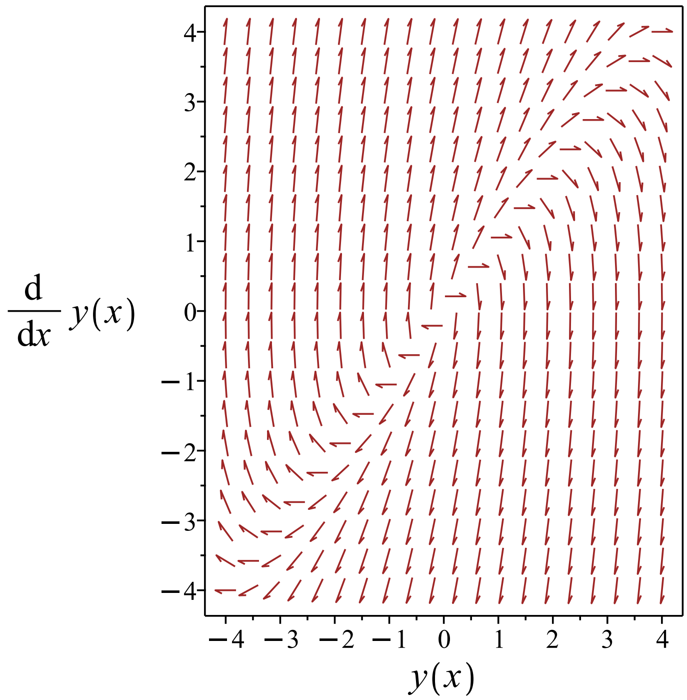


Figure 29: Slope field plot

Verification of solutions

$$y = e^{2x}c_1x + c_2e^{2x} - 2\cos(x)^2 + 1$$

Verified OK.

1.14.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 28: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\, dx} \\ &= z_1 e^{2x} \\ &= z_1 \left(e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{2x} x
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
-8\sin(2x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(2x), \sin(2x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{2x} x, \mathrm{e}^{2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(2x) + A_2 \sin(2x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
8A_1 \sin(2x) - 8A_2 \cos(2x) = -8\sin(2x)
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{2x} x\right) + (-\cos(2x)) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) - \cos(2x)$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) - \cos(2x) \tag{1}$$

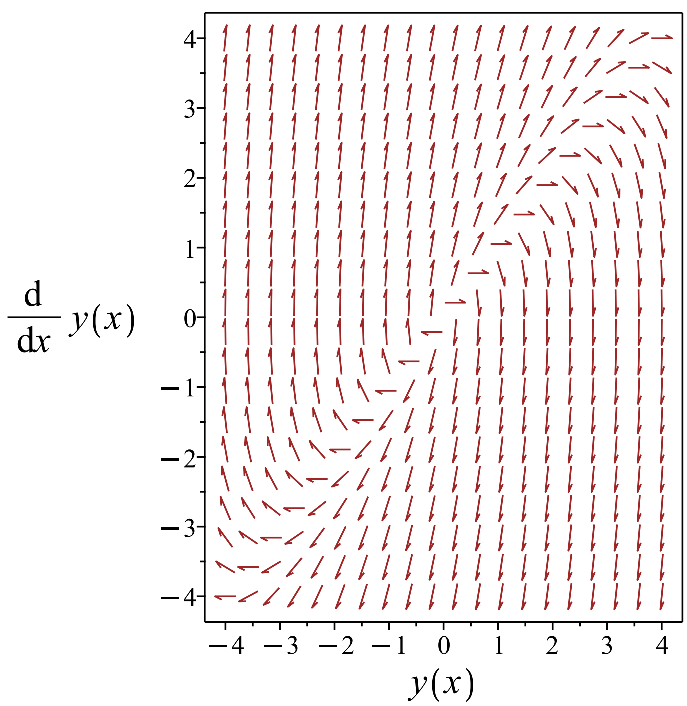


Figure 30: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2 x + c_1) - \cos(2x)$$

Verified OK.

1.14.4 Maple step by step solution

Let's solve

$y'' - 4y' + 4y = -8\sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^{2x}x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{2x} + c_2 e^{2x}x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = -8\sin(2x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & e^{2x}x \\ 2e^{2x} & 2e^{2x}x + e^{2x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{4x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = 8\,e^{2x}\left(\int \sin(2x)\, x\, e^{-2x} dx - x\left(\int e^{-2x} \sin(2x)\, dx\right)\right)$
- Compute integrals

  $y_p(x) = -\cos(2x)$

- Substitute particular solution into general solution to ODE

  $y = c_2 e^{2x} x + c_1 e^{2x} - \cos(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=-8*sin(2*x),y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2)\, e^{2x} - \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 25

```
DSolve[y''[x]-4*y'[x]+4*y[x]==-8*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\cos(2x) + e^{2x}(c_2 x + c_1)$$

1.15 problem 7(a)

Internal problem ID [11584]
Internal file name [`OUTPUT/10566_Thursday_May_18_2023_05_56_33_PM_7548041/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, Differential equations and their solutions. Exercises page 13
Problem number: 7(a).
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y'^2 - 4y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 2\sqrt{y} \tag{1}$$
$$y' = -2\sqrt{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{2\sqrt{y}} dy = \int dx$$
$$\sqrt{y} = x + c_1$$

Summary
The solution(s) found are the following

$$\sqrt{y} = x + c_1 \tag{1}$$

Verification of solutions

$$\sqrt{y} = x + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{2\sqrt{y}}dy = \int dx$$

$$-\sqrt{y} = x + c_2$$

Summary

The solution(s) found are the following

$$-\sqrt{y} = x + c_2 \tag{1}$$

Verification of solutions

$$-\sqrt{y} = x + c_2$$

Verified OK.

1.15.1 Maple step by step solution

Let's solve

$y'^2 - 4y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{y}} = 2$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y}}dx = \int 2dx + c_1$

- Evaluate integral

  $2\sqrt{y} = 2x + c_1$

- Solve for y

  $y = \frac{1}{4}c_1^2 + c_1x + x^2$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   <- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)^2-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = (x - c_1)^2$$

✓ Solution by Mathematica

Time used: 0.079 (sec). Leaf size: 38

```
DSolve[(y'[x])^2-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}(-2x + c_1)^2$$
$$y(x) \to \frac{1}{4}(2x + c_1)^2$$
$$y(x) \to 0$$

2 Chapter 1, section 1.3. Exercises page 22

2.1 problem 1

Internal problem ID [11585]
Internal file name [`OUTPUT/10567_Thursday_May_18_2023_05_56_33_PM_90907946/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, section 1.3. Exercises page 22
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' - 6y = 0}$$

With initial conditions

$$[y(0) = 6, y'(0) = 2]$$

2.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -6 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y' - 6y = 0$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.1.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda\, e^{\lambda x} - 6\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(-6)}\\ &= -\frac{1}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{5}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{5}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(-3)x}$$

Or

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{-3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 6$ and $x = 0$ in the above gives

$$6 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} - 3c_2 e^{-3x}$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = 2c_1 - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = 2$$

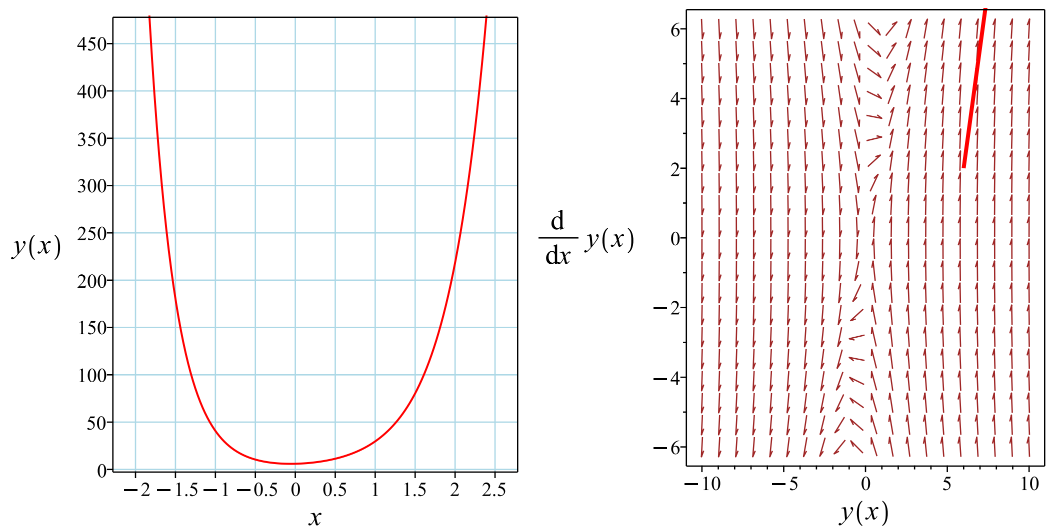
Substituting these values back in above solution results in

$$y = 4 e^{2x} + 2 e^{-3x}$$

Summary

The solution(s) found are the following

$$y = 4\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{-3x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 4\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{-3x}$$

Verified OK.

2.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 6y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 31: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{5x}}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-3x}\right) + c_2 \left(e^{-3x} \left(\frac{e^{5x}}{5}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 e^{-3x} + \frac{c_2 e^{2x}}{5} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 6$ and $x = 0$ in the above gives

$$
6 = c_1 + \frac{c_2}{5} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -3c_1 e^{-3x} + \frac{2c_2 e^{2x}}{5}
$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$
2 = -3c_1 + \frac{2c_2}{5} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= 20
\end{aligned}
$$

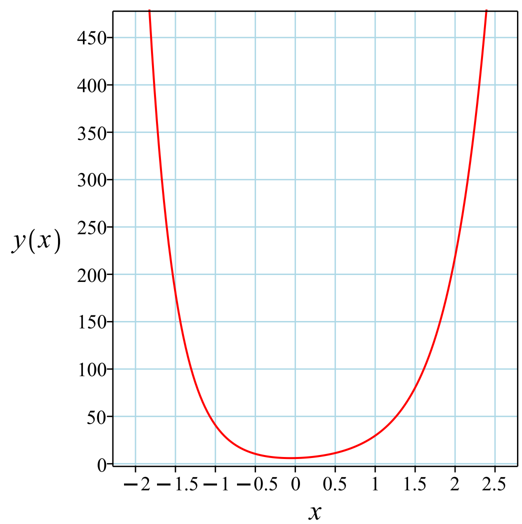
Substituting these values back in above solution results in

$$
y = 4\,e^{2x} + 2\,e^{-3x}
$$

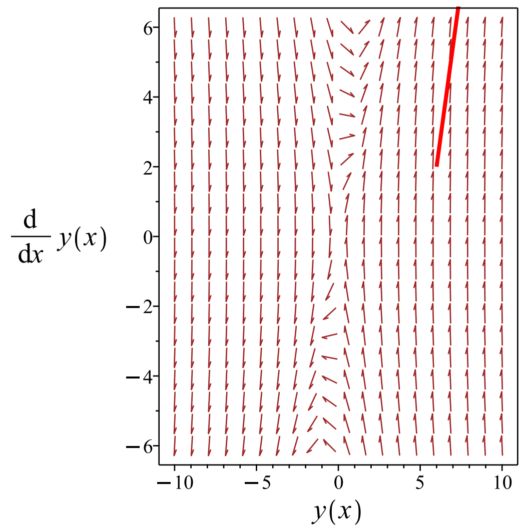
Summary

The solution(s) found are the following

$$
y = 4\,e^{2x} + 2\,e^{-3x} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 4\,e^{2x} + 2\,e^{-3x}$$

Verified OK.

2.1.4 Maple step by step solution

Let's solve

$$\left[y'' + y' - 6y = 0, y(0) = 6, y'\Big|_{\{x=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of ODE

  $r^2 + r - 6 = 0$
- Factor the characteristic polynomial

  $(r+3)(r-2) = 0$
- Roots of the characteristic polynomial

  $r = (-3, 2)$
- 1st solution of the ODE

$y_1(x) = e^{-3x}$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-3x} + c_2 e^{2x}$

□ Check validity of solution $y = c_1 e^{-3x} + c_2 e^{2x}$

  - Use initial condition $y(0) = 6$

    $6 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -3c_1 e^{-3x} + 2c_2 e^{2x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 2$

    $2 = -3c_1 + 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 2, c_2 = 4\}$

  - Substitute constant values into general solution and simplify

    $y = (4\,e^{5x} + 2)\,e^{-3x}$

- Solution to the IVP

  $y = (4\,e^{5x} + 2)\,e^{-3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2)+diff(y(x),x)-6*y(x)=0,y(0) = 6, D(y)(0) = 2],y(x), singsol=all)
```

$$y(x) = \left(4\,\mathrm{e}^{5x} + 2\right) \mathrm{e}^{-3x}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 20

```
DSolve[{y''[x]+y'[x]-6*y[x]==0,{y[0]==6,y'[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(4e^{5x} + 2\right)$$

2.2 problem 2(a)

Internal problem ID [11586]
Internal file name [`OUTPUT/10568_Thursday_May_18_2023_05_56_34_PM_55001789/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, section 1.3. Exercises page 22
Problem number: 2(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = 2x\,\mathrm{e}^{-x}}$$

With initial conditions

$$[y(0) = 2]$$

2.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 2x\,\mathrm{e}^{-x} \end{aligned}$$

Hence the ode is

$$y' + y = 2x\,e^{-x}$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2x\,e^{-x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.2.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1dx} \\ &= e^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(2x\,e^{-x}\right) \\ \frac{d}{dx}(e^{x}y) &= (e^{x})\left(2x\,e^{-x}\right) \\ d(e^{x}y) &= (2x)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{x}y &= \int 2x\,dx \\ e^{x}y &= x^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = x^2 e^{-x} + c_1 e^{-x}$$

which simplifies to

$$y = e^{-x}\left(x^2 + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

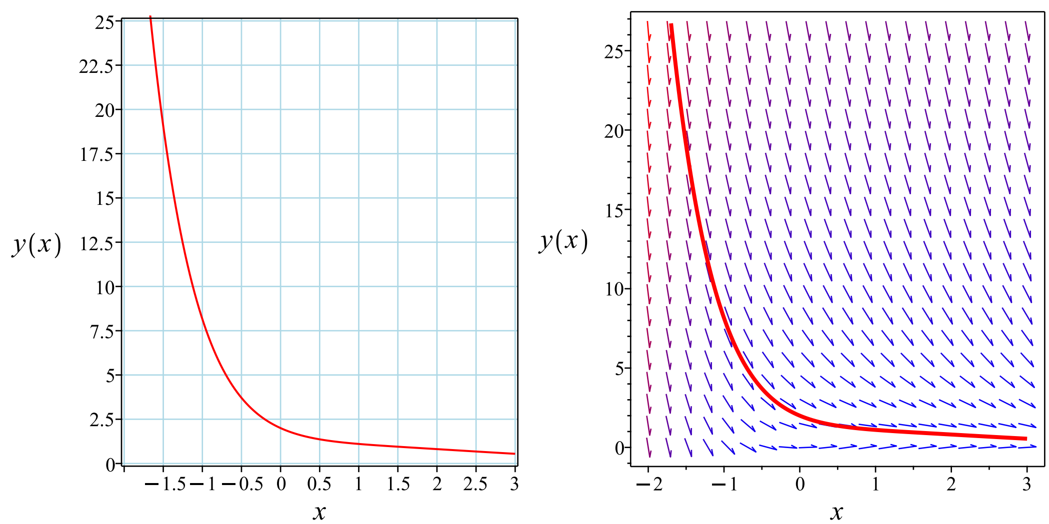
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$y = e^{-x}(x^2 + 2)$$

Summary
The solution(s) found are the following

$$y = e^{-x}(x^2 + 2) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = e^{-x}(x^2 + 2)$$

Verified OK.

2.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -y + 2x\,\mathrm{e}^{-x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 33: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{-x} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{e^{-x}} dy
\end{aligned}
$$

Which results in

$$
S = e^{x} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \qquad (2)
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -y + 2x\, e^{-x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= e^{x} y \\
S_y &= e^{x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{x} = x^2 + c_1$$

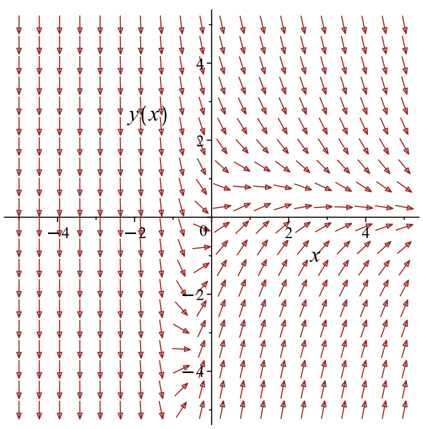
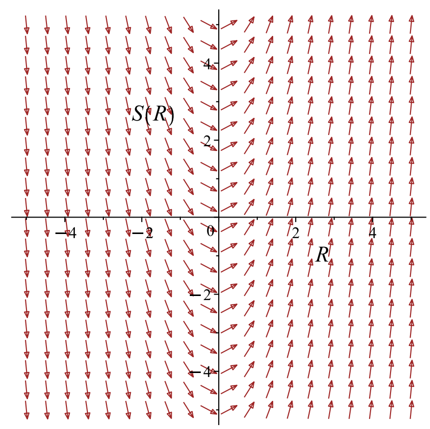
Which simplifies to

$$y\,\mathrm{e}^{x} = x^2 + c_1$$

Which gives

$$y = \mathrm{e}^{-x}\left(x^2 + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y + 2x\,\mathrm{e}^{-x}$ 	$R = x$ $S = \mathrm{e}^{x} y$	$\frac{dS}{dR} = 2R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

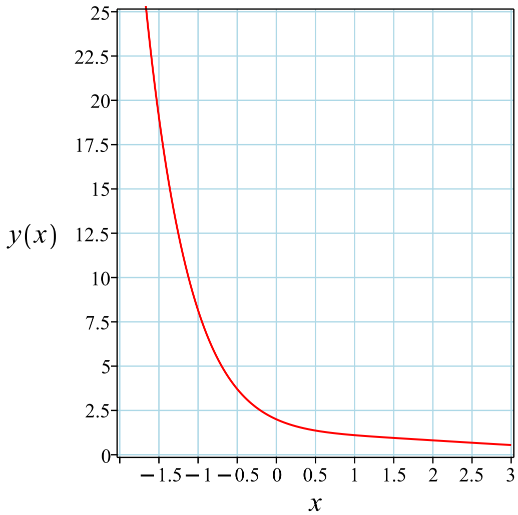
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

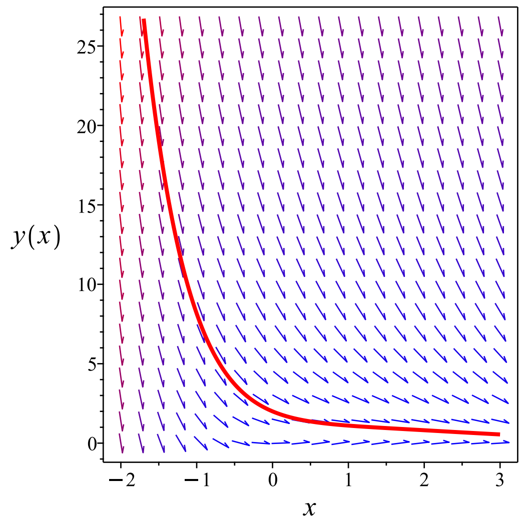
$$y = x^2 \mathrm{e}^{-x} + 2\,\mathrm{e}^{-x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = x^2 \mathrm{e}^{-x} + 2\,\mathrm{e}^{-x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = x^2 e^{-x} + 2 e^{-x}$$

Verified OK.

2.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= \left(-y + 2x\,\mathrm{e}^{-x}\right)\mathrm{d}x \\ \left(y - 2x\,\mathrm{e}^{-x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= y - 2x\,\mathrm{e}^{-x} \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y - 2x\,\mathrm{e}^{-x}\right) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((1) - (0)) \\ &= 1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x} \\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{x}\left(y - 2x\,\mathrm{e}^{-x}\right) \\ &= \mathrm{e}^{x} y - 2x\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{x}(1) \\ &= \mathrm{e}^{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (\mathrm{e}^{x} y - 2x) + (\mathrm{e}^{x})\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \mathrm{e}^{x} y - 2x \, \mathrm{d}x\end{aligned}$$

$$\phi = \mathrm{e}^{x} y - x^{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{x} y - x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{x} y - x^2$$

The solution becomes

$$y = \mathrm{e}^{-x}\left(x^2 + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

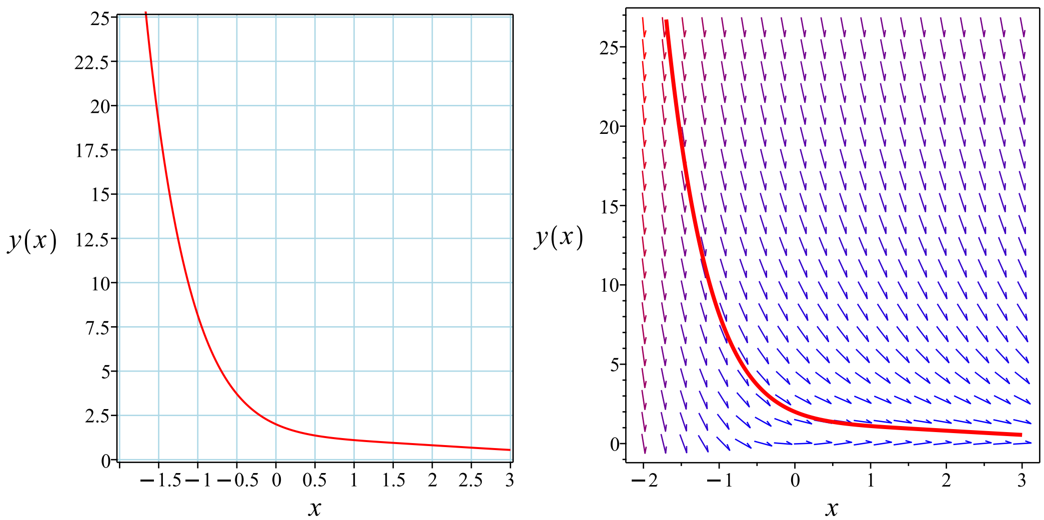
Substituting c_1 found above in the general solution gives

$$y = x^2 \mathrm{e}^{-x} + 2\,\mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = x^2 \mathrm{e}^{-x} + 2\,\mathrm{e}^{-x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x^2 \mathrm{e}^{-x} + 2\,\mathrm{e}^{-x}$$

Verified OK.

2.2.5 Maple step by step solution

Let's solve

$[y' + y = 2x\,\mathrm{e}^{-x}, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + 2x\,\mathrm{e}^{-x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = 2x\,\mathrm{e}^{-x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\,(y'+y) = 2\mu(x)\,x\,\mathrm{e}^{-x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'+y) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\,x\,\mathrm{e}^{-x} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 2\mu(x)\,x\,\mathrm{e}^{-x} dx + c_1$
- Solve for y

  $y = \frac{\int 2\mu(x)x\,\mathrm{e}^{-x} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int 2\,\mathrm{e}^{-x} x\,\mathrm{e}^{x} dx + c_1}{\mathrm{e}^{x}}$
- Evaluate the integrals on the rhs

  $y = \frac{x^2 + c_1}{\mathrm{e}^{x}}$
- Simplify

  $y = \mathrm{e}^{-x}(x^2 + c_1)$
- Use initial condition $y(0) = 2$

  $2 = c_1$
- Solve for c_1

  $c_1 = 2$
- Substitute $c_1 = 2$ into general solution and simplify

  $y = \mathrm{e}^{-x}(x^2 + 2)$
- Solution to the IVP

  $y = \mathrm{e}^{-x}(x^2 + 2)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)+y(x)=2*x*exp(-x),y(0) = 2],y(x), singsol=all)
```

$$y(x) = (x^2 + 2) e^{-x}$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 16

```
DSolve[{y'[x]+y[x]==2*x*Exp[-x],{y[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(x^2 + 2)$$

2.3 problem 2(b)

Internal problem ID [11587]
Internal file name [`OUTPUT/10569_Thursday_May_18_2023_05_56_36_PM_46001342/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, section 1.3. Exercises page 22
Problem number: 2(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = 2x\,\mathrm{e}^{-x}}$$

With initial conditions

$$[y(-1) = \mathrm{e} + 3]$$

2.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 2x\,\mathrm{e}^{-x} \end{aligned}$$

Hence the ode is

$$y' + y = 2x\,\mathrm{e}^{-x}$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is inside this domain. The domain of $q(x) = 2x\,\mathrm{e}^{-x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -1$ is also inside this domain. Hence solution exists and is unique.

2.3.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 1 dx}\\ &= \mathrm{e}^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(2x\,\mathrm{e}^{-x}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} y) &= (\mathrm{e}^{x})\left(2x\,\mathrm{e}^{-x}\right)\\ \mathrm{d}(\mathrm{e}^{x} y) &= (2x)\ \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{x} y &= \int 2x\,\mathrm{d}x\\ \mathrm{e}^{x} y &= x^{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$y = x^{2}\mathrm{e}^{-x} + c_1\mathrm{e}^{-x}$$

which simplifies to

$$y = \mathrm{e}^{-x}\left(x^{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \mathrm{e} + 3$ in the above solution gives an equation to solve for the constant of integration.

$$\mathrm{e} + 3 = \mathrm{e}(1 + c_1)$$

$$c_1 = 3\,\mathrm{e}^{-1}$$

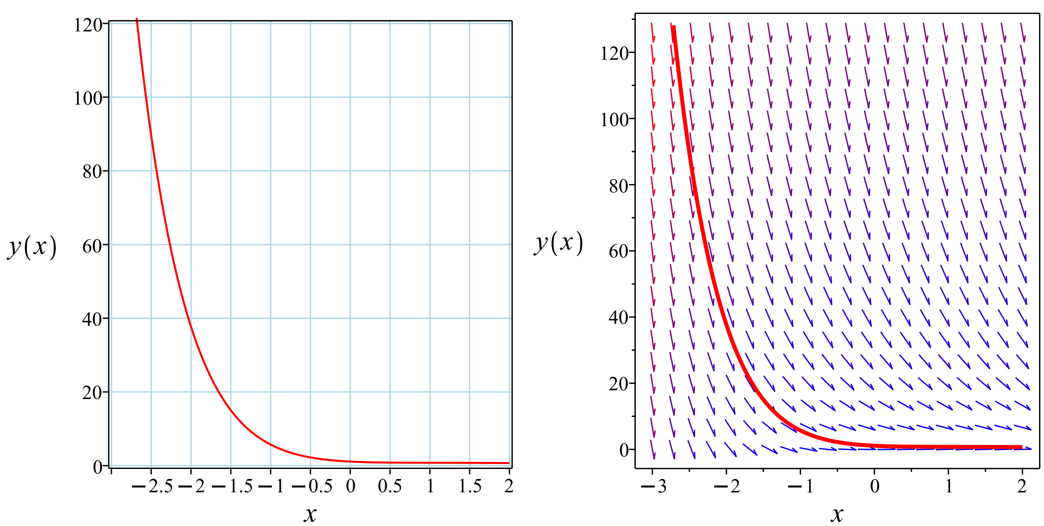
Substituting c_1 found above in the general solution gives

$$y = x^2\mathrm{e}^{-x} + 3\,\mathrm{e}^{-x-1}$$

Summary

The solution(s) found are the following

$$y = x^2\mathrm{e}^{-x} + 3\,\mathrm{e}^{-x-1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x^2\mathrm{e}^{-x} + 3\,\mathrm{e}^{-x-1}$$

Verified OK.

2.3.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -y + 2x\,\mathrm{e}^{-x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 36: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \mathrm{e}^{-x}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{\mathrm{e}^{-x}} dy\end{aligned}$$

Which results in

$$S = \mathrm{e}^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y + 2x\,\mathrm{e}^{-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= \mathrm{e}^{x} y\\ S_y &= \mathrm{e}^{x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{x} = x^2 + c_1$$

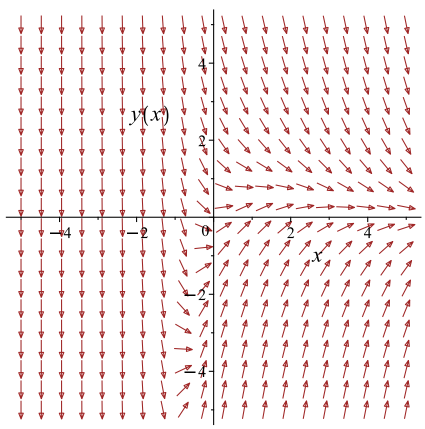
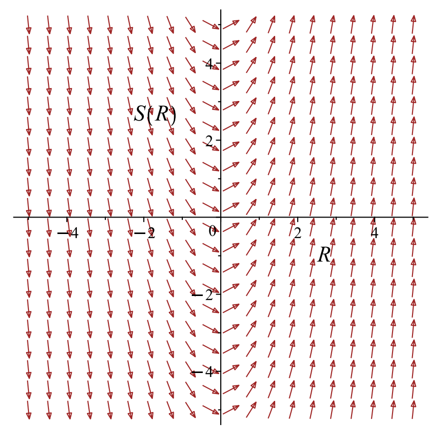
Which simplifies to

$$y\,\mathrm{e}^{x} = x^2 + c_1$$

Which gives

$$y = \mathrm{e}^{-x}\left(x^2 + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y + 2x\,\mathrm{e}^{-x}$ 	$R = x$ $S = \mathrm{e}^{x} y$	$\frac{dS}{dR} = 2R$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \mathrm{e} + 3$ in the above solution gives an equation to solve for the constant of integration.

$$\mathrm{e} + 3 = \mathrm{e} c_1 + \mathrm{e}$$

$$c_1 = 3\,\mathrm{e}^{-1}$$

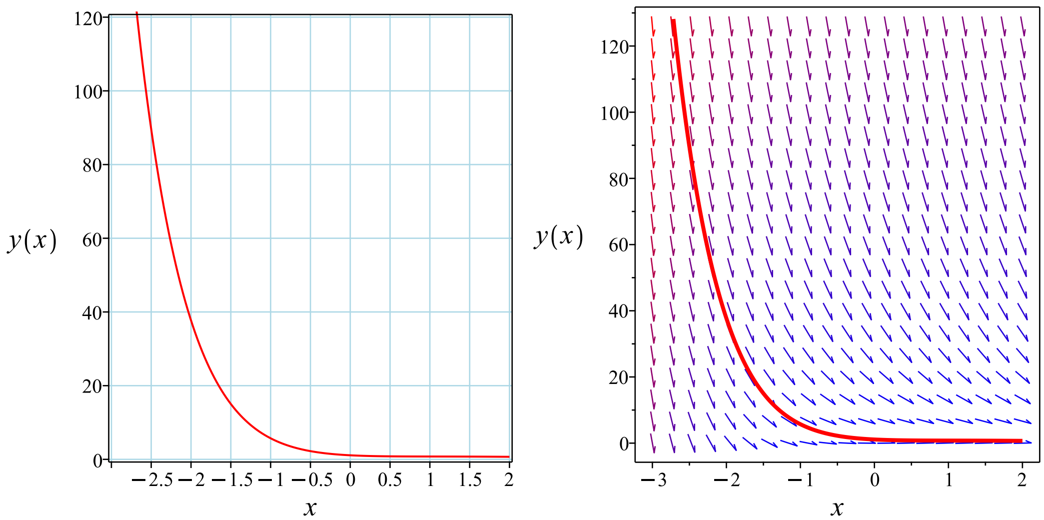
Substituting c_1 found above in the general solution gives

$$y = x^2 \mathrm{e}^{-x} + 3\,\mathrm{e}^{-x-1}$$

Summary

The solution(s) found are the following

$$y = x^2 \mathrm{e}^{-x} + 3\,\mathrm{e}^{-x-1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x^2 e^{-x} + 3\, e^{-x-1}$$

Verified OK.

2.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(-y + 2x\,\mathrm{e}^{-x}\right)\mathrm{d}x\\ \left(y - 2x\,\mathrm{e}^{-x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= y - 2x\,\mathrm{e}^{-x}\\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y - 2x\,\mathrm{e}^{-x}\right)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((1) - (0))\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x} \\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{x}\left(y - 2x\,\mathrm{e}^{-x}\right) \\ &= \mathrm{e}^{x} y - 2x\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{x}(1) \\ &= \mathrm{e}^{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\mathrm{e}^{x} y - 2x\right) + \left(\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \mathrm{e}^{x} y - 2x \, \mathrm{d}x$$

$$\phi = \mathrm{e}^{x} y - x^{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^x$. Therefore equation (4) becomes

$$\mathrm{e}^x = \mathrm{e}^x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^x y - x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^x y - x^2$$

The solution becomes

$$y = \mathrm{e}^{-x}\left(x^2 + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = \mathrm{e} + 3$ in the above solution gives an equation to solve for the constant of integration.

$$\mathrm{e} + 3 = \mathrm{e} c_1 + \mathrm{e}$$

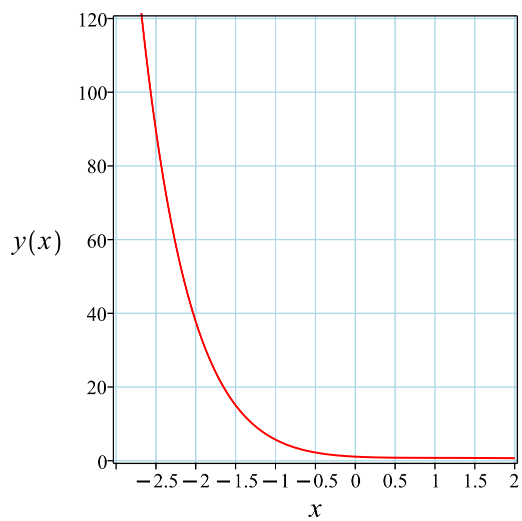
$$c_1 = 3\,\mathrm{e}^{-1}$$

Substituting c_1 found above in the general solution gives

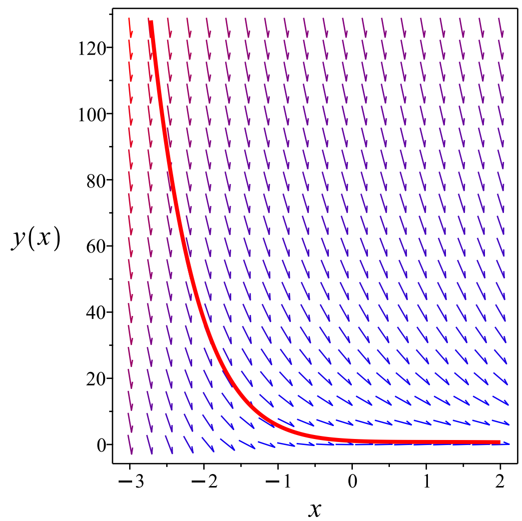
$$y = x^2 \mathrm{e}^{-x} + 3\,\mathrm{e}^{-x-1}$$

Summary
The solution(s) found are the following

$$y = x^2 e^{-x} + 3 e^{-x-1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^2 e^{-x} + 3 e^{-x-1}$$

Verified OK.

2.3.5 Maple step by step solution

Let's solve

$[y' + y = 2x e^{-x}, y(-1) = e + 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + 2x e^{-x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = 2x e^{-x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\,(y'+y) = 2\mu(x)\,x\,\mathrm{e}^{-x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'+y) = \mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\,x\,\mathrm{e}^{-x}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 2\mu(x)\,x\,\mathrm{e}^{-x}dx + c_1$

- Solve for y

  $y = \frac{\int 2\mu(x)x\,\mathrm{e}^{-x}dx+c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int 2\,\mathrm{e}^{-x}x\,\mathrm{e}^{x}dx+c_1}{\mathrm{e}^{x}}$

- Evaluate the integrals on the rhs

  $y = \frac{x^2+c_1}{\mathrm{e}^{x}}$

- Simplify

  $y = \mathrm{e}^{-x}(x^2+c_1)$

- Use initial condition $y(-1) = \mathrm{e}+3$

  $\mathrm{e}+3 = \mathrm{e}(1+c_1)$

- Solve for c_1

  $c_1 = \frac{3}{\mathrm{e}}$

- Substitute $c_1 = \frac{3}{\mathrm{e}}$ into general solution and simplify

  $y = (x^2+3\,\mathrm{e}^{-1})\,\mathrm{e}^{-x}$

- Solution to the IVP

  $y = (x^2+3\,\mathrm{e}^{-1})\,\mathrm{e}^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)+y(x)=2*x*exp(-x),y(-1) = exp(1)+3],y(x), singsol=all)
```

$$y(x) = (x^2 + 3\,e^{-1})\,e^{-x}$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 20

```
DSolve[{y'[x]+y[x]==2*x*Exp[-x],{y[-1]==Exp[1]+3}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x-1}(ex^2 + 3)$$

2.4 problem 3(a)

Internal problem ID [11588]
Internal file name [`OUTPUT/10570_Thursday_May_18_2023_05_56_36_PM_46263500/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, section 1.3. Exercises page 22
Problem number: 3(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' - 12y = 0}$$

With initial conditions

$$[y(0) = 5, y'(0) = 6]$$

2.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = -1$$
$$q(x) = -12$$
$$F = 0$$

Hence the ode is

$$y'' - y' - 12y = 0$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -12$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.4.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -12$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda\, e^{\lambda x} - 12\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 12 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -12$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-12)}\\ &= \frac{1}{2} \pm \frac{7}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{2} + \frac{7}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{7}{2}$$

Which simplifies to

$$\lambda_1 = 4$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(4)x} + c_2 e^{(-3)x}$$

Or

$$y = c_1 e^{4x} + c_2 e^{-3x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{4x} + c_2 e^{-3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 0$ in the above gives

$$5 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 4c_1 e^{4x} - 3c_2 e^{-3x}$$

substituting $y' = 6$ and $x = 0$ in the above gives

$$6 = 4c_1 - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

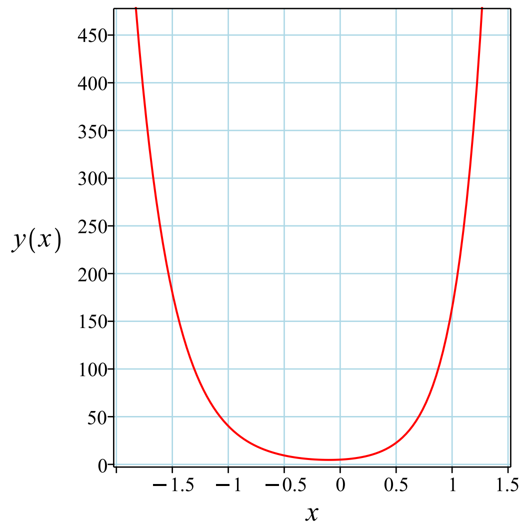
$$c_1 = 3$$
$$c_2 = 2$$

Substituting these values back in above solution results in

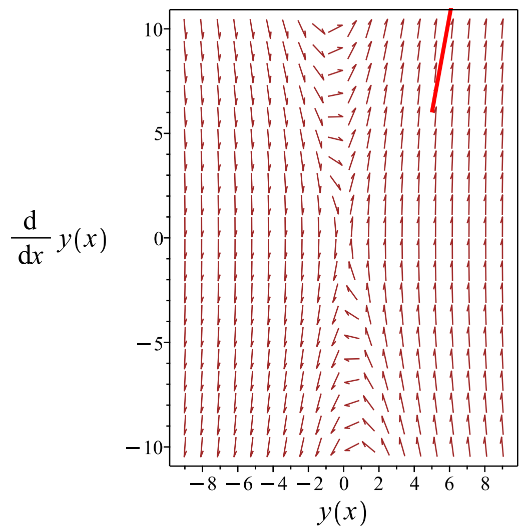
$$y = 3\,e^{4x} + 2\,e^{-3x}$$

Summary
The solution(s) found are the following

$$y = 3\,\mathrm{e}^{4x} + 2\,\mathrm{e}^{-3x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{4x} + 2\,\mathrm{e}^{-3x}$$

Verified OK.

2.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 12y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{49}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 49 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{49z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 39: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{49}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{7x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{7x}}{7}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{7x}}{7}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{4x}}{7} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 0$ in the above gives

$$
5 = c_1 + \frac{c_2}{7} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -3c_1 \mathrm{e}^{-3x} + \frac{4c_2 \mathrm{e}^{4x}}{7}
$$

substituting $y' = 6$ and $x = 0$ in the above gives

$$
6 = -3c_1 + \frac{4c_2}{7} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= 21
\end{aligned}
$$

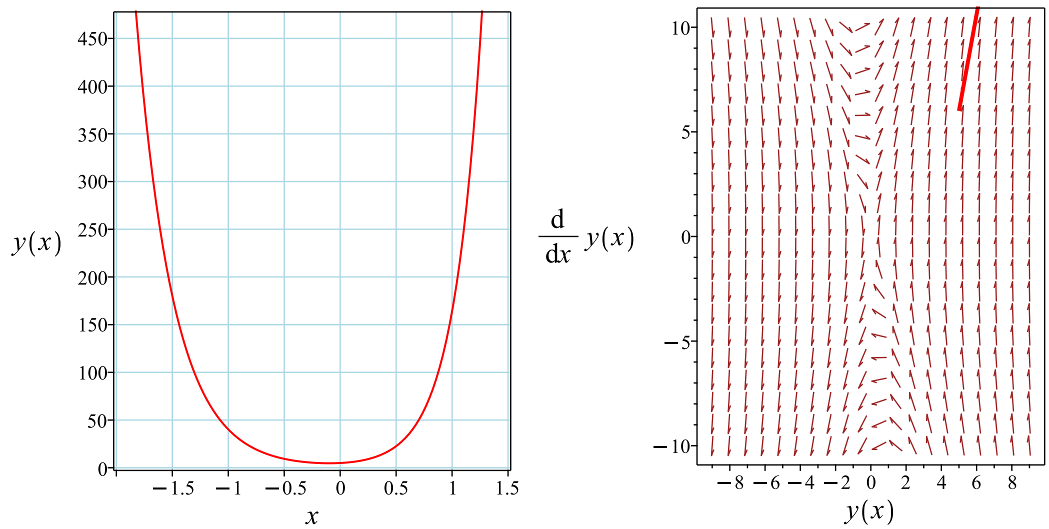
Substituting these values back in above solution results in

$$
y = 3\,\mathrm{e}^{4x} + 2\,\mathrm{e}^{-3x}
$$

Summary

The solution(s) found are the following

$$
y = 3\,\mathrm{e}^{4x} + 2\,\mathrm{e}^{-3x} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 3\,\mathrm{e}^{4x} + 2\,\mathrm{e}^{-3x}$$

Verified OK.

2.4.4 Maple step by step solution

Let's solve

$$\left[y'' - y' - 12y = 0, y(0) = 5, y'\Big|_{\{x=0\}} = 6\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - r - 12 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-4) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 4)$

- 1st solution of the ODE

$y_1(x) = e^{-3x}$

- 2nd solution of the ODE

  $y_2(x) = e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-3x} + c_2 e^{4x}$

☐ Check validity of solution $y = c_1 e^{-3x} + c_2 e^{4x}$

  - Use initial condition $y(0) = 5$

    $5 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -3c_1 e^{-3x} + 4c_2 e^{4x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 6$

    $6 = -3c_1 + 4c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 2, c_2 = 3\}$

  - Substitute constant values into general solution and simplify

    $y = (3\,e^{7x} + 2)\,e^{-3x}$

- Solution to the IVP

  $y = (3\,e^{7x} + 2)\,e^{-3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2)-diff(y(x),x)-12*y(x)=0,y(0) = 5, D(y)(0) = 6],y(x), singsol=all)
```

$$y(x) = (3\,e^{7x} + 2)\,e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 20

```
DSolve[{y''[x]-y'[x]-12*y[x]==0,{y[0]==5,y'[0]==6}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}(3e^{7x} + 2)$$

2.5 problem 4(a)

Internal problem ID [11589]
Internal file name [`OUTPUT/10571_Thursday_May_18_2023_05_56_37_PM_91740308/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, section 1.3. Exercises page 22
Problem number: 4(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + y = 0}$$

With initial conditions

$$\left[y(0) = 0, y'\left(\frac{\pi}{2}\right) = 1\right]$$

2.5.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sin(x) c_1 + c_2 \cos(x)$$

substituting $y' = 1$ and $x = \frac{\pi}{2}$ in the above gives

$$1 = -c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

2.5.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + yy')\, dx = 0$$
$$\frac{y'^2}{2} + \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2 + 2c_1}} dy = \int dx$$

$$\arctan\left(\frac{y}{\sqrt{-y^2 + 2c_1}}\right) = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-y^2 + 2c_1}} dy = \int dx$$

$$-\arctan\left(\frac{y}{\sqrt{-y^2 + 2c_1}}\right) = c_3 + x$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\arctan\left(\frac{y}{\sqrt{-y^2 + 2c_1}}\right) = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \left(\tan(x + c_2)^2 + 1\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(x + c_2)^2 + 1}} - \frac{\tan(x + c_2)^2\sqrt{2}\,c_1}{\sqrt{\frac{c_1}{\tan(x+c_2)^2+1}}\,\left(\tan(x + c_2)^2 + 1\right)}$$

substituting $y' = 1$ and $x = \frac{\pi}{2}$ in the above gives

$$1 = \frac{\sin(c_2)^2\sqrt{2}\,c_1}{\sqrt{\sin(c_2)^2\, c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$-\arctan\left(\frac{y}{\sqrt{-y^{2}+2c_{1}}}\right) = c_{3}+x \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\left(\tan\left(c_{3}+x\right)^{2}+1\right)\sqrt{2}\,\sqrt{\frac{c_{1}}{\tan\left(c_{3}+x\right)^{2}+1}}+\frac{\tan\left(c_{3}+x\right)^{2}\sqrt{2}\,c_{1}}{\sqrt{\frac{c_{1}}{\tan(c_{3}+x)^{2}+1}}\,\left(\tan\left(c_{3}+x\right)^{2}+1\right)}$$

substituting $y' = 1$ and $x = \frac{\pi}{2}$ in the above gives

$$1 = -\frac{\sin\left(c_{3}\right)^{2}\sqrt{2}\,c_{1}}{\sqrt{\sin\left(c_{3}\right)^{2}c_{1}}} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_{1}, c_{3}\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

2.5.3 Solving using Kovacic algorithm

Writing the ode as

$$\begin{align*} y''+y &= 0 \tag{1}\\ Ay''+By'+Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= 1\\ B &= 0 \tag{3}\\ C &= 1 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 41: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sin(x) c_1 + c_2 \cos(x)$$

substituting $y' = 1$ and $x = \frac{\pi}{2}$ in the above gives

$$1 = -c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

2.5.4 Maple step by step solution

Let's solve

$\left[y'' + y = 0, y(0) = 0, y'\Big|_{\left\{x=\frac{\pi}{2}\right\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 \cos(x) + c_2 \sin(x)$

□ Check validity of solution $y = c_1 \cos(x) + c_2 \sin(x)$
  - Use initial condition $y(0) = 0$

    $0 = c_1$
  - Compute derivative of the solution

    $y' = -\sin(x) c_1 + c_2 \cos(x)$
  - Use the initial condition $y'\Big|_{\{x=\frac{\pi}{2}\}} = 1$

    $1 = -c_1$
  - Solve for c_1 and c_2
  - The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✗ Solution by Maple

```
dsolve([diff(y(x),x$2)+y(x)=0,y(0) = 0, D(y)(1/2*Pi) = 1],y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]+y[x]==0,{y[0]==0,y'[Pi/2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

{}

2.6 problem 4(b)

Internal problem ID [11590]
Internal file name [`OUTPUT/10572_Thursday_May_18_2023_05_56_37_PM_6449168/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, section 1.3. Exercises page 22
Problem number: 4(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + y = 0}$$

With initial conditions

$$\left[y(0) = 0, y'\left(\frac{\pi}{2}\right) = -1\right]$$

2.6.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sin(x) c_1 + c_2 \cos(x)$$

substituting $y' = -1$ and $x = \frac{\pi}{2}$ in the above gives

$$-1 = -c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

2.6.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + yy')\, dx = 0$$
$$\frac{y'^2}{2} + \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2+2c_1}}dy = \int dx$$

$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-y^2+2c_1}}dy = \int dx$$

$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = c_3 + x$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \left(\tan(x+c_2)^2 + 1\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(x+c_2)^2+1}} - \frac{\tan(x+c_2)^2\sqrt{2}\,c_1}{\sqrt{\frac{c_1}{\tan(x+c_2)^2+1}}\left(\tan(x+c_2)^2+1\right)}$$

substituting $y' = -1$ and $x = \frac{\pi}{2}$ in the above gives

$$-1 = \frac{\sin(c_2)^2\sqrt{2}\,c_1}{\sqrt{\sin(c_2)^2\,c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = c_3 + x \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\left(\tan\left(c_3+x\right)^2+1\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan\left(c_3+x\right)^2+1}} + \frac{\tan\left(c_3+x\right)^2\sqrt{2}\,c_1}{\sqrt{\frac{c_1}{\tan(c_3+x)^2+1}}\,\left(\tan\left(c_3+x\right)^2+1\right)}$$

substituting $y' = -1$ and $x = \frac{\pi}{2}$ in the above gives

$$-1 = -\frac{\sin\left(c_3\right)^2\sqrt{2}\,c_1}{\sqrt{\sin\left(c_3\right)^2 c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

2.6.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 43: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sin(x) c_1 + c_2 \cos(x)$$

substituting $y' = -1$ and $x = \frac{\pi}{2}$ in the above gives

$$-1 = -c_1 \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

2.6.4 Maple step by step solution

Let's solve

$$\left[y'' + y = 0, y(0) = 0, y'\Big|_{\{x=\frac{\pi}{2}\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 \cos(x) + c_2 \sin(x)$

□ Check validity of solution $y = c_1 \cos(x) + c_2 \sin(x)$

- Use initial condition $y(0) = 0$

  $0 = c_1$
- Compute derivative of the solution

  $y' = -\sin(x) c_1 + c_2 \cos(x)$
- Use the initial condition $y'\Big|_{\{x=\frac{\pi}{2}\}} = -1$

  $-1 = -c_1$
- Solve for c_1 and c_2
- The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✗ Solution by Maple

```
dsolve([diff(y(x),x$2)+y(x)=0,y(0) = 0, D(y)(1/2*Pi) = -1],y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]+y[x]==0,{y[0]==0,y'[Pi/2]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

{}

2.7 problem 4(c)

Internal problem ID [11591]
Internal file name [`OUTPUT/10573_Thursday_May_18_2023_05_56_38_PM_91594257/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, section 1.3. Exercises page 22
Problem number: 4(c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y = 0}$$

With initial conditions

$$[y(0) = 0, y'(\pi) = 1]$$

2.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sin(x) c_1 + c_2 \cos(x)$$

substituting $y' = 1$ and $x = \pi$ in the above gives

$$1 = -c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = -\sin(x)$$

Summary
The solution(s) found are the following

$$y = -\sin(x) \tag{1}$$

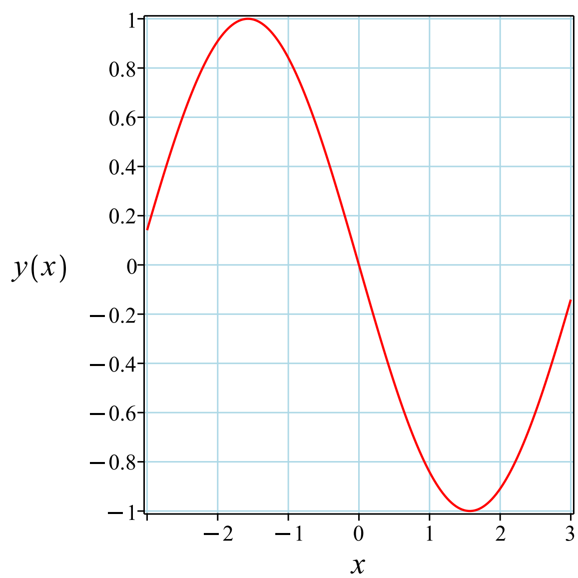


Figure 41: Solution plot

Verification of solutions

$$y = -\sin(x)$$

Verified OK.

2.7.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + yy')\, dx = 0$$
$$\frac{y'^2}{2} + \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-y^2 + 2c_1}} dy = \int dx$$
$$\arctan\left(\frac{y}{\sqrt{-y^2 + 2c_1}}\right) = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-y^2 + 2c_1}} dy = \int dx$$
$$-\arctan\left(\frac{y}{\sqrt{-y^2 + 2c_1}}\right) = c_3 + x$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \left(\tan(x+c_2)^2 + 1\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(x+c_2)^2+1}} - \frac{\tan(x+c_2)^2\sqrt{2}\,c_1}{\sqrt{\frac{c_1}{\tan(x+c_2)^2+1}}\,\left(\tan(x+c_2)^2+1\right)}$$

substituting $y' = 1$ and $x = \pi$ in the above gives

$$1 = \frac{\cos(c_2)^2\sqrt{2}\,c_1}{\sqrt{\cos(c_2)^2\,c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2}$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$\arctan\left(\frac{y}{\sqrt{1-y^2}}\right) = x$$

Looking at the Second solution

$$-\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = c_3 + x \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\left(\tan \left(c_3 + x\right)^2 + 1\right) \sqrt{2}\, \sqrt{\frac{c_1}{\tan \left(c_3 + x\right)^2 + 1}} + \frac{\tan \left(c_3 + x\right)^2 \sqrt{2}\, c_1}{\sqrt{\frac{c_1}{\tan(c_3+x)^2+1}}\, \left(\tan \left(c_3 + x\right)^2 + 1\right)}$$

substituting $y' = 1$ and $x = \pi$ in the above gives

$$1 = -\frac{\cos \left(c_3\right)^2 \sqrt{2}\, c_1}{\sqrt{\cos \left(c_3\right)^2 c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

Summary

The solution(s) found are the following

$$\arctan \left(\frac{y}{\sqrt{1 - y^2}}\right) = x \tag{1}$$

Verification of solutions

$$\arctan \left(\frac{y}{\sqrt{1 - y^2}}\right) = x$$

Verified OK.

2.7.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 45: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)(\tan(x))
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sin(x) c_1 + c_2 \cos(x)$$

substituting $y' = 1$ and $x = \pi$ in the above gives

$$1 = -c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= -1
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = -\sin(x)$$

Summary

The solution(s) found are the following

$$y = -\sin(x) \tag{1}$$

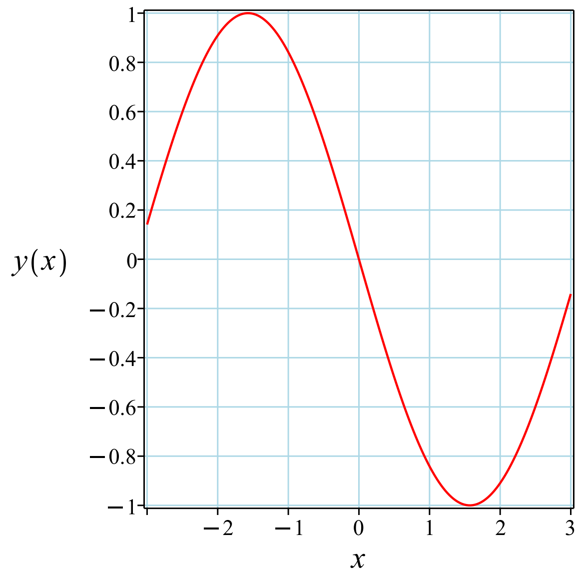


Figure 42: Solution plot

Verification of solutions

$$y = -\sin(x)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve([diff(y(x),x$2)+y(x)=0,y(0) = 0, D(y)(Pi) = 1],y(x), singsol=all)
```

$$y(x) = -\sin(x)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 9

```
DSolve[{y''[x]+y[x]==0,{y[0]==0,y'[Pi]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sin(x)$$

2.8 problem 5

Internal problem ID [11592]
Internal file name [`OUTPUT/10574_Thursday_May_18_2023_05_56_39_PM_4049348/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, section 1.3. Exercises page 22
Problem number: 5.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y''' - 3x^2y'' + 6y'x - 6y = 0}$$

With initial conditions

$$[y(2) = 0, y'(2) = 2, y''(2) = 6]$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' - 3x^2y'' + 6y'x - 6y = 0$$

gives

$$6x\lambda\, x^{\lambda-1} - 3x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} - 6x^\lambda = 0$$

Which simplifies to

$$6\lambda\, x^\lambda - 3\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda - 6x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$6\lambda - 3\lambda(\lambda-1) + \lambda(\lambda-1)(\lambda-2) - 6 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

This table summarises the result

root	multiplicity	type of root
1	1	real root
2	1	real root
3	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_3 x^3 + c_2 x^2 + c_1 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = x$$
$$y_2 = x^2$$
$$y_3 = x^3$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_3x^3 + c_2x^2 + c_1x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$0 = 8c_3 + 4c_2 + 2c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_3x^2 + 2c_2x + c_1$$

substituting $y' = 2$ and $x = 2$ in the above gives

$$2 = 12c_3 + 4c_2 + c_1 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = 6c_3x + 2c_2$$

substituting $y'' = 6$ and $x = 2$ in the above gives

$$6 = 12c_3 + 2c_2 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 2 \\ c_2 &= -3 \\ c_3 &= 1 \end{aligned}$$

Substituting these values back in above solution results in

$$y = x^3 - 3x^2 + 2x$$

Summary

The solution(s) found are the following

$$y = x^3 - 3x^2 + 2x \tag{1}$$

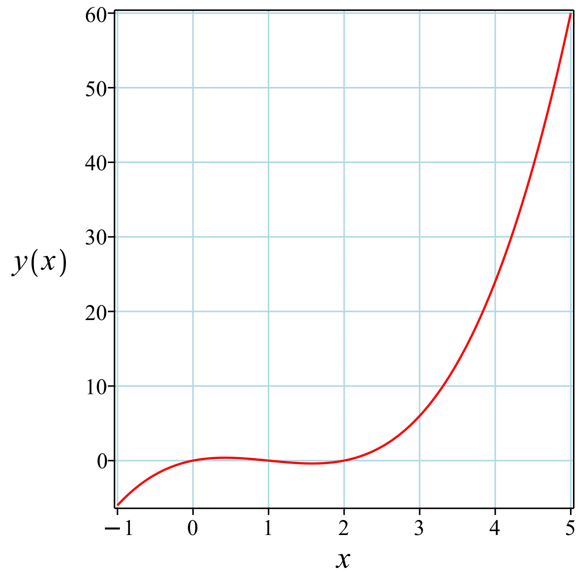


Figure 43: Solution plot

Verification of solutions

$$y = x^3 - 3x^2 + 2x$$

Verified OK.

2.8.1 Maple step by step solution

Let's solve

$$\left[x^3y''' - 3y''x^2 + 6y'x - 6y = 0, y(2) = 0, y'\Big|_{\{x=2\}} = 2, y''\Big|_{\{x=2\}} = 6\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{6y}{x^3} + \frac{3(y''x - 2y')}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{3y''}{x} + \frac{6y'}{x^2} - \frac{6y}{x^3} = 0$

- Multiply by denominators of the ODE

  $x^3y''' - 3y''x^2 + 6y'x - 6y = 0$

- Make a change of variables

$t = \ln(x)$

☐ Substitute the change of variables back into the ODE

- Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

- Calculate the 3rd derivative of y with respect to x , using the chain rule

  $y''' = \left(\frac{d^3}{dt^3}y(t)\right)t'(x)^3 + 3t'(x)\,t''(x)\left(\frac{d^2}{dt^2}y(t)\right) + t'''(x)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) - 3\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 + 6\frac{d}{dt}y(t) - 6y(t) = 0$

- Simplify

  $\frac{d^3}{dt^3}y(t) - 6\frac{d^2}{dt^2}y(t) + 11\frac{d}{dt}y(t) - 6y(t) = 0$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = 6y_3(t) - 11y_2(t) + 6y_1(t)$

Convert linear ODE into a system of first order ODEs

$\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 6y_3(t) - 11y_2(t) + 6y_1(t)\right]$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$

- Rewrite the system as

$\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^t \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$y(t) = c_1 e^t + \frac{c_2 e^{2t}}{4} + \frac{c_3 e^{3t}}{9}$

- Change variables back using $t = \ln(x)$

  $y = c_1 x + \frac{1}{4} c_2 x^2 + \frac{1}{9} c_3 x^3$
- Use the initial condition $y(2) = 0$

  $0 = 2c_1 + c_2 + \frac{8c_3}{9}$
- Calculate the 1st derivative of the solution

  $y' = c_1 + \frac{1}{2} c_2 x + \frac{1}{3} c_3 x^2$
- Use the initial condition $y'\Big|_{\{x=2\}} = 2$

  $2 = c_1 + c_2 + \frac{4c_3}{3}$
- Calculate the 2nd derivative of the solution

  $y'' = \frac{c_2}{2} + \frac{2c_3 x}{3}$
- Use the initial condition $y''\Big|_{\{x=2\}} = 6$

  $6 = \frac{c_2}{2} + \frac{4c_3}{3}$
- Solve for the unknown coefficients

  $\{c_1 = 2, c_2 = -12, c_3 = 9, x = x\}$
- Solution to the IVP

  $y = x^3 - 3x^2 + 2x$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([x^3*diff(y(x),x$3)-3*x^2*diff(y(x),x$2)+6*x*diff(y(x),x)-6*y(x)=0,y(2) = 0, D(y)(2)
```

$$y(x) = x^3 - 3x^2 + 2x$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 15

```
DSolve[{x^3*y'''[x]-3*x^2*y''[x]+6*x*y'[x]-6*y[x]==0,{y[2]==0,y'[2]==2,y''[2]==6}},y[x],x,In
```

$$y(x) \to x(x^2 - 3x + 2)$$

2.9 problem 6(a)

Internal problem ID [11593]
Internal file name [`OUTPUT/10575_Thursday_May_18_2023_05_56_39_PM_18318295/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, section 1.3. Exercises page 22
Problem number: 6(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - x^2 \sin(y) = 0}$$

With initial conditions

$$[y(1) = -2]$$

2.9.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= x^2 \sin(y) \end{aligned}$$

The x domain of $f(x, y)$ when $y = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 \sin(y)\right) \\ &= x^2 \cos(y)\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -2$ is inside this domain. Therefore solution exists and is unique.

2.9.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= x^2 \sin(y)\end{aligned}$$

Where $f(x) = x^2$ and $g(y) = \sin(y)$. Integrating both sides gives

$$\frac{1}{\sin(y)}\, dy = x^2\, dx$$

$$\int \frac{1}{\sin(y)}\, dy = \int x^2\, dx$$

$$\ln(\csc(y) - \cot(y)) = \frac{x^3}{3} + c_1$$

Raising both side to exponential gives

$$\csc(y) - \cot(y) = e^{\frac{x^3}{3} + c_1}$$

Which simplifies to

$$\csc(y) - \cot(y) = c_2 e^{\frac{x^3}{3}}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = -i \ln\left(\frac{-e^{c_1+\frac{1}{3}} c_2 + i}{e^{c_1+\frac{1}{3}} c_2 + i}\right)$$

$$c_1 = -\frac{1}{3} + \ln\left(-\frac{i(e^{-2i} - 1)}{c_2 (e^{-2i} + 1)}\right)$$

Substituting c_1 found above in the general solution gives

$$y = -i \ln\left(\frac{-c_2 e^{\frac{x^3}{3}-\frac{1}{3}+\ln\left(-\frac{i\left(e^{-2i}-1\right)}{c_2\left(e^{-2i}+1\right)}\right)} + i}{c_2 e^{\frac{x^3}{3}-\frac{1}{3}+\ln\left(-\frac{i\left(e^{-2i}-1\right)}{c_2\left(e^{-2i}+1\right)}\right)} + i}\right)$$

Summary

The solution(s) found are the following

$$y = -i \ln\left(\frac{-c_2 e^{\frac{x^3}{3}-\frac{1}{3}+\ln\left(-\frac{i\left(e^{-2i}-1\right)}{c_2\left(e^{-2i}+1\right)}\right)} + i}{c_2 e^{\frac{x^3}{3}-\frac{1}{3}+\ln\left(-\frac{i\left(e^{-2i}-1\right)}{c_2\left(e^{-2i}+1\right)}\right)} + i}\right) \tag{1}$$

Verification of solutions

$$y = -i \ln\left(\frac{-c_2 e^{\frac{x^3}{3}-\frac{1}{3}+\ln\left(-\frac{i\left(e^{-2i}-1\right)}{c_2\left(e^{-2i}+1\right)}\right)} + i}{c_2 e^{\frac{x^3}{3}-\frac{1}{3}+\ln\left(-\frac{i\left(e^{-2i}-1\right)}{c_2\left(e^{-2i}+1\right)}\right)} + i}\right)$$

Verified OK.

2.9.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= x^2 \sin(y) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 47: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= \frac{1}{x^2} \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x^2}} dx \end{aligned}$$

Which results in

$$S = \frac{x^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x^2 \sin(y)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x^2 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \csc(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \csc(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(\csc(R) + \cot(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^3}{3} = -\ln(\csc(y) + \cot(y)) + c_1$$

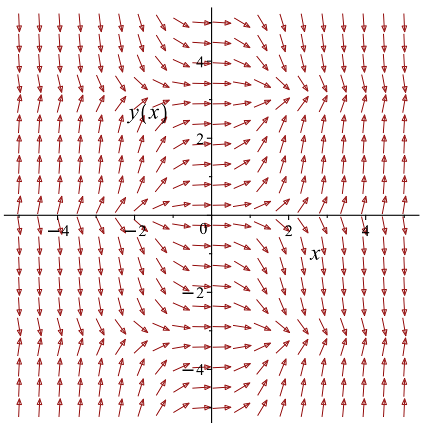
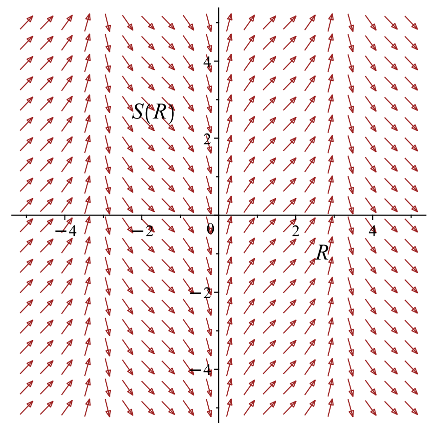
Which simplifies to

$$\frac{x^3}{3} = -\ln(\csc(y) + \cot(y)) + c_1$$

Which gives

$$y = \arctan\left(\frac{2\,\mathrm{e}^{-\frac{x^3}{3}+c_1}}{\mathrm{e}^{-\frac{2x^3}{3}+2c_1}+1}, \frac{\mathrm{e}^{-\frac{2x^3}{3}+2c_1}-1}{\mathrm{e}^{-\frac{2x^3}{3}+2c_1}+1}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = x^2 \sin(y)$
</td><td>$R = y$
$S = \frac{x^3}{3}$</td><td>$\frac{dS}{dR} = \csc(R)$
</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = -i \ln\left(\frac{-e^{-\frac{1}{3}+c_1} - i}{-e^{-\frac{1}{3}+c_1} + i}\right)$$

$$c_1 = \frac{1}{3} + \ln\left(\frac{i(e^{-2i} + 1)}{e^{-2i} - 1}\right)$$

Substituting c_1 found above in the general solution gives

$$y = -i \ln\left(\frac{-e^{-\frac{x^3}{3}+\frac{1}{3}+\ln\left(\frac{i(e^{-2i}+1)}{e^{-2i}-1}\right)} - i}{-e^{-\frac{x^3}{3}+\frac{1}{3}+\ln\left(\frac{i(e^{-2i}+1)}{e^{-2i}-1}\right)} + i}\right)$$

Summary

The solution(s) found are the following

$$y = -i \ln\left(\frac{-e^{-\frac{x^3}{3}+\frac{1}{3}+\ln\left(\frac{i(e^{-2i}+1)}{e^{-2i}-1}\right)} - i}{-e^{-\frac{x^3}{3}+\frac{1}{3}+\ln\left(\frac{i(e^{-2i}+1)}{e^{-2i}-1}\right)} + i}\right) \quad (1)$$

Verification of solutions

$$y = -i \ln\left(\frac{-\mathrm{e}^{-\frac{x^3}{3}+\frac{1}{3}+\ln\left(\frac{i\left(\mathrm{e}^{-2i}+1\right)}{\mathrm{e}^{-2i}-1}\right)} - i}{-\mathrm{e}^{-\frac{x^3}{3}+\frac{1}{3}+\ln\left(\frac{i\left(\mathrm{e}^{-2i}+1\right)}{\mathrm{e}^{-2i}-1}\right)} + i}\right)$$

Verified OK.

2.9.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{\sin(y)}\right) \mathrm{d}y = (x^2)\,\mathrm{d}x$$
$$(-x^2)\,\mathrm{d}x + \left(\frac{1}{\sin(y)}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -x^2$$
$$N(x,y) = \frac{1}{\sin(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x^2)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{\sin(y)}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -x^2\,\mathrm{d}x$$
$$\phi = -\frac{x^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sin(y)}$. Therefore equation (4) becomes

$$\frac{1}{\sin(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= \frac{1}{\sin(y)} \\ &= \csc(y) \end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned} \int f'(y) \,\mathrm{d}y &= \int (\csc(y)) \,\mathrm{d}y \\ f(y) &= -\ln(\csc(y) + \cot(y)) + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} - \ln(\csc(y) + \cot(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} - \ln(\csc(y) + \cot(y))$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{3} - \ln(\cos(2) + 1) + \ln(\sin(2)) - i\pi = c_1$$

$$c_1 = -\frac{1}{3} - \ln(\cos(2) + 1) + \ln(\sin(2)) - i\pi$$

Substituting c_1 found above in the general solution gives

$$-\frac{x^3}{3} - \ln(\csc(y) + \cot(y)) = -\frac{1}{3} - \ln(\cos(2) + 1) + \ln(\sin(2)) - i\pi$$

Summary

The solution(s) found are the following

$$-\frac{x^3}{3} - \ln(\csc(y) + \cot(y)) = -\frac{1}{3} - \ln(\cos(2) + 1) + \ln(\sin(2)) - i\pi \tag{1}$$

Verification of solutions

$$-\frac{x^3}{3} - \ln(\csc(y) + \cot(y)) = -\frac{1}{3} - \ln(\cos(2) + 1) + \ln(\sin(2)) - i\pi$$

Verified OK.

2.9.5 Maple step by step solution

Let's solve

$[y' - x^2 \sin(y) = 0, y(1) = -2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sin(y)} = x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sin(y)} dx = \int x^2 dx + c_1$

- Evaluate integral

  $\ln(\csc(y) - \cot(y)) = \frac{x^3}{3} + c_1$

- Solve for y

  $y = \arctan\left(\frac{2\,\mathrm{e}^{\frac{x^3}{3}+c_1}}{\left(\mathrm{e}^{\frac{x^3}{3}+c_1}\right)^2+1}, -\frac{\left(\mathrm{e}^{\frac{x^3}{3}+c_1}\right)^2-1}{\left(\mathrm{e}^{\frac{x^3}{3}+c_1}\right)^2+1}\right)$

- Use initial condition $y(1) = -2$

$$-2 = \arctan\left(\frac{2\,e^{c_1+\frac{1}{3}}}{\left(e^{c_1+\frac{1}{3}}\right)^2+1}, -\frac{\left(e^{c_1+\frac{1}{3}}\right)^2-1}{\left(e^{c_1+\frac{1}{3}}\right)^2+1}\right)$$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 1.844 (sec). Leaf size: 97

```
dsolve([diff(y(x),x)=x^2*sin(y(x)),y(1) = -2],y(x), singsol=all)
```

$$y(x) = \arctan\left(\frac{2\sin(2)\,e^{\frac{(-1+x)\left(x^2+x+1\right)}{3}}}{(-1+\cos(2))\,e^{\frac{2(-1+x)\left(x^2+x+1\right)}{3}}-1-\cos(2)}, \frac{(1-\cos(2))\,e^{\frac{2(-1+x)\left(x^2+x+1\right)}{3}}-1-\cos(2)}{(-1+\cos(2))\,e^{\frac{2(-1+x)\left(x^2+x+1\right)}{3}}-1-\cos(2)}\right)$$

✓ Solution by Mathematica

Time used: 0.68 (sec). Leaf size: 23

```
DSolve[{y'[x]==x^2*Sin[y[x]],{y[1]==-2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\arccos\left(\tanh\left(\operatorname{arctanh}(\cos(2)) - \frac{x^3}{3} + \frac{1}{3}\right)\right)$$

2.10 problem 6(b)

Internal problem ID [11594]
Internal file name [`OUTPUT/10576_Thursday_May_18_2023_05_59_06_PM_8400801/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, section 1.3. Exercises page 22
Problem number: 6(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y^2}{x-2} = 0}$$

With initial conditions

$$[y(1) = 0]$$

2.10.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y^2}{x-2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y^2}{x-2}\right)\\ &= \frac{2y}{x-2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

2.10.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= \frac{y^2}{x-2}\end{aligned}$$

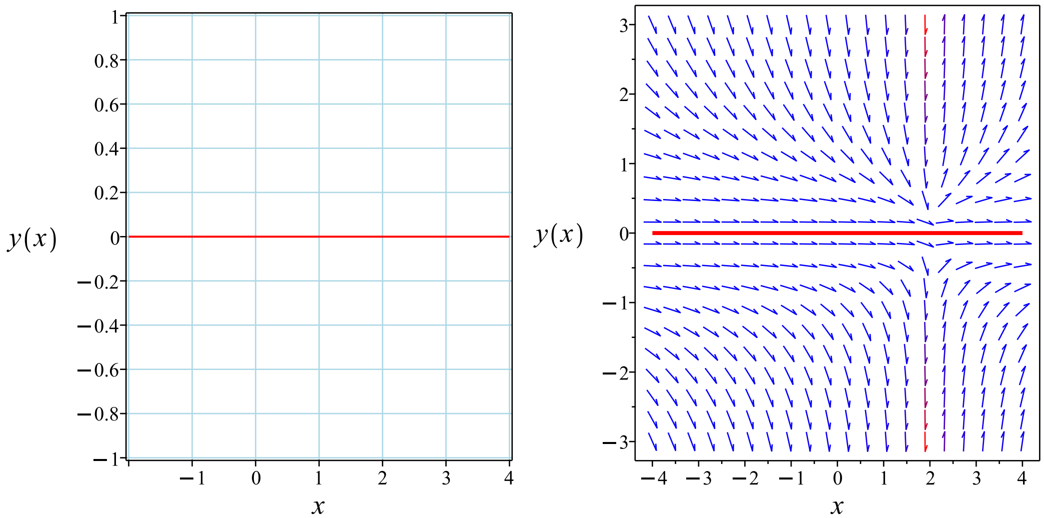
Where $f(x) = \frac{1}{x-2}$ and $g(y) = y^2$. Since unique solution exists and $g(y)$ evaluated at $y_0 = 0$ is zero, then the solution is

$$\begin{aligned}y &= y_0\\ &= 0\end{aligned}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 0$$

Verified OK.

2.10.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2}{x-2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 50: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x - 2 \\
\eta(x, y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x-2} dx \end{aligned}$$

Which results in

$$S = \ln(x-2)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2}{x-2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{x-2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x-2) = -\frac{1}{y} + c_1$$

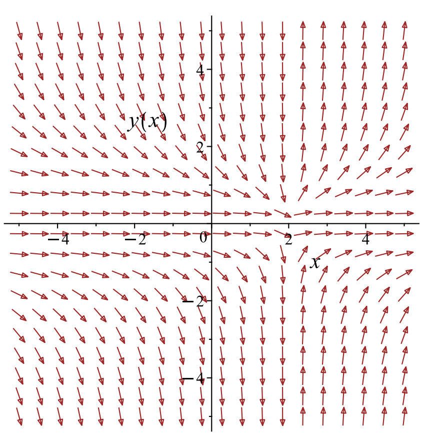
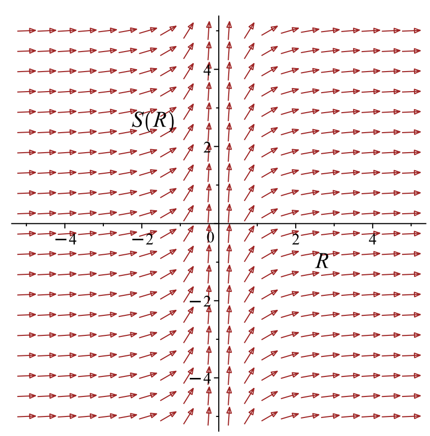
Which simplifies to

$$\ln(x-2) = -\frac{1}{y} + c_1$$

Which gives

$$y = -\frac{1}{\ln(x-2) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2}{x-2}$ 	$R = y$ $S = \ln(x-2)$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $x=1$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

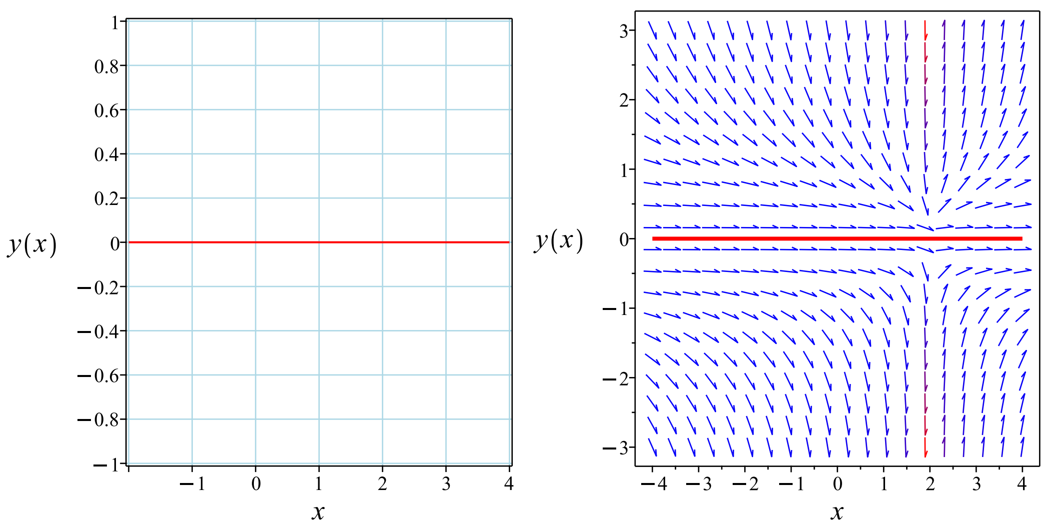
$$0=-\frac{1}{i\pi-c_1}$$

Unable to solve for constant of integration. Since $\lim_{c_1\to\infty}$ gives $y=-\frac{1}{\ln(x-2)-c_1}=y=0$ and this result satisfies the given initial condition.

Summary

The solution(s) found are the following

$$y=0$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y=0$$

Verified OK.

2.10.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2}\right)\mathrm{d}y&=\left(\frac{1}{x-2}\right)\mathrm{d}x\\ \left(-\frac{1}{x-2}\right)\mathrm{d}x+\left(\frac{1}{y^2}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{x-2}\\ N(x,y)&=\frac{1}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{x-2}\right)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x-2} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x-2) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$
\frac{1}{y^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^2}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y^2}\right) \mathrm{d}y \\
f(y) &= -\frac{1}{y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x-2) - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x-2) - \frac{1}{y}$$

The solution becomes

$$y = -\frac{1}{\ln(x-2) + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

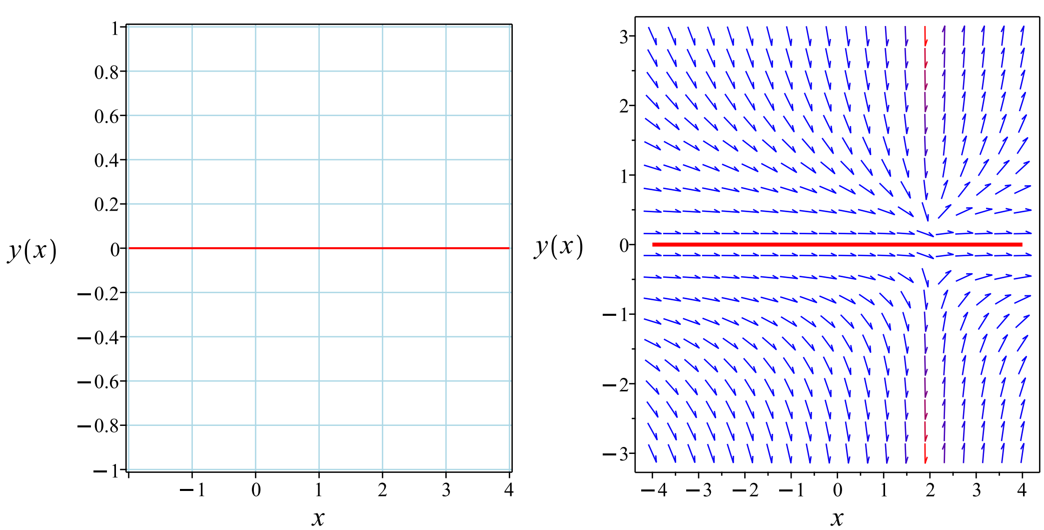
$$0 = -\frac{1}{i\pi + c_1}$$

Unable to solve for constant of integration. Since $\lim_{c_1 \to \infty}$ gives $y = -\frac{1}{\ln(x-2)+c_1} = y = 0$ and this result satisfies the given initial condition.

Summary

The solution(s) found are the following

$$y = 0$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 0$$

Verified OK.

2.10.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y^2}{x-2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x-2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{x-2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x-2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{(x-2)^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x-2} + \frac{u'(x)}{(x-2)^2} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 \ln(x-2) + c_1$$

The above shows that

$$u'(x) = \frac{c_2}{x-2}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2}{c_2 \ln(x-2) + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{\ln(x-2) + c_3}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

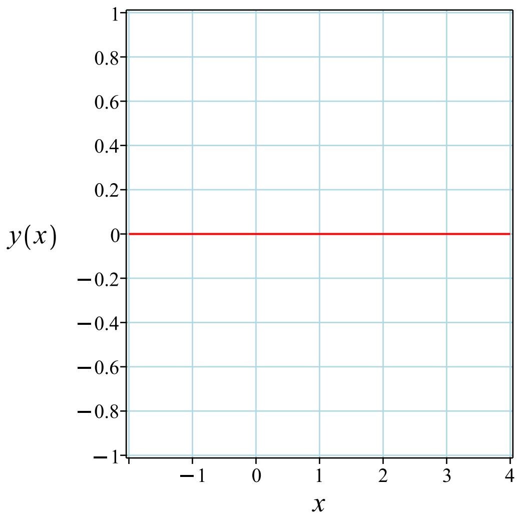
$$0 = -\frac{1}{i\pi + c_3}$$

Unable to solve for constant of integration. Since $\lim_{c_1 \to \infty}$ gives $y = -\frac{1}{\ln(x-2)+c_3} = y = 0$ and this result satisfies the given initial condition.

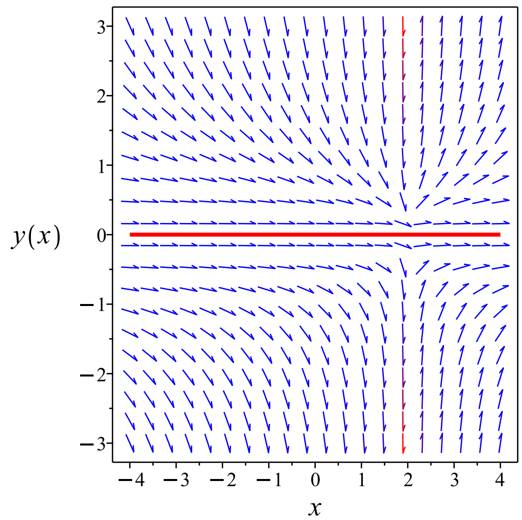
Summary

The solution(s) found are the following

$$y = 0$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 0$$

Verified OK.

2.10.6 Maple step by step solution

Let's solve

$\left[y' - \frac{y^2}{x-2} = 0, y(1) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = \frac{1}{x-2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int \frac{1}{x-2} dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = \ln(x-2) + c_1$

- Solve for y

  $y = -\frac{1}{\ln(x-2)+c_1}$

- Use initial condition $y(1) = 0$

  $0 = -\frac{1}{I\pi + c_1}$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=y(x)^2/(x-2),y(1) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 6

```
DSolve[{y'[x]==y[x]^2/(x-2),{y[1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

2.11 problem 8

Internal problem ID [11595]
Internal file name [`OUTPUT/10577_Thursday_May_18_2023_05_59_06_PM_93747806/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 1, section 1.3. Exercises page 22
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^{\frac{1}{3}} = 0}$$

With initial conditions

$$[y(0) = 0]$$

2.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^{\frac{1}{3}} \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{0 \leq y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^{\frac{1}{3}}\right) \\ &= \frac{1}{3y^{\frac{2}{3}}} \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x=0$ is

$$\{0<y\}$$

But the point $y_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

2.11.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^{\frac{1}{3}}}dy = \int dx$$

$$\frac{3y^{\frac{2}{3}}}{2} = x + c_1$$

Initial conditions are used to solve for c_1. Substituting $x=0$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$\frac{3y^{\frac{2}{3}}}{2} = x$$

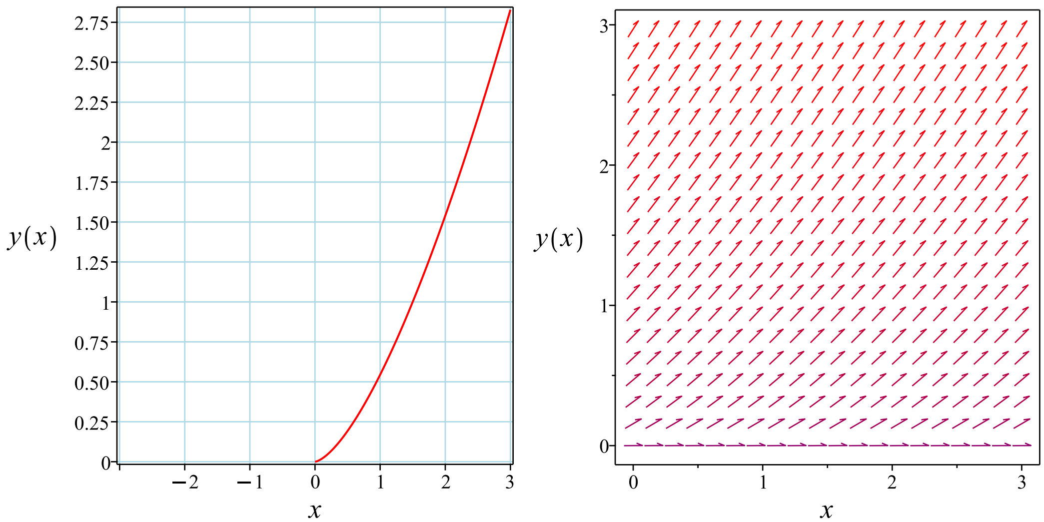
Solving for y from the above gives

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$$

Summary

The solution(s) found are the following

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$$

Verified OK.

2.11.3 Maple step by step solution

Let's solve

$\left[y' - y^{\frac{1}{3}} = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{1}{3}}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{1}{3}}}dx = \int 1dx + c_1$

- Evaluate integral

  $\frac{3y^{\frac{2}{3}}}{2} = x + c_1$

- Solve for y

$$y = \frac{(6x+6c_1)^{\frac{3}{2}}}{27}$$

- Use initial condition $y(0) = 0$

  $$0 = \frac{2\sqrt{6}\,c_1^{\frac{3}{2}}}{9}$$

- Solve for c_1

  $$c_1 = 0$$

- Substitute $c_1 = 0$ into general solution and simplify

  $$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$$

- Solution to the IVP

  $$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x)=y(x)^(1/3),y(0) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 21

```
DSolve[{y'[x]==y[x]^(1/3),{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{3}\sqrt{\frac{2}{3}}x^{3/2}$$

3 Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37

3.1 problem 1

Internal problem ID [11596]
Internal file name [`OUTPUT/10578_Thursday_May_18_2023_05_59_07_PM_71828352/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{2y + (2x + y)\, y' = -3x}$$

3.1.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$2u(x)\, x + (2x + u(x)\, x)\, (u'(x)\, x + u(x)) = -3x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{u^2 + 4u + 3}{x\,(u + 2)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+4u+3}{u+2}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+4u+3}{u+2}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2+4u+3}{u+2}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln\left(u^2+4u+3\right)}{2} = -\ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+4u+3} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u^2+4u+3} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u\left(x\right)^2 + 4u\left(x\right) + 3} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{u\left(x\right)^2 + 4u\left(x\right) + 3} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2} + \frac{4y}{x} + 3} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

$$\sqrt{\frac{y^2+4yx+3x^2}{x^2}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Which simplifies to

$$\sqrt{\frac{\left(3x+y\right)\left(y+x\right)}{x^2}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{\left(3x+y\right)\left(y+x\right)}{x^2}} = \frac{c_3\mathrm{e}^{c_2}}{x} \tag{1}$$

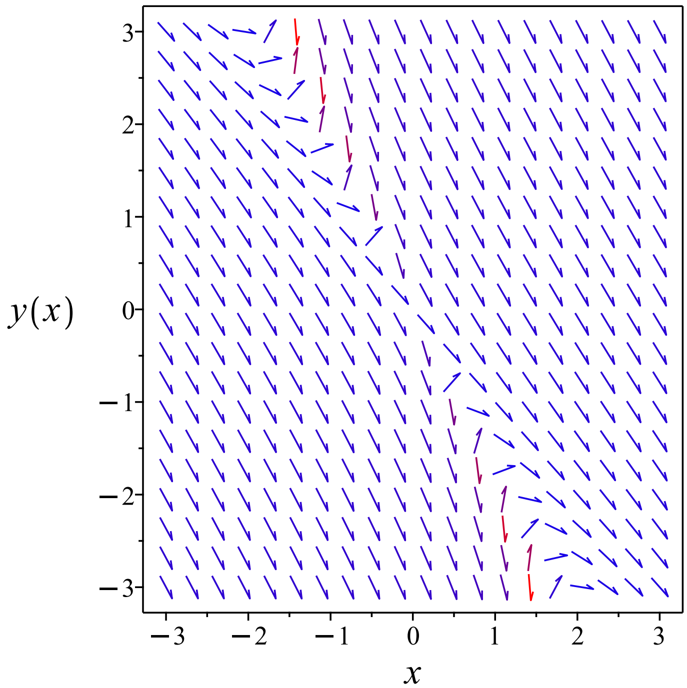


Figure 49: Slope field plot

Verification of solutions

$$\sqrt{\frac{(3x+y)(y+x)}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

3.1.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-3x-2y}{2x+y} \tag{1}$$

Which becomes

$$(y)\,dy = (-2x)\,dy + (-3x-2y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-2x)\,dy + (-3x-2y)\,dx = d\left(-\frac{3}{2}x^2 - 2xy\right)$$

Hence (2) becomes

$$(y)\,dy = d\left(-\frac{3}{2}x^2 - 2xy\right)$$

Integrating both sides gives gives these solutions

$$y = -2x + \sqrt{x^2 + 2c_1} + c_1$$
$$y = -2x - \sqrt{x^2 + 2c_1} + c_1$$

Summary

The solution(s) found are the following

$$y = -2x + \sqrt{x^2 + 2c_1} + c_1 \tag{1}$$
$$y = -2x - \sqrt{x^2 + 2c_1} + c_1 \tag{2}$$

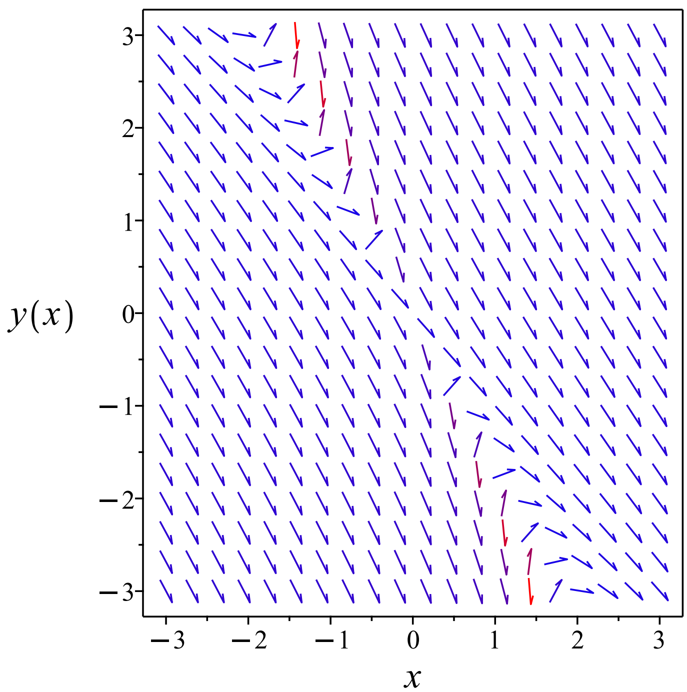


Figure 50: Slope field plot

Verification of solutions

$$y = -2x + \sqrt{x^2 + 2c_1} + c_1$$

Verified OK.

$$y = -2x - \sqrt{x^2 + 2c_1} + c_1$$

Verified OK.

3.1.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{3x+2y}{2x+y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{(3x+2y)(b_3 - a_2)}{2x+y} - \frac{(3x+2y)^2 a_3}{(2x+y)^2} \\
& - \left(-\frac{3}{2x+y} + \frac{6x+4y}{(2x+y)^2}\right)(x a_2 + y a_3 + a_1) \\
& - \left(-\frac{2}{2x+y} + \frac{3x+2y}{(2x+y)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{6x^2 a_2 - 9x^2 a_3 + 5x^2 b_2 - 6x^2 b_3 + 6xy a_2 - 12xy a_3 + 4xy b_2 - 6xy b_3 + 2y^2 a_2 - 5y^2 a_3 + y^2 b_2 - 2y^2 b_3 + x b_1}{(2x+y)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& 6x^2 a_2 - 9x^2 a_3 + 5x^2 b_2 - 6x^2 b_3 + 6xy a_2 - 12xy a_3 + 4xy b_2 \\
& - 6xy b_3 + 2y^2 a_2 - 5y^2 a_3 + y^2 b_2 - 2y^2 b_3 + x b_1 - y a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &6a_2v_1^2 + 6a_2v_1v_2 + 2a_2v_2^2 - 9a_3v_1^2 - 12a_3v_1v_2 - 5a_3v_2^2 + 5b_2v_1^2 \\ &+ 4b_2v_1v_2 + b_2v_2^2 - 6b_3v_1^2 - 6b_3v_1v_2 - 2b_3v_2^2 - a_1v_2 + b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(6a_2 - 9a_3 + 5b_2 - 6b_3)\, v_1^2 + (6a_2 - 12a_3 + 4b_2 - 6b_3)\, v_1v_2 \\ &+ b_1v_1 + (2a_2 - 5a_3 + b_2 - 2b_3)\, v_2^2 - a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -a_1 &= 0 \\ 2a_2 - 5a_3 + b_2 - 2b_3 &= 0 \\ 6a_2 - 12a_3 + 4b_2 - 6b_3 &= 0 \\ 6a_2 - 9a_3 + 5b_2 - 6b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 4a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 0 \\ b_2 &= -3a_3 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{3x + 2y}{2x + y}\right)(x) \\
&= \frac{3x^2 + 4xy + y^2}{2x + y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3x^2 + 4xy + y^2}{2x + y}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(3x^2 + 4xy + y^2)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3x + 2y}{2x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3x + 2y}{(y + x)(3x + y)} \\ S_y &= \frac{2x + y}{(y + x)(3x + y)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

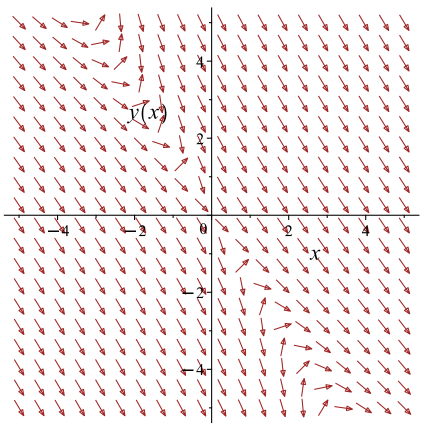
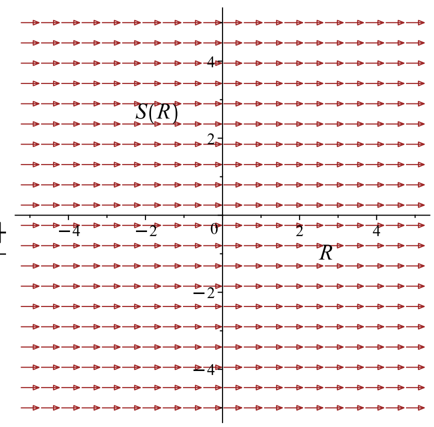
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y + x)}{2} + \frac{\ln(3x + y)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(y + x)}{2} + \frac{\ln(3x + y)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3x+2y}{2x+y}$ 	$R = x$ $S = \frac{\ln (y+x)}{2} + \frac{\ln (3x+}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln (y+x)}{2} + \frac{\ln (3x+y)}{2} = c_1 \tag{1}$$

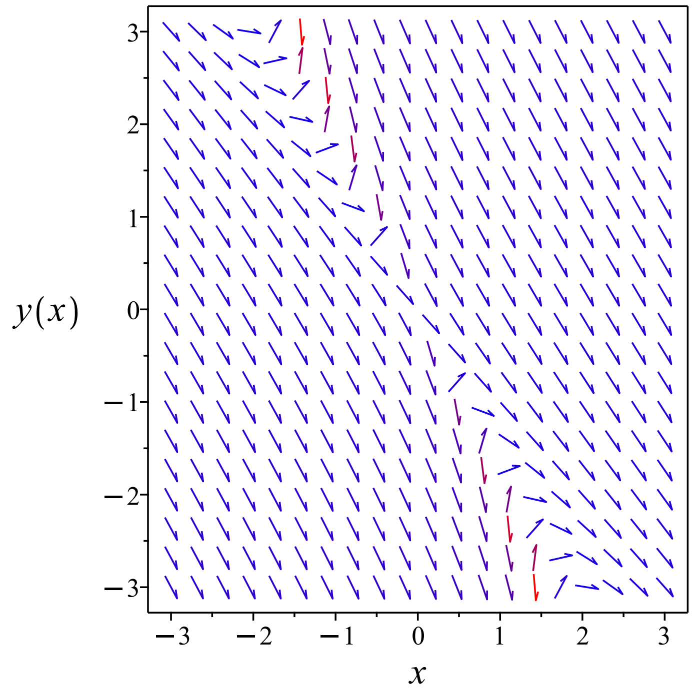


Figure 51: Slope field plot

Verification of solutions

$$\frac{\ln(y+x)}{2}+\frac{\ln(3x+y)}{2}=c_1$$

Verified OK.

3.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x + y)\, \mathrm{d}y &= (-3x - 2y)\, \mathrm{d}x \\ (3x + 2y)\, \mathrm{d}x + (2x + y)\, \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 3x + 2y \\ N(x, y) &= 2x + y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x + 2y) \\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x + y) \\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int 3x + 2y\,\mathrm{d}x$$
$$\phi = \frac{x(3x + 4y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x + y$. Therefore equation (4) becomes

$$2x + y = 2x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y)\,\mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(3x + 4y)}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(3x + 4y)}{2} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$\frac{x(3x + 4y)}{2} + \frac{y^2}{2} = c_1 \tag{1}$$

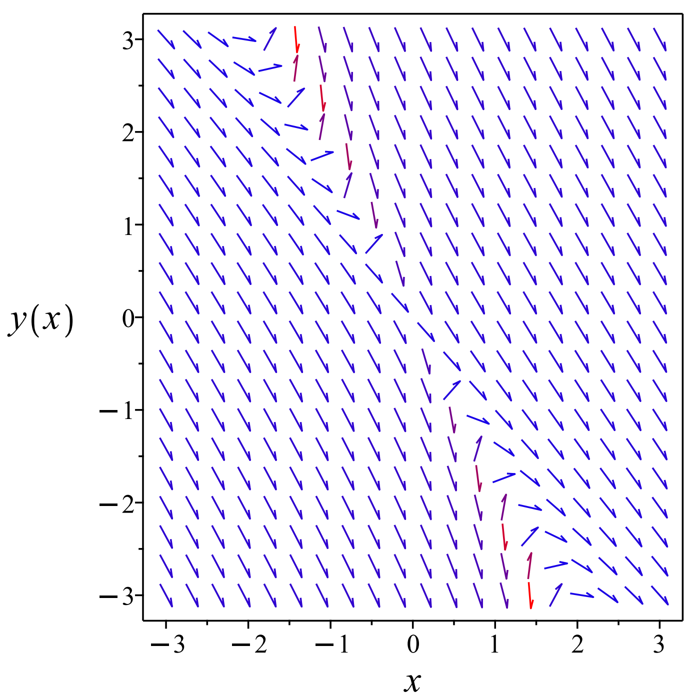


Figure 52: Slope field plot

Verification of solutions

$$\frac{x(3x + 4y)}{2} + \frac{y^2}{2} = c_1$$

Verified OK.

3.1.5 Maple step by step solution

Let's solve

$2y + (2x + y)\, y' = -3x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $2 = 2$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (3x + 2y)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{3x^2}{2} + 2xy + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $2x + y = 2x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = y$

- Solve for $f_1(y)$

  $f_1(y) = \frac{y^2}{2}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = \frac{3}{2}x^2 + 2xy + \frac{1}{2}y^2$$

- Substitute $F(x,y)$ into the solution of the ODE

  $$\frac{3}{2}x^2 + 2xy + \frac{1}{2}y^2 = c_1$$

- Solve for y

  $$\left\{y = -2x - \sqrt{x^2 + 2c_1}, y = -2x + \sqrt{x^2 + 2c_1}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 49

```
dsolve((3*x+2*y(x))+(2*x+y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-2c_1x - \sqrt{c_1^2x^2 + 1}}{c_1}$$

$$y(x) = \frac{-2c_1x + \sqrt{c_1^2x^2 + 1}}{c_1}$$

✓ Solution by Mathematica

Time used: 0.781 (sec). Leaf size: 79

```
DSolve[(3*x+2*y[x])+(2*x+y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x - \sqrt{x^2 + e^{2c_1}}$$
$$y(x) \to -2x + \sqrt{x^2 + e^{2c_1}}$$
$$y(x) \to -\sqrt{x^2} - 2x$$
$$y(x) \to \sqrt{x^2} - 2x$$

3.2 problem 2

Internal problem ID [11597]
Internal file name [`OUTPUT/10579_Thursday_May_18_2023_10_42_42_PM_75656588/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational, [_1st_order, `_with_symmetry_[F(x)*G(y)
   ,0]`], [_Abel, `2nd type`, `class B`]]
```

$$\boxed{y^2 + (2yx - 4)\, y' = -3}$$

3.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(2xy - 4)\, \mathrm{d}y &= \left(-y^2 - 3\right) \mathrm{d}x \\
\left(y^2 + 3\right) \mathrm{d}x + (2xy - 4)\, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y^2 + 3 \\
N(x, y) &= 2xy - 4
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 + 3\right) \\
&= 2y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy - 4) \\
&= 2y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$

$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int y^2+3\,\mathrm{d}x$$

$$\phi=\left(y^2+3\right)x+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=2xy+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=2xy-4$. Therefore equation (4) becomes

$$2xy-4=2xy+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=-4$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int(-4)\,\mathrm{d}y$$

$$f(y)=-4y+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=\left(y^2+3\right)x-4y+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y^2 + 3)\, x - 4y$$

Summary

The solution(s) found are the following

$$(y^2 + 3)\, x - 4y = c_1 \tag{1}$$

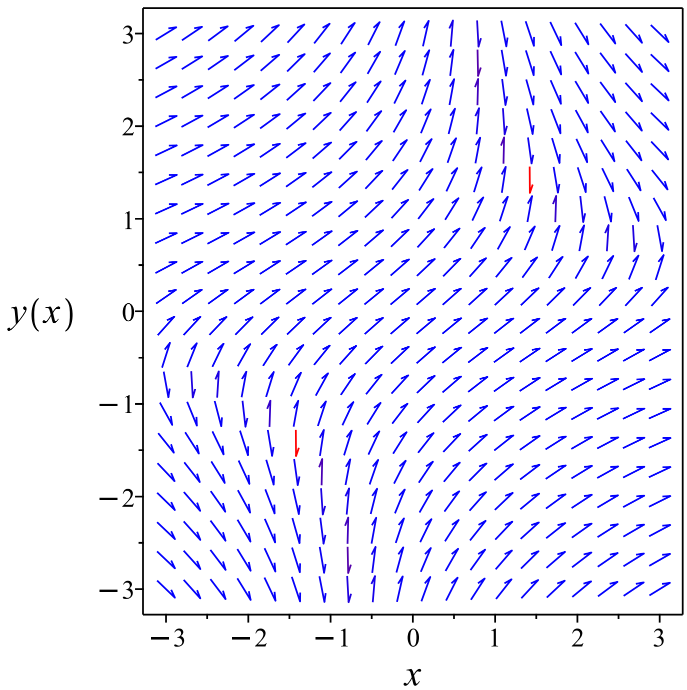


Figure 53: Slope field plot

Verification of solutions

$$(y^2 + 3)\, x - 4y = c_1$$

Verified OK.

3.2.2 Maple step by step solution

Let's solve

$y^2+(2yx-4)\,y'=-3$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ○ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - ○ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - ○ Evaluate derivatives

    $2y=2y$

  - ○ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int (y^2+3)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=(y^2+3)\,x+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $2xy-4=2xy+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=-4$

- Solve for $f_1(y)$

  $f_1(y)=-4y$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = (y^2+3)\,x - 4y$$

- Substitute $F(x,y)$ into the solution of the ODE

$$(y^2+3)\,x - 4y = c_1$$

- Solve for y

$$\left\{y = \frac{2+\sqrt{c_1x-3x^2+4}}{x}, y = -\frac{-2+\sqrt{c_1x-3x^2+4}}{x}\right\}$$

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Reducible group (found an exponential solution)
      <- Kovacics algorithm successful
   <- Abel AIR successful: ODE belongs to the 2F1 3-parameter class
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 62

```
dsolve((y(x)^2+3)+(2*x*y(x)-4)*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{-ic_1\left(y(x)^2\,x+3x-4y(x)\right)\sqrt{3}+12c_1+i}{\left(-y\left(x\right)\sqrt{3}\,x+4\sqrt{3}-3ix\right)\left(\sqrt{3}+iy\left(x\right)\right)} = 0$$

✓ Solution by Mathematica

Time used: 0.615 (sec). Leaf size: 79

```
DSolve[(y[x]^2+3)+(2*x*y[x]-4)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2-\sqrt{-3x^2+c_1x+4}}{x}$$
$$y(x) \to \frac{2+\sqrt{-3x^2+c_1x+4}}{x}$$
$$y(x) \to -i\sqrt{3}$$
$$y(x) \to i\sqrt{3}$$

3.3 problem 3

Internal problem ID [11598]
Internal file name [`OUTPUT/10580_Thursday_May_18_2023_10_42_44_PM_48988675/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

```
[_exact, _rational, [_1st_order, `_with_symmetry_[F(x),G(x)]`],
    [_Abel, `2nd type`, `class A`]]
```

$$\boxed{2yx + \left(x^2 + 4y\right) y' = -1}$$

3.3.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-2yx - 1}{x^2 + 4y} \tag{1}$$

Which becomes

$$(4y)\, dy = \left(-x^2\right) dy + (-2xy - 1)\, dx \tag{2}$$

But the RHS is complete differential because

$$\left(-x^2\right) dy + (-2xy - 1)\, dx = d\left(-x^2 y - x\right)$$

Hence (2) becomes

$$(4y)\, dy = d\left(-x^2 y - x\right)$$

Integrating both sides gives gives these solutions

$$
\begin{aligned}
y &= -\frac{x^2}{4} + \frac{\sqrt{x^4 + 8c_1 - 8x}}{4} + c_1 \\
y &= -\frac{x^2}{4} - \frac{\sqrt{x^4 + 8c_1 - 8x}}{4} + c_1
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{x^2}{4} + \frac{\sqrt{x^4 + 8c_1 - 8x}}{4} + c_1 \tag{1}
$$

$$
y = -\frac{x^2}{4} - \frac{\sqrt{x^4 + 8c_1 - 8x}}{4} + c_1 \tag{2}
$$

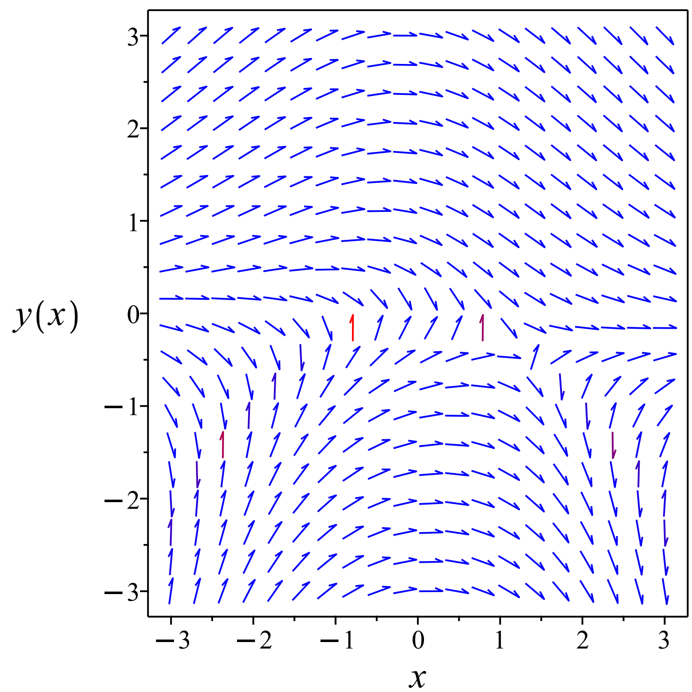


Figure 54: Slope field plot

Verification of solutions

$$y = -\frac{x^2}{4} + \frac{\sqrt{x^4 + 8c_1 - 8x}}{4} + c_1$$

Verified OK.

$$y = -\frac{x^2}{4} - \frac{\sqrt{x^4 + 8c_1 - 8x}}{4} + c_1$$

Verified OK.

3.3.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2+4y\right)\mathrm{d}y &= (-2xy-1)\,\mathrm{d}x\\ (2xy+1)\,\mathrm{d}x+\left(x^2+4y\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2xy+1\\ N(x,y) &= x^2+4y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2xy+1)\\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2+4y\right)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \qquad (1)$$

$$\frac{\partial\phi}{\partial y}=N \qquad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int 2xy+1\,\mathrm{d}x\\ \phi &= x^2y+x+f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 + 4y$. Therefore equation (4) becomes

$$x^2 + 4y = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (4y)\,\mathrm{d}y$$
$$f(y) = 2y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2y + 2y^2 + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2y + 2y^2 + x$$

Summary
The solution(s) found are the following

$$x^2y + 2y^2 + x = c_1 \tag{1}$$

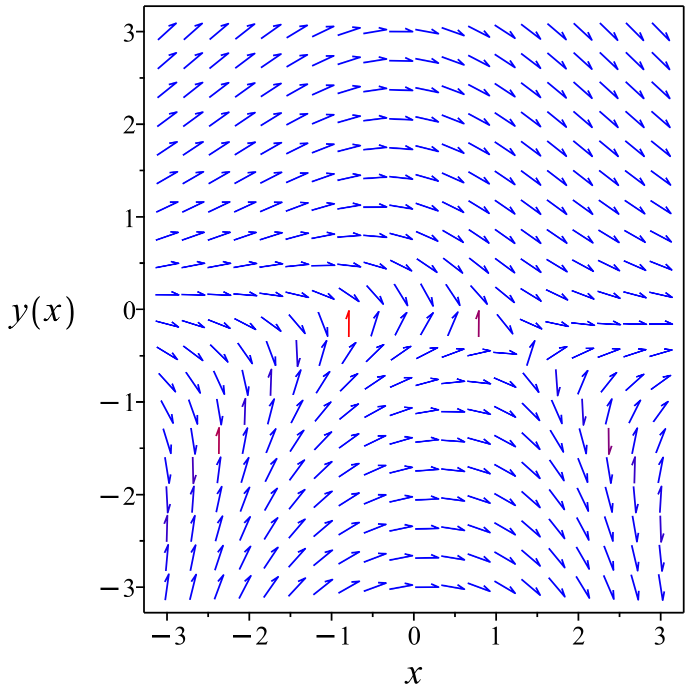


Figure 55: Slope field plot

Verification of solutions

$$x^2y + 2y^2 + x = c_1$$

Verified OK.

3.3.3 Maple step by step solution

Let's solve

$2yx + (x^2 + 4y)\, y' = -1$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $2x = 2x$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (2xy+1)\, dx + f_1(y)$
- Evaluate integral

  $F(x,y) = x^2y + x + f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $x^2 + 4y = x^2 + \frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 4y$
- Solve for $f_1(y)$

  $f_1(y) = 2y^2$
- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = x^2y + 2y^2 + x$
- Substitute $F(x,y)$ into the solution of the ODE

  $x^2y + 2y^2 + x = c_1$
- Solve for y

  $\left\{y = -\frac{x^2}{4} - \frac{\sqrt{x^4+8c_1-8x}}{4}, y = -\frac{x^2}{4} + \frac{\sqrt{x^4+8c_1-8x}}{4}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve((2*x*y(x)+1)+(x^2+4*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x^2}{4} - \frac{\sqrt{x^4 - 8c_1 - 8x}}{4}$$
$$y(x) = -\frac{x^2}{4} + \frac{\sqrt{x^4 - 8c_1 - 8x}}{4}$$

✓ Solution by Mathematica

Time used: 0.223 (sec). Leaf size: 61

```
DSolve[(2*x*y[x]+1)+(x^2+4*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(-x^2 - \sqrt{x^4 - 8x + 16c_1}\right)$$
$$y(x) \to \frac{1}{4}\left(-x^2 + \sqrt{x^4 - 8x + 16c_1}\right)$$

3.4 problem 4

Internal problem ID [11599]
Internal file name [`OUTPUT/10581_Thursday_May_18_2023_10_42_45_PM_55985488/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{3x^2y - (x^3 + y)\, y' = -2}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve((3*x^2*y(x)+2)-(x^3+y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(3*x^2+2)-(x^3+y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

3.5 problem 5

Internal problem ID [11600]
Internal file name [`OUTPUT/10582_Thursday_May_18_2023_10_42_46_PM_41313843/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{6yx + 2y^2 + \left(3x^2 + 4yx - 6\right) y' = 5}$$

3.5.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3x^2 + 4xy - 6)\,\mathrm{d}y &= (-6xy - 2y^2 + 5)\,\mathrm{d}x \\ (6xy + 2y^2 - 5)\,\mathrm{d}x + (3x^2 + 4xy - 6)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 6xy + 2y^2 - 5 \\ N(x, y) &= 3x^2 + 4xy - 6\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(6xy + 2y^2 - 5) \\ &= 6x + 4y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x^2 + 4xy - 6) \\ &= 6x + 4y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 6xy + 2y^2 - 5 \, \mathrm{d}x$$

$$\phi = 3x^2y + 2x\, y^2 - 5x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3x^2 + 4xy + f'(y) \tag{4}$$

$$= x(3x + 4y) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3x^2 + 4xy - 6$. Therefore equation (4) becomes

$$3x^2 + 4xy - 6 = x(3x + 4y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -6$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-6) \, \mathrm{d}y$$

$$f(y) = -6y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = 3x^2y + 2x\, y^2 - 5x - 6y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 3x^2y + 2x\,y^2 - 5x - 6y$$

Summary

The solution(s) found are the following

$$3x^2y + 2y^2x - 5x - 6y = c_1 \tag{1}$$

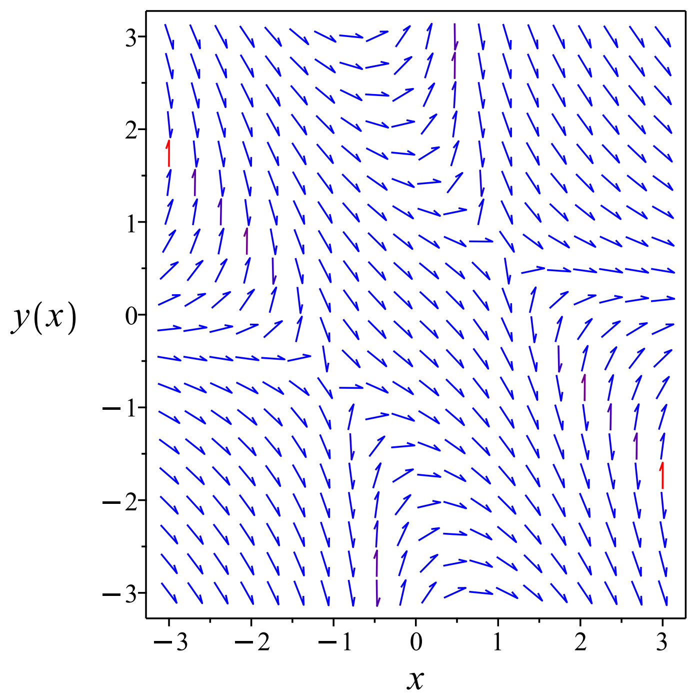


Figure 56: Slope field plot

Verification of solutions

$$3x^2y + 2y^2x - 5x - 6y = c_1$$

Verified OK.

3.5.2 Maple step by step solution

Let's solve

$6yx + 2y^2 + (3x^2 + 4yx - 6)\, y' = 5$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $6x + 4y = 6x + 4y$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (6xy + 2y^2 - 5)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = 3x^2y + 2x\, y^2 - 5x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $3x^2 + 4xy - 6 = 3x^2 + 4xy + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = -6$

- Solve for $f_1(y)$

  $f_1(y) = -6y$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = 3x^2y + 2x\,y^2 - 5x - 6y$$

- Substitute $F(x, y)$ into the solution of the ODE

$$3x^2y + 2x\,y^2 - 5x - 6y = c_1$$

- Solve for y

$$\left\{y = \frac{-3x^2+6+\sqrt{9x^4+8c_1x+4x^2+36}}{4x}, y = -\frac{3x^2+\sqrt{9x^4+8c_1x+4x^2+36}-6}{4x}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
dsolve((6*x*y(x)+2*y(x)^2-5)+(3*x^2+4*x*y(x)-6)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-3x^2 + 6 + \sqrt{9x^4 - 8c_1x + 4x^2 + 36}}{4x}$$

$$y(x) = \frac{-3x^2 + 6 - \sqrt{9x^4 - 8c_1x + 4x^2 + 36}}{4x}$$

✓ Solution by Mathematica

Time used: 0.709 (sec). Leaf size: 79

```
DSolve[(6*x*y[x]+2*y[x]^2-5)+(3*x^2+4*x*y[x]-6)*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\frac{3x^2 + \sqrt{9x^4 + 4x^2 + 16c_1x + 36} - 6}{4x}$$

$$y(x) \to \frac{-3x^2 + \sqrt{9x^4 + 4x^2 + 16c_1x + 36} + 6}{4x}$$

3.6 problem 7

Internal problem ID [11601]
Internal file name [`OUTPUT/10583_Thursday_May_18_2023_10_42_47_PM_64831531/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_Abel, `2nd type`, `class A`]]
```

$$\boxed{y\sec(x)^2 + (\tan(x) + 2y)\,y' = -\sec(x)\tan(x)}$$

3.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\tan(x) + 2y) \, \mathrm{d}y = \left(-y \sec(x)^2 - \sec(x) \tan(x)\right) \mathrm{d}x$$
$$\left(y \sec(x)^2 + \sec(x) \tan(x)\right) \mathrm{d}x + (\tan(x) + 2y) \, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y \sec(x)^2 + \sec(x) \tan(x)$$
$$N(x, y) = \tan(x) + 2y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(y \sec(x)^2 + \sec(x) \tan(x)\right)$$
$$= \sec(x)^2$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(\tan(x) + 2y)$$
$$= \sec(x)^2$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y \sec(x)^2 + \sec(x)\tan(x) \, \mathrm{d}x$$
$$\phi = y \tan(x) + \sec(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \tan(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \tan(x) + 2y$. Therefore equation (4) becomes

$$\tan(x) + 2y = \tan(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (2y) \, \mathrm{d}y$$
$$f(y) = y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y \tan(x) + \sec(x) + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\tan(x) + \sec(x) + y^2$$

Summary
The solution(s) found are the following

$$\tan(x)\, y + \sec(x) + y^2 = c_1 \tag{1}$$

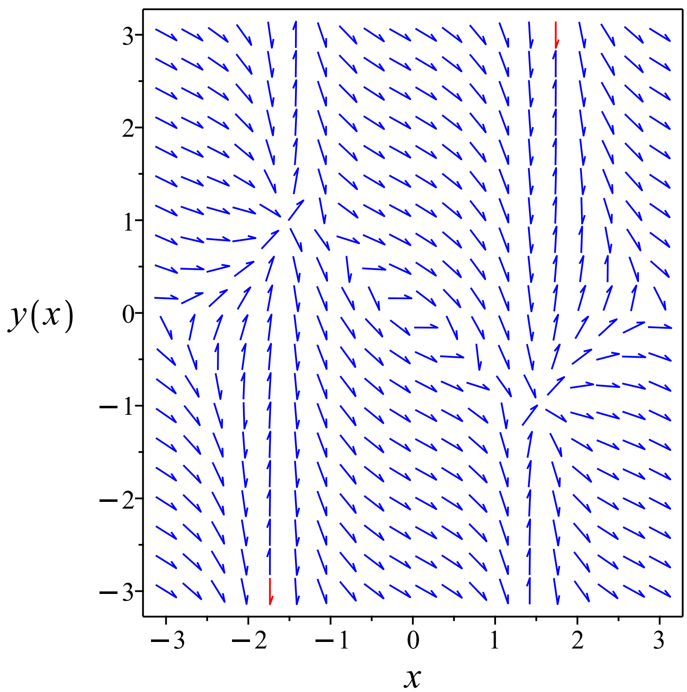


Figure 57: Slope field plot

Verification of solutions

$$\tan(x)\, y + \sec(x) + y^2 = c_1$$

Verified OK.

3.6.2 Maple step by step solution

Let's solve

$y \sec(x)^2 + (\tan(x) + 2y)\, y' = -\sec(x)\tan(x)$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $\sec(x)^2 = \tan(x)^2 + 1$

  - Simplify

    $\sec(x)^2 = \sec(x)^2$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \left(y \sec(x)^2 + \sec(x)\tan(x)\right) dx + f_1(y)$

- Evaluate integral

  $F(x, y) = y \tan(x) + \sec(x) + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $\tan(x) + 2y = \tan(x) + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 2y$

- Solve for $f_1(y)$

$$f_1(y) = y^2$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = y\tan(x) + \sec(x) + y^2$$

- Substitute $F(x,y)$ into the solution of the ODE

$$y\tan(x) + \sec(x) + y^2 = c_1$$

- Solve for y

$$\left\{y = -\frac{\sin(x) - \sqrt{4c_1\cos(x)^2 + \sin(x)^2 - 4\cos(x)}}{2\cos(x)}, y = -\frac{\sin(x) + \sqrt{4c_1\cos(x)^2 + \sin(x)^2 - 4\cos(x)}}{2\cos(x)}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 61

```
dsolve((y(x)*sec(x)^2+sec(x)*tan(x))+(tan(x)+2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\tan(x)}{2} + \frac{\sec(x)\sqrt{-4\cos(x)^2 c_1 + \sin(x)^2 - 4\cos(x)}}{2}$$

$$y(x) = -\frac{\tan(x)}{2} - \frac{\sec(x)\sqrt{-4\cos(x)^2 c_1 + \sin(x)^2 - 4\cos(x)}}{2}$$

✓ Solution by Mathematica

Time used: 1.831 (sec). Leaf size: 101

```
DSolve[(y[x]*Sec[x]^2+Sec[x]*Tan[x])+(Tan[x]+2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{1}{4}\left(-2\tan(x) - \sqrt{2}\sqrt{\sec^2(x)}\sqrt{-8\cos(x) + (-1+4c_1)\cos(2x) + 1 + 4c_1}\right)$$
$$y(x) \to \frac{1}{4}\left(-2\tan(x) + \sqrt{\sec^2(x)}\sqrt{-16\cos(x) + (-2+8c_1)\cos(2x) + 2 + 8c_1}\right)$$

3.7 problem 8

Internal problem ID [11602]
Internal file name [`OUTPUT/10584_Thursday_May_18_2023_10_42_49_PM_42100361/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{\frac{x}{y^2}+\left(\frac{x^2}{y^3}+y\right)y'=-x}$$

3.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{x^2}{y^3} + y\right) \mathrm{d}y = \left(-\frac{x}{y^2} - x\right) \mathrm{d}x$$
$$\left(\frac{x}{y^2} + x\right) \mathrm{d}x + \left(\frac{x^2}{y^3} + y\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{x}{y^2} + x$$
$$N(x, y) = \frac{x^2}{y^3} + y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x}{y^2} + x\right) \\ &= -\frac{2x}{y^3} \end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x^2}{y^3}+y\right) \\
&= \frac{2x}{y^3}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{\frac{x^2}{y^3}+y}\left(\left(-\frac{2x}{y^3}\right) - \left(\frac{2x}{y^3}\right)\right) \\
&= -\frac{4x}{y^4+x^2}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{\frac{x}{y^2}+x}\left(\left(\frac{2x}{y^3}\right) - \left(-\frac{2x}{y^3}\right)\right) \\
&= \frac{4}{y\left(y^2+1\right)}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int \frac{4}{y\left(y^2+1\right)} \,\mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{4\ln(y)-2\ln\left(y^2+1\right)} \\
&= \frac{y^4}{\left(y^2+1\right)^2}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{y^4}{\left(y^2+1\right)^2}\left(\frac{x}{y^2}+x\right) \\
&= \frac{x\,y^2}{y^2+1}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{y^4}{\left(y^2+1\right)^2}\left(\frac{x^2}{y^3}+y\right) \\
&= \frac{\left(y^4+x^2\right)y}{\left(y^2+1\right)^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x\,y^2}{y^2+1}\right)+\left(\frac{\left(y^4+x^2\right)y}{\left(y^2+1\right)^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\frac{x\,y^2}{y^2+1}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^2y^2}{2y^2+2}+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= \frac{2x^2y}{2y^2+2} - \frac{4x^2y^3}{(2y^2+2)^2} + f'(y) \\ &= \frac{x^2y}{(y^2+1)^2} + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{(y^4+x^2)y}{(y^2+1)^2}$. Therefore equation (4) becomes

$$\frac{(y^4+x^2)\,y}{(y^2+1)^2} = \frac{x^2y}{(y^2+1)^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y^5}{(y^2+1)^2}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{y^5}{(y^2+1)^2}\right) \mathrm{d}y \\ f(y) &= \frac{y^2}{2} - \ln\left(y^2+1\right) - \frac{1}{2\left(y^2+1\right)} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2y^2}{2y^2+2} + \frac{y^2}{2} - \ln\left(y^2+1\right) - \frac{1}{2\left(y^2+1\right)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2y^2}{2y^2+2} + \frac{y^2}{2} - \ln\left(y^2+1\right) - \frac{1}{2\left(y^2+1\right)}$$

Summary
The solution(s) found are the following

$$\frac{x^2y^2}{2y^2+2}+\frac{y^2}{2}-\ln\left(1+y^2\right)-\frac{1}{2\left(1+y^2\right)}=c_1 \tag{1}$$

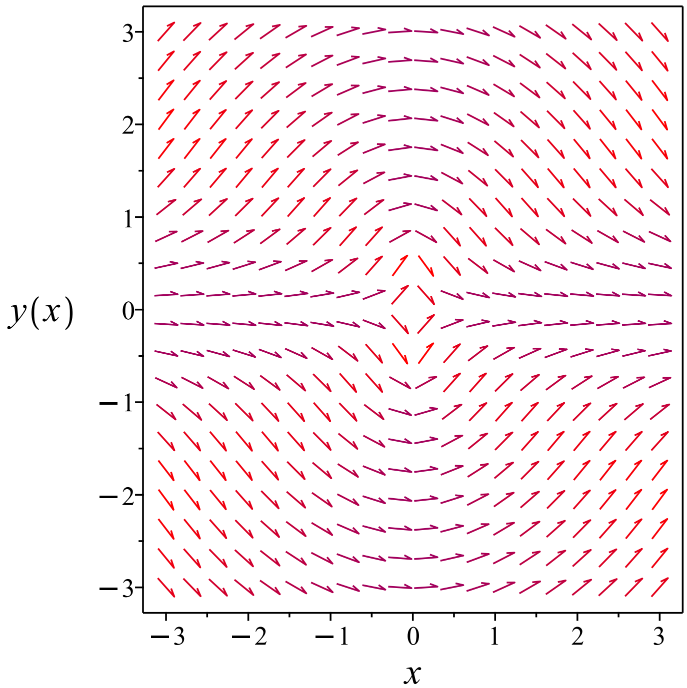


Figure 58: Slope field plot

Verification of solutions

$$\frac{x^2y^2}{2y^2+2}+\frac{y^2}{2}-\ln\left(1+y^2\right)-\frac{1}{2\left(1+y^2\right)}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 52

```
dsolve((x/y(x)^2+x)+(x^2/y(x)^3+y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{(-2y(x)^2-2)\ln(y(x)^2+1)+y(x)^4+(x^2+2c_1+1)\,y(x)^2+2c_1-1}{2y(x)^2+2}=0$$

✓ Solution by Mathematica

Time used: 0.4 (sec). Leaf size: 55

```
DSolve[(x/y[x]^2+x)+(x^2/y[x]^3+y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{x^2y(x)^2}{2\,(y(x)^2+1)}+\frac{y(x)^2}{2}-\frac{1}{2\,(y(x)^2+1)}-\log\left(y(x)^2+1\right)=c_1,y(x)\right]$$

3.8 problem 9

Internal problem ID [11603]
Internal file name [OUTPUT/10585_Thursday_May_18_2023_10_42_51_PM_54314826/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\frac{(2s-1)\,s'}{t}+\frac{s-s^2}{t^2}=0}$$

3.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
s' &= F(t,s)\\
&= f(t)g(s)\\
&= \frac{s(s-1)}{t\,(2s-1)}
\end{aligned}$$

Where $f(t)=\frac{1}{t}$ and $g(s)=\frac{s(s-1)}{2s-1}$. Integrating both sides gives

$$\frac{1}{\frac{s(s-1)}{2s-1}}\,ds=\frac{1}{t}\,dt$$

$$\int \frac{1}{\frac{s(s-1)}{2s-1}}\, ds = \int \frac{1}{t}\, dt$$

$$\ln\left(s(s-1)\right) = \ln\left(t\right) + c_1$$

Raising both side to exponential gives

$$s(s-1) = \mathrm{e}^{\ln(t)+c_1}$$

Which simplifies to

$$s(s-1) = c_2 t$$

Which simplifies to

$$s(s-1) = c_2 \mathrm{e}^{c_1} t$$

The solution is

$$s(s-1) = c_2 \mathrm{e}^{c_1} t$$

Summary

The solution(s) found are the following

$$s(s-1) = c_2 \mathrm{e}^{c_1} t \tag{1}$$

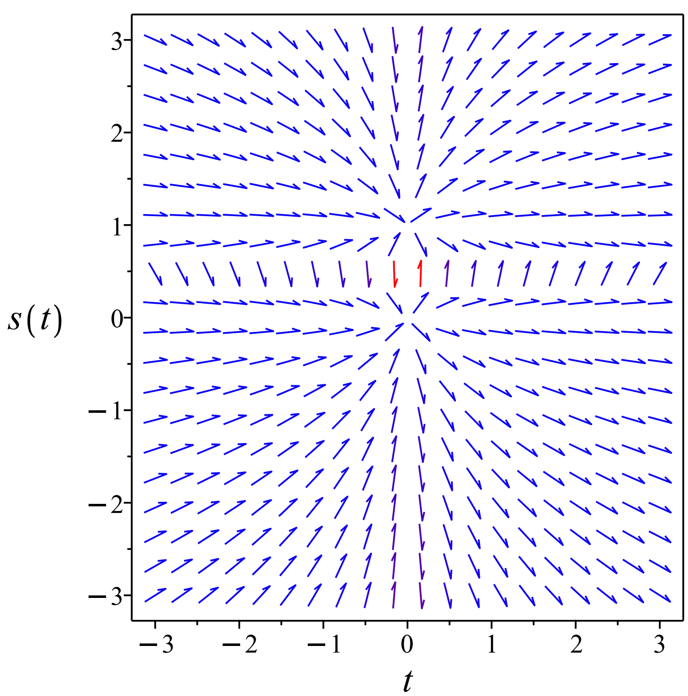


Figure 59: Slope field plot

Verification of solutions

$$s(s-1) = c_2 \mathrm{e}^{c_1} t$$

Verified OK.

3.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$s' = \frac{s(s-1)}{t(2s-1)}$$
$$s' = \omega(t, s)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_s - \xi_t) - \omega^2 \xi_s - \omega_t \xi - \omega_s \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 59: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, s) &= t \\
\eta(t, s) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, s) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{ds}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial s}\right) S(t, s) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = s$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{t} dt \end{aligned}$$

Which results in

$$S = \ln(t)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,s)S_s}{R_t + \omega(t,s)R_s} \tag{2}$$

Where in the above R_t, R_s, S_t, S_s are all partial derivatives and $\omega(t,s)$ is the right hand side of the original ode given by

$$\omega(t,s) = \frac{s(s-1)}{t(2s-1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_s &= 1 \\ S_t &= \frac{1}{t} \\ S_s &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2s-1}{s(s-1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, s in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2R-1}{R(R-1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln (R(R-1)) + c_1 \tag{4}$$

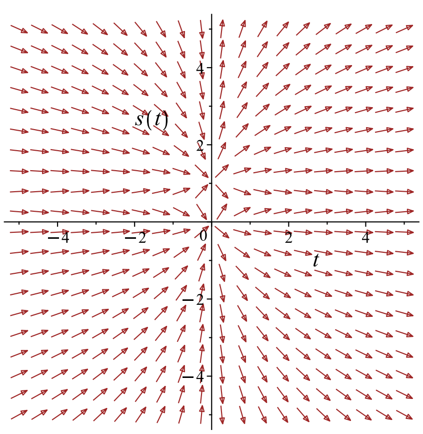
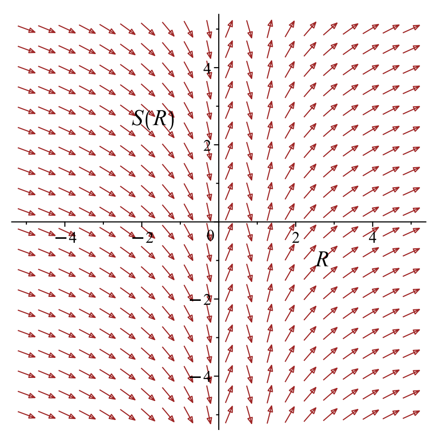
To complete the solution, we just need to transform (4) back to t, s coordinates. This results in

$$\ln (t) = \ln (s(s-1)) + c_1$$

Which simplifies to

$$\ln (t) = \ln (s(s-1)) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, s coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{ds}{dt} = \frac{s(s-1)}{t(2s-1)}$ 	$R = s$ $S = \ln (t)$	$\frac{dS}{dR} = \frac{2R-1}{R(R-1)}$ 

Summary

The solution(s) found are the following

$$\ln (t) = \ln (s(s-1)) + c_1 \tag{1}$$

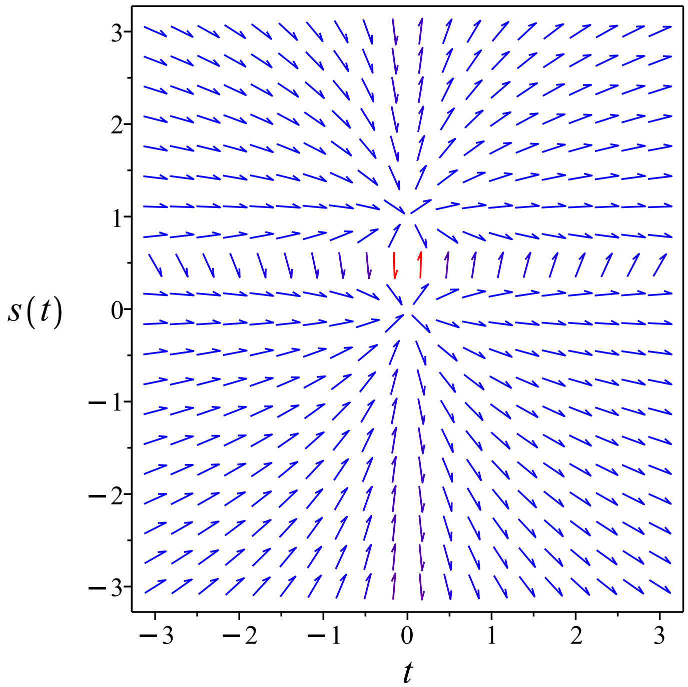


Figure 60: Slope field plot

Verification of solutions

$$\ln(t) = \ln(s(s-1)) + c_1$$

Verified OK.

3.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, s)\,\mathrm{d}t + N(t, s)\,\mathrm{d}s = 0 \tag{1A}$$

Therefore

$$\left(\frac{2s-1}{s\,(s-1)}\right)\mathrm{d}s = \left(\frac{1}{t}\right)\mathrm{d}t$$
$$\left(-\frac{1}{t}\right)\mathrm{d}t + \left(\frac{2s-1}{s\,(s-1)}\right)\mathrm{d}s = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, s) = -\frac{1}{t}$$
$$N(t, s) = \frac{2s-1}{s\,(s-1)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial s} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\frac{\partial M}{\partial s} = \frac{\partial}{\partial s}\left(-\frac{1}{t}\right)$$
$$= 0$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{2s-1}{s\,(s-1)}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial s} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,s)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial s} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int M\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int -\frac{1}{t}\,\mathrm{d}t$$
$$\phi = -\ln(t) + f(s) \tag{3}$$

Where $f(s)$ is used for the constant of integration since ϕ is a function of both t and s. Taking derivative of equation (3) w.r.t s gives

$$\frac{\partial \phi}{\partial s} = 0 + f'(s) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial s} = \frac{2s-1}{s(s-1)}$. Therefore equation (4) becomes

$$\frac{2s-1}{s\,(s-1)} = 0 + f'(s) \tag{5}$$

Solving equation (5) for $f'(s)$ gives

$$f'(s) = \frac{2s-1}{s\,(s-1)}$$

Integrating the above w.r.t s gives

$$\int f'(s)\,\mathrm{d}s = \int \left(\frac{2s-1}{s\,(s-1)}\right)\mathrm{d}s$$
$$f(s) = \ln\left(s(s-1)\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(s)$ into equation (3) gives ϕ

$$\phi = -\ln(t) + \ln(s(s-1)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(t) + \ln(s(s-1))$$

Summary

The solution(s) found are the following

$$-\ln(t) + \ln(s(s-1)) = c_1 \tag{1}$$

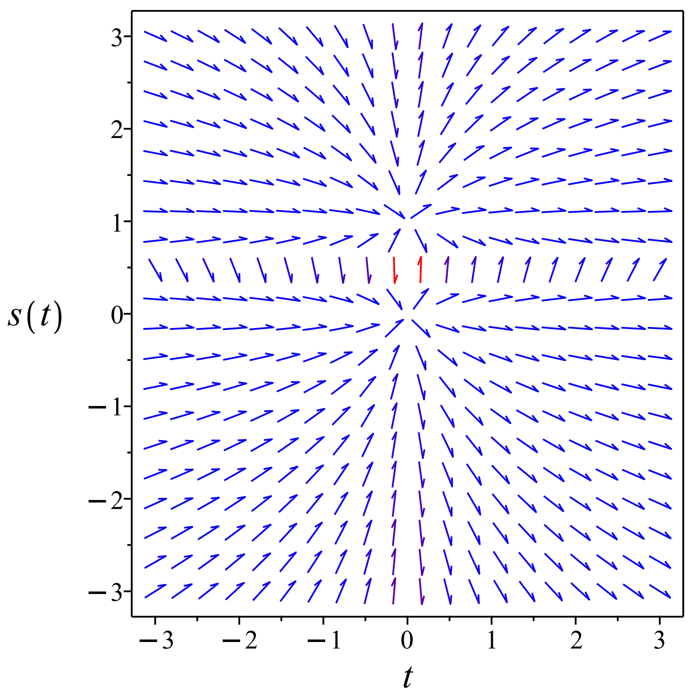


Figure 61: Slope field plot

Verification of solutions

$$-\ln(t) + \ln(s(s-1)) = c_1$$

Verified OK.

3.8.4 Maple step by step solution

Let's solve

$\frac{(2s-1)s'}{t}+\frac{s-s^2}{t^2}=0$

- Highest derivative means the order of the ODE is 1

  s'

- Integrate both sides with respect to t

  $\int\left(\frac{(2s-1)s'}{t}+\frac{s-s^2}{t^2}\right)dt=\int 0dt+c_1$

- Evaluate integral

  $\frac{s^2-s}{t}=c_1$

- Solve for s

  $\left\{s=\frac{1}{2}-\frac{\sqrt{4c_1t+1}}{2}, s=\frac{1}{2}+\frac{\sqrt{4c_1t+1}}{2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve((2*s(t)-1)/t*diff(s(t),t)+(s(t)-s(t)^2)/t^2=0,s(t), singsol=all)
```

$$s(t)=\frac{1}{2}-\frac{\sqrt{4c_1t+1}}{2}$$
$$s(t)=\frac{1}{2}+\frac{\sqrt{4c_1t+1}}{2}$$

✓ Solution by Mathematica

Time used: 0.682 (sec). Leaf size: 59

```
DSolve[(2*s[t]-1)/t*s'[t]+(s[t]-s[t]^2)/t^2==0,s[t],t,IncludeSingularSolutions -> True]
```

$$s(t) \to \frac{1}{2}\left(1 - \sqrt{1 - 4e^{c_1}t}\right)$$
$$s(t) \to \frac{1}{2}\left(1 + \sqrt{1 - 4e^{c_1}t}\right)$$
$$s(t) \to 0$$
$$s(t) \to 1$$

3.9 problem 10

Internal problem ID [11604]
Internal file name [`OUTPUT/10586_Thursday_May_18_2023_10_42_52_PM_67193455/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational]`

Unable to solve or complete the solution.

$$\boxed{\frac{2y^{\frac{3}{2}}+1}{x^{\frac{1}{3}}}+\left(3\sqrt{x}\,\sqrt{y}-1\right)y'=0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-(1/3)*y(x)/x, y(x)`      *** Sublevel 2 *
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   -> Calling odsolve with the ODE`, diff(y(x), x)-(5/6)*y(x)/x, y(x)`      *** Sublevel 2 *
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
`, `-> Computing symmetries using: way = HINT
   -> Calling odsolve with the ODE`, diff(y(x), x)-3*y(x)*x/(2*x^2+x^(1/2)), y(x)`      ***
      Methods for first order ODEs:
      --- Trying classification methods ---
```

✗ Solution by Maple

```
dsolve((2*y(x)^(3/2)+1)/x^(1/3)+(3*x^(1/2)*y(x)^(1/2)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(2*y[x]^(3/2)+1)/x^(1/3)+(3*x^(1/2)*y[x]^(1/2)-1)*y'[x]==0,y[x],x,IncludeSingularSolu
```

Timed out

3.10 problem 11

Internal problem ID [11605]
Internal file name [`OUTPUT/10587_Thursday_May_18_2023_10_42_53_PM_49590489/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType"**

Maple gives the following as the ode type

```
[_exact, _rational, [_1st_order, `_with_symmetry_[F(x),G(x)]`],
    [_Abel, `2nd type`, `class A`]]
```

$$\boxed{2yx+\left(x^{2}+4y\right)y'=3}$$

With initial conditions

$$[y(1)=2]$$

3.10.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{2xy-3}{x^{2}+4y} \end{aligned}$$

The x domain of $f(x,y)$ when $y=2$ is

$$\{-\infty<x<\infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\left\{y < -\frac{1}{4} \vee -\frac{1}{4} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2xy - 3}{x^2 + 4y}\right) \\ &= -\frac{2x}{x^2 + 4y} + \frac{8xy - 12}{(x^2 + 4y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\left\{y < -\frac{1}{4} \vee -\frac{1}{4} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

3.10.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-2yx + 3}{x^2 + 4y} \tag{1}$$

Which becomes

$$(4y)\, dy = \left(-x^2\right) dy + (-2xy + 3)\, dx \tag{2}$$

But the RHS is complete differential because

$$\left(-x^2\right) dy + (-2xy + 3)\, dx = d\left(-x^2 y + 3x\right)$$

Hence (2) becomes

$$(4y)\, dy = d\left(-x^2 y + 3x\right)$$

Integrating both sides gives gives these solutions

$$
\begin{aligned}
y &= -\frac{x^2}{4} + \frac{\sqrt{x^4 + 8c_1 + 24x}}{4} + c_1 \\
y &= -\frac{x^2}{4} - \frac{\sqrt{x^4 + 8c_1 + 24x}}{4} + c_1
\end{aligned}
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$
2 = -\frac{1}{4} - \frac{\sqrt{25 + 8c_1}}{4} + c_1
$$

$$
c_1 = \frac{5}{2} + \frac{\sqrt{11}}{2}
$$

Substituting c_1 found above in the general solution gives

$$
y = -\frac{x^2}{4} - \frac{\sqrt{x^4 + 20 + 4\sqrt{11} + 24x}}{4} + \frac{5}{2} + \frac{\sqrt{11}}{2}
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$
2 = -\frac{1}{4} + \frac{\sqrt{25 + 8c_1}}{4} + c_1
$$

$$
c_1 = \frac{5}{2} - \frac{\sqrt{11}}{2}
$$

Substituting c_1 found above in the general solution gives

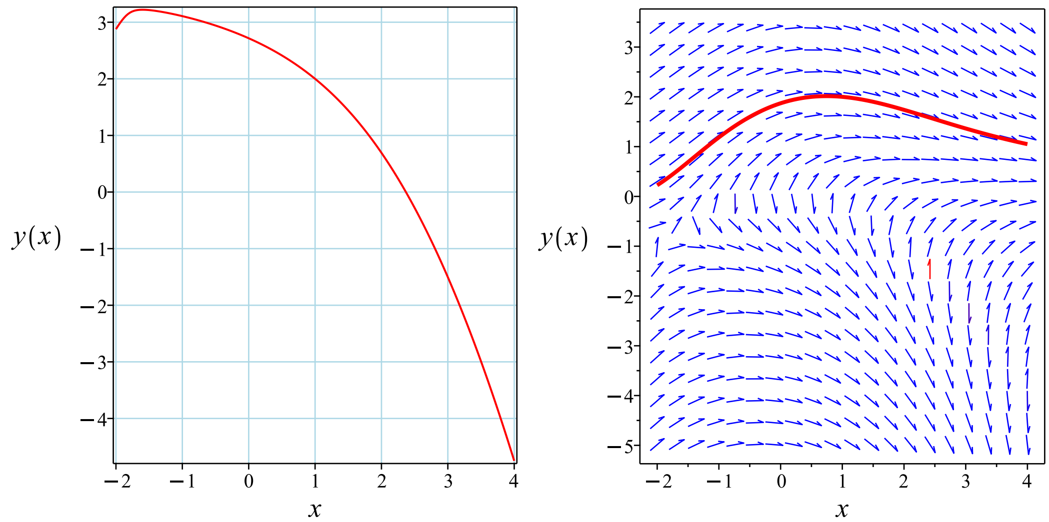
$$
y = -\frac{x^2}{4} + \frac{\sqrt{x^4 + 20 - 4\sqrt{11} + 24x}}{4} + \frac{5}{2} - \frac{\sqrt{11}}{2}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = -\frac{x^2}{4} + \frac{\sqrt{x^4 + 20 - 4\sqrt{11} + 24x}}{4} + \frac{5}{2} - \frac{\sqrt{11}}{2} \tag{1}
$$

$$
y = -\frac{x^2}{4} - \frac{\sqrt{x^4 + 20 + 4\sqrt{11} + 24x}}{4} + \frac{5}{2} + \frac{\sqrt{11}}{2} \tag{2}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{x^2}{4} + \frac{\sqrt{x^4 + 20 - 4\sqrt{11} + 24x}}{4} + \frac{5}{2} - \frac{\sqrt{11}}{2}$$

Verified OK.

$$y = -\frac{x^2}{4} - \frac{\sqrt{x^4 + 20 + 4\sqrt{11} + 24x}}{4} + \frac{5}{2} + \frac{\sqrt{11}}{2}$$

Verified OK.

3.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2 + 4y\right)\mathrm{d}y &= (-2xy + 3)\,\mathrm{d}x\\ (2xy - 3)\,\mathrm{d}x + \left(x^2 + 4y\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2xy - 3\\ N(x, y) &= x^2 + 4y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2xy - 3)\\ &= 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 + 4y\right)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int 2xy-3\,\mathrm{d}x$$
$$\phi=x^2y-3x+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x^2+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x^2+4y$. Therefore equation (4) becomes

$$x^2+4y=x^2+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=4y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int (4y)\,\mathrm{d}y$$
$$f(y)=2y^2+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=x^2y+2y^2-3x+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2y + 2y^2 - 3x$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$7 = c_1$$

$$c_1 = 7$$

Substituting c_1 found above in the general solution gives

$$x^2y + 2y^2 - 3x = 7$$

Summary
The solution(s) found are the following

$$x^2y + 2y^2 - 3x = 7 \tag{1}$$

Verification of solutions

$$x^2y + 2y^2 - 3x = 7$$

Verified OK.

3.10.4 Maple step by step solution

Let's solve

$[2yx + (x^2 + 4y)\, y' = 3, y(1) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $2x = 2x$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (2xy - 3)\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = x^2y - 3x + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x^2 + 4y = x^2 + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 4y$

- Solve for $f_1(y)$

  $f_1(y) = 2y^2$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = x^2y + 2y^2 - 3x$

- Substitute $F(x,y)$ into the solution of the ODE

  $x^2y + 2y^2 - 3x = c_1$

- Solve for y

  $\left\{y = -\frac{x^2}{4} - \frac{\sqrt{x^4+8c_1+24x}}{4}, y = -\frac{x^2}{4} + \frac{\sqrt{x^4+8c_1+24x}}{4}\right\}$

- Use initial condition $y(1) = 2$

  $2 = -\frac{1}{4} - \frac{\sqrt{25+8c_1}}{4}$

- Solution does not satisfy initial condition

- Use initial condition $y(1) = 2$

  $2 = -\frac{1}{4} + \frac{\sqrt{25+8c_1}}{4}$

- Solve for c_1

  $c_1 = 7$
- Substitute $c_1 = 7$ into general solution and simplify

  $y = -\frac{x^2}{4} + \frac{\sqrt{x^4+24x+56}}{4}$
- Solution to the IVP

  $y = -\frac{x^2}{4} + \frac{\sqrt{x^4+24x+56}}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 22

```
dsolve([(2*x*y(x)-3)+(x^2+4*y(x))*diff(y(x),x)=0,y(1) = 2],y(x), singsol=all)
```

$$y(x) = -\frac{x^2}{4} + \frac{\sqrt{x^4 + 24x + 56}}{4}$$

✓ Solution by Mathematica

Time used: 0.218 (sec). Leaf size: 27

```
DSolve[{(2*x*y[x]-3)+(x^2+4*y[x])*y'[x]==0,{y[1]==2}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{4}\left(\sqrt{x^4 + 24x + 56} - x^2\right)$$

3.11 problem 12

Internal problem ID [11606]
Internal file name [`OUTPUT/10588_Thursday_May_18_2023_10_42_54_PM_44971533/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{3x^2y^2 - y^3 + \left(2yx^3 - 3y^2x + 1\right) y' = -2x}$$

With initial conditions

$$[y(-2) = 1]$$

3.11.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(2y\, x^{3} - 3x\, y^{2} + 1\right) \mathrm{d}y &= \left(-3x^{2} y^{2} + y^{3} - 2x\right) \mathrm{d}x \\
\left(3x^{2} y^{2} - y^{3} + 2x\right) \mathrm{d}x + \left(2y\, x^{3} - 3x\, y^{2} + 1\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 3x^{2} y^{2} - y^{3} + 2x \\
N(x, y) &= 2y\, x^{3} - 3x\, y^{2} + 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3x^{2} y^{2} - y^{3} + 2x\right) \\
&= 6x^{2} y - 3y^{2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2y\, x^{3} - 3x\, y^{2} + 1\right) \\
&= 6x^{2} y - 3y^{2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 3x^2y^2 - y^3 + 2x \, \mathrm{d}x$$
$$\phi = x\left(x^2y^2 - y^3 + x\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x\left(2x^2y - 3y^2\right) + f'(y) \tag{4}$$
$$= xy\left(2x^2 - 3y\right) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y\,x^3 - 3x\,y^2 + 1$. Therefore equation (4) becomes

$$2y\,x^3 - 3x\,y^2 + 1 = xy\left(2x^2 - 3y\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (1) \, \mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(x^2y^2 - y^3 + x) + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(x^2y^2 - y^3 + x) + y$$

Initial conditions are used to solve for c_1. Substituting $x = -2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = c_1$$

$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$x(x^2y^2 - y^3 + x) + y = -1$$

Summary

The solution(s) found are the following

$$y^2x^3 - y^3x + x^2 + y = -1 \tag{1}$$

Verification of solutions

$$y^2x^3 - y^3x + x^2 + y = -1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 28.797 (sec). Leaf size: 210

```
dsolve([(3*x^2*y(x)^2-y(x)^3+2*x)+(2*x^3*y(x)-3*x*y(x)^2+1)*diff(y(x),x)=0,y(-2) = 1],y(x),
```

$$y(x) = \frac{-\frac{2^{\frac{2}{3}}\left(1+i\sqrt{3}\right)\left(\left(2x^7+3\sqrt{3}\,\sqrt{\frac{4x^{10}+4x^8+44x^5+72x^3+27x-4}{x}}+36x^2+27\right)x^2\right)^{\frac{2}{3}}}{2} + x\left(2x^2\left(\left(2x^7+3\sqrt{3}\,\sqrt{\frac{4x^{10}+4x^8+44x^5+72x^3+2}{x}}\right.\right.\right.}{6\left(\left(2x^7+3\sqrt{3}\,\sqrt{\frac{4x^{10}+4x^8+44x^5+72x^3+27x-4}{x}}+36x^2+27\right)x\right.}$$

✓ Solution by Mathematica

Time used: 60.368 (sec). Leaf size: 250

```
DSolve[{(3*x^2*y[x]^2-y[x]^3+2*x)+(2*x^3*y[x]-3*x*y[x]^2+1)*y'[x]==0,{y[-2]==1}},y[x],x,Incl
```

$$y(x) \to \frac{2\sqrt[3]{2}\left(1-i\sqrt{3}\right)x^6+4\sqrt[3]{-2x^9-36x^4-27x^2+3\sqrt{3}\sqrt{x^3\left(4x^{10}+4x^8+44x^5+72x^3+27x-4\right)}x^3}+(1}{12x\sqrt[3]{-2x^9-36x^4-27x^2+3\sqrt{3}\sqrt{x}}}$$

3.12 problem 13

Internal problem ID [11607]
Internal file name [`OUTPUT/10589_Thursday_May_18_2023_10_42_56_PM_94570214/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{2\sin(x)\cos(x)\,y+\sin(x)\,y^2+\left(\sin(x)^2-2y\cos(x)\right)y'=0}$$

With initial conditions

$$[y(0)=3]$$

3.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned}y'&=f(x,y)\\&=-\frac{\sin(x)\,y(2\cos(x)+y)}{\sin(x)^2-2\cos(x)\,y}\end{aligned}$$

The x domain of $f(x,y)$ when $y=3$ is

$$\left\{-\infty \le x \le -\arctan\left(\frac{\sqrt{6\sqrt{10}-18}}{-3+\sqrt{10}}\right)+2\pi\_Z90, -\arctan\left(\frac{\sqrt{6\sqrt{10}-18}}{-3+\sqrt{10}}\right)+2\pi\_Z90 \le x \le \arctan\left(\frac{\sqrt{6\sqrt{10}-18}}{-3+\sqrt{10}}\right) + 2\pi\_Z90, \arctan\left(-i\sqrt{6\sqrt{10}+18}, -3-\sqrt{10}\right) + 2\pi\_Z90 < x < \arctan\left(i\sqrt{6\sqrt{10}+18}, -3-\sqrt{10}\right)+2\pi\_Z90\right\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

3.12.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\sin(x)^2 - 2\cos(x)\,y\right)\mathrm{d}y = \left(-2\cos(x)\sin(x)\,y - \sin(x)\,y^2\right)\mathrm{d}x$$

$$\left(2\cos(x)\sin(x)\,y + \sin(x)\,y^2\right)\mathrm{d}x + \left(\sin(x)^2 - 2\cos(x)\,y\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 2\cos(x)\sin(x)\,y + \sin(x)\,y^2 \\
N(x, y) &= \sin(x)^2 - 2\cos(x)\,y
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2\cos(x)\sin(x)\,y + \sin(x)\,y^2\right) \\
&= 2\sin(x)\left(\cos(x) + y\right)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\sin(x)^2 - 2\cos(x)\,y\right) \\
&= 2\sin(x)\left(\cos(x) + y\right)
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int 2\cos(x)\sin(x)\, y + \sin(x)\, y^2 \mathrm{d}x$$
$$\phi = y\left(\sin(x)^2 - \cos(x)\, y\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(x)^2 - 2\cos(x)\, y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)^2 - 2\cos(x)\, y$. Therefore equation (4) becomes

$$\sin(x)^2 - 2\cos(x)\, y = \sin(x)^2 - 2\cos(x)\, y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y\left(\sin(x)^2 - \cos(x)\, y\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\left(\sin(x)^2 - \cos(x)\, y\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$-9 = c_1$$

$$c_1 = -9$$

Substituting c_1 found above in the general solution gives

$$y\left(\sin(x)^2 - \cos(x)\, y\right) = -9$$

Summary
The solution(s) found are the following

$$y\left(\sin(x)^2 - y\cos(x)\right) = -9 \tag{1}$$

Verification of solutions

$$y\left(\sin(x)^2 - y\cos(x)\right) = -9$$

Verified OK.

3.12.3 Maple step by step solution

Let's solve

$\left[2\sin(x)\cos(x)\, y + \sin(x)\, y^2 + \left(\sin(x)^2 - 2y\cos(x)\right) y' = 0, y(0) = 3\right]$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$
  - Evaluate derivatives

    $2\cos(x)\sin(x) + 2\sin(x)\, y = 2\cos(x)\sin(x) + 2\sin(x)\, y$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \left(2\cos(x)\sin(x)\, y + \sin(x)\, y^2\right) dx + f_1(y)$
- Evaluate integral

$$F(x,y) = -\cos(x)\,y^2 + \sin(x)^2\,y + f_1(y)$$

- Take derivative of $F(x,y)$ with respect to y

$$N(x,y) = \frac{\partial}{\partial y}F(x,y)$$

- Compute derivative

$$\sin(x)^2 - 2\cos(x)\,y = -2\cos(x)\,y + \sin(x)^2 + \frac{d}{dy}f_1(y)$$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y) = 0$$

- Solve for $f_1(y)$

$$f_1(y) = 0$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = -\cos(x)\,y^2 + \sin(x)^2\,y$$

- Substitute $F(x,y)$ into the solution of the ODE

$$-\cos(x)\,y^2 + \sin(x)^2\,y = c_1$$

- Solve for y

$$\left\{y = \frac{\sin(x)^2 - \sqrt{\sin(x)^4 - 4c_1\cos(x)}}{2\cos(x)}, y = \frac{\sin(x)^2 + \sqrt{\sin(x)^4 - 4c_1\cos(x)}}{2\cos(x)}\right\}$$

- Use initial condition $y(0) = 3$

$$3 = -\frac{\sqrt{-4c_1}}{2}$$

- Solution does not satisfy initial condition
- Use initial condition $y(0) = 3$

$$3 = \frac{\sqrt{-4c_1}}{2}$$

- Solve for c_1

$$c_1 = -9$$

- Substitute $c_1 = -9$ into general solution and simplify

$$y = \frac{\sec(x)\left(\sin(x)^2 + \sqrt{\sin(x)^4 + 36\cos(x)}\right)}{2}$$

- Solution to the IVP

$$y = \frac{\sec(x)\left(\sin(x)^2 + \sqrt{\sin(x)^4 + 36\cos(x)}\right)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 11.656 (sec). Leaf size: 24

```
dsolve([(2*y(x)*sin(x)*cos(x)+y(x)^2*sin(x))+(sin(x)^2-2*y(x)*cos(x))*diff(y(x),x)=0,y(0) =
```

$$y(x) = \frac{\sec(x)\left(\sin(x)^2 + \sqrt{\sin(x)^4 + 36\cos(x)}\right)}{2}$$

✓ Solution by Mathematica

Time used: 2.029 (sec). Leaf size: 34

```
DSolve[{(2*y[x]*Sin[x]*Cos[x]+y[x]^2*Sin[x])+(Sin[x]^2-2*y[x]*Cos[x])*y'[x]==0,{y[0]==3}},y[
```

$$y(x) \to \frac{1}{4}\sec(x)\left(-\cos(2x) + 2\sqrt{\sin^4(x) + 36\cos(x)} + 1\right)$$

3.13 problem 14

Internal problem ID [11608]
Internal file name [`OUTPUT/10590_Thursday_May_18_2023_10_42_58_PM_46650619/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{y\,\mathrm{e}^{x}+y^{2}+\left(\mathrm{e}^{x}+2yx\right)y'=-2\,\mathrm{e}^{x}}$$

With initial conditions

$$[y(0)=6]$$

3.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y)\\ &= -\frac{\mathrm{e}^{x}y+y^{2}+2\,\mathrm{e}^{x}}{\mathrm{e}^{x}+2xy} \end{aligned}$$

The x domain of $f(x,y)$ when $y=6$ is

$$\left\{x<-\mathrm{LambertW}\left(\_Z92,\frac{1}{12}\right)\vee-\mathrm{LambertW}\left(\_Z92,\frac{1}{12}\right)<x\right\}$$

But the point $x_0=0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

3.13.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^x+2xy)\,\mathrm{d}y&=\left(-\mathrm{e}^x y-2\,\mathrm{e}^x-y^2\right)\mathrm{d}x\\ \left(\mathrm{e}^x y+y^2+2\,\mathrm{e}^x\right)\mathrm{d}x+(\mathrm{e}^x+2xy)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\mathrm{e}^x y+y^2+2\,\mathrm{e}^x\\ N(x,y)&=\mathrm{e}^x+2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{x} y + y^{2} + 2\,\mathrm{e}^{x}\right) \\ &= \mathrm{e}^{x} + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^{x} + 2xy) \\ &= \mathrm{e}^{x} + 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \mathrm{e}^{x} y + y^{2} + 2\,\mathrm{e}^{x}\,\mathrm{d}x\end{aligned}$$

$$\phi = \mathrm{e}^{x}(y+2) + x\,y^{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + 2xy + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + 2xy$. Therefore equation (4) becomes

$$\mathrm{e}^{x} + 2xy = \mathrm{e}^{x} + 2xy + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = e^{x}(y + 2) + x\,y^{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{x}(y + 2) + x\,y^{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 6$ in the above solution gives an equation to solve for the constant of integration.

$$8 = c_1$$

$$c_1 = 8$$

Substituting c_1 found above in the general solution gives

$$e^{x}(y + 2) + x\,y^{2} = 8$$

<u>Summary</u>
The solution(s) found are the following

$$e^{x}(y + 2) + y^{2}x = 8 \tag{1}$$

<u>Verification of solutions</u>

$$e^{x}(y + 2) + y^{2}x = 8$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 29

```
dsolve([(y(x)*exp(x)+2*exp(x)+y(x)^2)+(exp(x)+2*x*y(x))*diff(y(x),x)=0,y(0) = 6],y(x), sings
```

$$y(x) = \frac{-e^{x} + \sqrt{e^{2x} - 8\,e^{x} x + 32x}}{2x}$$

✓ Solution by Mathematica

Time used: 32.264 (sec). Leaf size: 37

```
DSolve[{(y[x]*Exp[x]+2*Exp[x]+y[x]^2)+(Exp[x]+2*x*y[x])*y'[x]==0,{y[0]==6}},y[x],x,IncludeSi
```

$$y(x) \to \frac{\sqrt{-8e^{x}x + 32x + e^{2x}} - e^{x}}{2x}$$

3.14 problem 15

Internal problem ID [11609]
Internal file name [`OUTPUT/10591_Saturday_May_27_2023_03_00_55_AM_75656588/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_exact, _rational, [_1st_order, `_with_symmetry_[F(x)*G(y)
   ,0]`]]
```

$$\boxed{\frac{3-y}{x^2}+\frac{(y^2-2x)\,y'}{y^2x}=0}$$

With initial conditions

$$[y(-1)=2]$$

3.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y)\\ &= \frac{y^2(y-3)}{x\,(y^2-2x)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=2$ is

$$\{-\infty \leq x < 0, 0 < x < 2, 2 < x \leq \infty\}$$

And the point $x_0=-1$ is inside this domain. The y domain of $f(x,y)$ when $x=-1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0=2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y^2(y-3)}{x\,(y^2-2x)}\right)\\ &= \frac{2y(y-3)}{x\,(y^2-2x)} + \frac{y^2}{x\,(y^2-2x)} - \frac{2y^3(y-3)}{x\,(y^2-2x)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=2$ is

$$\{-\infty \leq x < 0, 0 < x < 2, 2 < x \leq \infty\}$$

And the point $x_0=-1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=-1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0=2$ is inside this domain. Therefore solution exists and is unique.

3.14.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= \frac{y^2(y-3)}{x\,(y^2-2x)}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&2xb_4 + yb_5 + b_2 + \frac{y^2(y-3)\left(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3\right)}{x\left(y^2-2x\right)} \\
&- \frac{y^4(y-3)^2\left(xa_5 + 2ya_6 + a_3\right)}{x^2\left(y^2-2x\right)^2} \\
&- \left(-\frac{y^2(y-3)}{x^2\left(y^2-2x\right)} + \frac{2y^2(y-3)}{x\left(y^2-2x\right)^2}\right)\left(x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1\right) \\
&- \left(\frac{2y(y-3)}{x\left(y^2-2x\right)} + \frac{y^2}{x\left(y^2-2x\right)} - \frac{2y^3(y-3)}{x\left(y^2-2x\right)^2}\right)\left(x^2b_4\right. \\
&\left. + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1\right) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^3y^4b_4 - x^2y^5a_4 + x^2y^5b_5 - x\,y^6a_5 + x\,y^6b_6 - y^7a_6 - 2x^4y^2b_4 + 3x^2y^4a_4 - 2x^2y^4a_5 - 3x^2y^4b_5 + 2x^2y^4b_6 +}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&x^3y^4b_4 - x^2y^5a_4 + x^2y^5b_5 - x\,y^6a_5 + x\,y^6b_6 - y^7a_6 - 2x^4y^2b_4 + 3x^2y^4a_4 \\
&- 2x^2y^4a_5 - 3x^2y^4b_5 + 2x^2y^4b_6 + 6x\,y^5a_5 - 4x\,y^5a_6 - 6x\,y^5b_6 + 9y^6a_6 \\
&+ 8x^5b_4 - 12x^4yb_4 + 4x^4yb_5 + 2x^3y^2b_2 - 6x^3y^2b_5 - 2x^2y^3a_2 + 6x^2y^3a_5 \\
&+ 4x^2y^3b_3 - 4x\,y^4a_3 - 9x\,y^4a_5 + 12x\,y^4a_6 - x\,y^4b_1 - 3x\,y^4b_3 + y^5a_1 \\
&+ 3y^5a_3 - 18y^5a_6 + 4x^4b_2 - 12x^3yb_2 + 6x^2y^2a_2 + 6x^2y^2b_1 - 6x^2y^2b_3 \\
&- 4x\,y^3a_1 + 12x\,y^3a_3 - 3y^4a_1 - 9y^4a_3 - 12x^2yb_1 + 12x\,y^2a_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{align*}
-a_4v_1^2v_2^5 - a_5v_1v_2^6 - a_6v_2^7 + b_4v_1^3v_2^4 + b_5v_1^2v_2^5 + b_6v_1v_2^6 + 3a_4v_1^2v_2^4 - 2a_5v_1^2v_2^4 \\
+ 6a_5v_1v_2^5 - 4a_6v_1v_2^5 + 9a_6v_2^6 - 2b_4v_1^4v_2^2 - 3b_5v_1^2v_2^4 + 2b_6v_1^2v_2^4 - 6b_6v_1v_2^5 \\
+ a_1v_2^5 - 2a_2v_1^2v_2^3 - 4a_3v_1v_2^4 + 3a_3v_2^5 + 6a_5v_1^2v_2^3 - 9a_5v_1v_2^4 + 12a_6v_1v_2^4 \\
- 18a_6v_2^5 - b_1v_1v_2^4 + 2b_2v_1^3v_2^2 + 4b_3v_1^2v_2^3 - 3b_3v_1v_2^4 + 8b_4v_1^5 - 12b_4v_1^4v_2 \\
+ 4b_5v_1^4v_2 - 6b_5v_1^3v_2^2 - 4a_1v_1v_2^3 - 3a_1v_2^4 + 6a_2v_1^2v_2^2 + 12a_3v_1v_2^3 - 9a_3v_2^4 \\
+ 6b_1v_1^2v_2^2 + 4b_2v_1^4 - 12b_2v_1^3v_2 - 6b_3v_1^2v_2^2 + 12a_1v_1v_2^2 - 12b_1v_1^2v_2 = 0
\end{align*} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{align*}
8b_4v_1^5 - 2b_4v_1^4v_2^2 + (-12b_4 + 4b_5)\,v_1^4v_2 + 4b_2v_1^4 + b_4v_1^3v_2^4 + (2b_2 - 6b_5)\,v_1^3v_2^2 \\
- 12b_2v_1^3v_2 + (-a_4 + b_5)\,v_1^2v_2^5 + (3a_4 - 2a_5 - 3b_5 + 2b_6)\,v_1^2v_2^4 \\
+ (-2a_2 + 6a_5 + 4b_3)\,v_1^2v_2^3 + (6a_2 + 6b_1 - 6b_3)\,v_1^2v_2^2 \\
- 12b_1v_1^2v_2 + (-a_5 + b_6)\,v_1v_2^6 + (6a_5 - 4a_6 - 6b_6)\,v_1v_2^5 \\
+ (-4a_3 - 9a_5 + 12a_6 - b_1 - 3b_3)\,v_1v_2^4 + (-4a_1 + 12a_3)\,v_1v_2^3 + 12a_1v_1v_2^2 \\
- a_6v_2^7 + 9a_6v_2^6 + (a_1 + 3a_3 - 18a_6)\,v_2^5 + (-3a_1 - 9a_3)\,v_2^4 = 0
\end{align*} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_4 &= 0 \\
12a_1 &= 0 \\
-a_6 &= 0 \\
9a_6 &= 0 \\
-12b_1 &= 0 \\
-12b_2 &= 0 \\
4b_2 &= 0 \\
-2b_4 &= 0 \\
8b_4 &= 0 \\
-4a_1 + 12a_3 &= 0 \\
-3a_1 - 9a_3 &= 0 \\
-a_4 + b_5 &= 0 \\
-a_5 + b_6 &= 0 \\
2b_2 - 6b_5 &= 0 \\
-12b_4 + 4b_5 &= 0 \\
a_1 + 3a_3 - 18a_6 &= 0 \\
-2a_2 + 6a_5 + 4b_3 &= 0 \\
6a_2 + 6b_1 - 6b_3 &= 0 \\
6a_5 - 4a_6 - 6b_6 &= 0 \\
3a_4 - 2a_5 - 3b_5 + 2b_6 &= 0 \\
-4a_3 - 9a_5 + 12a_6 - b_1 - 3b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= -3b_6 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= b_6 \\
a_6 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -3b_6 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= b_6
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= xy - 3x \\
\eta &= y^2 - 3y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y^2 - 3y - \left(\frac{y^2(y-3)}{x\,(y^2-2x)}\right)(xy - 3x) \\
&= \frac{2x^2y^2 - 3x\,y^3 - 6x^2y + 9x\,y^2}{-x\,y^2 + 2x^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}\right) S(x, y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} d y \\
&= \int \frac{1}{\frac{2x^2y^2-3x\,y^3-6x^2y+9x\,y^2}{-x\,y^2+2x^2}} d y
\end{aligned}$$

Which results in

$$S=\frac{\ln (y-3)}{3}-\frac{\ln (y)}{3}+\frac{\ln (-2x+3y)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{d S}{d R}=\frac{S_{x}+\omega(x, y) S_{y}}{R_{x}+\omega(x, y) R_{y}} \tag{2}$$

Where in the above $R_{x}, R_{y}, S_{x}, S_{y}$ are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y)=\frac{y^{2}(y-3)}{x\left(y^{2}-2x\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_{x} &= 1 \\
R_{y} &= 0 \\
S_{x} &= \frac{2}{6x-9y} \\
S_{y} &= \frac{-y^{2}+2x}{(y-3)\, y\,(2x-3y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{d S}{d R}=\frac{1}{3x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)}{3} + c_1 \tag{4}$$

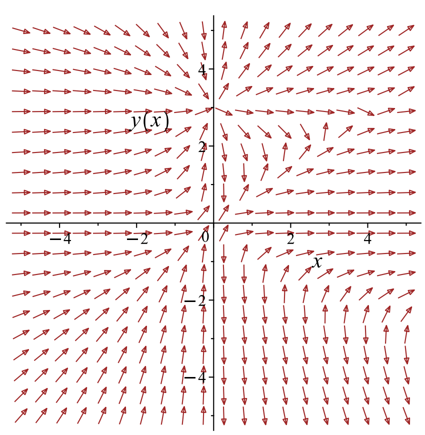
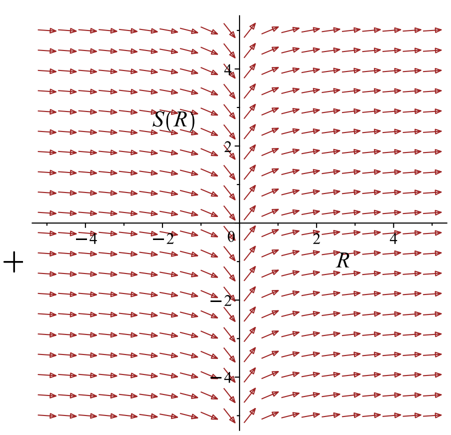
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y-3)}{3} - \frac{\ln(y)}{3} + \frac{\ln(-2x+3y)}{3} = \frac{\ln(x)}{3} + c_1$$

Which simplifies to

$$\frac{\ln(y-3)}{3} - \frac{\ln(y)}{3} + \frac{\ln(-2x+3y)}{3} = \frac{\ln(x)}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2(y-3)}{x(y^2-2x)}$ 	$R = x$ $S = \frac{\ln(y-3)}{3} - \frac{\ln(y)}{3} +$	$\frac{dS}{dR} = \frac{1}{3R}$ 

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{i\pi}{3} + \frac{2\ln(2)}{3} = \frac{i\pi}{3} + c_1$$

$$c_1 = \frac{2\ln(2)}{3}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(y-3)}{3} - \frac{\ln(y)}{3} + \frac{\ln(-2x+3y)}{3} = \frac{\ln(x)}{3} + \frac{2\ln(2)}{3}$$

Summary

The solution(s) found are the following

$$\frac{\ln(y-3)}{3} - \frac{\ln(y)}{3} + \frac{\ln(-2x+3y)}{3} = \frac{\ln(x)}{3} + \frac{2\ln(2)}{3} \tag{1}$$

Verification of solutions

$$\frac{\ln(y-3)}{3} - \frac{\ln(y)}{3} + \frac{\ln(-2x+3y)}{3} = \frac{\ln(x)}{3} + \frac{2\ln(2)}{3}$$

Verified OK.

3.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y^2 - 2x}{y^2 x}\right)\mathrm{d}y &= \left(-\frac{3 - y}{x^2}\right)\mathrm{d}x\\ \left(\frac{3 - y}{x^2}\right)\mathrm{d}x + \left(\frac{y^2 - 2x}{y^2 x}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= \frac{3 - y}{x^2}\\ N(x, y) &= \frac{y^2 - 2x}{y^2 x} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{3 - y}{x^2}\right)\\ &= -\frac{1}{x^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y^2 - 2x}{y^2 x}\right)\\ &= -\frac{1}{x^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int \frac{3-y}{x^2}\,\mathrm{d}x$$
$$\phi=\frac{y-3}{x}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=\frac{1}{x}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{y^2-2x}{y^2x}$. Therefore equation (4) becomes

$$\frac{y^2-2x}{y^2x}=\frac{1}{x}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=-\frac{2}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int\left(-\frac{2}{y^2}\right)\mathrm{d}y$$
$$f(y)=\frac{2}{y}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y-3}{x} + \frac{2}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y-3}{x} + \frac{2}{y}$$

Initial conditions are used to solve for c_1. Substituting $x = -1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$\frac{y-3}{x} + \frac{2}{y} = 2$$

The above simplifies to

$$-2xy + y^2 + 2x - 3y = 0$$

Summary

The solution(s) found are the following

$$y^2 + (-2x-3)\,y + 2x = 0 \tag{1}$$

Verification of solutions

$$y^2 + (-2x-3)\,y + 2x = 0$$

Verified OK.

3.14.4 Maple step by step solution

Let's solve

$\left[\frac{3-y}{x^2}+\frac{(y^2-2x)y'}{y^2x}=0, y(-1)=2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $-\frac{1}{x^2}=-\frac{2}{x\,y^2}-\frac{y^2-2x}{y^2x^2}$

  - Simplify

    $-\frac{1}{x^2}=-\frac{1}{x^2}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\frac{3-y}{x^2}dx+f_1(y)$

- Evaluate integral

  $F(x,y)=-\frac{3-y}{x}+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $\frac{y^2-2x}{y^2x}=\frac{1}{x}+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=-\frac{1}{x}+\frac{y^2-2x}{y^2x}$

- Solve for $f_1(y)$

$f_1(y) = \frac{y+\frac{2x}{y}}{x} - \frac{y}{x}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = -\frac{3-y}{x} + \frac{y+\frac{2x}{y}}{x} - \frac{y}{x}$
- Substitute $F(x, y)$ into the solution of the ODE

  $-\frac{3-y}{x} + \frac{y+\frac{2x}{y}}{x} - \frac{y}{x} = c_1$
- Solve for y

  $\left\{y = \frac{c_1 x}{2} + \frac{3}{2} - \frac{\sqrt{c_1^2 x^2 + 6c_1 x - 8x + 9}}{2}, y = \frac{c_1 x}{2} + \frac{3}{2} + \frac{\sqrt{c_1^2 x^2 + 6c_1 x - 8x + 9}}{2}\right\}$
- Use initial condition $y(-1) = 2$

  $2 = -\frac{c_1}{2} + \frac{3}{2} - \frac{\sqrt{c_1^2 - 6c_1 + 17}}{2}$
- Solution does not satisfy initial condition
- Use initial condition $y(-1) = 2$

  $2 = -\frac{c_1}{2} + \frac{3}{2} + \frac{\sqrt{c_1^2 - 6c_1 + 17}}{2}$
- Solve for c_1

  $c_1 = 2$
- Substitute $c_1 = 2$ into general solution and simplify

  $y = x + \frac{3}{2} + \frac{\sqrt{4x^2 + 4x + 9}}{2}$
- Solution to the IVP

  $y = x + \frac{3}{2} + \frac{\sqrt{4x^2 + 4x + 9}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 21

```
dsolve([(3-y(x))/x^2+((y(x)^2-2*x)/(x*y(x)^2))*diff(y(x),x)=0,y(-1) = 2],y(x), singsol=all)
```

$$y(x) = x + \frac{3}{2} + \frac{\sqrt{4x^2 + 4x + 9}}{2}$$

✓ Solution by Mathematica

Time used: 1.961 (sec). Leaf size: 28

```
DSolve[{(3-y[x])/x^2+( (y[x]^2-2*x)/(x*y[x]^2) )*y'[x]==0,{y[-1]==2}},y[x],x,IncludeSingular
```

$$y(x) \to \frac{1}{2}\left(\sqrt{4x^2 + 4x + 9} + 2x + 3\right)$$

3.15 problem 16

Internal problem ID [11610]
Internal file name [`OUTPUT/10592_Saturday_May_27_2023_03_00_57_AM_57356254/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _exact, _rational]
```

$$\boxed{\frac{1+8xy^{\frac{2}{3}}}{x^{\frac{2}{3}}y^{\frac{1}{3}}}+\frac{\left(2x^{\frac{4}{3}}y^{\frac{2}{3}}-x^{\frac{1}{3}}\right)y'}{y^{\frac{4}{3}}}=0}$$

With initial conditions

$$[y(1)=8]$$

3.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{8x\,y^{\frac{5}{3}}+y}{x\left(2x\,y^{\frac{2}{3}}-1\right)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=8$ is

$$\left\{\frac{1}{8}\le x\le \infty, -\infty \le x<0, 0<x\le \frac{1}{8}\right\}$$

And the point $x_0=1$ is inside this domain. The y domain of $f(x,y)$ when $x=1$ is

$$\left\{0\le y\le \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\le y\le \infty\right\}$$

And the point $y_0=8$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{8x\,y^{\frac{5}{3}}+y}{x\left(2x\,y^{\frac{2}{3}}-1\right)}\right)\\
&= -\frac{\frac{40x\,y^{\frac{2}{3}}}{3}+1}{x\left(2x\,y^{\frac{2}{3}}-1\right)}+\frac{\frac{32x\,y^{\frac{5}{3}}}{3}+\frac{4y}{3}}{\left(2x\,y^{\frac{2}{3}}-1\right)^{2}y^{\frac{1}{3}}}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=8$ is

$$\left\{\frac{1}{8}\le x\le \infty, -\infty \le x<0, 0<x\le \frac{1}{8}\right\}$$

And the point $x_0=1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=1$ is

$$\left\{\frac{\sqrt{2}}{4}\le y\le \infty, 0<y\le \frac{\sqrt{2}}{4}\right\}$$

And the point $y_0=8$ is inside this domain. Therefore solution exists and is unique.

3.15.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= -\frac{8x\,y^{\frac{5}{3}}+y}{x\left(2x\,y^{\frac{2}{3}}-1\right)}\\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{\left(8x\, y^{\frac{5}{3}} + y\right)(b_3 - a_2)}{x\left(2x\, y^{\frac{2}{3}} - 1\right)} - \frac{\left(8x\, y^{\frac{5}{3}} + y\right)^2 a_3}{x^2 \left(2x\, y^{\frac{2}{3}} - 1\right)^2} \\ &- \left(-\frac{8 y^{\frac{5}{3}}}{x\left(2x\, y^{\frac{2}{3}} - 1\right)} + \frac{8x\, y^{\frac{5}{3}} + y}{x^2\left(2x\, y^{\frac{2}{3}} - 1\right)} + \frac{2\left(8x\, y^{\frac{5}{3}} + y\right) y^{\frac{2}{3}}}{x\left(2x\, y^{\frac{2}{3}} - 1\right)^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(-\frac{\frac{40x\, y^{\frac{2}{3}}}{3} + 1}{x\left(2x\, y^{\frac{2}{3}} - 1\right)} + \frac{\frac{32x\, y^{\frac{5}{3}}}{3} + \frac{4y}{3}}{\left(2x\, y^{\frac{2}{3}} - 1\right)^2 y^{\frac{1}{3}}}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{240 y^{\frac{11}{3}} x^2 a_3 + 48 y^{\frac{8}{3}} x^2 a_1 - 60 y^{\frac{5}{3}} x^4 b_2 + 30 y^2 x^2 a_2 + 20 x^2 y^2 b_3 + 60 y^3 x a_3 - 48 y^{\frac{5}{3}} x^3 b_1 + 50 x^3 y b_2 + 38 x^2 y b_1}{3x^2 \left(2x\, y^{\frac{2}{3}} - 1\right)^2 y^{\frac{1}{3}}}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-240 y^{\frac{11}{3}} x^2 a_3 - 48 y^{\frac{8}{3}} x^2 a_1 + 60 y^{\frac{5}{3}} x^4 b_2 + 48 y^{\frac{5}{3}} x^3 b_1 + 3 y^{\frac{4}{3}} a_1 - 50 x^3 y b_2 \\ &- 30 y^2 x^2 a_2 - 20 x^2 y^2 b_3 - 60 y^3 x a_3 - 38 x^2 y b_1 - 12 y^2 x a_1 - 3 y^{\frac{1}{3}} x b_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, y^{\frac{1}{3}}, y^{\frac{4}{3}}, y^{\frac{5}{3}}, y^{\frac{8}{3}}, y^{\frac{11}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, y^{\frac{1}{3}} = v_3, y^{\frac{4}{3}} = v_4, y^{\frac{5}{3}} = v_5, y^{\frac{8}{3}} = v_6, y^{\frac{11}{3}} = v_7\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &60v_5v_1^4b_2 - 30v_2^2v_1^2a_2 - 60v_2^3v_1a_3 + 48v_5v_1^3b_1 - 50v_1^3v_2b_2 - 20v_1^2v_2^2b_3 \\ &- 48v_6v_1^2a_1 - 12v_2^2v_1a_1 - 240v_7v_1^2a_3 - 38v_1^2v_2b_1 - 3v_3v_1b_1 + 3v_4a_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$

Equation (7E) now becomes

$$\begin{aligned} &60v_5v_1^4b_2 - 50v_1^3v_2b_2 + 48v_5v_1^3b_1 + (-30a_2 - 20b_3)\, v_1^2v_2^2 - 38v_1^2v_2b_1 \\ &- 48v_6v_1^2a_1 - 240v_7v_1^2a_3 - 60v_2^3v_1a_3 - 12v_2^2v_1a_1 - 3v_3v_1b_1 + 3v_4a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -48a_1 &= 0 \\ -12a_1 &= 0 \\ 3a_1 &= 0 \\ -240a_3 &= 0 \\ -60a_3 &= 0 \\ -38b_1 &= 0 \\ -3b_1 &= 0 \\ 48b_1 &= 0 \\ -50b_2 &= 0 \\ 60b_2 &= 0 \\ -30a_2 - 20b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -\frac{3a_2}{2}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -\frac{3y}{2}
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= -\frac{3y}{2} - \left(-\frac{8x\, y^{\frac{5}{3}} + y}{x \left(2x\, y^{\frac{2}{3}} - 1\right)}\right) (x) \\
&= \frac{10x\, y^{\frac{5}{3}} + 5y}{4x\, y^{\frac{2}{3}} - 2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{10x\,y^{\frac{5}{3}}+5y}{4x\,y^{\frac{2}{3}}-2}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{6\ln\left(2x\,y^{\frac{2}{3}}+1\right)}{5} - \frac{2\ln\left(y\right)}{5}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{8x\,y^{\frac{5}{3}}+y}{x\left(2x\,y^{\frac{2}{3}}-1\right)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{12y^{\frac{2}{3}}}{10x\,y^{\frac{2}{3}}+5} \\
S_y &= \frac{8x}{y^{\frac{1}{3}}\left(10x\,y^{\frac{2}{3}}+5\right)} - \frac{2}{5y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{2}{5x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{2}{5R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{2\ln(R)}{5} + c_1 \tag{4}$$

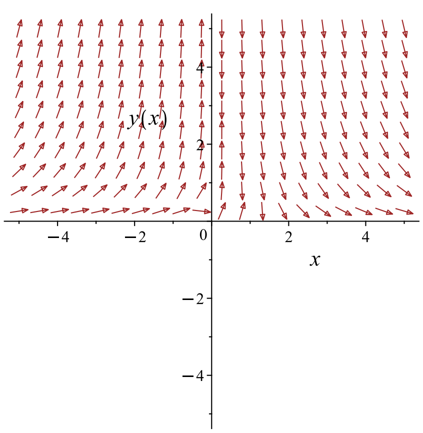
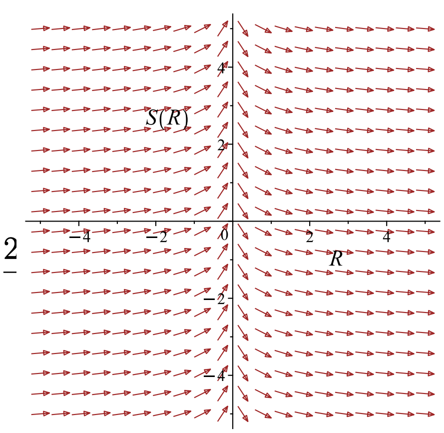
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{6\ln\left(2xy^{\frac{2}{3}}+1\right)}{5} - \frac{2\ln(y)}{5} = -\frac{2\ln(x)}{5} + c_1$$

Which simplifies to

$$\frac{6\ln\left(2xy^{\frac{2}{3}}+1\right)}{5} - \frac{2\ln(y)}{5} = -\frac{2\ln(x)}{5} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{8x\,y^{\frac{5}{3}}+y}{x\left(2x\,y^{\frac{2}{3}}-1\right)}$ 	$R = x$ $S = \frac{6\ln\left(2x\,y^{\frac{2}{3}}+1\right)}{5} - 2$	$\frac{dS}{dR} = -\frac{2}{5R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 8$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{12\ln(3)}{5} - \frac{6\ln(2)}{5} = c_1$$

$$c_1 = \frac{12 \ln (3)}{5} - \frac{6 \ln (2)}{5}$$

Substituting c_1 found above in the general solution gives

$$\frac{6 \ln \left(2x\, y^{\frac{2}{3}} + 1\right)}{5} - \frac{2 \ln (y)}{5} = -\frac{2 \ln (x)}{5} + \frac{12 \ln (3)}{5} - \frac{6 \ln (2)}{5}$$

Summary

The solution(s) found are the following

$$\frac{6 \ln \left(2x y^{\frac{2}{3}} + 1\right)}{5} - \frac{2 \ln (y)}{5} = -\frac{2 \ln (x)}{5} + \frac{12 \ln (3)}{5} - \frac{6 \ln (2)}{5} \tag{1}$$

Verification of solutions

$$\frac{6 \ln \left(2x y^{\frac{2}{3}} + 1\right)}{5} - \frac{2 \ln (y)}{5} = -\frac{2 \ln (x)}{5} + \frac{12 \ln (3)}{5} - \frac{6 \ln (2)}{5}$$

Verified OK.

3.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{2x^{\frac{4}{3}}y^{\frac{2}{3}} - x^{\frac{1}{3}}}{y^{\frac{4}{3}}}\right)\mathrm{d}y = \left(-\frac{1 + 8x\,y^{\frac{2}{3}}}{x^{\frac{2}{3}}y^{\frac{1}{3}}}\right)\mathrm{d}x$$

$$\left(\frac{1 + 8x\,y^{\frac{2}{3}}}{x^{\frac{2}{3}}y^{\frac{1}{3}}}\right)\mathrm{d}x + \left(\frac{2x^{\frac{4}{3}}y^{\frac{2}{3}} - x^{\frac{1}{3}}}{y^{\frac{4}{3}}}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{1 + 8x\,y^{\frac{2}{3}}}{x^{\frac{2}{3}}y^{\frac{1}{3}}}$$

$$N(x, y) = \frac{2x^{\frac{4}{3}}y^{\frac{2}{3}} - x^{\frac{1}{3}}}{y^{\frac{4}{3}}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{1 + 8x\,y^{\frac{2}{3}}}{x^{\frac{2}{3}}y^{\frac{1}{3}}}\right)\\ &= \frac{8x\,y^{\frac{2}{3}} - 1}{3x^{\frac{2}{3}}y^{\frac{4}{3}}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2x^{\frac{4}{3}}y^{\frac{2}{3}} - x^{\frac{1}{3}}}{y^{\frac{4}{3}}}\right)\\ &= \frac{8x\,y^{\frac{2}{3}} - 1}{3x^{\frac{2}{3}}y^{\frac{4}{3}}}\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x}=M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y}=N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x
$$

$$
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int \frac{1+8x\,y^{\frac{2}{3}}}{x^{\frac{2}{3}}y^{\frac{1}{3}}}\,\mathrm{d}x
$$

$$
\phi=\frac{6x^{\frac{1}{3}}\left(x\,y^{\frac{2}{3}}+\frac{1}{2}\right)}{y^{\frac{1}{3}}}+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y}&=-\frac{2x^{\frac{1}{3}}\left(x\,y^{\frac{2}{3}}+\frac{1}{2}\right)}{y^{\frac{4}{3}}}+\frac{4x^{\frac{4}{3}}}{y^{\frac{2}{3}}}+f'(y) \qquad (4)\\
&=\frac{x^{\frac{1}{3}}\left(2x\,y^{\frac{2}{3}}-1\right)}{y^{\frac{4}{3}}}+f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{2x^{\frac{4}{3}}y^{\frac{2}{3}}-x^{\frac{1}{3}}}{y^{\frac{4}{3}}}$. Therefore equation (4) becomes

$$
\frac{2x^{\frac{4}{3}}y^{\frac{2}{3}}-x^{\frac{1}{3}}}{y^{\frac{4}{3}}}=\frac{x^{\frac{1}{3}}\left(2x\,y^{\frac{2}{3}}-1\right)}{y^{\frac{4}{3}}}+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y)=0
$$

Therefore

$$
f(y)=c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{6x^{\frac{1}{3}}\left(x\,y^{\frac{2}{3}}+\frac{1}{2}\right)}{y^{\frac{1}{3}}}+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{6x^{\frac{1}{3}}\left(x\,y^{\frac{2}{3}}+\frac{1}{2}\right)}{y^{\frac{1}{3}}}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 8$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{27}{2} = c_1$$

$$c_1 = \frac{27}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{6x^{\frac{1}{3}}\left(x\,y^{\frac{2}{3}}+\frac{1}{2}\right)}{y^{\frac{1}{3}}} = \frac{27}{2}$$

The above simplifies to

$$12x^{\frac{4}{3}}y^{\frac{2}{3}} - 27y^{\frac{1}{3}} + 6x^{\frac{1}{3}} = 0$$

Summary

The solution(s) found are the following

$$12x^{\frac{4}{3}}y^{\frac{2}{3}} - 27y^{\frac{1}{3}} + 6x^{\frac{1}{3}} = 0 \tag{1}$$

Verification of solutions

$$12x^{\frac{4}{3}}y^{\frac{2}{3}} - 27y^{\frac{1}{3}} + 6x^{\frac{1}{3}} = 0$$

Verified OK.

3.15.4 Maple step by step solution

Let's solve

$$\left[\frac{1+8xy^{\frac{2}{3}}}{x^{\frac{2}{3}}y^{\frac{1}{3}}}+\frac{\left(2x^{\frac{4}{3}}y^{\frac{2}{3}}-x^{\frac{1}{3}}\right)y'}{y^{\frac{4}{3}}}=0, y(1)=8\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$
  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
  - Evaluate derivatives

    $$\frac{16x^{\frac{1}{3}}}{3y^{\frac{2}{3}}}-\frac{1+8xy^{\frac{2}{3}}}{3x^{\frac{2}{3}}y^{\frac{4}{3}}}=\frac{\frac{8x^{\frac{1}{3}}y^{\frac{2}{3}}}{3}-\frac{1}{3x^{\frac{2}{3}}}}{y^{\frac{4}{3}}}$$
  - Simplify

    $$\frac{8xy^{\frac{2}{3}}-1}{3x^{\frac{2}{3}}y^{\frac{4}{3}}}=\frac{8xy^{\frac{2}{3}}-1}{3x^{\frac{2}{3}}y^{\frac{4}{3}}}$$
  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $$\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $$F(x,y)=\int\frac{1+8xy^{\frac{2}{3}}}{x^{\frac{2}{3}}y^{\frac{1}{3}}}dx+f_1(y)$$

- Evaluate integral

  $$F(x,y)=\frac{3x^{\frac{1}{3}}+6x^{\frac{4}{3}}y^{\frac{2}{3}}}{y^{\frac{1}{3}}}+f_1(y)$$

- Take derivative of $F(x,y)$ with respect to y

  $$N(x,y)=\frac{\partial}{\partial y}F(x,y)$$

- Compute derivative

  $$\frac{2x^{\frac{4}{3}}y^{\frac{2}{3}}-x^{\frac{1}{3}}}{y^{\frac{4}{3}}}=-\frac{3x^{\frac{1}{3}}+6x^{\frac{4}{3}}y^{\frac{2}{3}}}{3y^{\frac{4}{3}}}+\frac{4x^{\frac{4}{3}}}{y^{\frac{2}{3}}}+\frac{d}{dy}f_1(y)$$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y) = \frac{2x^{\frac{4}{3}}y^{\frac{2}{3}} - x^{\frac{1}{3}}}{y^{\frac{4}{3}}} + \frac{3x^{\frac{1}{3}} + 6x^{\frac{4}{3}}y^{\frac{2}{3}}}{3y^{\frac{4}{3}}} - \frac{4x^{\frac{4}{3}}}{y^{\frac{2}{3}}}$$

- Solve for $f_1(y)$

$$f_1(y) = 0$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = \frac{3x^{\frac{1}{3}} + 6x^{\frac{4}{3}}y^{\frac{2}{3}}}{y^{\frac{1}{3}}}$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\frac{3x^{\frac{1}{3}} + 6x^{\frac{4}{3}}y^{\frac{2}{3}}}{y^{\frac{1}{3}}} = c_1$$

- Solve for y

$$\left\{y = -\frac{-\frac{3x^{\frac{4}{3}}\left(-c_1+\sqrt{c_1^2-72x^{\frac{5}{3}}}\right)}{2} + 3c_1x^{\frac{4}{3}} + \frac{c_1^2\left(-c_1+\sqrt{c_1^2-72x^{\frac{5}{3}}}\right)}{12x^{\frac{1}{3}}}}{36x^{\frac{11}{3}}}, y = -\frac{\frac{3x^{\frac{4}{3}}\left(c_1+\sqrt{c_1^2-72x^{\frac{5}{3}}}\right)}{2} + 3c_1x^{\frac{4}{3}} - \frac{c_1^2\left(c_1+\sqrt{c_1^2-72x^{\frac{5}{3}}}\right)}{12x^{\frac{1}{3}}}}{36x^{\frac{11}{3}}}\right\}$$

- Use initial condition $y(1) = 8$

$$8 = -\frac{c_1}{8} + \frac{\sqrt{c_1^2-72}}{24} - \frac{c_1^2\left(-c_1+\sqrt{c_1^2-72}\right)}{432}$$

- Solve for c_1

$$c_1 = \left(-\frac{27}{4} - \frac{21\,\mathrm{I}\sqrt{3}}{4}, -\frac{27}{4} + \frac{21\,\mathrm{I}\sqrt{3}}{4}\right)$$

- Substitute $c_1 = \left(-\frac{27}{4} - \frac{21\,\mathrm{I}\sqrt{3}}{4}, -\frac{27}{4} + \frac{21\,\mathrm{I}\sqrt{3}}{4}\right)$ into general solution and simplify

$$y = \frac{\left(16x^{\frac{5}{3}} - 63\,\mathrm{I}\sqrt{3} + 33\right)\sqrt{-66 + 126\,\mathrm{I}\sqrt{3} - 128x^{\frac{5}{3}} + 336\,\mathrm{I}\sqrt{3}\,x^{\frac{5}{3}}} - 336\,\mathrm{I}\sqrt{3} + 432x^{\frac{5}{3}} + 1620}{512x^4}$$

- Use initial condition $y(1) = 8$

$$8 = -\frac{c_1}{8} - \frac{\sqrt{c_1^2-72}}{24} + \frac{c_1^2\left(c_1+\sqrt{c_1^2-72}\right)}{432}$$

- Solve for c_1

$$c_1 = \frac{27}{2}$$

- Substitute $c_1 = \frac{27}{2}$ into general solution and simplify

$$y = \frac{\left(-8x^{\frac{5}{3}} + 81\right)\sqrt{81 - 32x^{\frac{5}{3}}} - 216x^{\frac{5}{3}} + 729}{128x^4}$$

- Solutions to the IVP

$$\left\{y = \frac{\left(-8x^{\frac{5}{3}}+81\right)\sqrt{81-32x^{\frac{5}{3}}}-216x^{\frac{5}{3}}+729}{128x^4}, y = \frac{\left(16x^{\frac{5}{3}}-63\,\mathrm{I}\sqrt{3}+33\right)\sqrt{-66+126\,\mathrm{I}\sqrt{3}-128x^{\frac{5}{3}}+336\,\mathrm{I}\sqrt{3}\,x^{\frac{5}{3}}-336\,\mathrm{I}\sqrt{3}+432x^{\frac{5}{3}}}}{512x^4}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 55

```
dsolve([(1+8*x*y(x)^(2/3))/(x^(2/3)*y(x)^(1/3))+((2*x^(4/3)*y(x)^(2/3)-x^(1/3))/(y(x)^(4/3))
```

$$y(x) = \mathrm{RootOf}\left(64\_Z^{\frac{7}{3}}x^4 + 96\_Z^{\frac{5}{3}}x^3 - 729\_Z^{\frac{4}{3}} + 48x^2\_Z + 8x\_Z^{\frac{1}{3}}\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{(1+8*x*y[x]^(2/3))/(x^(2/3)*y[x]^(1/3))+((2*x^(4/3)*y[x]^(2/3)-x^(1/3))/(y[x]^(4/3))
```

{}

3.16 problem 21

Internal problem ID [11611]
Internal file name [`OUTPUT/10593_Saturday_May_27_2023_03_01_00_AM_68786493/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{3y^2 + 2xyy' = -4x}$$

3.16.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{3y^2 + 4x}{2xy} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 66: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^3 y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^3 y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 x^3}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3y^2 + 4x}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3x^2 y^2}{2} \\ S_y &= y\, x^3 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -2x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -2R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R^4}{2} + c_1 \tag{4}$$

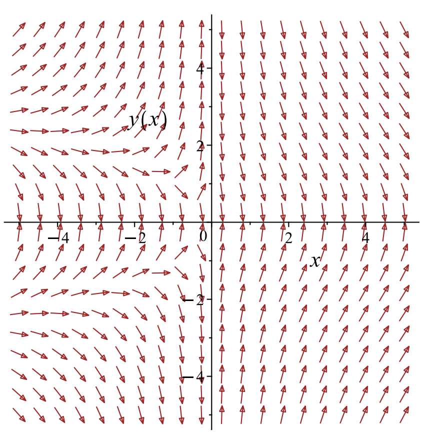
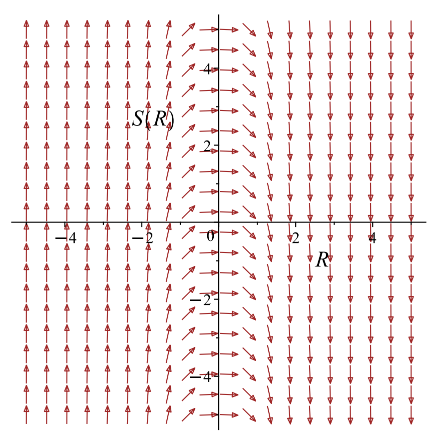
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2 x^3}{2} = -\frac{x^4}{2} + c_1$$

Which simplifies to

$$\frac{y^2 x^3}{2} = -\frac{x^4}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y^2+4x}{2xy}$ 	$R = x$ $S = \frac{y^2 x^3}{2}$	$\frac{dS}{dR} = -2R^3$ 

Summary
The solution(s) found are the following

$$\frac{y^2 x^3}{2} = -\frac{x^4}{2} + c_1 \tag{1}$$

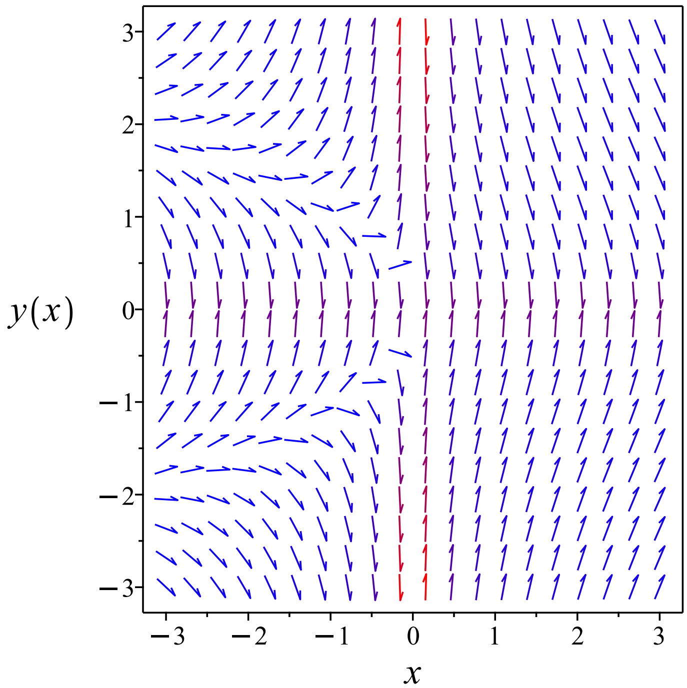


Figure 63: Slope field plot

Verification of solutions

$$\frac{y^2x^3}{2} = -\frac{x^4}{2} + c_1$$

Verified OK.

3.16.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{3y^2 + 4x}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{3}{2x}y - 2\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{3}{2x} \\ f_1(x) &= -2 \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{3y^2}{2x} - 2 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -\frac{3w(x)}{2x} - 2 \\ w' &= -\frac{3w}{x} - 4 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= -4 \end{aligned}$$

Hence the ode is

$$w'(x)+\frac{3w(x)}{x}=-4$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{3}{x}dx}\\ &= x^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(-4)\\ \frac{\mathrm{d}}{\mathrm{d}x}(x^3 w) &= (x^3)(-4)\\ \mathrm{d}(x^3 w) &= (-4x^3)\ \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}x^3 w &= \int -4x^3\,\mathrm{d}x\\ x^3 w &= -x^4+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$w(x)=-x+\frac{c_1}{x^3}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2=-x+\frac{c_1}{x^3}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{x\left(-x^4+c_1\right)}}{x^2}\\ y(x) &= -\frac{\sqrt{x\left(-x^4+c_1\right)}}{x^2}\end{aligned}$$

Summary

The solution(s) found are the following

$$y=\frac{\sqrt{x\left(-x^4+c_1\right)}}{x^2} \tag{1}$$

$$y=-\frac{\sqrt{x\left(-x^4+c_1\right)}}{x^2} \tag{2}$$

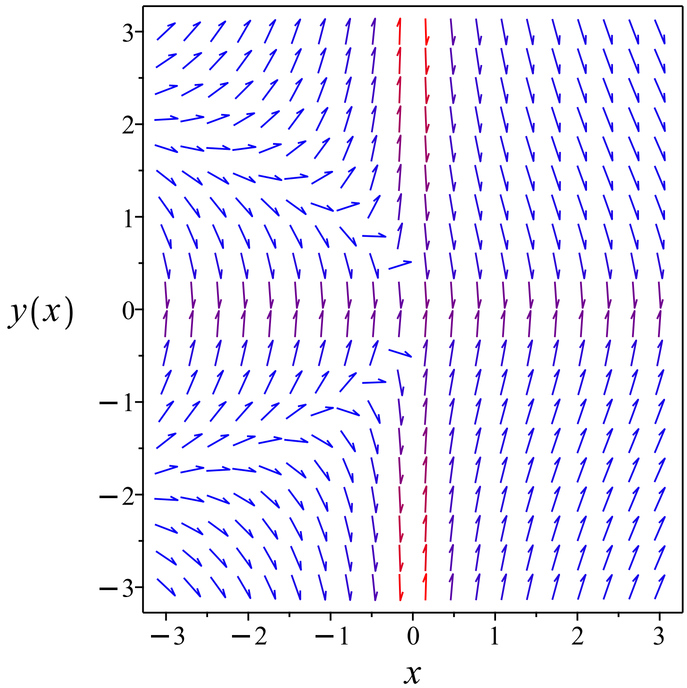


Figure 64: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x(-x^4 + c_1)}}{x^2}$$

Verified OK.

$$y = -\frac{\sqrt{x(-x^4 + c_1)}}{x^2}$$

Verified OK.

3.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy)\,\mathrm{d}y&=\left(-3y^2-4x\right)\mathrm{d}x\\ \left(3y^2+4x\right)\mathrm{d}x+(2xy)\,\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=3y^2+4x\\ N(x,y)&=2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(3y^2+4x\right)\\ &=6y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy) \\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{2xy}((6y) - (2y)) \\ &= \frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int \frac{2}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{2\ln(x)} \\ &= x^2\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x^2(3y^2 + 4x) \\ &= 3x^2y^2 + 4x^3\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x^2(2xy) \\ &= 2y\,x^3\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (3x^2y^2 + 4x^3) + (2y\,x^3)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \overline{M} \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int 3x^2y^2 + 4x^3 \, dx$$
$$\phi = x^3(y^2 + x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2y\,x^3 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y\,x^3$. Therefore equation (4) becomes

$$2y\,x^3 = 2y\,x^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x^3(y^2 + x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^3(y^2 + x)$$

Summary

The solution(s) found are the following

$$x^3(y^2+x) = c_1 \tag{1}$$

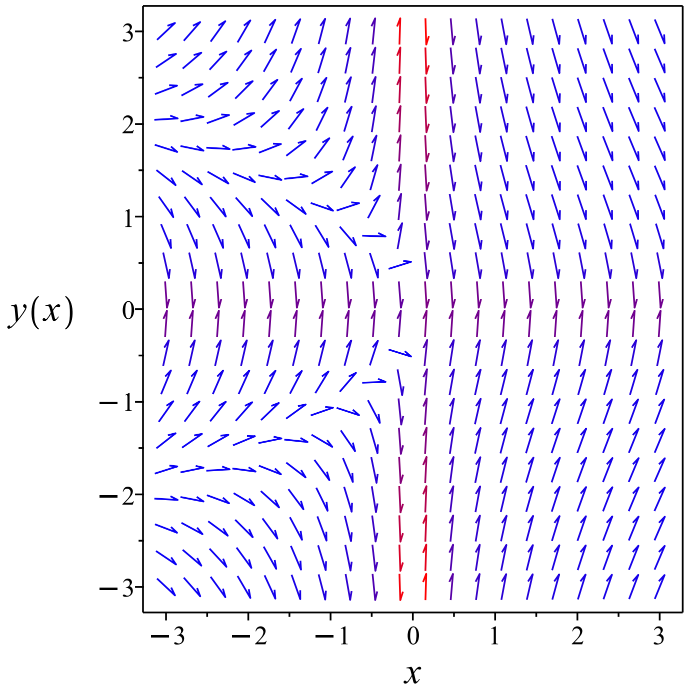


Figure 65: Slope field plot

Verification of solutions

$$x^3(y^2+x) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve((4*x+3*y(x)^2)+(2*x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{x\left(-x^4+c_1\right)}}{x^2}$$
$$y(x) = -\frac{\sqrt{x\left(-x^4+c_1\right)}}{x^2}$$

✓ Solution by Mathematica

Time used: 0.349 (sec). Leaf size: 46

```
DSolve[(4*x+3*y[x]^2)+(2*x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-x^4+c_1}}{x^{3/2}}$$
$$y(x) \to \frac{\sqrt{-x^4+c_1}}{x^{3/2}}$$

3.17 problem 22

Internal problem ID [11612]
Internal file name [`OUTPUT/10594_Saturday_May_27_2023_03_01_02_AM_9338345/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 + 2yx - x^2 y' = 0}$$

3.17.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 + 2u(x)\,x^2 - x^2(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(u+1)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u(u+1)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u(u+1)}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{u(u+1)}\,du &= \int \frac{1}{x}\,dx \\
-\ln(u+1) + \ln(u) &= \ln(x) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{-\ln(u+1)+\ln(u)} = e^{\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{u}{u+1} = c_3 x
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= -\frac{x^2 c_3}{c_3 x - 1}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{x^2 c_3}{c_3 x - 1} \tag{1}
$$

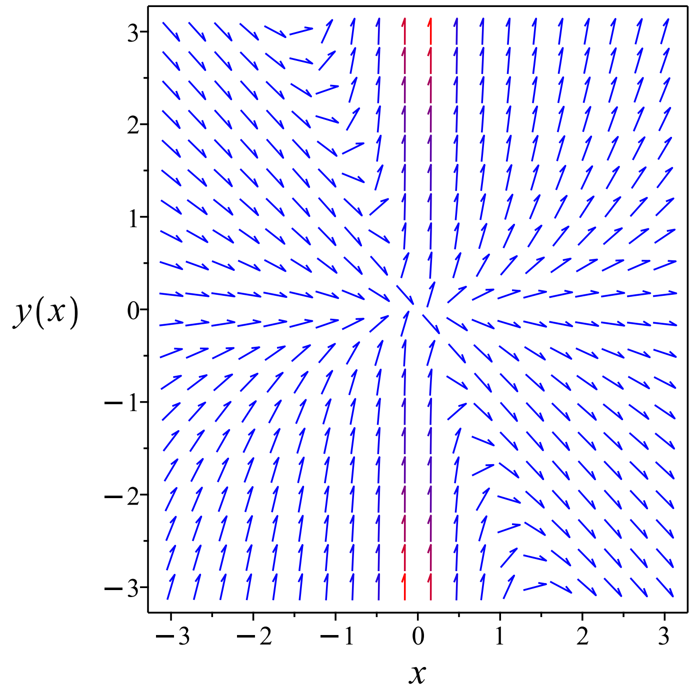


Figure 66: Slope field plot

Verification of solutions

$$y = -\frac{x^2 c_3}{c_3 x - 1}$$

Verified OK.

3.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(2x + y)}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 68: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{y^2}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^2}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(2x+y)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2x}{y} \\ S_y &= \frac{x^2}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{y} = x + c_1$$

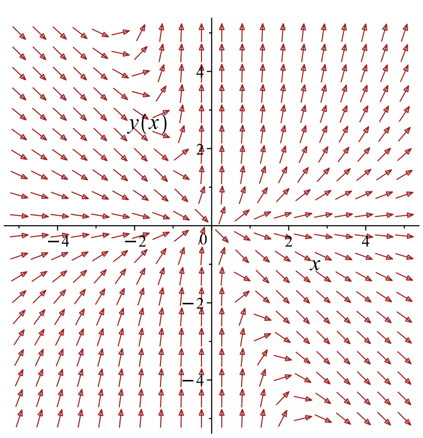
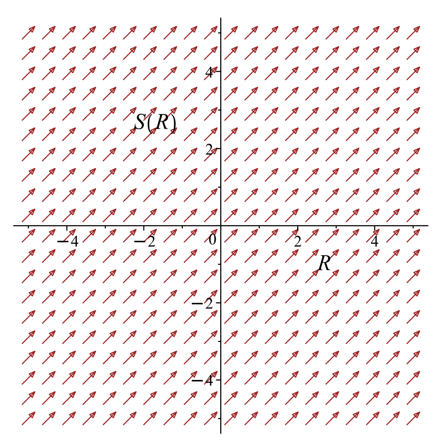
Which simplifies to

$$-\frac{x^2}{y} = x + c_1$$

Which gives

$$y = -\frac{x^2}{x + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(2x+y)}{x^2}$ 	$R = x$ $S = -\frac{x^2}{y}$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = -\frac{x^2}{x + c_1} \tag{1}$$

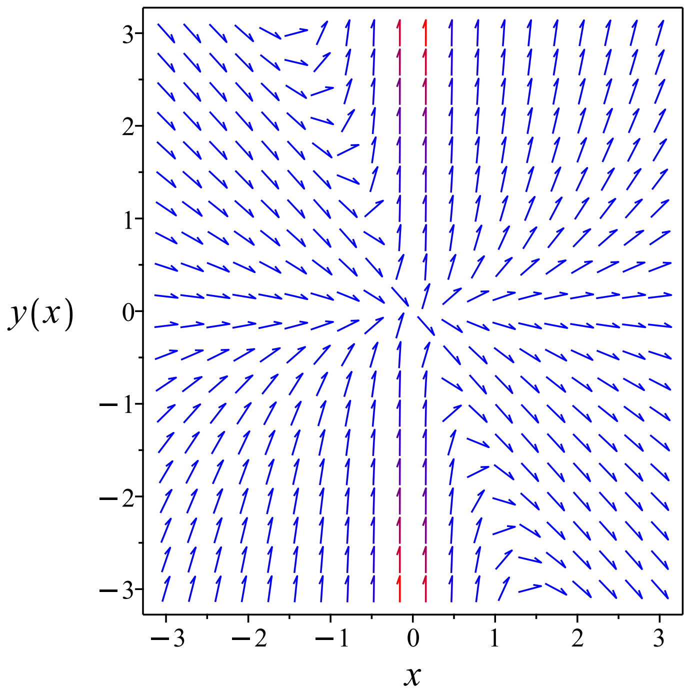


Figure 67: Slope field plot

Verification of solutions

$$y = -\frac{x^2}{x + c_1}$$

Verified OK.

3.17.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(2x + y)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{2}{x}y + \frac{1}{x^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{2}{x} \\ f_1(x) &= \frac{1}{x^2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{2}{xy} + \frac{1}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{2w(x)}{x} + \frac{1}{x^2} \\ w' &= -\frac{2w}{x} - \frac{1}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = -\frac{1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{2}{x} dx} \\ &= x^2 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{1}{x^2}\right) \\ \frac{d}{dx}(x^2 w) &= (x^2)\left(-\frac{1}{x^2}\right) \\ d(x^2 w) &= -1\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} x^2 w &= \int -1\, dx \\ x^2 w &= -x + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = -\frac{1}{x} + \frac{c_1}{x^2}$$

which simplifies to

$$w(x) = \frac{-x + c_1}{x^2}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{-x + c_1}{x^2}$$

Or

$$y = \frac{x^2}{-x + c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{-x + c_1} \tag{1}$$

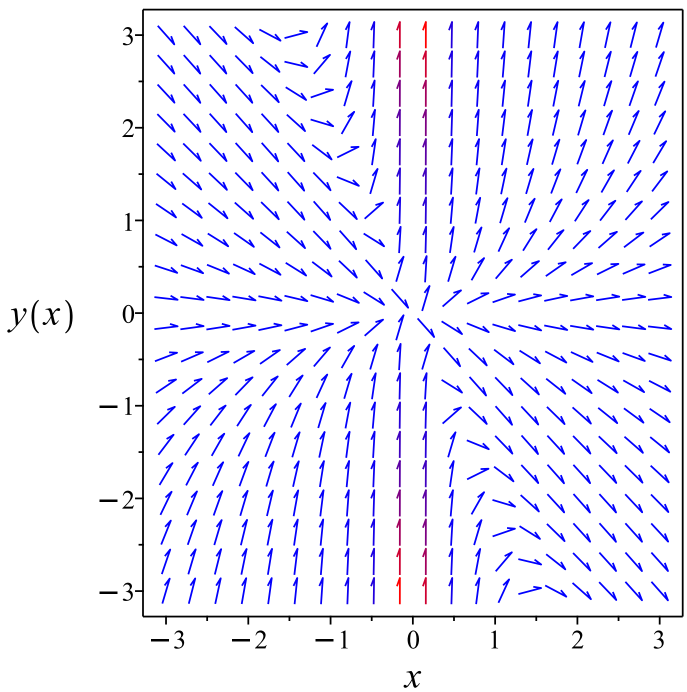


Figure 68: Slope field plot

Verification of solutions

$$y = \frac{x^2}{-x + c_1}$$

Verified OK.

3.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x^2)\,\mathrm{d}y&=(-2xy-y^2)\,\mathrm{d}x\\ (2xy+y^2)\,\mathrm{d}x+(-x^2)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=2xy+y^2\\ N(x,y)&=-x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2xy + y^2) \\ &= 2y + 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2) \\ &= -2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{x^2}((2y + 2x) - (-2x)) \\ &= \frac{-4x - 2y}{x^2}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{y(2x + y)}((-2x) - (2y + 2x)) \\ &= -\frac{2}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}y} \\ &= e^{\int -\frac{2}{y} \,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(y)}\\ &= \frac{1}{y^2}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{y^2}\left(2xy + y^2\right)\\ &= \frac{2x + y}{y}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{y^2}\left(-x^2\right)\\ &= -\frac{x^2}{y^2}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{2x + y}{y}\right) + \left(-\frac{x^2}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2x + y}{y}\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{x(y + x)}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{x}{y} - \frac{x(y+x)}{y^2} + f'(y) \qquad (4) \\
&= -\frac{x^2}{y^2} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x^2}{y^2}$. Therefore equation (4) becomes

$$
-\frac{x^2}{y^2} = -\frac{x^2}{y^2} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{x(y+x)}{y} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{x(y+x)}{y}
$$

The solution becomes

$$
y = \frac{x^2}{-x + c_1}
$$

Summary
The solution(s) found are the following

$$
y = \frac{x^2}{-x + c_1} \qquad (1)
$$

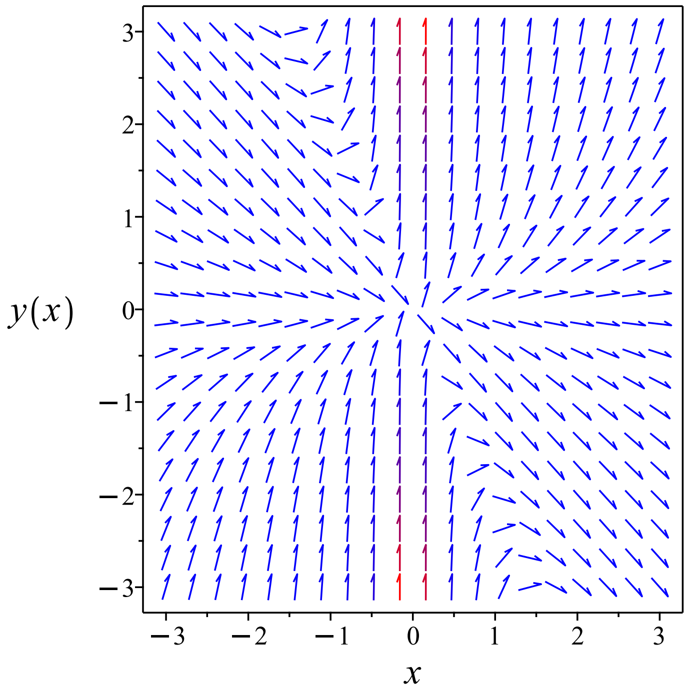


Figure 69: Slope field plot

Verification of solutions

$$y = \frac{x^2}{-x + c_1}$$

Verified OK.

3.17.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(2x + y)}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{2y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{2}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{x^2}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2}{x^3} \\
f_1 f_2 &= \frac{2}{x^3} \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{x^2} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 x + c_2
$$

The above shows that

$$
u'(x) = c_1
$$

Using the above in (1) gives the solution

$$
y = -\frac{c_1 x^2}{c_1 x + c_2}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = -\frac{c_3 x^2}{c_3 x + 1}
$$

Summary
The solution(s) found are the following

$$y = -\frac{c_3 x^2}{c_3 x + 1} \tag{1}$$

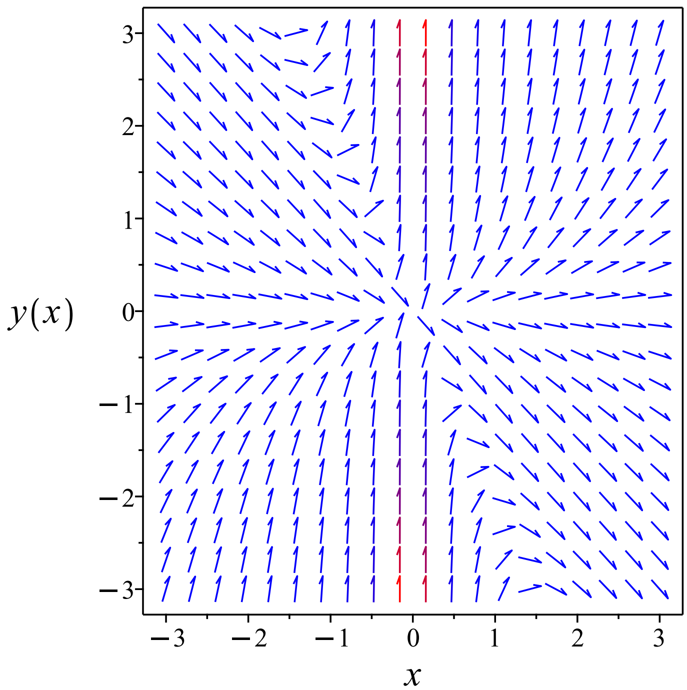


Figure 70: Slope field plot

Verification of solutions

$$y = -\frac{c_3 x^2}{c_3 x + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((y(x)^2+2*x*y(x))-x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{c_1 - x}$$

✓ Solution by Mathematica

Time used: 0.221 (sec). Leaf size: 23

```
DSolve[(y[x]^2+2*x*y[x])-x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2}{x - c_1}$$
$$y(x) \to 0$$

3.18 problem 24

Internal problem ID [11613]
Internal file name [`OUTPUT/10595_Saturday_May_27_2023_03_01_05_AM_38663670/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.1 (Exact differential equations and integrating factors). Exercises page 37
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational]
```

$$\boxed{y+x\left(x^{2}+y^{2}\right)^{2}+\left(y\left(x^{2}+y^{2}\right)^{2}-x\right)y'=0}$$

3.18.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{x^{5}+2y^{2}x^{3}+x\,y^{4}+y}{x^{4}y+2y^{3}x^{2}+y^{5}-x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(x^5+2y^2x^3+x\,y^4+y)\,(b_3-a_2)}{x^4y+2y^3x^2+y^5-x} - \frac{(x^5+2y^2x^3+x\,y^4+y)^2\,a_3}{(x^4y+2y^3x^2+y^5-x)^2} - \left(-\frac{5x^4+6x^2y^2+y^4}{x^4y+2y^3x^2+y^5-x} + \frac{(x^5+2y^2x^3+x\,y^4+y)\,(4y\,x^3+4x\,y^3-1)}{(x^4y+2y^3x^2+y^5-x)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{4y\,x^3+4x\,y^3+1}{x^4y+2y^3x^2+y^5-x} + \frac{(x^5+2y^2x^3+x\,y^4+y)\,(x^4+6x^2y^2+5y^4)}{(x^4y+2y^3x^2+y^5-x)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^{10}a_3+x^{10}b_2-2x^9ya_2+2x^9yb_3+3x^8y^2a_3+3x^8y^2b_2-8x^7y^3a_2+8x^7y^3b_3+2x^6y^4a_3+2x^6y^4b_2-12x^5}{}$$
$$= 0$$

Setting the numerator to zero gives

$$-x^{10}a_3-x^{10}b_2+2x^9ya_2-2x^9yb_3-3x^8y^2a_3-3x^8y^2b_2+8x^7y^3a_2-8x^7y^3b_3-2x^6y^4a_3-2x^6y^4b_2+12x^5y^5a_2-12x^5y^5b_3+2x^4y^6a_3+2x^4y^6b_2+8x^3y^7a_2-8x^3y^7b_3+3x^2y^8a_3+3x^2y^8b_2+2x\,y^9a_2-2x\,y^9b_3+y^{10}a_3+y^{10}b_2-x^9b_1+x^8ya_1-4x^7y^2b_1+4x^6y^3a_1-6x^5y^4b_1+6x^4y^5a_1-4x^3y^6b_1+4x^2y^7a_1-x\,y^8b_1+y^9a_1-5x^6a_2+x^6b_3-6x^5ya_3-6x^5yb_2-9x^4y^2a_2-3x^4y^2b_3-12x^3y^3a_3-12x^3y^3b_2-3x^2y^4a_2-9x^2y^4b_3-6x\,y^5a_3-6x\,y^5b_2+y^6a_2-5y^6b_3-4x^5a_1-4x^4yb_1-8x^3y^2a_1-8x^2y^3b_1-4x\,y^4a_1-4y^5b_1-xb_1+ya_1=0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&2a_2v_1^9v_2 + 8a_2v_1^7v_2^3 + 12a_2v_1^5v_2^5 + 8a_2v_1^3v_2^7 + 2a_2v_1v_2^9 - a_3v_1^{10} - 3a_3v_1^8v_2^2 \\
&- 2a_3v_1^6v_2^4 + 2a_3v_1^4v_2^6 + 3a_3v_1^2v_2^8 + a_3v_2^{10} - b_2v_1^{10} - 3b_2v_1^8v_2^2 - 2b_2v_1^6v_2^4 \\
&+ 2b_2v_1^4v_2^6 + 3b_2v_1^2v_2^8 + b_2v_2^{10} - 2b_3v_1^9v_2 - 8b_3v_1^7v_2^3 - 12b_3v_1^5v_2^5 \\
&- 8b_3v_1^3v_2^7 - 2b_3v_1v_2^9 + a_1v_1^8v_2 + 4a_1v_1^6v_2^3 + 6a_1v_1^4v_2^5 + 4a_1v_1^2v_2^7 + a_1v_2^9 \\
&- b_1v_1^9 - 4b_1v_1^7v_2^2 - 6b_1v_1^5v_2^4 - 4b_1v_1^3v_2^6 - b_1v_1v_2^8 - 5a_2v_1^6 - 9a_2v_1^4v_2^2 \\
&- 3a_2v_1^2v_2^4 + a_2v_2^6 - 6a_3v_1^5v_2 - 12a_3v_1^3v_2^3 - 6a_3v_1v_2^5 - 6b_2v_1^5v_2 \\
&- 12b_2v_1^3v_2^3 - 6b_2v_1v_2^5 + b_3v_1^6 - 3b_3v_1^4v_2^2 - 9b_3v_1^2v_2^4 - 5b_3v_2^6 - 4a_1v_1^5 \\
&- 8a_1v_1^3v_2^2 - 4a_1v_1v_2^4 - 4b_1v_1^4v_2 - 8b_1v_1^2v_2^3 - 4b_1v_2^5 + a_1v_2 - b_1v_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(2a_3+2b_2)\, v_1^4v_2^6 + (-9a_2-3b_3)\, v_1^4v_2^2 + (8a_2-8b_3)\, v_1^3v_2^7 + (-3a_3-3b_2)\, v_1^8v_2^2 + (8a_2-8b_3)\, v_1^7v_2^3 \\
&+ (-2a_3 - 2b_2)\, v_1^6v_2^4 + (12a_2 - 12b_3)\, v_1^5v_2^5 + (-6a_3 - 6b_2)\, v_1^5v_2 + (-12a_3 - 12b_2)\, v_1^3v_2^3 \\
&+ (3a_3 + 3b_2)\, v_1^2v_2^8 + (-3a_2 - 9b_3)\, v_1^2v_2^4 + (2a_2 - 2b_3)\, v_1v_2^9 + (-6a_3 - 6b_2)\, v_1v_2^5 \\
&+ (2a_2 - 2b_3)\, v_1^9v_2 + a_1v_2^9 - b_1v_1^9 - 4a_1v_1^5 - 4b_1v_2^5 + a_1v_2 - b_1v_1 + (a_3 + b_2)\, v_2^{10} \\
&+ (a_2 - 5b_3)\, v_2^6 + (-5a_2 + b_3)\, v_1^6 + (-a_3 - b_2)\, v_1^{10} + a_1v_1^8v_2 + 4a_1v_1^6v_2^3 + 6a_1v_1^4v_2^5 + 4a_1v_1^2v_2^7 \\
&- 4b_1v_1^7v_2^2 - 6b_1v_1^5v_2^4 - 4b_1v_1^3v_2^6 - b_1v_1v_2^8 - 8a_1v_1^3v_2^2 - 4a_1v_1v_2^4 - 4b_1v_1^4v_2 - 8b_1v_1^2v_2^3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
-8a_1 &= 0 \\
-4a_1 &= 0 \\
4a_1 &= 0 \\
6a_1 &= 0 \\
-8b_1 &= 0 \\
-6b_1 &= 0 \\
-4b_1 &= 0 \\
-b_1 &= 0 \\
-9a_2 - 3b_3 &= 0 \\
-5a_2 + b_3 &= 0 \\
-3a_2 - 9b_3 &= 0 \\
a_2 - 5b_3 &= 0 \\
2a_2 - 2b_3 &= 0 \\
8a_2 - 8b_3 &= 0 \\
12a_2 - 12b_3 &= 0 \\
-12a_3 - 12b_2 &= 0 \\
-6a_3 - 6b_2 &= 0 \\
-3a_3 - 3b_2 &= 0 \\
-2a_3 - 2b_2 &= 0 \\
-a_3 - b_2 &= 0 \\
a_3 + b_2 &= 0 \\
2a_3 + 2b_2 &= 0 \\
3a_3 + 3b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= -b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -y \\
\eta &= x
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= x - \left(-\frac{x^5 + 2y^2x^3 + x\,y^4 + y}{x^4y + 2y^3x^2 + y^5 - x}\right)(-y) \\
&= \frac{-x^2 - y^2}{x^4y + 2y^3x^2 + y^5 - x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2 - y^2}{x^4y + 2y^3x^2 + y^5 - x}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{y^4}{4} - \frac{x^2y^2}{2} + \arctan\left(\frac{y}{x}\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x^5 + 2y^2x^3 + x\,y^4 + y}{x^4y + 2y^3x^2 + y^5 - x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y(y\,x^3 + x\,y^3 + 1)}{x^2 + y^2} \\
S_y &= -y^3 - x^2y + \frac{x}{x^2 + y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^4}{4} + c_1 \tag{4}$$

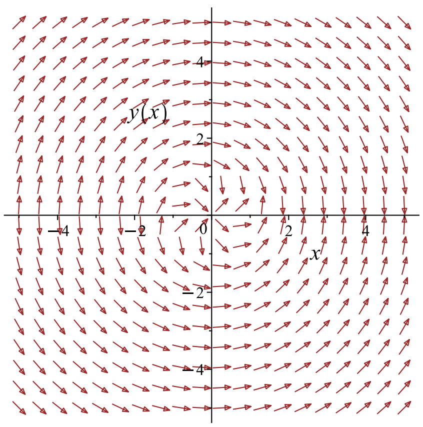
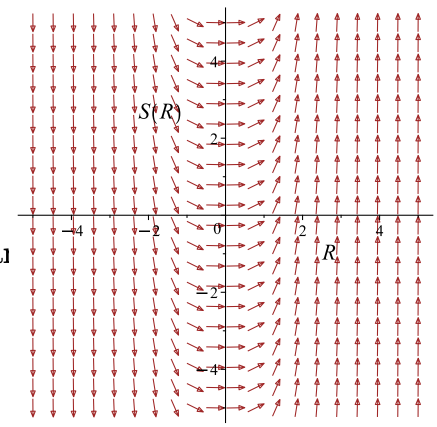
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{y^4}{4} - \frac{x^2y^2}{2} + \arctan\left(\frac{y}{x}\right) = \frac{x^4}{4} + c_1$$

Which simplifies to

$$-\frac{y^4}{4} - \frac{x^2y^2}{2} + \arctan\left(\frac{y}{x}\right) = \frac{x^4}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^5+2y^2x^3+x\,y^4+y}{x^4y+2y^3x^2+y^5-x}$ 	$R = x$ $S = -\frac{y^4}{4} - \frac{x^2y^2}{2} + \mathrm{arcta}$	$\frac{dS}{dR} = R^3$ 

Summary

The solution(s) found are the following

$$-\frac{y^4}{4} - \frac{x^2y^2}{2} + \arctan\left(\frac{y}{x}\right) = \frac{x^4}{4} + c_1 \tag{1}$$

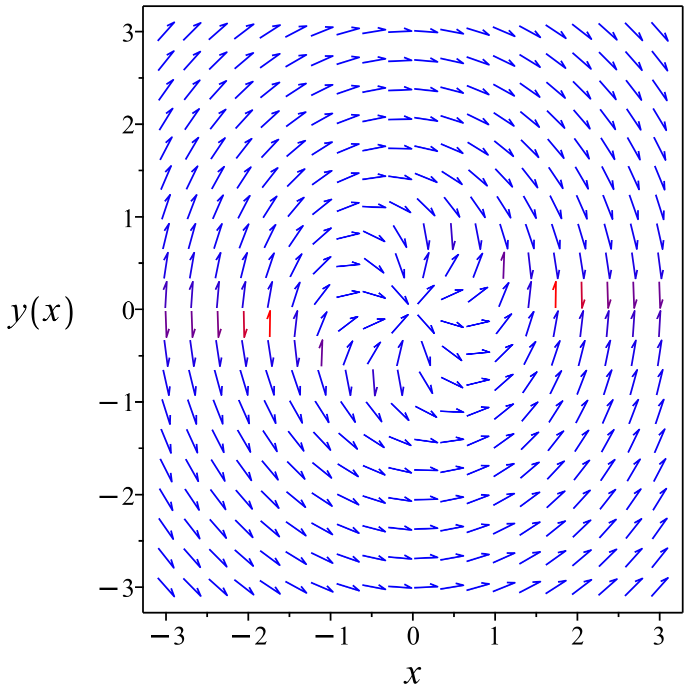


Figure 71: Slope field plot

Verification of solutions

$$-\frac{y^4}{4}-\frac{x^2y^2}{2}+\arctan\left(\frac{y}{x}\right)=\frac{x^4}{4}+c_1$$

Verified OK.

3.18.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(y\left(x^2+y^2\right)^2 - x\right)\mathrm{d}y &= \left(-y - \left(x^2+y^2\right)^2 x\right)\mathrm{d}x \\
\left(y + \left(x^2+y^2\right)^2 x\right)\mathrm{d}x + \left(y\left(x^2+y^2\right)^2 - x\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y + \left(x^2+y^2\right)^2 x \\
N(x, y) &= y\left(x^2+y^2\right)^2 - x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y + \left(x^2+y^2\right)^2 x\right) \\
&= 4y\,x^3 + 4x\,y^3 + 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y\left(x^2+y^2\right)^2 - x\right) \\
&= 4y\,x^3 + 4x\,y^3 - 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = y + x(x^2+y^2)^2$ and $N = y(x^2+y^2)^2 - x$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{y + x(x^2+y^2)^2}{x^2+y^2}$$
$$N = \frac{y(x^2+y^2)^2 - x}{x^2+y^2}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{y(x^2+y^2)^2-x}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{y+(x^2+y^2)^2 x}{x^2+y^2}\right)\mathrm{d}x$$

$$\left(\frac{y+(x^2+y^2)^2 x}{x^2+y^2}\right)\mathrm{d}x+\left(\frac{y(x^2+y^2)^2-x}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{y+(x^2+y^2)^2 x}{x^2+y^2}$$
$$N(x,y) = \frac{y(x^2+y^2)^2-x}{x^2+y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y+(x^2+y^2)^2 x}{x^2+y^2}\right)\\ &= \frac{2x^5y+4x^3y^3+2y^5x+x^2-y^2}{(x^2+y^2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y(x^2+y^2)^2-x}{x^2+y^2}\right)\\ &= \frac{2x^5y+4x^3y^3+2y^5x+x^2-y^2}{(x^2+y^2)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{y + (x^2+y^2)^2\,x}{x^2+y^2}\,\mathrm{d}x$$

$$\phi = \frac{x^4}{4} + \frac{x^2y^2}{2} + \arctan\left(\frac{x}{y}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = x^2y - \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{y(x^2+y^2)^2 - x}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{y(x^2+y^2)^2 - x}{x^2+y^2} = \frac{x(y\,x^3 + x\,y^3 - 1)}{x^2+y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^3$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y^3)\,\mathrm{d}y$$

$$f(y) = \frac{y^4}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^4}{4} + \frac{x^2y^2}{2} + \arctan\left(\frac{x}{y}\right) + \frac{y^4}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^4}{4} + \frac{x^2 y^2}{2} + \arctan\left(\frac{x}{y}\right) + \frac{y^4}{4}$$

Summary

The solution(s) found are the following

$$\frac{x^4}{4} + \frac{x^2 y^2}{2} + \arctan\left(\frac{x}{y}\right) + \frac{y^4}{4} = c_1 \tag{1}$$

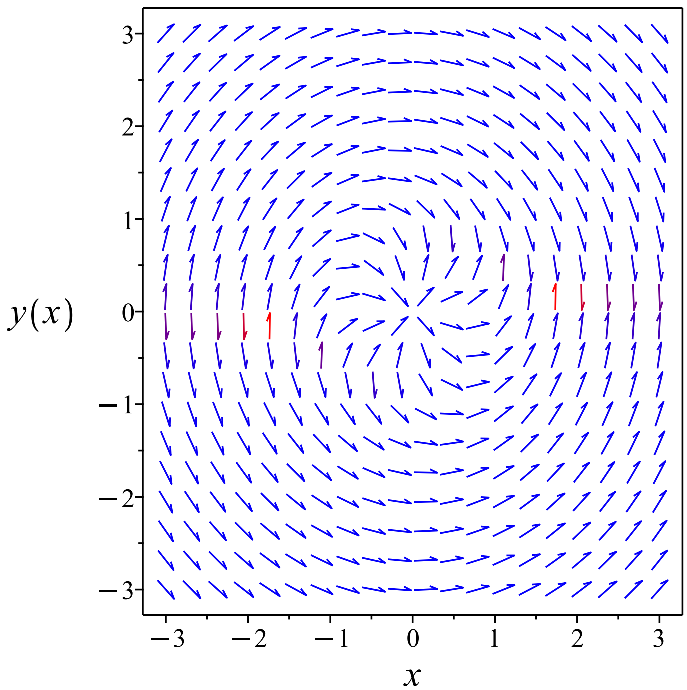


Figure 72: Slope field plot

Verification of solutions

$$\frac{x^4}{4} + \frac{x^2 y^2}{2} + \arctan\left(\frac{x}{y}\right) + \frac{y^4}{4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 28

```
dsolve((y(x)+x*(x^2+y(x)^2)^2)+(y(x)*(x^2+y(x)^2)^2-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \cot\left(\operatorname{RootOf}\left(4c_1 \sin\left(\_Z\right)^4 - 4\_Z \sin\left(\_Z\right)^4 - x^4\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.144 (sec). Leaf size: 40

```
DSolve[(y[x]+x*(x^2+y[x]^2)^2)+(y[x]*(x^2+y[x]^2)^2-x)*y'[x]==0,y[x],x,IncludeSingularSoluti
```

$$\text{Solve}\left[\arctan\left(\frac{x}{y(x)}\right) + \frac{x^4}{4} + \frac{1}{2}x^2 y(x)^2 + \frac{y(x)^4}{4} = c_1, y(x)\right]$$

4 Chapter 2, section 2.2 (Separable equations). Exercises page 47

4.1 problem 1

Internal problem ID [11614]
Internal file name [`OUTPUT/10596_Saturday_May_27_2023_03_01_06_AM_105870/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{4yx + \left(x^2+1\right) y' = 0}$$

4.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{4xy}{x^2+1} \end{aligned}$$

Where $f(x) = -\frac{4x}{x^2+1}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= -\frac{4x}{x^2+1}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{4x}{x^2+1}\,dx \\
\ln(y) &= -2\ln\left(x^2+1\right) + c_1 \\
y &= \mathrm{e}^{-2\ln\left(x^2+1\right)+c_1} \\
&= \frac{c_1}{\left(x^2+1\right)^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{\left(x^2+1\right)^2} \tag{1}
$$

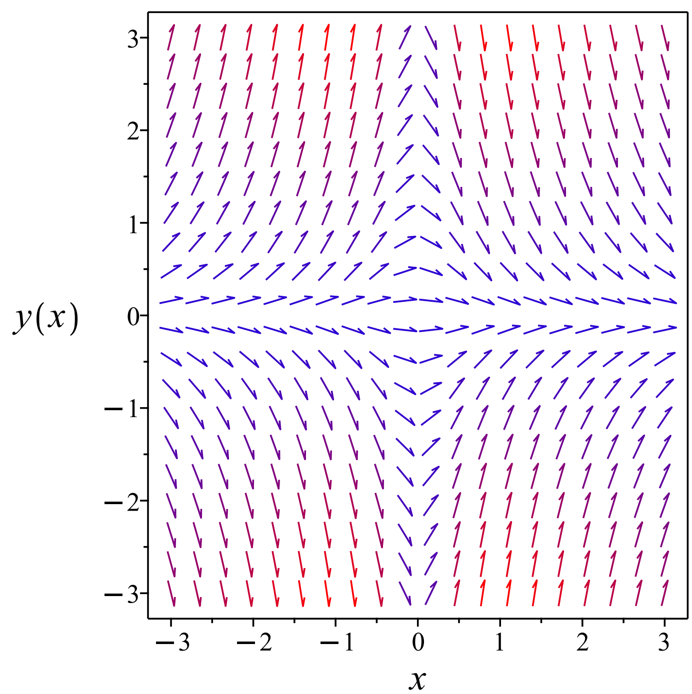


Figure 73: Slope field plot

Verification of solutions

$$
y = \frac{c_1}{\left(x^2+1\right)^2}
$$

Verified OK.

4.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{4x}{x^2+1} \\ q(x) &= 0\end{aligned}$$

Hence the ode is

$$y' + \frac{4xy}{x^2+1} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{4x}{x^2+1}dx} \\ &= \left(x^2+1\right)^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2+1\right)^2 y\right) &= 0\end{aligned}$$

Integrating gives

$$\left(x^2+1\right)^2 y = c_1$$

Dividing both sides by the integrating factor $\mu = \left(x^2+1\right)^2$ results in

$$y = \frac{c_1}{\left(x^2+1\right)^2}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{\left(x^2+1\right)^2} \tag{1}$$

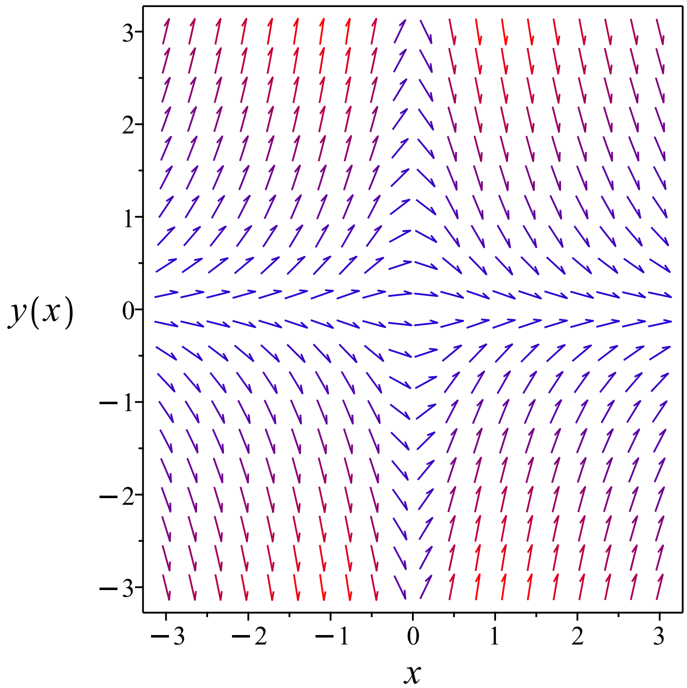


Figure 74: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\left(x^2+1\right)^2}$$

Verified OK.

4.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$4u(x)\, x^2 + \left(x^2+1\right)\left(u'(x)\, x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(5x^2+1)}{x\,(x^2+1)} \end{aligned}$$

Where $f(x) = -\frac{5x^2+1}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{5x^2+1}{x\,(x^2+1)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{5x^2+1}{x\,(x^2+1)}\,dx \\
\ln(u) &= -\ln(x) - 2\ln\left(x^2+1\right) + c_2 \\
u &= \mathrm{e}^{-\ln(x)-2\ln\left(x^2+1\right)+c_2} \\
&= c_2\mathrm{e}^{-\ln(x)-2\ln\left(x^2+1\right)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2}{x\,(x^2+1)^2}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= \frac{c_2}{(x^2+1)^2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2}{(x^2+1)^2} \tag{1}$$

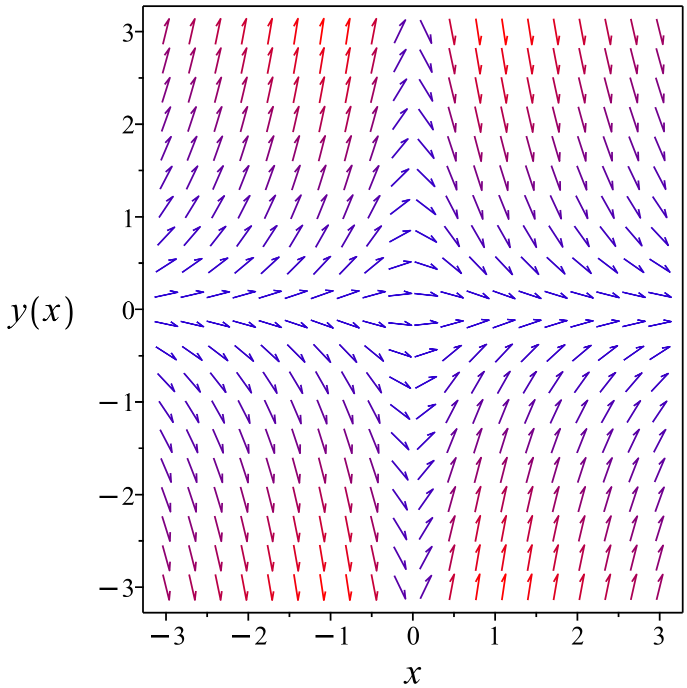


Figure 75: Slope field plot

Verification of solutions

$$y = \frac{c_2}{\left(x^2+1\right)^2}$$

Verified OK.

4.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{4xy}{x^2+1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 70: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{(x^2+1)^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(x^2+1)^2}} dy \end{aligned}$$

Which results in

$$S = (x^2 + 1)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{4xy}{x^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 4(x^2 + 1)\, yx \\ S_y &= (x^2 + 1)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\left(x^2+1\right)^2 y = c_1$$

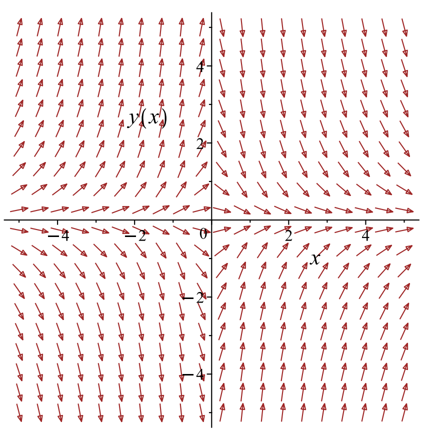
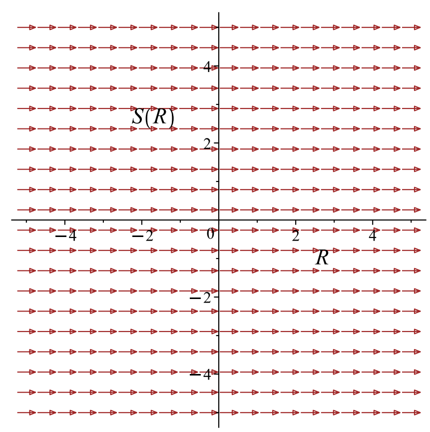
Which simplifies to

$$\left(x^2+1\right)^2 y = c_1$$

Which gives

$$y = \frac{c_1}{\left(x^2+1\right)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4xy}{x^2+1}$ 	$R = x$ $S = \left(x^2+1\right)^2 y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\left(x^2+1\right)^2} \tag{1}$$

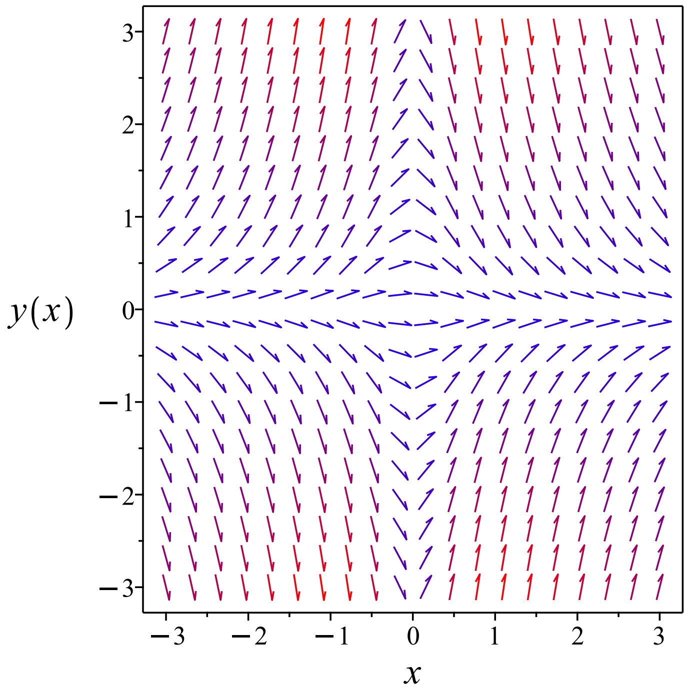


Figure 76: Slope field plot

Verification of solutions

$$y = \frac{c_1}{(x^2+1)^2}$$

Verified OK.

4.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{4y}\right) \mathrm{d}y &= \left(\frac{x}{x^2+1}\right) \mathrm{d}x \\
\left(-\frac{x}{x^2+1}\right) \mathrm{d}x + \left(-\frac{1}{4y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{x}{x^2+1} \\
N(x, y) &= -\frac{1}{4y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{4y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{x}{x^2+1}\, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln\left(x^2+1\right)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{4y}$. Therefore equation (4) becomes

$$
-\frac{1}{4y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{4y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{4y}\right)\mathrm{d}y$$

$$f(y) = -\frac{\ln(y)}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} - \frac{\ln(y)}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} - \frac{\ln(y)}{4}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-4c_1}}{(x^2+1)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-4c_1}}{(x^2+1)^2} \tag{1}$$

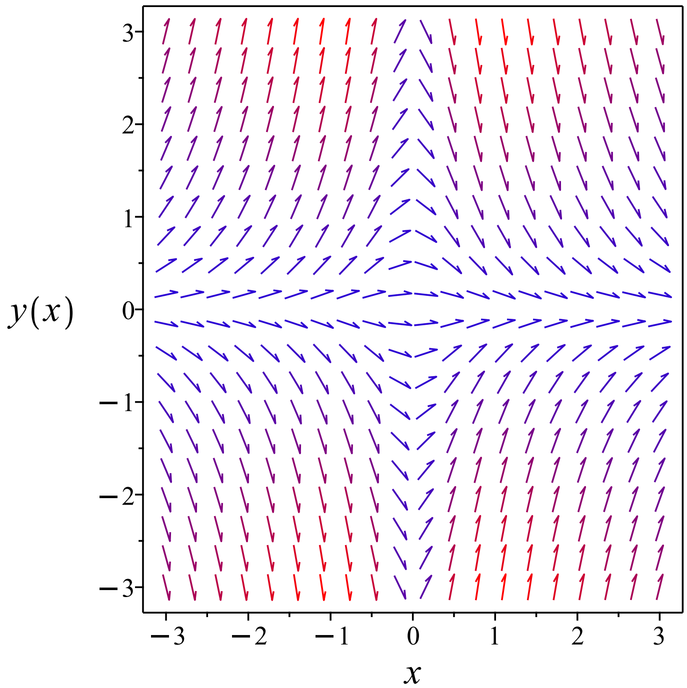


Figure 77: Slope field plot

Verification of solutions

$$y = \frac{e^{-4c_1}}{(x^2+1)^2}$$

Verified OK.

4.1.6 Maple step by step solution

Let's solve

$4yx + (x^2+1)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{4x}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\frac{4x}{x^2+1}dx + c_1$

- Evaluate integral

$$\ln(y) = -2\ln(x^2+1) + c_1$$

- Solve for y

$$y = \frac{e^{c_1}}{(x^2+1)^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve((4*x*y(x))+(x^2+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{(x^2+1)^2}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 20

```
DSolve[(4*x*y[x])+(x^2+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{(x^2+1)^2}$$
$$y(x) \to 0$$

4.2 problem 2

Internal problem ID [11615]
Internal file name [`OUTPUT/10597_Saturday_May_27_2023_03_01_08_AM_25265056/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{yx + y + \left(x^{2}+2x\right) y' = -2x - 2}$$

4.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{(1+x)\left(-y-2\right)}{x\left(x+2\right)} \end{aligned}$$

Where $f(x) = \frac{1+x}{x(x+2)}$ and $g(y) = -y-2$. Integrating both sides gives

$$\frac{1}{-y-2}\, dy = \frac{1+x}{x\left(x+2\right)}\, dx$$

$$\int \frac{1}{-y-2}\, dy = \int \frac{1+x}{x\left(x+2\right)}\, dx$$

$$-\ln\left(y+2\right) = \frac{\ln\left(x(x+2)\right)}{2} + c_1$$

Raising both side to exponential gives

$$\frac{1}{y+2} = \mathrm{e}^{\frac{\ln(x(x+2))}{2}+c_1}$$

Which simplifies to

$$\frac{1}{y+2} = c_2 \sqrt{x\left(x+2\right)}$$

Which simplifies to

$$y = -\frac{\left(2c_2\mathrm{e}^{c_1}\sqrt{x^2+2x}-1\right)\mathrm{e}^{-c_1}}{c_2\sqrt{x^2+2x}}$$

Summary

The solution(s) found are the following

$$y = -\frac{\left(2c_2\mathrm{e}^{c_1}\sqrt{x^2+2x}-1\right)\mathrm{e}^{-c_1}}{c_2\sqrt{x^2+2x}} \tag{1}$$

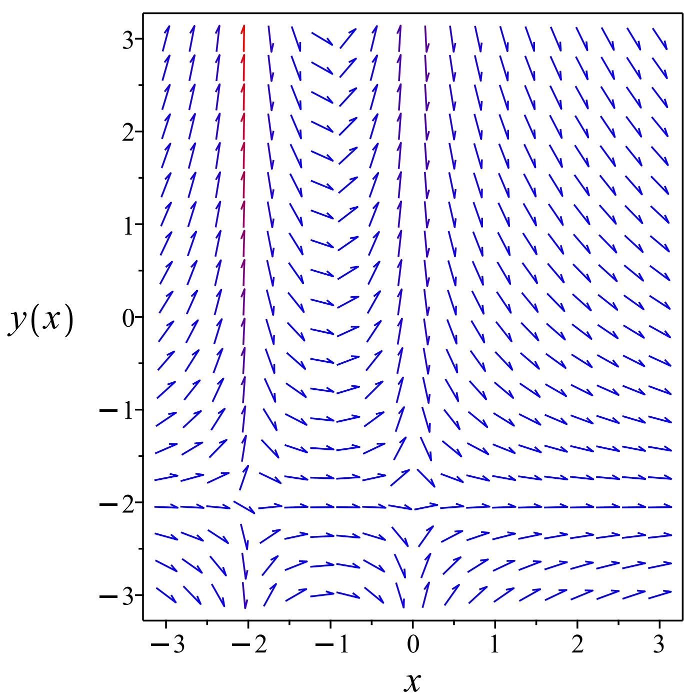


Figure 78: Slope field plot

Verification of solutions

$$y = -\frac{\left(2c_2 \mathrm{e}^{c_1} \sqrt{x^2+2x}-1\right) \mathrm{e}^{-c_1}}{c_2 \sqrt{x^2+2x}}$$

Verified OK.

4.2.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{-x-1}{x\,(x+2)} \\ q(x) &= \frac{-2x-2}{x\,(x+2)} \end{aligned}$$

Hence the ode is

$$y' - \frac{(-x-1)\,y}{x\,(x+2)} = \frac{-2x-2}{x\,(x+2)}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{-x-1}{x(x+2)}dx} \\ &= \sqrt{x\,(x+2)} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{-2x-2}{x\,(x+2)}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x\,(x+2)}\,y\right) &= \left(\sqrt{x\,(x+2)}\right)\left(\frac{-2x-2}{x\,(x+2)}\right) \\ \mathrm{d}\left(\sqrt{x\,(x+2)}\,y\right) &= \left(-\frac{2(1+x)\sqrt{x\,(x+2)}}{x\,(x+2)}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \sqrt{x\,(x+2)}\,y &= \int -\frac{2(1+x)\sqrt{x\,(x+2)}}{x\,(x+2)}\,\mathrm{d}x \\ \sqrt{x\,(x+2)}\,y &= -2\sqrt{x\,(x+2)} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{x(x+2)}$ results in

$$y = -2 + \frac{c_1}{\sqrt{x(x+2)}}$$

Summary

The solution(s) found are the following

$$y = -2 + \frac{c_1}{\sqrt{x(x+2)}} \tag{1}$$

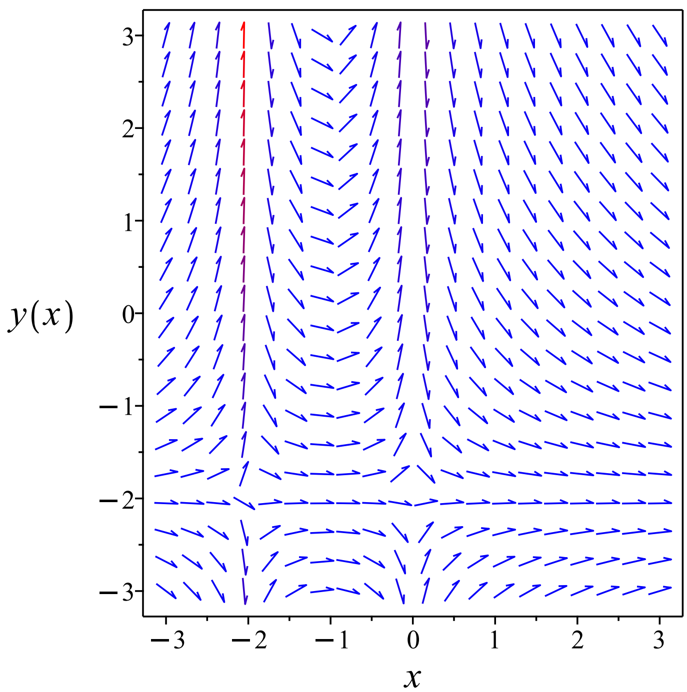


Figure 79: Slope field plot

Verification of solutions

$$y = -2 + \frac{c_1}{\sqrt{x(x+2)}}$$

Verified OK.

4.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{xy + 2x + y + 2}{x\,(x + 2)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 73: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \frac{1}{\sqrt{x(x+2)}} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sqrt{x(x+2)}}} dy \end{aligned}$$

Which results in

$$S = \sqrt{x(x+2)}\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{xy + 2x + y + 2}{x(x+2)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y(1+x)}{\sqrt{x}\,\sqrt{x+2}} \\
S_y &= \sqrt{x}\,\sqrt{x+2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2(1+x)}{\sqrt{x}\,\sqrt{x+2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2(1+R)}{\sqrt{R}\,\sqrt{R+2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\sqrt{R}\,\sqrt{R+2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{x}\,\sqrt{x+2}\,y = -2\sqrt{x}\,\sqrt{x+2} + c_1$$

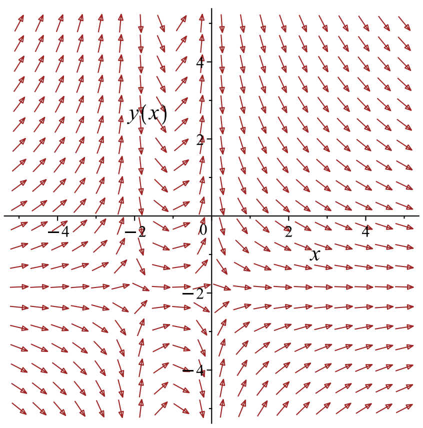
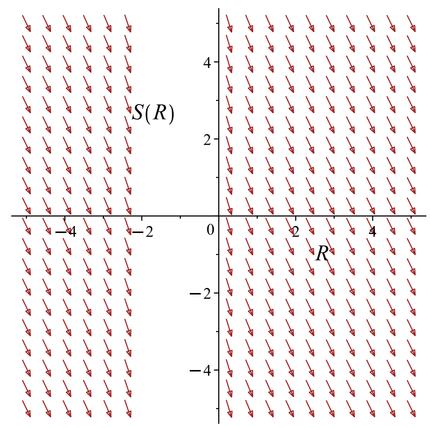
Which simplifies to

$$(y+2)\sqrt{x}\,\sqrt{x+2} - c_1 = 0$$

Which gives

$$y = -\frac{2\sqrt{x}\,\sqrt{x+2} - c_1}{\sqrt{x}\,\sqrt{x+2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{xy+2x+y+2}{x(x+2)}$ 	$R = x$ $S = \sqrt{x}\,\sqrt{x+2}\,y$	$\frac{dS}{dR} = -\frac{2(1+R)}{\sqrt{R}\,\sqrt{R+2}}$ 

Summary

The solution(s) found are the following

$$y = -\frac{2\sqrt{x}\,\sqrt{x+2} - c_1}{\sqrt{x}\,\sqrt{x+2}} \tag{1}$$

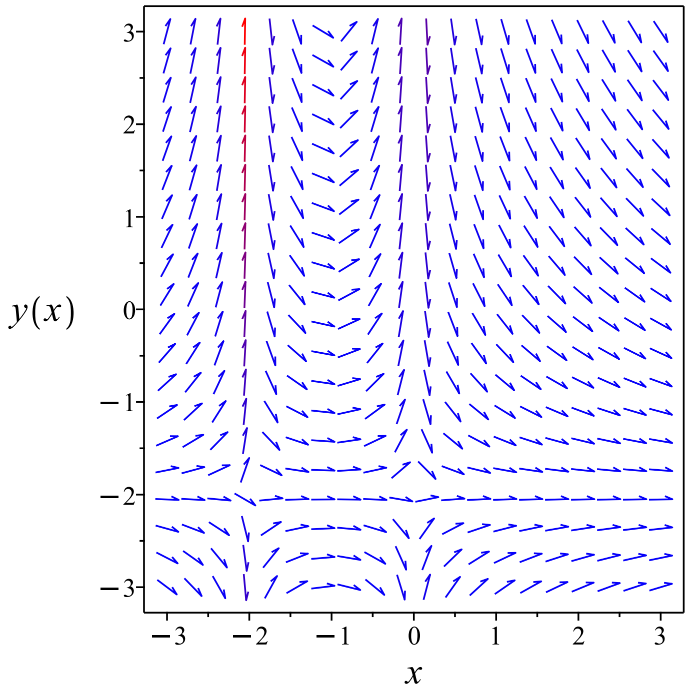


Figure 80: Slope field plot

Verification of solutions

$$y = -\frac{2\sqrt{x}\,\sqrt{x+2} - c_1}{\sqrt{x}\,\sqrt{x+2}}$$

Verified OK.

4.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{-y-2}\right) \mathrm{d}y &= \left(\frac{1+x}{x\,(x+2)}\right) \mathrm{d}x \\
\left(-\frac{1+x}{x\,(x+2)}\right) \mathrm{d}x + \left(\frac{1}{-y-2}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1+x}{x\,(x+2)} \\
N(x, y) &= \frac{1}{-y-2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1+x}{x\,(x+2)}\right) \\
&= 0
\end{aligned}
$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{-y-2}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{1+x}{x\,(x+2)}\,\mathrm{d}x$$
$$\phi = -\frac{\ln\,(x(x+2))}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-y-2}$. Therefore equation (4) becomes

$$\frac{1}{-y-2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y+2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y+2}\right)\mathrm{d}y$$
$$f(y) = -\ln\,(y+2) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x(x+2))}{2} - \ln(y+2) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x(x+2))}{2} - \ln(y+2)$$

The solution becomes

$$y = e^{-\frac{\ln(x(x+2))}{2} - c_1} - 2$$

Summary
The solution(s) found are the following

$$y = e^{-\frac{\ln(x(x+2))}{2} - c_1} - 2 \tag{1}$$

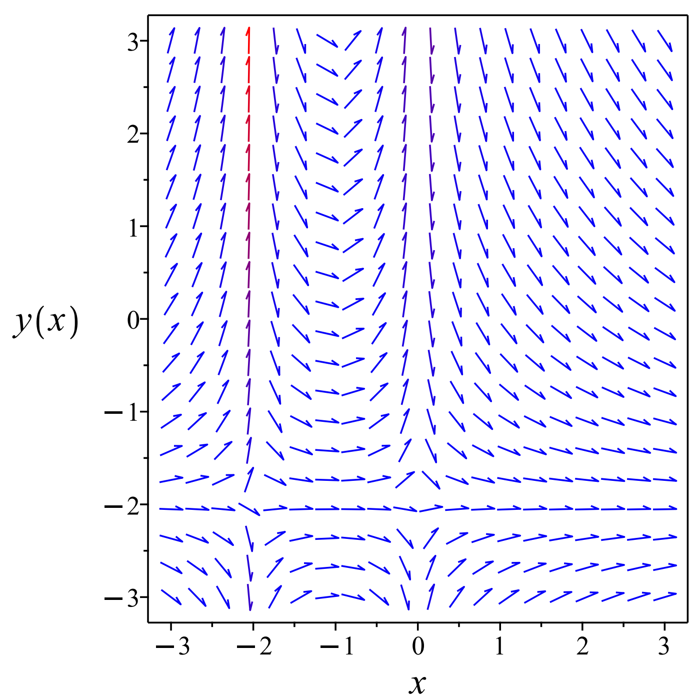


Figure 81: Slope field plot

Verification of solutions

$$y = e^{-\frac{\ln(x(x+2))}{2}-c_1} - 2$$

Verified OK.

4.2.5 Maple step by step solution

Let's solve

$yx + y + (x^2 + 2x)\, y' = -2x - 2$

- Highest derivative means the order of the ODE is 1

  y'
- Separate variables

  $\frac{y'}{y+2} = -\frac{1+x}{x(x+2)}$
- Integrate both sides with respect to x

  $\int \frac{y'}{y+2} dx = \int -\frac{1+x}{x(x+2)} dx + c_1$
- Evaluate integral

  $\ln(y+2) = -\frac{\ln(x(x+2))}{2} + c_1$
- Solve for y

  $y = e^{-\frac{\ln(x(x+2))}{2}+c_1} - 2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x*y(x)+2*x+y(x)+2)+(x^2+2*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -2 + \frac{c_1}{\sqrt{x(x+2)}}$$

✓ Solution by Mathematica

Time used: 0.149 (sec). Leaf size: 27

```
DSolve[(x*y[x]+2*x+y[x]+2)+(x^2+2*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2 + \frac{c_1}{\sqrt{x}\sqrt{x+2}}$$
$$y(x) \to -2$$

4.3 problem 3

Internal problem ID [11616]
Internal file name [`OUTPUT/10598_Saturday_May_27_2023_03_01_10_AM_98812696/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2r(s^2+1)+(r^4+1)\,s'=0}$$

4.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} s' &= F(r,s) \\ &= f(r)g(s) \\ &= \frac{r(-2s^2-2)}{r^4+1} \end{aligned}$$

Where $f(r)=\frac{r}{r^4+1}$ and $g(s)=-2s^2-2$. Integrating both sides gives

$$\frac{1}{-2s^2-2}\,ds = \frac{r}{r^4+1}\,dr$$

$$\int \frac{1}{-2s^2-2}\,ds = \int \frac{r}{r^4+1}\,dr$$

$$-\frac{\arctan(s)}{2} = \frac{\arctan(r^2)}{2} + c_1$$

Which results in

$$s = -\tan\left(\arctan\left(r^2\right) + 2c_1\right)$$

Summary

The solution(s) found are the following

$$s = -\tan\left(\arctan\left(r^2\right) + 2c_1\right) \tag{1}$$

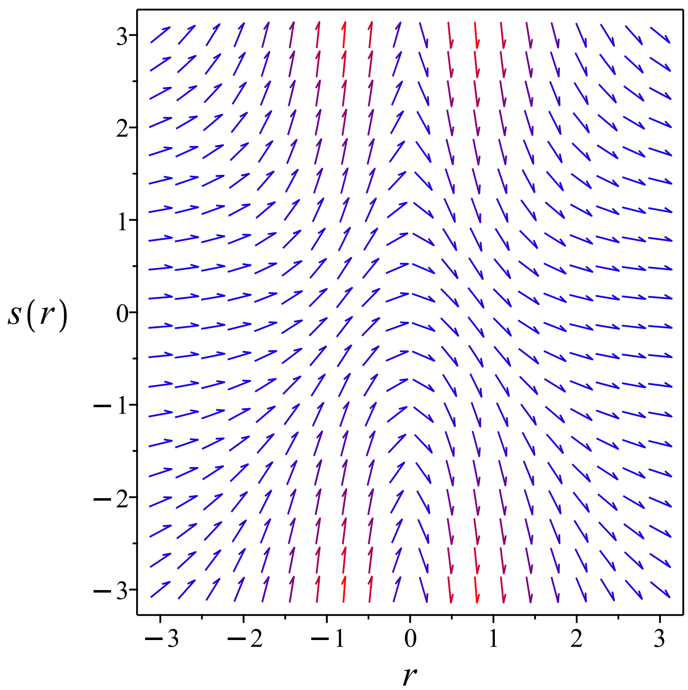


Figure 82: Slope field plot

Verification of solutions

$$s = -\tan\left(\arctan\left(r^2\right) + 2c_1\right)$$

Verified OK.

4.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$s' = -\frac{2r(s^2+1)}{r^4+1}$$
$$s' = \omega(r,s)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_r + \omega(\eta_s - \xi_r) - \omega^2\xi_s - \omega_r\xi - \omega_s\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 76: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(r,s) = \frac{r^4+1}{r}$$
$$\eta(r,s) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(r,s) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dr}{\xi} = \frac{ds}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial r} + \eta\frac{\partial}{\partial s}\right) S(r,s) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = s$$

S is found from

$$S = \int \frac{1}{\xi} dr$$
$$= \int \frac{1}{\frac{r^4+1}{r}} dr$$

Which results in

$$S = \frac{\arctan(r^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_r + \omega(r,s)S_s}{R_r + \omega(r,s)R_s} \tag{2}$$

Where in the above R_r, R_s, S_r, S_s are all partial derivatives and $\omega(r,s)$ is the right hand side of the original ode given by

$$\omega(r,s) = -\frac{2r(s^2+1)}{r^4+1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_r &= 0 \\
R_s &= 1 \\
S_r &= \frac{r}{r^4+1} \\
S_s &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{2s^2+2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for r, s in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{2R^2+2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\arctan(R)}{2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to r, s coordinates. This results in

$$
\frac{\arctan(r^2)}{2} = -\frac{\arctan(s)}{2} + c_1
$$

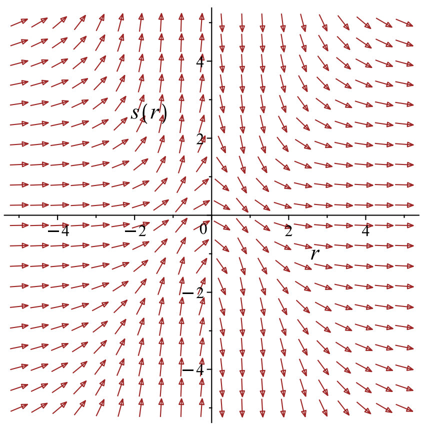
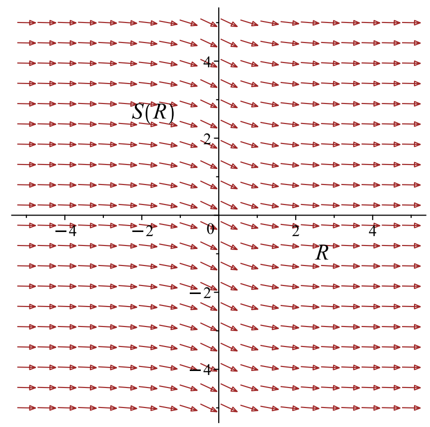
Which simplifies to

$$
\frac{\arctan(r^2)}{2} = -\frac{\arctan(s)}{2} + c_1
$$

Which gives

$$
s = \tan\left(-\arctan\left(r^2\right) + 2c_1\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in r, s coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{ds}{dr} = -\frac{2r(s^2+1)}{r^4+1}$ 	$R = s$ $S = \frac{\arctan(r^2)}{2}$	$\frac{dS}{dR} = -\frac{1}{2R^2+2}$ 

Summary

The solution(s) found are the following

$$s = \tan\left(-\arctan\left(r^2\right) + 2c_1\right) \tag{1}$$

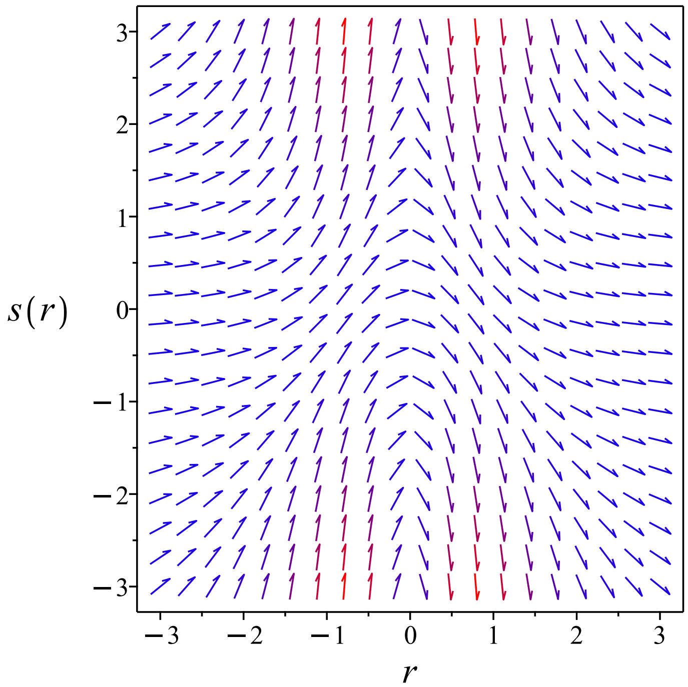


Figure 83: Slope field plot

<u>Verification of solutions</u>

$$s = \tan\left(-\arctan\left(r^2\right) + 2c_1\right)$$

Verified OK.

4.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(r,s)\,\mathrm{d}r + N(r,s)\,\mathrm{d}s = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{-2s^2-2}\right)\mathrm{d}s = \left(\frac{r}{r^4+1}\right)\mathrm{d}r$$
$$\left(-\frac{r}{r^4+1}\right)\mathrm{d}r + \left(\frac{1}{-2s^2-2}\right)\mathrm{d}s = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(r,s) = -\frac{r}{r^4+1}$$
$$N(r,s) = \frac{1}{-2s^2-2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial s} = \frac{\partial N}{\partial r}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial s} &= \frac{\partial}{\partial s}\left(-\frac{r}{r^4+1}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial r} &= \frac{\partial}{\partial r}\left(\frac{1}{-2s^2-2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial s} = \frac{\partial N}{\partial r}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(r, s)$

$$
\frac{\partial \phi}{\partial r} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial s} = N \tag{2}
$$

Integrating (1) w.r.t. r gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial r}\, \mathrm{d}r &= \int M\, \mathrm{d}r \\
\int \frac{\partial \phi}{\partial r}\, \mathrm{d}r &= \int -\frac{r}{r^4+1}\, \mathrm{d}r
\end{aligned}
$$

$$
\phi = -\frac{\arctan(r^2)}{2} + f(s) \tag{3}
$$

Where $f(s)$ is used for the constant of integration since ϕ is a function of both r and s. Taking derivative of equation (3) w.r.t s gives

$$
\frac{\partial \phi}{\partial s} = 0 + f'(s) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial s} = \frac{1}{-2s^2-2}$. Therefore equation (4) becomes

$$
\frac{1}{-2s^2-2} = 0 + f'(s) \tag{5}
$$

Solving equation (5) for $f'(s)$ gives

$$
f'(s) = -\frac{1}{2(s^2+1)}
$$

Integrating the above w.r.t s gives

$$
\begin{aligned}
\int f'(s)\, \mathrm{d}s &= \int \left(-\frac{1}{2s^2+2}\right) \mathrm{d}s \\
f(s) &= -\frac{\arctan(s)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(s)$ into equation (3) gives ϕ

$$\phi = -\frac{\arctan(r^2)}{2} - \frac{\arctan(s)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\arctan(r^2)}{2} - \frac{\arctan(s)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{\arctan(r^2)}{2} - \frac{\arctan(s)}{2} = c_1 \tag{1}$$

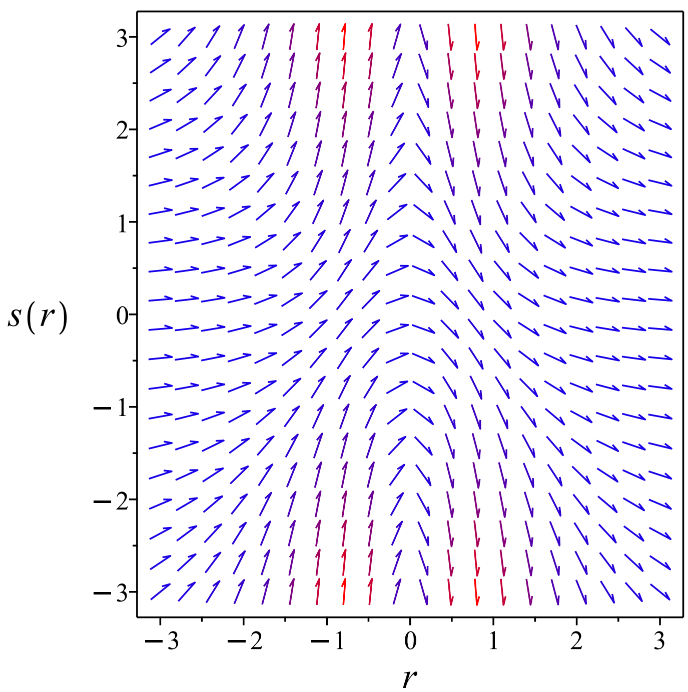


Figure 84: Slope field plot

Verification of solutions

$$-\frac{\arctan(r^2)}{2} - \frac{\arctan(s)}{2} = c_1$$

Verified OK.

4.3.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} s' &= F(r, s) \\ &= -\frac{2r(s^2+1)}{r^4+1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$s' = -\frac{2r\, s^2}{r^4+1} - \frac{2r}{r^4+1}$$

With Riccati ODE standard form

$$s' = f_0(r) + f_1(r)s + f_2(r)s^2$$

Shows that $f_0(r) = -\frac{2r}{r^4+1}$, $f_1(r) = 0$ and $f_2(r) = -\frac{2r}{r^4+1}$. Let

$$\begin{aligned} s &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{2ru}{r^4+1}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(r) - (f_2' + f_1 f_2)\, u'(r) + f_2^2 f_0 u(r) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{r^4+1} + \frac{8r^4}{(r^4+1)^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -\frac{8r^3}{(r^4+1)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{2ru''(r)}{r^4+1} - \left(-\frac{2}{r^4+1} + \frac{8r^4}{(r^4+1)^2}\right) u'(r) - \frac{8r^3 u(r)}{(r^4+1)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(r) = \frac{c_1 r^2 + c_2}{\sqrt{r^4+1}}$$

The above shows that

$$u'(r) = \frac{2r(-c_2 r^2 + c_1)}{(r^4+1)^{\frac{3}{2}}}$$

Using the above in (1) gives the solution

$$s = \frac{-c_2 r^2 + c_1}{c_1 r^2 + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$s = \frac{-r^2 + c_3}{c_3 r^2 + 1}$$

Summary

The solution(s) found are the following

$$s = \frac{-r^2 + c_3}{c_3 r^2 + 1} \tag{1}$$

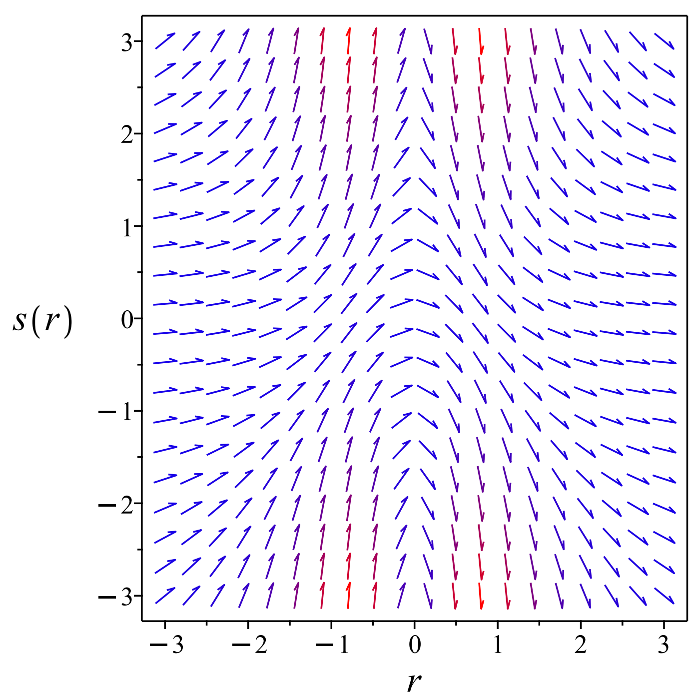


Figure 85: Slope field plot

Verification of solutions

$$s = \frac{-r^2 + c_3}{c_3 r^2 + 1}$$

Verified OK.

4.3.5 Maple step by step solution

Let's solve

$2r(s^2 + 1) + (r^4 + 1)\, s' = 0$

- Highest derivative means the order of the ODE is 1

  s'

- Separate variables

  $\frac{s'}{s^2+1} = -\frac{2r}{r^4+1}$

- Integrate both sides with respect to r

  $\int \frac{s'}{s^2+1} dr = \int -\frac{2r}{r^4+1} dr + c_1$

- Evaluate integral

  $\arctan(s) = -\arctan(r^2) + c_1$

- Solve for s

  $s = \tan(-\arctan(r^2) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(2*r*(s(r)^2+1)+(r^4+1)*diff(s(r),r)=0,s(r), singsol=all)
```

$$s(r) = -\tan\left(\arctan\left(r^2\right) + 2c_1\right)$$

✓ Solution by Mathematica

Time used: 0.478 (sec). Leaf size: 31

```
DSolve[2*r*(s[r]^2+1)+(r^4+1)*s'[r]==0,s[r],r,IncludeSingularSolutions -> True]
```

$$s(r) \to -\tan\left(\arctan\left(r^2\right) - c_1\right)$$
$$s(r) \to -i$$
$$s(r) \to i$$

4.4 problem 4

Internal problem ID [11617]
Internal file name [`OUTPUT/10599_Saturday_May_27_2023_03_01_11_AM_17840397/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\csc(y) + y' \sec(x) = 0}$$

4.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{\csc(y)}{\sec(x)} \end{aligned}$$

Where $f(x) = -\frac{1}{\sec(x)}$ and $g(y) = \csc(y)$. Integrating both sides gives

$$\frac{1}{\csc(y)}\, dy = -\frac{1}{\sec(x)}\, dx$$

$$\int \frac{1}{\csc(y)}\, dy = \int -\frac{1}{\sec(x)}\, dx$$

$$-\cos(y) = -\sin(x) + c_1$$

Which results in

$$y = \pi - \arccos(-\sin(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \pi - \arccos(-\sin(x) + c_1) \tag{1}$$

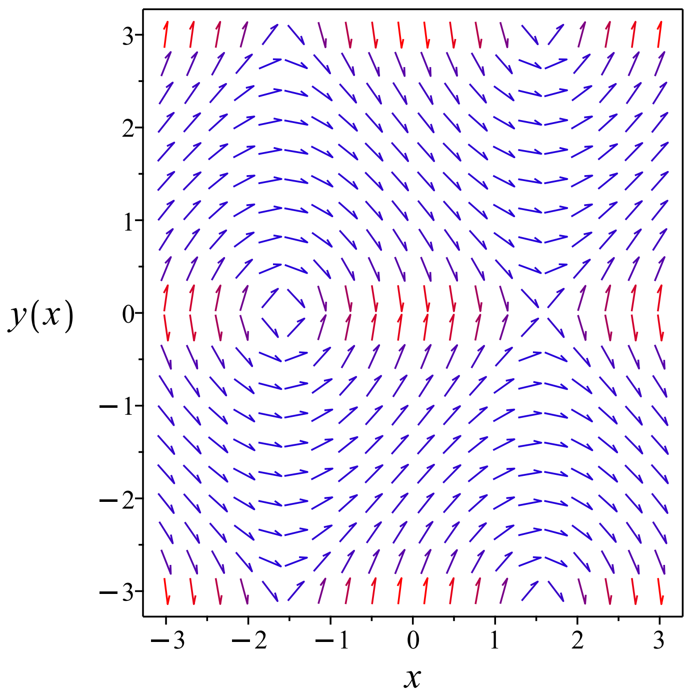


Figure 86: Slope field plot

Verification of solutions

$$y = \pi - \arccos(-\sin(x) + c_1)$$

Verified OK.

4.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\csc(y)}{\sec(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 79: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\sec(x) \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\sec(x)} dx
\end{aligned}
$$

Which results in

$$
S = -\sin(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{\csc(y)}{\sec(x)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\cos(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\cos(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\sin(x) = -\cos(y) + c_1$$

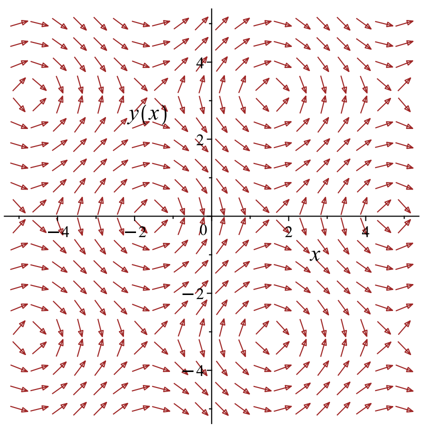
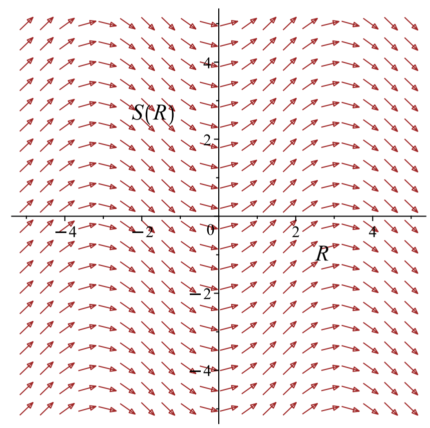
Which simplifies to

$$-\sin(x) = -\cos(y) + c_1$$

Which gives

$$y = \arccos(\sin(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\csc(y)}{\sec(x)}$ 	$R = y$ $S = -\sin(x)$	$\frac{dS}{dR} = \sin(R)$ 

Summary

The solution(s) found are the following

$$y = \arccos(\sin(x) + c_1) \tag{1}$$

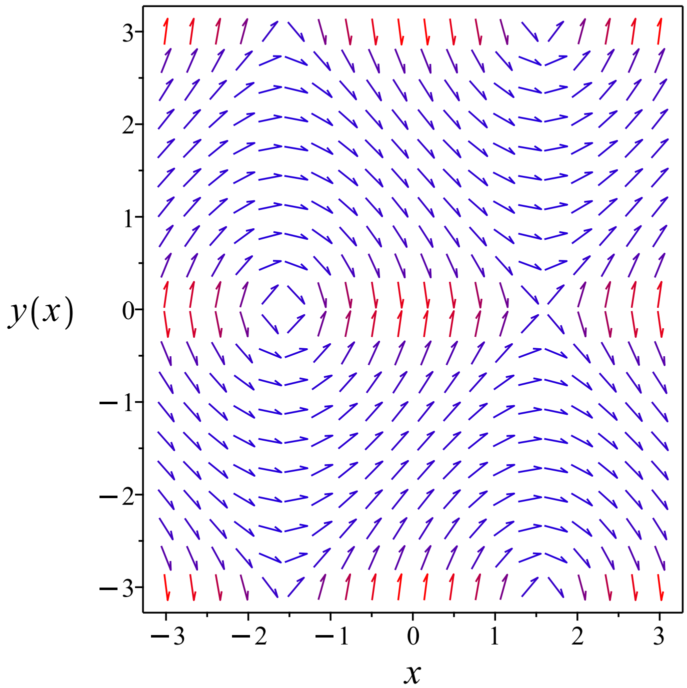


Figure 87: Slope field plot

Verification of solutions

$$y = \arccos\left(\sin\left(x\right) + c_1\right)$$

Verified OK.

4.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{\csc(y)}\right) \mathrm{d}y = \left(\frac{1}{\sec(x)}\right) \mathrm{d}x$$
$$\left(-\frac{1}{\sec(x)}\right) \mathrm{d}x + \left(-\frac{1}{\csc(y)}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{\sec(x)}$$
$$N(x, y) = -\frac{1}{\csc(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{\sec(x)}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{\csc\left(y\right)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{1}{\sec\left(x\right)} \mathrm{d}x \\
\phi &= -\sin\left(x\right) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{\csc(y)}$. Therefore equation (4) becomes

$$
-\frac{1}{\csc\left(y\right)} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= -\frac{1}{\csc\left(y\right)} \\
&= -\sin\left(y\right)
\end{aligned}
$$

Integrating the above w.r.t y results in

$$\int f'(y)\, dy = \int (-\sin(y))\, dy$$
$$f(y) = \cos(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(x) + \cos(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(x) + \cos(y)$$

Summary

The solution(s) found are the following

$$-\sin(x) + \cos(y) = c_1 \tag{1}$$

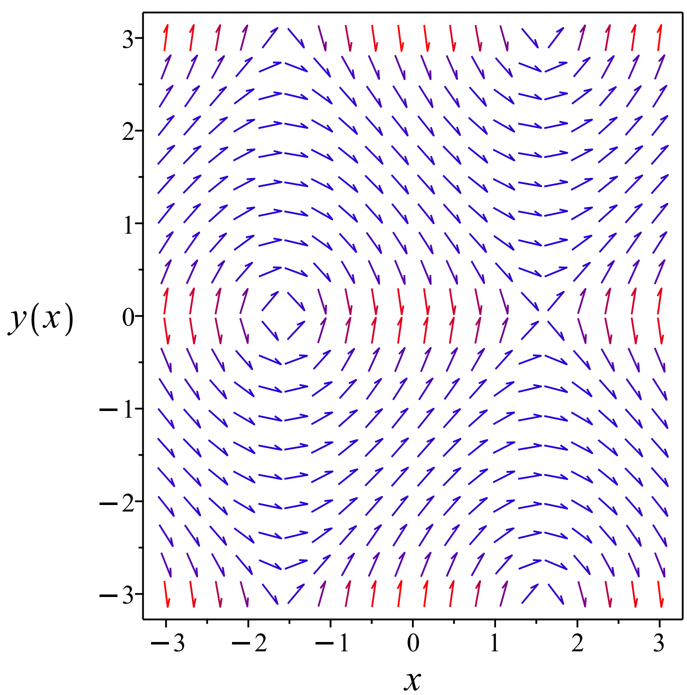


Figure 88: Slope field plot

Verification of solutions

$$-\sin(x) + \cos(y) = c_1$$

Verified OK.

4.4.4 Maple step by step solution

Let's solve

$\csc(y) + y' \sec(x) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\csc(y)} = -\frac{1}{\sec(x)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\csc(y)} dx = \int -\frac{1}{\sec(x)} dx + c_1$

- Evaluate integral

  $-\cos(y) = -\sin(x) + c_1$

- Solve for y

  $y = \pi - \arccos(-\sin(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(csc(y(x))+sec(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arccos(\sin(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.696 (sec). Leaf size: 27

```
DSolve[Csc[y[x]]+Sec[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\arccos(\sin(x) - c_1)$$
$$y(x) \to \arccos(\sin(x) - c_1)$$

4.5 problem 5

Internal problem ID [11618]
Internal file name [`OUTPUT/10600_Saturday_May_27_2023_03_01_13_AM_76976138/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\tan(\theta) + 2r\theta' = 0}$$

4.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}\theta' &= F(r,\theta)\\ &= f(r)g(\theta)\\ &= -\frac{\tan(\theta)}{2r}\end{aligned}$$

Where $f(r) = -\frac{1}{2r}$ and $g(\theta) = \tan(\theta)$. Integrating both sides gives

$$\frac{1}{\tan(\theta)}\,d\theta = -\frac{1}{2r}\,dr$$

$$\int \frac{1}{\tan(\theta)}\,d\theta = \int -\frac{1}{2r}\,dr$$

$$\ln(\sin(\theta)) = -\frac{\ln(r)}{2} + c_1$$

Raising both side to exponential gives

$$\sin(\theta) = e^{-\frac{\ln(r)}{2}+c_1}$$

Which simplifies to

$$\sin(\theta) = \frac{c_2}{\sqrt{r}}$$

Which simplifies to

$$\theta = \arcsin\left(\frac{c_2 e^{c_1}}{\sqrt{r}}\right)$$

Summary

The solution(s) found are the following

$$\theta = \arcsin\left(\frac{c_2 e^{c_1}}{\sqrt{r}}\right) \tag{1}$$

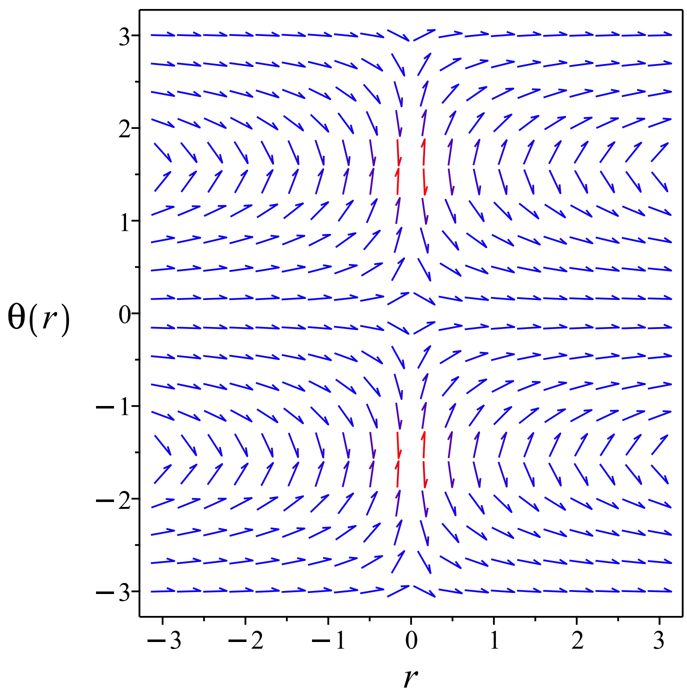


Figure 89: Slope field plot

Verification of solutions

$$\theta = \arcsin\left(\frac{c_2 e^{c_1}}{\sqrt{r}}\right)$$

Verified OK.

4.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\theta' = -\frac{\tan(\theta)}{2r}$$
$$\theta' = \omega(r, \theta)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_r + \omega(\eta_\theta - \xi_r) - \omega^2 \xi_\theta - \omega_r \xi - \omega_\theta \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 82: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(r,\theta) &= -2r \\ \eta(r,\theta) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(r,\theta) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dr}{\xi} = \frac{d\theta}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial r} + \eta\frac{\partial}{\partial \theta}\right) S(r,\theta) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = \theta$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dr \\ &= \int \frac{1}{-2r} dr\end{aligned}$$

Which results in

$$S = -\frac{\ln(r)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_r + \omega(r,\theta) S_\theta}{R_r + \omega(r,\theta) R_\theta} \tag{2}$$

Where in the above $R_r, R_\theta, S_r, S_\theta$ are all partial derivatives and $\omega(r,\theta)$ is the right hand side of the original ode given by

$$\omega(r,\theta) = -\frac{\tan(\theta)}{2r}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_r &= 0 \\ R_\theta &= 1 \\ S_r &= -\frac{1}{2r} \\ S_\theta &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cot(\theta) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for r, θ in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cot(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(\sin(R)) + c_1 \tag{4}$$

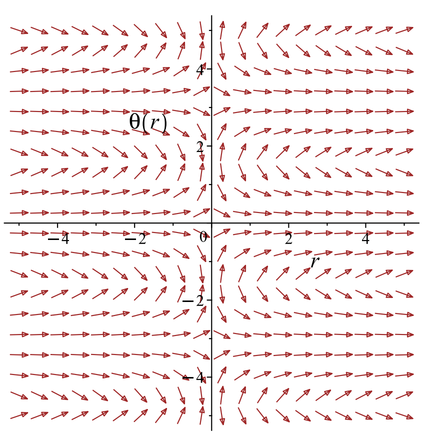
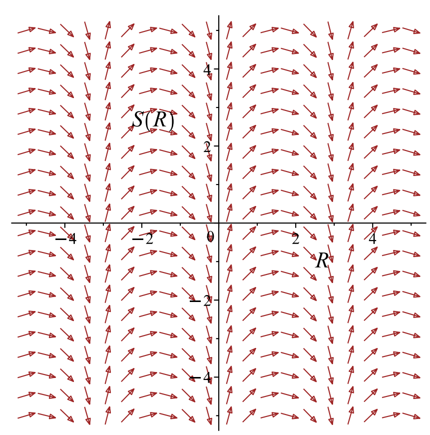
To complete the solution, we just need to transform (4) back to r, θ coordinates. This results in

$$-\frac{\ln(r)}{2} = \ln(\sin(\theta)) + c_1$$

Which simplifies to

$$-\frac{\ln(r)}{2} = \ln(\sin(\theta)) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in r, θ coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{d\theta}{dr} = -\frac{\tan(\theta)}{2r}$ 	$R = \theta$ $S = -\frac{\ln(r)}{2}$	$\frac{dS}{dR} = \cot(R)$ 

Summary

The solution(s) found are the following

$$-\frac{\ln(r)}{2} = \ln(\sin(\theta)) + c_1 \tag{1}$$

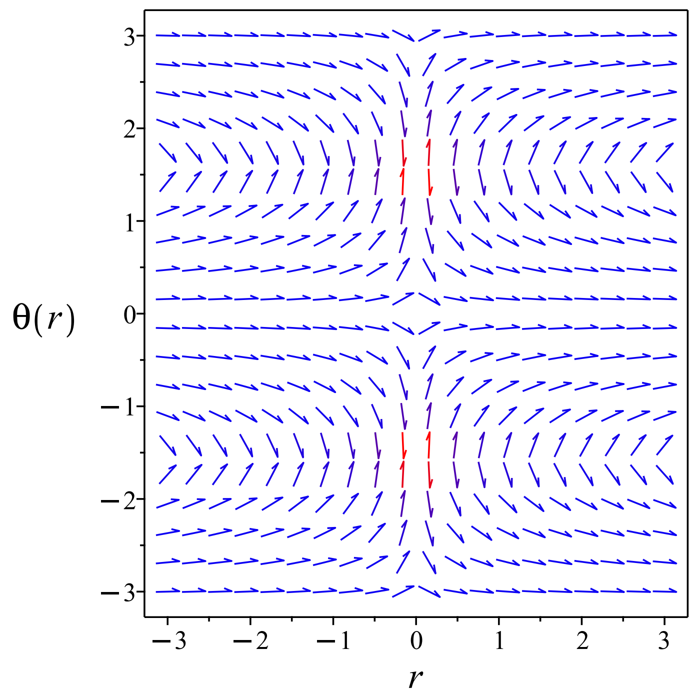


Figure 90: Slope field plot

Verification of solutions

$$-\frac{\ln(r)}{2} = \ln(\sin(\theta)) + c_1$$

Verified OK.

4.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(r,\theta)\,\mathrm{d}r + N(r,\theta)\,\mathrm{d}\theta = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{2}{\tan(\theta)}\right)\mathrm{d}\theta &= \left(\frac{1}{r}\right)\mathrm{d}r \\ \left(-\frac{1}{r}\right)\mathrm{d}r + \left(-\frac{2}{\tan(\theta)}\right)\mathrm{d}\theta &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(r,\theta) &= -\frac{1}{r} \\ N(r,\theta) &= -\frac{2}{\tan(\theta)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial r}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(-\frac{1}{r}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial r} &= \frac{\partial}{\partial r}\left(-\frac{2}{\tan(\theta)}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial r}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(r,\theta)$

$$
\frac{\partial \phi}{\partial r} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial \theta} = N \tag{2}
$$

Integrating (1) w.r.t. r gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial r}\, \mathrm{d}r &= \int M\, \mathrm{d}r \\
\int \frac{\partial \phi}{\partial r}\, \mathrm{d}r &= \int -\frac{1}{r}\, \mathrm{d}r \\
\phi &= -\ln(r) + f(\theta)
\end{aligned} \tag{3}
$$

Where $f(\theta)$ is used for the constant of integration since ϕ is a function of both r and θ. Taking derivative of equation (3) w.r.t θ gives

$$
\frac{\partial \phi}{\partial \theta} = 0 + f'(\theta) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial \theta} = -\frac{2}{\tan(\theta)}$. Therefore equation (4) becomes

$$
-\frac{2}{\tan(\theta)} = 0 + f'(\theta) \tag{5}
$$

Solving equation (5) for $f'(\theta)$ gives

$$
\begin{aligned}
f'(\theta) &= -\frac{2}{\tan(\theta)} \\
&= -2\cot(\theta)
\end{aligned}
$$

Integrating the above w.r.t θ results in

$$\int f'(\theta)\,d\theta = \int (-2\cot(\theta))\,d\theta$$
$$f(\theta) = -2\ln(\sin(\theta)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(\theta)$ into equation (3) gives ϕ

$$\phi = -\ln(r) - 2\ln(\sin(\theta)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(r) - 2\ln(\sin(\theta))$$

<u>Summary</u>
The solution(s) found are the following

$$-\ln(r) - 2\ln(\sin(\theta)) = c_1 \tag{1}$$

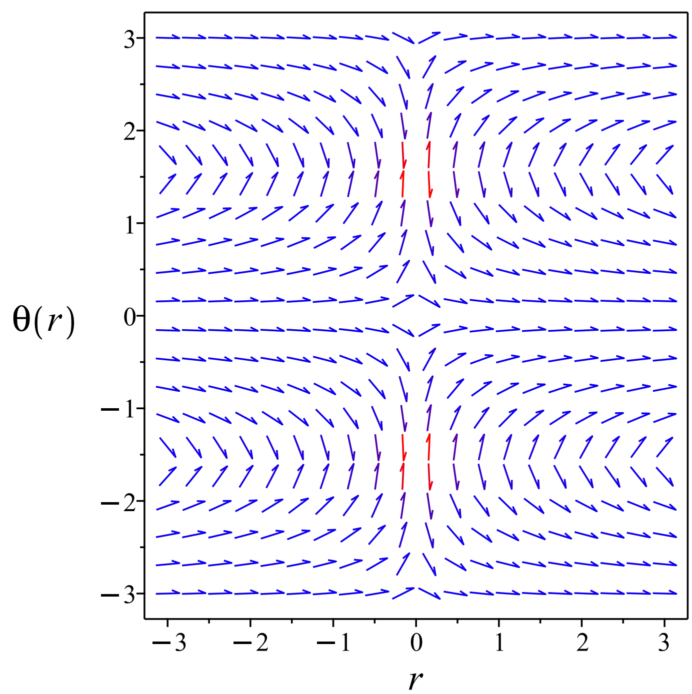


Figure 91: Slope field plot

Verification of solutions

$$-\ln(r) - 2\ln(\sin(\theta)) = c_1$$

Verified OK.

4.5.4 Maple step by step solution

Let's solve

$\tan(\theta) + 2r\theta' = 0$

- Highest derivative means the order of the ODE is 1

  θ'

- Separate variables

  $\frac{\theta'}{\tan(\theta)} = -\frac{1}{2r}$

- Integrate both sides with respect to r

  $\int \frac{\theta'}{\tan(\theta)}dr = \int -\frac{1}{2r}dr + c_1$

- Evaluate integral

  $\ln(\sin(\theta)) = -\frac{\ln(r)}{2} + c_1$

- Solve for θ

  $\left\{\theta = -\arcsin\left(\frac{\sqrt{r}\, e^{2c_1}}{r}\right), \theta = \arcsin\left(\frac{\sqrt{r}\, e^{2c_1}}{r}\right)\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.313 (sec). Leaf size: 21

```
dsolve(tan(theta(r))+2*r*diff(theta(r),r)=0,theta(r), singsol=all)
```

$$\theta(r) = \arcsin\left(\frac{1}{\sqrt{c_1 r}}\right)$$
$$\theta(r) = -\arcsin\left(\frac{1}{\sqrt{c_1 r}}\right)$$

✓ Solution by Mathematica

Time used: 15.319 (sec). Leaf size: 21

```
DSolve[Tan[theta[r]]+2*r*theta'[r]==0,theta[r],r,IncludeSingularSolutions -> True]
```

$$\theta(r) \to \arcsin\left(\frac{e^{c_1}}{\sqrt{r}}\right)$$
$$\theta(r) \to 0$$

4.6 problem 6

Internal problem ID [11619]
Internal file name [`OUTPUT/10601_Saturday_May_27_2023_03_01_15_AM_62894160/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(e^{v}+1)\cos(u)+e^{v}(1+\sin(u))\,v'=0}$$

4.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} v' &= F(u,v) \\ &= f(u)g(v) \\ &= -\frac{\cos(u)\,(1+e^{-v})}{1+\sin(u)} \end{aligned}$$

Where $f(u)=-\frac{\cos(u)}{1+\sin(u)}$ and $g(v)=1+e^{-v}$. Integrating both sides gives

$$\frac{1}{1+e^{-v}}\,dv=-\frac{\cos(u)}{1+\sin(u)}\,du$$

$$\int\frac{1}{1+e^{-v}}\,dv=\int-\frac{\cos(u)}{1+\sin(u)}\,du$$

$$\ln\left(1+e^{-v}\right)-\ln\left(e^{-v}\right)=-\ln(1+\sin(u))+c_1$$

Raising both side to exponential gives

$$e^{\ln(1+e^{-v})-\ln(e^{-v})} = e^{-\ln(1+\sin(u))+c_1}$$

Which simplifies to

$$e^{v} + 1 = \frac{c_2}{1 + \sin(u)}$$

Summary

The solution(s) found are the following

$$v = \ln\left(-\frac{1 + \sin(u) - c_2}{1 + \sin(u)}\right) \tag{1}$$

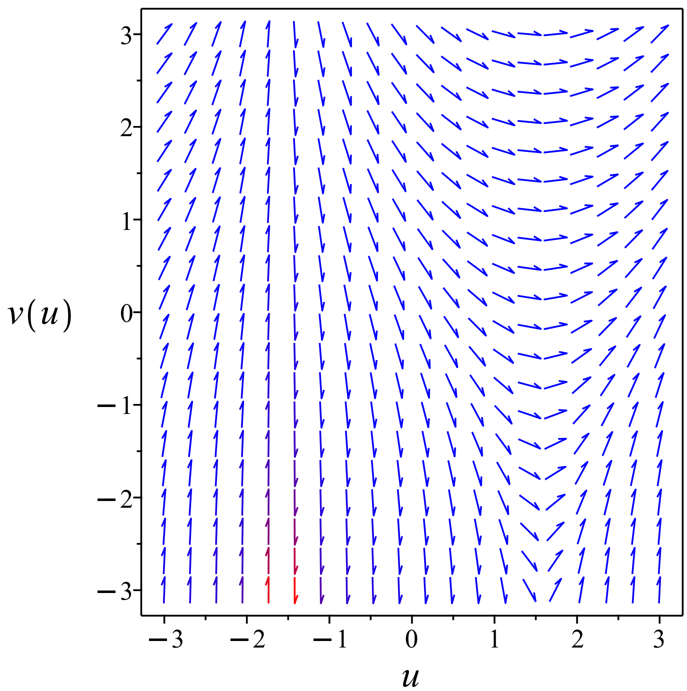


Figure 92: Slope field plot

Verification of solutions

$$v = \ln\left(-\frac{1 + \sin(u) - c_2}{1 + \sin(u)}\right)$$

Verified OK.

4.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
v' &= -\frac{(\mathrm{e}^{v}+1)\cos(u)\,\mathrm{e}^{-v}}{1+\sin(u)} \\
v' &= \omega(u,v)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_u + \omega(\eta_v - \xi_u) - \omega^2 \xi_v - \omega_u \xi - \omega_v \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 85: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(u, v) &= -\frac{1+\sin(u)}{\cos(u)} \\ \eta(u, v) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(u, v) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{du}{\xi} = \frac{dv}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial v}\right) S(u, v) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = v$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} du \\ &= \int \frac{1}{-\frac{1+\sin(u)}{\cos(u)}} du\end{aligned}$$

Which results in

$$S = -\ln(1+\sin(u))$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_u + \omega(u, v)S_v}{R_u + \omega(u, v)R_v} \tag{2}$$

Where in the above R_u, R_v, S_u, S_v are all partial derivatives and $\omega(u, v)$ is the right hand side of the original ode given by

$$\omega(u, v) = -\frac{(e^v + 1)\cos(u)\, e^{-v}}{1+\sin(u)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_u &= 0 \\
R_v &= 1 \\
S_u &= -\frac{\cos(u)}{1+\sin(u)} \\
S_v &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{e^v}{e^v + 1} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for u, v in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{e^R}{e^R + 1}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln\left(e^R + 1\right) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to u, v coordinates. This results in

$$
-\ln(1+\sin(u)) = \ln(e^v + 1) + c_1
$$

Which simplifies to

$$
-\ln(1+\sin(u)) = \ln(e^v + 1) + c_1
$$

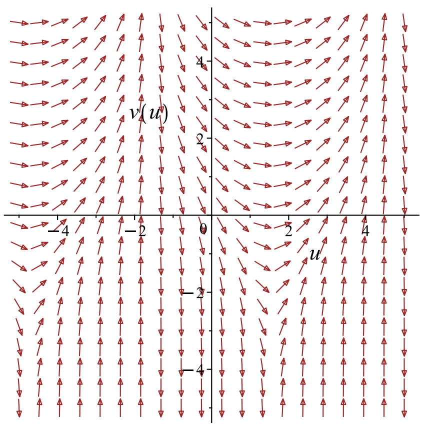
Which gives

$$
v = \ln\left(-\frac{-1+\sin(u)\,e^{c_1} + e^{c_1}}{1+\sin(u)}\right) - c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in u, v coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dv}{du} = -\frac{(e^{v}+1)\cos(u)e^{-v}}{1+\sin(u)}$



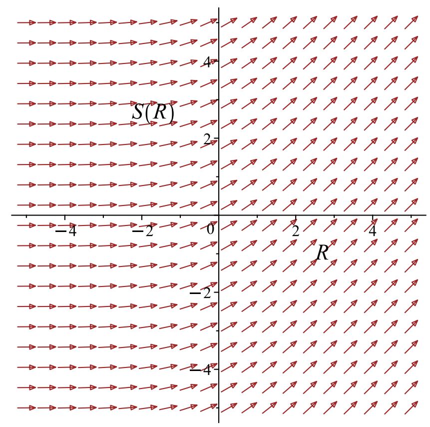
</td><td>

$R = v$

$S = -\ln(1+\sin(u))$

</td><td>

$\frac{dS}{dR} = \frac{e^{R}}{e^{R}+1}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$v = \ln\left(-\frac{-1+\sin(u)\,e^{c_1}+e^{c_1}}{1+\sin(u)}\right) - c_1 \tag{1}$$

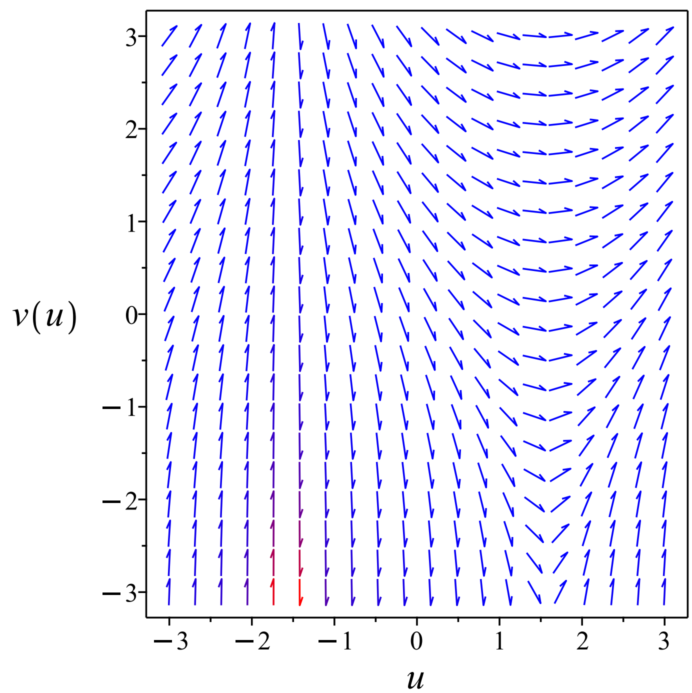


Figure 93: Slope field plot

Verification of solutions

$$v = \ln\left(-\frac{-1+\sin(u)\,e^{c_1}+e^{c_1}}{1+\sin(u)}\right) - c_1$$

Verified OK.

4.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(u,v)\,\mathrm{d}u + N(u,v)\,\mathrm{d}v = 0 \tag{1A}$$

Therefore

$$\left(-\frac{\mathrm{e}^v}{\mathrm{e}^v+1}\right)\mathrm{d}v = \left(\frac{\cos(u)}{1+\sin(u)}\right)\mathrm{d}u$$
$$\left(-\frac{\cos(u)}{1+\sin(u)}\right)\mathrm{d}u + \left(-\frac{\mathrm{e}^v}{\mathrm{e}^v+1}\right)\mathrm{d}v = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(u,v) = -\frac{\cos(u)}{1+\sin(u)}$$
$$N(u,v) = -\frac{\mathrm{e}^v}{\mathrm{e}^v+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u}$$

Using result found above gives

$$\frac{\partial M}{\partial v} = \frac{\partial}{\partial v}\left(-\frac{\cos(u)}{1+\sin(u)}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial u} &= \frac{\partial}{\partial u}\left(-\frac{e^v}{e^v+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(u, v)$

$$
\frac{\partial \phi}{\partial u} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial v} = N \tag{2}
$$

Integrating (1) w.r.t. u gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial u}\, du &= \int M\, du \\
\int \frac{\partial \phi}{\partial u}\, du &= \int -\frac{\cos(u)}{1+\sin(u)}\, du
\end{aligned}
$$

$$
\phi = -\ln(1+\sin(u)) + f(v) \tag{3}
$$

Where $f(v)$ is used for the constant of integration since ϕ is a function of both u and v. Taking derivative of equation (3) w.r.t v gives

$$
\frac{\partial \phi}{\partial v} = 0 + f'(v) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial v} = -\frac{e^v}{e^v+1}$. Therefore equation (4) becomes

$$
-\frac{e^v}{e^v+1} = 0 + f'(v) \tag{5}
$$

Solving equation (5) for $f'(v)$ gives

$$
f'(v) = -\frac{e^v}{e^v+1}
$$

Integrating the above w.r.t v gives

$$
\begin{aligned}
\int f'(v)\, dv &= \int \left(-\frac{e^v}{e^v+1}\right) dv \\
f(v) &= -\ln(e^v+1) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(v)$ into equation (3) gives ϕ

$$\phi = -\ln(1+\sin(u)) - \ln(e^v + 1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(1+\sin(u)) - \ln(e^v + 1)$$

The solution becomes

$$v = \ln\left(-\frac{-1+\sin(u)\,e^{c_1} + e^{c_1}}{1+\sin(u)}\right) - c_1$$

Summary

The solution(s) found are the following

$$v = \ln\left(-\frac{-1+\sin(u)\,e^{c_1} + e^{c_1}}{1+\sin(u)}\right) - c_1 \tag{1}$$

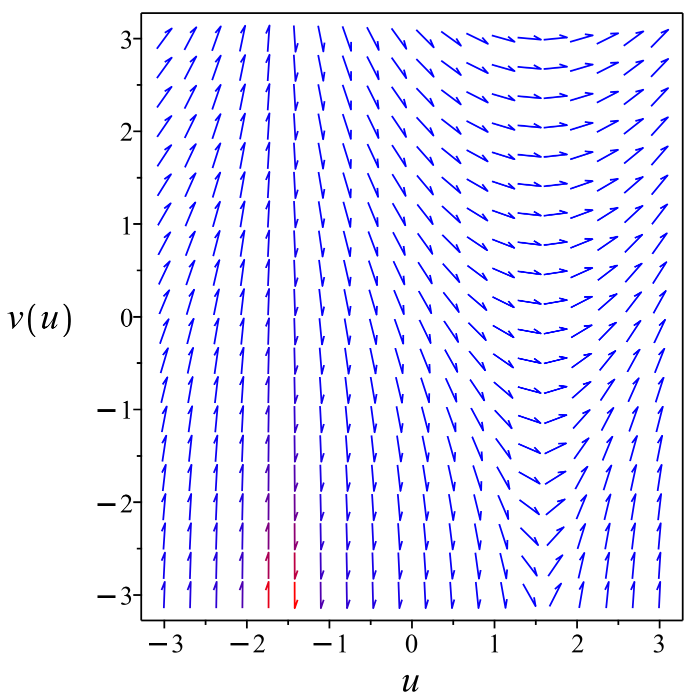


Figure 94: Slope field plot

Verification of solutions

$$v = \ln\left(-\frac{-1+\sin(u)\,e^{c_1}+e^{c_1}}{1+\sin(u)}\right)-c_1$$

Verified OK.

4.6.4 Maple step by step solution

Let's solve

$(e^{v}+1)\cos(u)+e^{v}(1+\sin(u))\,v'=0$

- Highest derivative means the order of the ODE is 1

  v'

- Integrate both sides with respect to u

  $\int((e^{v}+1)\cos(u)+e^{v}(1+\sin(u))\,v')\,du=\int 0du+c_1$

- Evaluate integral

  $e^{v}\sin(u)+\sin(u)+e^{v}=c_1$

- Solve for v

  $v=\ln\left(-\frac{\sin(u)-c_1}{1+\sin(u)}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.64 (sec). Leaf size: 29

```
dsolve((exp(v(u))+1)*cos(u) + exp(v(u))*(1+sin(u))*diff(v(u),u)=0,v(u), singsol=all)
```

$$v(u) = -\ln\left(\frac{-1-\sin(u)}{-1+(1+\sin(u))\,e^{c_1}}\right)-c_1$$

✓ Solution by Mathematica

Time used: 5.457 (sec). Leaf size: 37

```
DSolve[(Exp[v[u]]+1)*Cos[u] + Exp[v[u]]*(1+Sin[u])*v'[u]==0,v[u],u,IncludeSingularSolutions
```

$$v(u) \to \log \left(-1+\frac{e^{c_1}}{\left(\sin \left(\frac{u}{2}\right)+\cos \left(\frac{u}{2}\right)\right)^2}\right)$$
$$v(u) \to i\pi$$

4.7 problem 7

Internal problem ID [11620]
Internal file name [`OUTPUT/10602_Saturday_May_27_2023_03_01_16_AM_78210111/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(x+4)\left(1+y^2\right)+y\left(x^2+3x+2\right)y'=0}$$

4.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{(x+4)\left(-y^2-1\right)}{(1+x)\,(x+2)\,y} \end{aligned}$$

Where $f(x)=\frac{x+4}{(1+x)(x+2)}$ and $g(y)=\frac{-y^2-1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{-y^2-1}{y}}\,dy=\frac{x+4}{(1+x)\,(x+2)}\,dx$$

$$\int \frac{1}{\frac{-y^2-1}{y}} \, dy = \int \frac{x+4}{(1+x)(x+2)} \, dx$$

$$-\frac{\ln(y^2+1)}{2} = 3\ln(1+x) - 2\ln(x+2) + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{y^2+1}} = \mathrm{e}^{3\ln(1+x)-2\ln(x+2)+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{y^2+1}} = c_2 \mathrm{e}^{3\ln(1+x)-2\ln(x+2)}$$

The solution is

$$\frac{1}{\sqrt{1+y^2}} = c_2 \mathrm{e}^{3\ln(1+x)-2\ln(x+2)+c_1}$$

Summary

The solution(s) found are the following

$$\frac{1}{\sqrt{1+y^2}} = c_2 \mathrm{e}^{3\ln(1+x)-2\ln(x+2)+c_1} \tag{1}$$

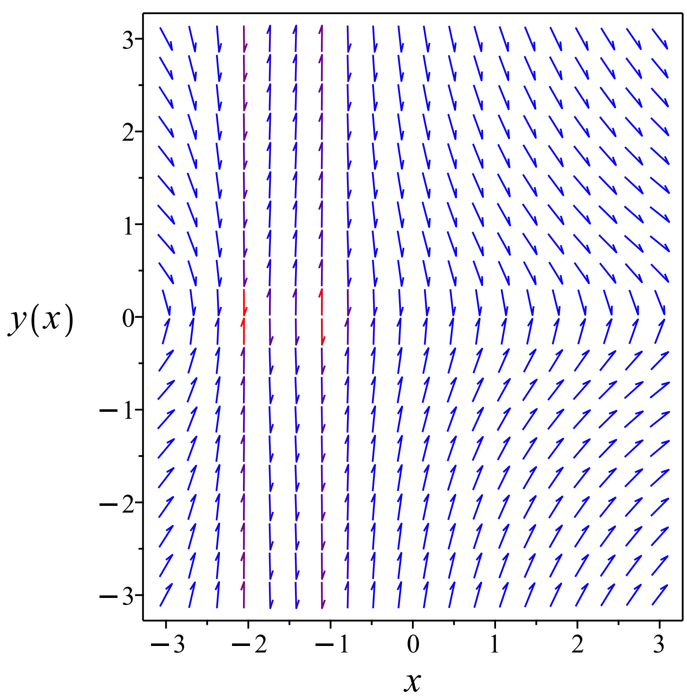


Figure 95: Slope field plot

Verification of solutions

$$\frac{1}{\sqrt{1+y^2}} = c_2 \mathrm{e}^{3\ln(1+x)-2\ln(x+2)+c_1}$$

Verified OK.

4.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x\,y^2+4y^2+x+4}{y\,(x^2+3x+2)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 88: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{(1 + x)\,(x + 2)}{x + 4} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{(1+x)(x+2)}{x+4}} dx \end{aligned}$$

Which results in

$$S = 3\ln(1+x) - 2\ln(x+2)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x\,y^2 + 4y^2 + x + 4}{y\,(x^2 + 3x + 2)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{x+4}{(1+x)(x+2)} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{y}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R^2+1)}{2} + c_1 \tag{4}$$

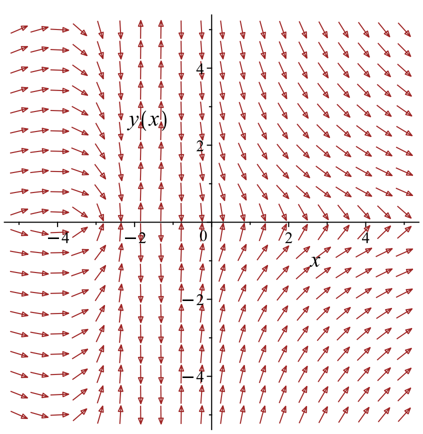
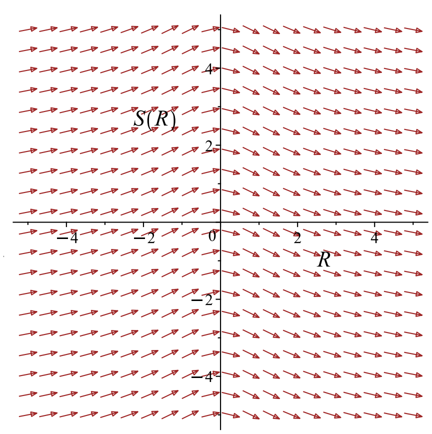
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$3\ln(1+x) - 2\ln(x+2) = -\frac{\ln(1+y^2)}{2} + c_1$$

Which simplifies to

$$3\ln(1+x) - 2\ln(x+2) = -\frac{\ln(1+y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x\,y^2+4y^2+x+4}{y(x^2+3x+2)}$ 	$R = y$ $S = 3\ln(1+x) - 2\ln(x$	$\frac{dS}{dR} = -\frac{R}{R^2+1}$ 

Summary

The solution(s) found are the following

$$3\ln(1+x) - 2\ln(x+2) = -\frac{\ln(1+y^2)}{2} + c_1 \tag{1}$$

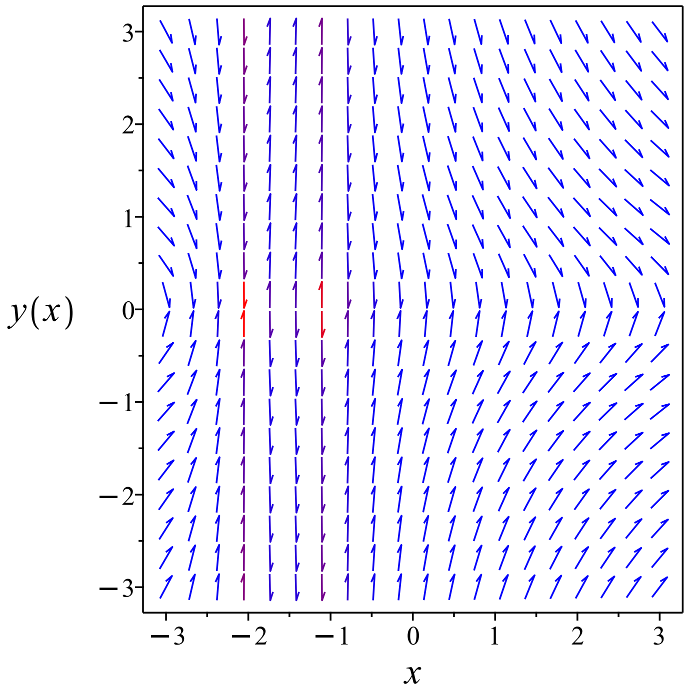


Figure 96: Slope field plot

Verification of solutions

$$3 \ln (1+x) - 2 \ln (x+2) = -\frac{\ln (1+y^2)}{2} + c_1$$

Verified OK.

4.7.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x\,y^2 + 4y^2 + x + 4}{y\,(x^2 + 3x + 2)} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{x+4}{x^2+3x+2} y - \frac{x+4}{x^2+3x+2} \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= -\frac{x+4}{x^2+3x+2} \\
f_1(x) &= -\frac{x+4}{x^2+3x+2} \\
n &= -1
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{(x+4)\,y^2}{x^2+3x+2} - \frac{x+4}{x^2+3x+2} \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
\frac{w'(x)}{2} &= -\frac{(x+4)\,w(x)}{x^2+3x+2} - \frac{x+4}{x^2+3x+2} \\
w' &= -\frac{2(x+4)\,w}{x^2+3x+2} - \frac{2(x+4)}{x^2+3x+2}
\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{-2x-8}{x^2+3x+2} \\
q(x) &= \frac{-2x-8}{x^2+3x+2}
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{(-2x-8)\,w(x)}{x^2+3x+2} = \frac{-2x-8}{x^2+3x+2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{-2x-8}{x^2+3x+2}dx} \\
&= \mathrm{e}^{6\ln(1+x)-4\ln(x+2)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{(1+x)^6}{(x+2)^4}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\frac{-2x-8}{x^2+3x+2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(1+x)^6 w}{(x+2)^4}\right) &= \left(\frac{(1+x)^6}{(x+2)^4}\right)\left(\frac{-2x-8}{x^2+3x+2}\right) \\
\mathrm{d}\left(\frac{(1+x)^6 w}{(x+2)^4}\right) &= \left(-\frac{2(1+x)^5(x+4)}{(x+2)^5}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{(1+x)^6 w}{(x+2)^4} &= \int -\frac{2(1+x)^5(x+4)}{(x+2)^5}\,\mathrm{d}x \\
\frac{(1+x)^6 w}{(x+2)^4} &= -x^2+2x+\frac{6}{(x+2)^3}-\frac{15}{(x+2)^2}+\frac{20}{x+2}-\frac{1}{(x+2)^4}+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{(1+x)^6}{(x+2)^4}$ results in

$$
w(x) = \frac{(x+2)^4\left(-x^2+2x+\frac{6}{(x+2)^3}-\frac{15}{(x+2)^2}+\frac{20}{x+2}-\frac{1}{(x+2)^4}\right)}{(1+x)^6} + \frac{c_1(x+2)^4}{(1+x)^6}
$$

which simplifies to

$$w(x) = \frac{-x^6 - 6x^5 + (c_1 - 8)\, x^4 + (8c_1 + 36)\, x^3 + (24c_1 + 153)\, x^2 + (32c_1 + 218)\, x + 16c_1 + 111}{(1 + x)^6}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{-x^6 - 6x^5 + (c_1 - 8)\, x^4 + (8c_1 + 36)\, x^3 + (24c_1 + 153)\, x^2 + (32c_1 + 218)\, x + 16c_1 + 111}{(1 + x)^6}$$

Solving for y gives

$$y(x) = \frac{\sqrt{-x^6 - 6x^5 + (c_1 - 8)\, x^4 + (8c_1 + 36)\, x^3 + (24c_1 + 153)\, x^2 + (32c_1 + 218)\, x + 16c_1 + 111}}{(1 + x)^3}$$
$$y(x) = -\frac{\sqrt{-x^6 - 6x^5 + (c_1 - 8)\, x^4 + (8c_1 + 36)\, x^3 + (24c_1 + 153)\, x^2 + (32c_1 + 218)\, x + 16c_1 + 111}}{(1 + x)^3}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\sqrt{-x^6 - 6x^5 + (c_1 - 8)\, x^4 + (8c_1 + 36)\, x^3 + (24c_1 + 153)\, x^2 + (32c_1 + 218)\, x + 16c_1 + 111}}{(1 + x)^3} \tag{1}$$

$$y = -\frac{\sqrt{-x^6 - 6x^5 + (c_1 - 8)\, x^4 + (8c_1 + 36)\, x^3 + (24c_1 + 153)\, x^2 + (32c_1 + 218)\, x + 16c_1 + 111}}{(1 + x)^3} \tag{2}$$

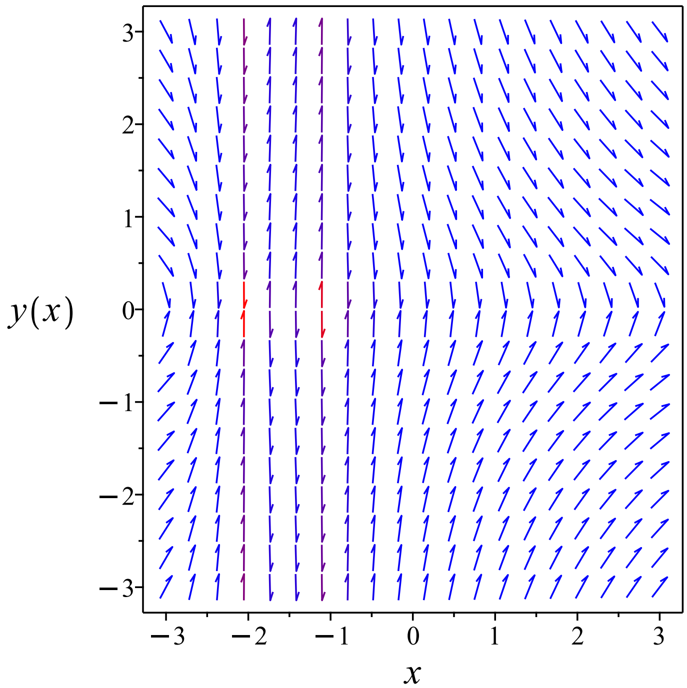


Figure 97: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\sqrt{-x^6 - 6x^5 + (c_1 - 8)x^4 + (8c_1 + 36)x^3 + (24c_1 + 153)x^2 + (32c_1 + 218)x + 16c_1 + 111}}{(1+x)^3}$$

Verified OK.

$$y = -\frac{\sqrt{-x^6 - 6x^5 + (c_1 - 8)x^4 + (8c_1 + 36)x^3 + (24c_1 + 153)x^2 + (32c_1 + 218)x + 16c_1 + 111}}{(1+x)^3}$$

Verified OK.

4.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{-y^2-1}\right)\mathrm{d}y&=\left(\frac{x+4}{(1+x)\,(x+2)}\right)\mathrm{d}x\\ \left(-\frac{x+4}{(1+x)\,(x+2)}\right)\mathrm{d}x+\left(\frac{y}{-y^2-1}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{x+4}{(1+x)\,(x+2)}\\ N(x,y)&=\frac{y}{-y^2-1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x+4}{(1+x)(x+2)}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{-y^2-1}\right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x+4}{(1+x)(x+2)} \, \mathrm{d}x \\ \phi &= -3\ln(1+x) + 2\ln(x+2) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{-y^2-1}$. Therefore equation (4) becomes

$$\frac{y}{-y^2-1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{y}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{y}{y^2+1}\right)\mathrm{d}y$$
$$f(y) = -\frac{\ln(y^2+1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -3\ln(1+x) + 2\ln(x+2) - \frac{\ln(y^2+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -3\ln(1+x) + 2\ln(x+2) - \frac{\ln(y^2+1)}{2}$$

Summary
The solution(s) found are the following

$$-3\ln(1+x) + 2\ln(x+2) - \frac{\ln(1+y^2)}{2} = c_1 \tag{1}$$

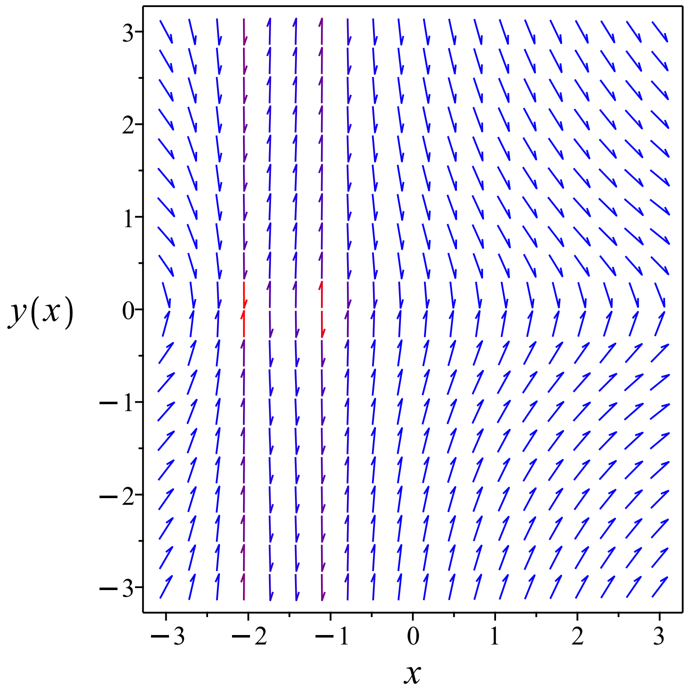


Figure 98: Slope field plot

Verification of solutions

$$-3\ln(1+x)+2\ln(x+2)-\frac{\ln(1+y^2)}{2}=c_1$$

Verified OK.

4.7.5 Maple step by step solution

Let's solve

$(x+4)(1+y^2)+y(x^2+3x+2)y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{yy'}{1+y^2}=-\frac{x+4}{x^2+3x+2}$

- Integrate both sides with respect to x

  $\int \frac{yy'}{1+y^2}dx=\int -\frac{x+4}{x^2+3x+2}dx+c_1$

- Evaluate integral

$$\frac{\ln(1+y^2)}{2} = -3\ln(1+x) + 2\ln(x+2) + c_1$$

- Solve for y

$$\left\{y = \frac{\sqrt{x^4\left(\mathrm{e}^{\frac{c_1}{3}}\right)^6 + 8x^3\left(\mathrm{e}^{\frac{c_1}{3}}\right)^6 + 24x^2\left(\mathrm{e}^{\frac{c_1}{3}}\right)^6 + 32x\left(\mathrm{e}^{\frac{c_1}{3}}\right)^6 + 16\left(\mathrm{e}^{\frac{c_1}{3}}\right)^6 - x^6 - 6x^5 - 15x^4 - 20x^3 - 15x^2 - 6x - 1}}{(1+x)^3}, y = -\frac{\sqrt{x^4\left(\mathrm{e}^{\frac{c_1}{3}}\right)^6}}{}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 114

```
dsolve((x+4)*(y(x)^2+1) + y(x)*(x^2+3*x+2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{-x^6 - 6x^5 + x^4c_1 + (8c_1+100)\,x^3 + (24c_1+345)\,x^2 + (32c_1+474)\,x + 16c_1 + 239}}{(1+x)^3}$$

$$y(x) = -\frac{\sqrt{-x^6 - 6x^5 + x^4c_1 + (8c_1+100)\,x^3 + (24c_1+345)\,x^2 + (32c_1+474)\,x + 16c_1 + 239}}{(1+x)^3}$$

✓ Solution by Mathematica

Time used: 5.501 (sec). Leaf size: 126

```
DSolve[(x+4)*(y[x]^2+1) + y[x]*(x^2+3*x+2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-(x+1)^6 + e^{2c_1}(x+2)^4}}{(x+1)^3}$$
$$y(x) \to \frac{\sqrt{-(x+1)^6 + e^{2c_1}(x+2)^4}}{(x+1)^3}$$
$$y(x) \to -i$$
$$y(x) \to i$$
$$y(x) \to \frac{(x+1)^3}{\sqrt{-(x+1)^6}}$$
$$y(x) \to \frac{\sqrt{-(x+1)^6}}{(x+1)^3}$$

4.8 problem 8

Internal problem ID [11621]
Internal file name [`OUTPUT/10603_Saturday_May_27_2023_03_01_19_AM_16407112/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y - y'x = -x}$$

4.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 1$$

Hence the ode is

$$y' - \frac{y}{x} = 1$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= \mu \\
\frac{d}{dx}\left(\frac{y}{x}\right) &= \frac{1}{x} \\
d\left(\frac{y}{x}\right) &= \frac{1}{x}dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int \frac{1}{x}\, dx \\
\frac{y}{x} &= \ln(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
y = x\ln(x) + c_1 x
$$

which simplifies to

$$
y = x(\ln(x) + c_1)
$$

Summary

The solution(s) found are the following

$$
y = x(\ln(x) + c_1) \tag{1}
$$

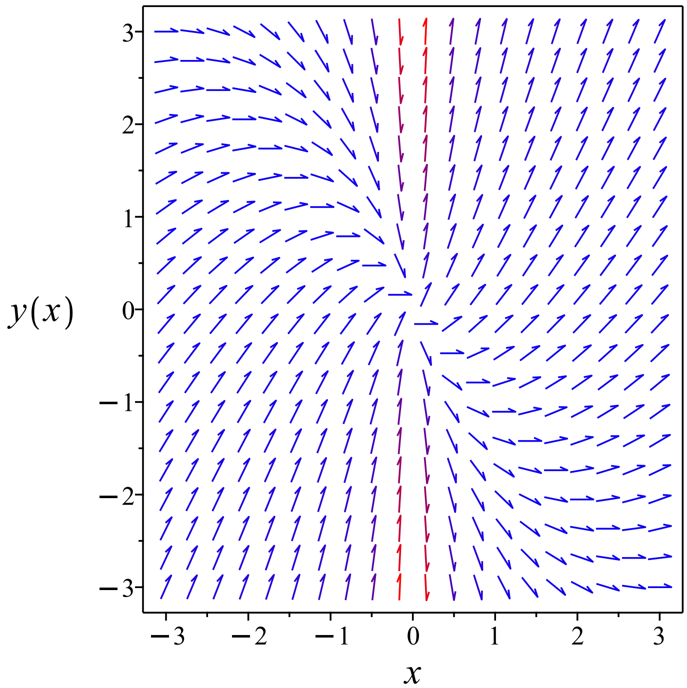


Figure 99: Slope field plot

Verification of solutions

$$y = x(\ln (x) + c_1)$$

Verified OK.

4.8.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x - (u'(x)\, x + u(x))\, x = -x$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{1}{x}\, \mathrm{d}x \\ &= \ln (x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x(\ln (x) + c_2) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x(\ln(x) + c_2) \tag{1}$$

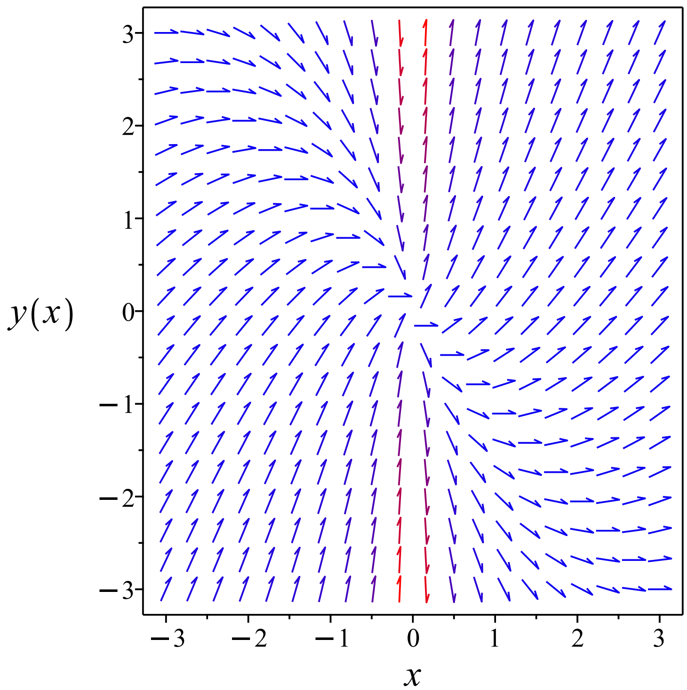


Figure 100: Slope field plot

Verification of solutions

$$y = x(\ln(x) + c_2)$$

Verified OK.

4.8.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + x}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 91: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + x}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \ln(x) + c_1$$

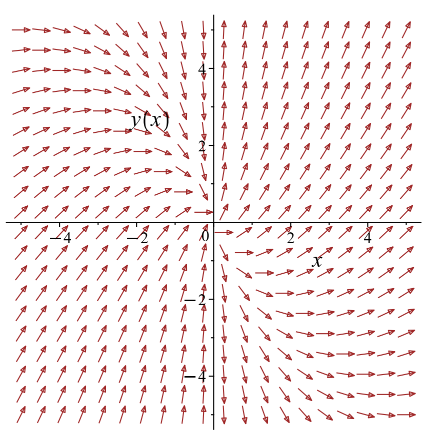
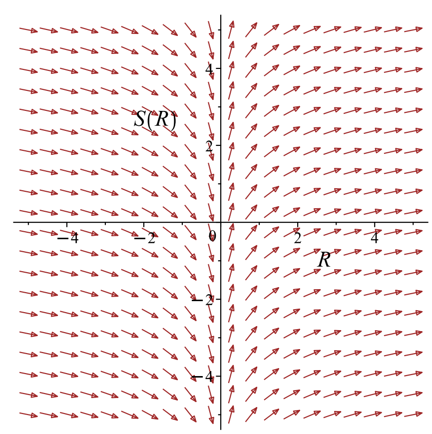
Which simplifies to

$$\frac{y}{x} = \ln(x) + c_1$$

Which gives

$$y = x(\ln(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+x}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = x(\ln(x) + c_1) \tag{1}$$

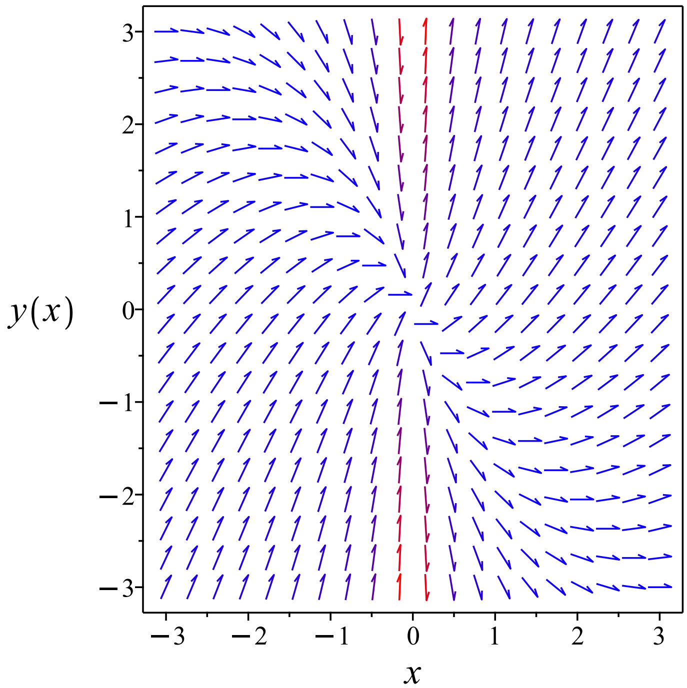


Figure 101: Slope field plot

Verification of solutions

$$y = x(\ln(x) + c_1)$$

Verified OK.

4.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x)\,\mathrm{d}y &= (-y - x)\,\mathrm{d}x\\ (y + x)\,\mathrm{d}x + (-x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y + x\\ N(x, y) &= -x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y + x)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x}((1) - (-1)) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}(y + x) \\
&= \frac{y + x}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(-x) \\
&= -\frac{1}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y + x}{x^2}\right) + \left(-\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{y + x}{x^2} \,\mathrm{d}x$$

$$\phi = -\frac{y}{x} + \ln (x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{x}$. Therefore equation (4) becomes

$$-\frac{1}{x} = -\frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y}{x} + \ln (x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y}{x} + \ln (x)$$

The solution becomes

$$y = x(\ln(x) - c_1)$$

Summary

The solution(s) found are the following

$$y = x(\ln(x) - c_1) \tag{1}$$

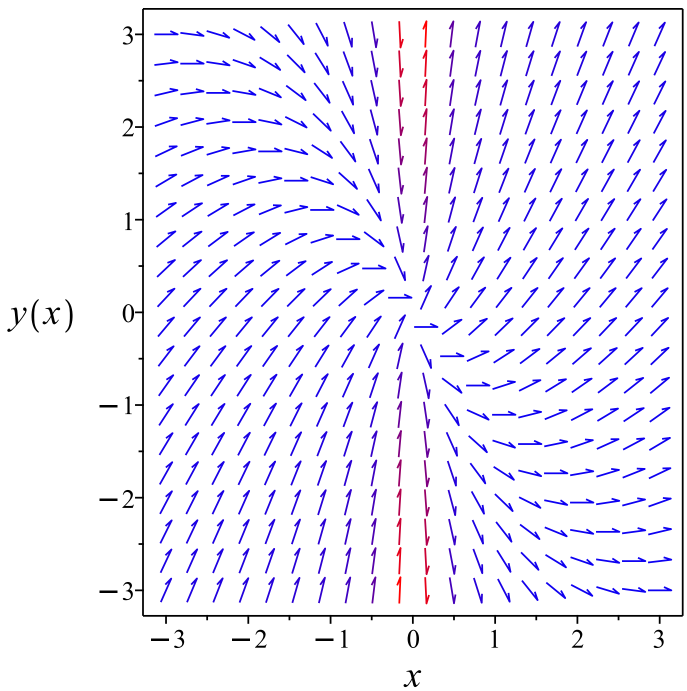


Figure 102: Slope field plot

Verification of solutions

$$y = x(\ln(x) - c_1)$$

Verified OK.

4.8.5 Maple step by step solution

Let's solve

$y - y'x = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \frac{1}{x} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(\ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve((x+y(x))- x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (\ln(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 12

```
DSolve[(x+y[x])- x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(\log(x) + c_1)$$

4.9 problem 9

Internal problem ID [11622]
Internal file name [`OUTPUT/10604_Saturday_May_27_2023_03_01_21_AM_5585912/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{2yx + 3y^2 - (2yx + x^2)\, y' = 0}$$

4.9.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$2u(x)\, x^2 + 3u(x)^2\, x^2 - (2u(x)\, x^2 + x^2)\, (u'(x)\, x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(u+1)}{(2u+1)\, x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u(u+1)}{2u+1}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u(u+1)}{2u+1}}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{\frac{u(u+1)}{2u+1}}\,du &= \int \frac{1}{x}\,dx \\
\ln\left(u(u+1)\right) &= \ln\left(x\right) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
u(u+1) = \mathrm{e}^{\ln(x)+c_2}
$$

Which simplifies to

$$
u(u+1) = c_3 x
$$

Which simplifies to

$$
u(x)\left(u(x)+1\right) = c_3 \mathrm{e}^{c_2} x
$$

The solution is

$$
u(x)\left(u(x)+1\right) = c_3 \mathrm{e}^{c_2} x
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\frac{y\left(\frac{y}{x}+1\right)}{x} &= c_3 \mathrm{e}^{c_2} x \\
\frac{y(y+x)}{x^2} &= c_3 \mathrm{e}^{c_2} x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\frac{y(y+x)}{x^2} = c_3 \mathrm{e}^{c_2} x \tag{1}
$$

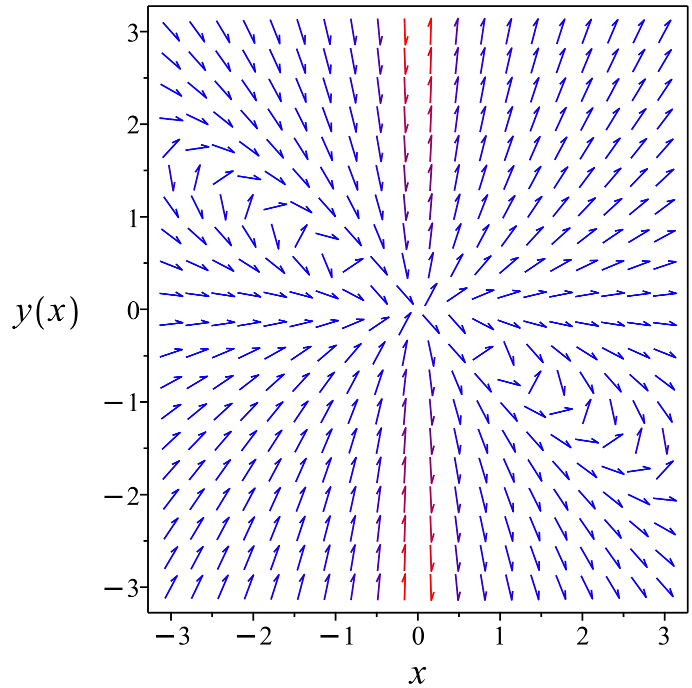


Figure 103: Slope field plot

Verification of solutions

$$\frac{y(y+x)}{x^2} = c_3 \mathrm{e}^{c_2} x$$

Verified OK.

4.9.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y(2x+3y)}{x\,(x+2y)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{y(2x+3y)\,(b_3-a_2)}{x\,(x+2y)} - \frac{y^2(2x+3y)^2\,a_3}{x^2\,(x+2y)^2} \\ &- \left(\frac{2y}{x\,(x+2y)} - \frac{y(2x+3y)}{x^2\,(x+2y)} - \frac{y(2x+3y)}{x\,(x+2y)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(\frac{2x+3y}{x\,(x+2y)} + \frac{3y}{x\,(x+2y)} - \frac{2y(2x+3y)}{x\,(x+2y)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4b_2+2x^3yb_2+x^2y^2a_2+2x^2y^2a_3+2x^2y^2b_2-x^2y^2b_3+6x\,y^3a_3+3y^4a_3+2x^3b_1-2x^2ya_1+6x^2yb_1-}{x^2\,(x+2y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^4b_2-2x^3yb_2-x^2y^2a_2-2x^2y^2a_3-2x^2y^2b_2+x^2y^2b_3-6x\,y^3a_3 \\ &-3y^4a_3-2x^3b_1+2x^2ya_1-6x^2yb_1+6x\,y^2a_1-6x\,y^2b_1+6y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-a_2v_1^2v_2^2-2a_3v_1^2v_2^2-6a_3v_1v_2^3-3a_3v_2^4-b_2v_1^4-2b_2v_1^3v_2-2b_2v_1^2v_2^2 \\ &+b_3v_1^2v_2^2+2a_1v_1^2v_2+6a_1v_1v_2^2+6a_1v_2^3-2b_1v_1^3-6b_1v_1^2v_2-6b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_2v_1^4 - 2b_2v_1^3v_2 - 2b_1v_1^3 + (-a_2 - 2a_3 - 2b_2 + b_3)\, v_1^2v_2^2 + (2a_1 - 6b_1)\, v_1^2v_2 - 6a_3v_1v_2^3 + (6a_1 - 6b_1)\, v_1v_2^2 - 3a_3v_2^4 + 6a_1v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
6a_1 &= 0\\
-6a_3 &= 0\\
-3a_3 &= 0\\
-2b_1 &= 0\\
-2b_2 &= 0\\
-b_2 &= 0\\
2a_1 - 6b_1 &= 0\\
6a_1 - 6b_1 &= 0\\
-a_2 - 2a_3 - 2b_2 + b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x\\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y(2x+3y)}{x\,(x+2y)}\right)(x) \\
&= \frac{-xy - y^2}{x+2y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-xy-y^2}{x+2y}} dy
\end{aligned}
$$

Which results in

$$
S = -\ln\left(y(y+x)\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{y(2x+3y)}{x\,(x+2y)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{y+x} \\
S_y &= \frac{-x-2y}{y\,(y+x)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{3}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{3}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -3\ln(R) + c_1 \tag{4}$$

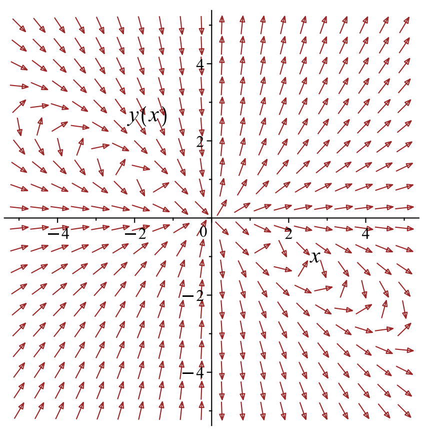
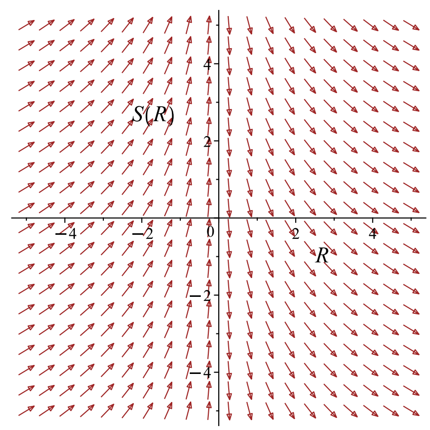
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(y) - \ln(y+x) = -3\ln(x) + c_1$$

Which simplifies to

$$-\ln(y) - \ln(y+x) = -3\ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(2x+3y)}{x(x+2y)}$ 	$R = x$ $S = -\ln(y) - \ln(y + x)$	$\frac{dS}{dR} = -\frac{3}{R}$ 

Summary

The solution(s) found are the following

$$-\ln(y) - \ln(y + x) = -3\ln(x) + c_1 \tag{1}$$

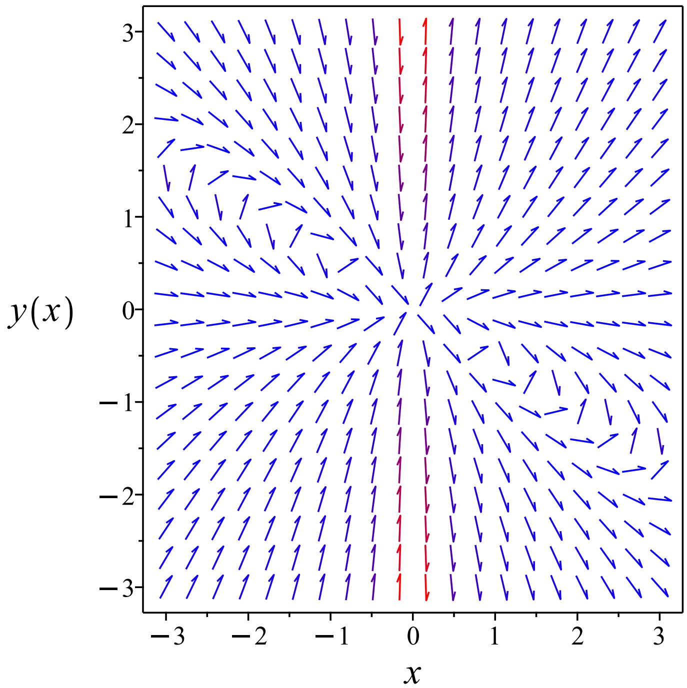


Figure 104: Slope field plot

Verification of solutions

$$-\ln(y) - \ln(y+x) = -3\ln(x) + c_1$$

Verified OK.

4.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
\left(-x^2 - 2xy\right) \mathrm{d}y &= \left(-2xy - 3y^2\right) \mathrm{d}x \\
\left(2xy + 3y^2\right) \mathrm{d}x + \left(-x^2 - 2xy\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 2xy + 3y^2 \\
N(x, y) &= -x^2 - 2xy
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2xy + 3y^2\right) \\
&= 6y + 2x
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^2 - 2xy\right) \\
&= -2x - 2y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{x\,(x+2y)}((6y+2x) - (-2x-2y))\\ &= -\frac{4}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{4}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4\ln(x)}\\ &= \frac{1}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^4}(2xy+3y^2)\\ &= \frac{y(2x+3y)}{x^4}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^4}(-x^2-2xy)\\ &= \frac{-x-2y}{x^3}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{y(2x+3y)}{x^4}\right) + \left(\frac{-x-2y}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \frac{y(2x+3y)}{x^4}\,\mathrm{d}x \\ \phi &= -\frac{y(y+x)}{x^3} + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= -\frac{y+x}{x^3} - \frac{y}{x^3} + f'(y) \qquad (4) \\ &= \frac{-x-2y}{x^3} + f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{-x-2y}{x^3}$. Therefore equation (4) becomes

$$\frac{-x-2y}{x^3} = \frac{-x-2y}{x^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y(y+x)}{x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y(y+x)}{x^3}$$

Summary

The solution(s) found are the following

$$-\frac{y(y+x)}{x^3} = c_1 \tag{1}$$

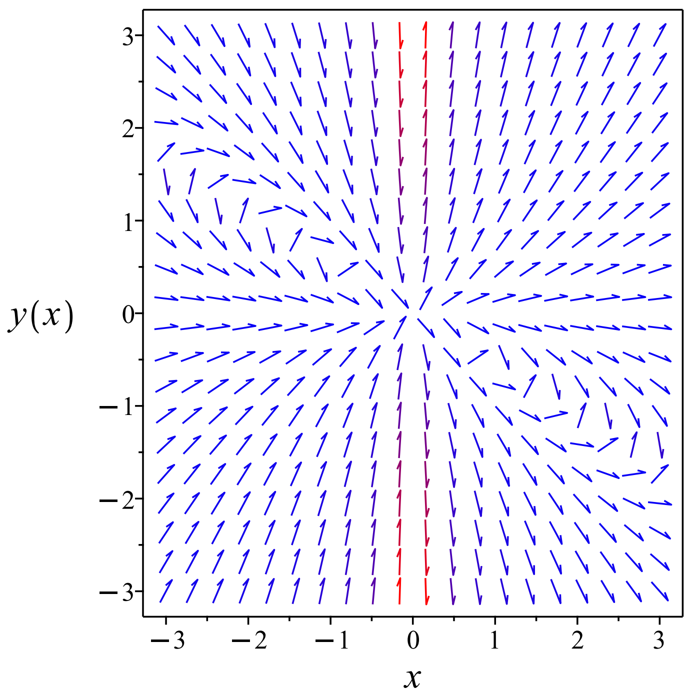


Figure 105: Slope field plot

Verification of solutions

$$-\frac{y(y+x)}{x^3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve((2*x*y(x)+3*y(x)^2)- (2*x*y(x)+x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\left(1+\sqrt{4c_1x+1}\right)x}{2}$$
$$y(x) = \frac{\left(-1+\sqrt{4c_1x+1}\right)x}{2}$$

✓ Solution by Mathematica

Time used: 0.618 (sec). Leaf size: 61

```
DSolve[(2*x*y[x]+3*y[x]^2)- (2*x*y[x]+x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}x\left(1+\sqrt{1+4e^{c_1}x}\right)$$
$$y(x) \to \frac{1}{2}x\left(-1+\sqrt{1+4e^{c_1}x}\right)$$
$$y(x) \to 0$$
$$y(x) \to -x$$

4.10 problem 10

Internal problem ID [11623]
Internal file name [`OUTPUT/10605_Saturday_May_27_2023_03_05_11_AM_66707145/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{v^3 + (u^3 - uv^2)\, v' = 0}$$

4.10.1 Solving as homogeneousTypeD2 ode

Using the change of variables $v = u(u)\, u$ on the above ode results in new ode in $u(u)$

$$u(u)^3 u^3 + \left(u^3 - u^3 u(u)^2\right)\left(u'(u)\, u + u(u)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(u, u) \\ &= f(u)g(u) \\ &= \frac{u}{(u^2 - 1)\, u} \end{aligned}$$

Where $f(u) = \frac{1}{u}$ and $g(u) = \frac{u}{u^2 - 1}$. Integrating both sides gives

$$\frac{1}{\frac{u}{u^2-1}}\, du = \frac{1}{u}\, du$$

$$\int \frac{1}{\frac{u}{u^2-1}}\,du = \int \frac{1}{u}\,du$$

$$\frac{u^2}{2} - \ln(u) = \ln(u) + c_2$$

The solution is

$$\frac{u(u)^2}{2} - \ln(u(u)) - \ln(u) - c_2 = 0$$

Replacing $u(u)$ in the above solution by $\frac{v}{u}$ results in the solution for v in implicit form

$$\frac{v^2}{2u^2} - \ln\left(\frac{v}{u}\right) - \ln(u) - c_2 = 0$$

$$\frac{v^2}{2u^2} - \ln\left(\frac{v}{u}\right) - \ln(u) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{v^2}{2u^2} - \ln\left(\frac{v}{u}\right) - \ln(u) - c_2 = 0 \tag{1}$$

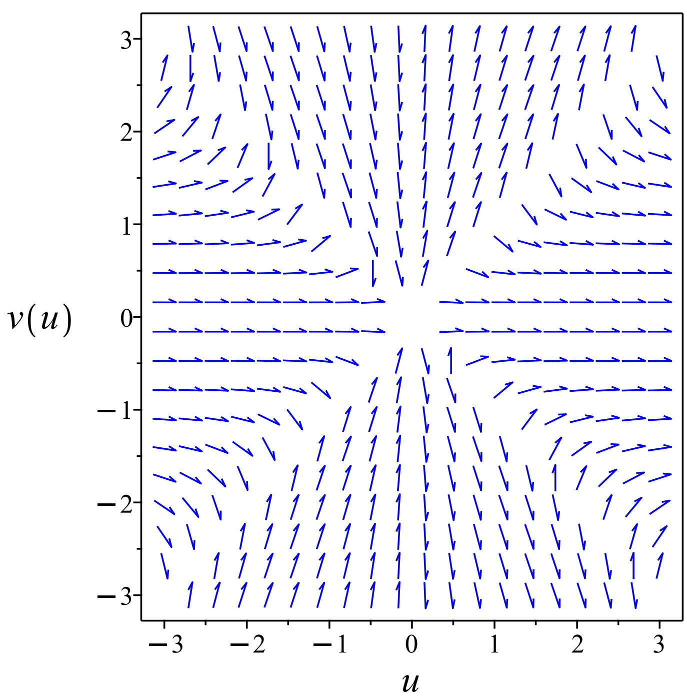


Figure 106: Slope field plot

Verification of solutions

$$\frac{v^2}{2u^2} - \ln\left(\frac{v}{u}\right) - \ln(u) - c_2 = 0$$

Verified OK.

4.10.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$v' = \frac{v^3}{u(-u^2 + v^2)}$$
$$v' = \omega(u, v)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_u + \omega(\eta_v - \xi_u) - \omega^2 \xi_v - \omega_u \xi - \omega_v \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = u a_2 + v a_3 + a_1 \tag{1E}$$
$$\eta = u b_2 + v b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{v^3(b_3 - a_2)}{u(-u^2 + v^2)} - \frac{v^6 a_3}{u^2 (-u^2 + v^2)^2} \\ &- \left(-\frac{v^3}{u^2(-u^2 + v^2)} + \frac{2v^3}{(-u^2 + v^2)^2}\right)(u a_2 + v a_3 + a_1) \\ &- \left(\frac{3v^2}{u(-u^2 + v^2)} - \frac{2v^4}{u(-u^2 + v^2)^2}\right)(u b_2 + v b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{u^6 b_2 + u^4 v^2 b_2 - 2u^3 v^3 a_2 + 2u^3 v^3 b_3 - 3u^2 v^4 a_3 + 3u^3 v^2 b_1 - 3u^2 v^3 a_1 - u v^4 b_1 + v^5 a_1}{u^2 (u^2 - v^2)^2} = 0$$

Setting the numerator to zero gives

$$u^6 b_2 + u^4 v^2 b_2 - 2u^3 v^3 a_2 + 2u^3 v^3 b_3 - 3u^2 v^4 a_3 + 3u^3 v^2 b_1 - 3u^2 v^3 a_1 - u v^4 b_1 + v^5 a_1 = 0 \quad \text{(6E)}$$

Looking at the above PDE shows the following are all the terms with $\{u, v\}$ in them.

$$\{u, v\}$$

The following substitution is now made to be able to collect on all terms with $\{u, v\}$ in them

$$\{u = v_1, v = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2 v_1^3 v_2^3 - 3a_3 v_1^2 v_2^4 + b_2 v_1^6 + b_2 v_1^4 v_2^2 + 2b_3 v_1^3 v_2^3 - 3a_1 v_1^2 v_2^3 + a_1 v_2^5 + 3b_1 v_1^3 v_2^2 - b_1 v_1 v_2^4 = 0 \quad \text{(7E)}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$b_2 v_1^6 + b_2 v_1^4 v_2^2 + (-2a_2 + 2b_3) v_1^3 v_2^3 + 3b_1 v_1^3 v_2^2 - 3a_3 v_1^2 v_2^4 - 3a_1 v_1^2 v_2^3 - b_1 v_1 v_2^4 + a_1 v_2^5 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_2 &= 0 \\ -3a_1 &= 0 \\ -3a_3 &= 0 \\ -b_1 &= 0 \\ 3b_1 &= 0 \\ -2a_2 + 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= u \\
\eta &= v
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(u, v)\,\xi \\
&= v - \left(\frac{v^3}{u\left(-u^2 + v^2\right)}\right)(u) \\
&= \frac{u^2 v}{u^2 - v^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(u, v) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{du}{\xi} = \frac{dv}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial v}\right) S(u, v) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = u
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{u^2 v}{u^2 - v^2}} dy
\end{aligned}
$$

Which results in

$$S = -\frac{v^2}{2u^2} + \ln(v)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_u + \omega(u, v)S_v}{R_u + \omega(u, v)R_v} \tag{2}$$

Where in the above R_u, R_v, S_u, S_v are all partial derivatives and $\omega(u, v)$ is the right hand side of the original ode given by

$$\omega(u, v) = \frac{v^3}{u(-u^2 + v^2)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_u &= 1 \\
R_v &= 0 \\
S_u &= \frac{v^2}{u^3} \\
S_v &= \frac{u^2 - v^2}{u^2 v}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for u, v in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to u, v coordinates. This results in

$$\frac{2\ln(v)\, u^2 - v^2}{2u^2} = c_1$$

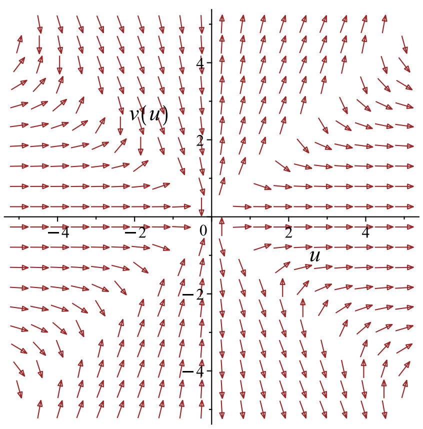
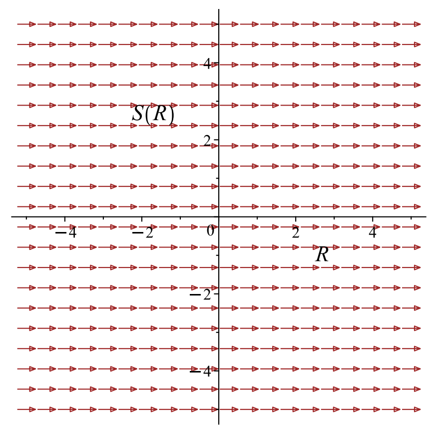
Which simplifies to

$$\frac{2\ln(v)\,u^2 - v^2}{2u^2} = c_1$$

Which gives

$$v = \mathrm{e}^{-\frac{\operatorname{LambertW}\left(-\frac{\mathrm{e}^{2c_1}}{u^2}\right)}{2} + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in u, v coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dv}{du} = \frac{v^3}{u(-u^2+v^2)}$ 	$R = u$ $S = \frac{2\ln(v)\,u^2 - v^2}{2u^2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$v = \mathrm{e}^{-\frac{\operatorname{LambertW}\left(-\frac{\mathrm{e}^{2c_1}}{u^2}\right)}{2} + c_1} \tag{1}$$

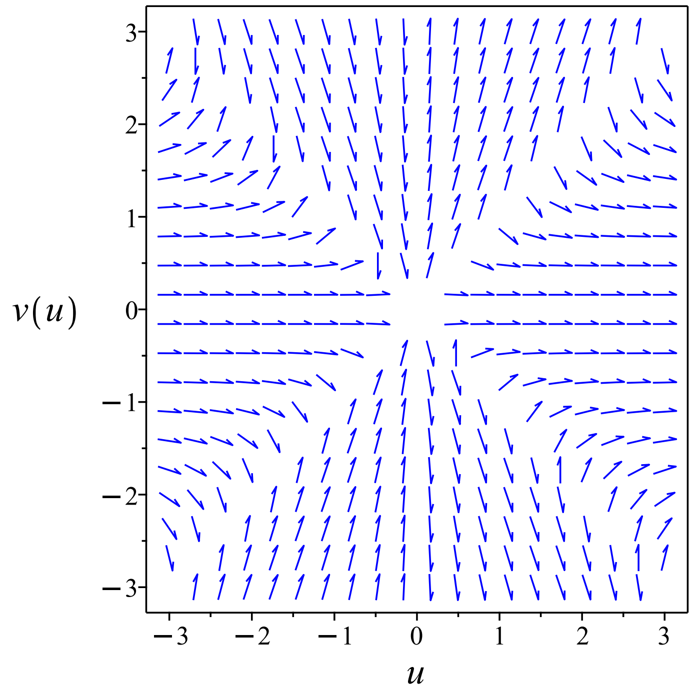


Figure 107: Slope field plot

Verification of solutions

$$v = e^{-\frac{\mathrm{LambertW}\left(-\frac{e^{2c_1}}{u^2}\right)}{2}+c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 32

```
dsolve(v(u)^3+ (u^3-u*v(u)^2)*diff(v(u),u)=0,v(u), singsol=all)
```

$$v(u) = \frac{e^{-c_1}}{\sqrt{-\frac{e^{-2c_1}}{u^2 \operatorname{LambertW}\left(-\frac{e^{-2c_1}}{u^2}\right)}}}$$

✓ Solution by Mathematica

Time used: 9.023 (sec). Leaf size: 56

```
DSolve[v[u]^3+ (u^3-u*v[u]^2)*v'[u]==0,v[u],u,IncludeSingularSolutions -> True]
```

$$v(u) \to -iu\sqrt{W\left(-\frac{e^{-2c_1}}{u^2}\right)}$$
$$v(u) \to iu\sqrt{W\left(-\frac{e^{-2c_1}}{u^2}\right)}$$
$$v(u) \to 0$$

4.11 problem 11

Internal problem ID [11624]
Internal file name [`OUTPUT/10606_Saturday_May_27_2023_03_05_11_AM_67096172/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{x \tan\left(\frac{y}{x}\right) + y - y'x = 0}$$

4.11.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \tan\left(\frac{y}{x}\right) + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$
\begin{aligned}
\frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\
u' &= \frac{1}{x} g(x)\, f(bu)^{\frac{n}{m}} \qquad (2)
\end{aligned}
$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$
\begin{aligned}
g(x) &= 1 \\
b &= 1 \\
f\left(\frac{bx}{y}\right) &= \tan\left(\frac{y}{x}\right)
\end{aligned}
$$

Substituting the above in (2) results in the $u(x)$ ode as

$$
u'(x) = \frac{\tan(u(x))}{x}
$$

Which is now solved as separable In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{\tan(u)}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \tan(u)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\tan(u)}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{\tan(u)}\, du &= \int \frac{1}{x}\, dx \\
\ln(\sin(u)) &= \ln(x) + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
\sin(u) = e^{\ln(x)+c_1}
$$

Which simplifies to

$$
\sin(u) = c_2 x
$$

Therefore the solution is

$$
\begin{aligned}
y &= ux \\
&= x \arcsin\left(c_2 x \,\mathrm{e}^{c_1}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = x \arcsin\left(c_2 x \,\mathrm{e}^{c_1}\right) \tag{1}
$$

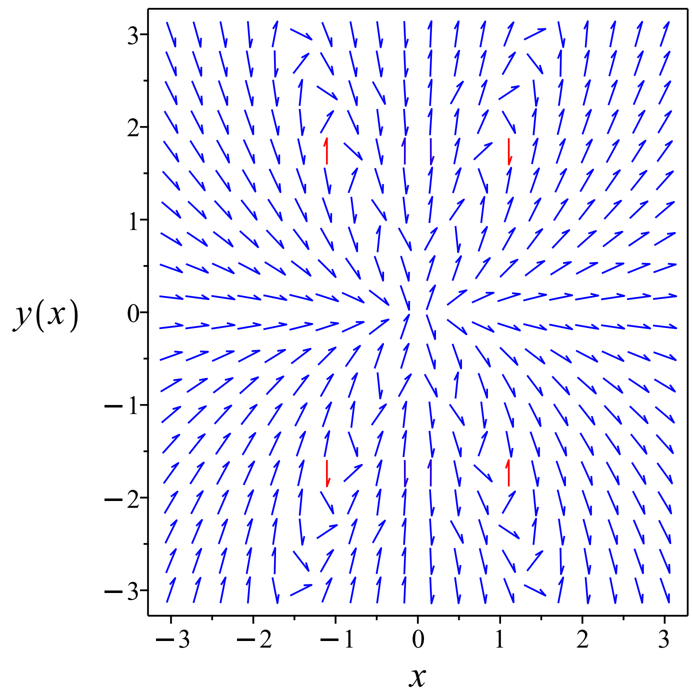


Figure 108: Slope field plot

Verification of solutions

$$
y = x \arcsin\left(c_2 x \,\mathrm{e}^{c_1}\right)
$$

Verified OK.

4.11.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x\tan(u(x)) + u(x)\,x - (u'(x)\,x + u(x))\,x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{\tan(u)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \tan(u)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\tan(u)}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{\tan(u)}\,du &= \int \frac{1}{x}\,dx \\ \ln(\sin(u)) &= \ln(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\sin(u) = e^{\ln(x)+c_2}$$

Which simplifies to

$$\sin(u) = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x\arcsin(c_3 e^{c_2} x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\arcsin(c_3 e^{c_2} x) \tag{1}$$

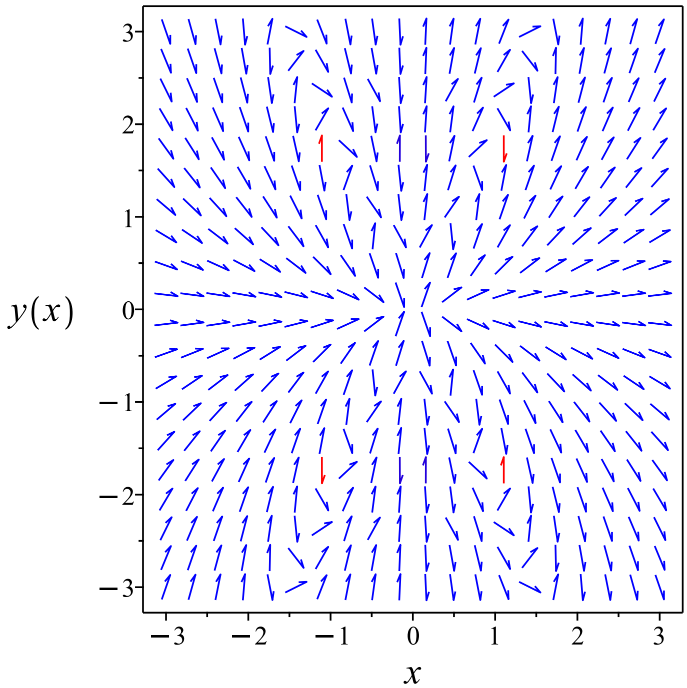


Figure 109: Slope field plot

Verification of solutions

$$y = x \arcsin\left(c_3 e^{c_2} x\right)$$

Verified OK.

4.11.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x \tan\left(\frac{y}{x}\right) + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 94: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= x^2 \\ \eta(x,y) &= xy \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{xy}{x^2} \\
&= \frac{y}{x}
\end{aligned}
$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{x^2}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= -\frac{1}{x}
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x \tan\left(\frac{y}{x}\right) + y}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{y}{x^2} \\
R_y &= \frac{1}{x} \\
S_x &= \frac{1}{x^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\cot\left(\frac{y}{x}\right)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\cot(R)\, S(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{c_1}{\sin(R)} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = \frac{c_1}{\sin\left(\frac{y}{x}\right)}$$

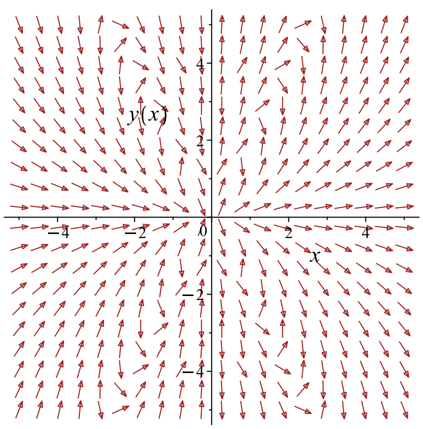
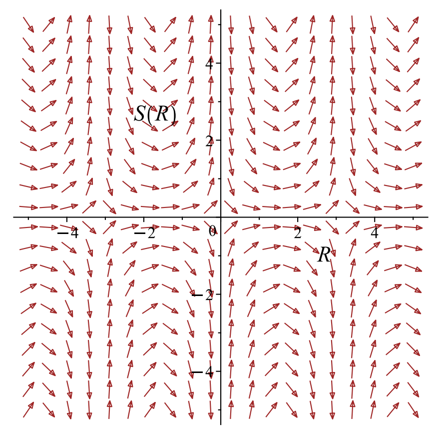
Which simplifies to

$$-\frac{1}{x} = \frac{c_1}{\sin\left(\frac{y}{x}\right)}$$

Which gives

$$y = -\arcsin(c_1 x)\, x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{x\tan\left(\frac{y}{x}\right)+y}{x}$

</td><td>$R = \frac{y}{x}$
$S = -\frac{1}{x}$</td><td>$\frac{dS}{dR} = -\cot(R)\, S(R)$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\arcsin(c_1 x)\, x \tag{1}$$

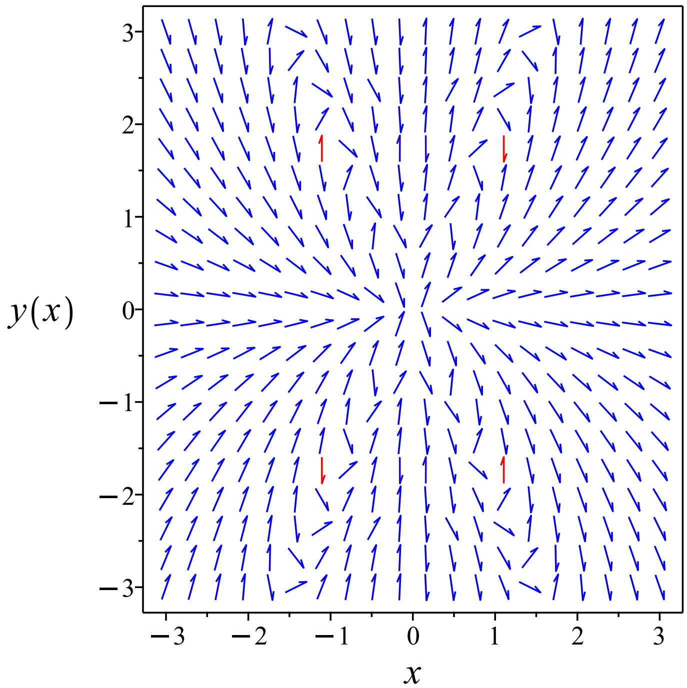


Figure 110: Slope field plot

Verification of solutions

$$y = -\arcsin(c_1 x)\, x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve((x*tan(y(x)/x)+y(x))- x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arcsin(c_1 x)\, x$$

✓ Solution by Mathematica

Time used: 8.002 (sec). Leaf size: 19

```
DSolve[(x*Tan[y[x]/x]+y[x])- x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \arcsin(e^{c_1} x)$$
$$y(x) \to 0$$

4.12 problem 12

Internal problem ID [11625]
Internal file name [`OUTPUT/10607_Saturday_May_27_2023_03_05_14_AM_78327122/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{\left(2s^2+2st+t^2\right)s'+s^2+2st=t^2}$$

4.12.1 Solving as homogeneousTypeD2 ode

Using the change of variables $s = u(t)\,t$ on the above ode results in new ode in $u(t)$

$$\left(2u(t)^2\,t^2+2u(t)\,t^2+t^2\right)\left(u'(t)\,t+u(t)\right)+u(t)^2\,t^2+2u(t)\,t^2=t^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(t,u) \\ &= f(t)g(u) \\ &= -\frac{2u^3+3u^2+3u-1}{t\,(2u^2+2u+1)} \end{aligned}$$

Where $f(t) = -\frac{1}{t}$ and $g(u) = \frac{2u^3+3u^2+3u-1}{2u^2+2u+1}$. Integrating both sides gives

$$\frac{1}{\frac{2u^3+3u^2+3u-1}{2u^2+2u+1}}\, du = -\frac{1}{t}\, dt$$

$$\int \frac{1}{\frac{2u^3+3u^2+3u-1}{2u^2+2u+1}}\, du = \int -\frac{1}{t}\, dt$$

$$\frac{\ln(2u^3+3u^2+3u-1)}{3} = -\ln(t) + c_2$$

Raising both side to exponential gives

$$(2u^3+3u^2+3u-1)^{\frac{1}{3}} = e^{-\ln(t)+c_2}$$

Which simplifies to

$$(2u^3+3u^2+3u-1)^{\frac{1}{3}} = \frac{c_3}{t}$$

Which simplifies to

$$\left(2u(t)^3+3u(t)^2+3u(t)-1\right)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{t}$$

The solution is

$$\left(2u(t)^3+3u(t)^2+3u(t)-1\right)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{t}$$

Replacing $u(t)$ in the above solution by $\frac{s}{t}$ results in the solution for s in implicit form

$$\left(\frac{2s^3}{t^3}+\frac{3s^2}{t^2}+\frac{3s}{t}-1\right)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{t}$$

$$\left(\frac{2s^3+3s^2t+3st^2-t^3}{t^3}\right)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{t}$$

Summary

The solution(s) found are the following

$$\left(\frac{2s^3+3s^2t+3st^2-t^3}{t^3}\right)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{t} \tag{1}$$

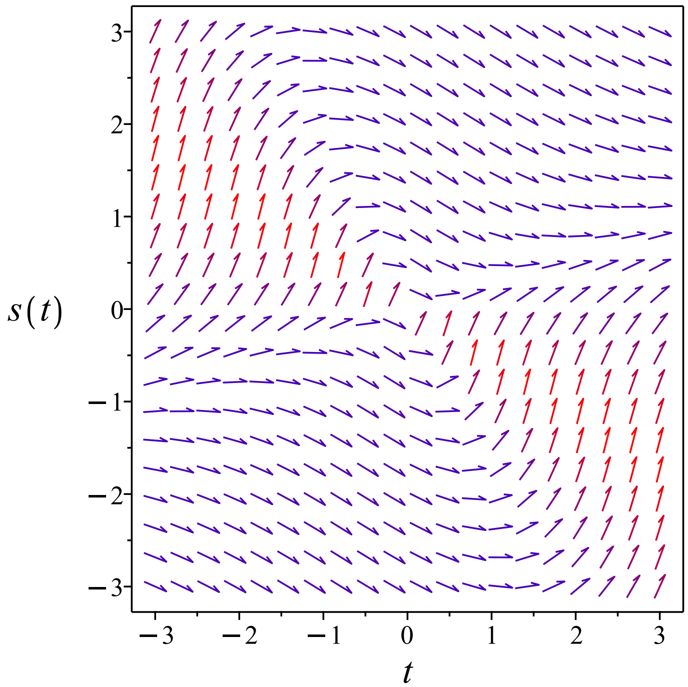


Figure 111: Slope field plot

Verification of solutions

$$\left(\frac{2s^3 + 3s^2t + 3st^2 - t^3}{t^3}\right)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{t}$$

Verified OK.

4.12.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$s' = -\frac{s^2 + 2st - t^2}{2s^2 + 2st + t^2}$$
$$s' = \omega(t, s)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_s - \xi_t) - \omega^2\xi_s - \omega_t\xi - \omega_s\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = sa_3 + ta_2 + a_1 \tag{1E}$$

$$\eta = sb_3 + tb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(s^2+2st-t^2)(b_3-a_2)}{2s^2+2st+t^2} - \frac{(s^2+2st-t^2)^2 a_3}{(2s^2+2st+t^2)^2} - \left(-\frac{2s-2t}{2s^2+2st+t^2} + \frac{(s^2+2st-t^2)(2s+2t)}{(2s^2+2st+t^2)^2}\right)(sa_3+ta_2+a_1) - \left(-\frac{2s+2t}{2s^2+2st+t^2} + \frac{(s^2+2st-t^2)(4s+2t)}{(2s^2+2st+t^2)^2}\right)(sb_3+tb_2+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2s^4a_2 + s^4a_3 + 4s^4b_2 - 2s^4b_3 + 8s^3ta_2 - 10s^3ta_3 + 8s^3tb_2 - 8s^3tb_3 - 3s^2t^2a_2 - 6s^2t^2a_3 + 6s^2t^2b_2 + 3s^2t^2b}{(}$$
$$= 0$$

Setting the numerator to zero gives

$$2s^4a_2 + s^4a_3 + 4s^4b_2 - 2s^4b_3 + 8s^3ta_2 - 10s^3ta_3 + 8s^3tb_2 - 8s^3tb_3 - 3s^2t^2a_2 - 6s^2t^2a_3 + 6s^2t^2b_2 + 3s^2t^2b_3 - 4s\,t^3a_2 + 4s\,t^3a_3 + 10s\,t^3b_2 + 4s\,t^3b_3 - t^4a_2 - t^4a_3 + 5t^4b_2 + t^4b_3 + 2s^3a_1 - 6s^2ta_1 - 2s^2tb_1 - 4s\,t^2a_1 + 6s\,t^2b_1 + 4t^3b_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{s,t\}$ in them.

$$\{s,t\}$$

The following substitution is now made to be able to collect on all terms with $\{s,t\}$ in them

$$\{s = v_1, t = v_2\}$$

The above PDE (6E) now becomes

$$2a_2v_1^4 + 8a_2v_1^3v_2 - 3a_2v_1^2v_2^2 - 4a_2v_1v_2^3 - a_2v_2^4 + a_3v_1^4 - 10a_3v_1^3v_2 - 6a_3v_1^2v_2^2 + 4a_3v_1v_2^3 - a_3v_2^4 + 4b_2v_1^4 + 8b_2v_1^3v_2 + 6b_2v_1^2v_2^2 + 10b_2v_1v_2^3 + 5b_2v_2^4 - 2b_3v_1^4 - 8b_3v_1^3v_2 + 3b_3v_1^2v_2^2 + 4b_3v_1v_2^3 + b_3v_2^4 + 2a_1v_1^3 - 6a_1v_1^2v_2 - 4a_1v_1v_2^2 - 2b_1v_1^2v_2 + 6b_1v_1v_2^2 + 4b_1v_2^3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(2a_2 + a_3 + 4b_2 - 2b_3)\, v_1^4 + (8a_2 - 10a_3 + 8b_2 - 8b_3)\, v_1^3 v_2 \\
&+ 2a_1 v_1^3 + (-3a_2 - 6a_3 + 6b_2 + 3b_3)\, v_1^2 v_2^2 \\
&+ (-6a_1 - 2b_1)\, v_1^2 v_2 + (-4a_2 + 4a_3 + 10b_2 + 4b_3)\, v_1 v_2^3 \\
&+ (-4a_1 + 6b_1)\, v_1 v_2^2 + (-a_2 - a_3 + 5b_2 + b_3)\, v_2^4 + 4b_1 v_2^3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
2a_1 &= 0 \\
4b_1 &= 0 \\
-6a_1 - 2b_1 &= 0 \\
-4a_1 + 6b_1 &= 0 \\
-4a_2 + 4a_3 + 10b_2 + 4b_3 &= 0 \\
-3a_2 - 6a_3 + 6b_2 + 3b_3 &= 0 \\
-a_2 - a_3 + 5b_2 + b_3 &= 0 \\
2a_2 + a_3 + 4b_2 - 2b_3 &= 0 \\
8a_2 - 10a_3 + 8b_2 - 8b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= t \\
\eta &= s
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(t, s)\,\xi \\
&= s - \left(-\frac{s^2 + 2st - t^2}{2s^2 + 2st + t^2}\right)(t) \\
&= \frac{2s^3 + 3s^2 t + 3s\,t^2 - t^3}{2s^2 + 2st + t^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, s) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{ds}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial s}\right) S(t, s) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2s^3 + 3s^2 t + 3s\,t^2 - t^3}{2s^2 + 2st + t^2}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln\left(2s^3 + 3s^2 t + 3s\,t^2 - t^3\right)}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t, s) S_s}{R_t + \omega(t, s) R_s} \tag{2}
$$

Where in the above R_t, R_s, S_t, S_s are all partial derivatives and $\omega(t, s)$ is the right hand side of the original ode given by

$$
\omega(t, s) = -\frac{s^2 + 2st - t^2}{2s^2 + 2st + t^2}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_s &= 0 \\ S_t &= \frac{s^2 + 2st - t^2}{2s^3 + 3s^2t + 3s\,t^2 - t^3} \\ S_s &= \frac{2s^2 + 2st + t^2}{2s^3 + 3s^2t + 3s\,t^2 - t^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, s in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

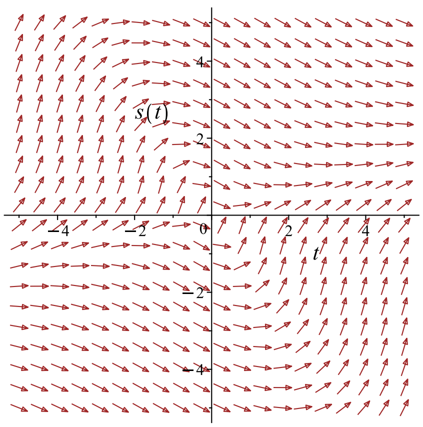
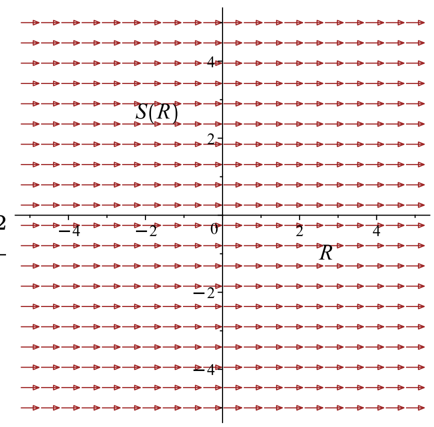
To complete the solution, we just need to transform (4) back to t, s coordinates. This results in

$$\frac{\ln\left(2s^3 + 3s^2t + 3st^2 - t^3\right)}{3} = c_1$$

Which simplifies to

$$\frac{\ln\left(2s^3 + 3s^2t + 3st^2 - t^3\right)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, s coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{ds}{dt} = -\frac{s^2+2st-t^2}{2s^2+2st+t^2}$
</td><td>$R = t$
$S = \frac{\ln(2s^3 + 3s^2t + 3s\,t^2}{3}$</td><td>$\frac{dS}{dR} = 0$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{\ln(2s^3 + 3s^2t + 3st^2 - t^3)}{3} = c_1 \tag{1}$$

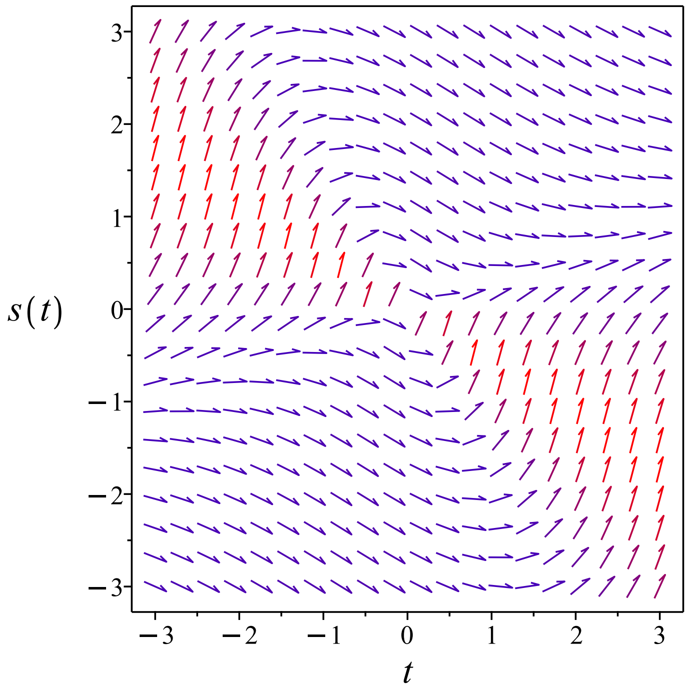


Figure 112: Slope field plot

<u>Verification of solutions</u>

$$\frac{\ln(2s^3+3s^2t+3st^2-t^3)}{3}=c_1$$

Verified OK.

4.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, s)\, \mathrm{d}t + N(t, s)\, \mathrm{d}s = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \left(2s^2 + 2st + t^2\right) \mathrm{d}s &= \left(-s^2 - 2st + t^2\right) \mathrm{d}t \\ \left(s^2 + 2st - t^2\right) \mathrm{d}t + \left(2s^2 + 2st + t^2\right) \mathrm{d}s &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t, s) &= s^2 + 2st - t^2 \\ N(t, s) &= 2s^2 + 2st + t^2 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial s} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial s} &= \frac{\partial}{\partial s}\left(s^2 + 2st - t^2\right) \\ &= 2s + 2t \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(2s^2 + 2st + t^2\right) \\ &= 2s + 2t \end{aligned}$$

Since $\frac{\partial M}{\partial s} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,s)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial s} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int M\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int s^2 + 2st - t^2\,\mathrm{d}t$$
$$\phi = s^2 t + s\,t^2 - \frac{1}{3}t^3 + f(s) \tag{3}$$

Where $f(s)$ is used for the constant of integration since ϕ is a function of both t and s. Taking derivative of equation (3) w.r.t s gives

$$\frac{\partial \phi}{\partial s} = 2st + t^2 + f'(s) \tag{4}$$
$$= t(2s+t) + f'(s)$$

But equation (2) says that $\frac{\partial \phi}{\partial s} = 2s^2 + 2st + t^2$. Therefore equation (4) becomes

$$2s^2 + 2st + t^2 = t(2s+t) + f'(s) \tag{5}$$

Solving equation (5) for $f'(s)$ gives

$$f'(s) = 2s^2$$

Integrating the above w.r.t s gives

$$\int f'(s)\,\mathrm{d}s = \int (2s^2)\,\mathrm{d}s$$
$$f(s) = \frac{2s^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(s)$ into equation (3) gives ϕ

$$\phi = s^2 t + s\,t^2 - \frac{1}{3}t^3 + \frac{2}{3}s^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = s^2 t + s\,t^2 - \frac{1}{3}t^3 + \frac{2}{3}s^3$$

<u>Summary</u>

The solution(s) found are the following

$$s^2 t + s t^2 - \frac{t^3}{3} + \frac{2 s^3}{3} = c_1 \tag{1}$$

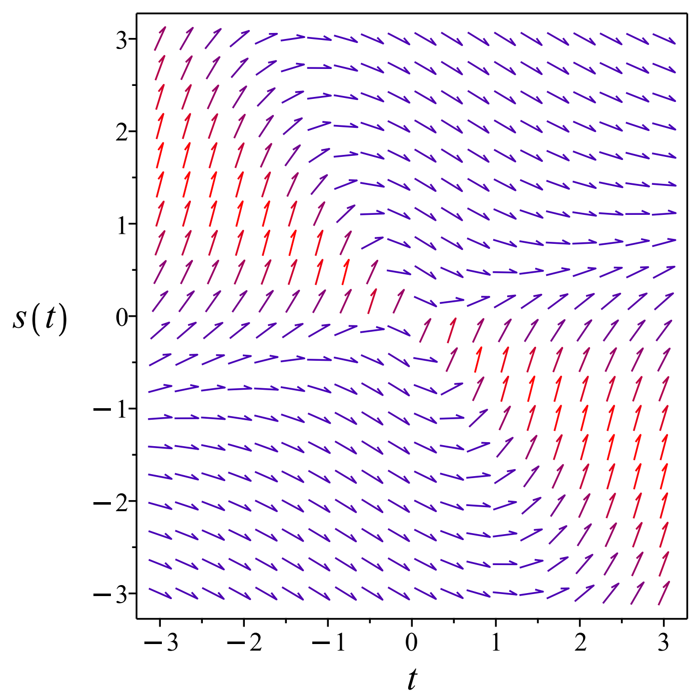


Figure 113: Slope field plot

<u>Verification of solutions</u>

$$s^2 t + s t^2 - \frac{t^3}{3} + \frac{2 s^3}{3} = c_1$$

Verified OK.

4.12.4 Maple step by step solution

Let's solve

$(2s^2 + 2st + t^2)\, s' + s^2 + 2st = t^2$

- Highest derivative means the order of the ODE is 1

  s'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(t, s) = 0$
  - Compute derivative of lhs

    $F'(t, s) + \left(\frac{\partial}{\partial s} F(t, s)\right) s' = 0$
  - Evaluate derivatives

    $2s + 2t = 2s + 2t$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(t, s) = c_1, M(t, s) = F'(t, s), N(t, s) = \frac{\partial}{\partial s} F(t, s)\right]$
- Solve for $F(t, s)$ by integrating $M(t, s)$ with respect to t

  $F(t, s) = \int (s^2 + 2st - t^2)\, dt + f_1(s)$
- Evaluate integral

  $F(t, s) = s^2 t + s\, t^2 - \frac{t^3}{3} + f_1(s)$
- Take derivative of $F(t, s)$ with respect to s

  $N(t, s) = \frac{\partial}{\partial s} F(t, s)$
- Compute derivative

  $2s^2 + 2st + t^2 = 2st + t^2 + \frac{d}{ds} f_1(s)$
- Isolate for $\frac{d}{ds} f_1(s)$

  $\frac{d}{ds} f_1(s) = 2s^2$
- Solve for $f_1(s)$

  $f_1(s) = \frac{2s^3}{3}$
- Substitute $f_1(s)$ into equation for $F(t, s)$

  $F(t, s) = s^2 t + s\, t^2 - \frac{1}{3} t^3 + \frac{2}{3} s^3$

- Substitute $F(t,s)$ into the solution of the ODE

  $s^2t + s\,t^2 - \frac{1}{3}t^3 + \frac{2}{3}s^3 = c_1$

- Solve for s

$$\left\{s = \frac{\left(4t^3+6c_1+\sqrt{17t^6+48t^3c_1+36c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{t^2}{2\left(4t^3+6c_1+\sqrt{17t^6+48t^3c_1+36c_1^2}\right)^{\frac{1}{3}}} - \frac{t}{2}, s = -\frac{\left(4t^3+6c_1+\sqrt{17t^6+48t^3c_1+\right.}{4}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 348

```
dsolve((2*s(t)^2+2*s(t)*t+t^2)*diff(s(t),t)+(s(t)^2+2*s(t)*t-t^2)=0,s(t), singsol=all)
```

$$s(t) = \frac{\left(4t^3c_1^3+2+\sqrt{17c_1^6t^6+16t^3c_1^3+4}\right)^{\frac{1}{3}} - \frac{t^2c_1^2}{\left(4t^3c_1^3+2+\sqrt{17c_1^6t^6+16t^3c_1^3+4}\right)^{\frac{1}{3}}} - c_1t}{2c_1}$$

$$s(t) = -\frac{\left(1+i\sqrt{3}\right)\left(4t^3c_1^3+2+\sqrt{17c_1^6t^6+16t^3c_1^3+4}\right)^{\frac{2}{3}} + c_1t\left(2\left(4t^3c_1^3+2+\sqrt{17c_1^6t^6+16t^3c_1^3+4}\right)^{\frac{1}{3}} + (i\sqrt{}\right.}{4\left(4t^3c_1^3+2+\sqrt{17c_1^6t^6+16t^3c_1^3+4}\right)^{\frac{1}{3}}c_1}$$

$$s(t) = \frac{\left(i\sqrt{3}-1\right)\left(4t^3c_1^3+2+\sqrt{17c_1^6t^6+16t^3c_1^3+4}\right)^{\frac{2}{3}} + \left(-2\left(4t^3c_1^3+2+\sqrt{17c_1^6t^6+16t^3c_1^3+4}\right)^{\frac{1}{3}} + c_1t(1-\right.}{4\left(4t^3c_1^3+2+\sqrt{17c_1^6t^6+16t^3c_1^3+4}\right)^{\frac{1}{3}}c_1}$$

✓ Solution by Mathematica

Time used: 48.03 (sec). Leaf size: 616

```
DSolve[(2*s[t]^2+2*s[t]*t+t^2)*s'[t]+(s[t]^2+2*s[t]*t-t^2)==0,s[t],t,IncludeSingularSolution
```

$$s(t) \to \frac{1}{2}\left(\sqrt[3]{4t^3+\sqrt{17t^6+16e^{3c_1}t^3+4e^{6c_1}}+2e^{3c_1}} - \frac{t^2}{\sqrt[3]{4t^3+\sqrt{17t^6+16e^{3c_1}t^3+4e^{6c_1}}+2e^{3c_1}}} - t\right)$$

$$s(t) \to \frac{1}{8}\left(2i\left(\sqrt{3}+i\right)\sqrt[3]{4t^3+\sqrt{17t^6+16e^{3c_1}t^3+4e^{6c_1}}+2e^{3c_1}} + \frac{2\left(1+i\sqrt{3}\right)t^2}{\sqrt[3]{4t^3+\sqrt{17t^6+16e^{3c_1}t^3+4e^{6c_1}}+2e^{3c_1}}} - 4t\right)$$

$$s(t) \to \frac{1}{8}\left(-2\left(1+i\sqrt{3}\right)\sqrt[3]{4t^3+\sqrt{17t^6+16e^{3c_1}t^3+4e^{6c_1}}+2e^{3c_1}} + \frac{2\left(1-i\sqrt{3}\right)t^2}{\sqrt[3]{4t^3+\sqrt{17t^6+16e^{3c_1}t^3+4e^{6c_1}}+2e^{3c_1}}} - 4t\right)$$

$$s(t) \to \frac{1}{2}\left(\sqrt[3]{\sqrt{17}\sqrt{t^6}+4t^3} - \frac{t^2}{\sqrt[3]{\sqrt{17}\sqrt{t^6}+4t^3}} - t\right)$$

$$s(t) \to \frac{1}{4}\left(\left(-1-i\sqrt{3}\right)\sqrt[3]{\sqrt{17}\sqrt{t^6}+4t^3} + \frac{\left(1-i\sqrt{3}\right)t^2}{\sqrt[3]{\sqrt{17}\sqrt{t^6}+4t^3}} - 2t\right)$$

$$s(t) \to \frac{1}{4}\left(i\left(\sqrt{3}+i\right)\sqrt[3]{\sqrt{17}\sqrt{t^6}+4t^3} + \frac{\left(1+i\sqrt{3}\right)t^2}{\sqrt[3]{\sqrt{17}\sqrt{t^6}+4t^3}} - 2t\right)$$

4.13 problem 13

Internal problem ID [11626]
Internal file name [`OUTPUT/10608_Saturday_May_27_2023_03_05_22_AM_44852541/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y^2\sqrt{x^2+y^2}-xy\sqrt{x^2+y^2}\,y'=-x^3}$$

4.13.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x^3+y^2\sqrt{x^2+y^2}}{xy\sqrt{x^2+y^2}}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$
$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{\left(x^3 + y^2\sqrt{x^2+y^2}\right)(b_3 - a_2)}{xy\sqrt{x^2+y^2}} - \frac{\left(x^3 + y^2\sqrt{x^2+y^2}\right)^2 a_3}{x^2y^2(x^2+y^2)} - \left(\frac{3x^2 + \frac{y^2x}{\sqrt{x^2+y^2}}}{xy\sqrt{x^2+y^2}} - \frac{x^3 + y^2\sqrt{x^2+y^2}}{x^2y\sqrt{x^2+y^2}} - \frac{x^3 + y^2\sqrt{x^2+y^2}}{y(x^2+y^2)^{\frac{3}{2}}}\right)(xa_2+ya_3+a_1) - \left(\frac{2y\sqrt{x^2+y^2} + \frac{y^3}{\sqrt{x^2+y^2}}}{xy\sqrt{x^2+y^2}} - \frac{x^3 + y^2\sqrt{x^2+y^2}}{x\,y^2\sqrt{x^2+y^2}} - \frac{x^3 + y^2\sqrt{x^2+y^2}}{x(x^2+y^2)^{\frac{3}{2}}}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(x^2+y^2)^{\frac{3}{2}} x^6 a_3 - (x^2+y^2)^{\frac{3}{2}} y^6 a_3 - (x^2+y^2)^{\frac{3}{2}} y^5 a_1 - 2x^8 y b_3 + 3x^4y^5a_2 + 4x^3y^6a_3 + 2x^3y^5a_1 - x^9b_2 -}{}$$

$$= 0$$

Setting the numerator to zero gives

$$-\left(x^2+y^2\right)^{\frac{3}{2}} x^6 a_3 + \left(x^2+y^2\right)^{\frac{3}{2}} y^6 a_3 + \left(x^2+y^2\right)^{\frac{3}{2}} y^5 a_1 + 2x^8 y b_3 - 3x^4y^5a_2 - 4x^3y^6a_3 - 2x^3y^5a_1 + x^9b_2 + x^8b_1 + \left(x^2+y^2\right)^{\frac{3}{2}} x^2y^4a_3 - \left(x^2+y^2\right)^{\frac{3}{2}} x^3y^2b_1 + \left(x^2+y^2\right)^{\frac{3}{2}} x^2y^3a_1 - \left(x^2+y^2\right)^{\frac{3}{2}} x\,y^4b_1 - 2x^8ya_2 - 3x^7y^2a_3 + 3x^7y^2b_2 - 5x^6y^3a_2 + 5x^6y^3b_3 - 7x^5y^4a_3 + 2x^5y^4b_2 + 3x^4y^5b_3 - x^7ya_1 + 3x^6y^2b_1 - 3x^5y^3a_1 + 2x^4y^4b_1 - \left(x^2+y^2\right)^{\frac{5}{2}} y^4a_3 = 0 \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&-\left(x^{2}+y^{2}\right)^{\frac{3}{2}} x^{6} a_{3}+\left(x^{2}+y^{2}\right)^{\frac{3}{2}} y^{6} a_{3}+\left(x^{2}+y^{2}\right) x^{7} b_{2} \\
&+\left(x^{2}+y^{2}\right)^{\frac{3}{2}} y^{5} a_{1}+\left(x^{2}+y^{2}\right) x^{6} b_{1}-2\left(x^{2}+y^{2}\right)^{2} x^{3} y^{2} a_{3} \\
&+\left(x^{2}+y^{2}\right)^{\frac{3}{2}} x^{2} y^{4} a_{3}-3\left(x^{2}+y^{2}\right) x^{6} y a_{2}+2\left(x^{2}+y^{2}\right) x^{6} y b_{3} \\
&-2\left(x^{2}+y^{2}\right) x^{5} y^{2} a_{3}+\left(x^{2}+y^{2}\right) x^{5} y^{2} b_{2}-3\left(x^{2}+y^{2}\right) x^{4} y^{3} a_{2} \\
&+2\left(x^{2}+y^{2}\right) x^{4} y^{3} b_{3}-2\left(x^{2}+y^{2}\right) x^{3} y^{4} a_{3}-\left(x^{2}+y^{2}\right)^{\frac{3}{2}} x^{3} y^{2} b_{1} \\
&+\left(x^{2}+y^{2}\right)^{\frac{3}{2}} x^{2} y^{3} a_{1}-\left(x^{2}+y^{2}\right)^{\frac{3}{2}} x y^{4} b_{1}-2\left(x^{2}+y^{2}\right) x^{5} y a_{1} \\
&+\left(x^{2}+y^{2}\right) x^{4} y^{2} b_{1}-2\left(x^{2}+y^{2}\right) x^{3} y^{3} a_{1}+x^{8} y a_{2}+x^{7} y^{2} a_{3} \\
&+x^{7} y^{2} b_{2}+x^{6} y^{3} a_{2}+x^{6} y^{3} b_{3}+x^{5} y^{4} a_{3}+x^{5} y^{4} b_{2}+x^{4} y^{5} b_{3} \\
&+x^{7} y a_{1}+x^{6} y^{2} b_{1}+x^{5} y^{3} a_{1}+x^{4} y^{4} b_{1}-\left(x^{2}+y^{2}\right)^{\frac{5}{2}} y^{4} a_{3}=0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-x^{8} \sqrt{x^{2}+y^{2}}\, a_{3}+\sqrt{x^{2}+y^{2}}\, y^{7} a_{1}-2 x^{3} \sqrt{x^{2}+y^{2}}\, y^{4} b_{1}+2 x^{2} \sqrt{x^{2}+y^{2}}\, y^{5} a_{1} \\
&-x \sqrt{x^{2}+y^{2}}\, y^{6} b_{1}+2 x^{8} y b_{3}-3 x^{4} y^{5} a_{2}-4 x^{3} y^{6} a_{3}-2 x^{3} y^{5} a_{1} \\
&-x^{6} \sqrt{x^{2}+y^{2}}\, y^{2} a_{3}-x^{5} \sqrt{x^{2}+y^{2}}\, y^{2} b_{1}+x^{4} \sqrt{x^{2}+y^{2}}\, y^{3} a_{1}+x^{9} b_{2} \\
&+x^{8} b_{1}-2 x^{8} y a_{2}-3 x^{7} y^{2} a_{3}+3 x^{7} y^{2} b_{2}-5 x^{6} y^{3} a_{2}+5 x^{6} y^{3} b_{3}-7 x^{5} y^{4} a_{3} \\
&+2 x^{5} y^{4} b_{2}+3 x^{4} y^{5} b_{3}-x^{7} y a_{1}+3 x^{6} y^{2} b_{1}-3 x^{5} y^{3} a_{1}+2 x^{4} y^{4} b_{1}=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^{2}+y^{2}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_{1}, y=v_{2}, \sqrt{x^{2}+y^{2}}=v_{3}\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2 v_{1}^{8} v_{2} a_{2}-5 v_{1}^{6} v_{2}^{3} a_{2}-3 v_{1}^{4} v_{2}^{5} a_{2}-v_{1}^{8} v_{3} a_{3}-3 v_{1}^{7} v_{2}^{2} a_{3}-v_{1}^{6} v_{3} v_{2}^{2} a_{3}-7 v_{1}^{5} v_{2}^{4} a_{3} \\
&-4 v_{1}^{3} v_{2}^{6} a_{3}+v_{1}^{9} b_{2}+3 v_{1}^{7} v_{2}^{2} b_{2}+2 v_{1}^{5} v_{2}^{4} b_{2}+2 v_{1}^{8} v_{2} b_{3}+5 v_{1}^{6} v_{2}^{3} b_{3}+3 v_{1}^{4} v_{2}^{5} b_{3} \\
&-v_{1}^{7} v_{2} a_{1}-3 v_{1}^{5} v_{2}^{3} a_{1}+v_{1}^{4} v_{3} v_{2}^{3} a_{1}-2 v_{1}^{3} v_{2}^{5} a_{1}+2 v_{1}^{2} v_{3} v_{2}^{5} a_{1}+v_{3} v_{2}^{7} a_{1} \\
&+v_{1}^{8} b_{1}+3 v_{1}^{6} v_{2}^{2} b_{1}-v_{1}^{5} v_{3} v_{2}^{2} b_{1}+2 v_{1}^{4} v_{2}^{4} b_{1}-2 v_{1}^{3} v_{3} v_{2}^{4} b_{1}-v_{1} v_{3} v_{2}^{6} b_{1}=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&v_1^9 b_2 + (-2a_2 + 2b_3)\, v_1^8 v_2 - v_1^8 v_3 a_3 + v_1^8 b_1 + (-3a_3 + 3b_2)\, v_1^7 v_2^2 - v_1^7 v_2 a_1 \\
&+ (-5a_2 + 5b_3)\, v_1^6 v_2^3 - v_1^6 v_3 v_2^2 a_3 + 3v_1^6 v_2^2 b_1 + (-7a_3 + 2b_2)\, v_1^5 v_2^4 \\
&- 3v_1^5 v_2^3 a_1 - v_1^5 v_3 v_2^2 b_1 + (-3a_2 + 3b_3)\, v_1^4 v_2^5 + 2v_1^4 v_2^4 b_1 + v_1^4 v_3 v_2^3 a_1 \\
&- 4v_1^3 v_2^6 a_3 - 2v_1^3 v_2^5 a_1 - 2v_1^3 v_3 v_2^4 b_1 + 2v_1^2 v_3 v_2^5 a_1 - v_1 v_3 v_2^6 b_1 + v_3 v_2^7 a_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_1 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
-3a_1 &= 0 \\
-2a_1 &= 0 \\
-a_1 &= 0 \\
2a_1 &= 0 \\
-4a_3 &= 0 \\
-a_3 &= 0 \\
-2b_1 &= 0 \\
-b_1 &= 0 \\
2b_1 &= 0 \\
3b_1 &= 0 \\
-5a_2 + 5b_3 &= 0 \\
-3a_2 + 3b_3 &= 0 \\
-2a_2 + 2b_3 &= 0 \\
-7a_3 + 2b_2 &= 0 \\
-3a_3 + 3b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{x^3 + y^2\sqrt{x^2 + y^2}}{xy\sqrt{x^2 + y^2}}\right)(x) \\
&= -\frac{x^3}{y\sqrt{x^2 + y^2}} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{x^3}{y\sqrt{x^2+y^2}}} dy
\end{aligned}
$$

Which results in

$$S = -\frac{(x^2+y^2)^{\frac{3}{2}}}{3x^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^3 + y^2\sqrt{x^2+y^2}}{xy\sqrt{x^2+y^2}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{\sqrt{x^2+y^2}\,y^2}{x^4}\\
S_y &= -\frac{y\sqrt{x^2+y^2}}{x^3}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

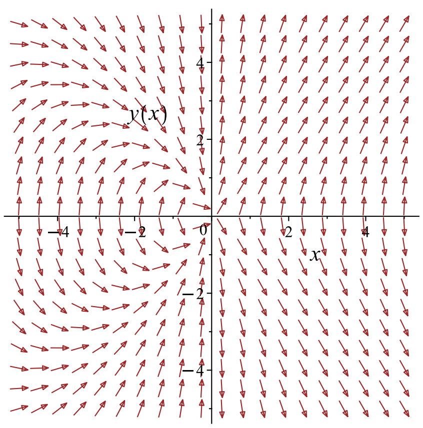
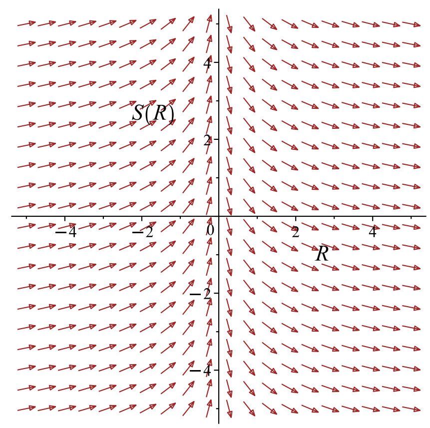
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{(x^2+y^2)^{\frac{3}{2}}}{3x^3} = -\ln(x) + c_1$$

Which simplifies to

$$-\frac{\left(x^{2}+y^{2}\right)^{\frac{3}{2}}}{3 x^{3}} = -\ln \left(x \right)+c_{1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^{3}+y^{2}\sqrt{x^{2}+y^{2}}}{x y \sqrt{x^{2}+y^{2}}}$ 	$R = x$ $S = -\frac{\left(x^{2}+y^{2}\right)^{\frac{3}{2}}}{3 x^{3}}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$-\frac{\left(x^{2}+y^{2}\right)^{\frac{3}{2}}}{3 x^{3}} = -\ln \left(x \right)+c_{1} \tag{1}$$

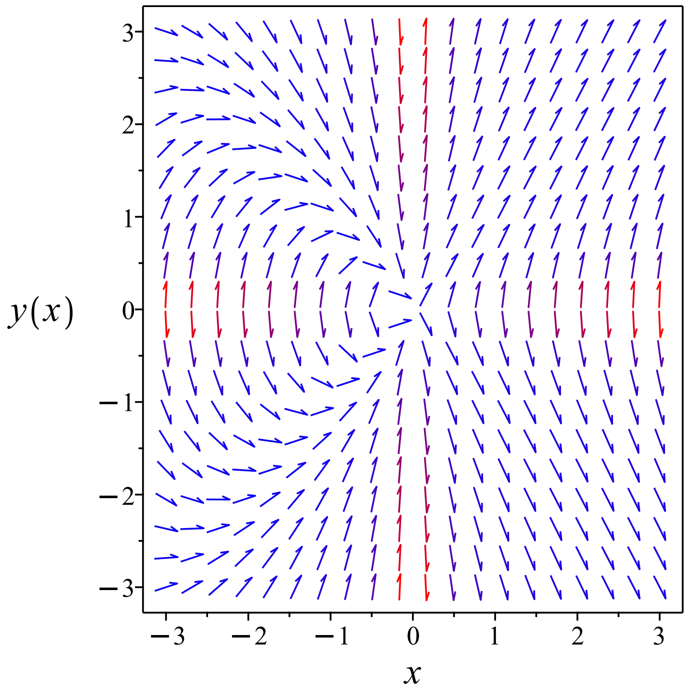


Figure 114: Slope field plot

<u>Verification of solutions</u>

$$-\frac{(x^2+y^2)^{\frac{3}{2}}}{3x^3} = -\ln(x) + c_1$$

Verified OK.

4.13.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-xy\sqrt{x^2+y^2}\right)\mathrm{d}y = \left(-x^3 - y^2\sqrt{x^2+y^2}\right)\mathrm{d}x$$
$$\left(x^3 + y^2\sqrt{x^2+y^2}\right)\mathrm{d}x + \left(-xy\sqrt{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = x^3 + y^2\sqrt{x^2+y^2}$$
$$N(x, y) = -xy\sqrt{x^2+y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^3 + y^2\sqrt{x^2+y^2}\right) \\ &= \frac{2x^2y + 3y^3}{\sqrt{x^2+y^2}}\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-xy\sqrt{x^2+y^2}\right)$$
$$= \frac{(-2x^2-y^2)\,y}{\sqrt{x^2+y^2}}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= -\frac{1}{xy\sqrt{x^2+y^2}}\left(\left(2y\sqrt{x^2+y^2}+\frac{y^3}{\sqrt{x^2+y^2}}\right)-\left(-y\sqrt{x^2+y^2}-\frac{x^2y}{\sqrt{x^2+y^2}}\right)\right) \\ &= -\frac{4}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int -\frac{4}{x}\,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-4\ln(x)} \\ &= \frac{1}{x^4} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^4}\left(x^3+y^2\sqrt{x^2+y^2}\right) \\ &= \frac{x^3+y^2\sqrt{x^2+y^2}}{x^4} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^4}\left(-xy\sqrt{x^2+y^2}\right) \\ &= -\frac{y\sqrt{x^2+y^2}}{x^3} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{x^3 + y^2\sqrt{x^2+y^2}}{x^4}\right) + \left(-\frac{y\sqrt{x^2+y^2}}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{x^3 + y^2\sqrt{x^2+y^2}}{x^4}\,\mathrm{d}x$$
$$\phi = \frac{3\ln(x)\,x^3 - x^2\sqrt{x^2+y^2} - y^2\sqrt{x^2+y^2}}{3x^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{-\frac{x^2 y}{\sqrt{x^2+y^2}} - 2y\sqrt{x^2+y^2} - \frac{y^3}{\sqrt{x^2+y^2}}}{3x^3} + f'(y) \tag{4}$$
$$= -\frac{y\sqrt{x^2+y^2}}{x^3} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y\sqrt{x^2+y^2}}{x^3}$. Therefore equation (4) becomes

$$-\frac{y\sqrt{x^2+y^2}}{x^3} = -\frac{y\sqrt{x^2+y^2}}{x^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3\ln(x)\,x^3 - x^2\sqrt{x^2+y^2} - y^2\sqrt{x^2+y^2}}{3x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3\ln(x)\,x^3 - x^2\sqrt{x^2+y^2} - y^2\sqrt{x^2+y^2}}{3x^3}$$

Summary

The solution(s) found are the following

$$\frac{3\ln(x)\,x^3 - x^2\sqrt{x^2+y^2} - y^2\sqrt{x^2+y^2}}{3x^3} = c_1 \tag{1}$$

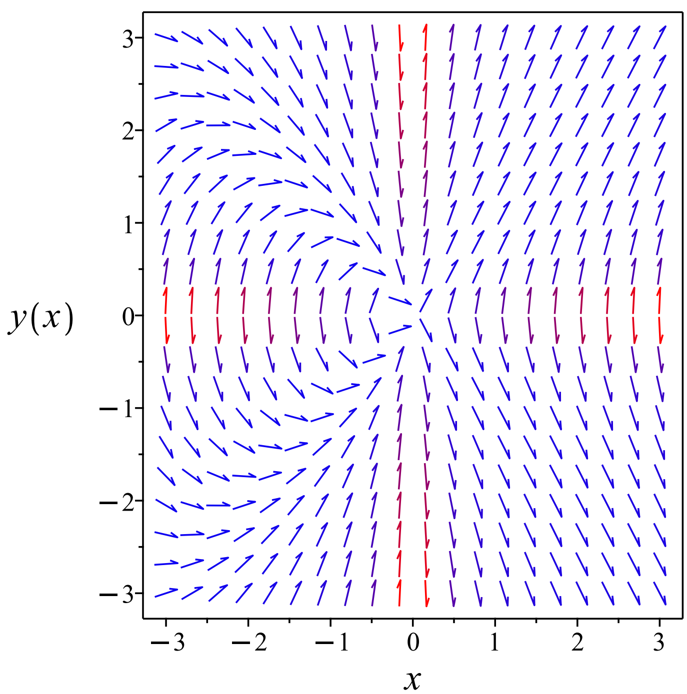


Figure 115: Slope field plot

Verification of solutions

$$\frac{3\ln(x)\, x^3 - x^2\sqrt{x^2+y^2} - y^2\sqrt{x^2+y^2}}{3x^3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve((x^3+y(x)^2*sqrt(x^2+y(x)^2))-x*y(x)*sqrt(x^2+y(x)^2)*diff(y(x),x)=0,y(x), singsol=al
```

$$\frac{\left(-y(x)^2 - x^2\right)\sqrt{y(x)^2 + x^2} - x^3(c_1 - 3\ln(x))}{x^3} = 0$$

✓ Solution by Mathematica

Time used: 28.664 (sec). Leaf size: 265

```
DSolve[(x^3+y[x]^2*Sqrt[x^2+y[x]^2])-x*y[x]*Sqrt[x^2+y[x]^2]*y'[x]==0,y[x],x,IncludeSingular
```

$$y(x) \to -\sqrt{-x^2 - \frac{1}{2}\sqrt[6]{3}\left(\sqrt{3}+3i\right)\sqrt[3]{x^6(\log(x)+c_1)^2}}$$

$$y(x) \to \sqrt{-x^2 - \frac{1}{2}\sqrt[6]{3}\left(\sqrt{3}+3i\right)\sqrt[3]{x^6(\log(x)+c_1)^2}}$$

$$y(x) \to -\sqrt{-x^2 - \frac{1}{2}\sqrt[6]{3}\left(\sqrt{3}-3i\right)\sqrt[3]{x^6(\log(x)+c_1)^2}}$$

$$y(x) \to \sqrt{-x^2 - \frac{1}{2}\sqrt[6]{3}\left(\sqrt{3}-3i\right)\sqrt[3]{x^6(\log(x)+c_1)^2}}$$

$$y(x) \to -\sqrt{-x^2 + 3^{2/3}\sqrt[3]{x^6(\log(x)+c_1)^2}}$$

$$y(x) \to \sqrt{-x^2 + 3^{2/3}\sqrt[3]{x^6(\log(x)+c_1)^2}}$$

4.14 problem 14

Internal problem ID [11627]
Internal file name [`OUTPUT/10609_Saturday_May_27_2023_03_05_26_AM_83606361/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{\sqrt{y+x}+\sqrt{-y+x}+\left(\sqrt{-y+x}-\sqrt{y+x}\right)y'=0}$$

4.14.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{\sqrt{y+x}+\sqrt{-y+x}}{\sqrt{-y+x}-\sqrt{y+x}} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{(\sqrt{y+x}+\sqrt{-y+x})\,(b_3-a_2)}{\sqrt{-y+x}-\sqrt{y+x}} \\
&- \frac{(\sqrt{y+x}+\sqrt{-y+x})^2\,a_3}{(\sqrt{-y+x}-\sqrt{y+x})^2} - \left(-\frac{\frac{1}{2\sqrt{y+x}}+\frac{1}{2\sqrt{-y+x}}}{\sqrt{-y+x}-\sqrt{y+x}}\right. \\
&\left.+ \frac{(\sqrt{y+x}+\sqrt{-y+x})\left(\frac{1}{2\sqrt{-y+x}}-\frac{1}{2\sqrt{y+x}}\right)}{(\sqrt{-y+x}-\sqrt{y+x})^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{\frac{1}{2\sqrt{y+x}}-\frac{1}{2\sqrt{-y+x}}}{\sqrt{-y+x}-\sqrt{y+x}}\right. \\
&\left.+ \frac{(\sqrt{y+x}+\sqrt{-y+x})\left(-\frac{1}{2\sqrt{-y+x}}-\frac{1}{2\sqrt{y+x}}\right)}{(\sqrt{-y+x}-\sqrt{y+x})^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(y+x)^{\frac{3}{2}}\sqrt{-y+x}\,a_2+(y+x)^{\frac{3}{2}}\sqrt{-y+x}\,a_3-(y+x)^{\frac{3}{2}}\sqrt{-y+x}\,b_2-(y+x)^{\frac{3}{2}}\sqrt{-y+x}\,b_3-\sqrt{y+x}\,(}{(-\sqrt{-}}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-(y+x)^{\frac{3}{2}}\sqrt{-y+x}\,a_2-(y+x)^{\frac{3}{2}}\sqrt{-y+x}\,a_3+(y+x)^{\frac{3}{2}}\sqrt{-y+x}\,b_2 \\
&+(y+x)^{\frac{3}{2}}\sqrt{-y+x}\,b_3+\sqrt{y+x}\,(-y+x)^{\frac{3}{2}}\,a_2-\sqrt{y+x}\,(-y+x)^{\frac{3}{2}}\,a_3 \\
&+\sqrt{y+x}\,(-y+x)^{\frac{3}{2}}\,b_2-\sqrt{y+x}\,(-y+x)^{\frac{3}{2}}\,b_3-2x^2a_3 \\
&-2xa_2y+2yb_3x+2y^2b_2+2b_1x-2a_1y = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&-(y+x)^{\frac{3}{2}} \sqrt{-y+x}\, a_2 - (y+x)^{\frac{3}{2}} \sqrt{-y+x}\, a_3 + (y+x)^{\frac{3}{2}} \sqrt{-y+x}\, b_2 \\
&+ (y+x)^{\frac{3}{2}} \sqrt{-y+x}\, b_3 - 2(y+x)(-y+x)\, a_3 - 2(y+x)(-y+x)\, b_2 \\
&+ \sqrt{y+x}\,(-y+x)^{\frac{3}{2}}\, a_2 - \sqrt{y+x}\,(-y+x)^{\frac{3}{2}}\, a_3 + \sqrt{y+x}\,(-y+x)^{\frac{3}{2}}\, b_2 \\
&- \sqrt{y+x}\,(-y+x)^{\frac{3}{2}}\, b_3 - (y+x)\, x a_2 + (y+x)\, x b_2 - (y+x)\, y a_3 \\
&+ (y+x)\, y b_3 + (-y+x)\, x a_2 + (-y+x)\, x b_2 + (-y+x)\, y a_3 \\
&+ (-y+x)\, y b_3 - (y+x)\, a_1 + (y+x)\, b_1 + (-y+x)\, a_1 + (-y+x)\, b_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-2\sqrt{y+x}\,\sqrt{-y+x}\, a_3 x + 2\sqrt{y+x}\,\sqrt{-y+x}\, b_2 x - 2\sqrt{y+x}\,\sqrt{-y+x}\, a_2 y \\
&+ 2\sqrt{y+x}\,\sqrt{-y+x}\, b_3 y - 2x^2 a_3 - 2x a_2 y + 2y b_3 x + 2y^2 b_2 + 2b_1 x - 2a_1 y = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y, \sqrt{-y+x}, \sqrt{y+x}\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2, \sqrt{-y+x} = v_3, \sqrt{y+x} = v_4\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2v_4 v_3 a_2 v_2 - 2v_4 v_3 a_3 v_1 + 2v_4 v_3 b_2 v_1 + 2v_4 v_3 b_3 v_2 - 2v_1 a_2 v_2 \\
&- 2v_1^2 a_3 + 2v_2^2 b_2 + 2v_2 b_3 v_1 - 2a_1 v_2 + 2b_1 v_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-2v_1^2 a_3 + (-2a_2 + 2b_3)\, v_1 v_2 + (-2a_3 + 2b_2)\, v_1 v_3 v_4 \\
&+ 2b_1 v_1 + 2v_2^2 b_2 + (-2a_2 + 2b_3)\, v_2 v_3 v_4 - 2a_1 v_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-2a_1 &= 0\\
-2a_3 &= 0\\
2b_1 &= 0\\
2b_2 &= 0\\
-2a_2 + 2b_3 &= 0\\
-2a_3 + 2b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x\\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi}\\
&= \frac{y}{x}\\
&= \frac{y}{x}
\end{aligned}
$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \ln(x) \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sqrt{y+x} + \sqrt{-y+x}}{\sqrt{-y+x} - \sqrt{y+x}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{y}{x^2} \\ R_y &= \frac{1}{x} \\ S_x &= \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left(-\sqrt{-y+x} + \sqrt{y+x}\right)x}{(y+x)\sqrt{-y+x} + \sqrt{y+x}\,(-y+x)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{-\sqrt{-1+R} - i\sqrt{1+R}}{(1+R)\sqrt{-1+R} + i\sqrt{1+R}\,(-1+R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{i\sqrt{-1+R}\,\sqrt{1+R}\,\arctan\left(\frac{1}{\sqrt{R^2-1}}\right)}{\sqrt{R^2-1}} - \ln(R) + c_1 \tag{4}$$

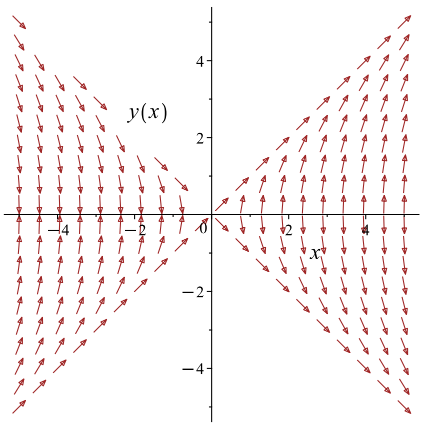
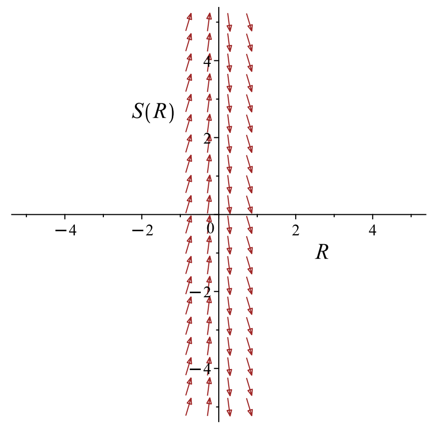
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = \frac{i\sqrt{\frac{y}{x}-1}\,\sqrt{\frac{y}{x}+1}\,\arctan\left(\frac{1}{\sqrt{\frac{y^2}{x^2}-1}}\right)}{\sqrt{\frac{y^2}{x^2}-1}} - \ln\left(\frac{y}{x}\right) + c_1$$

Which simplifies to

$$\ln(x) = \frac{i\sqrt{\frac{y}{x}-1}\,\sqrt{\frac{y}{x}+1}\,\arctan\left(\frac{1}{\sqrt{\frac{y^2}{x^2}-1}}\right)}{\sqrt{\frac{y^2}{x^2}-1}} - \ln\left(\frac{y}{x}\right) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sqrt{y+x}+\sqrt{-y+x}}{\sqrt{-y+x}-\sqrt{y+x}}$ 	$R = \frac{y}{x}$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{-\sqrt{-1+R}-i\sqrt{1+R}}{(1+R)\sqrt{-1+R}+i\sqrt{1+R}\,(-1+R)}$ 

Summary

The solution(s) found are the following

$$\ln(x) = \frac{i\sqrt{\frac{y}{x}-1}\,\sqrt{\frac{y}{x}+1}\,\arctan\left(\frac{1}{\sqrt{\frac{y^2}{x^2}-1}}\right)}{\sqrt{\frac{y^2}{x^2}-1}} - \ln\left(\frac{y}{x}\right) + c_1 \tag{1}$$

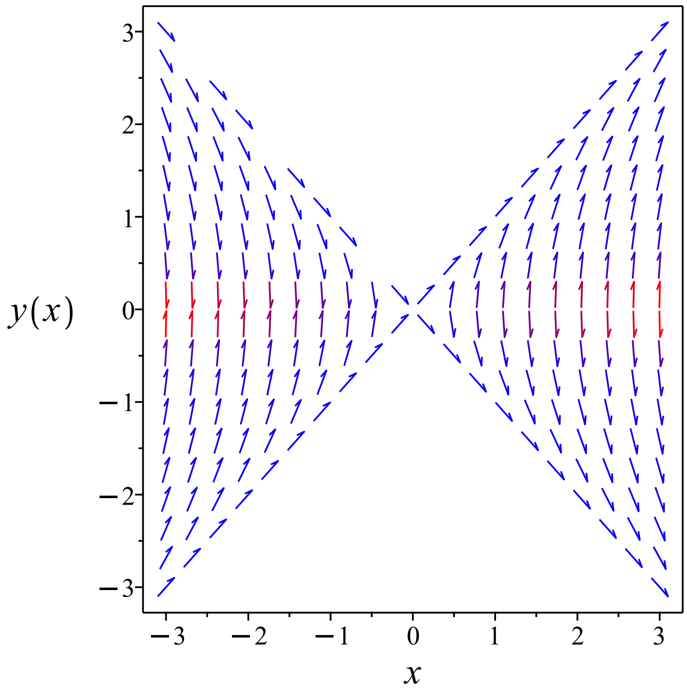


Figure 116: Slope field plot

Verification of solutions

$$\ln(x) = \frac{i\sqrt{\frac{y}{x}-1}\,\sqrt{\frac{y}{x}+1}\,\arctan\left(\frac{1}{\sqrt{\frac{y^2}{x^2}-1}}\right)}{\sqrt{\frac{y^2}{x^2}-1}} - \ln\left(\frac{y}{x}\right) + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = y(x)/x, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 2.796 (sec). Leaf size: 36

```
dsolve((sqrt(x+y(x))+sqrt(x-y(x)))+(sqrt(x-y(x))-sqrt(x+y(x)))*diff(y(x),x)=0,y(x), singsol=
```

$$\ln(x)+\ln\left(\frac{y(x)}{x}\right)-\operatorname{arctanh}\left(\frac{1}{\sqrt{-\frac{-x^2+y(x)^2}{x^2}}}\right)-c_1=0$$

✓ Solution by Mathematica

Time used: 2.828 (sec). Leaf size: 84

```
DSolve[(Sqrt[x+y[x]]+Sqrt[x-y[x]])+(Sqrt[x-y[x]]-Sqrt[x+y[x]])*y'[x]==0,y[x],x,IncludeSingul
```

$$y(x)\to -\frac{1}{4}\left(\cosh\left(\frac{c_1}{2}\right)+\sinh\left(\frac{c_1}{2}\right)\right)\sqrt{-8ix+\cosh(c_1)+\sinh(c_1)}$$
$$y(x)\to \frac{1}{4}\left(\cosh\left(\frac{c_1}{2}\right)+\sinh\left(\frac{c_1}{2}\right)\right)\sqrt{-8ix+\cosh(c_1)+\sinh(c_1)}$$
$$y(x)\to 0$$

4.15 problem 15

Internal problem ID [11628]
Internal file name [`OUTPUT/10610_Saturday_May_27_2023_03_05_35_AM_36326262/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y + y(x+4)\, y' = -2}$$

With initial conditions

$$[y(-3) = -1]$$

4.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{y+2}{y\,(x+4)} \end{aligned}$$

The x domain of $f(x,y)$ when $y = -1$ is

$$\{x < -4 \lor -4 < x\}$$

And the point $x_0 = -3$ is inside this domain. The y domain of $f(x, y)$ when $x = -3$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y+2}{y\,(x+4)}\right) \\
&= -\frac{1}{y\,(x+4)} + \frac{y+2}{y^2\,(x+4)}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\{x < -4 \lor -4 < x\}$$

And the point $x_0 = -3$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -3$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

4.15.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= -\frac{y+2}{y\,(x+4)}
\end{aligned}$$

Where $f(x) = -\frac{1}{x+4}$ and $g(y) = \frac{y+2}{y}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{y+2}{y}}\,dy &= -\frac{1}{x+4}\,dx \\
\int \frac{1}{\frac{y+2}{y}}\,dy &= \int -\frac{1}{x+4}\,dx \\
y - 2\ln(y+2) &= -\ln(x+4) + c_1
\end{aligned}$$

Which results in

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-1+\frac{\ln(x+4)}{2}-\frac{c_1}{2}}}{2}\right)-1+\frac{\ln(x+4)}{2}-\frac{c_1}{2}} - 2$$

Initial conditions are used to solve for c_1. Substituting $x = -3$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -2\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-1-\frac{c_1}{2}}}{2}\right) - 2$$

$$c_1 = -1$$

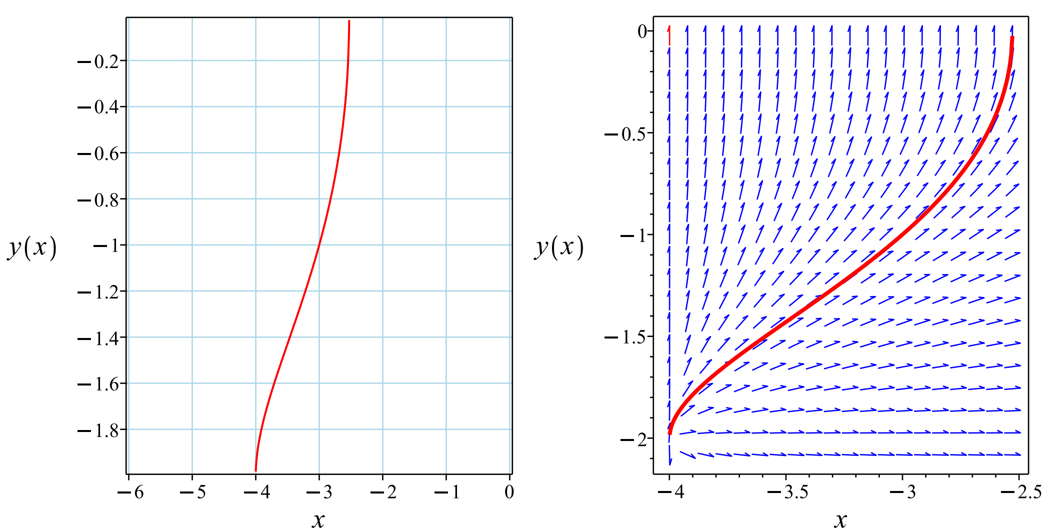
Substituting c_1 found above in the general solution gives

$$y = -2\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-\frac{1}{2}}\sqrt{x+4}}{2}\right) - 2$$

Summary

The solution(s) found are the following

$$y = -2\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-\frac{1}{2}}\sqrt{x+4}}{2}\right) - 2 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -2\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{1}{2}}\sqrt{x+4}}{2}\right) - 2$$

Verified OK.

4.15.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y+2}{y\,(x+4)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 97: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -x - 4 \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-x - 4} dx \end{aligned}$$

Which results in

$$S = -\ln(-x - 4)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y + 2}{y (x + 4)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{-x - 4} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y + 2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R + 2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R - 2\ln(R+2) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(-x-4) = y - 2\ln(y+2) + c_1$$

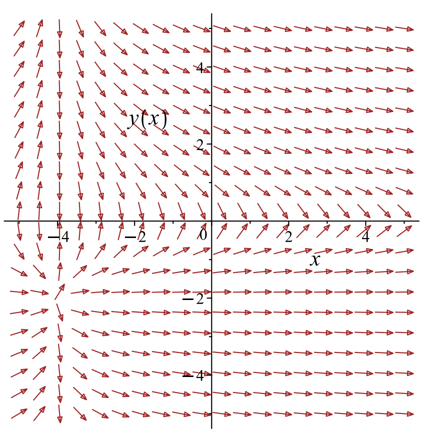
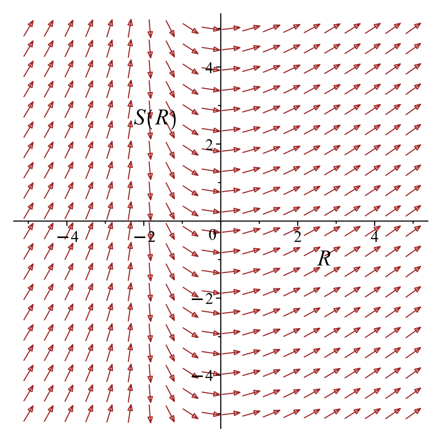
Which simplifies to

$$-\ln(-x-4) = y - 2\ln(y+2) + c_1$$

Which gives

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{\frac{\ln(-x-4)}{2}-1+\frac{c_1}{2}}}{2}\right)+\frac{\ln(-x-4)}{2}-1+\frac{c_1}{2}} - 2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y+2}{y(x+4)}$ 	$R = y$ $S = -\ln(-x-4)$	$\frac{dS}{dR} = \frac{R}{R+2}$ 

Initial conditions are used to solve for c_1. Substituting $x = -3$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -2\,\mathrm{LambertW}\left(-\frac{i\,\mathrm{e}^{-1+\frac{c_1}{2}}}{2}\right) - 2$$

$$c_1 = -i\pi + 1$$

Substituting c_1 found above in the general solution gives

$$y = -2\,\mathrm{LambertW}\left(\frac{i\sqrt{-x-4}\,\mathrm{e}^{-\frac{1}{2}}}{2}\right) - 2$$

Summary

The solution(s) found are the following

$$y = -2\,\mathrm{LambertW}\left(\frac{i\sqrt{-x-4}\,\mathrm{e}^{-\frac{1}{2}}}{2}\right) - 2 \tag{1}$$

Verification of solutions

$$y = -2\,\mathrm{LambertW}\left(\frac{i\sqrt{-x-4}\,\mathrm{e}^{-\frac{1}{2}}}{2}\right) - 2$$

Verified OK.

4.15.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{y}{y+2}\right)\mathrm{d}y &= \left(\frac{1}{x+4}\right)\mathrm{d}x \\
\left(-\frac{1}{x+4}\right)\mathrm{d}x + \left(-\frac{y}{y+2}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x+4} \\
N(x, y) &= -\frac{y}{y+2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x+4}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y}{y+2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x+4} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x+4) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y}{y+2}$. Therefore equation (4) becomes

$$
-\frac{y}{y+2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{y}{y+2}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{y}{y+2}\right) \mathrm{d}y \\
f(y) &= -y + 2\ln(y+2) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x+4) - y + 2\ln(y+2) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x+4) - y + 2\ln(y+2)$$

The solution becomes

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{\frac{\ln(x+4)}{2}-1+\frac{c_1}{2}}}{2}\right)+\frac{\ln(x+4)}{2}-1+\frac{c_1}{2}} - 2$$

Initial conditions are used to solve for c_1. Substituting $x = -3$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -2\operatorname{LambertW}\left(-\frac{e^{-1+\frac{c_1}{2}}}{2}\right) - 2$$

$$c_1 = 1$$

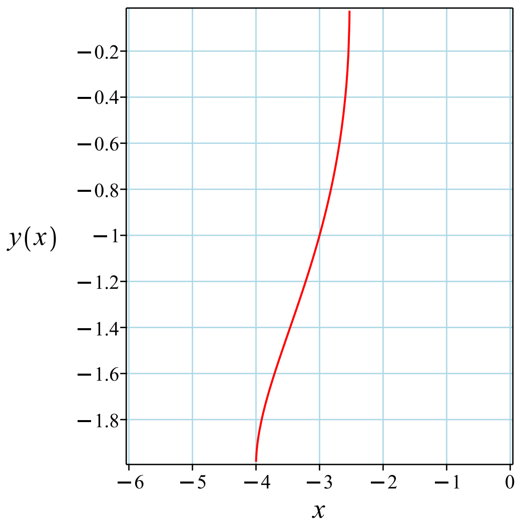
Substituting c_1 found above in the general solution gives

$$y = -2\operatorname{LambertW}\left(-\frac{e^{-\frac{1}{2}}\sqrt{x+4}}{2}\right) - 2$$

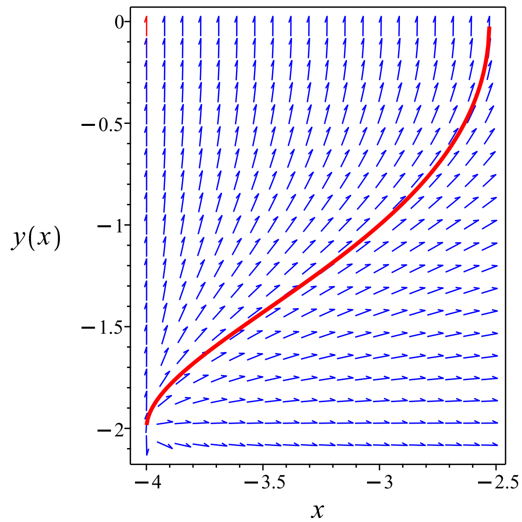
Summary

The solution(s) found are the following

$$y = -2\operatorname{LambertW}\left(-\frac{e^{-\frac{1}{2}}\sqrt{x+4}}{2}\right) - 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{1}{2}}\sqrt{x+4}}{2}\right) - 2$$

Verified OK.

4.15.5 Maple step by step solution

Let's solve

$[y + y(x+4)\,y' = -2, y(-3) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{-y-2} = \frac{1}{x+4}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{-y-2}dx = \int \frac{1}{x+4}dx + c_1$

- Evaluate integral

  $-y + 2\ln(y+2) = \ln(x+4) + c_1$

- Solve for y

$$y = e^{-LambertW\left(-\frac{e^{\frac{\ln(x+4)}{2}-1+\frac{c_1}{2}}}{2}\right)+\frac{\ln(x+4)}{2}-1+\frac{c_1}{2}} - 2$$

- Use initial condition $y(-3) = -1$

$$-1 = e^{-LambertW\left(-\frac{e^{-1+\frac{c_1}{2}}}{2}\right)-1+\frac{c_1}{2}} - 2$$

- Solve for c_1

$$c_1 = 1$$

- Substitute $c_1 = 1$ into general solution and simplify

$$y = -2LambertW\left(-\frac{e^{-\frac{1}{2}}\sqrt{x+4}}{2}\right) - 2$$

- Solution to the IVP

$$y = -2LambertW\left(-\frac{e^{-\frac{1}{2}}\sqrt{x+4}}{2}\right) - 2$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 18

```
dsolve([(y(x)+2)+(y(x)*(x+4))*diff(y(x),x)=0,y(-3) = -1],y(x), singsol=all)
```

$$y(x) = -2\,\mathrm{LambertW}\left(-\frac{\sqrt{x+4}\,e^{-\frac{1}{2}}}{2}\right) - 2$$

✓ Solution by Mathematica

Time used: 12.779 (sec). Leaf size: 26

```
DSolve[{(y[x]+2)+(y[x]*(x+4))*y'[x]==0,{y[-3]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2\left(W\left(-\frac{\sqrt{x+4}}{2\sqrt{e}}\right)+1\right)$$

4.16 problem 16

Internal problem ID [11629]
Internal file name [`OUTPUT/10611_Saturday_May_27_2023_03_05_45_AM_99833809/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{8\cos(y)^2 + \csc(x)^2 y' = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}\right]$$

4.16.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{8\cos(y)^2}{\csc(x)^2} \end{aligned}$$

The x domain of $f(x,y)$ when $y = \frac{\pi}{4}$ is

$$\{x < \pi\_Z95 \lor \pi\_Z95 < x\}$$

And the point $x_0 = \frac{\pi}{12}$ is inside this domain. The y domain of $f(x, y)$ when $x = \frac{\pi}{12}$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{\pi}{4}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{8\cos(y)^2}{\csc(x)^2}\right) \\
&= \frac{16\cos(y)\sin(y)}{\csc(x)^2}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{\pi}{4}$ is

$$\{x < \pi\_Z95 \lor \pi\_Z95 < x\}$$

And the point $x_0 = \frac{\pi}{12}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \frac{\pi}{12}$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{\pi}{4}$ is inside this domain. Therefore solution exists and is unique.

4.16.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= -\frac{8\cos(y)^2}{\csc(x)^2}
\end{aligned}$$

Where $f(x) = -\frac{8}{\csc(x)^2}$ and $g(y) = \cos(y)^2$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\cos(y)^2}\,dy &= -\frac{8}{\csc(x)^2}\,dx \\
\int \frac{1}{\cos(y)^2}\,dy &= \int -\frac{8}{\csc(x)^2}\,dx \\
\tan(y) &= 4\cos(x)\sin(x) - 4x + c_1
\end{aligned}$$

Which results in

$$y = \arctan\left(4\cos\left(x\right)\sin\left(x\right) - 4x + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{12}$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = \arctan\left(4\cos\left(\frac{\pi}{12}\right)\sin\left(\frac{\pi}{12}\right) - \frac{\pi}{3} + c_1\right)$$

$$c_1 = \frac{\pi}{3}$$

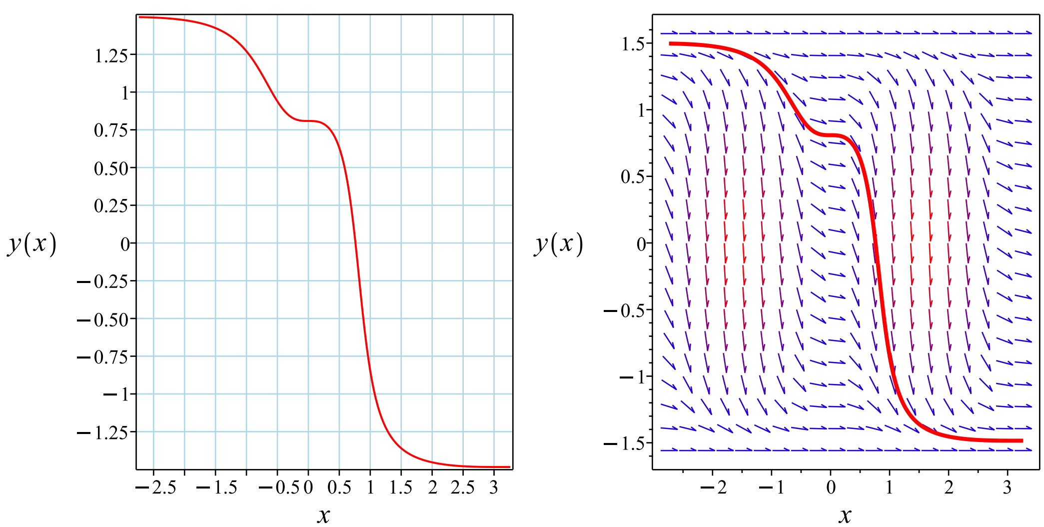
Substituting c_1 found above in the general solution gives

$$y = -\arctan\left(-4\cos\left(x\right)\sin\left(x\right) + 4x - \frac{\pi}{3}\right)$$

Summary

The solution(s) found are the following

$$y = -\arctan\left(-4\cos\left(x\right)\sin\left(x\right) + 4x - \frac{\pi}{3}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\arctan\left(-4\cos\left(x\right)\sin\left(x\right) + 4x - \frac{\pi}{3}\right)$$

Verified OK.

4.16.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{8\cos(y)^2}{\csc(x)^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 100: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{\csc\left(x\right)^{2}}{8} \\
\eta(x,y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{\csc(x)^{2}}{8}} dx
\end{aligned}
$$

Which results in

$$
S = 4\cos\left(x\right)\sin\left(x\right) - 4x
$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{8\cos\left(y\right)^{2}}{\csc\left(x\right)^{2}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -4 + 4\cos(2x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sec(y)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sec(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \tan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-4x + 2\sin(2x) = \tan(y) + c_1$$

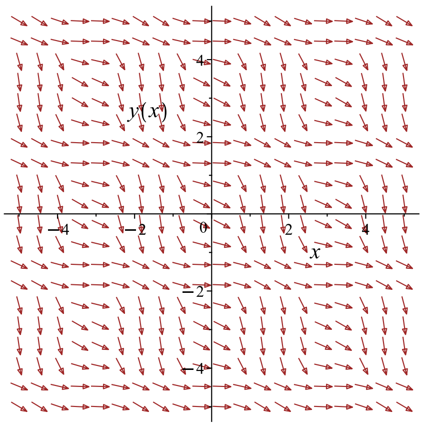
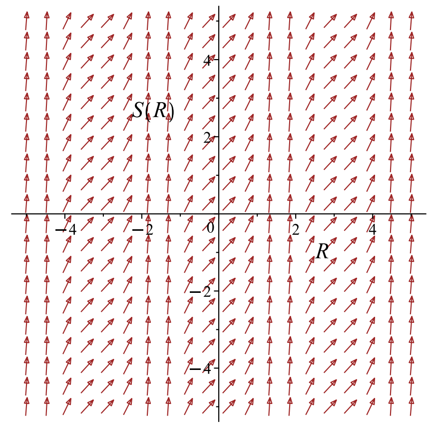
Which simplifies to

$$-4x + 2\sin(2x) = \tan(y) + c_1$$

Which gives

$$y = -\arctan(4x - 2\sin(2x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{8\cos(y)^2}{\csc(x)^2}$ 	$R = y$ $S = -4x + 2\sin(2x)$	$\frac{dS}{dR} = \sec(R)^2$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{12}$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = -\arctan\left(\frac{\pi}{3} - 1 + c_1\right)$$

$$c_1 = -\frac{\pi}{3}$$

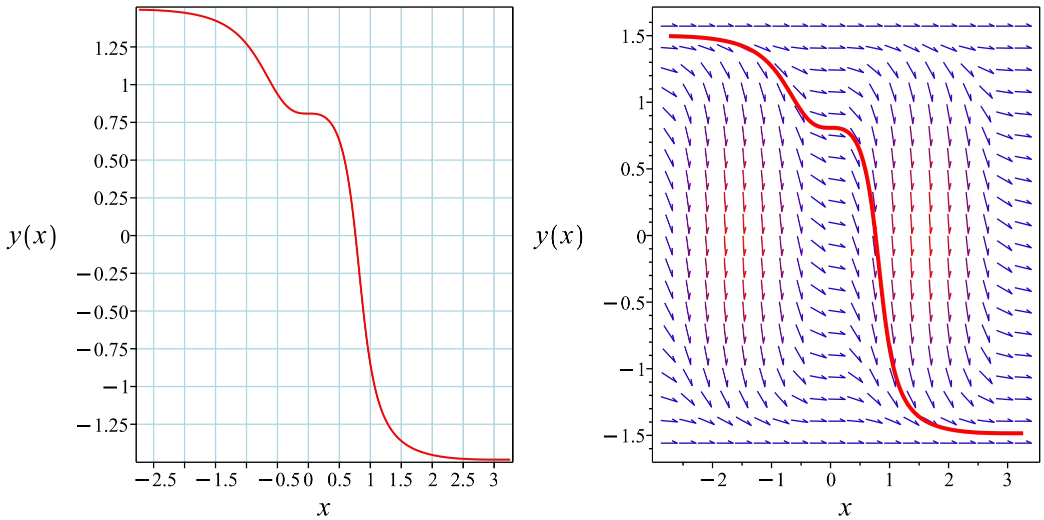
Substituting c_1 found above in the general solution gives

$$y = -\arctan\left(4x - \frac{\pi}{3} - 2\sin(2x)\right)$$

Summary

The solution(s) found are the following

$$y = -\arctan\left(4x - \frac{\pi}{3} - 2\sin(2x)\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\arctan\left(4x - \frac{\pi}{3} - 2\sin(2x)\right)$$

Verified OK.

4.16.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{8\cos(y)^2}\right)\mathrm{d}y=\left(\frac{1}{\csc(x)^2}\right)\mathrm{d}x$$
$$\left(-\frac{1}{\csc(x)^2}\right)\mathrm{d}x+\left(-\frac{1}{8\cos(y)^2}\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y)=-\frac{1}{\csc(x)^2}$$
$$N(x,y)=-\frac{1}{8\cos(y)^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{\csc(x)^2}\right)\\&=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(-\frac{1}{8\cos(y)^2}\right)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{\csc(x)^2} \, \mathrm{d}x$$
$$\phi = \frac{\sin(2x)}{4} - \frac{x}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{8\cos(y)^2}$. Therefore equation (4) becomes

$$-\frac{1}{8\cos(y)^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{8\cos(y)^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{\sec(y)^2}{8}\right) \mathrm{d}y$$
$$f(y) = -\frac{\tan(y)}{8} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\sin(2x)}{4} - \frac{x}{2} - \frac{\tan(y)}{8} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\sin(2x)}{4} - \frac{x}{2} - \frac{\tan(y)}{8}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{12}$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\pi}{24} = c_1$$

$$c_1 = -\frac{\pi}{24}$$

Substituting c_1 found above in the general solution gives

$$\frac{\sin(2x)}{4} - \frac{x}{2} - \frac{\tan(y)}{8} = -\frac{\pi}{24}$$

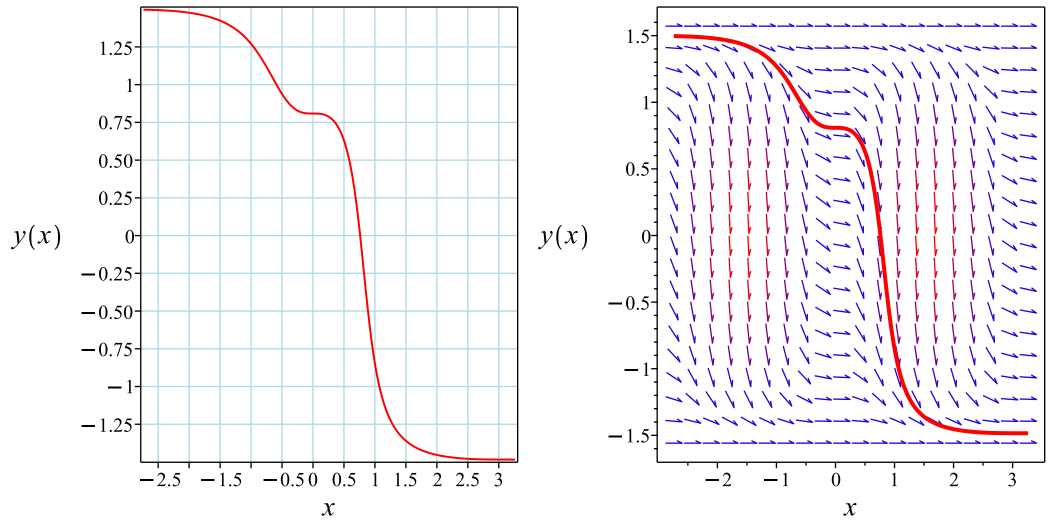
Solving for y from the above gives

$$y = -\arctan\left(4x - \frac{\pi}{3} - 2\sin(2x)\right)$$

Summary

The solution(s) found are the following

$$y = -\arctan\left(4x - \frac{\pi}{3} - 2\sin(2x)\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\arctan\left(4x - \frac{\pi}{3} - 2\sin(2x)\right)$$

Verified OK.

4.16.5 Maple step by step solution

Let's solve

$\left[8\cos(y)^2 + \csc(x)^2 y' = 0, y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\cos(y)^2} = -\frac{8}{\csc(x)^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\cos(y)^2} dx = \int -\frac{8}{\csc(x)^2} dx + c_1$

- Evaluate integral

  $\tan(y) = 4\cos(x)\sin(x) - 4x + c_1$

- Solve for y

  $y = \arctan(4\cos(x)\sin(x) - 4x + c_1)$

- Use initial condition $y\left(\frac{\pi}{12}\right)=\frac{\pi}{4}$

  $\frac{\pi}{4}=\arctan\left(4\cos\left(\frac{\pi}{12}\right)\sin\left(\frac{\pi}{12}\right)-\frac{\pi}{3}+c_1\right)$
- Solve for c_1

  $c_1=-4\cos\left(\frac{\pi}{12}\right)\sin\left(\frac{\pi}{12}\right)+\frac{\pi}{3}+1$
- Substitute $c_1=-4\cos\left(\frac{\pi}{12}\right)\sin\left(\frac{\pi}{12}\right)+\frac{\pi}{3}+1$ into general solution and simplify

  $y=-\arctan\left(4x-\frac{\pi}{3}-2\sin\left(2x\right)\right)$
- Solution to the IVP

  $y=-\arctan\left(4x-\frac{\pi}{3}-2\sin\left(2x\right)\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 20

```
dsolve([(8*cos(y(x))^2)+csc(x)^2*diff(y(x),x)=0,y(1/12*Pi) = 1/4*Pi],y(x), singsol=all)
```

$$y(x)=-\arctan\left(-\frac{\pi}{3}+4x-2\sin\left(2x\right)\right)$$

✓ Solution by Mathematica

Time used: 1.156 (sec). Leaf size: 21

```
DSolve[{(8*Cos[y[x]]^2)+Csc[x]^2*y'[x]==0,{y[Pi/12]==Pi/4}},y[x],x,IncludeSingularSolutions
```

$$y(x)\to\arctan\left(-4x+2\sin(2x)+\frac{\pi}{3}\right)$$

4.17 problem 17

Internal problem ID [11630]
Internal file name [`OUTPUT/10612_Saturday_May_27_2023_03_05_49_AM_45469359/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(3x+8)\left(y^2+4\right)-4y\left(x^2+5x+6\right)y'=0}$$

With initial conditions

$$[y(1)=2]$$

4.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{3x\,y^2+8y^2+12x+32}{4y\left(x^2+5x+6\right)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=2$ is

$$\{-\infty \le x < -3, -3 < x < -2, -2 < x \le \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{3x\,y^2 + 8y^2 + 12x + 32}{4y\,(x^2 + 5x + 6)}\right) \\
&= \frac{6xy + 16y}{4y\,(x^2 + 5x + 6)} - \frac{3x\,y^2 + 8y^2 + 12x + 32}{4y^2\,(x^2 + 5x + 6)}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

4.17.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= \frac{(y^2 + 4)\,(3x + 8)}{4y\,(x + 3)\,(x + 2)}
\end{aligned}$$

Where $f(x) = \frac{3x+8}{4(x+3)(x+2)}$ and $g(y) = \frac{y^2+4}{y}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{y^2+4}{y}}\,dy &= \frac{3x + 8}{4\,(x + 3)\,(x + 2)}\,dx \\
\int \frac{1}{\frac{y^2+4}{y}}\,dy &= \int \frac{3x + 8}{4\,(x + 3)\,(x + 2)}\,dx \\
\frac{\ln(y^2 + 4)}{2} &= \frac{\ln(x + 3)}{4} + \frac{\ln(x + 2)}{2} + c_1
\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{y^2+4} = \mathrm{e}^{\frac{\ln(x+3)}{4}+\frac{\ln(x+2)}{2}+c_1}$$

Which simplifies to

$$\sqrt{y^2+4} = c_2 \mathrm{e}^{\frac{\ln(x+3)}{4}+\frac{\ln(x+2)}{2}}$$

Which can be simplified to become

$$\sqrt{y^2+4} = c_2 (x+3)^{\frac{1}{4}} \sqrt{x+2}\, \mathrm{e}^{c_1}$$

The solution is

$$\sqrt{y^2+4} = c_2 (x+3)^{\frac{1}{4}} \sqrt{x+2}\, \mathrm{e}^{c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2\sqrt{2} = 4^{\frac{1}{4}} \sqrt{3}\, \mathrm{e}^{c_1} c_2$$

$$c_1 = \frac{\ln\left(\frac{4}{3c_2^2}\right)}{2}$$

Substituting c_1 found above in the general solution gives

$$\sqrt{y^2+4} = \frac{2c_2 \sqrt{x+2}\,(x+3)^{\frac{1}{4}} \sqrt{3}\, \sqrt{\frac{1}{c_2^2}}}{3}$$

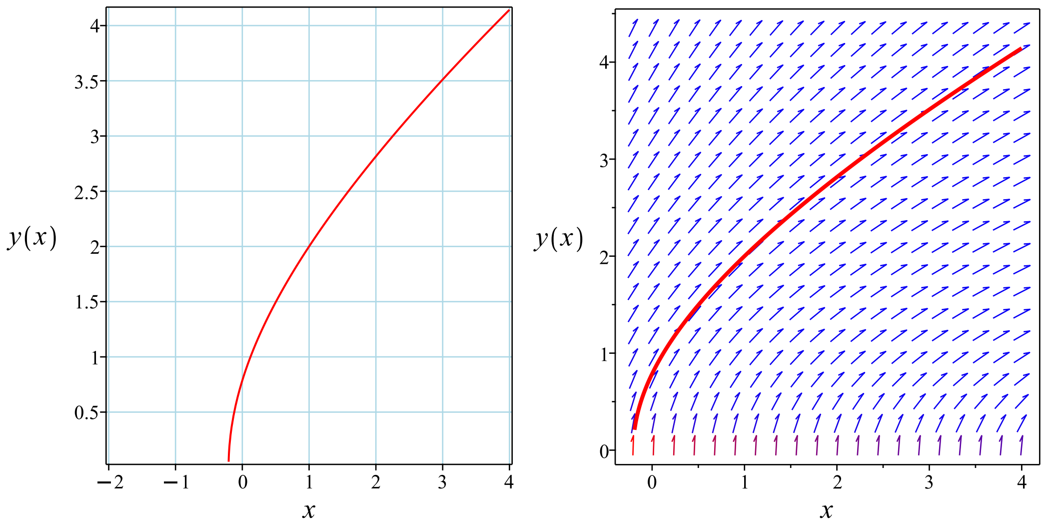
Solving for y from the above gives

$$y = \frac{2\sqrt{-9+(3x+6)\sqrt{x+3}}}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{2\sqrt{-9+(3x+6)\sqrt{x+3}}}{3} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{2\sqrt{-9+(3x+6)\sqrt{x+3}}}{3}$$

Verified OK. `{positive}`

4.17.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{3x\,y^2+8y^2+12x+32}{4y\,(x^2+5x+6)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 103: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{4(x + 3)\,(x + 2)}{3x + 8} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{4(x+3)(x+2)}{3x+8}} dx \end{aligned}$$

Which results in

$$S = \frac{\ln(x+3)}{4} + \frac{\ln(x+2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{3x\,y^2 + 8y^2 + 12x + 32}{4y\,(x^2 + 5x + 6)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{3x+8}{4\,(x+3)\,(x+2)} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2+4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2+4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R^2+4)}{2} + c_1 \tag{4}$$

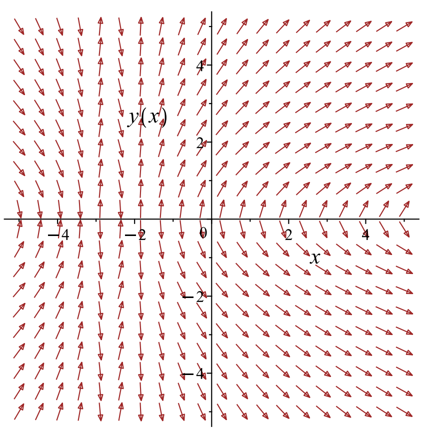
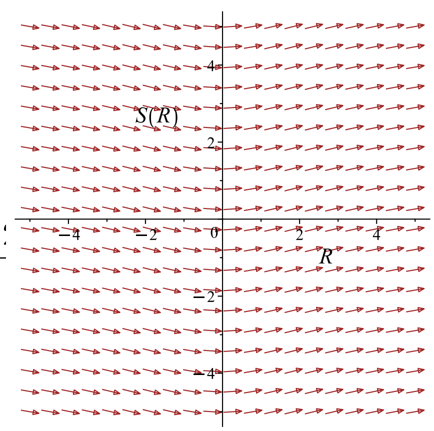
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x+3)}{4} + \frac{\ln(x+2)}{2} = \frac{\ln(y^2+4)}{2} + c_1$$

Which simplifies to

$$\frac{\ln(x+3)}{4} + \frac{\ln(x+2)}{2} = \frac{\ln(y^2+4)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x\,y^2+8y^2+12x+32}{4y(x^2+5x+6)}$ 	$R = y$ $S = \frac{\ln(x+3)}{4} + \frac{\ln(x+2)}{2}$	$\frac{dS}{dR} = \frac{R}{R^2+4}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\ln(2)}{2} + \frac{\ln(3)}{2} = \frac{3\ln(2)}{2} + c_1$$

$$c_1 = -\ln(2) + \frac{\ln(3)}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(x+3)}{4} + \frac{\ln(x+2)}{2} = \frac{\ln(y^2+4)}{2} - \ln(2) + \frac{\ln(3)}{2}$$

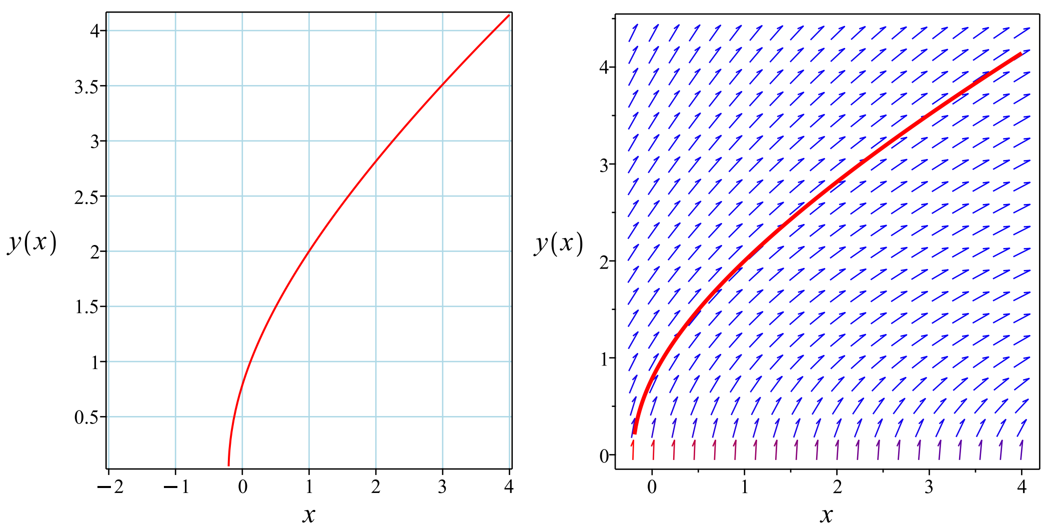
Solving for y from the above gives

$$y = \frac{2\sqrt{-9 + (3x+6)\sqrt{x+3}}}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{2\sqrt{-9 + (3x+6)\sqrt{x+3}}}{3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{2\sqrt{-9 + (3x+6)\sqrt{x+3}}}{3}$$

Verified OK. `{positive}`

4.17.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= \frac{3x\,y^2 + 8y^2 + 12x + 32}{4y\,(x^2 + 5x + 6)}
\end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{3x + 8}{4x^2 + 20x + 24}y + \frac{12x + 32}{4x^2 + 20x + 24}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= \frac{3x + 8}{4x^2 + 20x + 24} \\
f_1(x) &= \frac{12x + 32}{4x^2 + 20x + 24} \\
n &= -1
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{(3x + 8)\,y^2}{4x^2 + 20x + 24} + \frac{12x + 32}{4x^2 + 20x + 24} \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{(3x+8)\, w(x)}{4x^2+20x+24} + \frac{12x+32}{4x^2+20x+24} \\
w' &= \frac{(3x+8)\, w}{2x^2+10x+12} + \frac{12x+32}{2x^2+10x+12} \qquad (7)
\end{aligned}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3x+8}{2x^2+10x+12} \\
q(x) &= \frac{6x+16}{x^2+5x+6}
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{(3x+8)\, w(x)}{2x^2+10x+12} = \frac{6x+16}{x^2+5x+6}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{3x+8}{2x^2+10x+12}dx} \\
&= e^{-\frac{\ln(x+3)}{2}-\ln(x+2)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{1}{\sqrt{x+3}\,(x+2)}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{6x+16}{x^2+5x+6}\right) \\
\frac{d}{dx}\left(\frac{w}{\sqrt{x+3}\,(x+2)}\right) &= \left(\frac{1}{\sqrt{x+3}\,(x+2)}\right)\left(\frac{6x+16}{x^2+5x+6}\right) \\
d\left(\frac{w}{\sqrt{x+3}\,(x+2)}\right) &= \left(\frac{6x+16}{(x+3)^{\frac{3}{2}}\,(x+2)^2}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{\sqrt{x+3}\,(x+2)} &= \int \frac{6x+16}{(x+3)^{\frac{3}{2}}\,(x+2)^2}\,\mathrm{d}x \\
\frac{w}{\sqrt{x+3}\,(x+2)} &= -\frac{4}{\sqrt{x+3}\,(x+2)} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x+3}\,(x+2)}$ results in

$$
w(x) = -4 + c_1(x+2)\sqrt{x+3}
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$
y^2 = -4 + c_1(x+2)\sqrt{x+3}
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$
4 = -4 + 6c_1
$$

$$
c_1 = \frac{4}{3}
$$

Substituting c_1 found above in the general solution gives

$$
y^2 = -4 + \frac{4\sqrt{x+3}\,x}{3} + \frac{8\sqrt{x+3}}{3}
$$

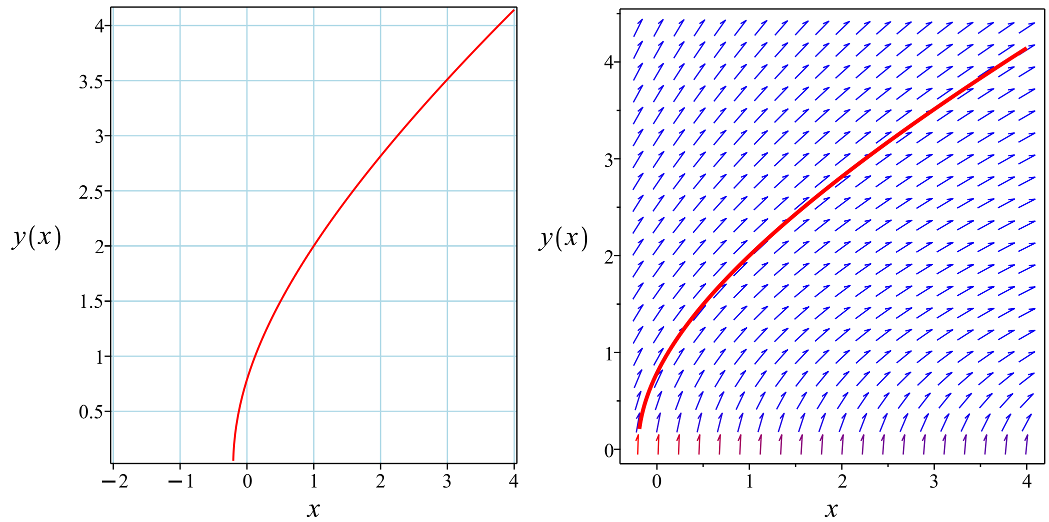
Solving for y from the above gives

$$
y = \frac{2\sqrt{-9+(3x+6)\sqrt{x+3}}}{3}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2\sqrt{-9+(3x+6)\sqrt{x+3}}}{3} \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{2\sqrt{-9+(3x+6)\sqrt{x+3}}}{3}$$

Verified OK. `{positive}`

4.17.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{4y}{y^2+4}\right)\mathrm{d}y = \left(\frac{3x+8}{(x+3)(x+2)}\right)\mathrm{d}x$$
$$\left(-\frac{3x+8}{(x+3)(x+2)}\right)\mathrm{d}x + \left(\frac{4y}{y^2+4}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{3x+8}{(x+3)(x+2)}$$
$$N(x, y) = \frac{4y}{y^2+4}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{3x+8}{(x+3)(x+2)}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{4y}{y^2+4}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{3x+8}{(x+3)(x+2)} \, \mathrm{d}x \\
\phi &= -\ln(x+3) - 2\ln(x+2) + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{4y}{y^2+4}$. Therefore equation (4) becomes

$$
\frac{4y}{y^2+4} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{4y}{y^2+4}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{4y}{y^2+4}\right) \mathrm{d}y \\
f(y) &= 2\ln\left(y^2+4\right) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x+3) - 2\ln(x+2) + 2\ln(y^2+4) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x+3) - 2\ln(x+2) + 2\ln(y^2+4)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$4\ln(2) - 2\ln(3) = c_1$$

$$c_1 = 4\ln(2) - 2\ln(3)$$

Substituting c_1 found above in the general solution gives

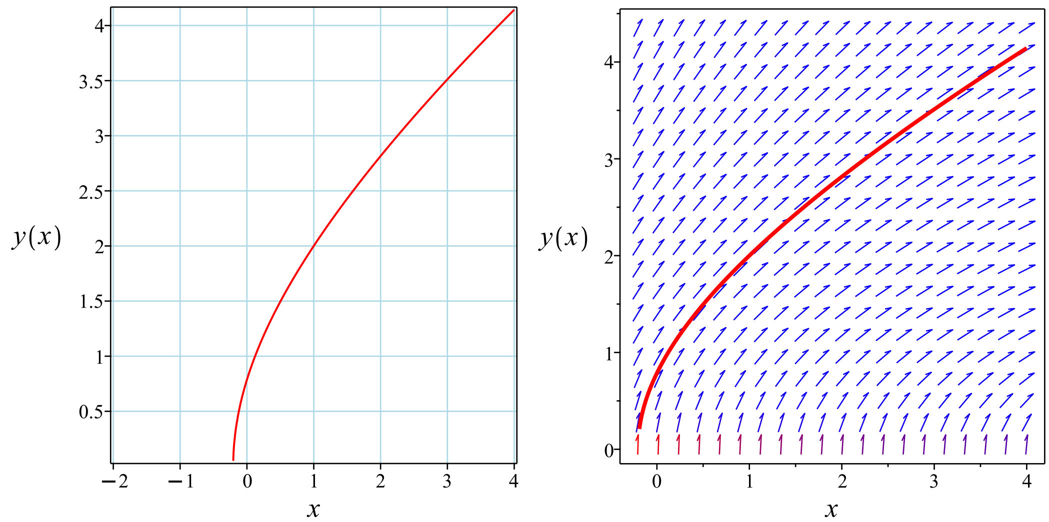
$$-\ln(x+3) - 2\ln(x+2) + 2\ln(y^2+4) = 4\ln(2) - 2\ln(3)$$

Solving for y from the above gives

$$y = \frac{2\sqrt{-9 + (3x+6)\sqrt{x+3}}}{3}$$

Summary
The solution(s) found are the following

$$y = \frac{2\sqrt{-9 + (3x+6)\sqrt{x+3}}}{3} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{2\sqrt{-9+(3x+6)\sqrt{x+3}}}{3}$$

Verified OK. {positive}

4.17.6 Maple step by step solution

Let's solve

$[(3x+8)(y^2+4) - 4y(x^2+5x+6)y' = 0, y(1) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{y^2+4} = \frac{3x+8}{4(x^2+5x+6)}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{y^2+4}dx = \int \frac{3x+8}{4(x^2+5x+6)}dx + c_1$

- Evaluate integral

  $\frac{\ln(y^2+4)}{2} = \frac{\ln(x+3)}{4} + \frac{\ln(x+2)}{2} + c_1$

- Solve for y

$$\left\{y = \frac{\sqrt{-\mathrm{e}^{-\frac{\ln(x+3)}{2}-2c_1}\left(4\,\mathrm{e}^{-\frac{\ln(x+3)}{2}-2c_1}-x-2\right)}}{\mathrm{e}^{-\frac{\ln(x+3)}{2}-2c_1}}, y = -\frac{\sqrt{-\mathrm{e}^{-\frac{\ln(x+3)}{2}-2c_1}\left(4\,\mathrm{e}^{-\frac{\ln(x+3)}{2}-2c_1}-x-2\right)}}{\mathrm{e}^{-\frac{\ln(x+3)}{2}-2c_1}}\right\}$$

- Use initial condition $y(1) = 2$

$$2 = \frac{\sqrt{-\mathrm{e}^{-\ln(2)-2c_1}\left(4\,\mathrm{e}^{-\ln(2)-2c_1}-3\right)}}{\mathrm{e}^{-\ln(2)-2c_1}}$$

- Solve for c_1

$$c_1 = -\frac{\ln(2)}{2} - \frac{\ln\left(\frac{3}{8}\right)}{2}$$

- Substitute $c_1 = -\frac{\ln(2)}{2} - \frac{\ln\left(\frac{3}{8}\right)}{2}$ into general solution and simplify

$$y = \frac{2\sqrt{3}\,\sqrt{x+3}\,\sqrt{\frac{-3+(x+2)\sqrt{x+3}}{x+3}}}{3}$$

- Use initial condition $y(1) = 2$

$$2 = -\frac{\sqrt{-\mathrm{e}^{-\ln(2)-2c_1}\left(4\,\mathrm{e}^{-\ln(2)-2c_1}-3\right)}}{\mathrm{e}^{-\ln(2)-2c_1}}$$

- Solution does not satisfy initial condition
- Solution to the IVP

$$y = \frac{2\sqrt{3}\,\sqrt{x+3}\,\sqrt{\frac{-3+(x+2)\sqrt{x+3}}{x+3}}}{3}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 21

```
dsolve([(3*x+8)*(y(x)^2+4)-4*y(x)*(x^2+5*x+6)*diff(y(x),x)=0,y(1) = 2],y(x), singsol=all)
```

$$y(x) = \frac{2\sqrt{-9 + (3x+6)\sqrt{x+3}}}{3}$$

✓ Solution by Mathematica

Time used: 4.88 (sec). Leaf size: 36

```
DSolve[{(3*x+8)*(y[x]^2+4)-4*y[x]*(x^2+5*x+6)*y'[x]==0,{y[1]==2}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \frac{2\sqrt{\sqrt{x+3}x + 2\sqrt{x+3} - 3}}{\sqrt{3}}$$

4.18 problem 18

Internal problem ID [11631]
Internal file name [`OUTPUT/10613_Saturday_May_27_2023_03_05_51_AM_85233479/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{3y^2 - 2xyy' = -x^2}$$

With initial conditions

$$[y(2) = 6]$$

4.18.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{x^2 + 3y^2}{2xy} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 6$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x,y)$ when $x = 2$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 6$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x^2+3y^2}{2xy}\right)\\ &= \frac{3}{x} - \frac{x^2+3y^2}{2x\,y^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 6$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 2$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 6$ is inside this domain. Therefore solution exists and is unique.

4.18.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$3u(x)^2\,x^2 - 2x^2u(x)\,(u'(x)\,x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{u^2+1}{2ux}\end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u}}\,du = \frac{1}{2x}\,dx$$

$$\int \frac{1}{\frac{u^2+1}{u}}\,du = \int \frac{1}{2x}\,dx$$

$$\frac{\ln(u^2+1)}{2} = \frac{\ln(x)}{2} + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+1} = e^{\frac{\ln(x)}{2}+c_2}$$

Which simplifies to

$$\sqrt{u^2+1} = c_3\sqrt{x}$$

Which simplifies to

$$\sqrt{u(x)^2+1} = c_3\sqrt{x}\,e^{c_2}$$

The solution is

$$\sqrt{u(x)^2+1} = c_3\sqrt{x}\,e^{c_2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}+1} = c_3\sqrt{x}\,e^{c_2}$$
$$\sqrt{\frac{x^2+y^2}{x^2}} = c_3\sqrt{x}\,e^{c_2}$$

Substituting initial conditions and solving for c_2 gives $c_2 = \frac{\ln\left(\frac{5}{c_3^2}\right)}{2}$. Hence the solution becomes Initial conditions are used to solve for c_3. Substituting $x = 2$ and $y = 6$ in the above solution gives an equation to solve for the constant of integration.

$$\sqrt{10} = \sqrt{2}\,c_3\sqrt{5}\,\sqrt{\frac{1}{c_3^2}}$$

This solution is valid for any c_3. Hence there are infinite number of solutions.

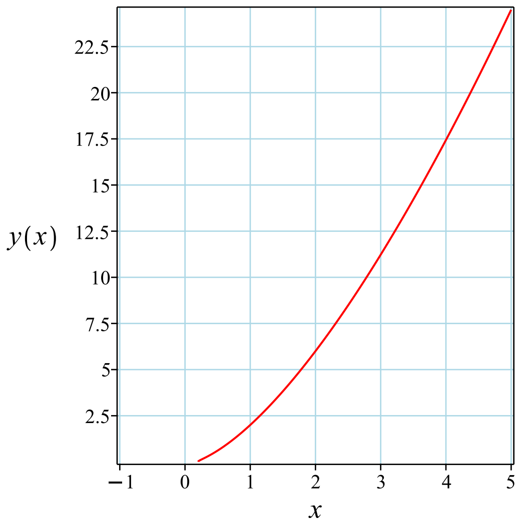
Solving for y from the above gives

$$y = \sqrt{5x-1}\,x$$

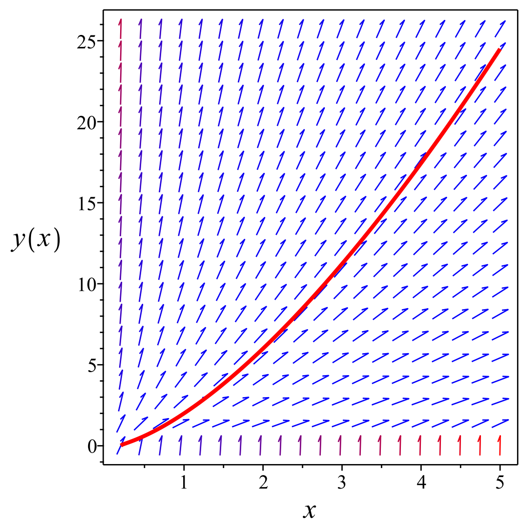
Summary

The solution(s) found are the following

$$y = \sqrt{5x-1}\,x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{5x-1}\, x$$

Verified OK. `{positive}`

4.18.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2+3y^2}{2xy}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 106: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x^3}{y}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^3}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 + 3y^2}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3y^2}{2x^4} \\ S_y &= \frac{y}{x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x^3} = -\frac{1}{2x} + c_1$$

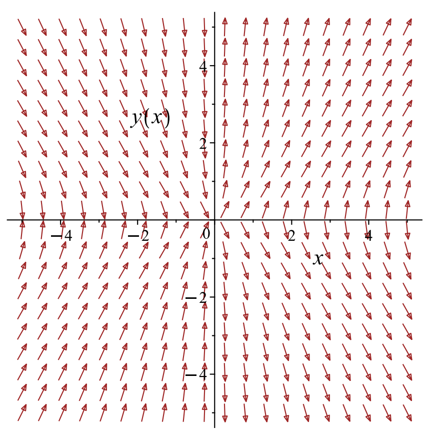
Which simplifies to

$$\frac{y^2}{2x^3} = -\frac{1}{2x} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$



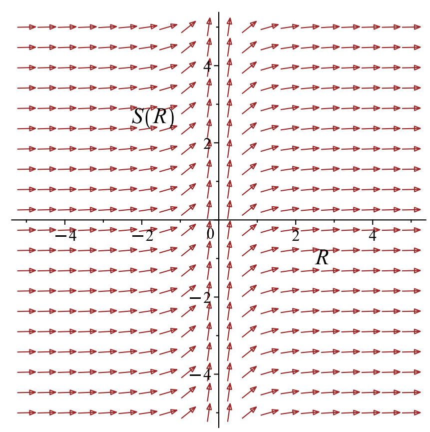
</td><td>

$R = x$

$S = \frac{y^2}{2x^3}$

</td><td>

$\frac{dS}{dR} = \frac{1}{2R^2}$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 6$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{9}{4} = c_1 - \frac{1}{4}$$

$$c_1 = \frac{5}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{y^2}{2x^3} = \frac{5x-1}{2x}$$

The above simplifies to

$$-5x^3 + x^2 + y^2 = 0$$

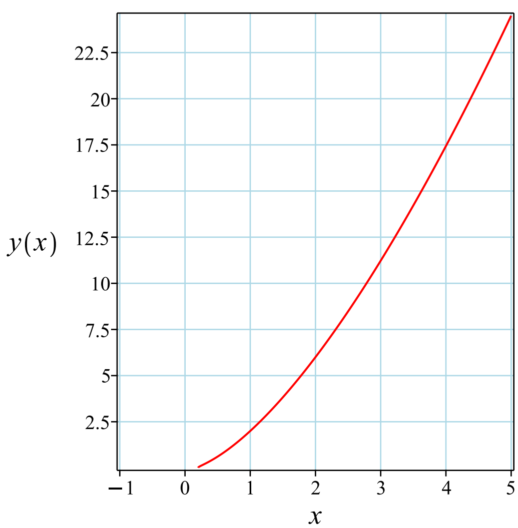
Solving for y from the above gives

$$y = \sqrt{5x-1}\, x$$

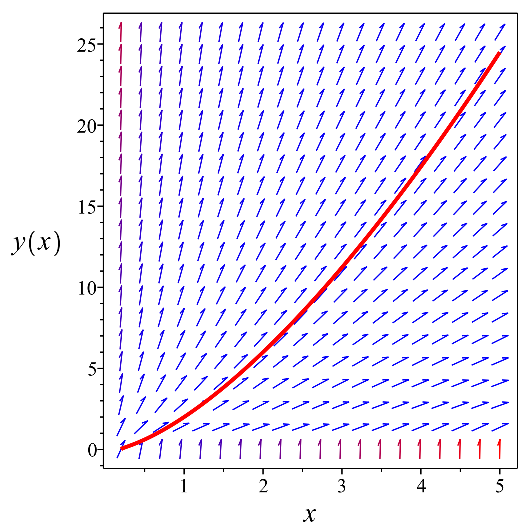
Summary

The solution(s) found are the following

$$y = \sqrt{5x-1}\, x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{5x-1}\, x$$

Verified OK. `{positive}`

4.18.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= \frac{x^2+3y^2}{2xy}
\end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{3}{2x}y + \frac{x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= \frac{3}{2x} \\
f_1(x) &= \frac{x}{2} \\
n &= -1
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{3y^2}{2x} + \frac{x}{2} \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{3w(x)}{2x} + \frac{x}{2} \\
w' &= \frac{3w}{x} + x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= x
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{3w(x)}{x} = x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{3}{x}dx} \\
&= \frac{1}{x^3}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)(x) \\
\frac{d}{dx}\left(\frac{w}{x^3}\right) &= \left(\frac{1}{x^3}\right)(x) \\
d\left(\frac{w}{x^3}\right) &= \frac{1}{x^2}\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x^3} &= \int \frac{1}{x^2}\, dx \\
\frac{w}{x^3} &= -\frac{1}{x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$
w(x) = c_1 x^3 - x^2
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = c_1 x^3 - x^2$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 6$ in the above solution gives an equation to solve for the constant of integration.

$$36 = 8c_1 - 4$$

$$c_1 = 5$$

Substituting c_1 found above in the general solution gives

$$y^2 = 5x^3 - x^2$$

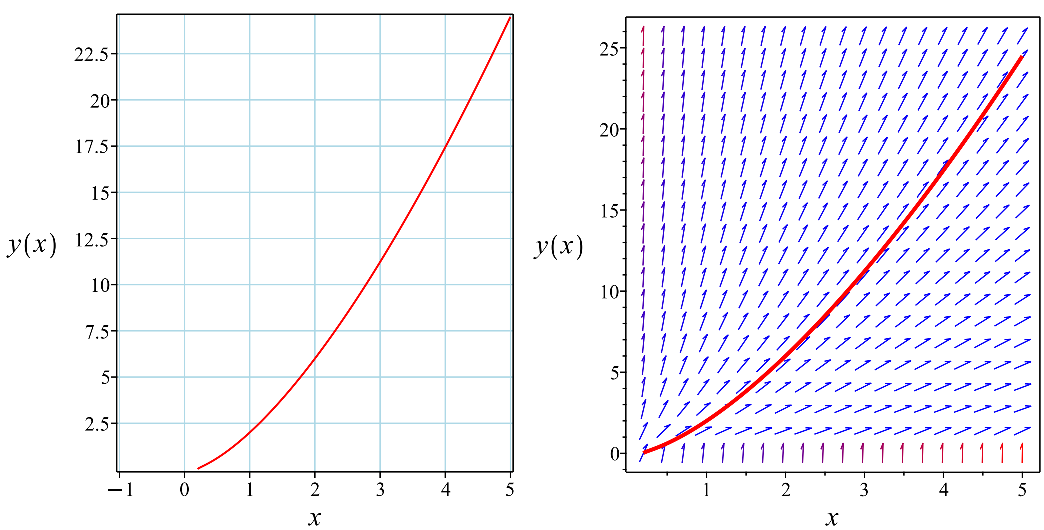
Solving for y from the above gives

$$y = \sqrt{5x - 1}\, x$$

Summary

The solution(s) found are the following

$$y = \sqrt{5x - 1}\, x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \sqrt{5x - 1}\, x$$

Verified OK. `{positive}`

4.18.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2xy)\,\mathrm{d}y&=\left(-x^2-3y^2\right)\mathrm{d}x\\ \left(x^2+3y^2\right)\mathrm{d}x+(-2xy)\,\mathrm{d}y&=0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x^2+3y^2\\ N(x,y)&=-2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(x^2+3y^2)\\&=6y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(-2xy)\\&=-2y\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\&=-\frac{1}{2xy}((6y)-(-2y))\\&=-\frac{4}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu&=e^{\int A\,\mathrm{d}x}\\&=e^{\int-\frac{4}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu&=e^{-4\ln(x)}\\&=\frac{1}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M}&=\mu M\\&=\frac{1}{x^4}(x^2+3y^2)\\&=\frac{x^2+3y^2}{x^4}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^4}(-2xy) \\
&= -\frac{2y}{x^3}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^2+3y^2}{x^4}\right) + \left(-\frac{2y}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{x^2+3y^2}{x^4} \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{-x^2-y^2}{x^3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{2y}{x^3} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{x^3}$. Therefore equation (4) becomes

$$
-\frac{2y}{x^3} = -\frac{2y}{x^3} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x^2 - y^2}{x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x^2 - y^2}{x^3}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 6$ in the above solution gives an equation to solve for the constant of integration.

$$-5 = c_1$$

$$c_1 = -5$$

Substituting c_1 found above in the general solution gives

$$\frac{-x^2 - y^2}{x^3} = -5$$

The above simplifies to

$$5x^3 - x^2 - y^2 = 0$$

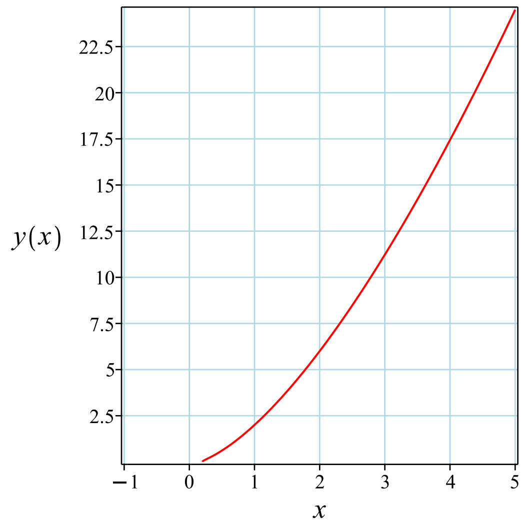
Solving for y from the above gives

$$y = \sqrt{5x - 1}\, x$$

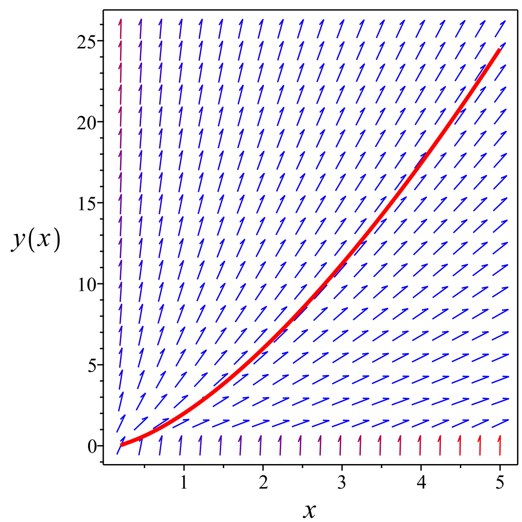
Summary

The solution(s) found are the following

$$y = \sqrt{5x - 1}\, x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{5x-1}\, x$$

Verified OK. {positive}

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 13

```
dsolve([(x^2+3*y(x)^2)-2*x*y(x)*diff(y(x),x)=0,y(2) = 6],y(x), singsol=all)
```

$$y(x) = \sqrt{5x-1}\, x$$

✓ Solution by Mathematica

Time used: 0.455 (sec). Leaf size: 16

```
DSolve[{(x^2+3*y[x]^2)-2*x*y[x]*y'[x]==0,{y[2]==6}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\sqrt{5x-1}$$

4.19 problem 19

Internal problem ID [11632]
Internal file name [`OUTPUT/10614_Saturday_May_27_2023_03_06_24_AM_45212028/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-5y + (4x - y)\, y' = -2x}$$

With initial conditions

$$[y(1) = 4]$$

4.19.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{-2x + 5y}{-4x + y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 4$ is

$$\{x < 1 \lor 1 < x\}$$

But the point $x_0 = 1$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

4.19.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-5u(x)\,x + (4x - u(x)\,x)\,(u'(x)\,x + u(x)) = -2x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2+u-2}{(u-4)\,x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+u-2}{u-4}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2+u-2}{u-4}}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{\frac{u^2+u-2}{u-4}}\,du &= \int -\frac{1}{x}\,dx \\ -\ln(u-1) + 2\ln(u+2) &= -\ln(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$e^{-\ln(u-1)+2\ln(u+2)} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{(u+2)^2}{u-1} = \frac{c_3}{x}$$

The solution is

$$\frac{(u(x)+2)^2}{u(x)-1} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\begin{aligned} \frac{\left(\frac{y}{x}+2\right)^2}{\frac{y}{x}-1} &= \frac{c_3}{x} \\ \frac{(2x+y)^2}{(y-x)\,x} &= \frac{c_3}{x} \end{aligned}$$

Which simplifies to

$$-\frac{(2x+y)^2}{-y+x} = c_3$$

Substituting initial conditions and solving for c_3 gives $c_3 = 12$. Hence the solution be-
Summary
The solution(s) found are the following
comes

$$-\frac{(2x+y)^2}{-y+x} = 12 \tag{1}$$

Verification of solutions

$$-\frac{(2x+y)^2}{-y+x} = 12$$

Verified OK.

4.19.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= -\frac{-2x+5y}{-4x+y} \\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(-2x+5y)(b_3-a_2)}{-4x+y} - \frac{(-2x+5y)^2 a_3}{(-4x+y)^2} \\
&- \left(\frac{2}{-4x+y} - \frac{4(-2x+5y)}{(-4x+y)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{5}{-4x+y} + \frac{-2x+5y}{(-4x+y)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
\frac{8x^2a_2 - 4x^2a_3 - 2x^2b_2 - 8x^2b_3 - 4xya_2 + 20xya_3 - 8xyb_2 + 4xyb_3 + 5y^2a_2 - 7y^2a_3 + y^2b_2 - 5y^2b_3 - 18x}{(4x-y)^2}
$$
$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&8x^2a_2 - 4x^2a_3 - 2x^2b_2 - 8x^2b_3 - 4xya_2 + 20xya_3 - 8xyb_2 \\
&+ 4xyb_3 + 5y^2a_2 - 7y^2a_3 + y^2b_2 - 5y^2b_3 - 18xb_1 + 18ya_1 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&8a_2v_1^2 - 4a_2v_1v_2 + 5a_2v_2^2 - 4a_3v_1^2 + 20a_3v_1v_2 - 7a_3v_2^2 - 2b_2v_1^2 \\
&- 8b_2v_1v_2 + b_2v_2^2 - 8b_3v_1^2 + 4b_3v_1v_2 - 5b_3v_2^2 + 18a_1v_2 - 18b_1v_1 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(8a_2 - 4a_3 - 2b_2 - 8b_3) v_1^2 + (-4a_2 + 20a_3 - 8b_2 + 4b_3) v_1 v_2 - 18b_1 v_1 + (5a_2 - 7a_3 + b_2 - 5b_3) v_2^2 + 18a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
18a_1 &= 0 \\
-18b_1 &= 0 \\
-4a_2 + 20a_3 - 8b_2 + 4b_3 &= 0 \\
5a_2 - 7a_3 + b_2 - 5b_3 &= 0 \\
8a_2 - 4a_3 - 2b_2 - 8b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= 2a_3 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{-2x + 5y}{-4x + y}\right)(x) \\
&= \frac{2x^2 - xy - y^2}{4x - y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2x^2-xy-y^2}{4x-y}} dy \end{aligned}$$

Which results in

$$S = -\ln(y-x) + 2\ln(2x+y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-2x+5y}{-4x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y-x} + \frac{4}{2x+y} \\ S_y &= \frac{1}{-y+x} + \frac{2}{2x+y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

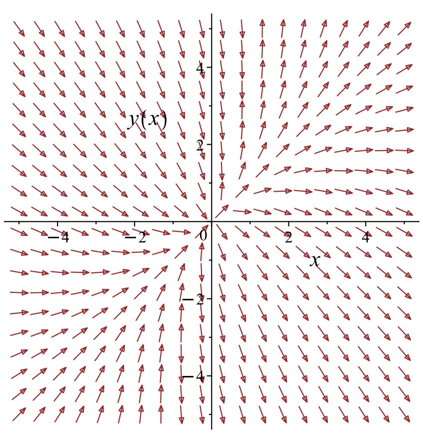
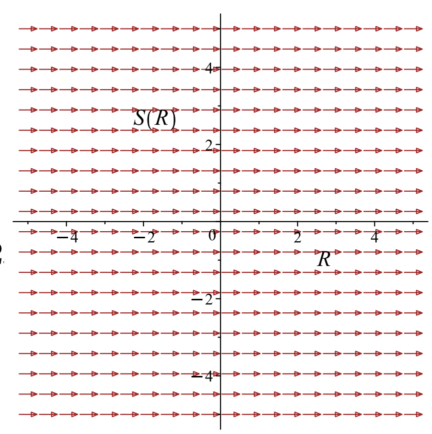
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(y - x) + 2\ln(2x + y) = c_1$$

Which simplifies to

$$-\ln(y - x) + 2\ln(2x + y) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-2x+5y}{-4x+y}$ 	$R = x$ $S = -\ln(y - x) + 2\ln(2$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$2\ln(2) + \ln(3) = c_1$$

$$c_1 = 2\ln(2) + \ln(3)$$

Substituting c_1 found above in the general solution gives

$$-\ln(y - x) + 2\ln(2x + y) = 2\ln(2) + \ln(3)$$

Summary

The solution(s) found are the following

$$-\ln(y - x) + 2\ln(2x + y) = 2\ln(2) + \ln(3) \tag{1}$$

Verification of solutions

$$-\ln(y - x) + 2\ln(2x + y) = 2\ln(2) + \ln(3)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.218 (sec). Leaf size: 35

```
dsolve([(2*x-5*y(x))+(4*x-y(x))*diff(y(x),x)=0,y(1) = 4],y(x), singsol=all)
```

$$y(x) = 6 - 2x - 6\sqrt{1 - x}$$
$$y(x) = 6 - 2x + 6\sqrt{1 - x}$$

✓ Solution by Mathematica

Time used: 2.199 (sec). Leaf size: 41

```
DSolve[{(2*x-5*y[x])+(4*x-y[x])*y'[x]==0,{y[1]==4}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x - 6i\sqrt{x-1} + 6$$
$$y(x) \to -2x + 6i\sqrt{x-1} + 6$$

4.20 problem 20

Internal problem ID [11633]
Internal file name [`OUTPUT/10615_Saturday_May_27_2023_03_06_35_AM_56839210/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{9yx + 5y^2 - \left(6x^2 + 4yx\right) y' = -3x^2}$$

With initial conditions

$$[y(2) = -6]$$

4.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{3x^2 + 9xy + 5y^2}{2x\,(3x + 2y)} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -6$ is

$$\{-\infty \le x < 0, 0 < x < 4, 4 < x \le \infty\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x,y)$ when $x = 2$ is

$$\{y < -3 \vee -3 < y\}$$

And the point $y_0 = -6$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{3x^2 + 9xy + 5y^2}{2x\,(3x + 2y)}\right) \\ &= \frac{10y + 9x}{2x\,(3x + 2y)} - \frac{3x^2 + 9xy + 5y^2}{x\,(3x + 2y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -6$ is

$$\{-\infty \le x < 0, 0 < x < 4, 4 < x \le \infty\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 2$ is

$$\{y < -3 \vee -3 < y\}$$

And the point $y_0 = -6$ is inside this domain. Therefore solution exists and is unique.

4.20.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$9u(x)\,x^2 + 5u(x)^2\,x^2 - \left(6x^2 + 4u(x)\,x^2\right)\left(u'(x)\,x + u(x)\right) = -3x^2$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2 + 3u + 3}{2x\,(2u + 3)}\end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = \frac{u^2+3u+3}{2u+3}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+3u+3}{2u+3}}\,du = \frac{1}{2x}\,dx$$

$$\int \frac{1}{\frac{u^2+3u+3}{2u+3}}\,du = \int \frac{1}{2x}\,dx$$

$$\ln\left(u^2 + 3u + 3\right) = \frac{\ln(x)}{2} + c_2$$

Raising both side to exponential gives

$$u^2 + 3u + 3 = e^{\frac{\ln(x)}{2} + c_2}$$

Which simplifies to

$$u^2 + 3u + 3 = c_3\sqrt{x}$$

Which simplifies to

$$u(x)^2 + 3u(x) + 3 = c_3\sqrt{x}\, e^{c_2}$$

The solution is

$$u(x)^2 + 3u(x) + 3 = c_3\sqrt{x}\, e^{c_2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2} + \frac{3y}{x} + 3 = c_3\sqrt{x}\, e^{c_2}$$
$$\frac{y^2}{x^2} + \frac{3y}{x} + 3 = c_3\sqrt{x}\, e^{c_2}$$

Substituting initial conditions and solving for c_2 gives $c_2 = \frac{\ln\left(\frac{9}{2c_3^2}\right)}{2}$. Hence the solution becomes Initial conditions are used to solve for c_3. Substituting $x = 2$ and $y = -6$ in the above solution gives an equation to solve for the constant of integration.

$$3 = 3c_3\sqrt{\frac{1}{c_3^2}}$$

This solution is valid for any c_3. Hence there are infinite number of solutions.

Summary

The solution(s) found are the following

$$\frac{y^2}{x^2} + \frac{3y}{x} + 3 = \frac{3c_3\sqrt{x}\,\sqrt{2}\,\sqrt{\frac{1}{c_3^2}}}{2} \tag{1}$$

Verification of solutions

$$\frac{y^2}{x^2} + \frac{3y}{x} + 3 = \frac{3c_3\sqrt{x}\,\sqrt{2}\,\sqrt{\frac{1}{c_3^2}}}{2}$$

Verified OK.

4.20.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{3x^2 + 9xy + 5y^2}{2x(3x + 2y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(3x^2 + 9xy + 5y^2)(b_3 - a_2)}{2x(3x + 2y)} - \frac{(3x^2 + 9xy + 5y^2)^2 a_3}{4x^2 (3x + 2y)^2} - \left(\frac{9y + 6x}{2x(3x + 2y)} - \frac{3x^2 + 9xy + 5y^2}{2x^2 (3x + 2y)} - \frac{3(3x^2 + 9xy + 5y^2)}{2x(3x + 2y)^2}\right)(x a_2 + y a_3 + a_1) - \left(\frac{10y + 9x}{2x(3x + 2y)} - \frac{3x^2 + 9xy + 5y^2}{x(3x + 2y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{18x^4 a_2 + 9x^4 a_3 + 6x^4 b_2 - 18x^4 b_3 + 24x^3 y a_2 + 54x^3 y a_3 + 12x^3 y b_2 - 24x^3 y b_3 + 6x^2 y^2 a_2 + 69x^2 y^2 a_3 + 4}{4x^2 (3x +} = 0$$

Setting the numerator to zero gives

$$-18x^4 a_2 - 9x^4 a_3 - 6x^4 b_2 + 18x^4 b_3 - 24x^3 y a_2 - 54x^3 y a_3 - 12x^3 y b_2 + 24x^3 y b_3 - 6x^2 y^2 a_2 - 69x^2 y^2 a_3 - 4x^2 y^2 b_2 + 6x^2 y^2 b_3 - 30x y^3 a_3 - 5y^4 a_3 - 42x^3 b_1 + 42x^2 y a_1 - 60x^2 y b_1 + 60x y^2 a_1 - 20x y^2 b_1 + 20y^3 a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-18a_2v_1^4 - 24a_2v_1^3v_2 - 6a_2v_1^2v_2^2 - 9a_3v_1^4 - 54a_3v_1^3v_2 - 69a_3v_1^2v_2^2 - 30a_3v_1v_2^3 \\ &- 5a_3v_2^4 - 6b_2v_1^4 - 12b_2v_1^3v_2 - 4b_2v_1^2v_2^2 + 18b_3v_1^4 + 24b_3v_1^3v_2 + 6b_3v_1^2v_2^2 \\ &+ 42a_1v_1^2v_2 + 60a_1v_1v_2^2 + 20a_1v_2^3 - 42b_1v_1^3 - 60b_1v_1^2v_2 - 20b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-18a_2 - 9a_3 - 6b_2 + 18b_3)\, v_1^4 + (-24a_2 - 54a_3 - 12b_2 + 24b_3)\, v_1^3v_2 \\ &- 42b_1v_1^3 + (-6a_2 - 69a_3 - 4b_2 + 6b_3)\, v_1^2v_2^2 + (42a_1 - 60b_1)\, v_1^2v_2 \\ &- 30a_3v_1v_2^3 + (60a_1 - 20b_1)\, v_1v_2^2 - 5a_3v_2^4 + 20a_1v_2^3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 20a_1 &= 0 \\ -30a_3 &= 0 \\ -5a_3 &= 0 \\ -42b_1 &= 0 \\ 42a_1 - 60b_1 &= 0 \\ 60a_1 - 20b_1 &= 0 \\ -24a_2 - 54a_3 - 12b_2 + 24b_3 &= 0 \\ -18a_2 - 9a_3 - 6b_2 + 18b_3 &= 0 \\ -6a_2 - 69a_3 - 4b_2 + 6b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{3x^2 + 9xy + 5y^2}{2x\,(3x + 2y)}\right)(x) \\
&= \frac{-3x^2 - 3xy - y^2}{6x + 4y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-3x^2 - 3xy - y^2}{6x + 4y}} dy
\end{aligned}
$$

Which results in

$$S = -2\ln\left(3x^2 + 3xy + y^2\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{3x^2 + 9xy + 5y^2}{2x\,(3x + 2y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-12x - 6y}{3x^2 + 3xy + y^2} \\
S_y &= \frac{-6x - 4y}{3x^2 + 3xy + y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{5}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{5}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -5\ln(R) + c_1 \tag{4}$$

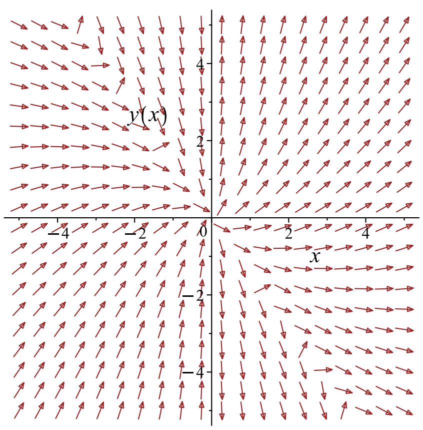
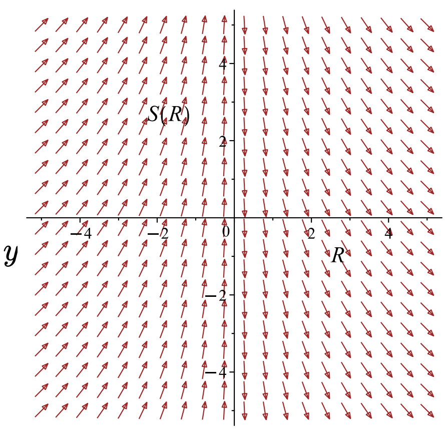
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-2\ln\left(y^2 + 3yx + 3x^2\right) = -5\ln(x) + c_1$$

Which simplifies to

$$-2\ln\left(y^2+3yx+3x^2\right) = -5\ln(x)+c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x^2+9xy+5y^2}{2x(3x+2y)}$ 	$R = x$ $S = -2\ln\left(3x^2+3xy+y\right.$	$\frac{dS}{dR} = -\frac{5}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = -6$ in the above solution gives an equation to solve for the constant of integration.

$$-2\ln(3) - 4\ln(2) = -5\ln(2) + c_1$$

$$c_1 = \ln(2) - 2\ln(3)$$

Substituting c_1 found above in the general solution gives

$$-2\ln\left(3x^2+3xy+y^2\right) = -5\ln(x)+\ln(2)-2\ln(3)$$

Summary

The solution(s) found are the following

$$-2\ln\left(y^2+3yx+3x^2\right) = -5\ln(x)+\ln(2)-2\ln(3) \tag{1}$$

Verification of solutions

$$-2\ln\left(y^2+3yx+3x^2\right) = -5\ln(x)+\ln(2)-2\ln(3)$$

Verified OK.

4.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-6x^2-4xy\right)\mathrm{d}y&=\left(-3x^2-9xy-5y^2\right)\mathrm{d}x\\ \left(3x^2+9xy+5y^2\right)\mathrm{d}x+\left(-6x^2-4xy\right)\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=3x^2+9xy+5y^2\\ N(x,y)&=-6x^2-4xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x^2 + 9xy + 5y^2)\\ &= 10y + 9x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-6x^2 - 4xy)\\ &= -12x - 4y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{6x^2 + 4xy}((10y + 9x) - (-12x - 4y))\\ &= -\frac{7}{2x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{7}{2x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{7\ln(x)}{2}}\\ &= \frac{1}{x^{\frac{7}{2}}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^{\frac{7}{2}}}(3x^2 + 9xy + 5y^2)\\ &= \frac{3x^2 + 9xy + 5y^2}{x^{\frac{7}{2}}}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^{\frac{7}{2}}}\left(-6x^2 - 4xy\right) \\
&= \frac{-6x - 4y}{x^{\frac{5}{2}}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{3x^2 + 9xy + 5y^2}{x^{\frac{7}{2}}}\right) + \left(\frac{-6x - 4y}{x^{\frac{5}{2}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{3x^2 + 9xy + 5y^2}{x^{\frac{7}{2}}}\,\mathrm{d}x \\
\phi &= -\frac{6\left(x^2 + xy + \frac{1}{3}y^2\right)}{x^{\frac{5}{2}}} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{6\left(x + \frac{2y}{3}\right)}{x^{\frac{5}{2}}} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-6x-4y}{x^{\frac{5}{2}}}$. Therefore equation (4) becomes

$$
\frac{-6x - 4y}{x^{\frac{5}{2}}} = \frac{-6x - 4y}{x^{\frac{5}{2}}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{6\left(x^2 + xy + \frac{1}{3}y^2\right)}{x^{\frac{5}{2}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{6\left(x^2 + xy + \frac{1}{3}y^2\right)}{x^{\frac{5}{2}}}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = -6$ in the above solution gives an equation to solve for the constant of integration.

$$-3\sqrt{2} = c_1$$

$$c_1 = -3\sqrt{2}$$

Substituting c_1 found above in the general solution gives

$$-\frac{6\left(x^2 + xy + \frac{1}{3}y^2\right)}{x^{\frac{5}{2}}} = -3\sqrt{2}$$

The above simplifies to

$$3\sqrt{2}\, x^{\frac{5}{2}} - 6x^2 - 6xy - 2y^2 = 0$$

Summary

The solution(s) found are the following

$$3\sqrt{2}\, x^{\frac{5}{2}} - 6x^2 - 6yx - 2y^2 = 0 \tag{1}$$

Verification of solutions

$$3\sqrt{2}\, x^{\frac{5}{2}} - 6x^2 - 6yx - 2y^2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.265 (sec). Leaf size: 21

```
dsolve([(3*x^2+9*x*y(x)+5*y(x)^2)-(6*x^2+4*x*y(x))*diff(y(x),x)=0,y(2) = -6],y(x), singsol=a
```

$$y(x) = -\frac{\left(3+\sqrt{-3+6\sqrt{2}\,\sqrt{x}}\right)x}{2}$$

✓ Solution by Mathematica

Time used: 37.251 (sec). Leaf size: 30

```
DSolve[{(3*x^2+9*x*y[x]+5*y[x]^2)-(6*x^2+4*x*y[x])*y'[x]==0,{y[2]==-6}},y[x],x,IncludeSingul
```

$$y(x) \to -\frac{1}{2}\left(\sqrt{6\sqrt{2}\sqrt{x}-3}+3\right)x$$

4.21 problem 22(a)

Internal problem ID [11634]
Internal file name [`OUTPUT/10616_Saturday_May_27_2023_04_14_52_AM_90704318/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 22(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{2y+(2x-y)\,y'=-x}$$

4.21.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2u(x)\,x+(2x-u(x)\,x)\,(u'(x)\,x+u(x))=-x$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{u^2-4u-1}{(u-2)\,x}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-4u-1}{u-2}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-4u-1}{u-2}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2-4u-1}{u-2}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln (u^2-4u-1)}{2} = -\ln (x) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2-4u-1} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u^2-4u-1} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u(x)^2-4u(x)-1} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{u(x)^2-4u(x)-1} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}-\frac{4y}{x}-1} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

$$\sqrt{\frac{y^2-4yx-x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{y^2-4yx-x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x} \tag{1}$$

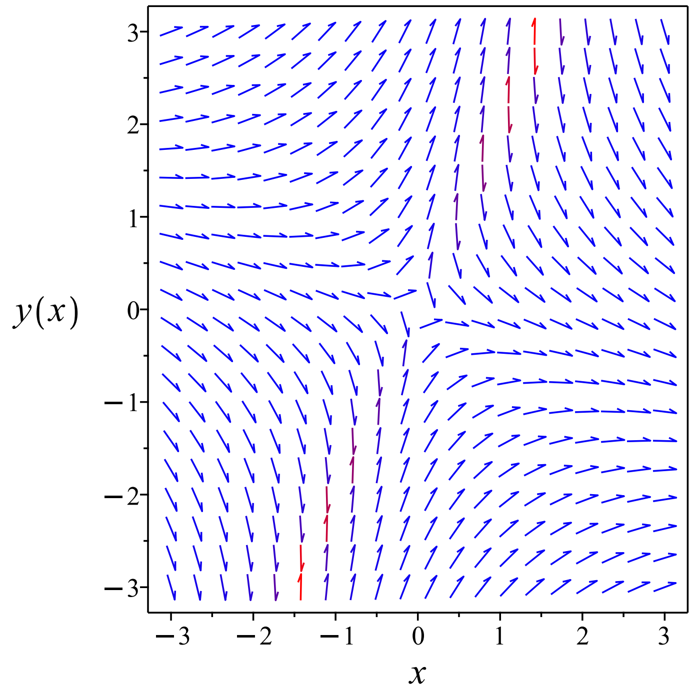


Figure 130: Slope field plot

Verification of solutions

$$\sqrt{\frac{y^2-4yx-x^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

4.21.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-x-2y}{2x-y} \tag{1}$$

Which becomes

$$(-y)\,dy = (-2x)\,dy + (-x-2y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-2x)\,dy + (-x-2y)\,dx = d\left(-\frac{1}{2}x^2 - 2xy\right)$$

Hence (2) becomes

$$(-y)\,dy = d\left(-\frac{1}{2}x^2 - 2xy\right)$$

Integrating both sides gives gives these solutions

$$y = 2x + \sqrt{5x^2 - 2c_1} + c_1$$
$$y = 2x - \sqrt{5x^2 - 2c_1} + c_1$$

Summary
The solution(s) found are the following

$$y = 2x + \sqrt{5x^2 - 2c_1} + c_1 \tag{1}$$
$$y = 2x - \sqrt{5x^2 - 2c_1} + c_1 \tag{2}$$

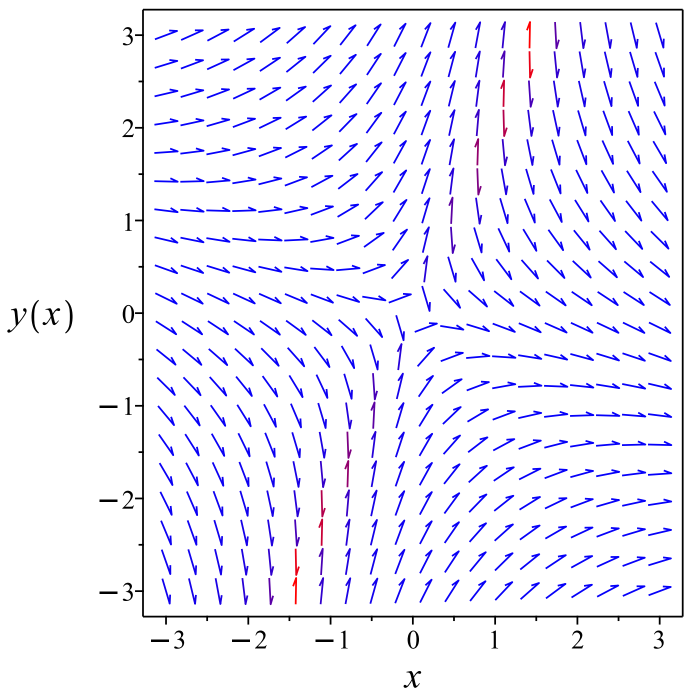


Figure 131: Slope field plot

Verification of solutions

$$y = 2x + \sqrt{5x^2 - 2c_1} + c_1$$

Verified OK.

$$y = 2x - \sqrt{5x^2 - 2c_1} + c_1$$

Verified OK.

4.21.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x+2y}{-2x+y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{(x+2y)(b_3-a_2)}{-2x+y} - \frac{(x+2y)^2 a_3}{(-2x+y)^2} \\
&- \left(\frac{1}{-2x+y} + \frac{2x+4y}{(-2x+y)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(\frac{2}{-2x+y} - \frac{x+2y}{(-2x+y)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{2x^2a_2 - x^2a_3 + 9x^2b_2 - 2x^2b_3 - 2xya_2 - 4xya_3 - 4xyb_2 + 2xyb_3 - 2y^2a_2 - 9y^2a_3 + y^2b_2 + 2y^2b_3 + 5xb_1 -}{(2x-y)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&2x^2a_2 - x^2a_3 + 9x^2b_2 - 2x^2b_3 - 2xya_2 - 4xya_3 - 4xyb_2 \\
&+ 2xyb_3 - 2y^2a_2 - 9y^2a_3 + y^2b_2 + 2y^2b_3 + 5xb_1 - 5ya_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$2a_2v_1^2 - 2a_2v_1v_2 - 2a_2v_2^2 - a_3v_1^2 - 4a_3v_1v_2 - 9a_3v_2^2 + 9b_2v_1^2 - 4b_2v_1v_2 + b_2v_2^2 - 2b_3v_1^2 + 2b_3v_1v_2 + 2b_3v_2^2 - 5a_1v_2 + 5b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(2a_2 - a_3 + 9b_2 - 2b_3)\, v_1^2 + (-2a_2 - 4a_3 - 4b_2 + 2b_3)\, v_1v_2 + 5b_1v_1 + (-2a_2 - 9a_3 + b_2 + 2b_3)\, v_2^2 - 5a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-5a_1 &= 0 \\
5b_1 &= 0 \\
-2a_2 - 9a_3 + b_2 + 2b_3 &= 0 \\
-2a_2 - 4a_3 - 4b_2 + 2b_3 &= 0 \\
2a_2 - a_3 + 9b_2 - 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= -4a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= 0 \\
b_2 &= a_3 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{x+2y}{-2x+y}\right)(x) \\
&= \frac{x^2 + 4xy - y^2}{2x - y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2+4xy-y^2}{2x-y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(-x^2 - 4xy + y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x+2y}{-2x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x+2y}{x^2+4xy-y^2} \\
S_y &= \frac{2x-y}{x^2+4xy-y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

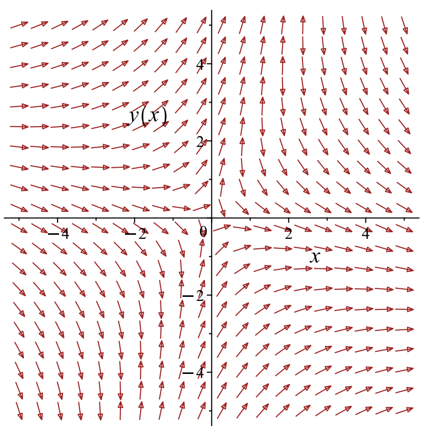
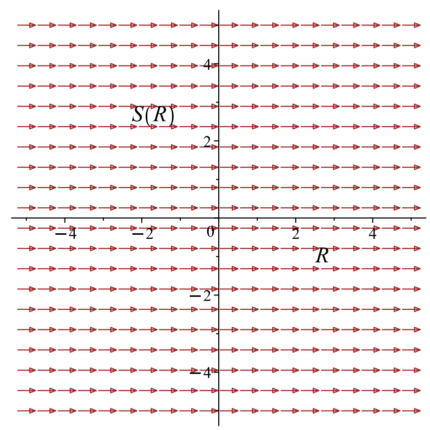
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2-4yx-x^2)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(y^2-4yx-x^2)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x+2y}{-2x+y}$ 	$R = x$ $S = \frac{\ln(-x^2 - 4xy + y^2)}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y^2 - 4yx - x^2)}{2} = c_1 \tag{1}$$

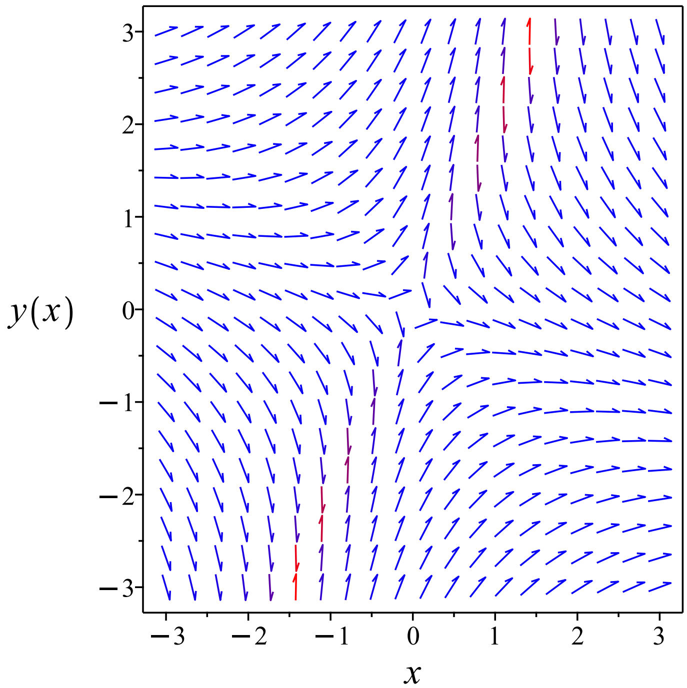


Figure 132: Slope field plot

Verification of solutions

$$\frac{\ln (y^2 - 4yx - x^2)}{2} = c_1$$

Verified OK.

4.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(2x - y)\, \mathrm{d}y &= (-x - 2y)\, \mathrm{d}x \\
(x + 2y)\, \mathrm{d}x + (2x - y)\, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= x + 2y \\
N(x, y) &= 2x - y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x + 2y) \\
&= 2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x - y) \\
&= 2
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int x + 2y \, dx$$
$$\phi = \frac{x(x + 4y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x - y$. Therefore equation (4) becomes

$$2x - y = 2x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int (-y) \, dy$$
$$f(y) = -\frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x+4y)}{2} - \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x+4y)}{2} - \frac{y^2}{2}$$

Summary
The solution(s) found are the following

$$\frac{x(4y+x)}{2} - \frac{y^2}{2} = c_1 \tag{1}$$

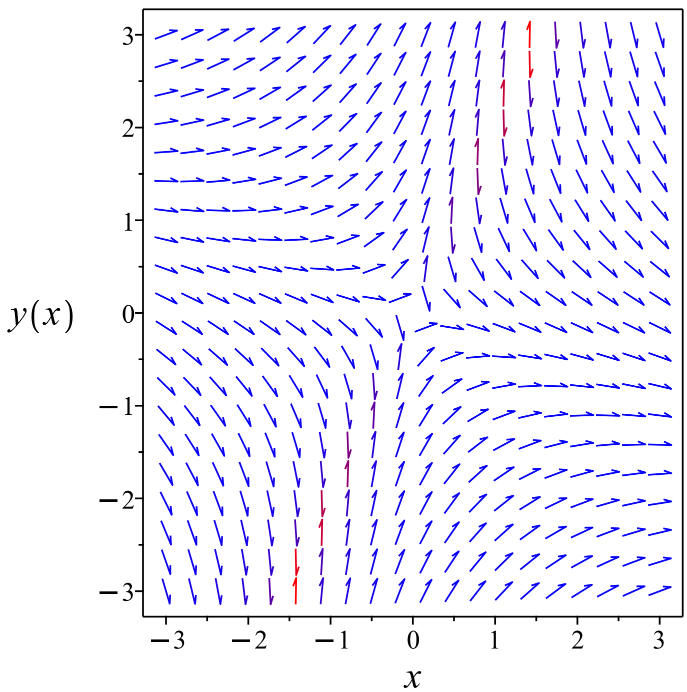


Figure 133: Slope field plot

Verification of solutions

$$\frac{x(4y+x)}{2} - \frac{y^2}{2} = c_1$$

Verified OK.

4.21.5 Maple step by step solution

Let's solve

$2y + (2x - y)\, y' = -x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $2 = 2$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (x + 2y)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{x^2}{2} + 2xy + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $2x - y = 2x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = -y$

- Solve for $f_1(y)$

  $f_1(y) = -\frac{y^2}{2}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = \frac{1}{2}x^2 + 2xy - \frac{1}{2}y^2$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\frac{1}{2}x^2 + 2xy - \frac{1}{2}y^2 = c_1$$

- Solve for y

$$\left\{y = 2x - \sqrt{5x^2 - 2c_1}, y = 2x + \sqrt{5x^2 - 2c_1}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 51

```
dsolve((x+2*y(x))+(2*x-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{2c_1x - \sqrt{5c_1^2x^2 + 1}}{c_1}$$
$$y(x) = \frac{2c_1x + \sqrt{5c_1^2x^2 + 1}}{c_1}$$

✓ Solution by Mathematica

Time used: 0.777 (sec). Leaf size: 94

```
DSolve[(x+2*y[x])+(2*x-y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2x - \sqrt{5x^2 + e^{2c_1}}$$
$$y(x) \to 2x + \sqrt{5x^2 + e^{2c_1}}$$
$$y(x) \to 2x - \sqrt{5}\sqrt{x^2}$$
$$y(x) \to \sqrt{5}\sqrt{x^2} + 2x$$

4.22 problem 22(b)

Internal problem ID [11635]
Internal file name [`OUTPUT/10617_Saturday_May_27_2023_04_14_56_AM_12227773/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 22(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{-y-(y+x)\,y'=-3x}$$

4.22.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-u(x)\,x-(u(x)\,x+x)\,(u'(x)\,x+u(x))=-3x$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u^2+2u-3}{(u+1)\,x}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+2u-3}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+2u-3}{u+1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2+2u-3}{u+1}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln\left(u^2+2u-3\right)}{2} = -\ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+2u-3} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u^2+2u-3} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u\left(x\right)^2 + 2u\left(x\right) - 3} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{u\left(x\right)^2 + 2u\left(x\right) - 3} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2} + \frac{2y}{x} - 3} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

$$\sqrt{\frac{y^2+2yx-3x^2}{x^2}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Which simplifies to

$$\sqrt{-\frac{\left(3x+y\right)\left(-y+x\right)}{x^2}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{-\frac{\left(3x+y\right)\left(-y+x\right)}{x^2}} = \frac{c_3\mathrm{e}^{c_2}}{x} \tag{1}$$

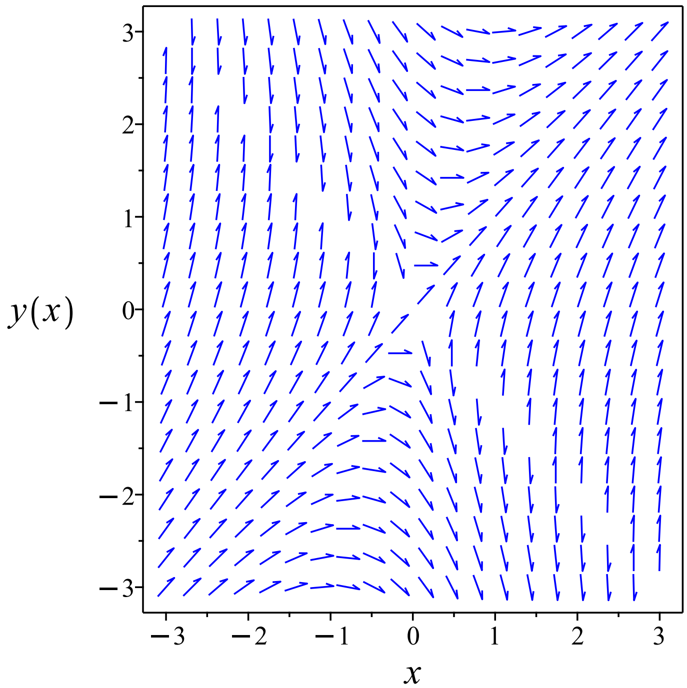


Figure 134: Slope field plot

Verification of solutions

$$\sqrt{-\frac{(3x+y)(-y+x)}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

4.22.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-3x+y}{-y-x} \tag{1}$$

Which becomes

$$(y)\,dy = (-x)\,dy + (3x-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (3x-y)\,dx = d\left(\frac{3}{2}x^2 - xy\right)$$

Hence (2) becomes

$$(y)\,dy = d\left(\frac{3}{2}x^2 - xy\right)$$

Integrating both sides gives gives these solutions

$$y = -x + \sqrt{4x^2 + 2c_1} + c_1$$
$$y = -x - \sqrt{4x^2 + 2c_1} + c_1$$

Summary

The solution(s) found are the following

$$y = -x + \sqrt{4x^2 + 2c_1} + c_1 \tag{1}$$
$$y = -x - \sqrt{4x^2 + 2c_1} + c_1 \tag{2}$$

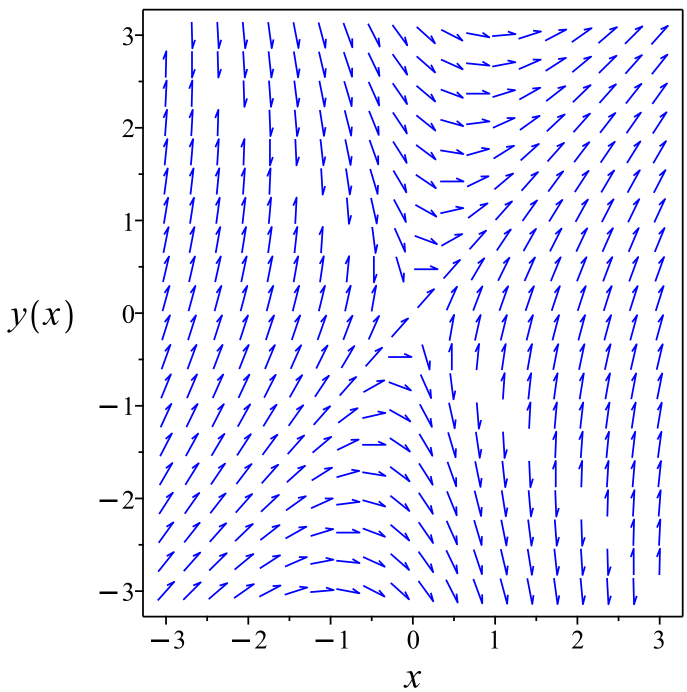


Figure 135: Slope field plot

Verification of solutions

$$y = -x + \sqrt{4x^2 + 2c_1} + c_1$$

Verified OK.

$$y = -x - \sqrt{4x^2 + 2c_1} + c_1$$

Verified OK.

4.22.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{-3x+y}{y+x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2 + ya_3 + a_1 \tag{1E}
$$

$$
\eta = xb_2 + yb_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 &- \frac{(-3x+y)(b_3-a_2)}{y+x} - \frac{(-3x+y)^2 a_3}{(y+x)^2} \\
&- \left(\frac{3}{y+x} + \frac{-3x+y}{(y+x)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{1}{y+x} + \frac{-3x+y}{(y+x)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{3x^2a_2 + 9x^2a_3 - 5x^2b_2 - 3x^2b_3 + 6xya_2 - 6xya_3 - 2xyb_2 - 6xyb_3 - y^2a_2 + 5y^2a_3 - y^2b_2 + y^2b_3 - 4xb_1}{(y+x)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-3x^2a_2 - 9x^2a_3 + 5x^2b_2 + 3x^2b_3 - 6xya_2 + 6xya_3 + 2xyb_2 \\
&+ 6xyb_3 + y^2a_2 - 5y^2a_3 + y^2b_2 - y^2b_3 + 4xb_1 - 4ya_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-3a_2v_1^2 - 6a_2v_1v_2 + a_2v_2^2 - 9a_3v_1^2 + 6a_3v_1v_2 - 5a_3v_2^2 + 5b_2v_1^2 \\ + 2b_2v_1v_2 + b_2v_2^2 + 3b_3v_1^2 + 6b_3v_1v_2 - b_3v_2^2 - 4a_1v_2 + 4b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-3a_2 - 9a_3 + 5b_2 + 3b_3) v_1^2 + (-6a_2 + 6a_3 + 2b_2 + 6b_3) v_1v_2 \\ + 4b_1v_1 + (a_2 - 5a_3 + b_2 - b_3) v_2^2 - 4a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -4a_1 &= 0 \\ 4b_1 &= 0 \\ -6a_2 + 6a_3 + 2b_2 + 6b_3 &= 0 \\ -3a_2 - 9a_3 + 5b_2 + 3b_3 &= 0 \\ a_2 - 5a_3 + b_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 0 \\ b_2 &= 3a_3 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{-3x + y}{y + x}\right)(x) \\
&= \frac{-3x^2 + 2xy + y^2}{y + x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-3x^2 + 2xy + y^2}{y + x}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(-3x^2 + 2xy + y^2)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-3x + y}{y + x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3x - y}{(3x + y)(-y + x)} \\ S_y &= \frac{-y - x}{(3x + y)(-y + x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

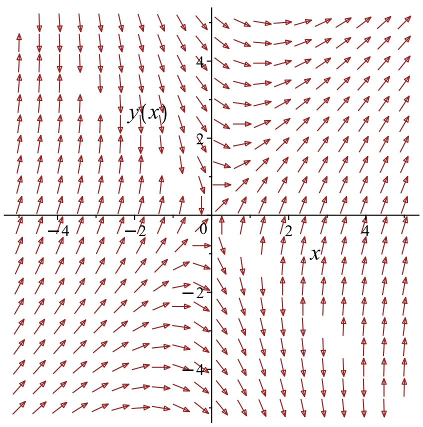
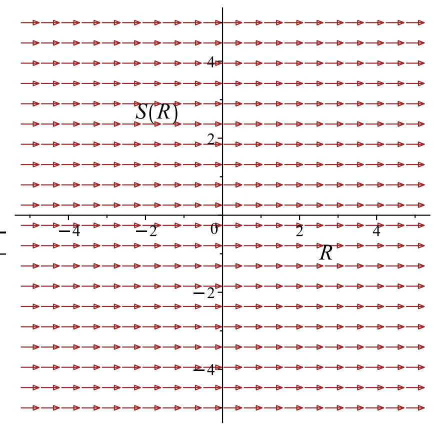
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(-3x - y)}{2} + \frac{\ln(-y + x)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(-3x - y)}{2} + \frac{\ln(-y + x)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-3x+y}{y+x}$ 	$R = x$ $S = \frac{\ln(-3x-y)}{2} + \frac{\ln(-}{}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(-3x-y)}{2} + \frac{\ln(-y+x)}{2} = c_1 \tag{1}$$

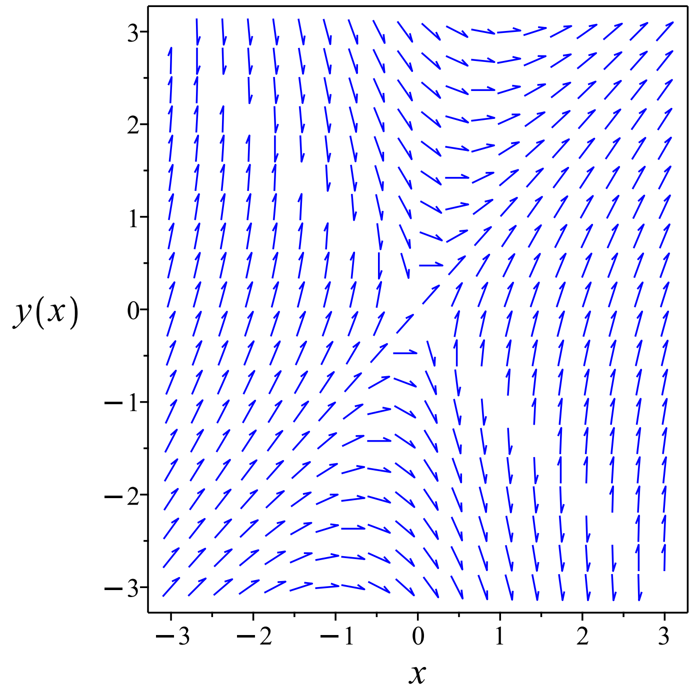


Figure 136: Slope field plot

Verification of solutions

$$\frac{\ln(-3x-y)}{2}+\frac{\ln(-y+x)}{2}=c_1$$

Verified OK.

4.22.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(-y - x) \, \mathrm{d}y &= (-3x + y) \, \mathrm{d}x \\
(3x - y) \, \mathrm{d}x + (-y - x) \, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 3x - y \\
N(x, y) &= -y - x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x - y) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y - x) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 3x - y \, \mathrm{d}x$$
$$\phi = \frac{x(3x - 2y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y - x$. Therefore equation (4) becomes

$$-y - x = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-y) \, \mathrm{d}y$$
$$f(y) = -\frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(3x-2y)}{2} - \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(3x-2y)}{2} - \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$\frac{x(3x-2y)}{2} - \frac{y^2}{2} = c_1 \tag{1}$$

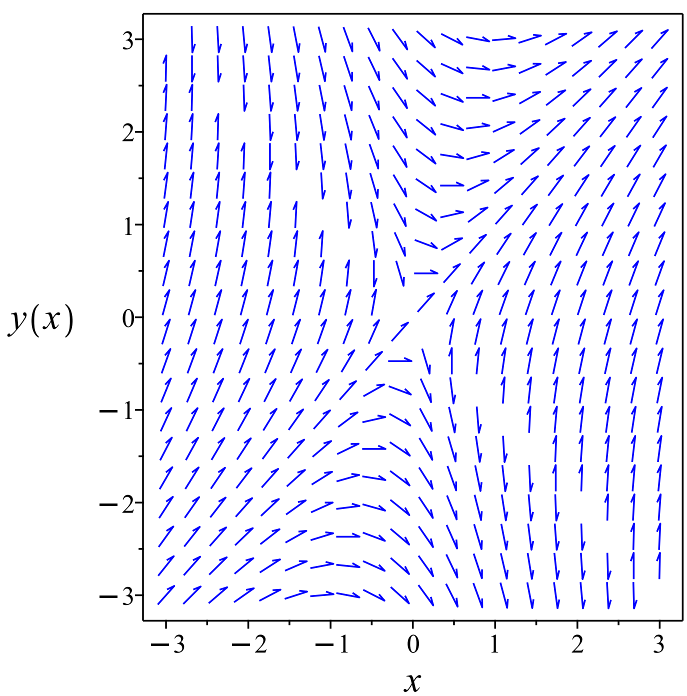


Figure 137: Slope field plot

Verification of solutions

$$\frac{x(3x-2y)}{2} - \frac{y^2}{2} = c_1$$

Verified OK.

4.22.5 Maple step by step solution

Let's solve

$-y-(y+x)\,y'=-3x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$

  ◦ Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  ◦ Evaluate derivatives

  $-1=-1$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(3x-y)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{3x^2}{2}-xy+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $-y-x=-x+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=-y$

- Solve for $f_1(y)$

  $f_1(y)=-\frac{y^2}{2}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x, y) = \frac{3}{2}x^2 - xy - \frac{1}{2}y^2$$

- Substitute $F(x, y)$ into the solution of the ODE

$$\frac{3}{2}x^2 - xy - \frac{1}{2}y^2 = c_1$$

- Solve for y

$$\left\{y = -x - \sqrt{4x^2 - 2c_1}, y = -x + \sqrt{4x^2 - 2c_1}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 51

```
dsolve((3*x-y(x))-(x+y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-c_1 x - \sqrt{4c_1^2 x^2 + 1}}{c_1}$$

$$y(x) = \frac{-c_1 x + \sqrt{4c_1^2 x^2 + 1}}{c_1}$$

✓ Solution by Mathematica

Time used: 0.752 (sec). Leaf size: 85

```
DSolve[(3*x-y[x])-(x+y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x - \sqrt{4x^2 + e^{2c_1}}$$
$$y(x) \to -x + \sqrt{4x^2 + e^{2c_1}}$$
$$y(x) \to -2\sqrt{x^2} - x$$
$$y(x) \to 2\sqrt{x^2} - x$$

4.23 problem 23(a)

Internal problem ID [11636]
Internal file name [`OUTPUT/10618_Saturday_May_27_2023_04_15_00_AM_83412974/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 23(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{2y^2 + (4yx - y^2)\, y' = -x^2}$$

4.23.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$2u(x)^2\, x^2 + (4u(x)\, x^2 - u(x)^2\, x^2)\, (u'(x)\, x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^3 - 6u^2 - 1}{ux\,(u-4)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^3-6u^2-1}{(u-4)u}$. Integrating both sides gives

$$\frac{1}{\frac{u^3-6u^2-1}{(u-4)u}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^3-6u^2-1}{(u-4)u}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln(u^3-6u^2-1)}{3} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$\left(u^3-6u^2-1\right)^{\frac{1}{3}} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\left(u^3-6u^2-1\right)^{\frac{1}{3}} = \frac{c_3}{x}$$

Which simplifies to

$$\left(u(x)^3-6u(x)^2-1\right)^{\frac{1}{3}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

The solution is

$$\left(u(x)^3-6u(x)^2-1\right)^{\frac{1}{3}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{y^3}{x^3}-\frac{6y^2}{x^2}-1\right)^{\frac{1}{3}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

$$\left(\frac{y^3-6y^2x-x^3}{x^3}\right)^{\frac{1}{3}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

<u>Summary</u>

The solution(s) found are the following

$$\left(\frac{y^3-6y^2x-x^3}{x^3}\right)^{\frac{1}{3}} = \frac{c_3\mathrm{e}^{c_2}}{x} \tag{1}$$

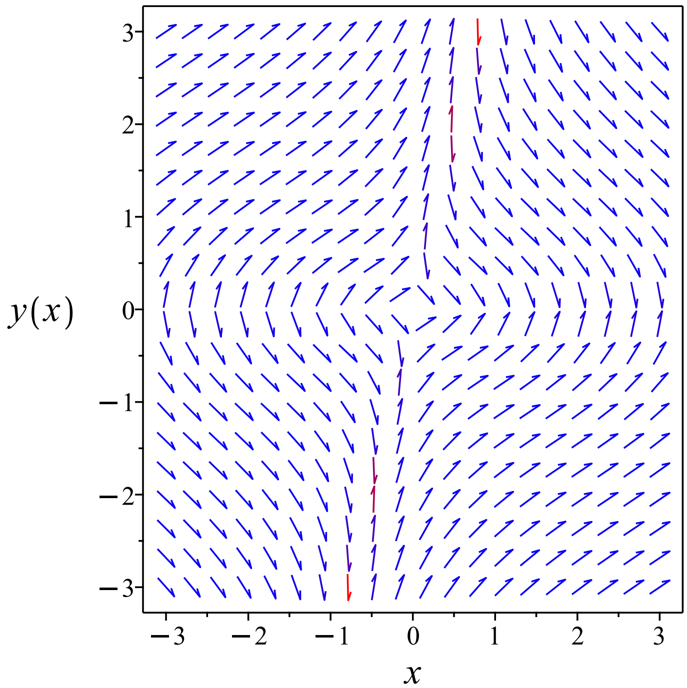


Figure 138: Slope field plot

Verification of solutions

$$\left(\frac{y^3 - 6y^2x - x^3}{x^3}\right)^{\frac{1}{3}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

4.23.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x^2 + 2y^2}{y\left(-4x + y\right)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(x^2+2y^2)(b_3-a_2)}{y(-4x+y)} - \frac{(x^2+2y^2)^2 a_3}{y^2(-4x+y)^2} - \left(\frac{2x}{y(-4x+y)} + \frac{4x^2+8y^2}{y(-4x+y)^2}\right)(xa_2+ya_3+a_1) - \left(\frac{4}{-4x+y} - \frac{x^2+2y^2}{y^2(-4x+y)} - \frac{x^2+2y^2}{y(-4x+y)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4a_3+4x^4b_2-8x^3ya_2-2x^3yb_2+8x^3yb_3+3x^2y^2a_2-24x^2y^2b_2-3x^2y^2b_3+2x\,y^3a_3+8x\,y^3b_2+2y^4a_2}{y^2(4x-y)^2} = 0$$

Setting the numerator to zero gives

$$-x^4a_3-4x^4b_2+8x^3ya_2+2x^3yb_2-8x^3yb_3-3x^2y^2a_2+24x^2y^2b_2+3x^2y^2b_3-2x\,y^3a_3-8x\,y^3b_2-2y^4a_2-12y^4a_3+y^4b_2+2y^4b_3-4x^3b_1+4x^2ya_1+2x^2yb_1-2x\,y^2a_1+8x\,y^2b_1-8y^3a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$8a_2v_1^3v_2-3a_2v_1^2v_2^2-2a_2v_2^4-a_3v_1^4-2a_3v_1v_2^3-12a_3v_2^4-4b_2v_1^4+2b_2v_1^3v_2+24b_2v_1^2v_2^2-8b_2v_1v_2^3+b_2v_2^4-8b_3v_1^3v_2+3b_3v_1^2v_2^2+2b_3v_2^4+4a_1v_1^2v_2-2a_1v_1v_2^2-8a_1v_2^3-4b_1v_1^3+2b_1v_1^2v_2+8b_1v_1v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_3 - 4b_2)\, v_1^4 + (8a_2 + 2b_2 - 8b_3)\, v_1^3 v_2 - 4b_1 v_1^3 + (-3a_2 + 24b_2 + 3b_3)\, v_1^2 v_2^2 + (4a_1 + 2b_1)\, v_1^2 v_2 + (-2a_3 - 8b_2)\, v_1 v_2^3 + (-2a_1 + 8b_1)\, v_1 v_2^2 + (-2a_2 - 12a_3 + b_2 + 2b_3)\, v_2^4 - 8a_1 v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-8a_1 &= 0 \\
-4b_1 &= 0 \\
-2a_1 + 8b_1 &= 0 \\
4a_1 + 2b_1 &= 0 \\
-2a_3 - 8b_2 &= 0 \\
-a_3 - 4b_2 &= 0 \\
-3a_2 + 24b_2 + 3b_3 &= 0 \\
8a_2 + 2b_2 - 8b_3 &= 0 \\
-2a_2 - 12a_3 + b_2 + 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{x^2+2y^2}{y\,(-4x+y)}\right)(x) \\
&= \frac{x^3+6x\,y^2-y^3}{4xy-y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^3+6x\,y^2-y^3}{4xy-y^2}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(-x^3-6x\,y^2+y^3\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2+2y^2}{y\,(-4x+y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x^2 + 2y^2}{x^3 + 6x\,y^2 - y^3} \\
S_y &= \frac{y(4x - y)}{x^3 + 6x\,y^2 - y^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

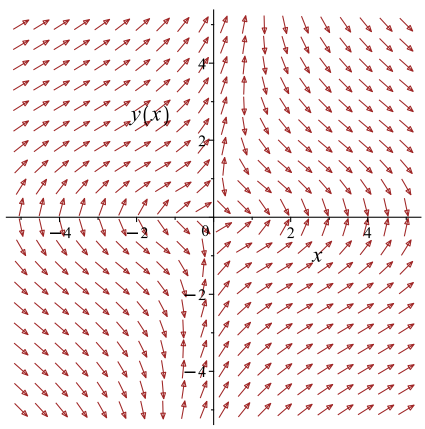
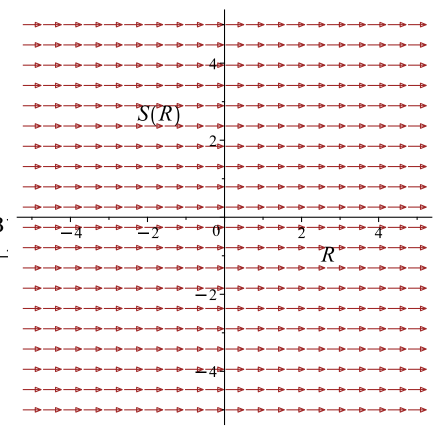
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(y^3 - 6y^2x - x^3\right)}{3} = c_1$$

Which simplifies to

$$\frac{\ln\left(y^3 - 6y^2x - x^3\right)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+2y^2}{y(-4x+y)}$ 	$R = x$ $S = \frac{\ln(-x^3 - 6x\,y^2 + y^3)}{3}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y^3 - 6y^2x - x^3)}{3} = c_1 \tag{1}$$

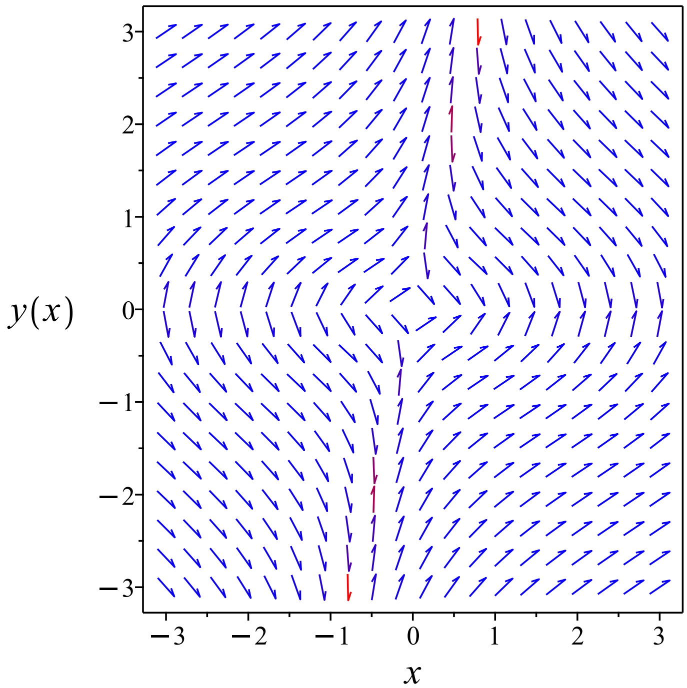


Figure 139: Slope field plot

Verification of solutions

$$\frac{\ln\left(y^3 - 6y^2x - x^3\right)}{3} = c_1$$

Verified OK.

4.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(4xy - y^2\right)\mathrm{d}y = \left(-x^2 - 2y^2\right)\mathrm{d}x$$
$$\left(x^2 + 2y^2\right)\mathrm{d}x + \left(4xy - y^2\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = x^2 + 2y^2$$
$$N(x, y) = 4xy - y^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + 2y^2\right)\\ &= 4y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(4xy - y^2\right)\\ &= 4y\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int x^2+2y^2\,\mathrm{d}x$$
$$\phi=\frac{1}{3}x^3+2x\,y^2+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=4xy+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=4xy-y^2$. Therefore equation (4) becomes

$$4xy-y^2=4xy+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=-y^2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int \left(-y^2\right)\mathrm{d}y$$
$$f(y)=-\frac{y^3}{3}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{3}x^3 + 2x\,y^2 - \frac{1}{3}y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3}x^3 + 2x\,y^2 - \frac{1}{3}y^3$$

Summary

The solution(s) found are the following

$$\frac{x^3}{3} + 2y^2x - \frac{y^3}{3} = c_1 \tag{1}$$

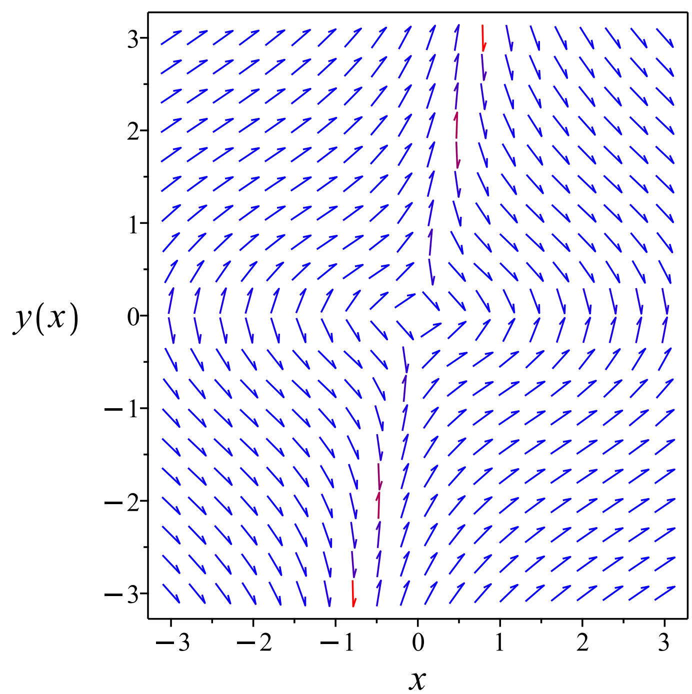


Figure 140: Slope field plot

Verification of solutions

$$\frac{x^3}{3} + 2y^2x - \frac{y^3}{3} = c_1$$

Verified OK.

4.23.4 Maple step by step solution

Let's solve

$2y^2 + (4yx - y^2)\, y' = -x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  - Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y} F(x,y)\right) y' = 0$

  - Evaluate derivatives

    $4y = 4y$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y} F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (x^2 + 2y^2)\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \frac{x^3}{3} + 2x\, y^2 + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y} F(x,y)$

- Compute derivative

  $4xy - y^2 = 4xy + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = -y^2$

- Solve for $f_1(y)$

  $f_1(y) = -\frac{y^3}{3}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$F(x,y) = \frac{1}{3}x^3 + 2x\,y^2 - \frac{1}{3}y^3$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{1}{3}x^3 + 2x\,y^2 - \frac{1}{3}y^3 = c_1$

- Solve for y

$$\left\{y = \frac{\left(68x^3-12c_1+4\sqrt{33x^6-102c_1x^3+9c_1^2}\right)^{\frac{1}{3}}}{2} + \frac{8x^2}{\left(68x^3-12c_1+4\sqrt{33x^6-102c_1x^3+9c_1^2}\right)^{\frac{1}{3}}} + 2x, y = -\frac{\left(68x^3-12c_1+4\sqrt{\right.}}{}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 439

```
dsolve((x^2+2*y(x)^2)+(4*x*y(x)-y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\frac{\left(4+68c_1^3x^3+4\sqrt{33c_1^6x^6+34c_1^3x^3+1}\right)^{\frac{1}{3}}}{2} + \frac{8x^2c_1^2}{\left(4+68c_1^3x^3+4\sqrt{33c_1^6x^6+34c_1^3x^3+1}\right)^{\frac{1}{3}}} + 2c_1x}{c_1}$$

$$y(x) = \frac{-\frac{\left(4+68c_1^3x^3+4\sqrt{33c_1^6x^6+34c_1^3x^3+1}\right)^{\frac{1}{3}}}{4} - \frac{4x^2c_1^2}{\left(4+68c_1^3x^3+4\sqrt{33c_1^6x^6+34c_1^3x^3+1}\right)^{\frac{1}{3}}} + 2c_1x - \frac{i\sqrt{3}\left(-16c_1^2x^2+\left(4+68c_1^3x^3+4\sqrt{33c_1^6x^6 \cdots}\right.\right.}{4\left(4+68c_1^3x^3+4\sqrt{33c_1^6x^6+34c_1^3 \cdots}\right.}}{c_1}$$

$$y(x) = -\frac{16i\sqrt{3}\,c_1^2x^2 - i\sqrt{3}\left(4+68c_1^3x^3+4\sqrt{33c_1^6x^6+34c_1^3x^3+1}\right)^{\frac{2}{3}} + 16c_1^2x^2 - 8c_1x\left(4+68c_1^3x^3+4\sqrt{33c_1^6x \cdots}\right.}{4\left(4+68c_1^3x^3+4\sqrt{33c_1^6x^6+34c_1^3x^3+1}\right)^{\cdots}}$$

✓ Solution by Mathematica

Time used: 33.481 (sec). Leaf size: 731

```
DSolve[(x^2+2*y[x]^2)+(4*x*y[x]-y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{17x^3 + \sqrt{33x^6 + 34e^{3c_1}x^3 + e^{6c_1}} + e^{3c_1}}}{\sqrt[3]{2}} + \frac{4\sqrt[3]{2}x^2}{\sqrt[3]{17x^3 + \sqrt{33x^6 + 34e^{3c_1}x^3 + e^{6c_1}} + e^{3c_1}}} + 2x$$

$$y(x) \to -\frac{\left(1 - i\sqrt{3}\right)\sqrt[3]{17x^3 + \sqrt{33x^6 + 34e^{3c_1}x^3 + e^{6c_1}} + e^{3c_1}}}{2\sqrt[3]{2}} - \frac{2\sqrt[3]{2}\left(1 + i\sqrt{3}\right)x^2}{\sqrt[3]{17x^3 + \sqrt{33x^6 + 34e^{3c_1}x^3 + e^{6c_1}} + e^{3c_1}}} + 2x$$

$$y(x) \to -\frac{\left(1 + i\sqrt{3}\right)\sqrt[3]{17x^3 + \sqrt{33x^6 + 34e^{3c_1}x^3 + e^{6c_1}} + e^{3c_1}}}{2\sqrt[3]{2}} - \frac{2\sqrt[3]{2}\left(1 - i\sqrt{3}\right)x^2}{\sqrt[3]{17x^3 + \sqrt{33x^6 + 34e^{3c_1}x^3 + e^{6c_1}} + e^{3c_1}}} + 2x$$

$$y(x) \to \frac{8\sqrt[3]{2}x^2 + 4\sqrt[3]{\sqrt{33}\sqrt{x^6} + 17x^3}\,x + 2^{2/3}\left(\sqrt{33}\sqrt{x^6} + 17x^3\right)^{2/3}}{2\sqrt[3]{\sqrt{33}\sqrt{x^6} + 17x^3}}$$

$$y(x) \to \frac{8i\sqrt[3]{2}\sqrt{3}x^2 - 8\sqrt[3]{2}x^2 + 8\sqrt[3]{\sqrt{33}\sqrt{x^6} + 17x^3}\,x - i2^{2/3}\sqrt{3}\left(\sqrt{33}\sqrt{x^6} + 17x^3\right)^{2/3} - 2^{2/3}\left(\sqrt{33}\sqrt{x^6} + 17x^3\right)}{4\sqrt[3]{\sqrt{33}\sqrt{x^6} + 17x^3}}$$

$$y(x) \to \frac{\left(\sqrt{33}\sqrt{x^6} + 17x^3\right)^{2/3}\text{Root}\left[2\#1^3 - 1\&, 3\right] - 4\sqrt[3]{-2}x^2 + 2\sqrt[3]{\sqrt{33}\sqrt{x^6} + 17x^3}\,x}{\sqrt[3]{\sqrt{33}\sqrt{x^6} + 17x^3}}$$

4.24 problem 23(b)

Internal problem ID [11637]
Internal file name [`OUTPUT/10619_Saturday_May_27_2023_04_15_05_AM_75348991/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.2 (Separable equations). Exercises page 47
Problem number: 23(b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class B`]]
```

$$\boxed{2yx + y^2 + (2yx + x^2)\, y' = -2x^2}$$

4.24.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$2u(x)\, x^2 + u(x)^2\, x^2 + (2u(x)\, x^2 + x^2)\, (u'(x)\, x + u(x)) = -2x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u^2 + 3u + 2}{x\,(2u + 1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{3u^2+3u+2}{2u+1}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{3u^2+3u+2}{2u+1}}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{\frac{3u^2+3u+2}{2u+1}}\, du &= \int -\frac{1}{x}\, dx \\
\frac{\ln\left(3u^2+3u+2\right)}{3} &= -\ln\left(x\right) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\left(3u^2+3u+2\right)^{\frac{1}{3}} = \mathrm{e}^{-\ln(x)+c_2}
$$

Which simplifies to

$$
\left(3u^2+3u+2\right)^{\frac{1}{3}} = \frac{c_3}{x}
$$

Which simplifies to

$$
\left(3u(x)^2+3u(x)+2\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{x}
$$

The solution is

$$
\left(3u(x)^2+3u(x)+2\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{x}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\left(\frac{3y^2}{x^2}+\frac{3y}{x}+2\right)^{\frac{1}{3}} &= \frac{c_3 \mathrm{e}^{c_2}}{x} \\
\left(\frac{3y^2+3yx+2x^2}{x^2}\right)^{\frac{1}{3}} &= \frac{c_3 \mathrm{e}^{c_2}}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\left(\frac{3y^2+3yx+2x^2}{x^2}\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{x} \tag{1}
$$

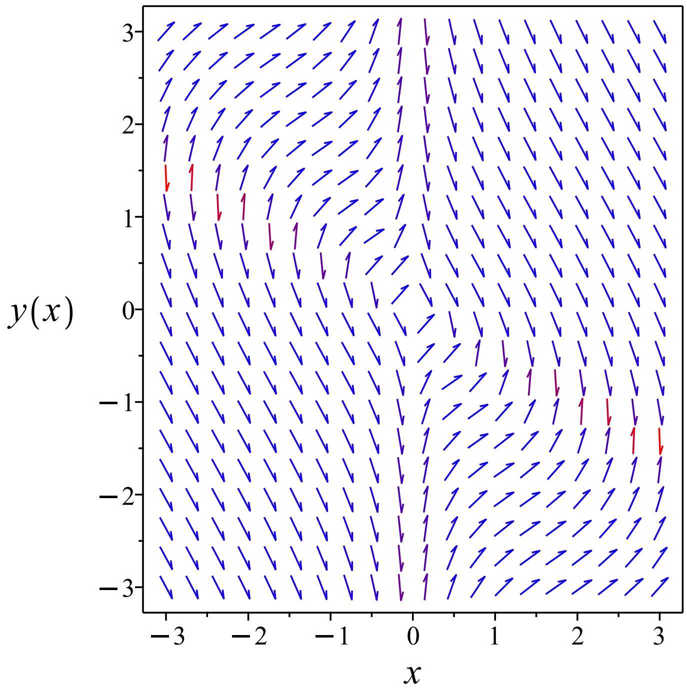


Figure 141: Slope field plot

Verification of solutions

$$\left(\frac{3y^2 + 3yx + 2x^2}{x^2}\right)^{\frac{1}{3}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Verified OK.

4.24.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2x^2 + 2xy + y^2}{x(x + 2y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(2x^2+2xy+y^2)(b_3-a_2)}{x(x+2y)} - \frac{(2x^2+2xy+y^2)^2 a_3}{x^2(x+2y)^2} \\ & - \left(-\frac{4x+2y}{x(x+2y)} + \frac{2x^2+2xy+y^2}{x^2(x+2y)} + \frac{2x^2+2xy+y^2}{x(x+2y)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{2y+2x}{x(x+2y)} + \frac{4x^2+4xy+2y^2}{x(x+2y)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^4a_2 - 4x^4a_3 - x^4b_2 - 2x^4b_3 + 8x^3ya_2 - 8x^3ya_3 + 6x^3yb_2 - 8x^3yb_3 + 3x^2y^2a_2 - 6x^2y^2a_3 + 6x^2y^2b_2 - 3x}{x^2(x+2y)^2}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 2x^4a_2 - 4x^4a_3 - x^4b_2 - 2x^4b_3 + 8x^3ya_2 - 8x^3ya_3 + 6x^3yb_2 - 8x^3yb_3 \\ & + 3x^2y^2a_2 - 6x^2y^2a_3 + 6x^2y^2b_2 - 3x^2y^2b_3 - 6x\,y^3a_3 - 3y^4a_3 \\ & - 2x^3b_1 + 2x^2ya_1 + 2x^2yb_1 - 2x\,y^2a_1 + 2x\,y^2b_1 - 2y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 2a_2v_1^4 + 8a_2v_1^3v_2 + 3a_2v_1^2v_2^2 - 4a_3v_1^4 - 8a_3v_1^3v_2 - 6a_3v_1^2v_2^2 - 6a_3v_1v_2^3 \\ & - 3a_3v_2^4 - b_2v_1^4 + 6b_2v_1^3v_2 + 6b_2v_1^2v_2^2 - 2b_3v_1^4 - 8b_3v_1^3v_2 - 3b_3v_1^2v_2^2 \\ & + 2a_1v_1^2v_2 - 2a_1v_1v_2^2 - 2a_1v_2^3 - 2b_1v_1^3 + 2b_1v_1^2v_2 + 2b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(2a_2 - 4a_3 - b_2 - 2b_3)\, v_1^4 + (8a_2 - 8a_3 + 6b_2 - 8b_3)\, v_1^3 v_2 \\
&\quad - 2b_1 v_1^3 + (3a_2 - 6a_3 + 6b_2 - 3b_3)\, v_1^2 v_2^2 + (2a_1 + 2b_1)\, v_1^2 v_2 \\
&\quad - 6a_3 v_1 v_2^3 + (-2a_1 + 2b_1)\, v_1 v_2^2 - 3a_3 v_2^4 - 2a_1 v_2^3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2a_1 &= 0 \\
-6a_3 &= 0 \\
-3a_3 &= 0 \\
-2b_1 &= 0 \\
-2a_1 + 2b_1 &= 0 \\
2a_1 + 2b_1 &= 0 \\
2a_2 - 4a_3 - b_2 - 2b_3 &= 0 \\
3a_2 - 6a_3 + 6b_2 - 3b_3 &= 0 \\
8a_2 - 8a_3 + 6b_2 - 8b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{2x^2+2xy+y^2}{x\,(x+2y)}\right)(x) \\
&= \frac{2x^2+3xy+3y^2}{x+2y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2x^2+3xy+3y^2}{x+2y}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln(2x^2+3xy+3y^2)}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{2x^2+2xy+y^2}{x\,(x+2y)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{4x+3y}{6x^2+9xy+9y^2} \\
S_y &= \frac{x+2y}{2x^2+3xy+3y^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{3x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{3R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln(R)}{3} + c_1 \tag{4}
$$

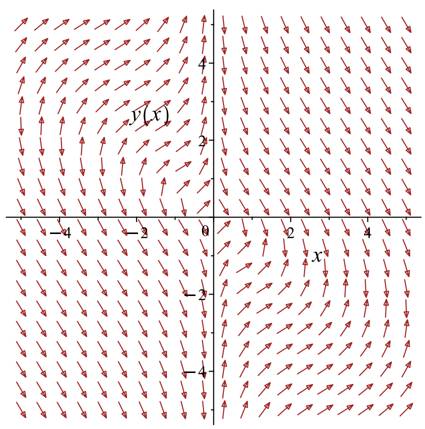
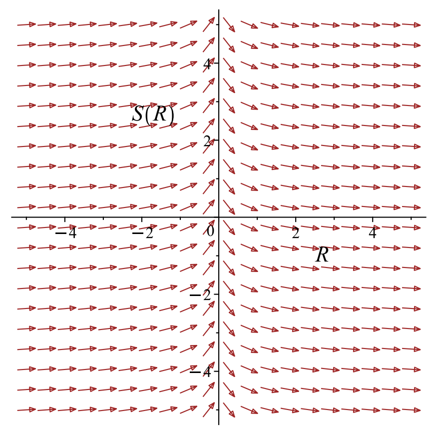
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(3y^2+3yx+2x^2)}{3} = -\frac{\ln(x)}{3} + c_1
$$

Which simplifies to

$$
\frac{\ln(3y^2+3yx+2x^2)}{3} = -\frac{\ln(x)}{3} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x^2+2xy+y^2}{x(x+2y)}$ 	$R = x$ $S = \frac{\ln(2x^2+3xy+3y^2)}{3}$	$\frac{dS}{dR} = -\frac{1}{3R}$ 

Summary

The solution(s) found are the following

$$\frac{\ln(3y^2+3yx+2x^2)}{3} = -\frac{\ln(x)}{3} + c_1 \tag{1}$$

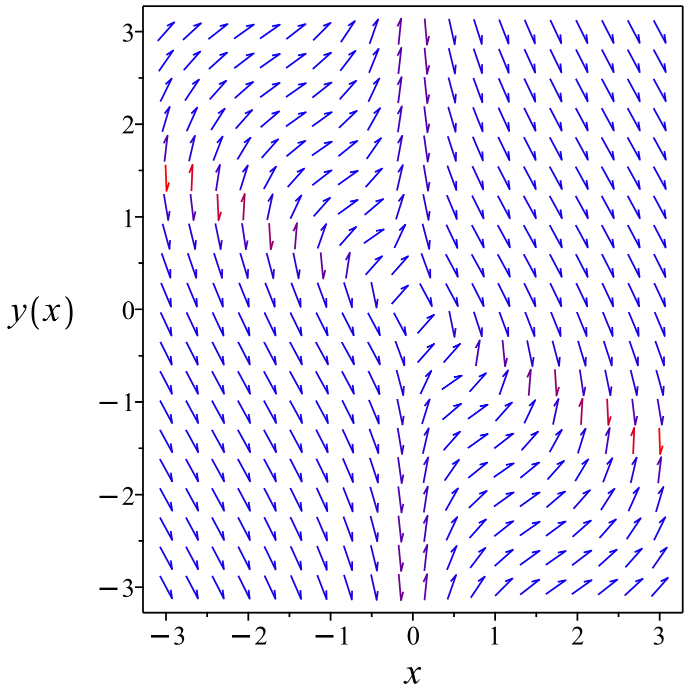


Figure 142: Slope field plot

Verification of solutions

$$\frac{\ln (3y^2 + 3yx + 2x^2)}{3} = -\frac{\ln (x)}{3} + c_1$$

Verified OK.

4.24.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x^2 + 2xy\right)\mathrm{d}y = \left(-2x^2 - 2xy - y^2\right)\mathrm{d}x$$
$$\left(2x^2 + 2xy + y^2\right)\mathrm{d}x + \left(x^2 + 2xy\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 2x^2 + 2xy + y^2$$
$$N(x, y) = x^2 + 2xy$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x^2 + 2xy + y^2\right)\\ &= 2y + 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 + 2xy\right)\\ &= 2y + 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int 2x^2+2xy+y^2\,\mathrm{d}x$$
$$\phi=\frac{2}{3}x^3+x^2y+x\,y^2+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x^2+2xy+f'(y) \tag{4}$$
$$=x(x+2y)+f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x^2+2xy$. Therefore equation (4) becomes

$$x^2+2xy=x(x+2y)+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=\frac{2}{3}x^3+x^2y+x\,y^2+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{2}{3}x^3 + x^2y + x\,y^2$$

Summary
The solution(s) found are the following

$$\frac{2x^3}{3} + x^2y + y^2x = c_1 \tag{1}$$

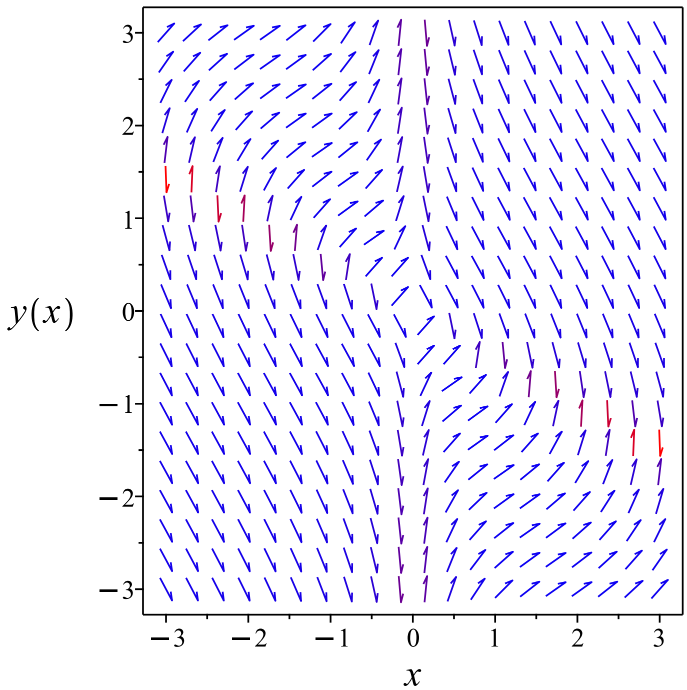


Figure 143: Slope field plot

Verification of solutions

$$\frac{2x^3}{3} + x^2y + y^2x = c_1$$

Verified OK.

4.24.4 Maple step by step solution

Let's solve

$2yx + y^2 + (2yx + x^2)\, y' = -2x^2$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$
  - Evaluate derivatives

    $2y + 2x = 2y + 2x$
  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (2x^2 + 2xy + y^2)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{2x^3}{3} + x^2 y + x\, y^2 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $x^2 + 2xy = x^2 + 2xy + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$F(x, y) = \frac{2}{3}x^3 + x^2y + x\,y^2$

- Substitute $F(x, y)$ into the solution of the ODE

  $\frac{2}{3}x^3 + x^2y + x\,y^2 = c_1$

- Solve for y

  $\left\{y = \frac{-3x^2+\sqrt{-15x^4+36c_1x}}{6x}, y = -\frac{3x^2+\sqrt{-15x^4+36c_1x}}{6x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 80

```
dsolve((2*x^2+2*x*y(x)+y(x)^2)+(x^2+2*x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-3c_1^2x^2 + \sqrt{3}\,\sqrt{-5\left(c_1^3x^3 - \frac{4}{5}\right)c_1x}}{6c_1^2x}$$
$$y(x) = \frac{-3c_1^2x^2 - \sqrt{3}\,\sqrt{-5\left(c_1^3x^3 - \frac{4}{5}\right)c_1x}}{6c_1^2x}$$

✓ Solution by Mathematica

Time used: 1.277 (sec). Leaf size: 150

```
DSolve[(2*x^2+2*x*y[x]+y[x]^2)+(x^2+2*x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{6}\left(-3x - \frac{\sqrt{3}\sqrt{-5x^3 + 4e^{3c_1}}}{\sqrt{x}}\right)$$
$$y(x) \to \frac{1}{6}\left(-3x + \frac{\sqrt{3}\sqrt{-5x^3 + 4e^{3c_1}}}{\sqrt{x}}\right)$$
$$y(x) \to \frac{1}{6}x\left(\frac{\sqrt{15}x^{3/2}}{\sqrt{-x^3}} - 3\right)$$
$$y(x) \to \frac{\sqrt{\frac{5}{3}}\sqrt{-x^3}}{2\sqrt{x}} - \frac{x}{2}$$

5 Chapter 2, section 2.3 (Linear equations). Exercises page 56

5.1 problem 1

Internal problem ID [11638]
Internal file name [`OUTPUT/11647_Tuesday_April_09_2024_02_04_51_AM_19754612/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{3y}{x} = 6x^2}$$

5.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= 6x^2 \end{aligned}$$

Hence the ode is

$$y' + \frac{3y}{x} = 6x^2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{3}{x}dx} \\
&= x^3
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(6x^2\right) \\
\frac{d}{dx}\left(y\, x^3\right) &= \left(x^3\right)\left(6x^2\right) \\
d\left(y\, x^3\right) &= \left(6x^5\right)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\, x^3 &= \int 6x^5\, dx \\
y\, x^3 &= x^6 + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$
y = x^3 + \frac{c_1}{x^3}
$$

Summary

The solution(s) found are the following

$$
y = x^3 + \frac{c_1}{x^3} \tag{1}
$$

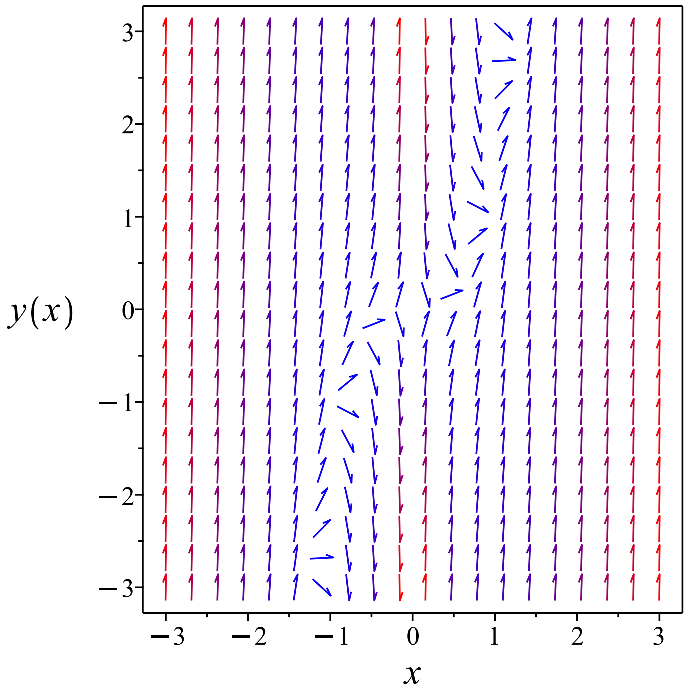


Figure 144: Slope field plot

Verification of solutions

$$y = x^3 + \frac{c_1}{x^3}$$

Verified OK.

5.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3(-2x^3 + y)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 112: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^3}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^3}} dy \end{aligned}$$

Which results in

$$S = y\,x^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3(-2x^3+y)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3x^2y \\ S_y &= x^3 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 6x^5 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 6R^5$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^6 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx^3 = x^6 + c_1$$

Which simplifies to

$$yx^3 = x^6 + c_1$$

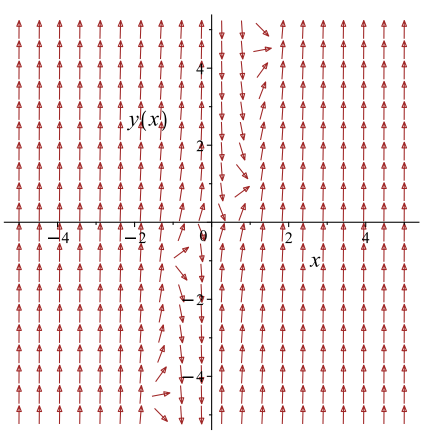
Which gives

$$y = \frac{x^6 + c_1}{x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{3(-2x^3+y)}{x}$



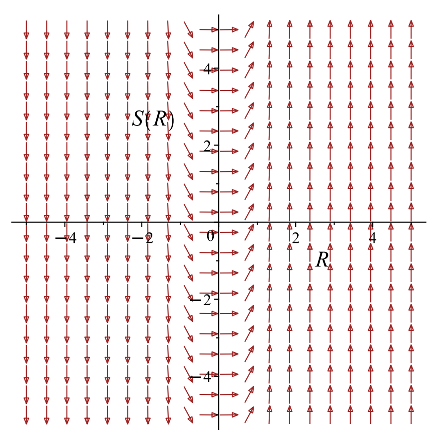
</td><td>

$R = x$

$S = y\,x^3$

</td><td>

$\frac{dS}{dR} = 6R^5$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{x^6 + c_1}{x^3} \tag{1}$$

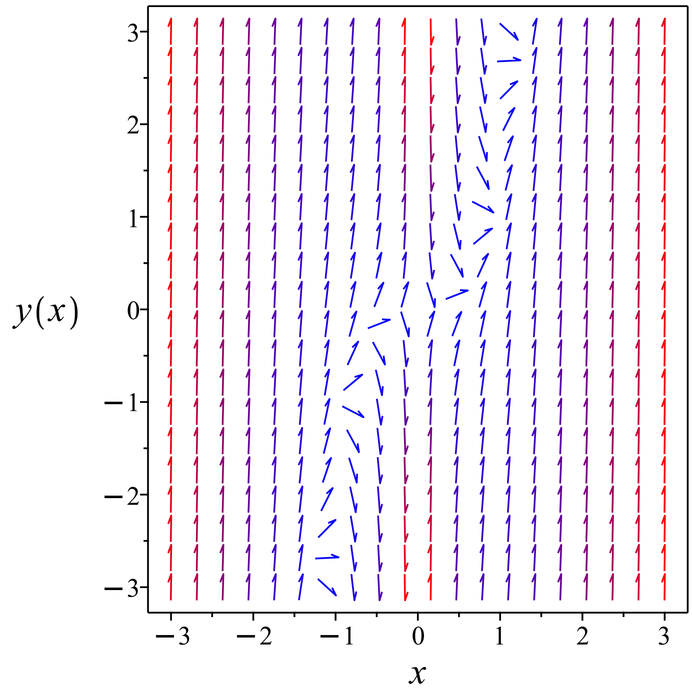


Figure 145: Slope field plot

Verification of solutions

$$y = \frac{x^6 + c_1}{x^3}$$

Verified OK.

5.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\mathrm{d}y &= \left(-\frac{3y}{x} + 6x^2\right) \mathrm{d}x \\
\left(\frac{3y}{x} - 6x^2\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= \frac{3y}{x} - 6x^2 \\
N(x, y) &= 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{3y}{x} - 6x^2\right) \\
&= \frac{3}{x}
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{3}{x}\right) - (0)\right) \\
&= \frac{3}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{3}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{3\ln(x)} \\
&= x^3
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x^3\left(\frac{3y}{x} - 6x^2\right) \\
&= -6x^5 + 3x^2 y
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x^3(1) \\
&= x^3
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-6x^5 + 3x^2 y\right) + \left(x^3\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -6x^5 + 3x^2 y\,\mathrm{d}x$$
$$\phi = -\frac{\left(2x^3 - y\right)^2}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 - \frac{y}{2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3$. Therefore equation (4) becomes

$$x^3 = x^3 - \frac{y}{2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y}{2}\right)\mathrm{d}y$$
$$f(y) = \frac{y^2}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(2x^3 - y)^2}{4} + \frac{y^2}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(2x^3 - y)^2}{4} + \frac{y^2}{4}$$

The solution becomes

$$y = \frac{x^6 + c_1}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{x^6 + c_1}{x^3} \tag{1}$$

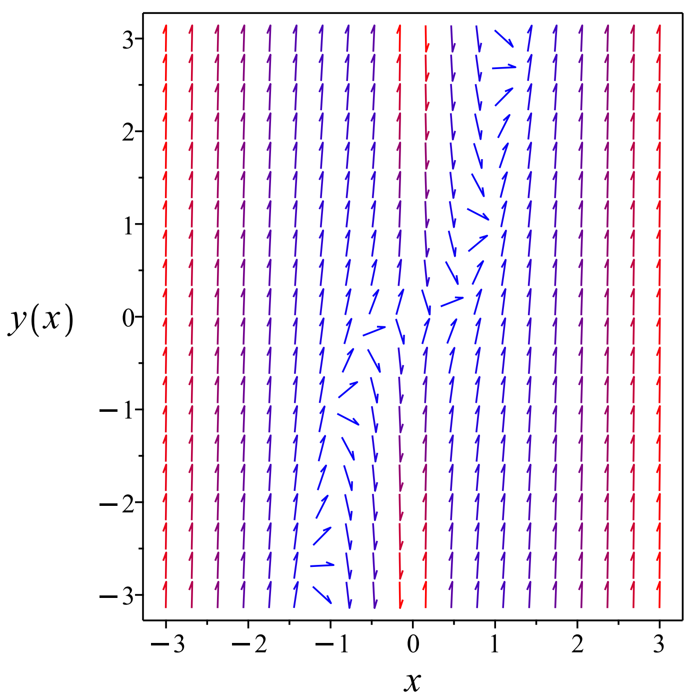


Figure 146: Slope field plot

Verification of solutions

$$y = \frac{x^6 + c_1}{x^3}$$

Verified OK.

5.1.4 Maple step by step solution

Let's solve

$y' + \frac{3y}{x} = 6x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{3y}{x} + 6x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{3y}{x} = 6x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = 6\mu(x)\,x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^3$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int 6\mu(x)\,x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 6\mu(x)\,x^2 dx + c_1$

- Solve for y

  $y = \frac{\int 6\mu(x)x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^3$

$$y = \frac{\int 6x^5 dx + c_1}{x^3}$$

- Evaluate the integrals on the rhs

$$y = \frac{x^6 + c_1}{x^3}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)+3*y(x)/x=6*x^2,y(x), singsol=all)
```

$$y(x) = \frac{x^6 + c_1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 15

```
DSolve[y'[x]+3*y[x]/x==6*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^6 + c_1}{x^3}$$

5.2 problem 2

Internal problem ID [11639]
Internal file name [`OUTPUT/11648_Tuesday_April_09_2024_02_04_53_AM_58150107/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x^4y' + 2yx^3 = 1}$$

5.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{1}{x^4}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{1}{x^4}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{2}{x}dx} \\
&= x^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{x^4}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(x^2 y) &= (x^2)\left(\frac{1}{x^4}\right) \\
\mathrm{d}(x^2 y) &= \frac{1}{x^2}\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^2 y &= \int \frac{1}{x^2}\,\mathrm{d}x \\
x^2 y &= -\frac{1}{x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$
y = -\frac{1}{x^3} + \frac{c_1}{x^2}
$$

which simplifies to

$$
y = \frac{c_1 x - 1}{x^3}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 x - 1}{x^3} \tag{1}
$$

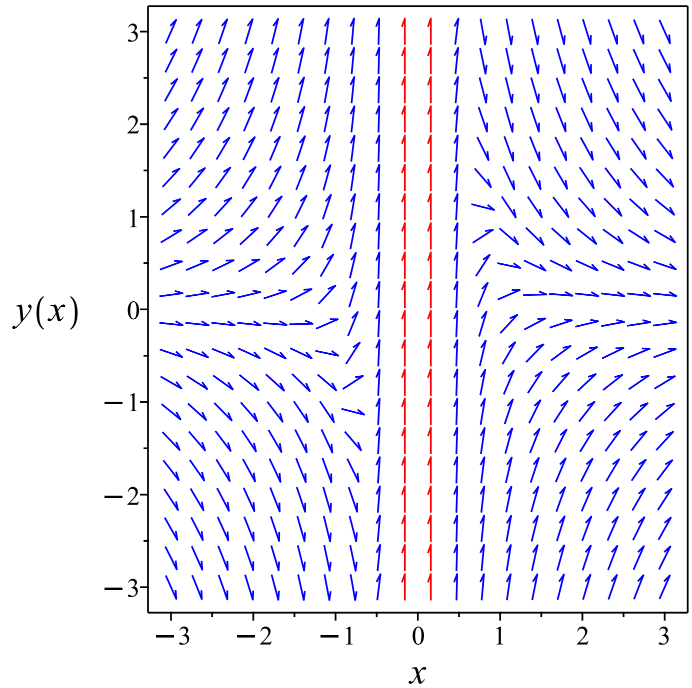


Figure 147: Slope field plot

Verification of solutions

$$y = \frac{c_1 x - 1}{x^3}$$

Verified OK.

5.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{2y\, x^3 - 1}{x^4} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 115: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy \end{aligned}$$

Which results in

$$S = x^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y\, x^3 - 1}{x^4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2xy \\ S_y &= x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 y = -\frac{1}{x} + c_1$$

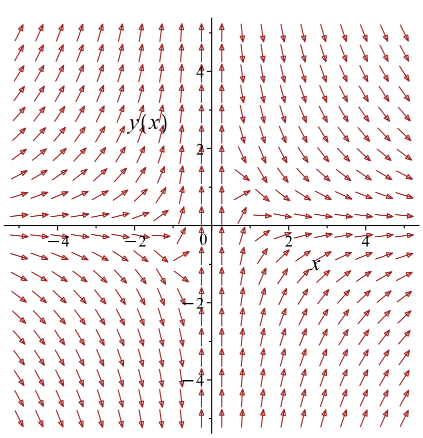
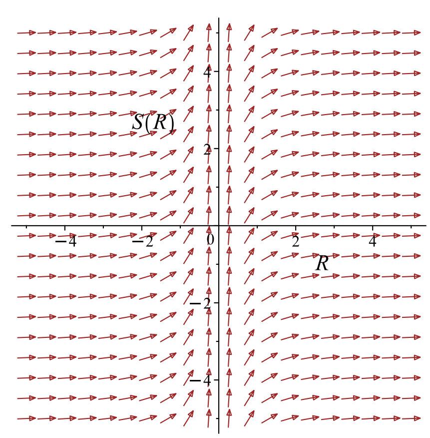
Which simplifies to

$$x^2 y = -\frac{1}{x} + c_1$$

Which gives

$$y = \frac{c_1 x - 1}{x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{2y\,x^3-1}{x^4}$

</td><td>$R = x$
$S = x^2 y$</td><td>$\frac{dS}{dR} = \frac{1}{R^2}$

</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = \frac{c_1 x - 1}{x^3} \tag{1}$$

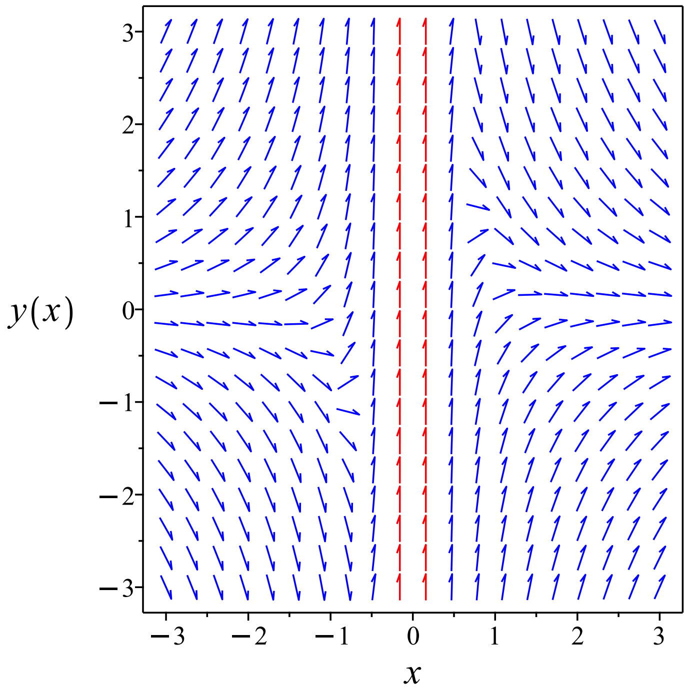


Figure 148: Slope field plot

Verification of solutions

$$y = \frac{c_1 x - 1}{x^3}$$

Verified OK.

5.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^4)\,\mathrm{d}y &= (-2y\,x^3 + 1)\,\mathrm{d}x \\ (2y\,x^3 - 1)\,\mathrm{d}x + (x^4)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 2y\,x^3 - 1$$
$$N(x,y) = x^4$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y\,x^3 - 1) \\ &= 2x^3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^4) \\ &= 4x^3\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^4}\left((2x^3) - (4x^3)\right) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}(2y\,x^3 - 1) \\
&= \frac{2y\,x^3 - 1}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(x^4) \\
&= x^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2y\,x^3 - 1}{x^2}\right) + (x^2)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{2y\, x^3 - 1}{x^2} \, \mathrm{d}x$$
$$\phi = x^2 y + \frac{1}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2 y + \frac{1}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2 y + \frac{1}{x}$$

The solution becomes

$$y = \frac{c_1 x - 1}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x - 1}{x^3} \tag{1}$$

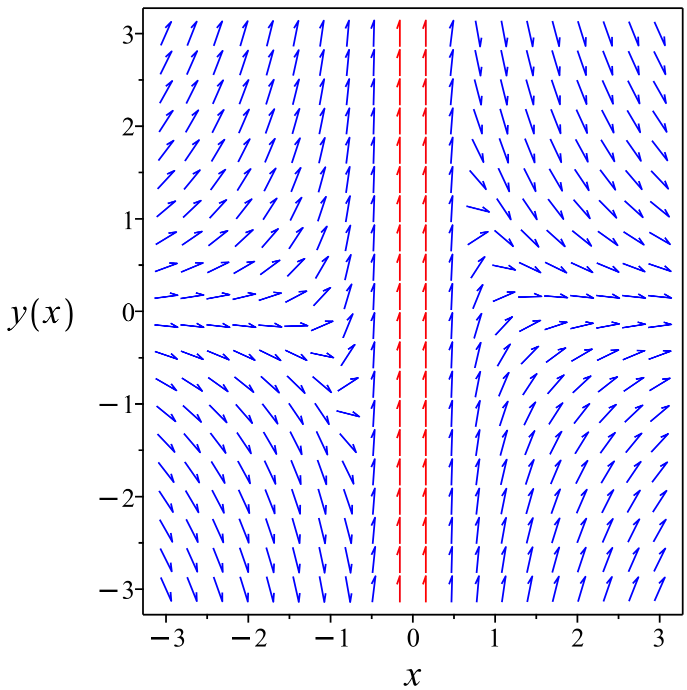


Figure 149: Slope field plot

Verification of solutions

$$y = \frac{c_1 x - 1}{x^3}$$

Verified OK.

5.2.4 Maple step by step solution

Let's solve

$x^4 y' + 2 y x^3 = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + \frac{1}{x^4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = \frac{1}{x^4}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' + \frac{2y}{x}\right) = \frac{\mu(x)}{x^4}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' + \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{x^4} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{x^4} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{x^4} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

  $y = \frac{\int \frac{1}{x^2} dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{1}{x} + c_1}{x^2}$

- Simplify

$y = \frac{c_1 x - 1}{x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 13

```
dsolve(x^4*diff(y(x),x)+2*x^3*y(x)=1,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x - 1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 15

```
DSolve[x^4*y'[x]+2*x^3*y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-1 + c_1 x}{x^3}$$

5.3 problem 3

Internal problem ID [11640]
Internal file name [`OUTPUT/11649_Tuesday_April_09_2024_02_04_54_AM_84307621/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 3y = 3x^2 \mathrm{e}^{-3x}}$$

5.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 3$$
$$q(x) = 3x^2 \mathrm{e}^{-3x}$$

Hence the ode is

$$y' + 3y = 3x^2 \mathrm{e}^{-3x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 3dx} \\ &= e^{3x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(3x^2 e^{-3x}\right) \\ \frac{d}{dx}\left(e^{3x} y\right) &= \left(e^{3x}\right)\left(3x^2 e^{-3x}\right) \\ d\left(e^{3x} y\right) &= \left(3x^2\right)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{3x} y &= \int 3x^2\, dx \\ e^{3x} y &= x^3 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{3x}$ results in

$$y = e^{-3x} x^3 + c_1 e^{-3x}$$

which simplifies to

$$y = e^{-3x}\left(x^3 + c_1\right)$$

Summary
The solution(s) found are the following

$$y = e^{-3x}\left(x^3 + c_1\right) \tag{1}$$

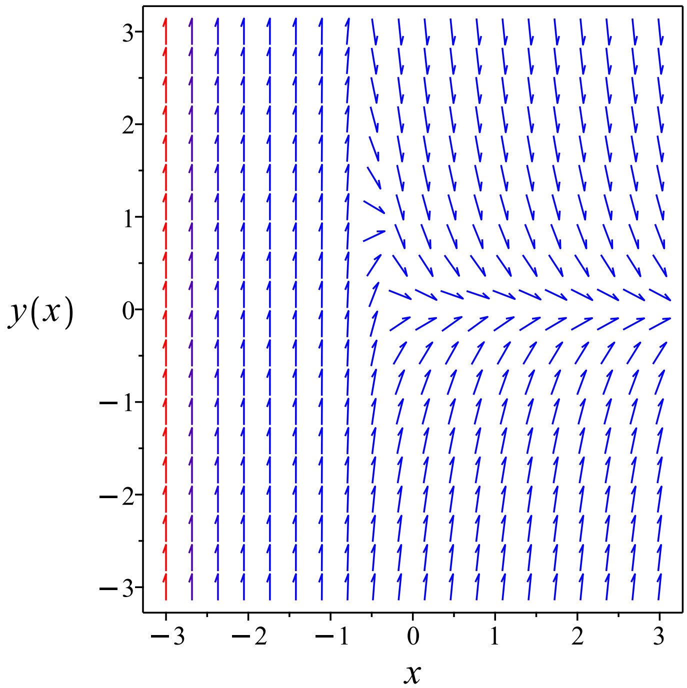


Figure 150: Slope field plot

Verification of solutions

$$y = e^{-3x}(x^3 + c_1)$$

Verified OK.

5.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -3y + 3x^2 e^{-3x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 118: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= e^{-3x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-3x}} dy \end{aligned}$$

Which results in

$$S = e^{3x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -3y + 3x^2 e^{-3x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3 e^{3x} y \\ S_y &= e^{3x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^3 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{3x} y = x^3 + c_1$$

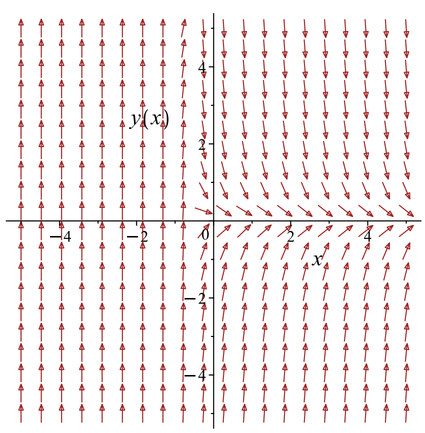
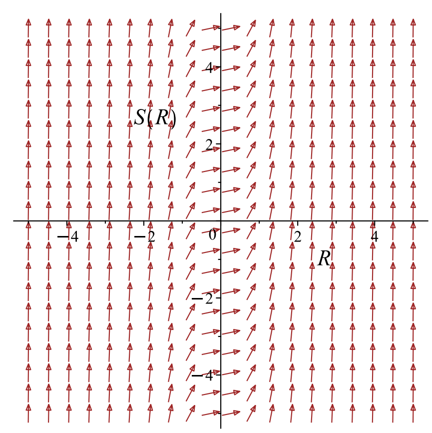
Which simplifies to

$$e^{3x} y = x^3 + c_1$$

Which gives

$$y = e^{-3x}(x^3 + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -3y + 3x^2 e^{-3x}$ 	$R = x$ $S = e^{3x} y$	$\frac{dS}{dR} = 3R^2$ 

Summary

The solution(s) found are the following

$$y = e^{-3x}(x^3 + c_1) \tag{1}$$

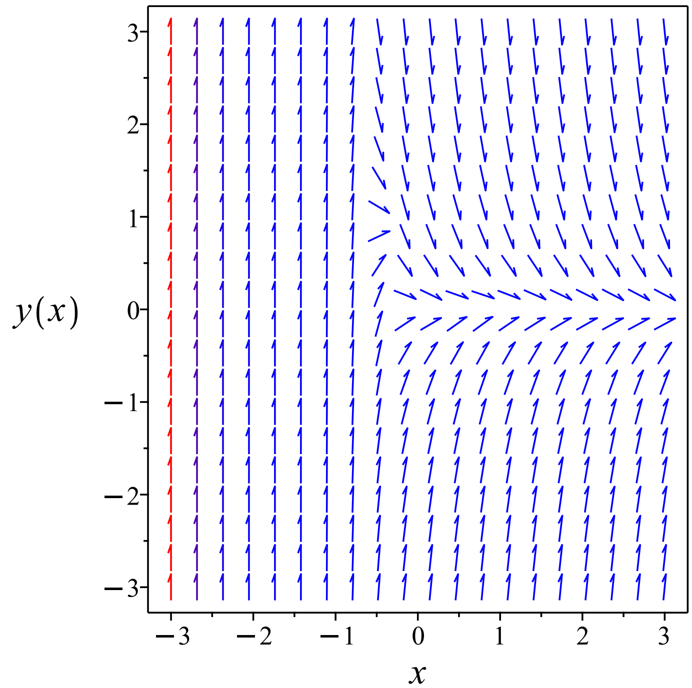


Figure 151: Slope field plot

Verification of solutions

$$y = e^{-3x}(x^3 + c_1)$$

Verified OK.

5.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(-3y + 3x^2 \mathrm{e}^{-3x}\right) \mathrm{d}x \\
\left(3y - 3x^2 \mathrm{e}^{-3x}\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2\mathrm{A})
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 3y - 3x^2 \mathrm{e}^{-3x} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y - 3x^2 \mathrm{e}^{-3x}\right) \\
&= 3
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((3) - (0)) \\
&= 3
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 3 \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{3x} \\
&= \mathrm{e}^{3x}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{3x}\left(3y - 3x^2 \mathrm{e}^{-3x}\right) \\
&= 3\,\mathrm{e}^{3x} y - 3x^2
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{3x}(1) \\
&= \mathrm{e}^{3x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(3\,\mathrm{e}^{3x} y - 3x^2\right) + \left(\mathrm{e}^{3x}\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 3\, \mathrm{e}^{3x} y - 3x^2 \, \mathrm{d}x$$

$$\phi = \mathrm{e}^{3x} y - x^3 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{3x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{3x}$. Therefore equation (4) becomes

$$\mathrm{e}^{3x} = \mathrm{e}^{3x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{3x} y - x^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{3x} y - x^3$$

The solution becomes

$$y = \mathrm{e}^{-3x} (x^3 + c_1)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-3x} (x^3 + c_1) \tag{1}$$

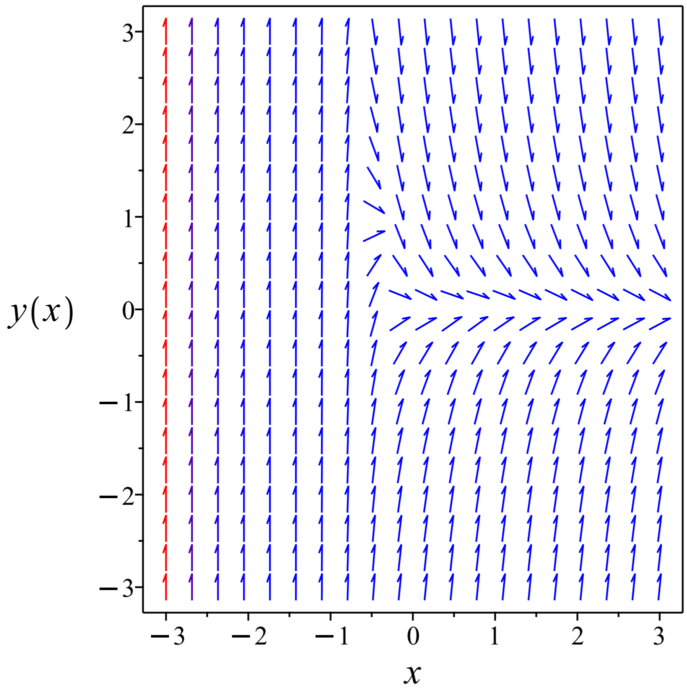


Figure 152: Slope field plot

Verification of solutions

$$y = e^{-3x}\left(x^3 + c_1\right)$$

Verified OK.

5.3.4 Maple step by step solution

Let's solve

$y' + 3y = 3x^2 e^{-3x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -3y + 3x^2 e^{-3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 3y = 3x^2 e^{-3x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + 3y\right) = 3\mu(x)\, x^2 e^{-3x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'+3y)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=3\mu(x)$
- Solve to find the integrating factor

  $\mu(x)=\mathrm{e}^{3x}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int 3\mu(x)\,x^2\mathrm{e}^{-3x}dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int 3\mu(x)\,x^2\mathrm{e}^{-3x}dx+c_1$
- Solve for y

  $y=\frac{\int 3\mu(x)x^2\mathrm{e}^{-3x}dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\mathrm{e}^{3x}$

  $y=\frac{\int 3x^2\mathrm{e}^{-3x}\mathrm{e}^{3x}dx+c_1}{\mathrm{e}^{3x}}$
- Evaluate the integrals on the rhs

  $y=\frac{x^3+c_1}{\mathrm{e}^{3x}}$
- Simplify

  $y=\mathrm{e}^{-3x}(x^3+c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+3*y(x)=3*x^2*exp(-3*x),y(x), singsol=all)
```

$$y(x) = (x^3 + c_1) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 17

```
DSolve[y'[x]+3*y[x]==3*x^2*Exp[-3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}(x^3 + c_1)$$

5.4 problem 4

Internal problem ID [11641]
Internal file name [`OUTPUT/11650_Tuesday_April_09_2024_02_04_54_AM_54658325/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + 4yx = 8x}$$

5.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x(-4y+8) \end{aligned}$$

Where $f(x) = x$ and $g(y) = -4y + 8$. Integrating both sides gives

$$\frac{1}{-4y+8}\, dy = x\, dx$$

$$\int \frac{1}{-4y+8}\, dy = \int x\, dx$$

$$-\frac{\ln(y-2)}{4} = \frac{x^2}{2} + c_1$$

Raising both side to exponential gives

$$\frac{1}{(y-2)^{\frac{1}{4}}} = e^{\frac{x^2}{2}+c_1}$$

Which simplifies to

$$\frac{1}{(y-2)^{\frac{1}{4}}} = c_2 e^{\frac{x^2}{2}}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(2c_2^4 e^{2x^2+4c_1} + 1\right) e^{-2x^2-4c_1}}{c_2^4} \tag{1}$$

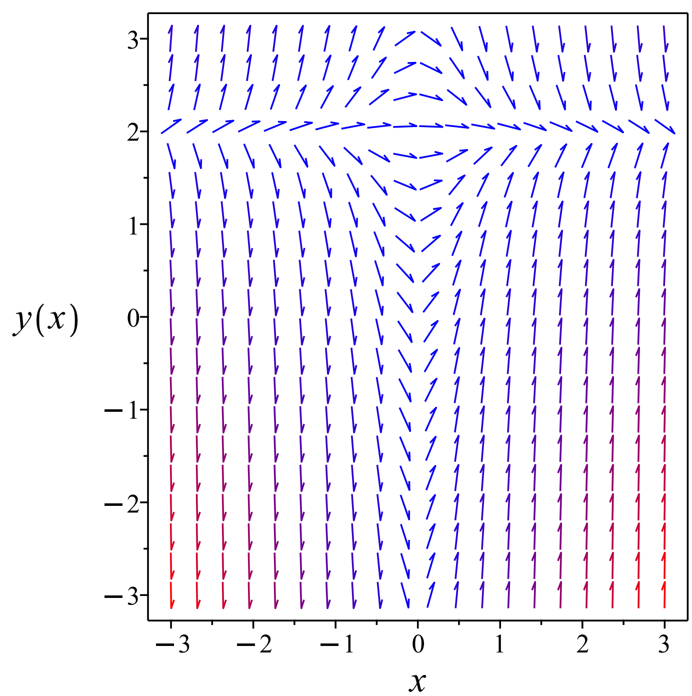


Figure 153: Slope field plot

Verification of solutions

$$y = \frac{\left(2c_2^4 e^{2x^2+4c_1} + 1\right) e^{-2x^2-4c_1}}{c_2^4}$$

Verified OK.

5.4.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= 4x\\ q(x) &= 8x\end{aligned}$$

Hence the ode is

$$y' + 4yx = 8x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 4x dx}\\ &= e^{2x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(8x)\\ \frac{d}{dx}\left(e^{2x^2}y\right) &= \left(e^{2x^2}\right)(8x)\\ d\left(e^{2x^2}y\right) &= \left(8x\, e^{2x^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{2x^2}y &= \int 8x\, e^{2x^2} dx\\ e^{2x^2}y &= 2\, e^{2x^2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2x^2}$ results in

$$y = 2\, e^{-2x^2} e^{2x^2} + c_1 e^{-2x^2}$$

which simplifies to

$$y = 2 + c_1 e^{-2x^2}$$

Summary

The solution(s) found are the following

$$y = 2 + c_1 e^{-2x^2} \tag{1}$$

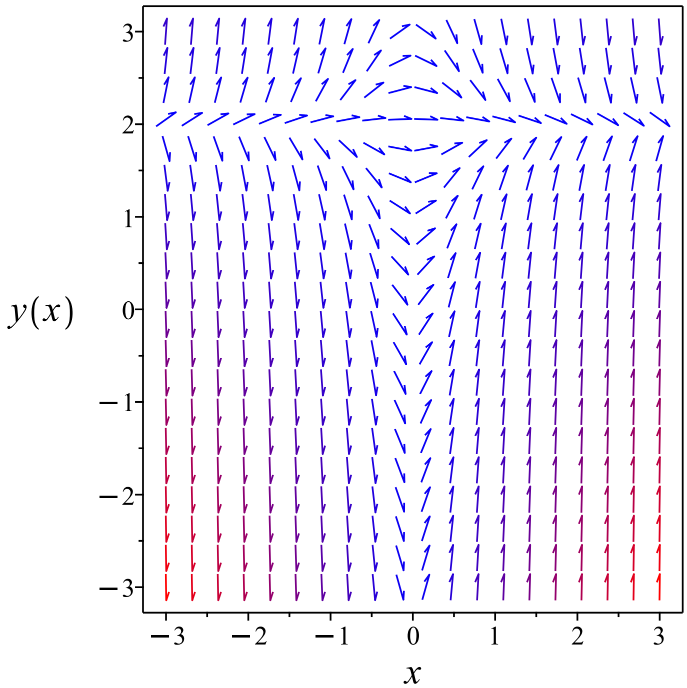


Figure 154: Slope field plot

Verification of solutions

$$y = 2 + c_1 e^{-2x^2}$$

Verified OK.

5.4.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -4xy + 8x \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 121: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-2x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-2x^2}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{2x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -4xy + 8x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 4x\,\mathrm{e}^{2x^2} y \\ S_y &= \mathrm{e}^{2x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 8x\,\mathrm{e}^{2x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 8R\,\mathrm{e}^{2R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\,\mathrm{e}^{2R^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{2x^2} y = 2\,\mathrm{e}^{2x^2} + c_1$$

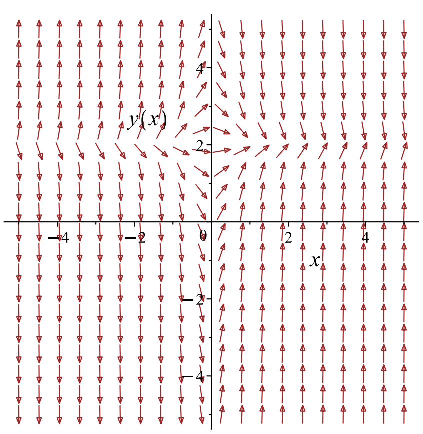
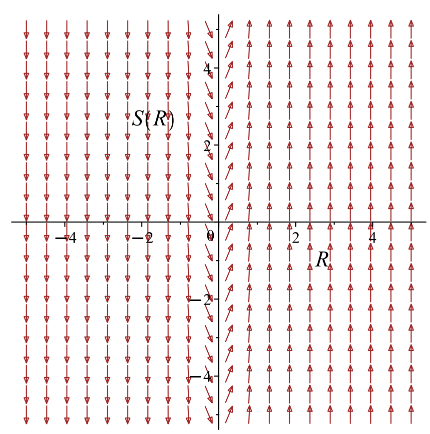
Which simplifies to

$$(y-2)\,\mathrm{e}^{2x^2} - c_1 = 0$$

Which gives

$$y = \left(2\,\mathrm{e}^{2x^2} + c_1\right)\mathrm{e}^{-2x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -4xy + 8x$ 	$R = x$ $S = \mathrm{e}^{2x^2} y$	$\frac{dS}{dR} = 8R\,\mathrm{e}^{2R^2}$ 

Summary
The solution(s) found are the following

$$y = \left(2\,\mathrm{e}^{2x^2} + c_1\right)\mathrm{e}^{-2x^2} \tag{1}$$

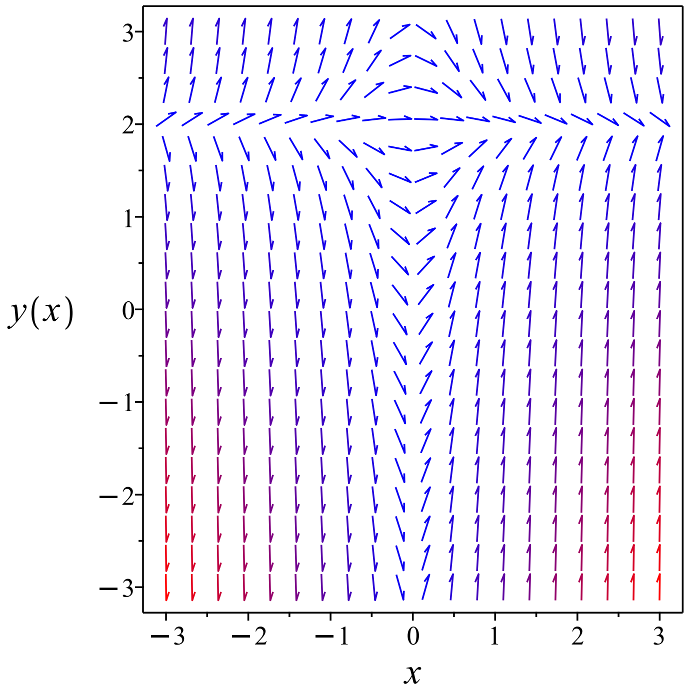


Figure 155: Slope field plot

Verification of solutions

$$y = \left(2\,\mathrm{e}^{2x^2} + c_1\right)\mathrm{e}^{-2x^2}$$

Verified OK.

5.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{-4y+8}\right)\mathrm{d}y &= (x)\,\mathrm{d}x \\
(-x)\,\mathrm{d}x + \left(\frac{1}{-4y+8}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -x \\
N(x,y) &= \frac{1}{-4y+8}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-4y+8}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -x \,\mathrm{d}x \\
\phi &= -\frac{x^2}{2} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-4y+8}$. Therefore equation (4) becomes

$$
\frac{1}{-4y+8} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{4(y-2)}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{4y-8}\right)\mathrm{d}y$$

$$f(y) = -\frac{\ln(y-2)}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{\ln(y-2)}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{\ln(y-2)}{4}$$

The solution becomes

$$y = \mathrm{e}^{-2x^2-4c_1} + 2$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x^2-4c_1} + 2 \tag{1}$$

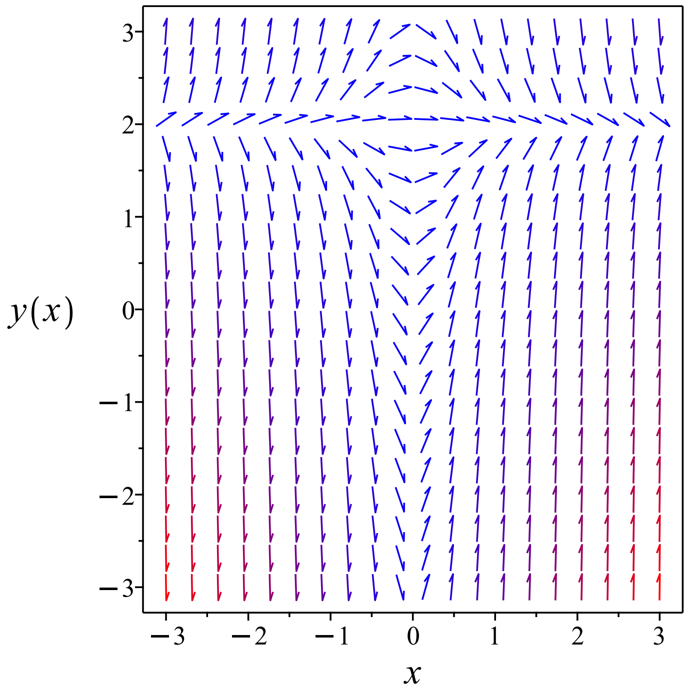


Figure 156: Slope field plot

Verification of solutions

$$y = e^{-2x^2-4c_1} + 2$$

Verified OK.

5.4.5 Maple step by step solution

Let's solve

$y' + 4yx = 8x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y-2} = -4x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y-2}dx = \int -4xdx + c_1$

- Evaluate integral

$$\ln(y-2) = -2x^2 + c_1$$

- Solve for y

$$y = e^{-2x^2+c_1} + 2$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+4*x*y(x)=8*x,y(x), singsol=all)
```

$$y(x) = 2 + e^{-2x^2} c_1$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 22

```
DSolve[y'[x]+4*x*y[x]==8*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2 + c_1 e^{-2x^2}$$
$$y(x) \to 2$$

5.5 problem 5

Internal problem ID [11642]
Internal file name [`OUTPUT/11651_Tuesday_April_09_2024_02_04_55_AM_68847940/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' + \frac{x}{t^2} = \frac{1}{t^2}}$$

5.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t, x) \\ &= f(t)g(x) \\ &= \frac{1-x}{t^2} \end{aligned}$$

Where $f(t) = \frac{1}{t^2}$ and $g(x) = 1 - x$. Integrating both sides gives

$$\frac{1}{1-x}\,dx = \frac{1}{t^2}\,dt$$

$$\int \frac{1}{1-x}\,dx = \int \frac{1}{t^2}\,dt$$

$$-\ln\left(x-1\right) = -\frac{1}{t} + c_1$$

Raising both side to exponential gives

$$\frac{1}{x-1} = \mathrm{e}^{-\frac{1}{t}+c_1}$$

Which simplifies to

$$\frac{1}{x-1} = c_2 \mathrm{e}^{-\frac{1}{t}}$$

Summary

The solution(s) found are the following

$$x = \frac{\left(c_2 \mathrm{e}^{\frac{c_1 t-1}{t}} + 1\right) \mathrm{e}^{-\frac{c_1 t-1}{t}}}{c_2} \tag{1}$$

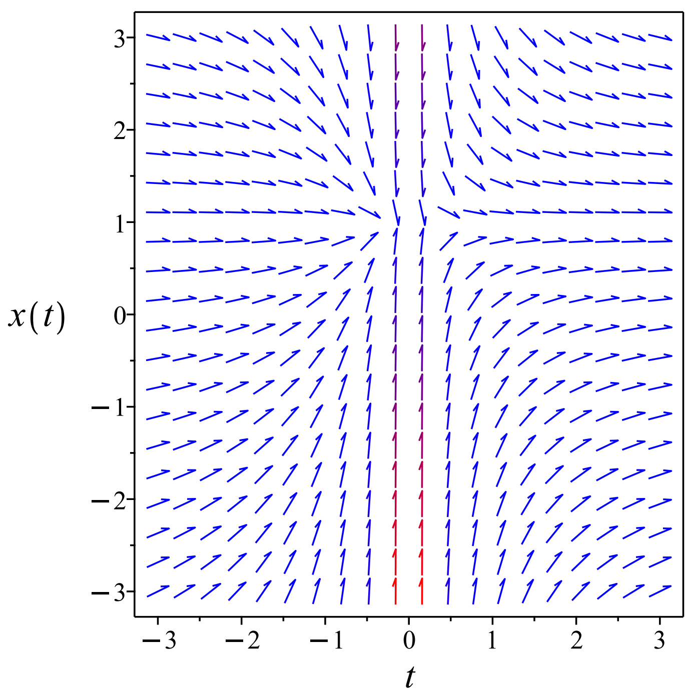


Figure 157: Slope field plot

Verification of solutions

$$x = \frac{\left(c_2 \mathrm{e}^{\frac{c_1 t-1}{t}}+1\right) \mathrm{e}^{-\frac{c_1 t-1}{t}}}{c_2}$$

Verified OK.

5.5.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= \frac{1}{t^2} \\ q(t) &= \frac{1}{t^2} \end{aligned}$$

Hence the ode is

$$x' + \frac{x}{t^2} = \frac{1}{t^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{1}{t^2} dt} \\ &= \mathrm{e}^{-\frac{1}{t}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\frac{1}{t^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\frac{1}{t}} x\right) &= \left(\mathrm{e}^{-\frac{1}{t}}\right)\left(\frac{1}{t^2}\right) \\ \mathrm{d}\left(\mathrm{e}^{-\frac{1}{t}} x\right) &= \left(\frac{\mathrm{e}^{-\frac{1}{t}}}{t^2}\right) \mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{-\frac{1}{t}} x &= \int \frac{\mathrm{e}^{-\frac{1}{t}}}{t^2} \mathrm{d}t \\ \mathrm{e}^{-\frac{1}{t}} x &= \mathrm{e}^{-\frac{1}{t}} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{1}{t}}$ results in

$$x = \mathrm{e}^{\frac{1}{t}} \mathrm{e}^{-\frac{1}{t}} + c_1 \mathrm{e}^{\frac{1}{t}}$$

which simplifies to

$$x = 1 + c_1 \mathrm{e}^{\frac{1}{t}}$$

Summary

The solution(s) found are the following

$$x = 1 + c_1 \mathrm{e}^{\frac{1}{t}} \tag{1}$$

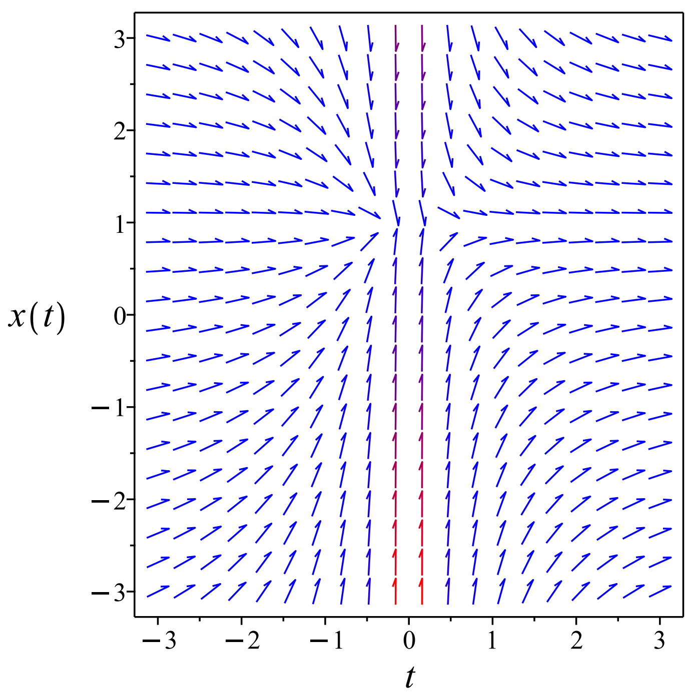


Figure 158: Slope field plot

Verification of solutions

$$x = 1 + c_1 \mathrm{e}^{\frac{1}{t}}$$

Verified OK.

5.5.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= -\frac{x-1}{t^2} \\
x' &= \omega(t,x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 124: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= 0\\
\eta(t,x) &= \mathrm{e}^{\frac{1}{t}} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right)S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = t
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy\\
&= \int \frac{1}{\mathrm{e}^{\frac{1}{t}}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{-\frac{1}{t}} x
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}
$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$
\omega(t,x) = -\frac{x-1}{t^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1\\
R_x &= 0\\
S_t &= \frac{\mathrm{e}^{-\frac{1}{t}} x}{t^2}\\
S_x &= \mathrm{e}^{-\frac{1}{t}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{-\frac{1}{t}}}{t^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{-\frac{1}{R}}}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{-\frac{1}{R}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$e^{-\frac{1}{t}} x = e^{-\frac{1}{t}} + c_1$$

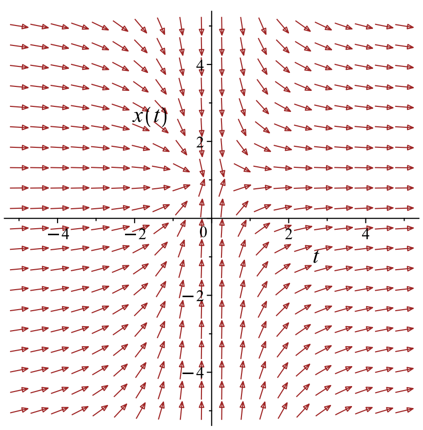
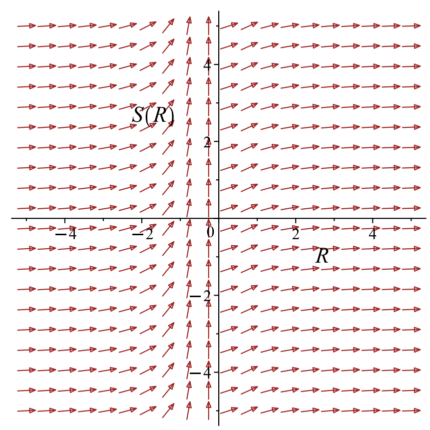
Which simplifies to

$$e^{-\frac{1}{t}}(x-1) - c_1 = 0$$

Which gives

$$x = \left(e^{-\frac{1}{t}} + c_1\right) e^{\frac{1}{t}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, x coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dx}{dt} = -\frac{x-1}{t^2}$
</td><td>$R = t$
$S = \mathrm{e}^{-\frac{1}{t}} x$</td><td>$\frac{dS}{dR} = \frac{\mathrm{e}^{-\frac{1}{R}}}{R^2}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$x = \left(\mathrm{e}^{-\frac{1}{t}} + c_1\right) \mathrm{e}^{\frac{1}{t}} \tag{1}$$

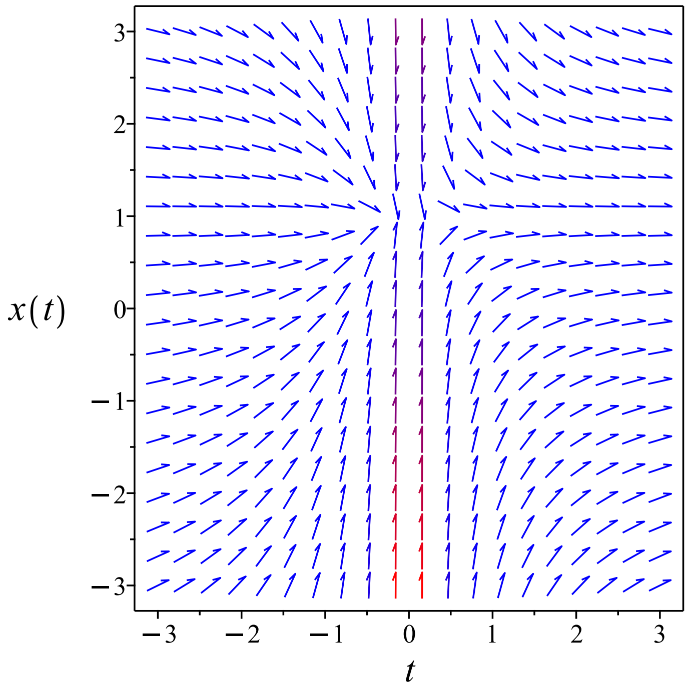


Figure 159: Slope field plot

<u>Verification of solutions</u>

$$x = \left(e^{-\frac{1}{t}} + c_1\right) e^{\frac{1}{t}}$$

Verified OK.

5.5.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x) \, \mathrm{d}t + N(t, x) \, \mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{1-x}\right) \mathrm{d}x = \left(\frac{1}{t^2}\right) \mathrm{d}t$$
$$\left(-\frac{1}{t^2}\right) \mathrm{d}t + \left(\frac{1}{1-x}\right) \mathrm{d}x = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, x) = -\frac{1}{t^2}$$
$$N(t, x) = \frac{1}{1-x}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{t^2}\right) \\ &= 0 \end{aligned}$$

And

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{1-x}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$
$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int M\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int -\frac{1}{t^2}\,\mathrm{d}t$$
$$\phi = \frac{1}{t} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{1-x}$. Therefore equation (4) becomes

$$\frac{1}{1-x} = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\frac{1}{x-1}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\frac{1}{x-1}\right)\mathrm{d}x$$
$$f(x) = -\ln(x-1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{1}{t} - \ln(x-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{t} - \ln(x-1)$$

The solution becomes

$$x = \mathrm{e}^{-\frac{c_1 t - 1}{t}} + 1$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{-\frac{c_1 t - 1}{t}} + 1 \tag{1}$$

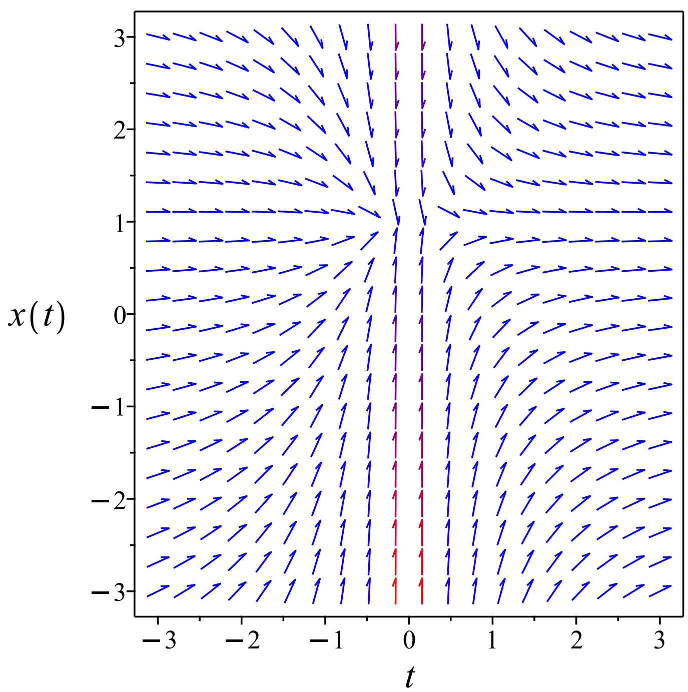


Figure 160: Slope field plot

Verification of solutions

$$x = e^{-\frac{c_1 t - 1}{t}} + 1$$

Verified OK.

5.5.5 Maple step by step solution

Let's solve

$x' + \frac{x}{t^2} = \frac{1}{t^2}$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'}{x-1} = -\frac{1}{t^2}$

- Integrate both sides with respect to t

  $\int \frac{x'}{x-1} dt = \int -\frac{1}{t^2} dt + c_1$

- Evaluate integral

  $\ln(x-1) = \frac{1}{t} + c_1$

- Solve for x

  $x = e^{\frac{c_1 t + 1}{t}} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(x(t),t)+x(t)/t^2=1/t^2,x(t), singsol=all)
```

$$x(t) = 1 + e^{\frac{1}{t}} c_1$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 20

```
DSolve[x'[t]+x[t]/t^2==1/t^2,x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to 1 + c_1 e^{\frac{1}{t}}$$
$$x(t) \to 1$$

5.6 problem 6

Internal problem ID [11643]
Internal file name [`OUTPUT/11652_Tuesday_April_09_2024_02_04_56_AM_51780437/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(u^2+1)\, v'+4vu=3u}$$

5.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} v' &= F(u,v) \\ &= f(u)g(v) \\ &= \frac{u(-4v+3)}{u^2+1} \end{aligned}$$

Where $f(u)=\frac{u}{u^2+1}$ and $g(v)=-4v+3$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-4v+3}\, dv &= \frac{u}{u^2+1}\, du \\ \int \frac{1}{-4v+3}\, dv &= \int \frac{u}{u^2+1}\, du \end{aligned}$$

$$
-\frac{\ln \left(-4 v+3\right)}{4} = \frac{\ln \left(u^{2}+1\right)}{2}+c_{1}
$$

Raising both side to exponential gives

$$
\frac{1}{\left(-4 v+3\right)^{\frac{1}{4}}} = {\mathrm e}^{\frac{\ln \left(u^{2}+1\right)}{2}+c_{1}}
$$

Which simplifies to

$$
\frac{1}{\left(-4 v+3\right)^{\frac{1}{4}}} = c_{2} \sqrt{u^{2}+1}
$$

Which simplifies to

$$
v = \frac{\left(3 c_{2}^{4} {\mathrm e}^{4 c_{1}} \left(u^{2}+1\right)^{2}-1\right) {\mathrm e}^{-4 c_{1}}}{4 c_{2}^{4} \left(u^{2}+1\right)^{2}}
$$

Summary

The solution(s) found are the following

$$
v = \frac{\left(3 c_{2}^{4} {\mathrm e}^{4 c_{1}} \left(u^{2}+1\right)^{2}-1\right) {\mathrm e}^{-4 c_{1}}}{4 c_{2}^{4} \left(u^{2}+1\right)^{2}} \tag{1}
$$

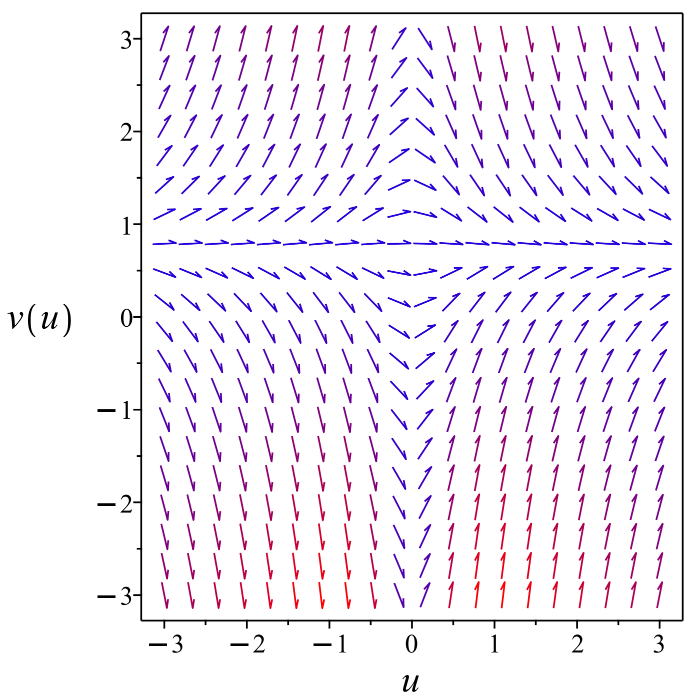


Figure 161: Slope field plot

Verification of solutions

$$v = \frac{\left(3c_2^4 e^{4c_1} \left(u^2+1\right)^2 - 1\right) e^{-4c_1}}{4c_2^4 \left(u^2+1\right)^2}$$

Verified OK.

5.6.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$v' + p(u)v = q(u)$$

Where here

$$\begin{aligned} p(u) &= \frac{4u}{u^2+1} \\ q(u) &= \frac{3u}{u^2+1} \end{aligned}$$

Hence the ode is

$$v' + \frac{4uv}{u^2+1} = \frac{3u}{u^2+1}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{4u}{u^2+1} du} \\ &= \left(u^2+1\right)^2 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{du}(\mu v) &= (\mu)\left(\frac{3u}{u^2+1}\right) \\ \frac{d}{du}\left(\left(u^2+1\right)^2 v\right) &= \left(\left(u^2+1\right)^2\right)\left(\frac{3u}{u^2+1}\right) \\ d\left(\left(u^2+1\right)^2 v\right) &= \left(3u\left(u^2+1\right)\right) du \end{aligned}$$

Integrating gives

$$\begin{aligned} \left(u^2+1\right)^2 v &= \int 3u\left(u^2+1\right) du \\ \left(u^2+1\right)^2 v &= \frac{3\left(u^2+1\right)^2}{4} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = (u^2+1)^2$ results in

$$v = \frac{3}{4} + \frac{c_1}{(u^2+1)^2}$$

Summary

The solution(s) found are the following

$$v = \frac{3}{4} + \frac{c_1}{(u^2+1)^2} \tag{1}$$

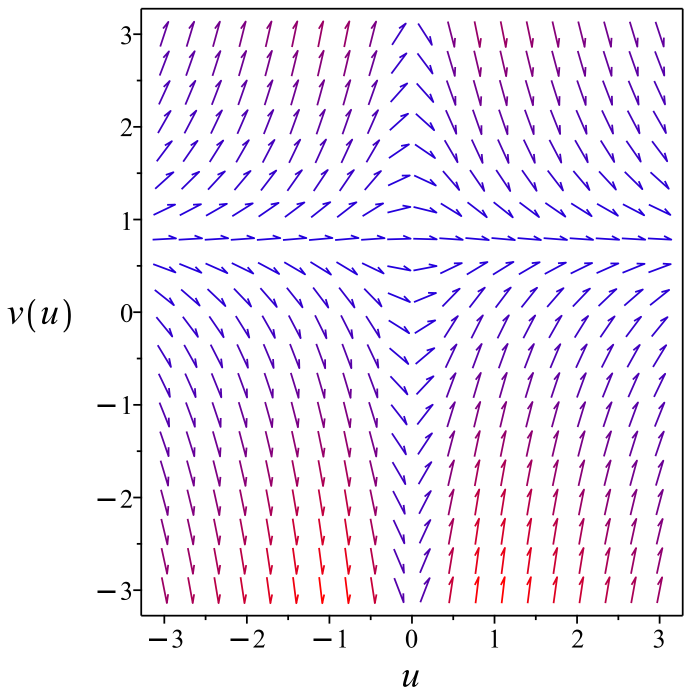


Figure 162: Slope field plot

Verification of solutions

$$v = \frac{3}{4} + \frac{c_1}{(u^2+1)^2}$$

Verified OK.

5.6.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
v' &= -\frac{u(4v-3)}{u^2+1} \\
v' &= \omega(u,v)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_u + \omega(\eta_v - \xi_u) - \omega^2 \xi_v - \omega_u \xi - \omega_v \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 127: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(u,v) &= 0 \\
\eta(u,v) &= \frac{1}{\left(u^2+1\right)^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(u,v) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{du}{\xi} = \frac{dv}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial v}\right) S(u,v) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = u
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{\left(u^2+1\right)^2}} dy
\end{aligned}
$$

Which results in

$$
S = \left(u^2+1\right)^2 v
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_u + \omega(u,v) S_v}{R_u + \omega(u,v) R_v} \tag{2}
$$

Where in the above R_u, R_v, S_u, S_v are all partial derivatives and $\omega(u,v)$ is the right hand side of the original ode given by

$$
\omega(u,v) = -\frac{u(4v-3)}{u^2+1}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_u &= 1 \\
R_v &= 0 \\
S_u &= 4(u^2+1)\, vu \\
S_v &= (u^2+1)^2
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3u^3 + 3u \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for u, v in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R^3 + 3R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{3(R^2+1)^2}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to u, v coordinates. This results in

$$(u^2+1)^2 v = \frac{3(u^2+1)^2}{4} + c_1$$

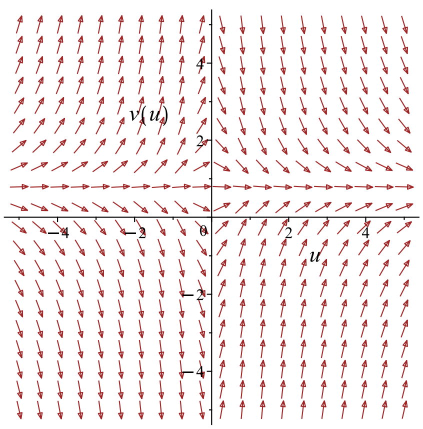
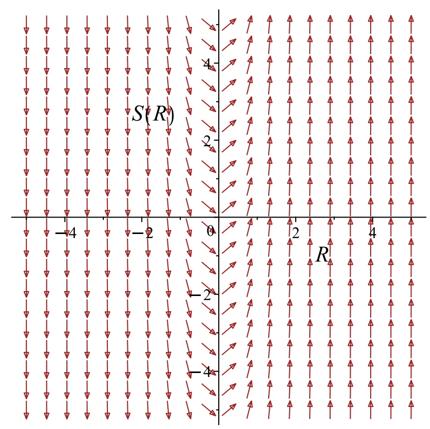
Which simplifies to

$$(u^2+1)^2 v = \frac{3(u^2+1)^2}{4} + c_1$$

Which gives

$$v = \frac{3u^4 + 6u^2 + 4c_1 + 3}{4(u^2+1)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in u, v coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dv}{du} = -\frac{u(4v-3)}{u^2+1}$ 	$R = u$ $S = (u^2+1)^2 v$	$\frac{dS}{dR} = 3R^3 + 3R$ 

Summary
The solution(s) found are the following

$$v = \frac{3u^4 + 6u^2 + 4c_1 + 3}{4(u^2+1)^2} \tag{1}$$

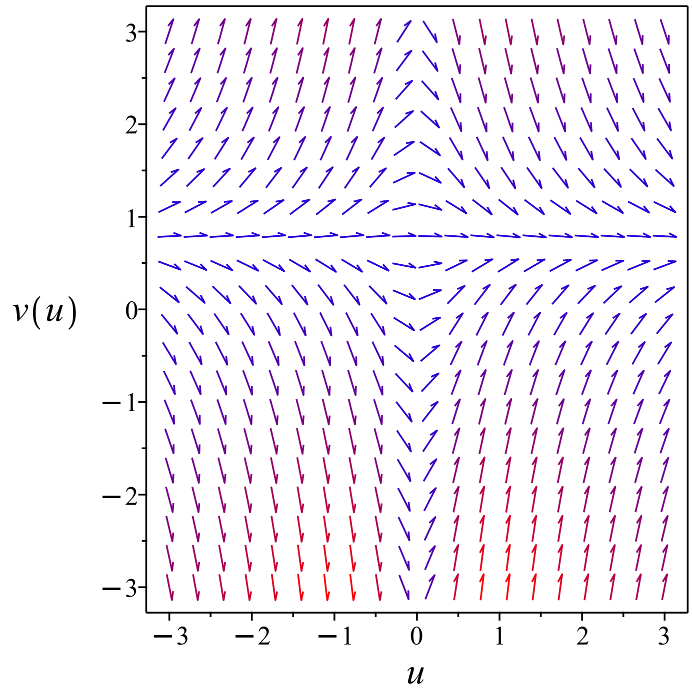


Figure 163: Slope field plot

Verification of solutions

$$v = \frac{3u^4 + 6u^2 + 4c_1 + 3}{4\left(u^2 + 1\right)^2}$$

Verified OK.

5.6.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(u, v)\,\mathrm{d}u + N(u, v)\,\mathrm{d}v = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-4v+3}\right)\mathrm{d}v &= \left(\frac{u}{u^2+1}\right)\mathrm{d}u\\ \left(-\frac{u}{u^2+1}\right)\mathrm{d}u + \left(\frac{1}{-4v+3}\right)\mathrm{d}v &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(u, v) &= -\frac{u}{u^2+1}\\ N(u, v) &= \frac{1}{-4v+3}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial v} &= \frac{\partial}{\partial v}\left(-\frac{u}{u^2+1}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial u} &= \frac{\partial}{\partial u}\left(\frac{1}{-4v+3}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(u, v)$

$$
\frac{\partial \phi}{\partial u} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial v} = N \tag{2}
$$

Integrating (1) w.r.t. u gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial u} \,\mathrm{d}u &= \int M \,\mathrm{d}u \\
\int \frac{\partial \phi}{\partial u} \,\mathrm{d}u &= \int -\frac{u}{u^2+1} \,\mathrm{d}u
\end{aligned}
$$

$$
\phi = -\frac{\ln(u^2+1)}{2} + f(v) \tag{3}
$$

Where $f(v)$ is used for the constant of integration since ϕ is a function of both u and v. Taking derivative of equation (3) w.r.t v gives

$$
\frac{\partial \phi}{\partial v} = 0 + f'(v) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial v} = \frac{1}{-4v+3}$. Therefore equation (4) becomes

$$
\frac{1}{-4v+3} = 0 + f'(v) \tag{5}
$$

Solving equation (5) for $f'(v)$ gives

$$
f'(v) = -\frac{1}{4v-3}
$$

Integrating the above w.r.t v gives

$$
\begin{aligned}
\int f'(v) \,\mathrm{d}v &= \int \left(-\frac{1}{4v-3}\right) \mathrm{d}v \\
f(v) &= -\frac{\ln(4v-3)}{4} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(v)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(u^2+1)}{2} - \frac{\ln(4v-3)}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(u^2+1)}{2} - \frac{\ln(4v-3)}{4}$$

The solution becomes

$$v = \frac{3u^4 + 6u^2 + e^{-4c_1} + 3}{4u^4 + 8u^2 + 4}$$

Summary

The solution(s) found are the following

$$v = \frac{3u^4 + 6u^2 + e^{-4c_1} + 3}{4u^4 + 8u^2 + 4} \tag{1}$$

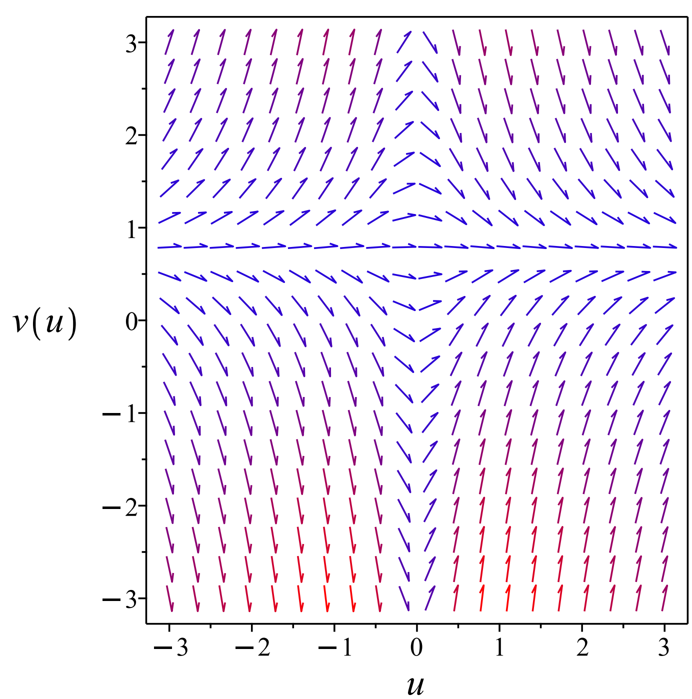


Figure 164: Slope field plot

Verification of solutions

$$v = \frac{3u^4 + 6u^2 + e^{-4c_1} + 3}{4u^4 + 8u^2 + 4}$$

Verified OK.

5.6.5 Maple step by step solution

Let's solve

$(u^2 + 1)\, v' + 4vu = 3u$

- Highest derivative means the order of the ODE is 1

  v'

- Separate variables

  $\frac{v'}{4v-3} = -\frac{u}{u^2+1}$

- Integrate both sides with respect to u

  $\int \frac{v'}{4v-3} du = \int -\frac{u}{u^2+1} du + c_1$

- Evaluate integral

  $\frac{\ln(4v-3)}{4} = -\frac{\ln(u^2+1)}{2} + c_1$

- Solve for v

  $v = \frac{3u^4+6u^2+e^{4c_1}+3}{4(u^4+2u^2+1)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((u^2+1)*diff(v(u),u)+4*u*v(u)=3*u,v(u), singsol=all)
```

$$v(u) = \frac{3}{4} + \frac{c_1}{(u^2+1)^2}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 38

```
DSolve[(u^2+1)*v'[u]+4*u*v[u]==3*u,v[u],u,IncludeSingularSolutions -> True]
```

$$v(u) \to \frac{3u^4 + 6u^2 + 4c_1}{4(u^2+1)^2}$$
$$v(u) \to \frac{3}{4}$$

5.7 problem 7

Internal problem ID [11644]
Internal file name [`OUTPUT/11653_Tuesday_April_09_2024_02_04_57_AM_21312822/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x + \frac{(2x+1)\,y}{1+x} = x-1}$$

5.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-2x-1}{x\,(1+x)}$$
$$q(x) = \frac{x-1}{x}$$

Hence the ode is

$$y' - \frac{(-2x-1)\,y}{x\,(1+x)} = \frac{x-1}{x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{-2x-1}{x(1+x)}dx} \\
&= x(1+x)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x-1}{x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(x(1+x)\,y) &= (x(1+x))\left(\frac{x-1}{x}\right) \\
\mathrm{d}(x(1+x)\,y) &= \left(x^2-1\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x(1+x)\,y &= \int x^2-1\,\mathrm{d}x \\
x(1+x)\,y &= \frac{1}{3}x^3-x+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x(1+x)$ results in

$$
y = \frac{\frac{1}{3}x^3-x}{x\,(1+x)}+\frac{c_1}{x\,(1+x)}
$$

which simplifies to

$$
y = \frac{x^3+3c_1-3x}{3x\,(1+x)}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^3+3c_1-3x}{3x\,(1+x)} \tag{1}
$$

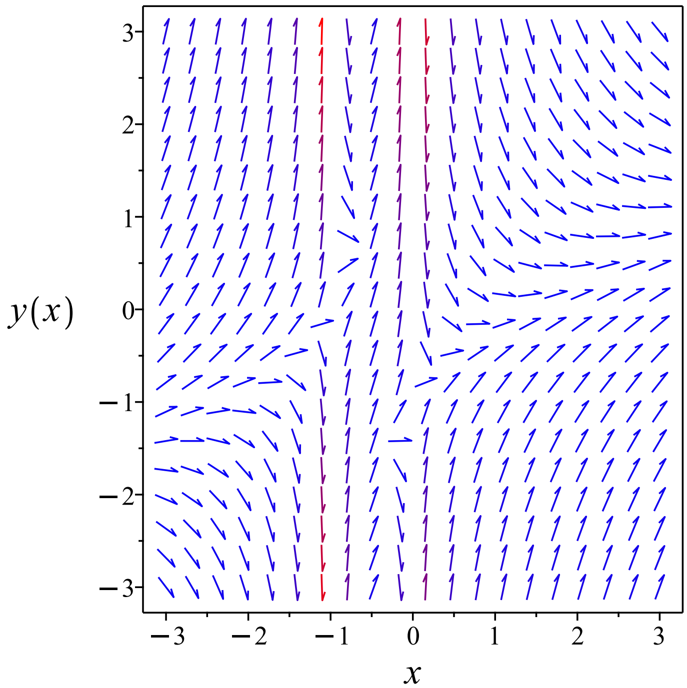


Figure 165: Slope field plot

Verification of solutions

$$y = \frac{x^3 + 3c_1 - 3x}{3x(1+x)}$$

Verified OK.

5.7.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-\frac{(2x+1)y}{1+x} + x - 1}{x} \tag{1}$$

Which becomes

$$0 = (-x^2 - x)\,dy + (x^2 - 2xy - y - 1)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x^2 - x)\,dy + (x^2 - 2xy - y - 1)\,dx = d\left(\frac{1}{3}x^3 - x^2y - xy - x\right)$$

Hence (2) becomes

$$0 = d\left(\frac{1}{3}x^3 - x^2y - xy - x\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{x^3 + 3c_1 - 3x}{3x(1+x)} + c_1$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 + 3c_1 - 3x}{3x(1+x)} + c_1 \tag{1}$$

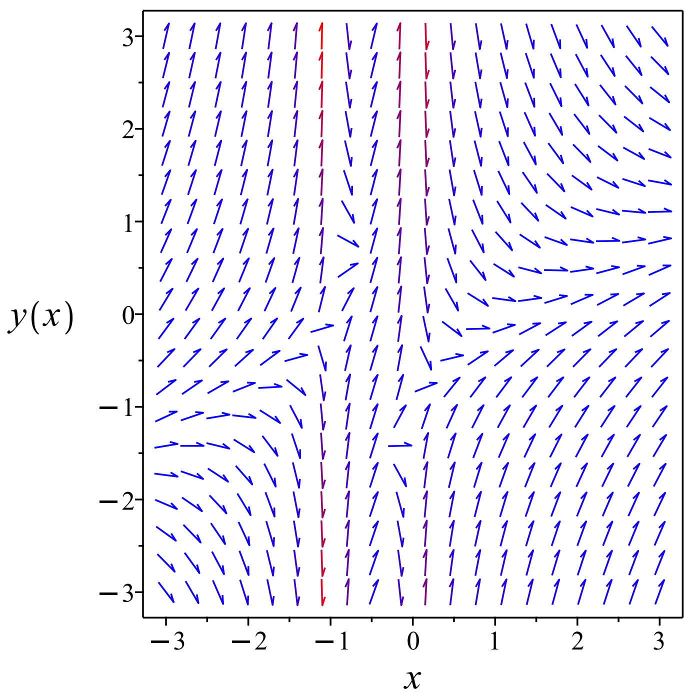


Figure 166: Slope field plot

Verification of solutions

$$y = \frac{x^3 + 3c_1 - 3x}{3x(1+x)} + c_1$$

Verified OK.

5.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{-x^2 + 2xy + y + 1}{(1 + x)\, x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 130: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= \frac{1}{x\,(1+x)}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta}dy\\ &= \int \frac{1}{\frac{1}{x(1+x)}}dy\end{aligned}$$

Which results in

$$S = x(1+x)\,y$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-x^2+2xy+y+1}{(1+x)\,x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= (2x+1)\,y \\
S_y &= x(1+x)
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^2 - 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2 - 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{3}R^3 - R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x(1+x)\,y = \frac{1}{3}x^3 - x + c_1$$

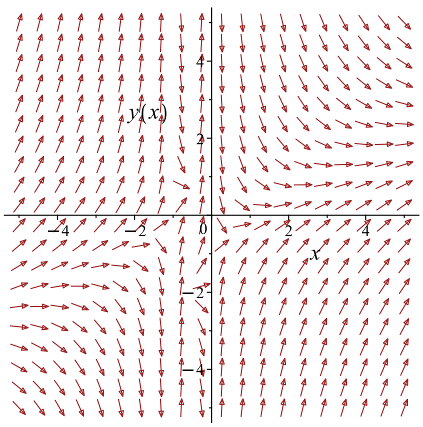
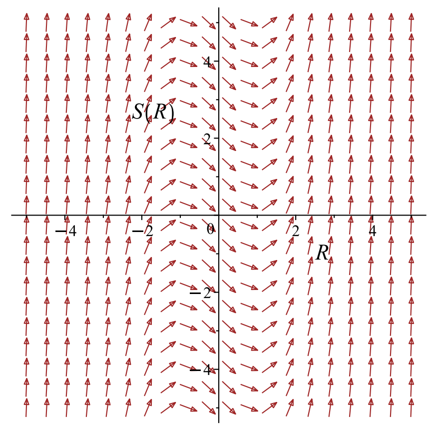
Which simplifies to

$$x(1+x)\,y = \frac{1}{3}x^3 - x + c_1$$

Which gives

$$y = \frac{x^3 + 3c_1 - 3x}{3x\,(1+x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^2+2xy+y+1}{(1+x)x}$ 	$R = x$ $S = x(1+x)\,y$	$\frac{dS}{dR} = R^2 - 1$ 

Summary

The solution(s) found are the following

$$y = \frac{x^3 + 3c_1 - 3x}{3x\,(1+x)} \tag{1}$$

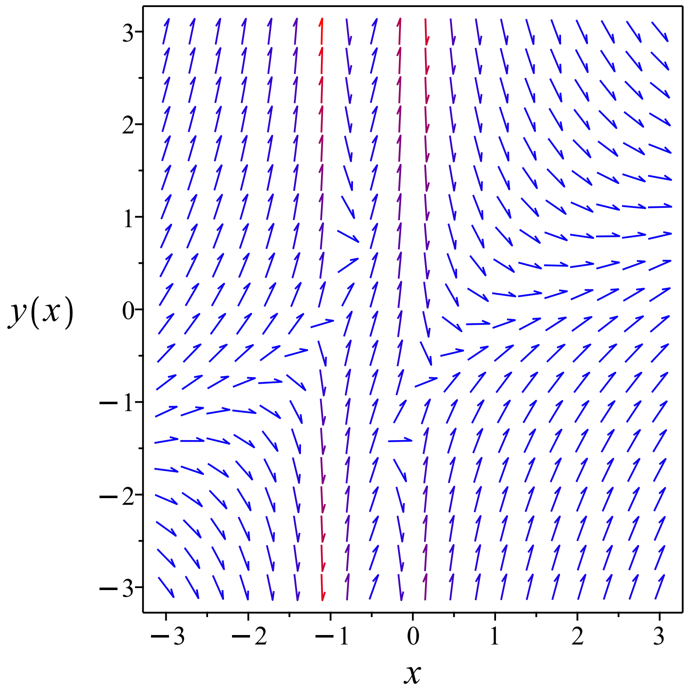


Figure 167: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{x^3 + 3c_1 - 3x}{3x(1 + x)}$$

Verified OK.

5.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x(1 + x))\,\mathrm{d}y = \left(x^2 - 2xy - y - 1\right)\mathrm{d}x$$
$$\left(-x^2 + 2xy + y + 1\right)\mathrm{d}x + (x(1 + x))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^2 + 2xy + y + 1$$
$$N(x, y) = x(1 + x)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 + 2xy + y + 1\right)\\ &= 2x + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x(1 + x))\\ &= 2x + 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x^2 + 2xy + y + 1 \, \mathrm{d}x$$

$$\phi = -\frac{1}{3}x^3 + x^2y + xy + x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x(1 + x)$. Therefore equation (4) becomes

$$x(1 + x) = x^2 + x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{3}x^3 + x^2y + xy + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{3}x^3 + x^2y + xy + x$$

The solution becomes

$$y = \frac{x^3 + 3c_1 - 3x}{3x(1+x)}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 + 3c_1 - 3x}{3x(1+x)} \tag{1}$$

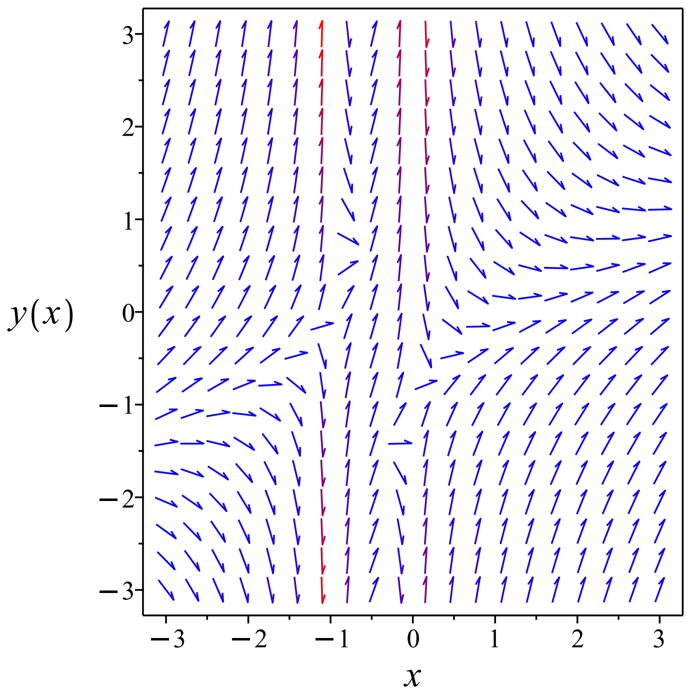


Figure 168: Slope field plot

Verification of solutions

$$y = \frac{x^3 + 3c_1 - 3x}{3x(1+x)}$$

Verified OK.

5.7.5 Maple step by step solution

Let's solve

$y'x+\frac{(2x+1)y}{1+x}=x-1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-\frac{(2x+1)y}{x(1+x)}+\frac{x-1}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'+\frac{(2x+1)y}{x(1+x)}=\frac{x-1}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{(2x+1)y}{x(1+x)}\right)=\frac{\mu(x)(x-1)}{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{(2x+1)y}{x(1+x)}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)(2x+1)}{x(1+x)}$

- Solve to find the integrating factor

  $\mu(x)=x(1+x)$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)(x-1)}{x}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{\mu(x)(x-1)}{x}dx+c_1$

- Solve for y

  $y=\frac{\int\frac{\mu(x)(x-1)}{x}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=x(1+x)$

  $y=\frac{\int(x-1)(1+x)dx+c_1}{x(1+x)}$

- Evaluate the integrals on the rhs

  $y=\frac{\frac{1}{3}x^3-x+c_1}{x(1+x)}$

- Simplify

  $y = \frac{x^3+3c_1-3x}{3x(1+x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(x*diff(y(x),x)+(2*x+1)/(x+1)*y(x)=x-1,y(x), singsol=all)
```

$$y(x) = \frac{x^3 + 3c_1 - 3x}{3x\,(1 + x)}$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 28

```
DSolve[x*y'[x]+(2*x+1)/(x+1)*y[x]==x-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3 - 3x + 3c_1}{3x(x + 1)}$$

5.8 problem 8

Internal problem ID [11645]
Internal file name [`OUTPUT/11654_Tuesday_April_09_2024_02_04_58_AM_13102016/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x^2+x-2)\,y'+3y(1+x)=x-1}$$

5.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=-\frac{-3x-3}{x^2+x-2}$$
$$q(x)=\frac{1}{x+2}$$

Hence the ode is

$$y'-\frac{(-3x-3)\,y}{x^2+x-2}=\frac{1}{x+2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-3x-3}{x^2+x-2}dx}$$
$$= e^{2\ln(x-1)+\ln(x+2)}$$

Which simplifies to

$$\mu = x^3 - 3x + 2$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{1}{x+2}\right)$$
$$\frac{d}{dx}\left(\left(x^3 - 3x + 2\right)y\right) = \left(x^3 - 3x + 2\right)\left(\frac{1}{x+2}\right)$$
$$d\left(\left(x^3 - 3x + 2\right)y\right) = (x-1)^2\ dx$$

Integrating gives

$$\left(x^3 - 3x + 2\right)y = \int (x-1)^2\ dx$$
$$\left(x^3 - 3x + 2\right)y = \frac{(x-1)^3}{3} + c_1$$

Dividing both sides by the integrating factor $\mu = x^3 - 3x + 2$ results in

$$y = \frac{(x-1)^3}{3x^3 - 9x + 6} + \frac{c_1}{x^3 - 3x + 2}$$

which simplifies to

$$y = \frac{x^3 - 3x^2 + 3c_1 + 3x - 1}{3x^3 - 9x + 6}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 - 3x^2 + 3c_1 + 3x - 1}{3x^3 - 9x + 6} \tag{1}$$

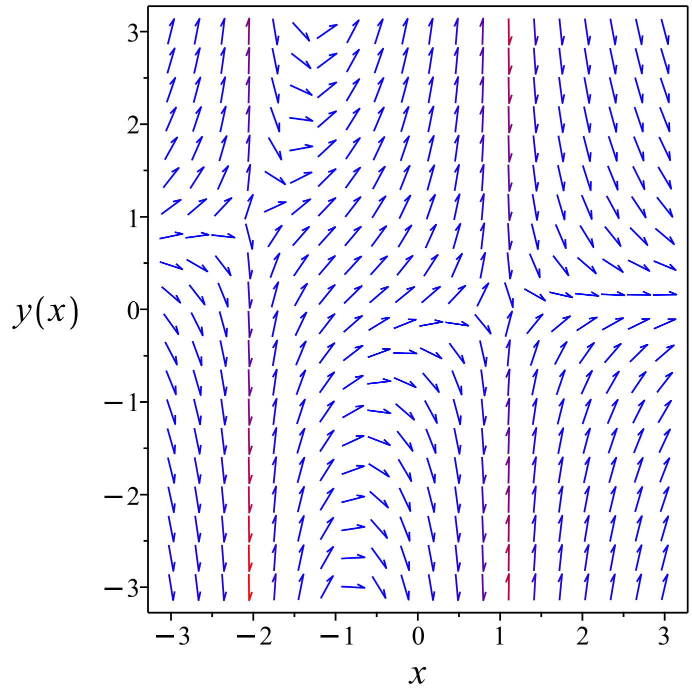


Figure 169: Slope field plot

Verification of solutions

$$y = \frac{x^3 - 3x^2 + 3c_1 + 3x - 1}{3x^3 - 9x + 6}$$

Verified OK.

5.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3xy - x + 3y + 1}{x^2 + x - 2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 133: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-2\ln(x-1)-\ln(x+2)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-2\ln(x-1)-\ln(x+2)}} dy \end{aligned}$$

Which results in

$$S = (x-1)^2 (x+2)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3xy - x + 3y + 1}{x^2 + x - 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3x^2 y - 3y \\ S_y &= (x-1)^2 (x+2) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (x-1)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (R-1)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{(R-1)^3}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x-1)^2 (x+2) y = \frac{(x-1)^3}{3} + c_1$$

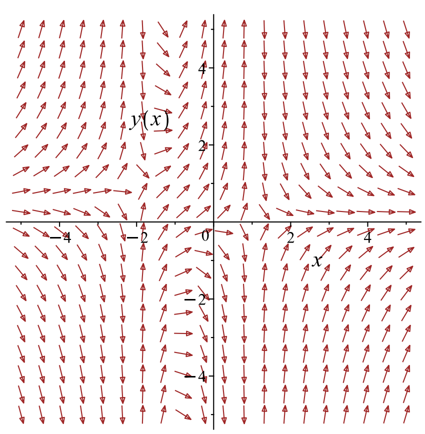
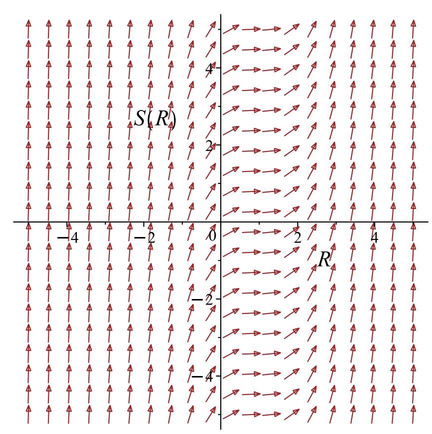
Which simplifies to

$$(x-1)^2 (x+2) y = \frac{(x-1)^3}{3} + c_1$$

Which gives

$$y = \frac{x^3 - 3x^2 + 3c_1 + 3x - 1}{3 (x-1)^2 (x+2)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{3xy-x+3y+1}{x^2+x-2}$

</td><td>$R = x$
$S = (x-1)^2 (x+2) y$</td><td>$\frac{dS}{dR} = (R-1)^2$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{x^3 - 3x^2 + 3c_1 + 3x - 1}{3 (x-1)^2 (x+2)} \tag{1}$$

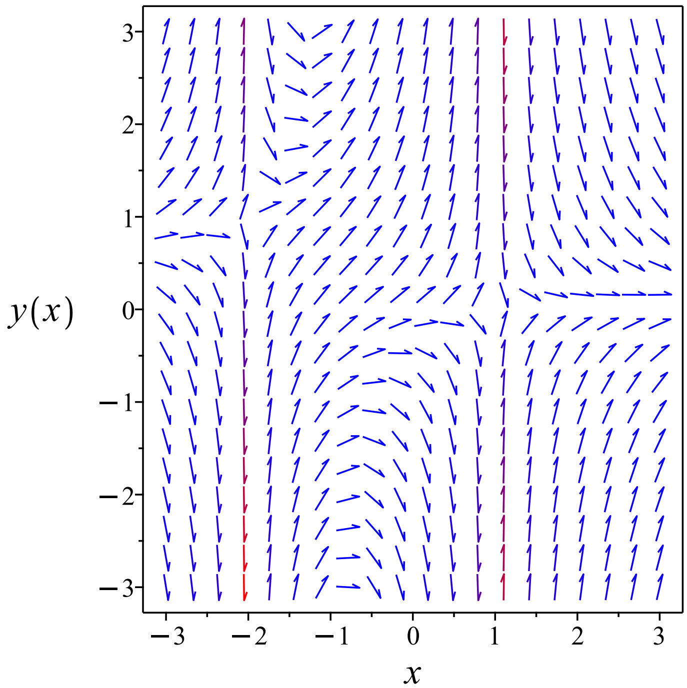


Figure 170: Slope field plot

Verification of solutions

$$y = \frac{x^3 - 3x^2 + 3c_1 + 3x - 1}{3(x-1)^2 (x+2)}$$

Verified OK.

5.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(x^2 + x - 2\right)\mathrm{d}y &= (-3y(1 + x) + x - 1)\,\mathrm{d}x \\
(3y(1 + x) - x + 1)\,\mathrm{d}x + \left(x^2 + x - 2\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 3y(1 + x) - x + 1 \\
N(x, y) &= x^2 + x - 2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3y(1 + x) - x + 1) \\
&= 3x + 3
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 + x - 2\right) \\
&= 2x + 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^2 + x - 2}((3x + 3) - (2x + 1)) \\
&= \frac{1}{x - 1}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{x-1} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(x-1)} \\
&= x - 1
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x - 1(3y(1 + x) - x + 1) \\
&= (3y - 1)\, x^2 + 2x - 1 - 3y
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x - 1\left(x^2 + x - 2\right) \\
&= (x - 1)^2\,(x + 2)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left((3y - 1)\, x^2 + 2x - 1 - 3y\right) + \left((x - 1)^2\,(x + 2)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int (3y - 1) \, x^2 + 2x - 1 - 3y \, \mathrm{d}x$$
$$\phi = \frac{x(3x^2 y - x^2 + 3x - 9y - 3)}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x(3x^2 - 9)}{3} + f'(y) \tag{4}$$
$$= x\left(x^2 - 3\right) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x - 1)^2 \, (x + 2)$. Therefore equation (4) becomes

$$(x - 1)^2 \, (x + 2) = x\left(x^2 - 3\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (2) \, \mathrm{d}y$$
$$f(y) = 2y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(3x^2y - x^2 + 3x - 9y - 3)}{3} + 2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(3x^2y - x^2 + 3x - 9y - 3)}{3} + 2y$$

The solution becomes

$$y = \frac{x^3 - 3x^2 + 3c_1 + 3x}{3x^3 - 9x + 6}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 - 3x^2 + 3c_1 + 3x}{3x^3 - 9x + 6} \tag{1}$$

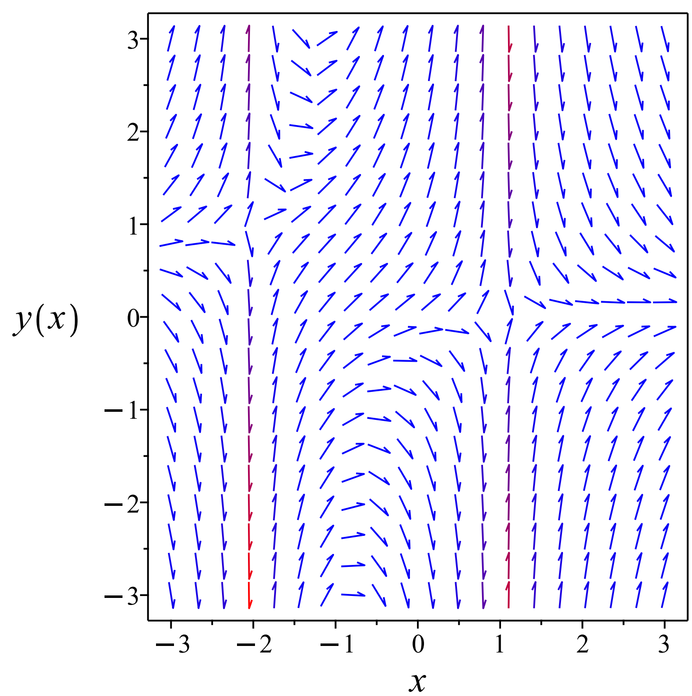


Figure 171: Slope field plot

Verification of solutions

$$y = \frac{x^3 - 3x^2 + 3c_1 + 3x}{3x^3 - 9x + 6}$$

Verified OK.

5.8.4 Maple step by step solution

Let's solve

$(x^2 + x - 2) y' + 3y(1 + x) = x - 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{3(1+x)y}{x^2+x-2} + \frac{1}{x+2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{3(1+x)y}{x^2+x-2} = \frac{1}{x+2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{3(1+x)y}{x^2+x-2}\right) = \frac{\mu(x)}{x+2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{3(1+x)y}{x^2+x-2}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)(1+x)}{x^2+x-2}$

- Solve to find the integrating factor

  $\mu(x) = (x-1)^2 (x+2)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{x+2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{x+2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{x+2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = (x-1)^2 (x+2)$

$$y = \frac{\int (x-1)^2 dx + c_1}{(x-1)^2 (x+2)}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{(x-1)^3}{3} + c_1}{(x-1)^2 (x+2)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve((x^2+x-2)*diff(y(x),x)+3*(x+1)*y(x)=x-1,y(x), singsol=all)
```

$$y(x) = \frac{\frac{(-1+x)^3}{3} + c_1}{(x+2)(-1+x)^2}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 34

```
DSolve[(x^2+x-2)*y'[x]+3*(x+1)*y[x]==x-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3 - 3x^2 + 3x + 3c_1}{3x^3 - 9x + 6}$$

5.9 problem 9

Internal problem ID [11646]
Internal file name [`OUTPUT/11655_Tuesday_April_09_2024_02_04_59_AM_70744530/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x + yx + y = 1}$$

5.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-x-1}{x}$$
$$q(x) = \frac{1}{x}$$

Hence the ode is

$$y' - \frac{(-x-1)\,y}{x} = \frac{1}{x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-x-1}{x}dx}$$
$$= e^{x+\ln(x)}$$

Which simplifies to

$$\mu = x\,e^{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{1}{x}\right)$$
$$\frac{d}{dx}(x\,e^{x}y) = (x\,e^{x})\left(\frac{1}{x}\right)$$
$$d(x\,e^{x}y) = e^{x}\,dx$$

Integrating gives

$$x\,e^{x}y = \int e^{x}\,dx$$
$$x\,e^{x}y = e^{x} + c_1$$

Dividing both sides by the integrating factor $\mu = x\,e^{x}$ results in

$$y = \frac{e^{-x}e^{x}}{x} + \frac{c_1 e^{-x}}{x}$$

which simplifies to

$$y = \frac{c_1 e^{-x} + 1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 e^{-x} + 1}{x} \tag{1}$$

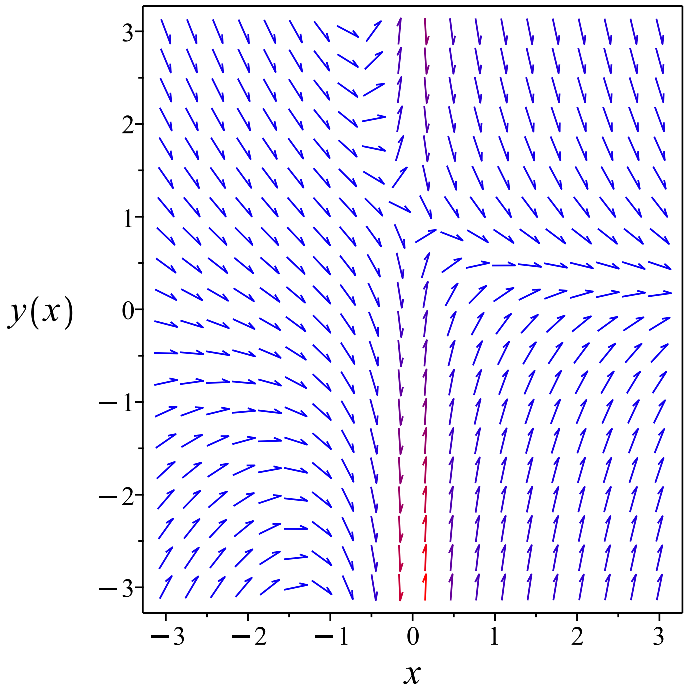


Figure 172: Slope field plot

Verification of solutions

$$y = \frac{c_1 e^{-x} + 1}{x}$$

Verified OK.

5.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{xy + y - 1}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 136: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x-\ln(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x-\ln(x)}} dy \end{aligned}$$

Which results in

$$S = x\,e^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{xy + y - 1}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{x} y(1+x) \\ S_y &= x\,e^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx\,e^{x} = e^{x} + c_1$$

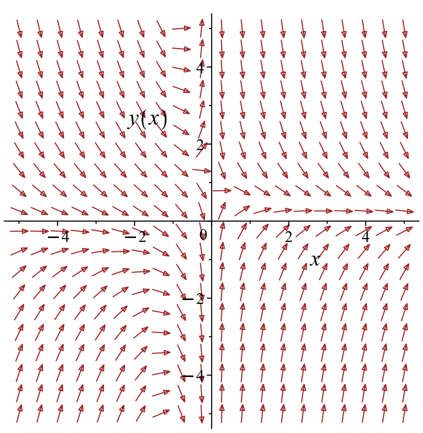
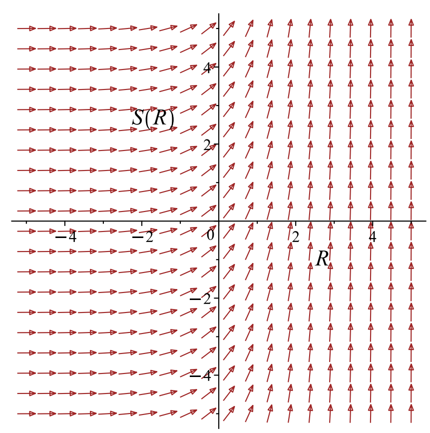
Which simplifies to

$$yx\,e^{x} = e^{x} + c_1$$

Which gives

$$y = \frac{(e^{x} + c_1)\,e^{-x}}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{xy+y-1}{x}$ 	$R = x$ $S = x\,e^{x} y$	$\frac{dS}{dR} = e^{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{(e^{x} + c_1)\,e^{-x}}{x} \tag{1}$$

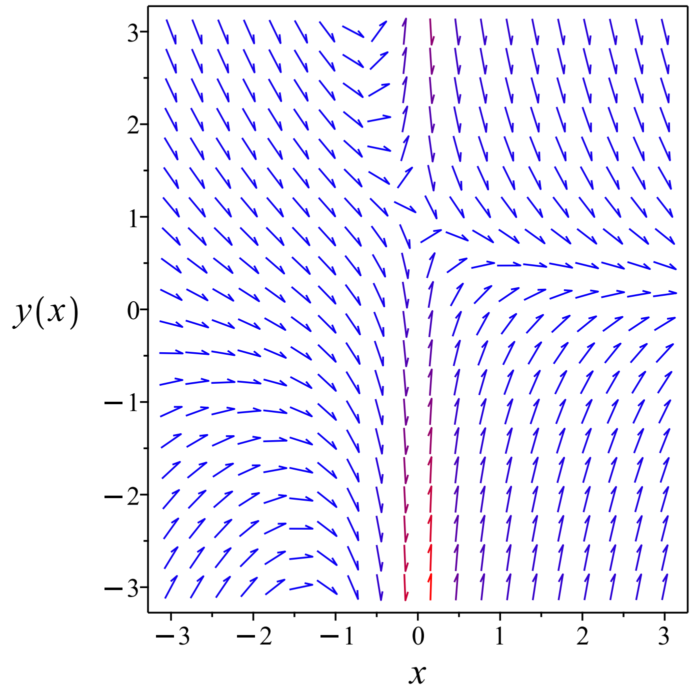


Figure 173: Slope field plot

Verification of solutions

$$y = \frac{(e^x + c_1) e^{-x}}{x}$$

Verified OK.

5.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x) \, \mathrm{d}y &= (-xy - y + 1) \, \mathrm{d}x \\
(xy + y - 1) \, \mathrm{d}x + (x) \, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= xy + y - 1 \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(xy + y - 1) \\
&= 1 + x
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((1+x)-(1))\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int 1 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x}\\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{x}(xy+y-1)\\ &= \mathrm{e}^{x}(xy+y-1)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{x}(x)\\ &= x\,\mathrm{e}^{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ (\mathrm{e}^{x}(xy+y-1)) + (x\,\mathrm{e}^{x})\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \mathrm{e}^{x}(xy + y - 1) \,\mathrm{d}x \\
\phi &= (xy - 1)\,\mathrm{e}^{x} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^{x} + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x\,\mathrm{e}^{x}$. Therefore equation (4) becomes

$$
x\,\mathrm{e}^{x} = x\,\mathrm{e}^{x} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = (xy - 1)\,\mathrm{e}^{x} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = (xy - 1)\,\mathrm{e}^{x}
$$

The solution becomes

$$
y = \frac{(\mathrm{e}^{x} + c_1)\,\mathrm{e}^{-x}}{x}
$$

Summary
The solution(s) found are the following

$$y = \frac{(e^{x} + c_1)\, e^{-x}}{x} \tag{1}$$

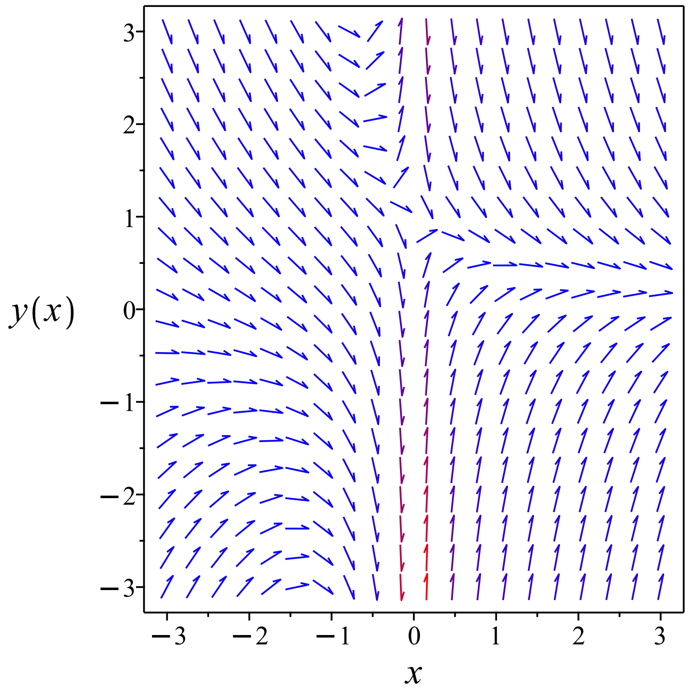


Figure 174: Slope field plot

Verification of solutions

$$y = \frac{(e^{x} + c_1)\, e^{-x}}{x}$$

Verified OK.

5.9.4 Maple step by step solution

Let's solve

$y'x + yx + y = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -\frac{(1+x)y}{x} + \frac{1}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{(1+x)y}{x} = \frac{1}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{(1+x)y}{x}\right) = \frac{\mu(x)}{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{(1+x)y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)(1+x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x\,\mathrm{e}^{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{x} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x\mathrm{e}^{x}$

  $y = \frac{\int \mathrm{e}^{x} dx + c_1}{x\,\mathrm{e}^{x}}$

- Evaluate the integrals on the rhs

  $y = \frac{\mathrm{e}^{x} + c_1}{\mathrm{e}^{x} x}$

- Simplify

  $y = \frac{c_1 \mathrm{e}^{-x} + 1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x*diff(y(x),x)+(x*y(x)+y(x)-1)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{-x} + 1}{x}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 19

```
DSolve[x*y'[x]+(x*y[x]+y[x]-1)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1 + c_1 e^{-x}}{x}$$

5.10 problem 10

Internal problem ID [11647]
Internal file name [`OUTPUT/11656_Tuesday_April_09_2024_02_04_59_AM_24934280/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{y + (y^2x + x - y)\, y' = 0}$$

5.10.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x\,y^2+x-y\right)\mathrm{d}y&=(-y)\,\mathrm{d}x\\ (y)\,\mathrm{d}x+\left(x\,y^2+x-y\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y\\ N(x,y)&=x\,y^2+x-y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(y)\\ &=1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(x\,y^2+x-y\right)\\ &=y^2+1\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=\frac{1}{x\,y^2+x-y}\left((1)-\left(y^2+1\right)\right)\\ &=-\frac{y^2}{x\,y^2+x-y}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y}\left(\left(y^2+1\right) - (1)\right) \\
&= y
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int y \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\frac{y^2}{2}} \\
&= \mathrm{e}^{\frac{y^2}{2}}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{\frac{y^2}{2}}(y) \\
&= y\,\mathrm{e}^{\frac{y^2}{2}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\frac{y^2}{2}}\left(x\,y^2 + x - y\right) \\
&= \left(x\,y^2 + x - y\right)\mathrm{e}^{\frac{y^2}{2}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(y\,\mathrm{e}^{\frac{y^2}{2}}\right) + \left(\left(x\,y^2 + x - y\right)\mathrm{e}^{\frac{y^2}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int y \,\mathrm{e}^{\frac{y^2}{2}} \,\mathrm{d}x$$

$$\phi = y \,\mathrm{e}^{\frac{y^2}{2}} x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{y^2}{2}} x + y^2 \mathrm{e}^{\frac{y^2}{2}} x + f'(y) \tag{4}$$

$$= \mathrm{e}^{\frac{y^2}{2}} x\left(y^2 + 1\right) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (x\,y^2 + x - y)\,\mathrm{e}^{\frac{y^2}{2}}$. Therefore equation (4) becomes

$$\left(x\,y^2 + x - y\right) \mathrm{e}^{\frac{y^2}{2}} = \mathrm{e}^{\frac{y^2}{2}} x\left(y^2 + 1\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y \,\mathrm{e}^{\frac{y^2}{2}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(-y \,\mathrm{e}^{\frac{y^2}{2}}\right) \mathrm{d}y$$

$$f(y) = -\mathrm{e}^{\frac{y^2}{2}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,\mathrm{e}^{\frac{y^2}{2}} x - \mathrm{e}^{\frac{y^2}{2}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,\mathrm{e}^{\frac{y^2}{2}} x - \mathrm{e}^{\frac{y^2}{2}}$$

Summary

The solution(s) found are the following

$$y\,\mathrm{e}^{\frac{y^2}{2}} x - \mathrm{e}^{\frac{y^2}{2}} = c_1 \tag{1}$$

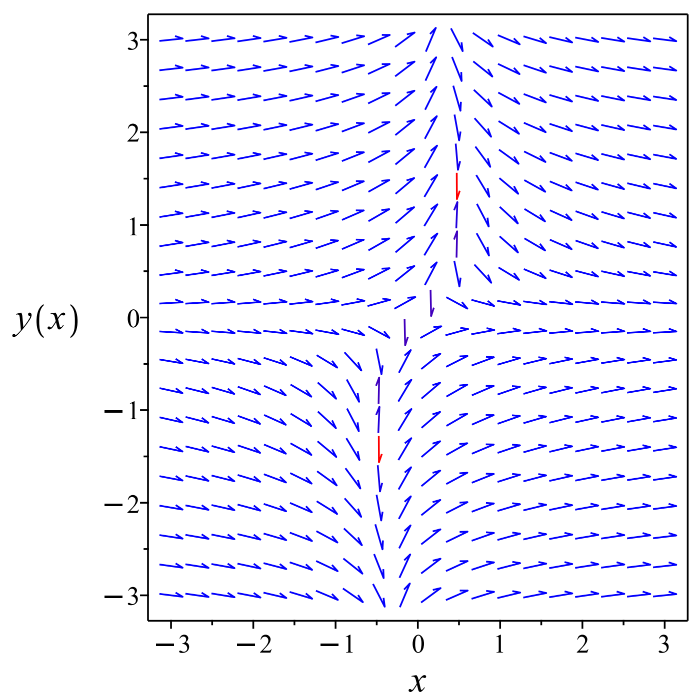


Figure 175: Slope field plot

Verification of solutions

$$y\,\mathrm{e}^{\frac{y^2}{2}} x - \mathrm{e}^{\frac{y^2}{2}} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
dsolve(y(x)+(x*y(x)^2+x-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{\mathrm{RootOf}\left(c_1^2\mathrm{e}^{2\_Z}+2x^2\_Z+2c_1\mathrm{e}^{\_Z}+1\right)}c_1 + 1}{x}$$

✓ Solution by Mathematica

Time used: 0.206 (sec). Leaf size: 27

```
DSolve[y[x]+(x*y[x]^2+x-y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[x = \frac{1}{y(x)} + \frac{c_1 e^{-\frac{1}{2}y(x)^2}}{y(x)}, y(x)\right]$$

5.11 problem 11

Internal problem ID [11648]
Internal file name [`OUTPUT/11657_Tuesday_April_09_2024_02_05_00_AM_74603805/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{r' + r\tan(t) = \cos(t)}$$

5.11.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r' + p(t)r = q(t)$$

Where here

$$p(t) = \tan(t)$$
$$q(t) = \cos(t)$$

Hence the ode is

$$r' + r\tan(t) = \cos(t)$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int \tan(t)dt}$$
$$= \frac{1}{\cos(t)}$$

Which simplifies to

$$\mu = \sec(t)$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu r) = (\mu)(\cos(t))$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\sec(t) r) = (\sec(t))(\cos(t))$$
$$\mathrm{d}(\sec(t) r) = \mathrm{d}t$$

Integrating gives

$$\sec(t) r = \int \mathrm{d}t$$
$$\sec(t) r = t + c_1$$

Dividing both sides by the integrating factor $\mu = \sec(t)$ results in

$$r = \cos(t) t + c_1 \cos(t)$$

which simplifies to

$$r = \cos(t)(t + c_1)$$

Summary
The solution(s) found are the following

$$r = \cos(t)(t + c_1) \tag{1}$$

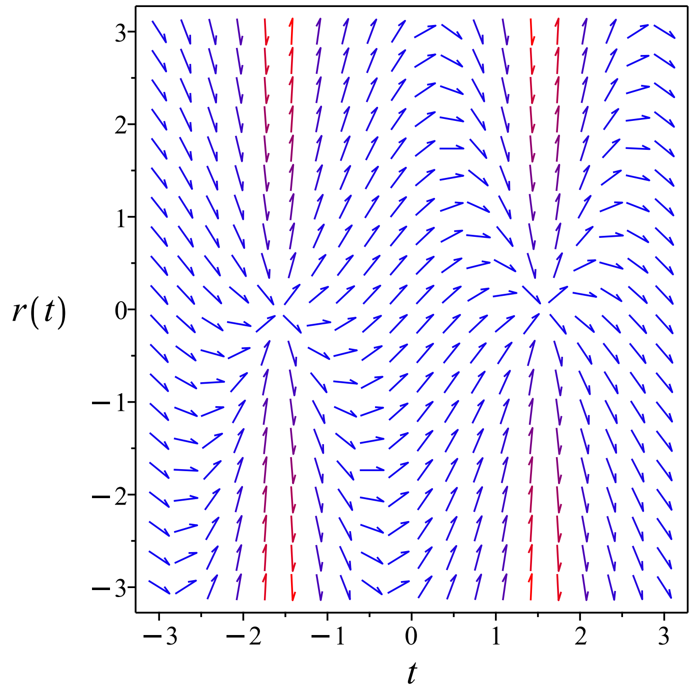


Figure 176: Slope field plot

Verification of solutions

$$r = \cos(t)(t + c_1)$$

Verified OK.

5.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$r' = -r\tan(t) + \cos(t)$$
$$r' = \omega(t, r)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_r - \xi_t) - \omega^2 \xi_r - \omega_t \xi - \omega_r \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 139: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, r) &= 0 \\
\eta(t, r) &= \cos(t)
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, r) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dr}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial r}\right) S(t, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\cos(t)} dy \end{aligned}$$

Which results in

$$S = \frac{r}{\cos(t)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, r) S_r}{R_t + \omega(t, r) R_r} \tag{2}$$

Where in the above R_t, R_r, S_t, S_r are all partial derivatives and $\omega(t, r)$ is the right hand side of the original ode given by

$$\omega(t, r) = -r \tan(t) + \cos(t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_r &= 0 \\ S_t &= \tan(t) \sec(t) r \\ S_r &= \sec(t) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, r coordinates. This results in

$$\sec(t)\, r = t + c_1$$

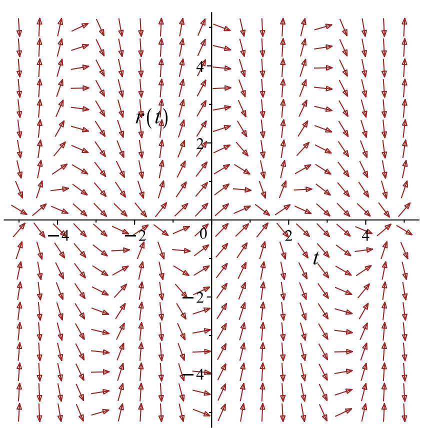
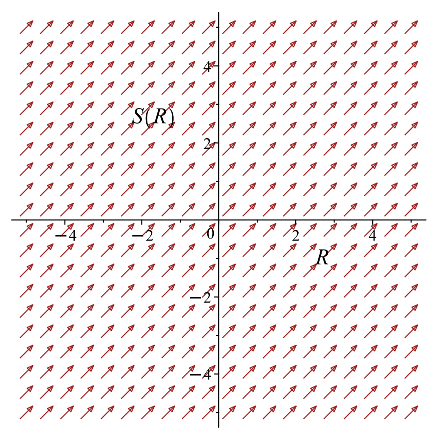
Which simplifies to

$$\sec(t)\, r = t + c_1$$

Which gives

$$r = \frac{t + c_1}{\sec(t)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, r coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dr}{dt} = -r\tan(t) + \cos(t)$ 	$R = t$ $S = \sec(t)\, r$	$\frac{dS}{dR} = 1$ 

Summary
The solution(s) found are the following

$$r = \frac{t + c_1}{\sec(t)} \tag{1}$$

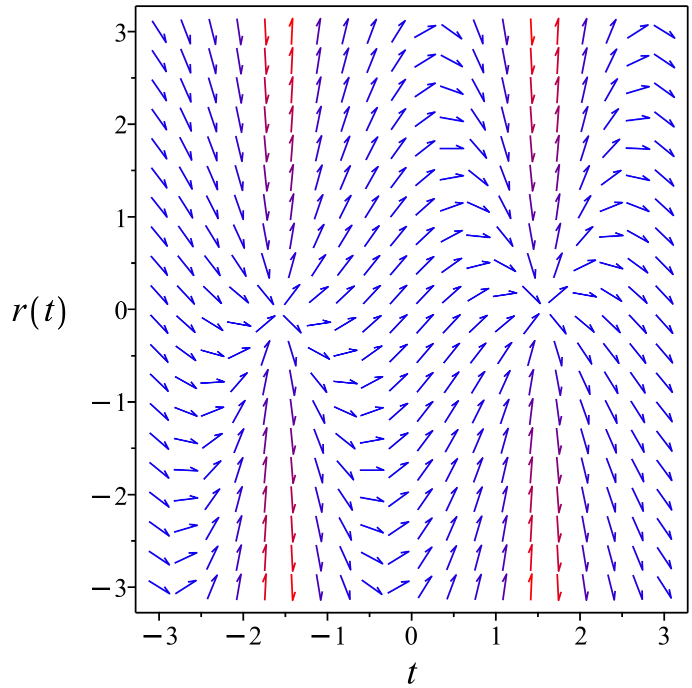


Figure 177: Slope field plot

Verification of solutions

$$r = \frac{t + c_1}{\sec(t)}$$

Verified OK.

5.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, r)\,\mathrm{d}t + N(t, r)\,\mathrm{d}r = 0 \tag{1A}$$

Therefore

$$\mathrm{d}r = (-r \tan(t) + \cos(t))\,\mathrm{d}t$$
$$(r \tan(t) - \cos(t))\,\mathrm{d}t + \mathrm{d}r = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(t, r) = r \tan(t) - \cos(t)$$
$$N(t, r) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}(r \tan(t) - \cos(t)) \\ &= \tan(t)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial r} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial r} - \frac{\partial N}{\partial t}\right) \\ &= 1((\tan(t)) - (0)) \\ &= \tan(t) \end{aligned}$$

Since A does not depend on r, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}t} \\ &= e^{\int \tan(t) \, \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(\cos(t))} \\ &= \sec(t) \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \sec(t)(r\tan(t) - \cos(t)) \\ &= -1 + \tan(t)\sec(t) r \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \sec(t)(1) \\ &= \sec(t) \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}r}{\mathrm{d}t} &= 0 \\ (-1 + \tan(t)\sec(t) r) + (\sec(t))\frac{\mathrm{d}r}{\mathrm{d}t} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, r)$

$$\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial r} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int \overline{M} \, \mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -1 + \tan(t)\sec(t) r \, \mathrm{d}t$$
$$\phi = -t + \sec(t) r + f(r) \tag{3}$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both t and r. Taking derivative of equation (3) w.r.t r gives

$$\frac{\partial \phi}{\partial r} = \sec(t) + f'(r) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = \sec(t)$. Therefore equation (4) becomes

$$\sec(t) = \sec(t) + f'(r) \tag{5}$$

Solving equation (5) for $f'(r)$ gives

$$f'(r) = 0$$

Therefore

$$f(r) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(r)$ into equation (3) gives ϕ

$$\phi = -t + \sec(t) r + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t + \sec(t) r$$

The solution becomes

$$r = \frac{t + c_1}{\sec(t)}$$

Summary
The solution(s) found are the following

$$r = \frac{t + c_1}{\sec(t)} \tag{1}$$

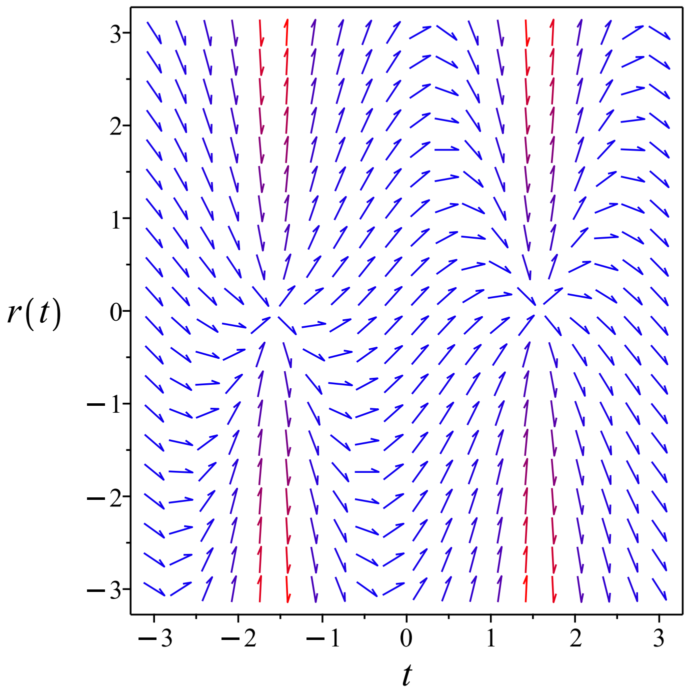


Figure 178: Slope field plot

Verification of solutions

$$r = \frac{t + c_1}{\sec(t)}$$

Verified OK.

5.11.4 Maple step by step solution

Let's solve

$r' + r\tan(t) = \cos(t)$

- Highest derivative means the order of the ODE is 1

  r'

- Isolate the derivative

$r' = -r \tan(t) + \cos(t)$

- Group terms with r on the lhs of the ODE and the rest on the rhs of the ODE

  $r' + r \tan(t) = \cos(t)$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t) (r' + r \tan(t)) = \mu(t) \cos(t)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t) r)$

  $\mu(t) (r' + r \tan(t)) = \mu'(t) r + \mu(t) r'$
- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t) \tan(t)$
- Solve to find the integrating factor

  $\mu(t) = \frac{1}{\cos(t)}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t) r)\right) dt = \int \mu(t) \cos(t) dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t) r = \int \mu(t) \cos(t) dt + c_1$
- Solve for r

  $r = \frac{\int \mu(t) \cos(t) dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \frac{1}{\cos(t)}$

  $r = \cos(t) \left(\int 1 dt + c_1\right)$
- Evaluate the integrals on the rhs

  $r = \cos(t) (t + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(r(t),t)+r(t)*tan(t)=cos(t),r(t), singsol=all)
```

$$r(t) = (t + c_1) \cos(t)$$

✓ Solution by Mathematica

Time used: 0.06 (sec). Leaf size: 12

```
DSolve[r'[t]+r[t]*Tan[t]==Cos[t],r[t],t,IncludeSingularSolutions -> True]
```

$$r(t) \to (t + c_1) \cos(t)$$

5.12 problem 12

Internal problem ID [11649]
Internal file name [`OUTPUT/11658_Tuesday_April_09_2024_02_05_01_AM_43799744/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\cos(t) r' + r \sin(t) = \cos(t)^4}$$

5.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r' + p(t)r = q(t)$$

Where here

$$p(t) = \tan(t)$$
$$q(t) = \cos(t)^3$$

Hence the ode is

$$r' + r\tan(t) = \cos(t)^3$$

The integrating factor μ is

$$\mu = e^{\int \tan(t)dt}$$
$$= \frac{1}{\cos(t)}$$

Which simplifies to

$$\mu = \sec(t)$$

The ode becomes

$$\frac{d}{dt}(\mu r) = (\mu)\left(\cos(t)^3\right)$$
$$\frac{d}{dt}(\sec(t)\, r) = (\sec(t))\left(\cos(t)^3\right)$$
$$d(\sec(t)\, r) = \cos(t)^2\, dt$$

Integrating gives

$$\sec(t)\, r = \int \cos(t)^2\, dt$$
$$\sec(t)\, r = \frac{\cos(t)\sin(t)}{2} + \frac{t}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = \sec(t)$ results in

$$r = \cos(t)\left(\frac{\cos(t)\sin(t)}{2} + \frac{t}{2}\right) + c_1\cos(t)$$

which simplifies to

$$r = \frac{\cos(t)\left(\cos(t)\sin(t) + 2c_1 + t\right)}{2}$$

Summary
The solution(s) found are the following

$$r = \frac{\cos(t)\left(\cos(t)\sin(t) + 2c_1 + t\right)}{2} \tag{1}$$

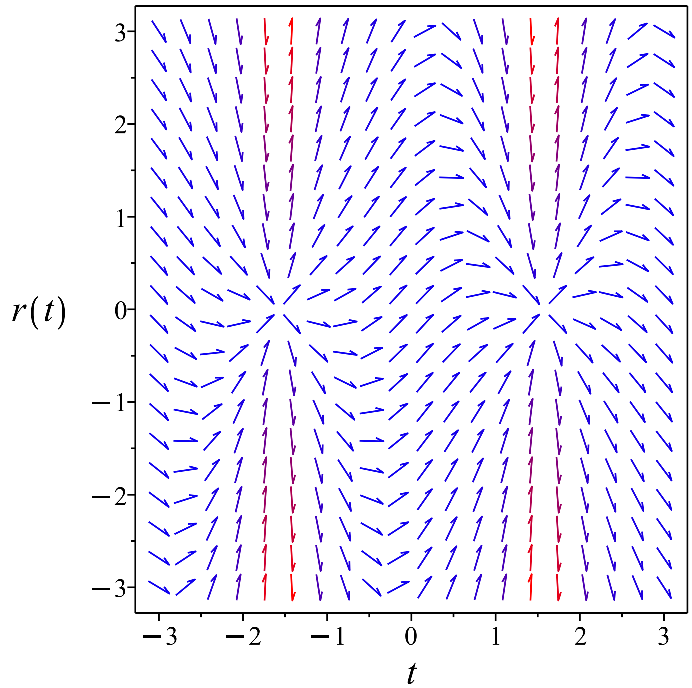


Figure 179: Slope field plot

Verification of solutions

$$r = \frac{\cos(t)\,(\cos(t)\sin(t) + 2c_1 + t)}{2}$$

Verified OK.

5.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$r' = \frac{-r\sin(t) + \cos(t)^4}{\cos(t)}$$
$$r' = \omega(t, r)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_r - \xi_t) - \omega^2\xi_r - \omega_t\xi - \omega_r\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 142: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t, r) &= 0 \\
\eta(t, r) &= \cos(t) \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, r) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dr}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial r}\right) S(t, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\cos(t)} dy \end{aligned}$$

Which results in

$$S = \frac{r}{\cos(t)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, r) S_r}{R_t + \omega(t, r) R_r} \tag{2}$$

Where in the above R_t, R_r, S_t, S_r are all partial derivatives and $\omega(t, r)$ is the right hand side of the original ode given by

$$\omega(t, r) = \frac{-r \sin(t) + \cos(t)^4}{\cos(t)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_r &= 0 \\ S_t &= \tan(t) \sec(t) r \\ S_r &= \sec(t) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(t)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 + \frac{\sin(2R)}{4} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, r coordinates. This results in

$$\sec(t)\, r = \frac{t}{2} + c_1 + \frac{\sin(2t)}{4}$$

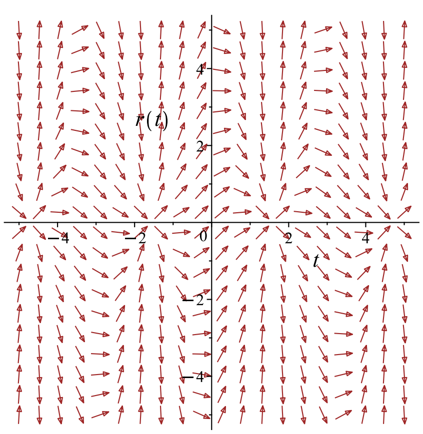
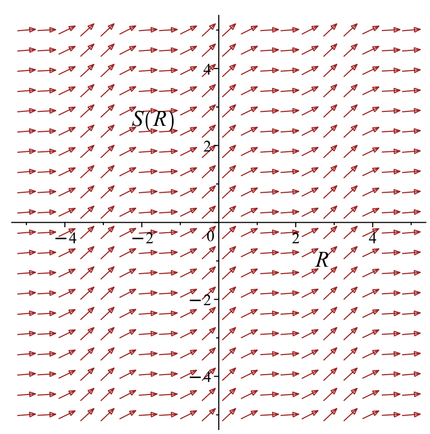
Which simplifies to

$$\sec(t)\, r = \frac{t}{2} + c_1 + \frac{\sin(2t)}{4}$$

Which gives

$$r = \frac{4c_1 + 2t + \sin(2t)}{4\sec(t)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, r coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dr}{dt} = \frac{-r\sin(t)+\cos(t)^4}{\cos(t)}$ 	$R = t$ $S = \sec(t)\, r$	$\frac{dS}{dR} = \cos(R)^2$ 

Summary
The solution(s) found are the following

$$r = \frac{4c_1 + 2t + \sin(2t)}{4\sec(t)} \tag{1}$$

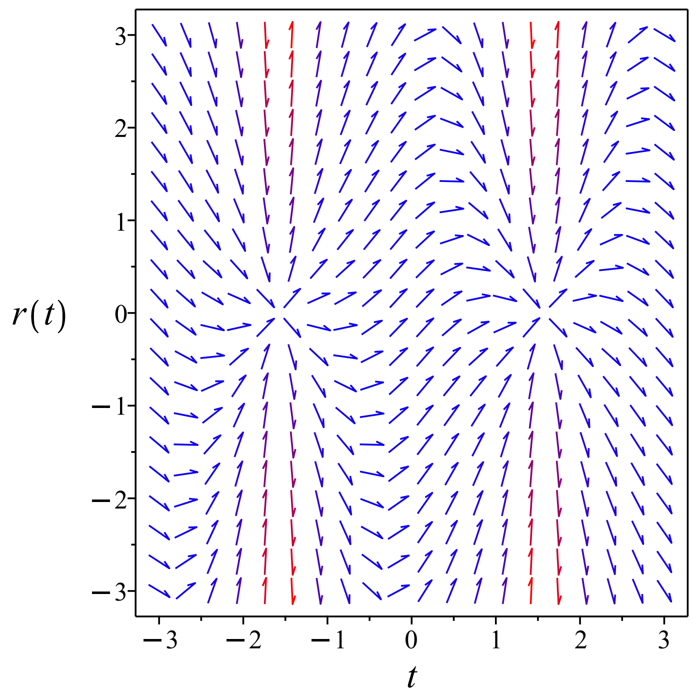


Figure 180: Slope field plot

Verification of solutions

$$r = \frac{4c_1 + 2t + \sin(2t)}{4\sec(t)}$$

Verified OK.

5.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,r)\,\mathrm{d}t + N(t,r)\,\mathrm{d}r = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(t))\,\mathrm{d}r &= \left(-r\sin(t) + \cos(t)^4\right)\mathrm{d}t \\ \left(r\sin(t) - \cos(t)^4\right)\mathrm{d}t + (\cos(t))\,\mathrm{d}r &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,r) &= r\sin(t) - \cos(t)^4 \\ N(t,r) &= \cos(t)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}\left(r\sin(t)-\cos(t)^4\right)\\ &= \sin(t)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(\cos(t))\\ &= -\sin(t)\end{aligned}$$

Since $\frac{\partial M}{\partial r} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial r}-\frac{\partial N}{\partial t}\right)\\ &= \sec(t)\left((\sin(t))-(-\sin(t))\right)\\ &= 2\tan(t)\end{aligned}$$

Since A does not depend on r, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int 2\tan(t)\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(\cos(t))}\\ &= \sec(t)^2\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \sec(t)^2\left(r\sin(t)-\cos(t)^4\right)\\ &= \tan(t)\sec(t)\,r-\cos(t)^2\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \sec(t)^2(\cos(t))\\ &= \sec(t)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}r}{\mathrm{d}t} = 0$$
$$\left(\tan(t)\sec(t)\,r - \cos(t)^2\right) + (\sec(t))\frac{\mathrm{d}r}{\mathrm{d}t} = 0$$

The following equations are now set up to solve for the function $\phi(t, r)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial r} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$
$$\int \frac{\partial\phi}{\partial t}\,\mathrm{d}t = \int \tan(t)\sec(t)\,r - \cos(t)^2\,\mathrm{d}t$$
$$\phi = \sec(t)\,r - \frac{\cos(t)\sin(t)}{2} - \frac{t}{2} + f(r) \tag{3}$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both t and r. Taking derivative of equation (3) w.r.t r gives

$$\frac{\partial\phi}{\partial r} = \sec(t) + f'(r) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial r} = \sec(t)$. Therefore equation (4) becomes

$$\sec(t) = \sec(t) + f'(r) \tag{5}$$

Solving equation (5) for $f'(r)$ gives

$$f'(r) = 0$$

Therefore

$$f(r) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(r)$ into equation (3) gives ϕ

$$\phi = \sec(t)\, r - \frac{\cos(t)\sin(t)}{2} - \frac{t}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sec(t)\, r - \frac{\cos(t)\sin(t)}{2} - \frac{t}{2}$$

The solution becomes

$$r = \frac{\cos(t)\sin(t) + 2c_1 + t}{2\sec(t)}$$

Summary

The solution(s) found are the following

$$r = \frac{\cos(t)\sin(t) + 2c_1 + t}{2\sec(t)} \tag{1}$$

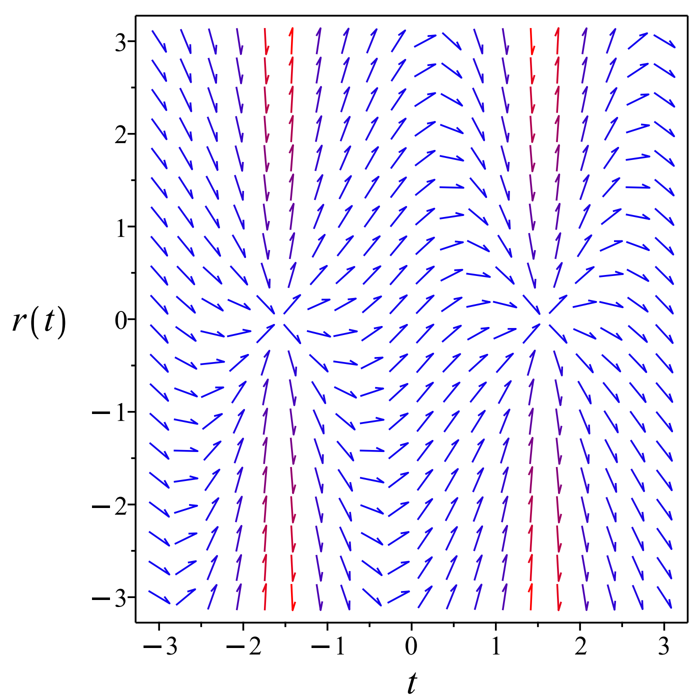


Figure 181: Slope field plot

Verification of solutions

$$r = \frac{\cos(t)\sin(t) + 2c_1 + t}{2\sec(t)}$$

Verified OK.

5.12.4 Maple step by step solution

Let's solve

$\cos(t)\, r' + r\sin(t) = \cos(t)^4$

- Highest derivative means the order of the ODE is 1

  r'

- Isolate the derivative

  $r' = -\frac{\sin(t) r}{\cos(t)} + \cos(t)^3$

- Group terms with r on the lhs of the ODE and the rest on the rhs of the ODE

  $r' + \frac{\sin(t) r}{\cos(t)} = \cos(t)^3$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(r' + \frac{\sin(t) r}{\cos(t)}\right) = \mu(t)\cos(t)^3$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, r)$

  $\mu(t)\left(r' + \frac{\sin(t) r}{\cos(t)}\right) = \mu'(t)\, r + \mu(t)\, r'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t)\sin(t)}{\cos(t)}$

- Solve to find the integrating factor

  $\mu(t) = \frac{1}{\cos(t)}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, r)\right) dt = \int \mu(t)\cos(t)^3\, dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, r = \int \mu(t)\cos(t)^3\, dt + c_1$

- Solve for r

  $r = \frac{\int \mu(t)\cos(t)^3 dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \frac{1}{\cos(t)}$

  $r = \cos(t)\left(\int \cos(t)^2 \, dt + c_1\right)$

- Evaluate the integrals on the rhs

  $r = \cos(t)\left(\frac{\cos(t)\sin(t)}{2} + \frac{t}{2} + c_1\right)$

- Simplify

  $r = \frac{\cos(t)(\cos(t)\sin(t)+2c_1+t)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(cos(t)*diff(r(t),t)+(r(t)*sin(t)-cos(t)^4)=0,r(t), singsol=all)
```

$$r(t) = \frac{(2t + \sin(2t) + 4c_1)\cos(t)}{4}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 22

```
DSolve[Cos[t]*r'[t]+(r[t]*Sin[t]-Cos[t]^4)==0,r[t],t,IncludeSingularSolutions -> True]
```

$$r(t) \to \frac{1}{2}\cos(t)(t + \sin(t)\cos(t) + 2c_1)$$

5.13 problem 13

Internal problem ID [11650]
Internal file name [`OUTPUT/11659_Tuesday_April_09_2024_02_05_03_AM_81013467/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-y\cos(x) - (1+\sin(x))\, y' = -\cos(x)^2}$$

5.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{\cos(x)}{1+\sin(x)}$$
$$q(x) = -\sin(x) + 1$$

Hence the ode is

$$y' + \frac{\cos(x)\, y}{1+\sin(x)} = -\sin(x) + 1$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{\cos(x)}{1+\sin(x)}dx} \\
&= 1+\sin(x)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(-\sin(x)+1) \\
\frac{\mathrm{d}}{\mathrm{d}x}((1+\sin(x))\,y) &= (1+\sin(x))\,(-\sin(x)+1) \\
\mathrm{d}((1+\sin(x))\,y) &= \cos(x)^2\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
(1+\sin(x))\,y &= \int \cos(x)^2\,\mathrm{d}x \\
(1+\sin(x))\,y &= \frac{\cos(x)\sin(x)}{2}+\frac{x}{2}+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = 1+\sin(x)$ results in

$$
y = \frac{\frac{\cos(x)\sin(x)}{2}+\frac{x}{2}}{1+\sin(x)}+\frac{c_1}{1+\sin(x)}
$$

which simplifies to

$$
y = \frac{\cos(x)\sin(x)+2c_1+x}{2+2\sin(x)}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\cos(x)\sin(x)+2c_1+x}{2+2\sin(x)} \tag{1}
$$

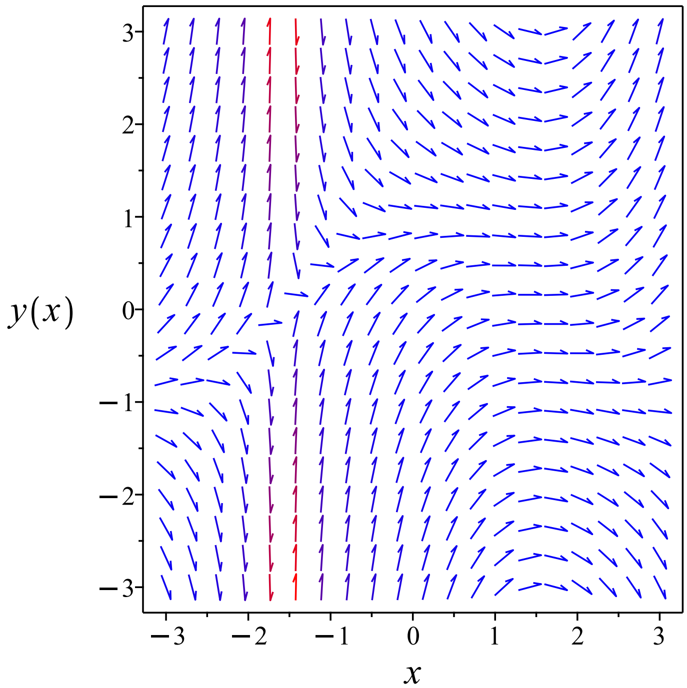


Figure 182: Slope field plot

Verification of solutions

$$y = \frac{\cos(x)\sin(x) + 2c_1 + x}{2 + 2\sin(x)}$$

Verified OK.

5.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\cos(x)(\cos(x) - y)}{1 + \sin(x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 145: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{1+\sin(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{1+\sin(x)}} dy \end{aligned}$$

Which results in

$$S = (1 + \sin(x))\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\cos(x)(\cos(x) - y)}{1 + \sin(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x)\, y \\ S_y &= 1 + \sin(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(x)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 + \frac{\sin(2R)}{4} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(1 + \sin(x))\, y = \frac{x}{2} + c_1 + \frac{\sin(2x)}{4}$$

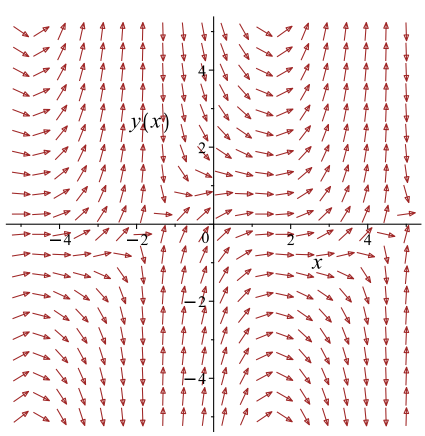
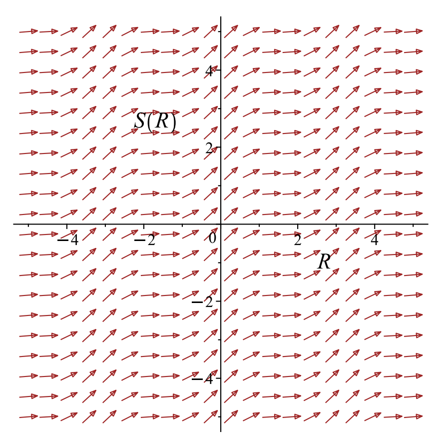
Which simplifies to

$$(1 + \sin(x))\, y = \frac{x}{2} + c_1 + \frac{\sin(2x)}{4}$$

Which gives

$$y = \frac{4c_1 + 2x + \sin(2x)}{4 + 4\sin(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\cos(x)(\cos(x)-y)}{1+\sin(x)}$ 	$R = x$ $S = (1 + \sin(x))\, y$	$\frac{dS}{dR} = \cos(R)^2$ 

Summary

The solution(s) found are the following

$$y = \frac{4c_1 + 2x + \sin(2x)}{4 + 4\sin(x)} \tag{1}$$

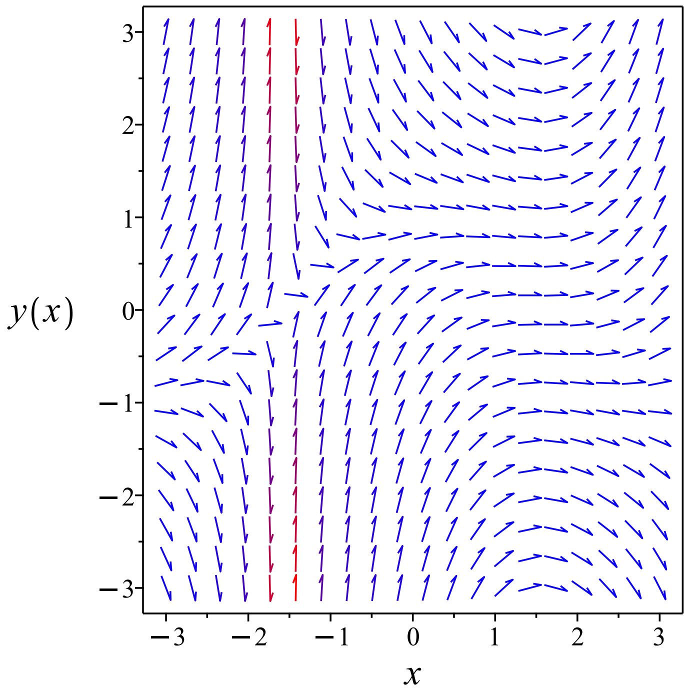


Figure 183: Slope field plot

Verification of solutions

$$y = \frac{4c_1 + 2x + \sin(2x)}{4 + 4\sin(x)}$$

Verified OK.

5.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-1-\sin(x))\,\mathrm{d}y &= \left(-\cos(x)^2+\cos(x)\,y\right)\mathrm{d}x \\ \left(\cos(x)^2-\cos(x)\,y\right)\mathrm{d}x + (-1-\sin(x))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \cos(x)^2 - \cos(x)\,y \\ N(x,y) &= -1-\sin(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\cos\left(x\right)^2-\cos\left(x\right)y\right)\\ &= -\cos\left(x\right)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-1-\sin\left(x\right)\right)\\ &= -\cos\left(x\right)\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \cos\left(x\right)^2-\cos\left(x\right)y\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{\sin\left(x\right)\left(\cos\left(x\right)-2y\right)}{2}+\frac{x}{2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\sin\left(x\right)+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=-1-\sin\left(x\right)$. Therefore equation (4) becomes

$$-1-\sin\left(x\right) = -\sin\left(x\right)+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-1)\,\mathrm{d}y$$
$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\sin(x)(\cos(x) - 2y)}{2} + \frac{x}{2} - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\sin(x)(\cos(x) - 2y)}{2} + \frac{x}{2} - y$$

The solution becomes

$$y = \frac{\cos(x)\sin(x) - 2c_1 + x}{2 + 2\sin(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{\cos(x)\sin(x) - 2c_1 + x}{2 + 2\sin(x)} \tag{1}$$

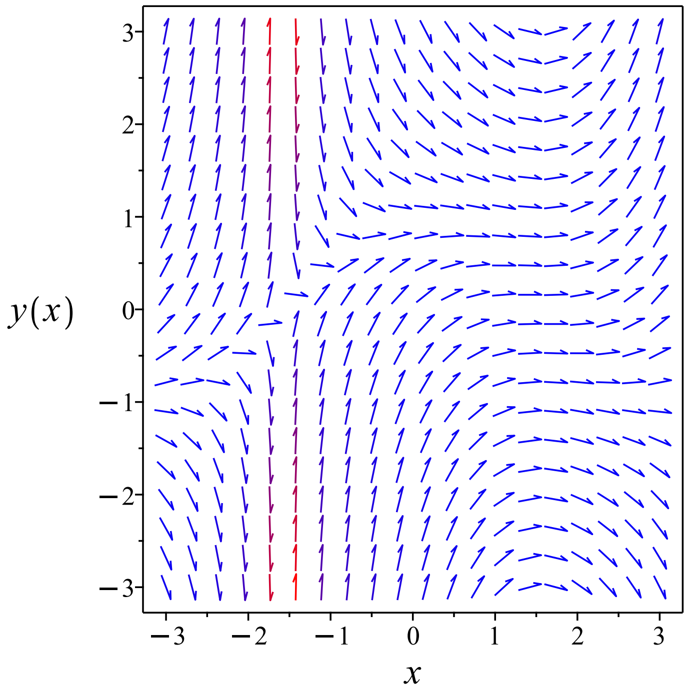


Figure 184: Slope field plot

Verification of solutions

$$y = \frac{\cos(x)\sin(x) - 2c_1 + x}{2 + 2\sin(x)}$$

Verified OK.

5.13.4 Maple step by step solution

Let's solve

$-y\cos(x) - (1+\sin(x))\, y' = -\cos(x)^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{\cos(x) y}{1+\sin(x)} + \frac{\cos(x)^2}{1+\sin(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{\cos(x) y}{1+\sin(x)} = \frac{\cos(x)^2}{1+\sin(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + \frac{\cos(x) y}{1+\sin(x)}\right) = \frac{\mu(x)\cos(x)^2}{1+\sin(x)}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x)\left(y' + \frac{\cos(x) y}{1+\sin(x)}\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{\mu(x)\cos(x)}{1+\sin(x)}$$

- Solve to find the integrating factor

$$\mu(x) = 1 + \sin(x)$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)\cos(x)^2}{1+\sin(x)} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \frac{\mu(x)\cos(x)^2}{1+\sin(x)} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x)\cos(x)^2}{1+\sin(x)} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = 1 + \sin(x)$

$$y = \frac{\int \cos(x)^2 dx + c_1}{1+\sin(x)}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{\cos(x)\sin(x)}{2} + \frac{x}{2} + c_1}{1+\sin(x)}$$

- Simplify

$$y = \frac{\cos(x)\sin(x) + 2c_1 + x}{2 + 2\sin(x)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve((cos(x)^2-y(x)*cos(x))-(1+sin(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sin(x)\cos(x) + 2c_1 + x}{2 + 2\sin(x)}$$

✓ Solution by Mathematica

Time used: 0.314 (sec). Leaf size: 25

```
DSolve[(Cos[x]^2-y[x]*Cos[x])-(1+Sin[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x + \sin(x)\cos(x) + 2c_1}{2\sin(x) + 2}$$

5.14 problem 14

Internal problem ID [11651]
Internal file name [`OUTPUT/11660_Tuesday_April_09_2024_02_05_05_AM_97015726/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y \sin(2x) + \left(1 + \sin(x)^2\right) y' = \cos(x)}$$

5.14.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2\sin(2x)}{-3 + \cos(2x)}$$
$$q(x) = -\frac{\cos(x)}{\cos(x)^2 - 2}$$

Hence the ode is

$$y' - \frac{2\sin(2x)\, y}{-3 + \cos(2x)} = -\frac{\cos(x)}{\cos(x)^2 - 2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2\sin(2x)}{-3+\cos(2x)}dx}$$
$$= -3 + \cos(2x)$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(-\frac{\cos(x)}{\cos(x)^2 - 2}\right)$$
$$\frac{d}{dx}((-3 + \cos(2x))\, y) = (-3 + \cos(2x))\left(-\frac{\cos(x)}{\cos(x)^2 - 2}\right)$$
$$d((-3 + \cos(2x))\, y) = (-2\cos(x))\, dx$$

Integrating gives

$$(-3 + \cos(2x))\, y = \int -2\cos(x)\, dx$$
$$(-3 + \cos(2x))\, y = -2\sin(x) + c_1$$

Dividing both sides by the integrating factor $\mu = -3 + \cos(2x)$ results in

$$y = -\frac{2\sin(x)}{-3 + \cos(2x)} + \frac{c_1}{-3 + \cos(2x)}$$

which simplifies to

$$y = \frac{-2\sin(x) + c_1}{2\cos(x)^2 - 4}$$

Summary

The solution(s) found are the following

$$y = \frac{-2\sin(x) + c_1}{2\cos(x)^2 - 4} \tag{1}$$

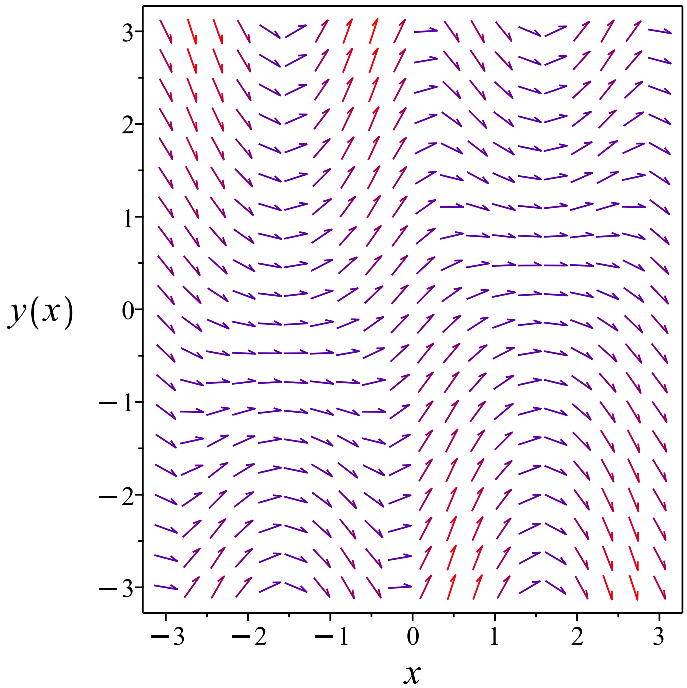


Figure 185: Slope field plot

Verification of solutions

$$y = \frac{-2\sin(x) + c_1}{2\cos(x)^2 - 4}$$

Verified OK.

5.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-y\sin(2x) + \cos(x)}{1 + \sin(x)^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 148: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{-3+\cos(2x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{-3+\cos(2x)}} dy \end{aligned}$$

Which results in

$$S = (-3 + \cos(2x))\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-y \sin(2x) + \cos(x)}{1 + \sin(x)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -2y \sin(2x) \\ S_y &= -3 + \cos(2x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -2\cos(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -2\cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -2\sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(-3 + \cos(2x))\, y = -2\sin(x) + c_1$$

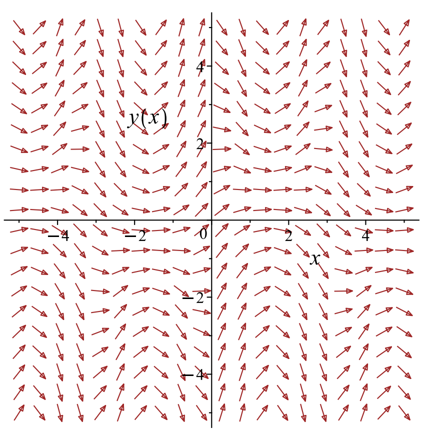
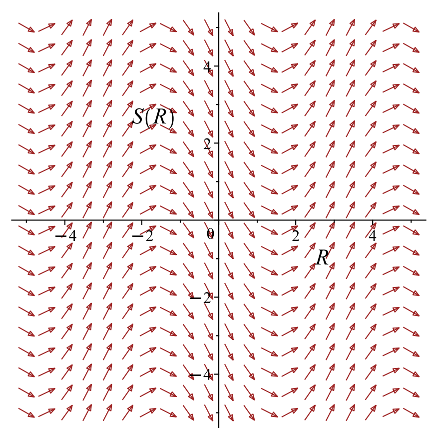
Which simplifies to

$$(-3 + \cos(2x))\, y = -2\sin(x) + c_1$$

Which gives

$$y = -\frac{2\sin(x) - c_1}{-3 + \cos(2x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y\sin(2x)+\cos(x)}{1+\sin(x)^2}$ 	$R = x$ $S = (-3 + \cos(2x))\, y$	$\frac{dS}{dR} = -2\cos(R)$ 

Summary

The solution(s) found are the following

$$y = -\frac{2\sin(x) - c_1}{-3 + \cos(2x)} \tag{1}$$

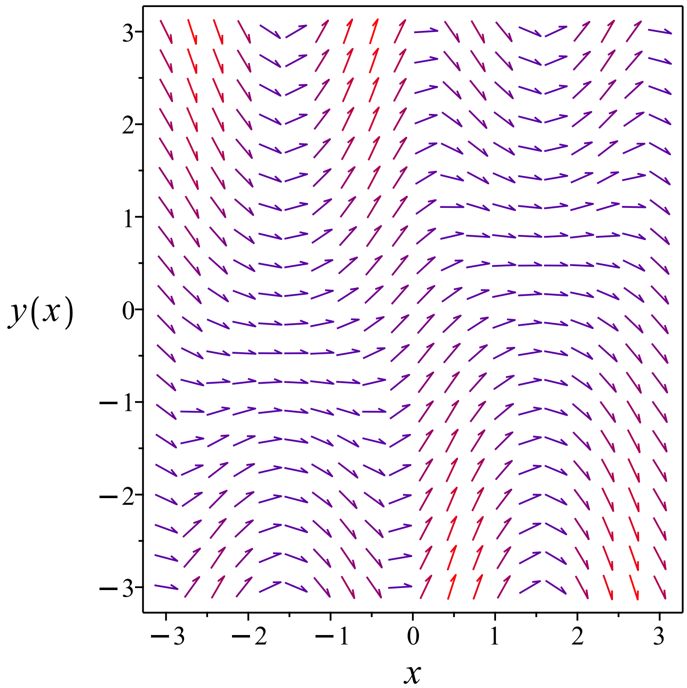


Figure 186: Slope field plot

Verification of solutions

$$y = -\frac{2\sin(x) - c_1}{-3 + \cos(2x)}$$

Verified OK.

5.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(1 + \sin(x)^2\right) \mathrm{d}y = (-y \sin(2x) + \cos(x)) \, \mathrm{d}x$$
$$(y \sin(2x) - \cos(x)) \, \mathrm{d}x + \left(1 + \sin(x)^2\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y \sin(2x) - \cos(x)$$
$$N(x, y) = 1 + \sin(x)^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y \sin(2x) - \cos(x))$$
$$= \sin(2x)$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(1 + \sin(x)^2\right)$$
$$= \sin(2x)$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$
$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int y\sin(2x)-\cos(x)\,\mathrm{d}x$$
$$\phi=-y\cos(x)^2-\sin(x)+\frac{y}{2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=-\cos(x)^2+\frac{1}{2}+f'(y) \tag{4}$$
$$=-\frac{\cos(2x)}{2}+f'(y)$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=1+\sin(x)^2$. Therefore equation (4) becomes

$$1+\sin(x)^2=-\frac{\cos(2x)}{2}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=\frac{\cos(2x)}{2}+1+\sin(x)^2$$
$$=\frac{3}{2}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y=\int\left(\frac{3}{2}\right)\mathrm{d}y$$
$$f(y)=\frac{3y}{2}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -y\cos(x)^2 - \sin(x) + 2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -y\cos(x)^2 - \sin(x) + 2y$$

The solution becomes

$$y = -\frac{\sin(x) + c_1}{\cos(x)^2 - 2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sin(x) + c_1}{\cos(x)^2 - 2} \tag{1}$$

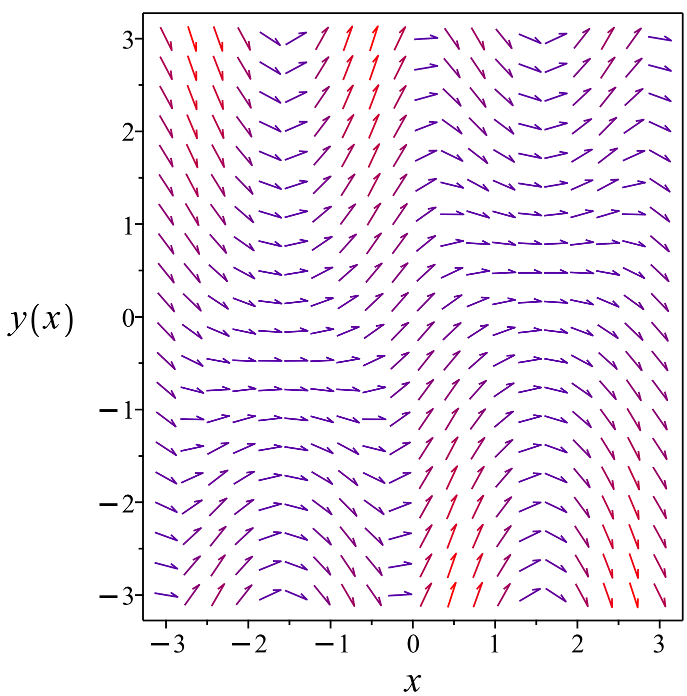


Figure 187: Slope field plot

Verification of solutions

$$y = -\frac{\sin(x) + c_1}{\cos(x)^2 - 2}$$

Verified OK.

5.14.4 Maple step by step solution

Let's solve

$y\sin(2x) + (1 + \sin(x)^2)\, y' = \cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{\sin(2x)y}{1+\sin(x)^2} + \frac{\cos(x)}{1+\sin(x)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{\sin(2x)y}{1+\sin(x)^2} = \frac{\cos(x)}{1+\sin(x)^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{\sin(2x)y}{1+\sin(x)^2}\right) = \frac{\mu(x)\cos(x)}{1+\sin(x)^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{\sin(2x)y}{1+\sin(x)^2}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)\sin(2x)}{1+\sin(x)^2}$

- Solve to find the integrating factor

  $\mu(x) = \cos(x)^2 - 2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)\cos(x)}{1+\sin(x)^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)\cos(x)}{1+\sin(x)^2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\cos(x)}{1+\sin(x)^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \cos(x)^2 - 2$

  $$y = \frac{\int \frac{(\cos(x)^2-2)\cos(x)}{1+\sin(x)^2}dx + c_1}{\cos(x)^2 - 2}$$

- Evaluate the integrals on the rhs

  $$y = \frac{-\sin(x) + c_1}{\cos(x)^2 - 2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve((y(x)*sin(2*x)-cos(x))+(1+sin(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-\sin(x) - c_1}{\cos(x)^2 - 2}$$

✓ Solution by Mathematica

Time used: 0.173 (sec). Leaf size: 21

```
DSolve[(y[x]*Sin[2*x]-Cos[x])+(1+Sin[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-2\sin(x) + c_1}{\cos(2x) - 3}$$

5.15 problem 15

Internal problem ID [11652]
Internal file name [`OUTPUT/11661_Tuesday_April_09_2024_02_05_08_AM_34006736/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "bernoulli", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \frac{y}{x} + \frac{y^2}{x} = 0}$$

5.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y(y-1)}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(y) = y(y-1)$. Integrating both sides gives

$$\frac{1}{y(y-1)}\,dy = -\frac{1}{x}\,dx$$
$$\int \frac{1}{y(y-1)}\,dy = \int -\frac{1}{x}\,dx$$
$$\ln(y-1) - \ln(y) = -\ln(x) + c_1$$

Raising both side to exponential gives

$$e^{\ln(y-1)-\ln(y)} = e^{-\ln(x)+c_1}$$

Which simplifies to

$$\frac{y-1}{y} = \frac{c_2}{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{-x + c_2} \tag{1}$$

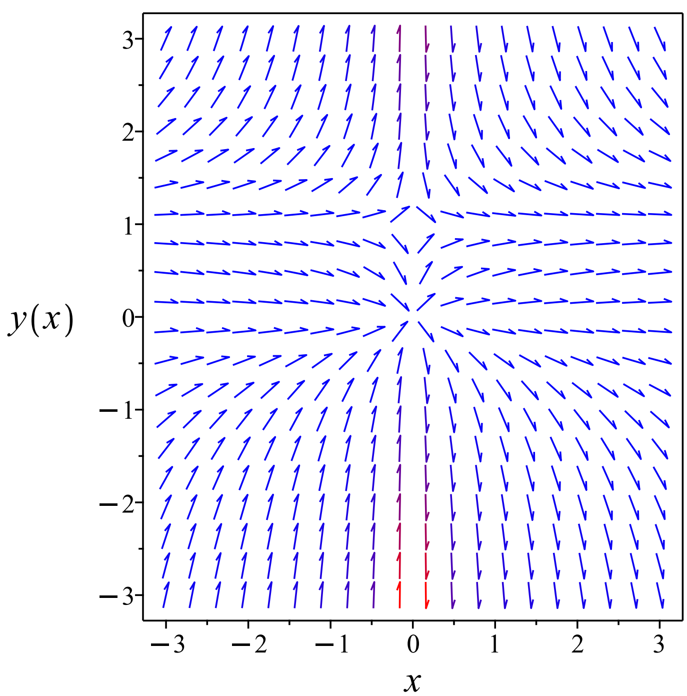


Figure 188: Slope field plot

Verification of solutions

$$y = -\frac{x}{-x + c_2}$$

Verified OK.

5.15.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x)^2\, x = 0$$

Integrating both sides gives

$$\int -\frac{1}{u^2} du = x + c_2$$
$$\frac{1}{u} = x + c_2$$

Solving for u gives these solutions

$$u_1 = \frac{1}{x + c_2}$$

Therefore the solution y is

$$y = xu$$
$$= \frac{x}{x + c_2}$$

Summary
The solution(s) found are the following

$$y = \frac{x}{x + c_2} \tag{1}$$

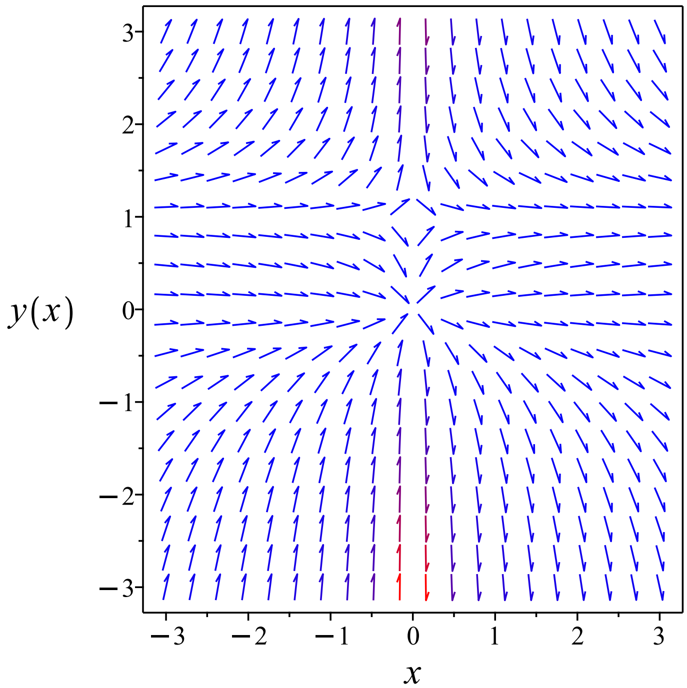


Figure 189: Slope field plot

Verification of solutions

$$y = \frac{x}{x + c_2}$$

Verified OK.

5.15.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(y-1)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 151: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -x \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-x} dx \end{aligned}$$

Which results in

$$S = -\ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(y-1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -\frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y(y-1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R(R-1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R-1) - \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(x) = \ln(y-1) - \ln(y) + c_1$$

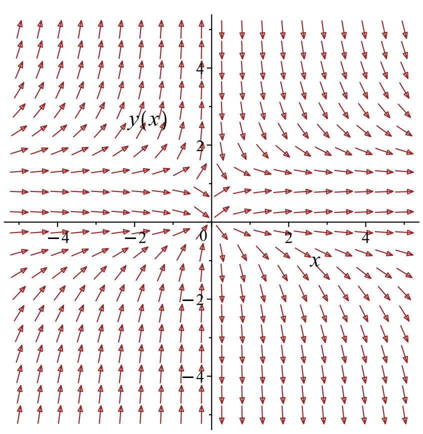
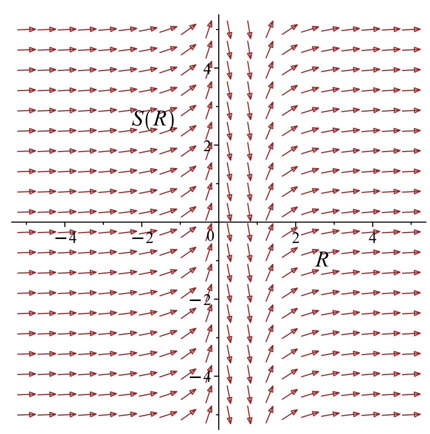
Which simplifies to

$$-\ln(x) = \ln(y-1) - \ln(y) + c_1$$

Which gives

$$y = \frac{x\,e^{c_1}}{e^{c_1}x - 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(y-1)}{x}$ 	$R = y$ $S = -\ln(x)$	$\frac{dS}{dR} = \frac{1}{R(R-1)}$ 

Summary

The solution(s) found are the following

$$y = \frac{x\,e^{c_1}}{e^{c_1}x - 1} \tag{1}$$

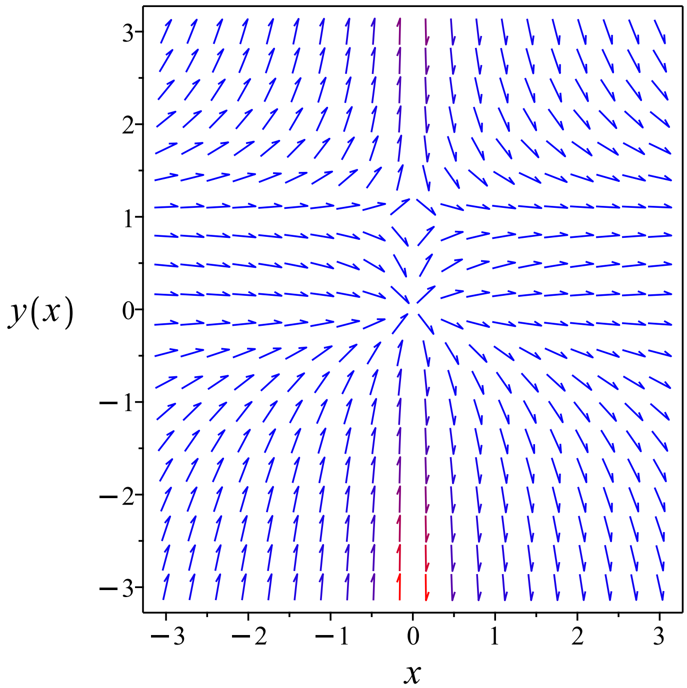


Figure 190: Slope field plot

Verification of solutions

$$y = \frac{x\,e^{c_1}}{e^{c_1}x - 1}$$

Verified OK.

5.15.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(y-1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y - \frac{1}{x}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{x} \\
f_1(x) &= -\frac{1}{x} \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y' \frac{1}{y^2} = \frac{1}{xy} - \frac{1}{x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{y^2} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= \frac{w(x)}{x} - \frac{1}{x} \\
w' &= -\frac{w}{x} + \frac{1}{x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{1}{x}
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = \frac{1}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{1}{x}\right) \\ \frac{d}{dx}(xw) &= (x)\left(\frac{1}{x}\right) \\ d(xw) &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xw &= \int dx \\ xw &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = 1 + \frac{c_1}{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = 1 + \frac{c_1}{x}$$

Or

$$y = \frac{1}{1 + \frac{c_1}{x}}$$

Which is simplified to

$$y = \frac{x}{x + c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{x + c_1} \tag{1}$$

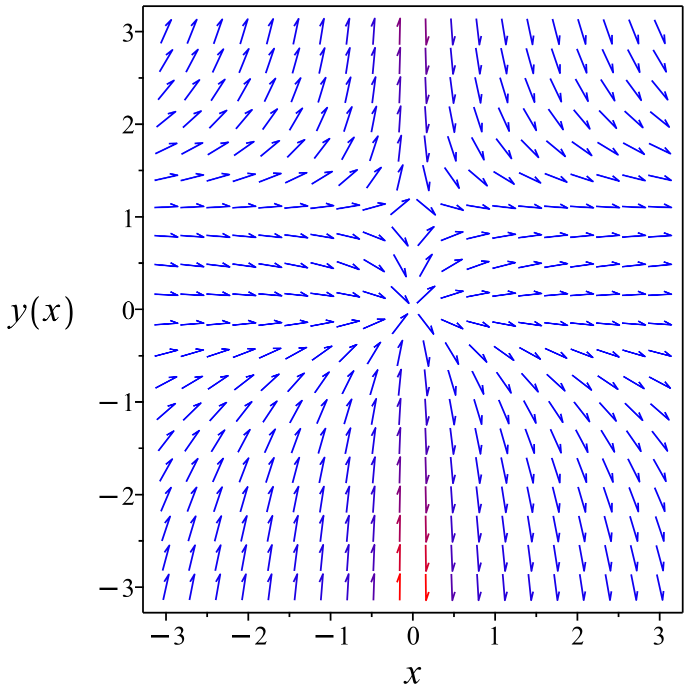


Figure 191: Slope field plot

Verification of solutions

$$y = \frac{x}{x + c_1}$$

Verified OK.

5.15.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y\,(y-1)}\right)\mathrm{d}y = \left(\frac{1}{x}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x}\right)\mathrm{d}x + \left(-\frac{1}{y\,(y-1)}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x}$$
$$N(x, y) = -\frac{1}{y\,(y-1)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{y\left(y-1\right)}\right) \\ = 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int M\, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int -\frac{1}{x}\, \mathrm{d}x$$

$$\phi = -\ln\left(x\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y(y-1)}$. Therefore equation (4) becomes

$$-\frac{1}{y\left(y-1\right)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y\left(y-1\right)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(-\frac{1}{y\left(y-1\right)}\right) \mathrm{d}y$$

$$f(y) = -\ln\left(y-1\right) + \ln\left(y\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \ln(y-1) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \ln(y-1) + \ln(y)$$

The solution becomes

$$y = \frac{x\,e^{c_1}}{e^{c_1}x - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{x\,e^{c_1}}{e^{c_1}x - 1} \tag{1}$$

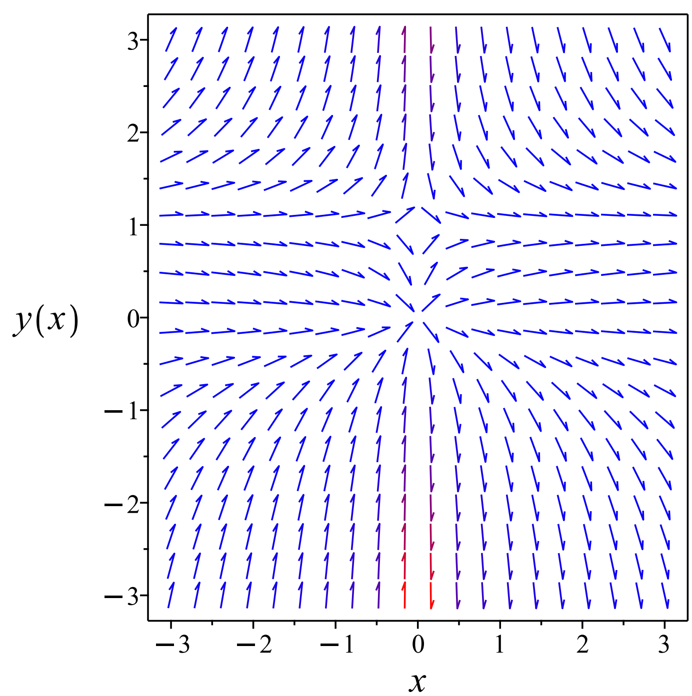


Figure 192: Slope field plot

Verification of solutions

$$y = \frac{x\,e^{c_1}}{e^{c_1}x - 1}$$

Verified OK.

5.15.6 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(y-1)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x} - \frac{y^2}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = -\frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{1}{x^2} \\ f_1 f_2 &= -\frac{1}{x^2} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 x + c_2$$

The above shows that

$$u'(x) = c_1$$

Using the above in (1) gives the solution

$$y = \frac{c_1 x}{c_1 x + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 x}{c_3 x + 1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_3 x}{c_3 x + 1} \tag{1}$$

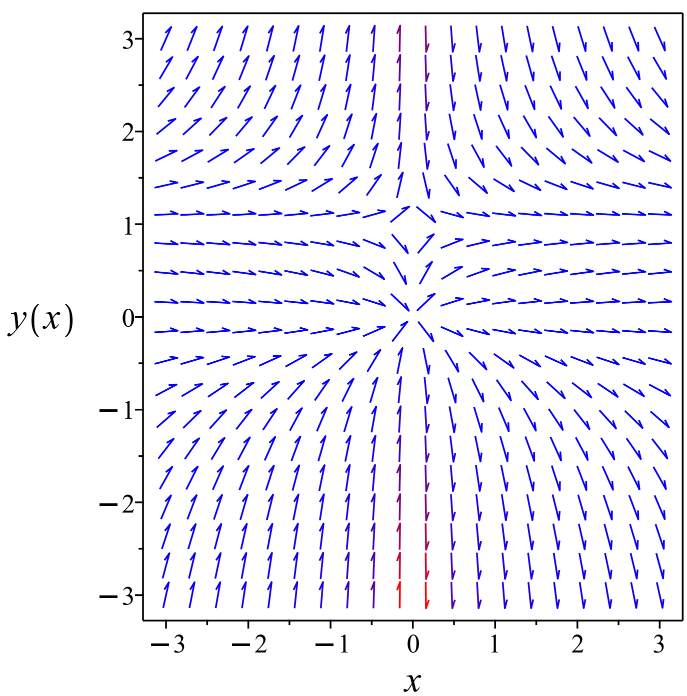


Figure 193: Slope field plot

Verification of solutions

$$y = \frac{c_3 x}{c_3 x + 1}$$

Verified OK.

5.15.7 Maple step by step solution

Let's solve

$y' - \frac{y}{x} + \frac{y^2}{x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(y-1)} = -\frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(y-1)} dx = \int -\frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y-1) - \ln(y) = -\ln(x) + c_1$

- Solve for y

  $y = -\frac{x}{e^{c_1} - x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x)-y(x)/x=-y(x)^2/x,y(x), singsol=all)
```

$$y(x) = \frac{x}{c_1 + x}$$

✓ Solution by Mathematica

Time used: 0.238 (sec). Leaf size: 25

```
DSolve[y'[x]-y[x]/x==-y[x]^2/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{x + e^{c_1}}$$
$$y(x) \to 0$$
$$y(x) \to 1$$

5.16 problem 16

Internal problem ID [11653]
Internal file name [`OUTPUT/11662_Tuesday_April_09_2024_02_05_09_AM_12694163/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y'x + y + 2x^6y^4 = 0}$$

5.16.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(2x^6y^3 + 1)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 154: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^4 x^3
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{y^4 x^3} dy
\end{aligned}$$

Which results in

$$S = -\frac{1}{3x^3 y^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(2x^6 y^3 + 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{x^4 y^3} \\
S_y &= \frac{1}{x^3 y^4}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -2x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -2R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{2R^3}{3} + c_1 \tag{4}$$

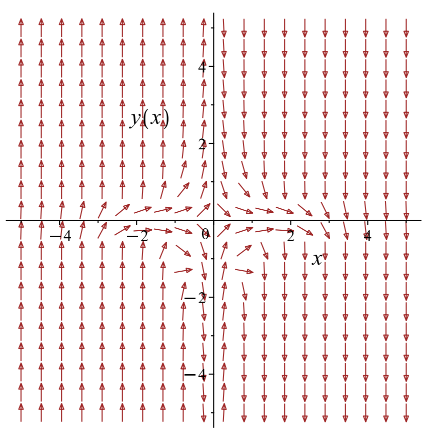
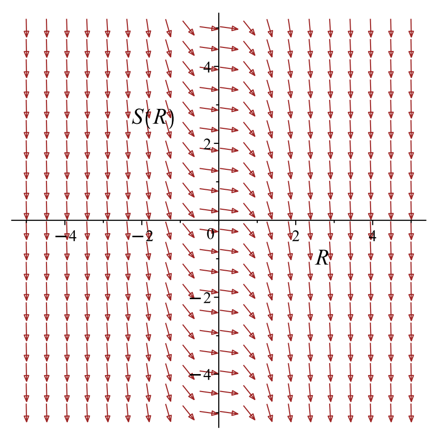
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{3x^3y^3} = -\frac{2x^3}{3} + c_1$$

Which simplifies to

$$-\frac{1}{3x^3y^3} = -\frac{2x^3}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(2x^6y^3+1)}{x}$ 	$R = x$ $S = -\frac{1}{3x^3y^3}$	$\frac{dS}{dR} = -2R^2$ 

Summary

The solution(s) found are the following

$$-\frac{1}{3x^3y^3} = -\frac{2x^3}{3} + c_1 \tag{1}$$

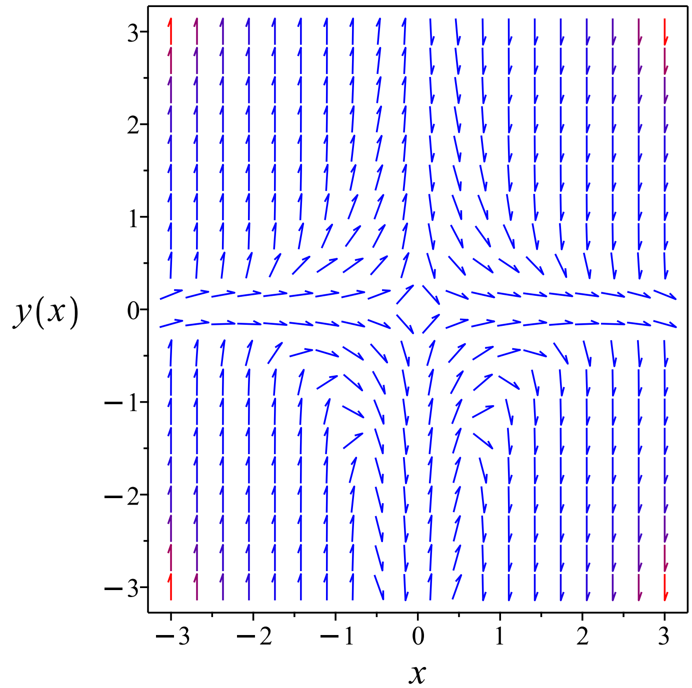


Figure 194: Slope field plot

Verification of solutions

$$-\frac{1}{3x^3y^3} = -\frac{2x^3}{3} + c_1$$

Verified OK.

5.16.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(2x^6y^3 + 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - 2x^5y^4 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= -2x^5 \\
n &= 4
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^4$ gives

$$
y' \frac{1}{y^4} = -\frac{1}{x\, y^3} - 2x^5 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^3}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{3}{y^4} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{3} &= -\frac{w(x)}{x} - 2x^5 \\
w' &= \frac{3w}{x} + 6x^5
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= 6x^5
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{3w(x)}{x} = 6x^5$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{3}{x}dx} \\ &= \frac{1}{x^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(6x^5\right) \\ \frac{d}{dx}\left(\frac{w}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(6x^5\right) \\ d\left(\frac{w}{x^3}\right) &= \left(6x^2\right)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^3} &= \int 6x^2\, dx \\ \frac{w}{x^3} &= 2x^3 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$w(x) = 2x^6 + c_1 x^3$$

Replacing w in the above by $\frac{1}{y^3}$ using equation (5) gives the final solution.

$$\frac{1}{y^3} = 2x^6 + c_1 x^3$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{1}{\left(2x^3 + c_1\right)^{\frac{1}{3}} x} \\ y(x) &= \frac{i\sqrt{3} - 1}{2\left(2x^3 + c_1\right)^{\frac{1}{3}} x} \\ y(x) &= -\frac{1 + i\sqrt{3}}{2\left(2x^3 + c_1\right)^{\frac{1}{3}} x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\left(2x^3+c_1\right)^{\frac{1}{3}} x} \tag{1}$$

$$y = \frac{i\sqrt{3}-1}{2\left(2x^3+c_1\right)^{\frac{1}{3}} x} \tag{2}$$

$$y = -\frac{1+i\sqrt{3}}{2\left(2x^3+c_1\right)^{\frac{1}{3}} x} \tag{3}$$

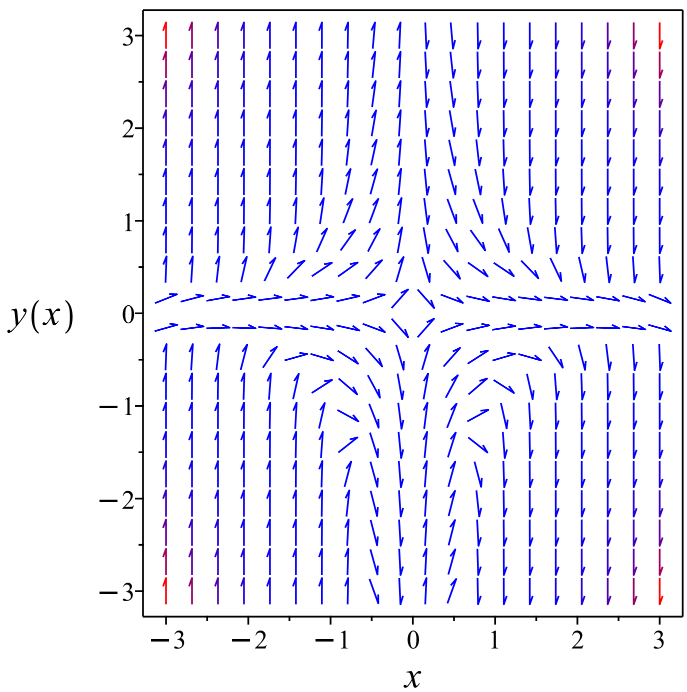


Figure 195: Slope field plot

Verification of solutions

$$y = \frac{1}{\left(2x^3 + c_1\right)^{\frac{1}{3}} x}$$

Verified OK.

$$y = \frac{i\sqrt{3} - 1}{2\left(2x^3 + c_1\right)^{\frac{1}{3}} x}$$

Verified OK.

$$y = -\frac{1 + i\sqrt{3}}{2\left(2x^3 + c_1\right)^{\frac{1}{3}} x}$$

Verified OK.

5.16.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x)\,\mathrm{d}y &= \left(-2x^6y^4 - y\right)\mathrm{d}x \\
\left(2x^6y^4 + y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= 2x^6y^4 + y \\
N(x,y) &= x
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x^6y^4 + y\right) \\
&= 8x^6y^3 + 1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}\left(\left(8x^6y^3 + 1\right) - (1)\right) \\
&= 8x^5y^3
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{2x^6y^4 + y}\left((1) - \left(8x^6y^3 + 1\right)\right) \\
&= -\frac{8x^6y^2}{2x^6y^3 + 1}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - (8x^6y^3 + 1)}{x\left(2x^6y^4 + y\right) - y\left(x\right)} \\
&= -\frac{4}{xy}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{4}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{4}{t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-4\ln(t)} \\
&= \frac{1}{t^4}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{x^4y^4}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^4 y^4}\left(2x^6 y^4 + y\right) \\
&= \frac{2x^6 y^3 + 1}{x^4 y^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^4 y^4}(x) \\
&= \frac{1}{x^3 y^4}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2x^6 y^3 + 1}{x^4 y^3}\right) + \left(\frac{1}{x^3 y^4}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2x^6 y^3 + 1}{x^4 y^3}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{2x^6 y^3 - 1}{3x^3 y^3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= \frac{2x^3}{y} - \frac{2x^6y^3 - 1}{x^3y^4} + f'(y) \\ &= \frac{1}{x^3y^4} + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^3y^4}$. Therefore equation (4) becomes

$$\frac{1}{x^3y^4} = \frac{1}{x^3y^4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{2x^6y^3 - 1}{3x^3y^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{2x^6y^3 - 1}{3x^3y^3}$$

Summary

The solution(s) found are the following

$$\frac{2y^3x^6 - 1}{3x^3y^3} = c_1 \tag{1}$$

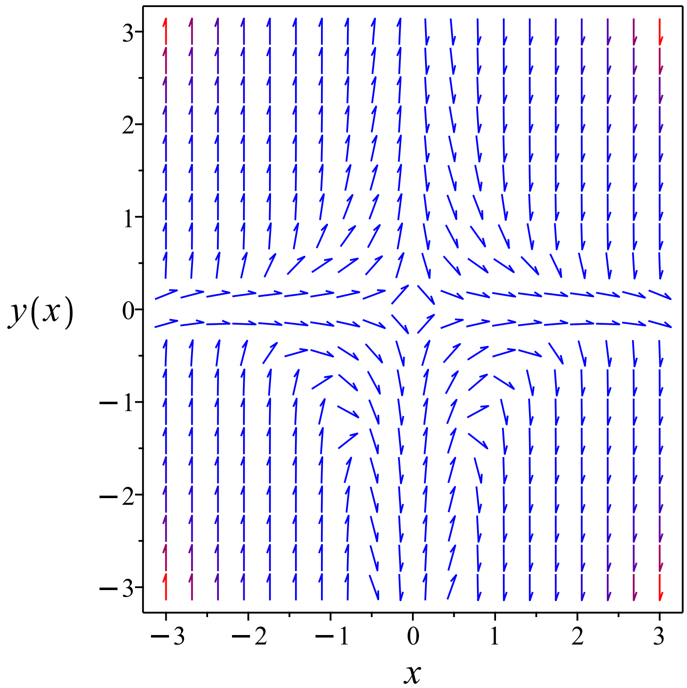


Figure 196: Slope field plot

Verification of solutions

$$\frac{2y^3x^6 - 1}{3x^3y^3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 67

```
dsolve(x*diff(y(x),x)+y(x)=-2*x^6*y(x)^4,y(x), singsol=all)
```

$$y(x) = \frac{1}{(2x^3 + c_1)^{\frac{1}{3}} x}$$
$$y(x) = -\frac{1 + i\sqrt{3}}{2 (2x^3 + c_1)^{\frac{1}{3}} x}$$
$$y(x) = \frac{i\sqrt{3} - 1}{2 (2x^3 + c_1)^{\frac{1}{3}} x}$$

✓ Solution by Mathematica

Time used: 0.87 (sec). Leaf size: 79

```
DSolve[x*y'[x]+y[x]==-2*x^6*y[x]^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{\sqrt[3]{x^3 (2x^3 + c_1)}}$$
$$y(x) \to -\frac{\sqrt[3]{-1}}{\sqrt[3]{x^3 (2x^3 + c_1)}}$$
$$y(x) \to \frac{(-1)^{2/3}}{\sqrt[3]{x^3 (2x^3 + c_1)}}$$
$$y(x) \to 0$$

5.17 problem 17

Internal problem ID [11654]
Internal file name [`OUTPUT/11663_Tuesday_April_09_2024_02_05_10_AM_56841430/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + \left(4y - \frac{8}{y^3}\right)x = 0}$$

5.17.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{4x(y^4-2)}{y^3} \end{aligned}$$

Where $f(x) = -4x$ and $g(y) = \frac{y^4-2}{y^3}$. Integrating both sides gives

$$\frac{1}{\frac{y^4-2}{y^3}}\,dy = -4x\,dx$$

$$\int \frac{1}{\frac{y^4-2}{y^3}}\, dy = \int -4x\, dx$$

$$\frac{\ln\left(y^4-2\right)}{4} = -2x^2 + c_1$$

Raising both side to exponential gives

$$\left(y^4-2\right)^{\frac{1}{4}} = \mathrm{e}^{-2x^2+c_1}$$

Which simplifies to

$$\left(y^4-2\right)^{\frac{1}{4}} = c_2 \mathrm{e}^{-2x^2}$$

The solution is

$$\left(y^4-2\right)^{\frac{1}{4}} = c_2 \mathrm{e}^{-2x^2+c_1}$$

Summary
The solution(s) found are the following

$$\left(y^4-2\right)^{\frac{1}{4}} = c_2 \mathrm{e}^{-2x^2+c_1} \tag{1}$$

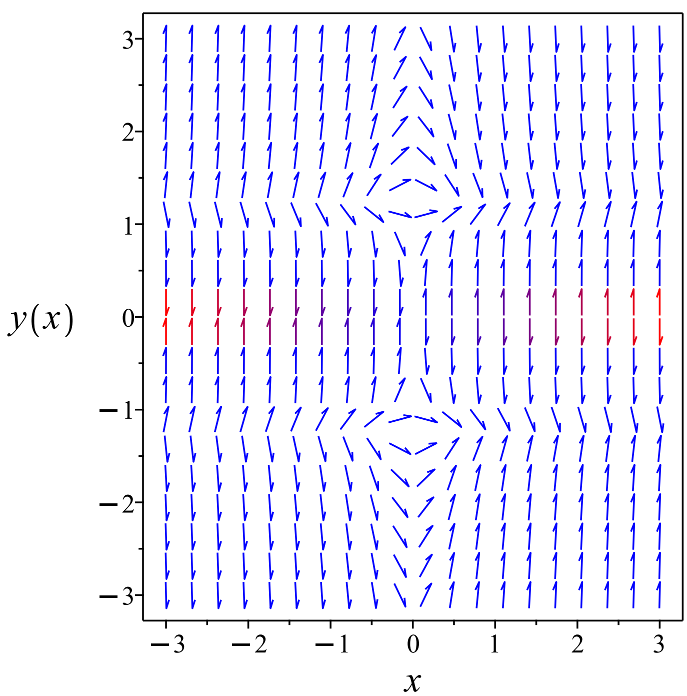


Figure 197: Slope field plot

Verification of solutions

$$\left(y^4-2\right)^{\frac{1}{4}} = c_2 \mathrm{e}^{-2x^2+c_1}$$

Verified OK.

5.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{4x(y^4-2)}{y^3} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 156: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{1}{4x} \\
\eta(x,y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{4x}} dx \end{aligned}$$

Which results in

$$S = -2x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{4x(y^4 - 2)}{y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -4x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y^3}{y^4 - 2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^3}{R^4 - 2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln (R^4 - 2)}{4} + c_1 \tag{4}$$

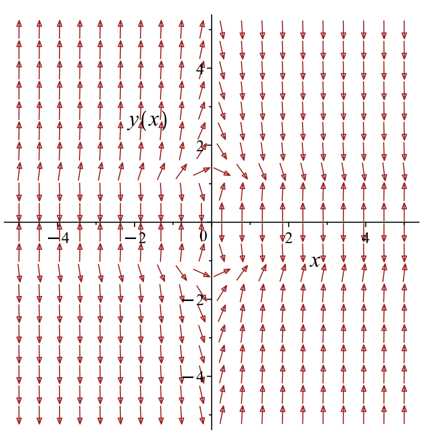
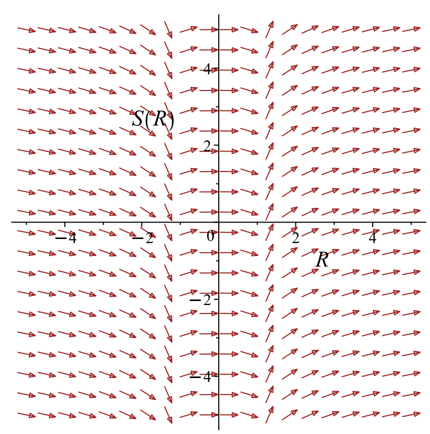
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-2x^2 = \frac{\ln (y^4 - 2)}{4} + c_1$$

Which simplifies to

$$-2x^2 = \frac{\ln (y^4 - 2)}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4x(y^4-2)}{y^3}$ 	$R = y$ $S = -2x^2$	$\frac{dS}{dR} = \frac{R^3}{R^4-2}$ 

Summary

The solution(s) found are the following

$$-2x^2 = \frac{\ln (y^4 - 2)}{4} + c_1 \tag{1}$$

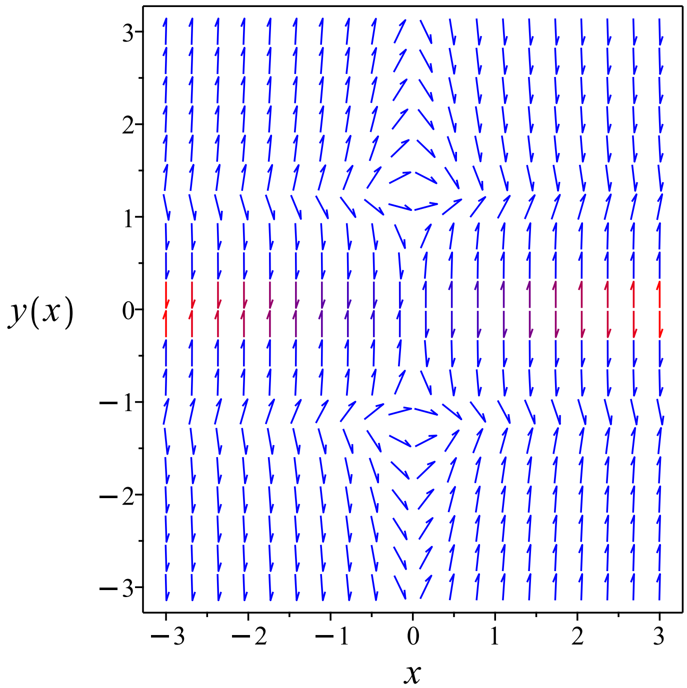


Figure 198: Slope field plot

Verification of solutions

$$-2x^2 = \frac{\ln(y^4 - 2)}{4} + c_1$$

Verified OK.

5.17.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{4x(y^4 - 2)}{y^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -4xy + 8x\frac{1}{y^3} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -4x \\ f_1(x) &= 8x \\ n &= -3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^3}$ gives

$$y'y^3 = -4x\, y^4 + 8x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^4 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 4y^3y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{4} &= -4w(x)\, x + 8x \\ w' &= -16xw + 32x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 16x \\ q(x) &= 32x \end{aligned}$$

Hence the ode is

$$w'(x) + 16w(x)\, x = 32x$$

The integrating factor μ is

$$\mu = e^{\int 16x dx}$$
$$= e^{8x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)(32x)$$
$$\frac{d}{dx}\left(e^{8x^2} w\right) = \left(e^{8x^2}\right)(32x)$$
$$d\left(e^{8x^2} w\right) = \left(32x\, e^{8x^2}\right) dx$$

Integrating gives

$$e^{8x^2} w = \int 32x\, e^{8x^2}\, dx$$
$$e^{8x^2} w = 2\, e^{8x^2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{8x^2}$ results in

$$w(x) = 2\, e^{-8x^2} e^{8x^2} + c_1 e^{-8x^2}$$

which simplifies to

$$w(x) = 2 + c_1 e^{-8x^2}$$

Replacing w in the above by y^4 using equation (5) gives the final solution.

$$y^4 = 2 + c_1 e^{-8x^2}$$

Solving for y gives

$$y(x) = \left(2\, e^{8x^2} + c_1\right)^{\frac{1}{4}} e^{-2x^2}$$
$$y(x) = i\left(2\, e^{8x^2} + c_1\right)^{\frac{1}{4}} e^{-2x^2}$$
$$y(x) = -\left(2\, e^{8x^2} + c_1\right)^{\frac{1}{4}} e^{-2x^2}$$
$$y(x) = -i\left(2\, e^{8x^2} + c_1\right)^{\frac{1}{4}} e^{-2x^2}$$

Summary

The solution(s) found are the following

$$y = \left(2\,\mathrm{e}^{8x^2}+c_1\right)^{\frac{1}{4}} \mathrm{e}^{-2x^2} \tag{1}$$

$$y = i\left(2\,\mathrm{e}^{8x^2}+c_1\right)^{\frac{1}{4}} \mathrm{e}^{-2x^2} \tag{2}$$

$$y = -\left(2\,\mathrm{e}^{8x^2}+c_1\right)^{\frac{1}{4}} \mathrm{e}^{-2x^2} \tag{3}$$

$$y = -i\left(2\,\mathrm{e}^{8x^2}+c_1\right)^{\frac{1}{4}} \mathrm{e}^{-2x^2} \tag{4}$$

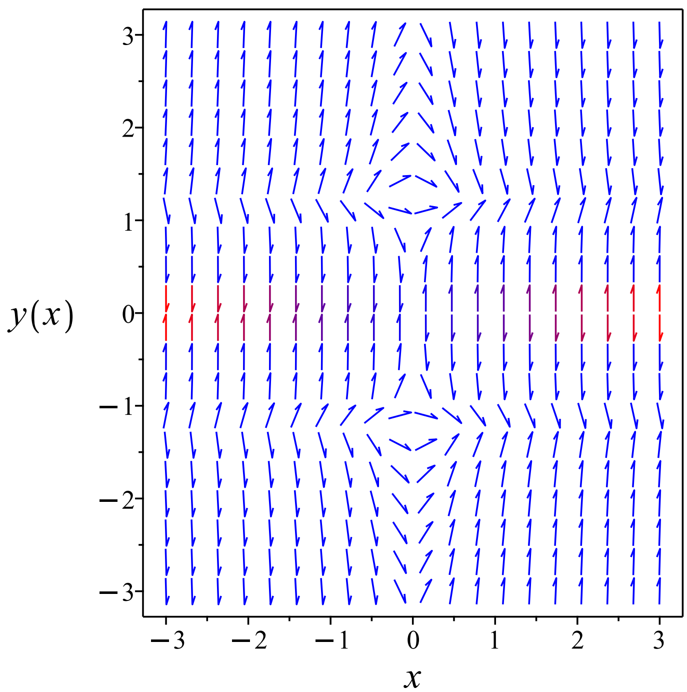


Figure 199: Slope field plot

Verification of solutions

$$y = \left(2\,\mathrm{e}^{8x^2} + c_1\right)^{\frac{1}{4}} \mathrm{e}^{-2x^2}$$

Verified OK.

$$y = i\left(2\,\mathrm{e}^{8x^2} + c_1\right)^{\frac{1}{4}} \mathrm{e}^{-2x^2}$$

Verified OK.

$$y = -\left(2\,\mathrm{e}^{8x^2} + c_1\right)^{\frac{1}{4}} \mathrm{e}^{-2x^2}$$

Verified OK.

$$y = -i\left(2\,\mathrm{e}^{8x^2} + c_1\right)^{\frac{1}{4}} \mathrm{e}^{-2x^2}$$

Verified OK.

5.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{y^3}{4\,(y^4-2)}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\(-x)\,\mathrm{d}x+\left(-\frac{y^3}{4\,(y^4-2)}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\\N(x,y)&=-\frac{y^3}{4\,(y^4-2)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-x)\\&=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(-\frac{y^3}{4\,(y^4-2)}\right)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -x \mathrm{d}x \\
\phi &= -\frac{x^2}{2} + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y^3}{4(y^4-2)}$. Therefore equation (4) becomes

$$-\frac{y^3}{4(y^4-2)} = 0 + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{y^3}{4(y^4-2)}$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(-\frac{y^3}{4y^4-8}\right) \mathrm{d}y \\
f(y) &= -\frac{\ln(y^4-2)}{16} + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{\ln(y^4-2)}{16} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{\ln(y^4 - 2)}{16}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} - \frac{\ln(y^4 - 2)}{16} = c_1 \tag{1}$$

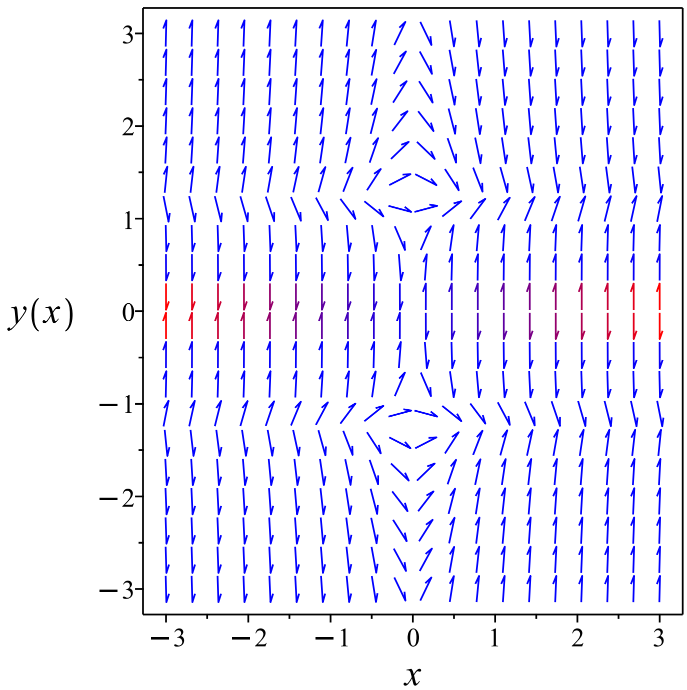


Figure 200: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} - \frac{\ln(y^4 - 2)}{16} = c_1$$

Verified OK.

5.17.5 Maple step by step solution

Let's solve

$y' + \left(4y - \frac{8}{y^3}\right)x = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{4y-\frac{8}{y^3}} = -x$

- Integrate both sides with respect to x

  $\int \frac{y'}{4y-\frac{8}{y^3}}dx = \int -xdx + c_1$

- Evaluate integral

  $\frac{\ln(y^4-2)}{16} = -\frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = \frac{\left((\mathrm{e}^{c_1})^{16}+2\left(\mathrm{e}^{x^2}\right)^8\right)^{\frac{1}{4}}}{\left(\mathrm{e}^{x^2}\right)^2}, y = -\frac{\left((\mathrm{e}^{c_1})^{16}+2\left(\mathrm{e}^{x^2}\right)^8\right)^{\frac{1}{4}}}{\left(\mathrm{e}^{x^2}\right)^2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 94

```
dsolve(diff(y(x),x)+(4*y(x)-8/y(x)^3)*x=0,y(x), singsol=all)
```

$$y(x) = \left(2\,\mathrm{e}^{8x^2}+c_1\right)^{\frac{1}{4}}\mathrm{e}^{-2x^2}$$
$$y(x) = -\left(2\,\mathrm{e}^{8x^2}+c_1\right)^{\frac{1}{4}}\mathrm{e}^{-2x^2}$$
$$y(x) = -i\left(2\,\mathrm{e}^{8x^2}+c_1\right)^{\frac{1}{4}}\mathrm{e}^{-2x^2}$$
$$y(x) = i\left(2\,\mathrm{e}^{8x^2}+c_1\right)^{\frac{1}{4}}\mathrm{e}^{-2x^2}$$

✓ Solution by Mathematica

Time used: 1.939 (sec). Leaf size: 145

```
DSolve[y'[x]+(4*y[x]-8/y[x]^3)*x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\sqrt[4]{2+e^{-8x^2+4c_1}}$$
$$y(x) \rightarrow -i\sqrt[4]{2+e^{-8x^2+4c_1}}$$
$$y(x) \rightarrow i\sqrt[4]{2+e^{-8x^2+4c_1}}$$
$$y(x) \rightarrow \sqrt[4]{2+e^{-8x^2+4c_1}}$$
$$y(x) \rightarrow -\sqrt[4]{2}$$
$$y(x) \rightarrow -i\sqrt[4]{2}$$
$$y(x) \rightarrow i\sqrt[4]{2}$$
$$y(x) \rightarrow \sqrt[4]{2}$$

5.18 problem 18

Internal problem ID [11655]
Internal file name [`OUTPUT/11664_Tuesday_April_09_2024_02_05_12_AM_38753072/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x' + \frac{(1+t)\,x}{2t} - \frac{1+t}{xt} = 0}$$

5.18.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} x' &= F(t,x) \\ &= f(t)g(x) \\ &= \frac{(1+t)\,(-x^2+2)}{2tx} \end{aligned}$$

Where $f(t) = \frac{1+t}{2t}$ and $g(x) = \frac{-x^2+2}{x}$. Integrating both sides gives

$$\frac{1}{\frac{-x^2+2}{x}}\,dx = \frac{1+t}{2t}\,dt$$

$$\int \frac{1}{\frac{-x^2+2}{x}} dx = \int \frac{1+t}{2t} dt$$

$$-\frac{\ln(x^2-2)}{2} = \frac{t}{2} + \frac{\ln(t)}{2} + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{x^2-2}} = e^{\frac{t}{2}+\frac{\ln(t)}{2}+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{x^2-2}} = c_2 e^{\frac{t}{2}+\frac{\ln(t)}{2}}$$

The solution is

$$\frac{1}{\sqrt{x^2-2}} = c_2 e^{\frac{t}{2}+\frac{\ln(t)}{2}+c_1}$$

Summary
The solution(s) found are the following

$$\frac{1}{\sqrt{x^2-2}} = c_2 e^{\frac{t}{2}+\frac{\ln(t)}{2}+c_1} \tag{1}$$

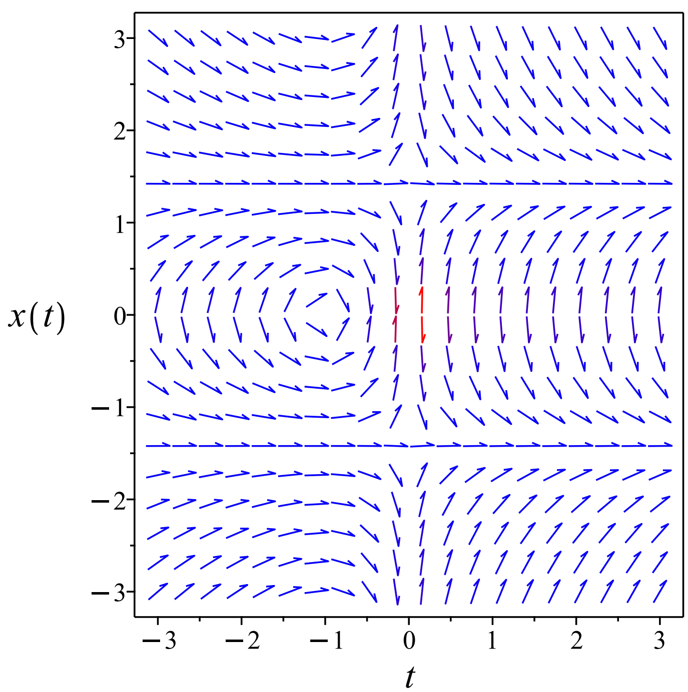


Figure 201: Slope field plot

Verification of solutions

$$\frac{1}{\sqrt{x^2-2}} = c_2 \mathrm{e}^{\frac{t}{2}+\frac{\ln(t)}{2}+c_1}$$

Verified OK.

5.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = -\frac{t\,x^2+x^2-2t-2}{2tx}$$
$$x' = \omega(t,x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 159: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,x) &= \frac{2t}{1+t} \\
\eta(t,x) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{\frac{2t}{1+t}} dt \end{aligned}$$

Which results in

$$S = \frac{t}{2} + \frac{\ln(t)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -\frac{t\,x^2 + x^2 - 2t - 2}{2tx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_x &= 1 \\ S_t &= \frac{1+t}{2t} \\ S_x &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{x}{x^2 - 2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R}{R^2 - 2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R^2 - 2)}{2} + c_1 \tag{4}$$

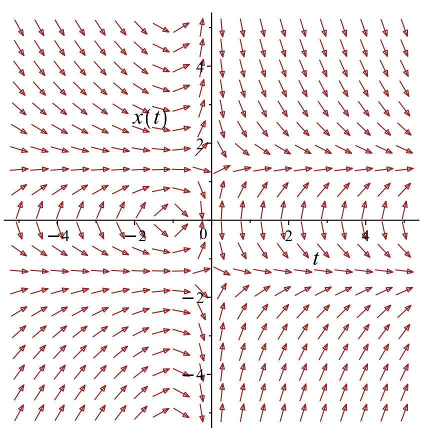
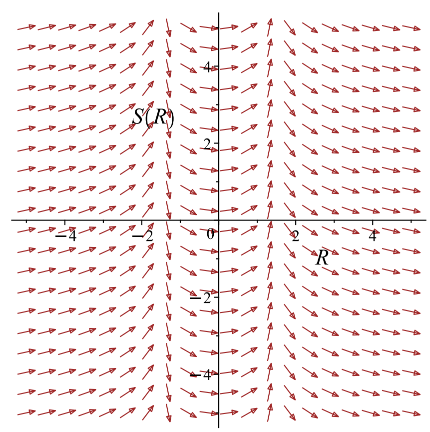
To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\frac{t}{2} + \frac{\ln(t)}{2} = -\frac{\ln(x^2 - 2)}{2} + c_1$$

Which simplifies to

$$\frac{t}{2} + \frac{\ln(t)}{2} = -\frac{\ln(x^2 - 2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{t\,x^2 + x^2 - 2t - 2}{2tx}$ 	$R = x$ $S = \frac{t}{2} + \frac{\ln(t)}{2}$	$\frac{dS}{dR} = -\frac{R}{R^2 - 2}$ 

Summary

The solution(s) found are the following

$$\frac{t}{2} + \frac{\ln(t)}{2} = -\frac{\ln(x^2 - 2)}{2} + c_1 \tag{1}$$

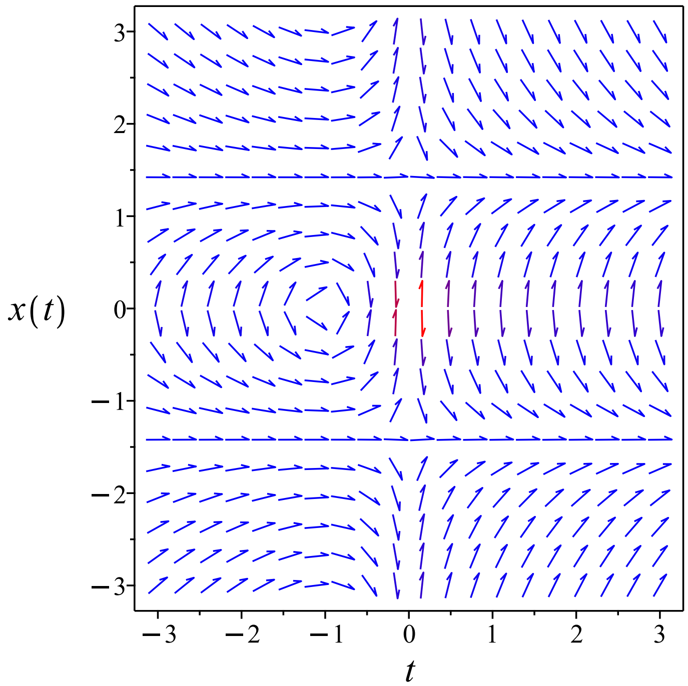


Figure 202: Slope field plot

Verification of solutions

$$\frac{t}{2}+\frac{\ln \left(t \right)}{2} = -\frac{\ln \left(x^{2}-2\right)}{2}+c_{1}$$

Verified OK.

5.18.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} x' &= F(t,x)\\ &= -\frac{t\,x^{2}+x^{2}-2t-2}{2tx} \end{aligned}$$

This is a Bernoulli ODE.

$$x' = -\frac{1+t}{2t}x-\frac{-2t-2}{2t}\frac{1}{x} \tag{1}$$

The standard Bernoulli ODE has the form

$$x' = f_0(t)x+f_1(t)x^n \tag{2}$$

The first step is to divide the above equation by x^n which gives

$$\frac{x'}{x^n} = f_0(t)x^{1-n}+f_1(t) \tag{3}$$

The next step is use the substitution $w = x^{1-n}$ in equation (3) which generates a new ODE in $w(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $x(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(t) &= -\frac{1+t}{2t} \\
f_1(t) &= -\frac{-2t-2}{2t} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $x^n = \frac{1}{x}$ gives

$$
x'x = -\frac{(1+t)\,x^2}{2t} - \frac{-2t-2}{2t} \tag{4}
$$

Let

$$
\begin{aligned}
w &= x^{1-n} \\
&= x^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t t gives

$$
w' = 2xx' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(t)}{2} &= -\frac{(1+t)\,w(t)}{2t} - \frac{-2t-2}{2t} \\
w' &= -\frac{(1+t)\,w}{t} - \frac{-2t-2}{t}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(t)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(t) + p(t)w(t) = q(t)
$$

Where here

$$
\begin{aligned}
p(t) &= -\frac{-1-t}{t} \\
q(t) &= \frac{2t+2}{t}
\end{aligned}
$$

Hence the ode is

$$w'(t) - \frac{(-1-t)\,w(t)}{t} = \frac{2t+2}{t}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{-1-t}{t}dt} \\ &= e^{t+\ln(t)}\end{aligned}$$

Which simplifies to

$$\mu = t\,e^{t}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu w) &= (\mu)\left(\frac{2t+2}{t}\right) \\ \frac{d}{dt}\left(t\,e^{t} w\right) &= \left(t\,e^{t}\right)\left(\frac{2t+2}{t}\right) \\ d\left(t\,e^{t} w\right) &= \left(2\,e^{t}(1+t)\right)\,dt\end{aligned}$$

Integrating gives

$$\begin{aligned}t\,e^{t} w &= \int 2\,e^{t}(1+t)\,dt \\ t\,e^{t} w &= 2t\,e^{t} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = t\,e^{t}$ results in

$$w(t) = 2\,e^{-t}e^{t} + \frac{c_1 e^{-t}}{t}$$

which simplifies to

$$w(t) = \frac{c_1 e^{-t} + 2t}{t}$$

Replacing w in the above by x^2 using equation (5) gives the final solution.

$$x^2 = \frac{c_1 e^{-t} + 2t}{t}$$

Solving for x gives

$$\begin{aligned}x(t) &= \frac{\sqrt{t\,e^{-t}c_1 + 2t^2}}{t} \\ x(t) &= -\frac{\sqrt{t\,e^{-t}c_1 + 2t^2}}{t}\end{aligned}$$

Summary

The solution(s) found are the following

$$x = \frac{\sqrt{t\,\mathrm{e}^{-t}c_1 + 2t^2}}{t} \tag{1}$$

$$x = -\frac{\sqrt{t\,\mathrm{e}^{-t}c_1 + 2t^2}}{t} \tag{2}$$

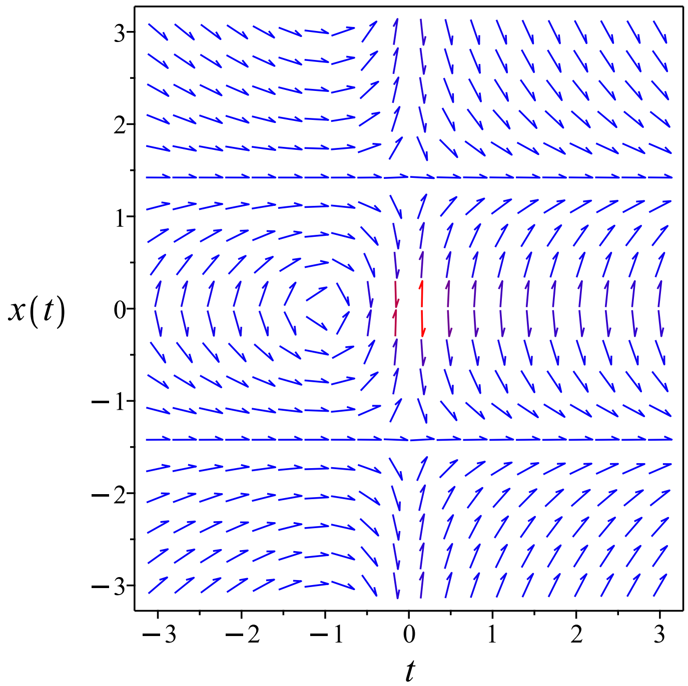


Figure 203: Slope field plot

Verification of solutions

$$x = \frac{\sqrt{t\,\mathrm{e}^{-t}c_1 + 2t^2}}{t}$$

Verified OK.

$$x = -\frac{\sqrt{t\,\mathrm{e}^{-t}c_1 + 2t^2}}{t}$$

Verified OK.

5.18.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{2x}{-x^2+2}\right)\mathrm{d}x&=\left(\frac{1+t}{t}\right)\mathrm{d}t\\ \left(-\frac{1+t}{t}\right)\mathrm{d}t+\left(\frac{2x}{-x^2+2}\right)\mathrm{d}x&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t,x) &= -\frac{1+t}{t} \\ N(t,x) &= \frac{2x}{-x^2+2} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1+t}{t}\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{2x}{-x^2+2}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial x} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t \\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\frac{1+t}{t}\,\mathrm{d}t \end{aligned}$$

$$\phi = -t - \ln(t) + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = 0 + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{2x}{-x^2+2}$. Therefore equation (4) becomes

$$\frac{2x}{-x^2+2} = 0 + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\frac{2x}{x^2-2}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\frac{2x}{x^2-2}\right)\mathrm{d}x$$
$$f(x) = -\ln\left(x^2-2\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -t - \ln(t) - \ln\left(x^2-2\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -t - \ln(t) - \ln\left(x^2-2\right)$$

Summary
The solution(s) found are the following

$$-t - \ln(t) - \ln\left(x^2-2\right) = c_1 \tag{1}$$

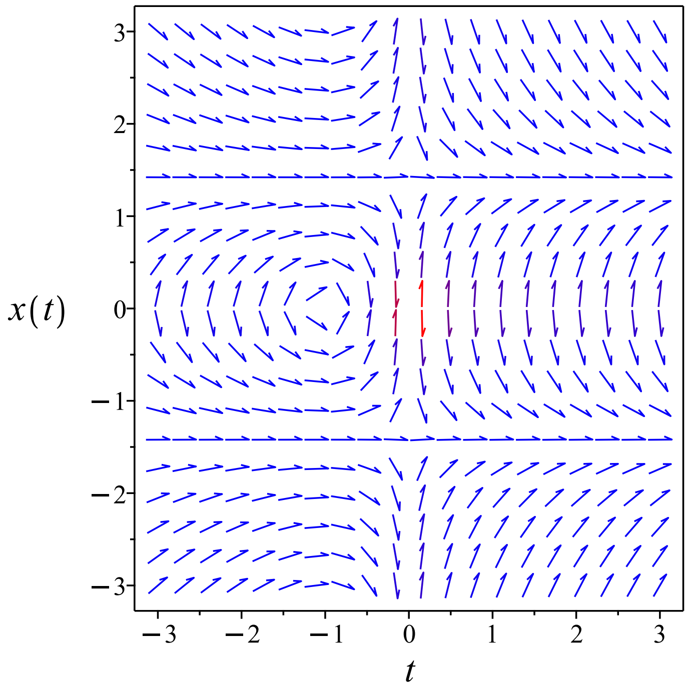


Figure 204: Slope field plot

Verification of solutions

$$-t - \ln(t) - \ln\left(x^2 - 2\right) = c_1$$

Verified OK.

5.18.5 Maple step by step solution

Let's solve

$x' + \frac{(1+t)x}{2t} - \frac{1+t}{xt} = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Separate variables

  $\frac{x'x}{x^2-2} = -\frac{1+t}{2t}$

- Integrate both sides with respect to t

  $\int \frac{x'x}{x^2-2} dt = \int -\frac{1+t}{2t} dt + c_1$

- Evaluate integral

$$\frac{\ln(x^2-2)}{2} = -\frac{t}{2} - \frac{\ln(t)}{2} + c_1$$

- Solve for x

$$\left\{x = \frac{\sqrt{t(e^{-t+2c_1}+2t)}}{t}, x = -\frac{\sqrt{t(e^{-t+2c_1}+2t)}}{t}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 46

```
dsolve(diff(x(t),t)+(t+1)/(2*t)*x(t)=(t+1)/(x(t)*t),x(t), singsol=all)
```

$$x(t) = \frac{\sqrt{t\,e^{-t}c_1 + 2t^2}}{t}$$
$$x(t) = -\frac{\sqrt{t\,e^{-t}c_1 + 2t^2}}{t}$$

✓ Solution by Mathematica

Time used: 3.335 (sec). Leaf size: 78

```
DSolve[x'[t]+(t+1)/(2*t)*x[t]==(t+1)/(x[t]*t),x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{\sqrt{2t + e^{-t+2c_1}}}{\sqrt{t}}$$
$$x(t) \to \frac{\sqrt{2t + e^{-t+2c_1}}}{\sqrt{t}}$$
$$x(t) \to -\sqrt{2}$$
$$x(t) \to \sqrt{2}$$

5.19 problem 19

Internal problem ID [11656]
Internal file name [`OUTPUT/11665_Wednesday_April_10_2024_04_53_59_PM_19754612/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x - 2y = 2x^4}$$

With initial conditions

$$[y(2) = 8]$$

5.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = 2x^3$$

Hence the ode is

$$y' - \frac{2y}{x} = 2x^3$$

The domain of $p(x) = -\frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = 2x^3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

5.19.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(2x^3\right) \\ \frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(2x^3\right) \\ d\left(\frac{y}{x^2}\right) &= (2x)\ dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2} &= \int 2x\,dx \\ \frac{y}{x^2} &= x^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^4 + c_1 x^2$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 8$ in the above solution gives an equation to solve for the constant of integration.

$$8 = 16 + 4c_1$$

$$c_1 = -2$$

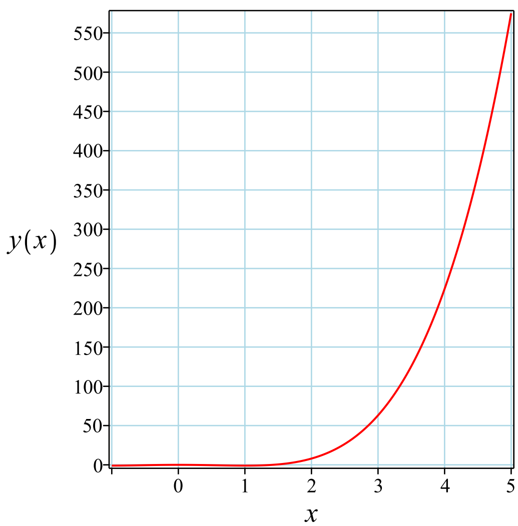
Substituting c_1 found above in the general solution gives

$$y = x^4 - 2x^2$$

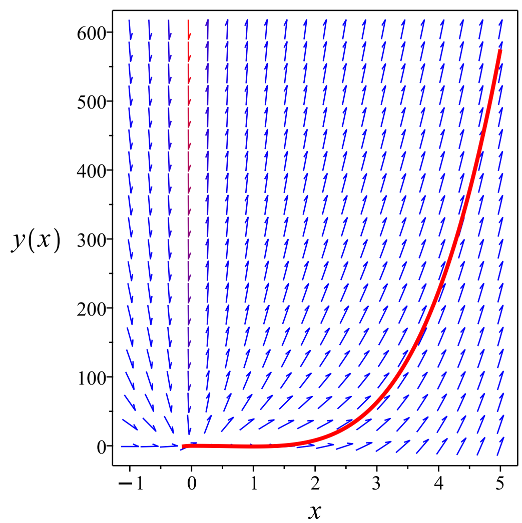
Summary

The solution(s) found are the following

$$y = x^4 - 2x^2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^4 - 2x^2$$

Verified OK.

5.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2x^4 + 2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 162: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^2
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S&=\int\frac{1}{\eta}dy\\&=\int\frac{1}{x^2}dy\end{aligned}$$

Which results in

$$S=\frac{y}{x^2}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{2x^4+2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=1\\R_y&=0\\S_x&=-\frac{2y}{x^3}\\S_y&=\frac{1}{x^2}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2} = x^2 + c_1$$

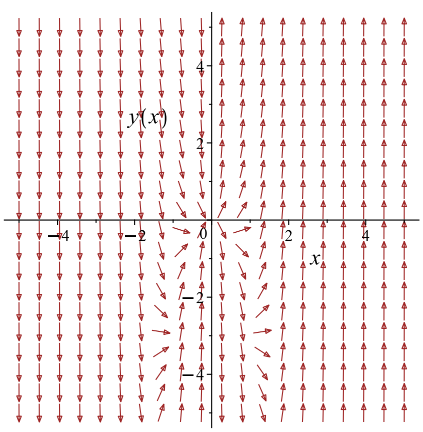
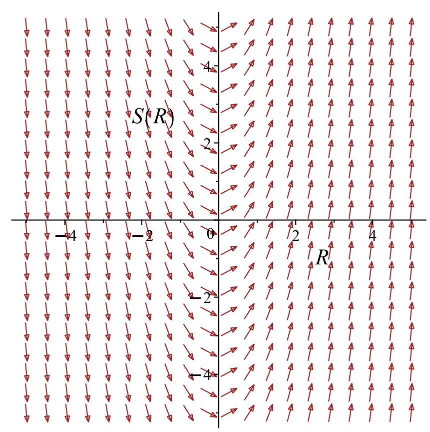
Which simplifies to

$$\frac{y}{x^2} = x^2 + c_1$$

Which gives

$$y = x^2(x^2 + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x^4+2y}{x}$ 	$R = x$ $S = \frac{y}{x^2}$	$\frac{dS}{dR} = 2R$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 8$ in the above solution gives an equation to solve for the constant of integration.

$$8 = 16 + 4c_1$$

$$c_1 = -2$$

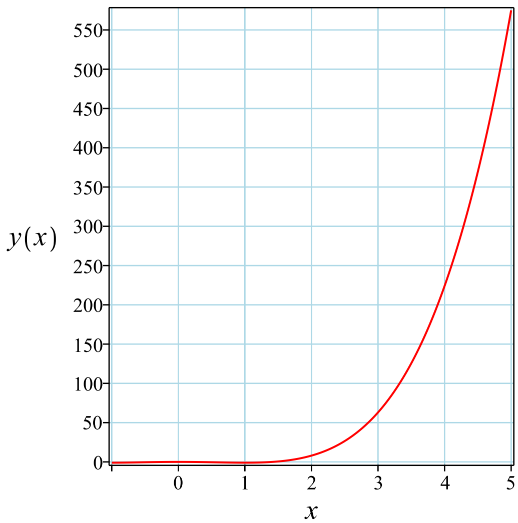
Substituting c_1 found above in the general solution gives

$$y = x^2(x^2 - 2)$$

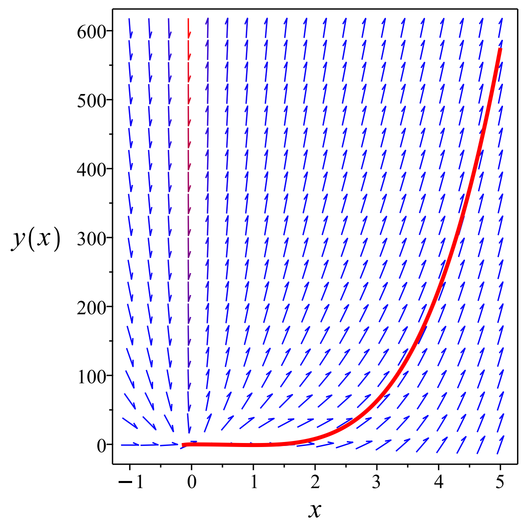
Summary

The solution(s) found are the following

$$y = x^2(x^2 - 2) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^2(x^2 - 2)$$

Verified OK.

5.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x) \mathrm{d} y &= \left(2 x^4+2 y\right) \mathrm{d} x \\
\left(-2 x^4-2 y\right) \mathrm{d} x+(x) \mathrm{d} y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -2 x^4-2 y \\
N(x, y) &= x
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2 x^4-2 y\right) \\
&= -2
\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((-2) - (1))\\ &= -\frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{3}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(x)}\\ &= \frac{1}{x^3}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^3}(-2x^4 - 2y)\\ &= \frac{-2x^4 - 2y}{x^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^3}(x)\\ &= \frac{1}{x^2}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-2x^4 - 2y}{x^3}\right) + \left(\frac{1}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{-2x^4 - 2y}{x^3}\,\mathrm{d}x$$
$$\phi = \frac{-x^4 + y}{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{1}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{x^2}$. Therefore equation (4) becomes

$$\frac{1}{x^2} = \frac{1}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x^4 + y}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x^4 + y}{x^2}$$

The solution becomes

$$y = x^2(x^2 + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 8$ in the above solution gives an equation to solve for the constant of integration.

$$8 = 16 + 4c_1$$

$$c_1 = -2$$

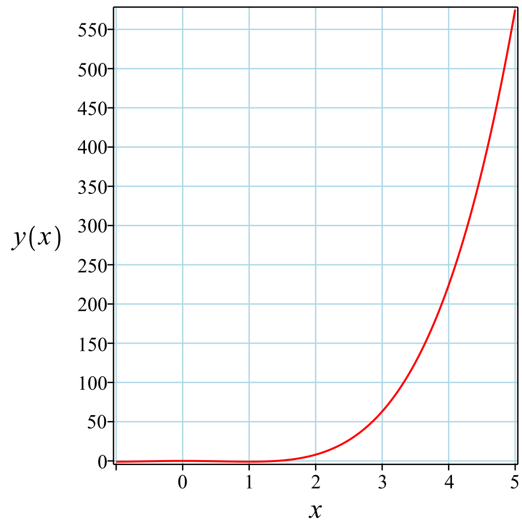
Substituting c_1 found above in the general solution gives

$$y = x^2(x^2 - 2)$$

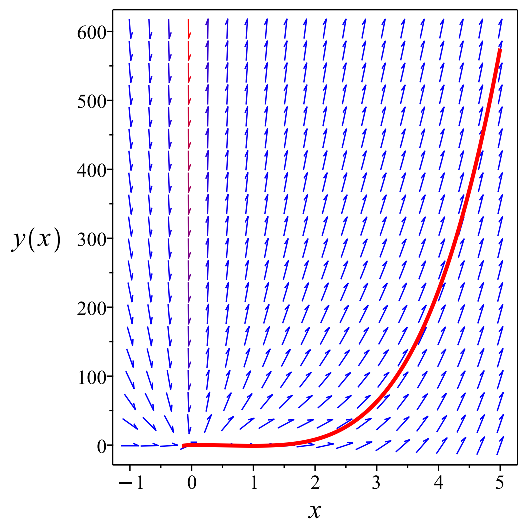
Summary

The solution(s) found are the following

$$y = x^2(x^2 - 2) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^2(x^2 - 2)$$

Verified OK.

5.19.5 Maple step by step solution

Let's solve

$[y'x - 2y = 2x^4, y(2) = 8]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{x} + 2x^3$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{x} = 2x^3$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = 2\mu(x)\,x^3$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{2\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^2}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 2\mu(x)\, x^3 dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 2\mu(x)\, x^3 dx + c_1$
- Solve for y

  $y = \frac{\int 2\mu(x) x^3 dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \frac{1}{x^2}$

  $y = x^2 \left(\int 2x dx + c_1\right)$
- Evaluate the integrals on the rhs

  $y = x^2(x^2 + c_1)$
- Use initial condition $y(2) = 8$

  $8 = 16 + 4c_1$
- Solve for c_1

  $c_1 = -2$
- Substitute $c_1 = -2$ into general solution and simplify

  $y = x^2(x^2 - 2)$
- Solution to the IVP

  $y = x^2(x^2 - 2)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([x*diff(y(x),x)-2*y(x)=2*x^4,y(2) = 8],y(x), singsol=all)
```

$$y(x) = (x^2 - 2)\, x^2$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 14

```
DSolve[{x*y'[x]-2*y[x]==2*x^4,{y[2]==8}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 (x^2 - 2)$$

5.20 problem 20

Internal problem ID [11657]
Internal file name [`OUTPUT/11666_Wednesday_April_10_2024_04_54_01_PM_33212483/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + 3x^2 y = x^2}$$

With initial conditions

$$[y(0) = 2]$$

5.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 3x^2$$
$$q(x) = x^2$$

Hence the ode is

$$y' + 3x^2y = x^2$$

The domain of $p(x) = 3x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

5.20.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x^2(-3y+1) \end{aligned}$$

Where $f(x) = x^2$ and $g(y) = -3y + 1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-3y+1}\,dy &= x^2\,dx \\ \int \frac{1}{-3y+1}\,dy &= \int x^2\,dx \\ -\frac{\ln(-3y+1)}{3} &= \frac{x^3}{3} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{(-3y+1)^{\frac{1}{3}}} = e^{\frac{x^3}{3}+c_1}$$

Which simplifies to

$$\frac{1}{(-3y+1)^{\frac{1}{3}}} = c_2 e^{\frac{x^3}{3}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{e^{-3c_1} e^{3c_1} c_2^3 - e^{-3c_1}}{3c_2^3}$$

$$c_1 = -\frac{\ln(-5c_2^3)}{3}$$

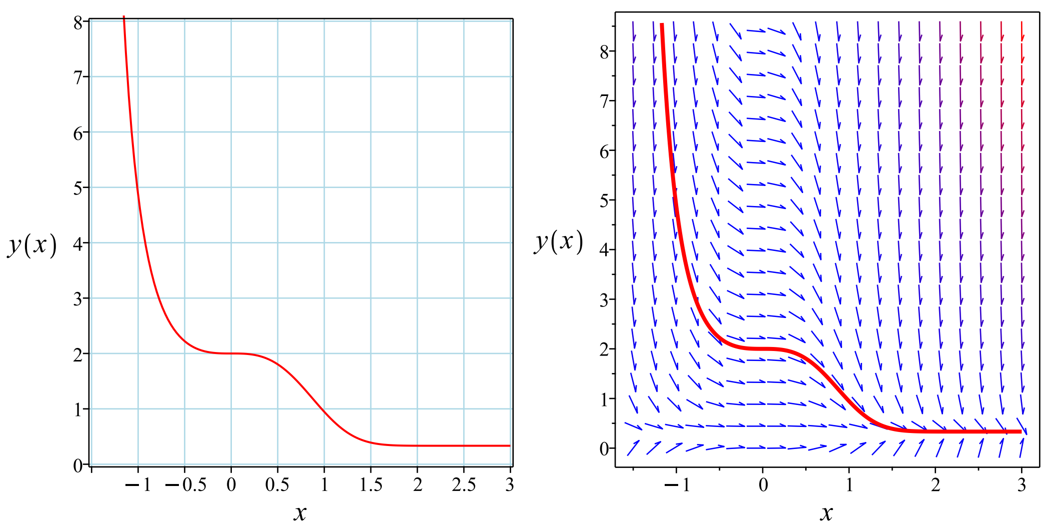
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{3} + \frac{5\,e^{-x^3}}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{3} + \frac{5\,e^{-x^3}}{3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{1}{3} + \frac{5\,e^{-x^3}}{3}$$

Verified OK.

5.20.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int 3x^2 dx} = e^{x^3}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x^2\right)$$
$$\frac{d}{dx}\left(e^{x^3} y\right) = \left(e^{x^3}\right)\left(x^2\right)$$
$$d\left(e^{x^3} y\right) = \left(x^2 e^{x^3}\right) dx$$

Integrating gives

$$e^{x^3} y = \int x^2 e^{x^3} dx$$
$$e^{x^3} y = \frac{e^{x^3}}{3} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{x^3}$ results in

$$y = \frac{e^{-x^3} e^{x^3}}{3} + c_1 e^{-x^3}$$

which simplifies to

$$y = \frac{1}{3} + c_1 e^{-x^3}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1 + \frac{1}{3}$$

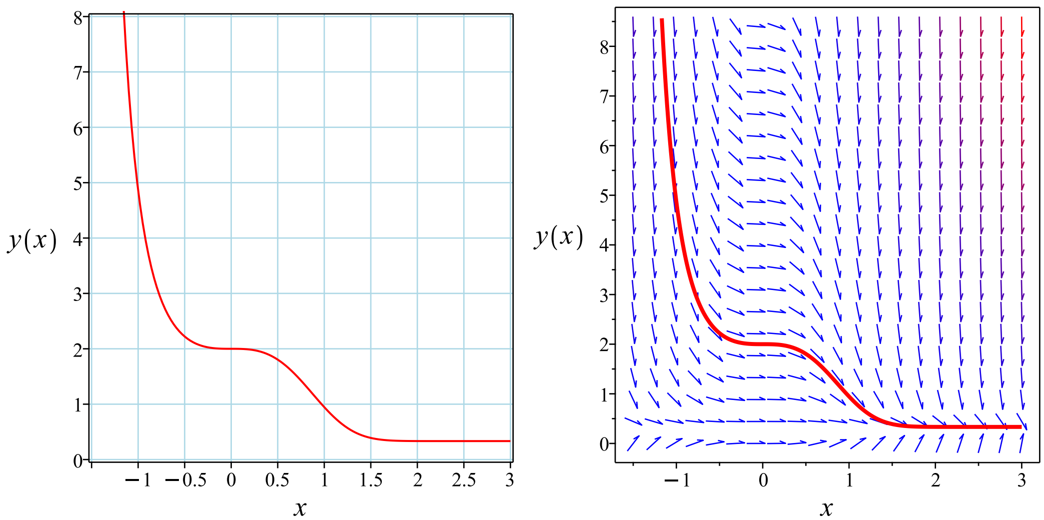
$$c_1 = \frac{5}{3}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{1}{3} + \frac{5 e^{-x^3}}{3}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{3} + \frac{5\,\mathrm{e}^{-x^3}}{3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{1}{3} + \frac{5\,\mathrm{e}^{-x^3}}{3}$$

Verified OK.

5.20.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -3x^2y + x^2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 165: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x^3}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x^3}} dy \end{aligned}$$

Which results in

$$S = e^{x^3} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -3x^2 y + x^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3x^2 e^{x^3} y \\ S_y &= e^{x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^2 e^{x^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2 e^{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{R^3}}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x^3} y = \frac{e^{x^3}}{3} + c_1$$

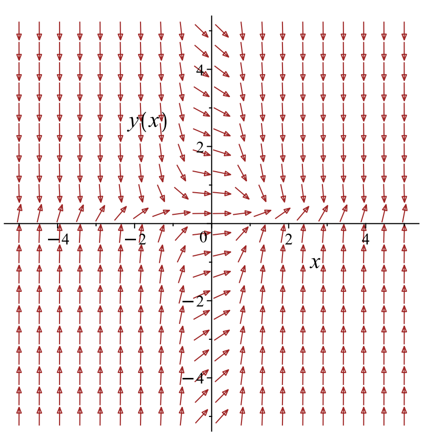
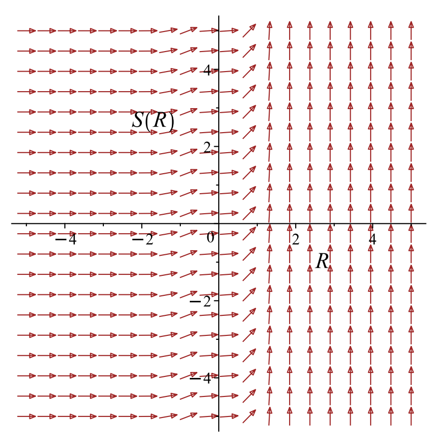
Which simplifies to

$$e^{x^3} y = \frac{e^{x^3}}{3} + c_1$$

Which gives

$$y = \frac{\left(e^{x^3} + 3c_1\right) e^{-x^3}}{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -3x^2y + x^2$ 	$R = x$ $S = e^{x^3} y$	$\frac{dS}{dR} = R^2 e^{R^3}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1 + \frac{1}{3}$$

$$c_1 = \frac{5}{3}$$

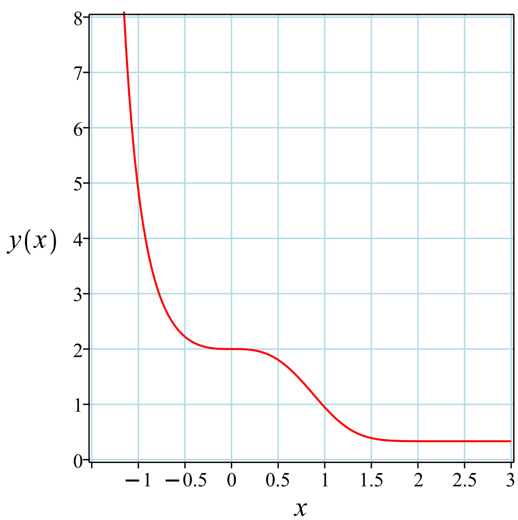
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{3} + \frac{5\,\mathrm{e}^{-x^3}}{3}$$

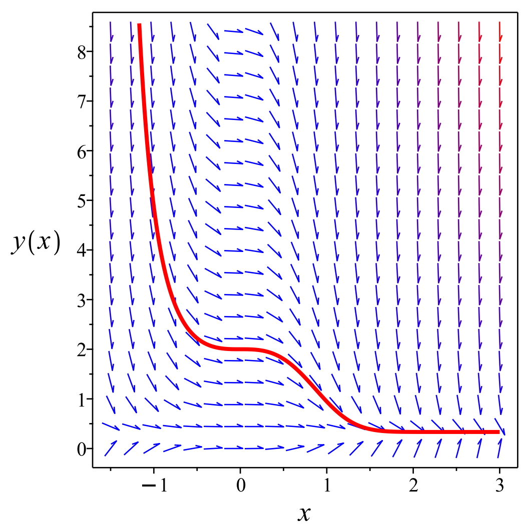
Summary

The solution(s) found are the following

$$y = \frac{1}{3} + \frac{5\,\mathrm{e}^{-x^3}}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{3} + \frac{5\,\mathrm{e}^{-x^3}}{3}$$

Verified OK.

5.20.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-3y+1}\right)\mathrm{d}y&=\left(x^2\right)\mathrm{d}x\\ \left(-x^2\right)\mathrm{d}x+\left(\frac{1}{-3y+1}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -x^2 \\
N(x,y) &= \frac{1}{-3y+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^2) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-3y+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, dx &= \int M \, dx \\
\int \frac{\partial \phi}{\partial x} \, dx &= \int -x^2 \, dx
\end{aligned}
$$

$$
\phi = -\frac{x^3}{3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-3y+1}$. Therefore equation (4) becomes

$$\frac{1}{-3y+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{3y-1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{3y-1}\right) \mathrm{d}y$$

$$f(y) = -\frac{\ln(3y-1)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} - \frac{\ln(3y-1)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} - \frac{\ln(3y-1)}{3}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-x^3-3c_1}}{3} + \frac{1}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{\mathrm{e}^{-3c_1}}{3} + \frac{1}{3}$$

$$c_1 = -\frac{\ln(5)}{3}$$

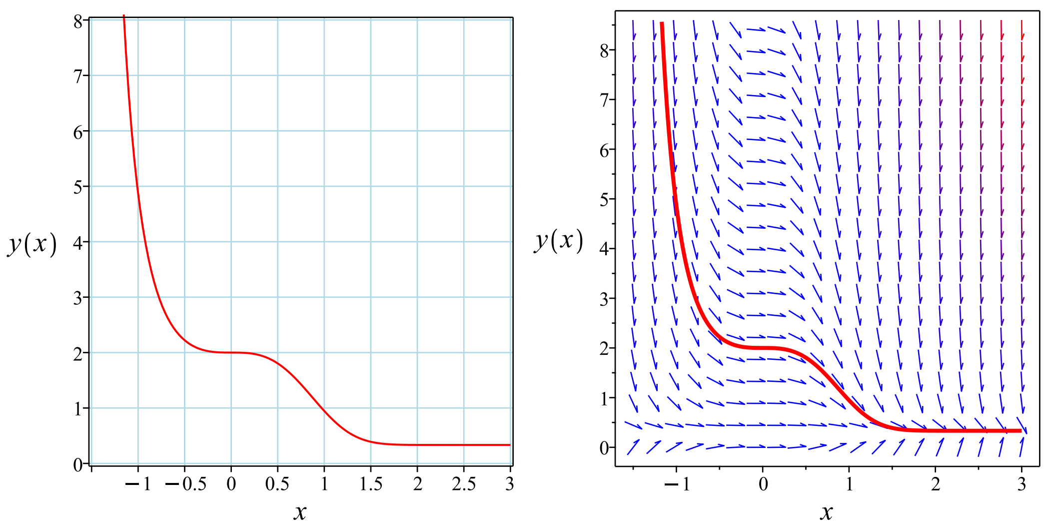
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{3} + \frac{5\,\mathrm{e}^{-x^3}}{3}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{3} + \frac{5\,\mathrm{e}^{-x^3}}{3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{1}{3} + \frac{5\,\mathrm{e}^{-x^3}}{3}$$

Verified OK.

5.20.6 Maple step by step solution

Let's solve

$[y' + 3x^2 y = x^2, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{3y-1} = -x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{3y-1} dx = \int -x^2 dx + c_1$

- Evaluate integral

  $\frac{\ln(3y-1)}{3} = -\frac{x^3}{3} + c_1$

- Solve for y

  $y = \frac{e^{-x^3+3c_1}}{3} + \frac{1}{3}$

- Use initial condition $y(0) = 2$

  $2 = \frac{e^{3c_1}}{3} + \frac{1}{3}$

- Solve for c_1

  $c_1 = \frac{\ln(5)}{3}$

- Substitute $c_1 = \frac{\ln(5)}{3}$ into general solution and simplify

  $y = \frac{1}{3} + \frac{5\,e^{-x^3}}{3}$

- Solution to the IVP

  $y = \frac{1}{3} + \frac{5\,e^{-x^3}}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)+3*x^2*y(x)=x^2,y(0) = 2],y(x), singsol=all)
```

$$y(x) = \frac{1}{3} + \frac{5\,e^{-x^3}}{3}$$

✓ Solution by Mathematica

Time used: 2.884 (sec). Leaf size: 20

```
DSolve[{y'[x]+3*x^2*y[x]==x^2,{y[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{5e^{-x^3}}{3} + \frac{1}{3}$$

5.21 problem 21

Internal problem ID [11658]
Internal file name [`OUTPUT/11667_Wednesday_April_10_2024_04_54_02_PM_91574622/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{e^{x} \left(y-3(e^{x}+1)^{2}\right)+\left(e^{x}+1\right) y^{\prime}=0}$$

With initial conditions

$$[y(0)=4]$$

5.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y^{\prime}+p(x)y=q(x)$$

Where here

$$\begin{aligned}p(x)&=\frac{e^{x}}{e^{x}+1}\\ q(x)&=3\,e^{x}(e^{x}+1)\end{aligned}$$

Hence the ode is

$$y' + \frac{e^x y}{e^x + 1} = 3\,e^x(e^x + 1)$$

The domain of $p(x) = \frac{e^x}{e^x+1}$ is

$$\{2i\pi\_Z100 + i\pi < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

5.21.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{e^x}{e^x+1}dx}\\ &= e^x + 1\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(3\,e^x(e^x + 1)\right)\\ \frac{d}{dx}\left((e^x + 1)\,y\right) &= (e^x + 1)\left(3\,e^x(e^x + 1)\right)\\ d\left((e^x + 1)\,y\right) &= \left(3\,e^x(e^x + 1)^2\right)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}(e^x + 1)\,y &= \int 3\,e^x(e^x + 1)^2\,dx\\ (e^x + 1)\,y &= (e^x + 1)^3 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x + 1$ results in

$$y = (e^x + 1)^2 + \frac{c_1}{e^x + 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = 4 + \frac{c_1}{2}$$

$$c_1 = 0$$

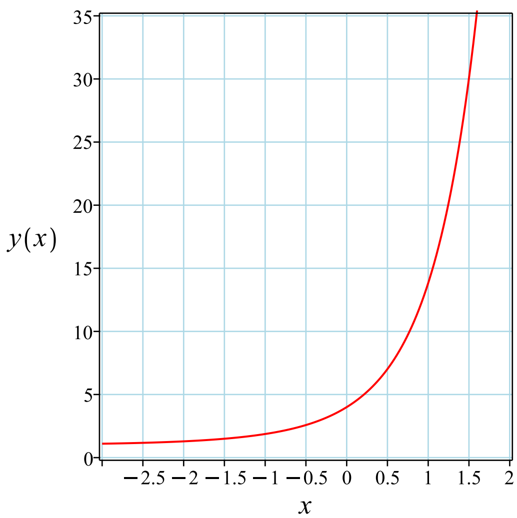
Substituting c_1 found above in the general solution gives

$$y = e^{2x} + 2 e^{x} + 1$$

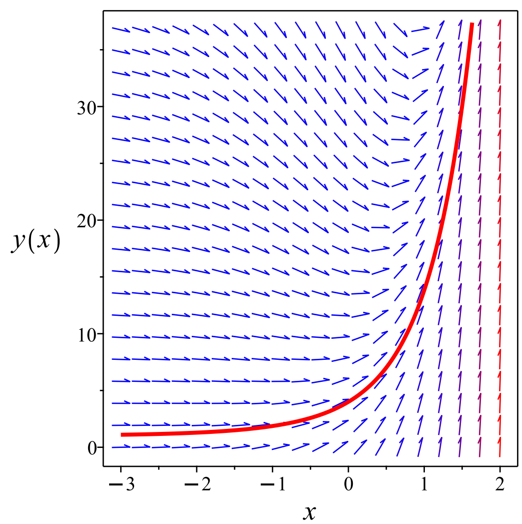
Summary

The solution(s) found are the following

$$y = e^{2x} + 2 e^{x} + 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{2x} + 2 e^{x} + 1$$

Verified OK.

5.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{e^{x}(3 e^{2x} + 6 e^{x} - y + 3)}{e^{x} + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 168: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{e^x + 1}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{e^x+1}} dy \end{aligned}$$

Which results in

$$S = (e^x + 1)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{e^x (3\, e^{2x} + 6\, e^x - y + 3)}{e^x + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^x y \\ S_y &= e^x + 1 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 6\, e^{2x} + 3\, e^{3x} + 3\, e^x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 6\, e^{2R} + 3\, e^{3R} + 3\, e^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 3\,\mathrm{e}^{2R} + 3\,\mathrm{e}^{R} + \mathrm{e}^{3R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(\mathrm{e}^{x} + 1)\,y = \mathrm{e}^{3x} + 3\,\mathrm{e}^{2x} + 3\,\mathrm{e}^{x} + c_1$$

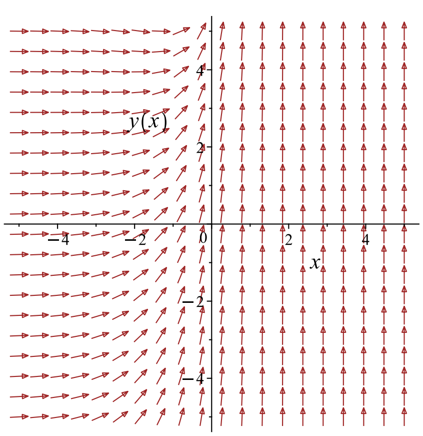
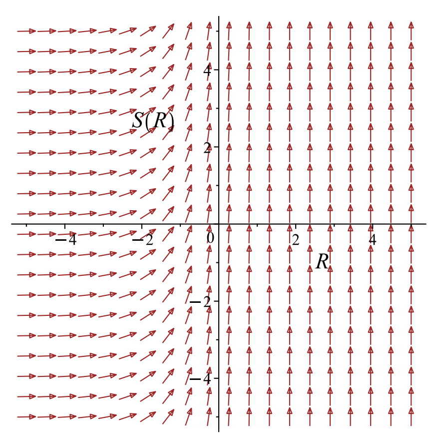
Which simplifies to

$$(\mathrm{e}^{x} + 1)\,y = \mathrm{e}^{3x} + 3\,\mathrm{e}^{2x} + 3\,\mathrm{e}^{x} + c_1$$

Which gives

$$y = \frac{\mathrm{e}^{3x} + 3\,\mathrm{e}^{2x} + 3\,\mathrm{e}^{x} + c_1}{\mathrm{e}^{x} + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{\mathrm{e}^{x}(3\,\mathrm{e}^{2x}+6\,\mathrm{e}^{x}-y+3)}{\mathrm{e}^{x}+1}$
</td><td>$R = x$
$S = (\mathrm{e}^{x} + 1)\,y$</td><td>$\frac{dS}{dR} = 6\,\mathrm{e}^{2R} + 3\,\mathrm{e}^{3R} + 3\,\mathrm{e}^{R}$
</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = \frac{7}{2} + \frac{c_1}{2}$$

$$c_1 = 1$$

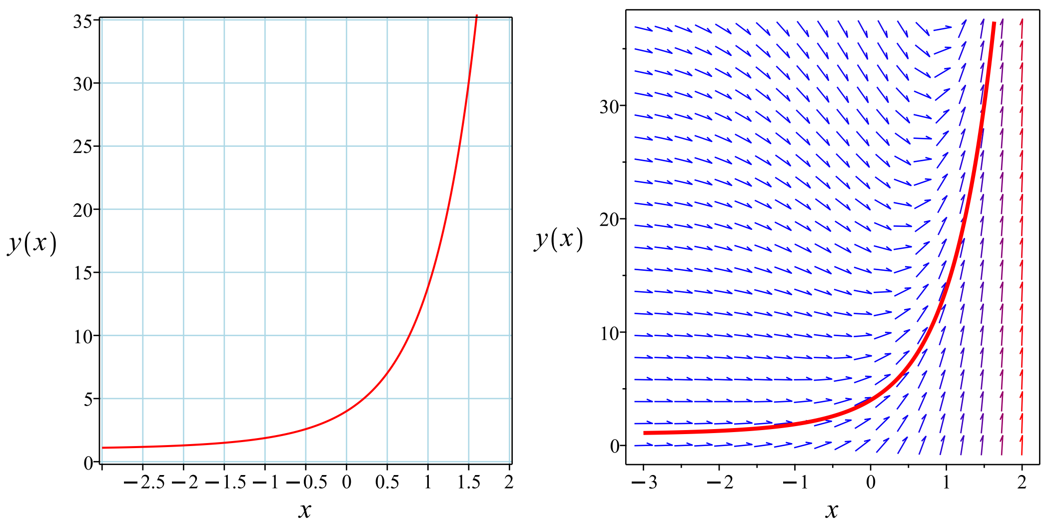
Substituting c_1 found above in the general solution gives

$$y = \frac{e^{3x} + 3\,e^{2x} + 3\,e^{x} + 1}{e^{x} + 1}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{e^{3x} + 3\,e^{2x} + 3\,e^{x} + 1}{e^{x} + 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{3x} + 3\,e^{2x} + 3\,e^{x} + 1}{e^{x} + 1}$$

Verified OK.

5.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^{x}+1)\,\mathrm{d}y&=\left(-\mathrm{e}^{x}\left(y-3(\mathrm{e}^{x}+1)^{2}\right)\right)\mathrm{d}x\\ \left(\mathrm{e}^{x}\left(y-3(\mathrm{e}^{x}+1)^{2}\right)\right)\mathrm{d}x+(\mathrm{e}^{x}+1)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\mathrm{e}^{x}\left(y-3(\mathrm{e}^{x}+1)^{2}\right)\\ N(x,y)&=\mathrm{e}^{x}+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{x}\left(y - 3(\mathrm{e}^{x}+1)^{2}\right)\right)\\ &= \mathrm{e}^{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^{x}+1)\\ &= \mathrm{e}^{x}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \mathrm{e}^{x}\left(y - 3(\mathrm{e}^{x}+1)^{2}\right)\mathrm{d}x\\ \phi &= -3\,\mathrm{e}^{2x} - \mathrm{e}^{3x} + (y-3)\,\mathrm{e}^{x} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + 1$. Therefore equation (4) becomes

$$\mathrm{e}^{x} + 1 = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (1) \,\mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -3\,\mathrm{e}^{2x} - \mathrm{e}^{3x} + (y-3)\,\mathrm{e}^{x} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -3\,\mathrm{e}^{2x} - \mathrm{e}^{3x} + (y-3)\,\mathrm{e}^{x} + y$$

The solution becomes

$$y = \frac{\mathrm{e}^{3x} + 3\,\mathrm{e}^{2x} + 3\,\mathrm{e}^{x} + c_1}{\mathrm{e}^{x} + 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = \frac{7}{2} + \frac{c_1}{2}$$

$$c_1 = 1$$

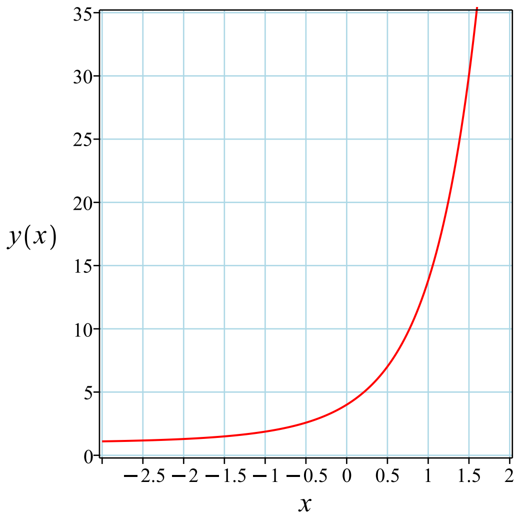
Substituting c_1 found above in the general solution gives

$$y = \frac{\mathrm{e}^{3x} + 3\,\mathrm{e}^{2x} + 3\,\mathrm{e}^{x} + 1}{\mathrm{e}^{x} + 1}$$

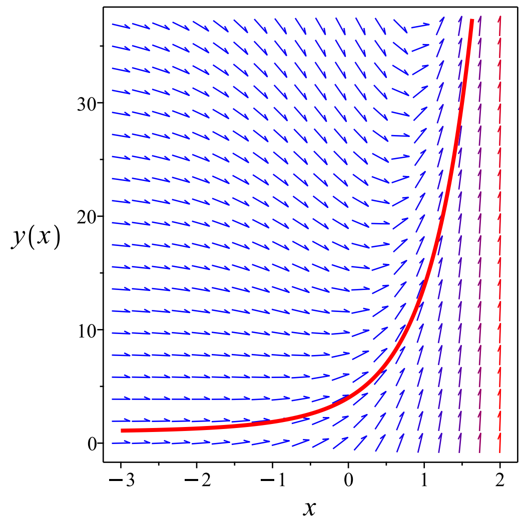
Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{3x} + 3\,\mathrm{e}^{2x} + 3\,\mathrm{e}^{x} + 1}{\mathrm{e}^{x} + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{3x} + 3\,e^{2x} + 3\,e^{x} + 1}{e^{x} + 1}$$

Verified OK.

5.21.5 Maple step by step solution

Let's solve

$\left[e^{x}\left(y - 3(e^{x}+1)^{2}\right) + (e^{x}+1)\,y' = 0, y(0) = 4\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{e^{x}y}{e^{x}+1} + 3\,e^{x}(e^{x}+1)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{e^{x}y}{e^{x}+1} = 3\,e^{x}(e^{x}+1)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{e^{x}y}{e^{x}+1}\right) = 3\mu(x)\,e^{x}(e^{x}+1)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$\mu(x)\left(y'+\frac{e^x y}{e^x+1}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)e^x}{e^x+1}$

- Solve to find the integrating factor

  $\mu(x)=e^x+1$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int 3\mu(x)\,e^x(e^x+1)\,dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int 3\mu(x)\,e^x(e^x+1)\,dx+c_1$

- Solve for y

  $y=\frac{\int 3\mu(x)e^x(e^x+1)dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=e^x+1$

  $y=\frac{\int 3\,e^x(e^x+1)^2dx+c_1}{e^x+1}$

- Evaluate the integrals on the rhs

  $y=\frac{(e^x+1)^3+c_1}{e^x+1}$

- Simplify

  $y=\frac{e^{3x}+3\,e^{2x}+3\,e^x+c_1+1}{e^x+1}$

- Use initial condition $y(0)=4$

  $4=4+\frac{c_1}{2}$

- Solve for c_1

  $c_1=0$

- Substitute $c_1=0$ into general solution and simplify

  $y=e^{2x}+2\,e^x+1$

- Solution to the IVP

  $y=e^{2x}+2\,e^x+1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([exp(x)*(y(x)-3*(exp(x)+1)^2)+(exp(x)+1)*diff(y(x),x)=0,y(0) = 4],y(x), singsol=all)
```

$$y(x) = e^{2x} + 2\,e^{x} + 1$$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 12

```
DSolve[{Exp[x]*(y[x]-3*(Exp[x]+1)^2)+(Exp[x]+1)*y'[x]==0,{y[0]==4}},y[x],x,IncludeSingularSo
```

$$y(x) \to (e^x + 1)^2$$

5.22 problem 22

Internal problem ID [11659]
Internal file name [`OUTPUT/11668_Wednesday_April_10_2024_04_54_03_PM_15034228/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2(y+1)\,x-\left(x^2+1\right)y'=0}$$

With initial conditions

$$[y(1)=-5]$$

5.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=-\frac{2x}{x^2+1}$$
$$q(x)=\frac{2x}{x^2+1}$$

Hence the ode is

$$y' - \frac{2xy}{x^2+1} = \frac{2x}{x^2+1}$$

The domain of $p(x) = -\frac{2x}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{2x}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

5.22.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(2y+2)}{x^2+1} \end{aligned}$$

Where $f(x) = \frac{x}{x^2+1}$ and $g(y) = 2y+2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{2y+2}\, dy &= \frac{x}{x^2+1}\, dx \\ \int \frac{1}{2y+2}\, dy &= \int \frac{x}{x^2+1}\, dx \\ \frac{\ln(y+1)}{2} &= \frac{\ln(x^2+1)}{2} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{y+1} = e^{\frac{\ln(x^2+1)}{2}+c_1}$$

Which simplifies to

$$\sqrt{y+1} = c_2\sqrt{x^2+1}$$

Which can be simplified to become

$$y = c_2^2 (x^2 + 1) e^{2c_1} - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -5$ in the above solution gives an equation to solve for the constant of integration.

$$-5 = -1 + 2c_2^2 e^{2c_1}$$

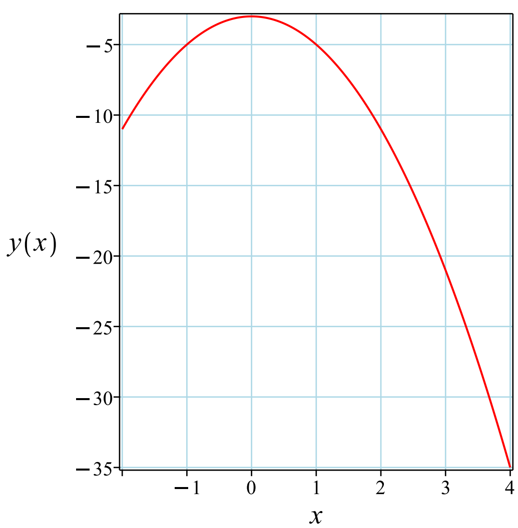
$$c_1 = \frac{\ln\left(-\frac{2}{c_2^2}\right)}{2}$$

Substituting c_1 found above in the general solution gives

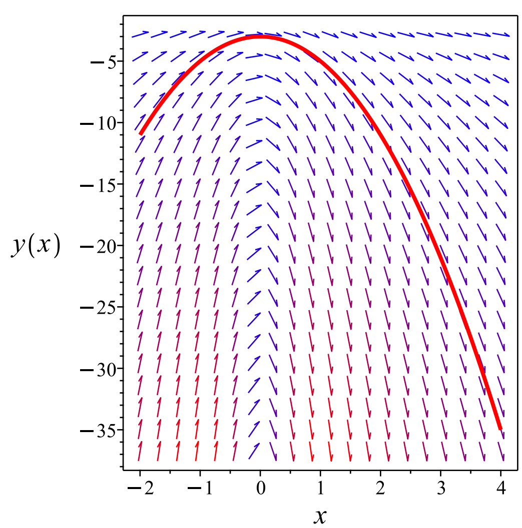
$$y = -2x^2 - 3$$

Summary
The solution(s) found are the following

$$y = -2x^2 - 3 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2x^2 - 3$$

Verified OK.

5.22.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2x}{x^2+1}dx} \\ &= \frac{1}{x^2+1}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{2x}{x^2+1}\right) \\ \frac{d}{dx}\left(\frac{y}{x^2+1}\right) &= \left(\frac{1}{x^2+1}\right)\left(\frac{2x}{x^2+1}\right) \\ d\left(\frac{y}{x^2+1}\right) &= \left(\frac{2x}{(x^2+1)^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2+1} &= \int \frac{2x}{(x^2+1)^2} dx \\ \frac{y}{x^2+1} &= -\frac{1}{x^2+1} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2+1}$ results in

$$y = -1 + c_1(x^2+1)$$

which simplifies to

$$y = c_1x^2 + c_1 - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -5$ in the above solution gives an equation to solve for the constant of integration.

$$-5 = 2c_1 - 1$$

$$c_1 = -2$$

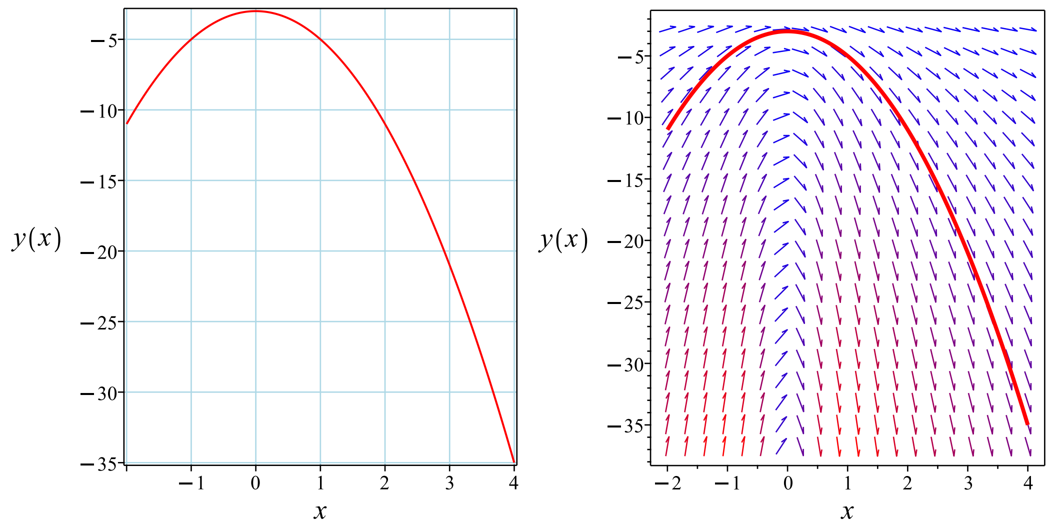
Substituting c_1 found above in the general solution gives

$$y = -2x^2 - 3$$

Summary

The solution(s) found are the following

$$y = -2x^2 - 3 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -2x^2 - 3$$

Verified OK.

5.22.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2(y+1)\,x}{x^2+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 171: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= x^2 + 1 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2 + 1} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2 + 1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2(y + 1) x}{x^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2yx}{(x^2 + 1)^2} \\ S_y &= \frac{1}{x^2 + 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2x}{(x^2 + 1)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2R}{(R^2 + 1)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R^2+1} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2+1} = -\frac{1}{x^2+1} + c_1$$

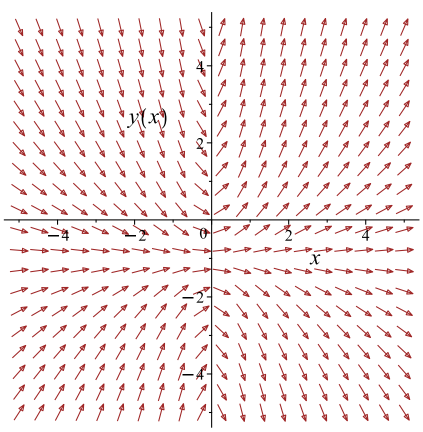
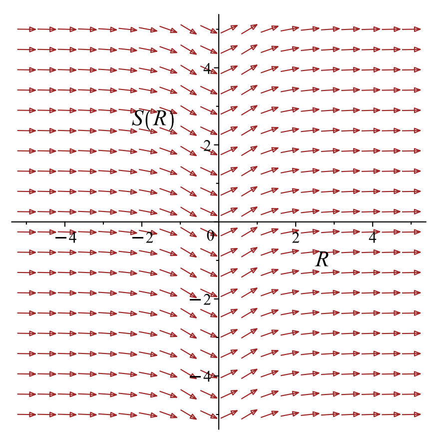
Which simplifies to

$$\frac{y}{x^2+1} = -\frac{1}{x^2+1} + c_1$$

Which gives

$$y = c_1 x^2 + c_1 - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2(y+1)x}{x^2+1}$ 	$R = x$ $S = \frac{y}{x^2+1}$	$\frac{dS}{dR} = \frac{2R}{(R^2+1)^2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -5$ in the above solution gives an equation to solve for the constant of integration.

$$-5 = 2c_1 - 1$$

$$c_1 = -2$$

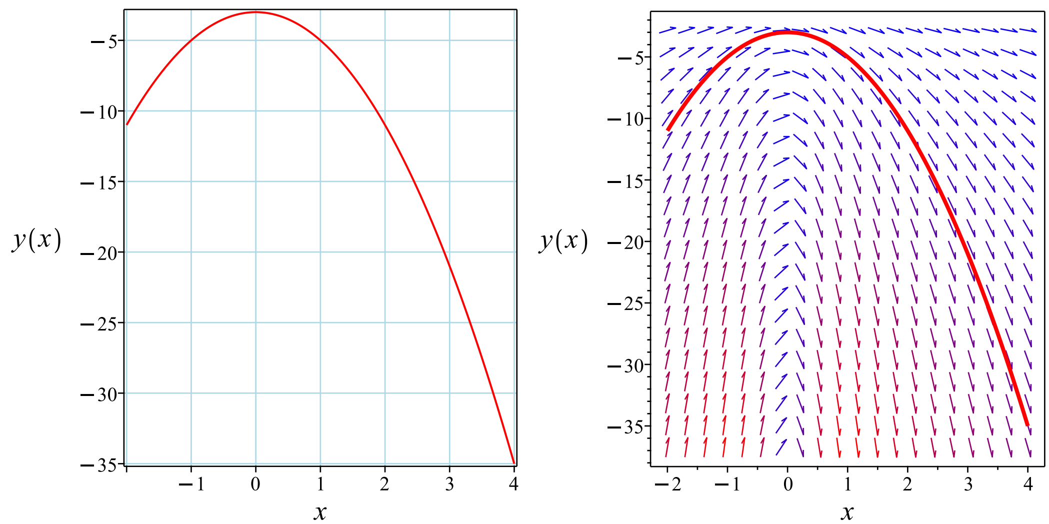
Substituting c_1 found above in the general solution gives

$$y = -2x^2 - 3$$

Summary

The solution(s) found are the following

$$y = -2x^2 - 3 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -2x^2 - 3$$

Verified OK.

5.22.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{2y+2}\right)\mathrm{d}y &= \left(\frac{x}{x^2+1}\right)\mathrm{d}x\\ \left(-\frac{x}{x^2+1}\right)\mathrm{d}x + \left(\frac{1}{2y+2}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{x}{x^2+1}\\ N(x, y) &= \frac{1}{2y+2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y+2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{x}{x^2+1} \, \mathrm{d}x$$

$$\phi = -\frac{\ln(x^2+1)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y+2}$. Therefore equation (4) becomes

$$\frac{1}{2y+2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2y+2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2y+2}\right) \mathrm{d}y$$
$$f(y) = \frac{\ln(y+1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} + \frac{\ln(y+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} + \frac{\ln(y+1)}{2}$$

The solution becomes

$$y = x^2 \mathrm{e}^{2c_1} + \mathrm{e}^{2c_1} - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -5$ in the above solution gives an equation to solve for the constant of integration.

$$-5 = -1 + 2\,\mathrm{e}^{2c_1}$$

$$c_1 = \frac{\ln(2)}{2} + \frac{i\pi}{2}$$

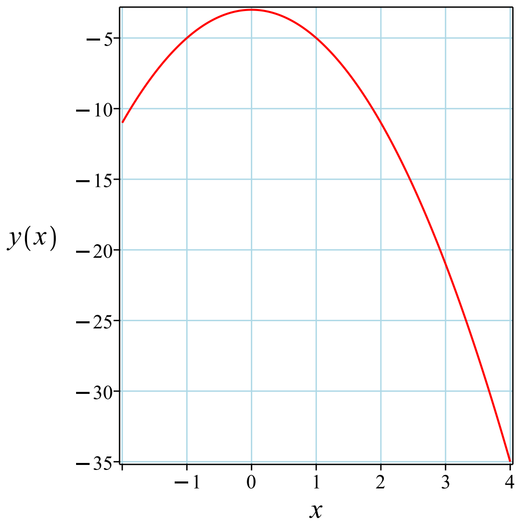
Substituting c_1 found above in the general solution gives

$$y = -2x^2 - 3$$

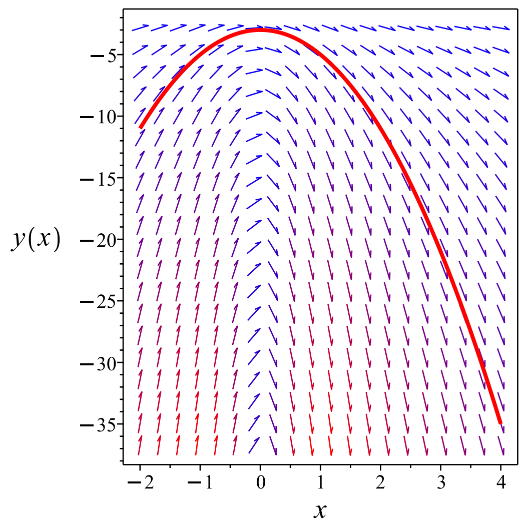
Summary

The solution(s) found are the following

$$y = -2x^2 - 3 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2x^2 - 3$$

Verified OK.

5.22.6 Maple step by step solution

Let's solve

$\left[2(y+1)\,x-\left(x^{2}+1\right)y'=0, y(1)=-5\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y+1}=\frac{2x}{x^{2}+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y+1}dx=\int \frac{2x}{x^{2}+1}dx+c_{1}$

- Evaluate integral

  $\ln\left(y+1\right)=\ln\left(x^{2}+1\right)+c_{1}$

- Solve for y

  $y=x^{2}\mathrm{e}^{c_{1}}+\mathrm{e}^{c_{1}}-1$

- Use initial condition $y(1) = -5$

  $-5 = 2\,e^{c_1} - 1$
- Solve for c_1

  $c_1 = \ln(2) + I\pi$
- Substitute $c_1 = \ln(2) + I\pi$ into general solution and simplify

  $y = -2x^2 - 3$
- Solution to the IVP

  $y = -2x^2 - 3$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve([2*x*(y(x)+1)-(x^2+1)*diff(y(x),x)=0,y(1) = -5],y(x), singsol=all)
```

$$y(x) = -2x^2 - 3$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 12

```
DSolve[{2*x*(y[x]+1)-(x^2+1)*y'[x]==0,{y[1]==-5}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x^2 - 3$$

5.23 problem 23

Internal problem ID [11660]
Internal file name [`OUTPUT/11669_Wednesday_April_10_2024_04_54_04_PM_9334433/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{r' + r\tan(t) = \cos(t)^2}$$

With initial conditions

$$\left[r\left(\frac{\pi}{4}\right) = 1\right]$$

5.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$r' + p(t)r = q(t)$$

Where here

$$p(t) = \tan(t)$$
$$q(t) = \cos(t)^2$$

Hence the ode is

$$r' + r \tan(t) = \cos(t)^2$$

The domain of $p(t) = \tan(t)$ is

$$\left\{t < \frac{1}{2}\pi + \pi\_Z101 \lor \frac{1}{2}\pi + \pi\_Z101 < t\right\}$$

And the point $t_0 = \frac{\pi}{4}$ is inside this domain. The domain of $q(t) = \cos(t)^2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = \frac{\pi}{4}$ is also inside this domain. Hence solution exists and is unique.

5.23.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \tan(t) dt} \\ &= \frac{1}{\cos(t)} \end{aligned}$$

Which simplifies to

$$\mu = \sec(t)$$

The ode becomes

$$\begin{aligned} \frac{d}{dt}(\mu r) &= (\mu)\left(\cos(t)^2\right) \\ \frac{d}{dt}(\sec(t)\, r) &= (\sec(t))\left(\cos(t)^2\right) \\ d(\sec(t)\, r) &= \cos(t)\, dt \end{aligned}$$

Integrating gives

$$\begin{aligned} \sec(t)\, r &= \int \cos(t)\, dt \\ \sec(t)\, r &= \sin(t) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(t)$ results in

$$r = \cos(t)\sin(t) + c_1 \cos(t)$$

which simplifies to

$$r = \cos(t)(\sin(t) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $t = \frac{\pi}{4}$ and $r = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{2} + \frac{\sqrt{2}\, c_1}{2}$$

$$c_1 = \frac{\sqrt{2}}{2}$$

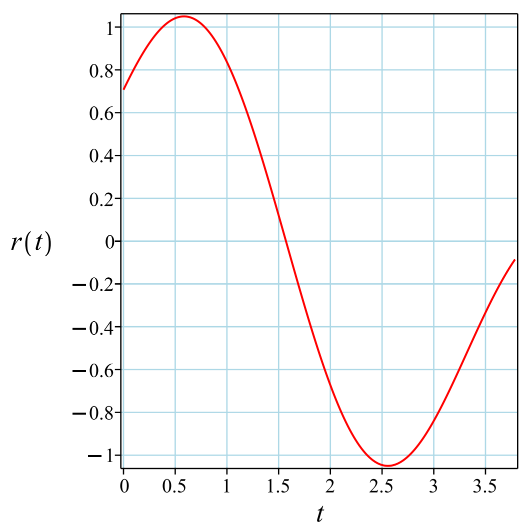
Substituting c_1 found above in the general solution gives

$$r = \cos(t)\sin(t) + \frac{\cos(t)\sqrt{2}}{2}$$

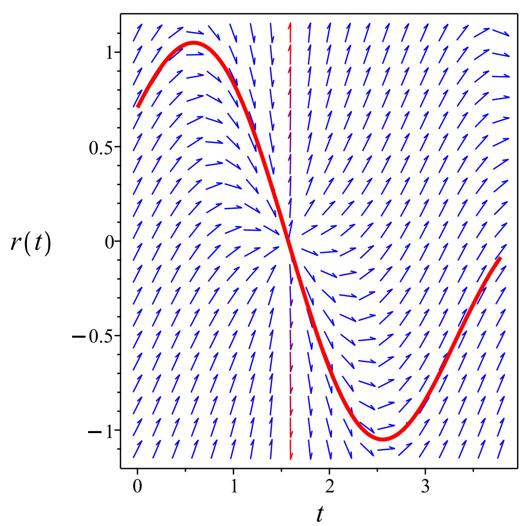
Summary

The solution(s) found are the following

$$r = \cos(t)\sin(t) + \frac{\cos(t)\sqrt{2}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$r = \cos(t)\sin(t) + \frac{\cos(t)\sqrt{2}}{2}$$

Verified OK.

5.23.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} r' &= -r\tan(t) + \cos(t)^2 \\ r' &= \omega(t, r) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_r - \xi_t) - \omega^2\xi_r - \omega_t\xi - \omega_r\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 174: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,r) &= 0 \\
\eta(t,r) &= \cos(t)
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,r) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{dr}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial r}\right) S(t,r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\cos(t)} dy \end{aligned}$$

Which results in

$$S = \frac{r}{\cos(t)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,r) S_r}{R_t + \omega(t,r) R_r} \tag{2}$$

Where in the above R_t, R_r, S_t, S_r are all partial derivatives and $\omega(t,r)$ is the right hand side of the original ode given by

$$\omega(t,r) = -r\tan(t) + \cos(t)^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_r &= 0 \\ S_t &= \tan(t)\sec(t)\, r \\ S_r &= \sec(t) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to t, r coordinates. This results in

$$\sec(t)\, r = \sin(t) + c_1$$

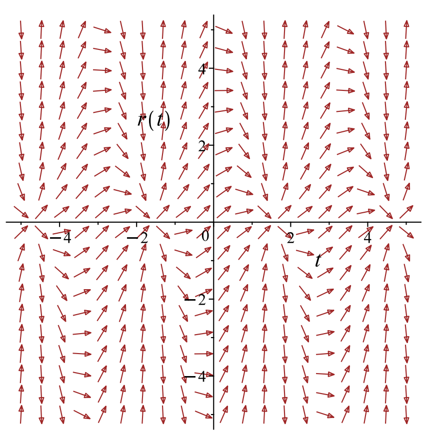
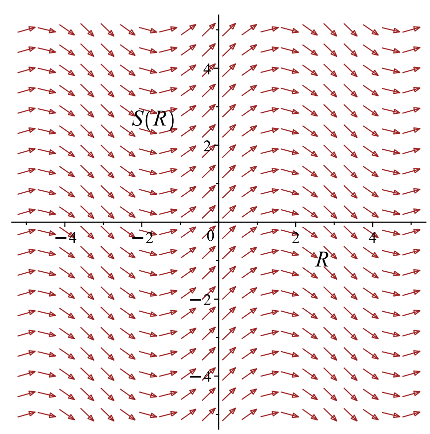
Which simplifies to

$$\sec(t)\, r = \sin(t) + c_1$$

Which gives

$$r = \frac{\sin(t) + c_1}{\sec(t)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, r coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dr}{dt} = -r\tan(t) + \cos(t)^2$ 	$R = t$ $S = \sec(t)\, r$	$\frac{dS}{dR} = \cos(R)$ 

Initial conditions are used to solve for c_1. Substituting $t = \frac{\pi}{4}$ and $r = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{2} + \frac{\sqrt{2}\, c_1}{2}$$

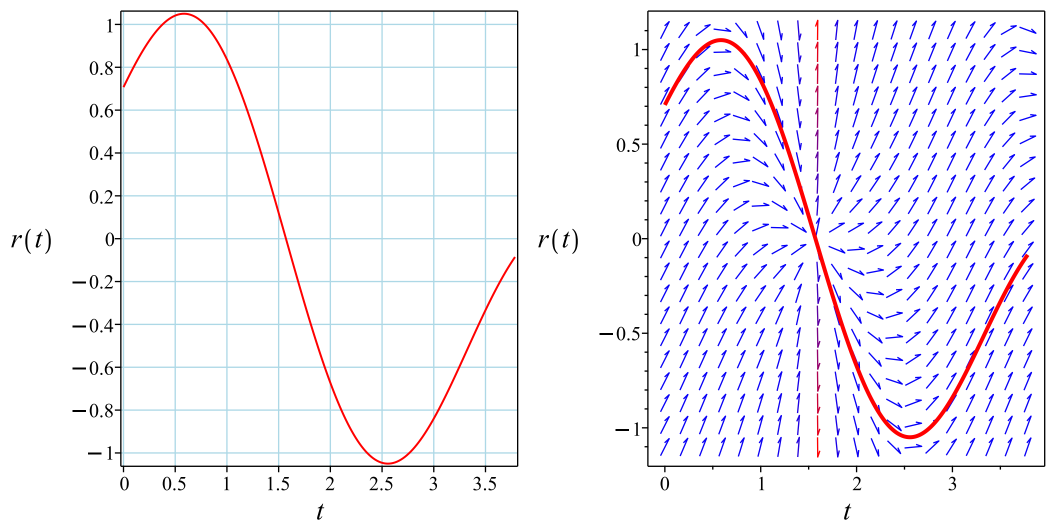
$$c_1 = \frac{\sqrt{2}}{2}$$

Substituting c_1 found above in the general solution gives

$$r = \cos(t)\sin(t) + \frac{\cos(t)\sqrt{2}}{2}$$

Summary
The solution(s) found are the following

$$r = \cos(t)\sin(t) + \frac{\cos(t)\sqrt{2}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$r = \cos(t)\sin(t) + \frac{\cos(t)\sqrt{2}}{2}$$

Verified OK.

5.23.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,r)\,\mathrm{d}t + N(t,r)\,\mathrm{d}r = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}r &= \left(-r\tan(t) + \cos(t)^2\right)\mathrm{d}t\\ \left(r\tan(t) - \cos(t)^2\right)\mathrm{d}t + \mathrm{d}r &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,r) &= r\tan(t) - \cos(t)^2\\ N(t,r) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r}=\frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial r}&=\frac{\partial}{\partial r}\left(r\tan(t)-\cos(t)^2\right)\\&=\tan(t)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t}&=\frac{\partial}{\partial t}(1)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial r}\neq\frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial r}-\frac{\partial N}{\partial t}\right)\\&=1((\tan(t))-(0))\\&=\tan(t)\end{aligned}$$

Since A does not depend on r, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu&=e^{\int A\,\mathrm{d}t}\\&=e^{\int\tan(t)\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu&=e^{-\ln(\cos(t))}\\&=\sec(t)\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M}&=\mu M\\&=\sec(t)\left(r\tan(t)-\cos(t)^2\right)\\&=-\cos(t)+\tan(t)\sec(t)\,r\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \sec(t)(1) \\
&= \sec(t)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}r}{\mathrm{d}t} &= 0 \\
(-\cos(t) + \tan(t)\sec(t)\, r) + (\sec(t))\frac{\mathrm{d}r}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, r)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial r} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\cos(t) + \tan(t)\sec(t)\, r\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = \sec(t)\, r - \sin(t) + f(r) \tag{3}
$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both t and r. Taking derivative of equation (3) w.r.t r gives

$$
\frac{\partial \phi}{\partial r} = \sec(t) + f'(r) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = \sec(t)$. Therefore equation (4) becomes

$$
\sec(t) = \sec(t) + f'(r) \tag{5}
$$

Solving equation (5) for $f'(r)$ gives

$$
f'(r) = 0
$$

Therefore

$$f(r) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(r)$ into equation (3) gives ϕ

$$\phi = \sec(t) r - \sin(t) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sec(t) r - \sin(t)$$

The solution becomes

$$r = \frac{\sin(t) + c_1}{\sec(t)}$$

Initial conditions are used to solve for c_1. Substituting $t = \frac{\pi}{4}$ and $r = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{2} + \frac{\sqrt{2}\, c_1}{2}$$

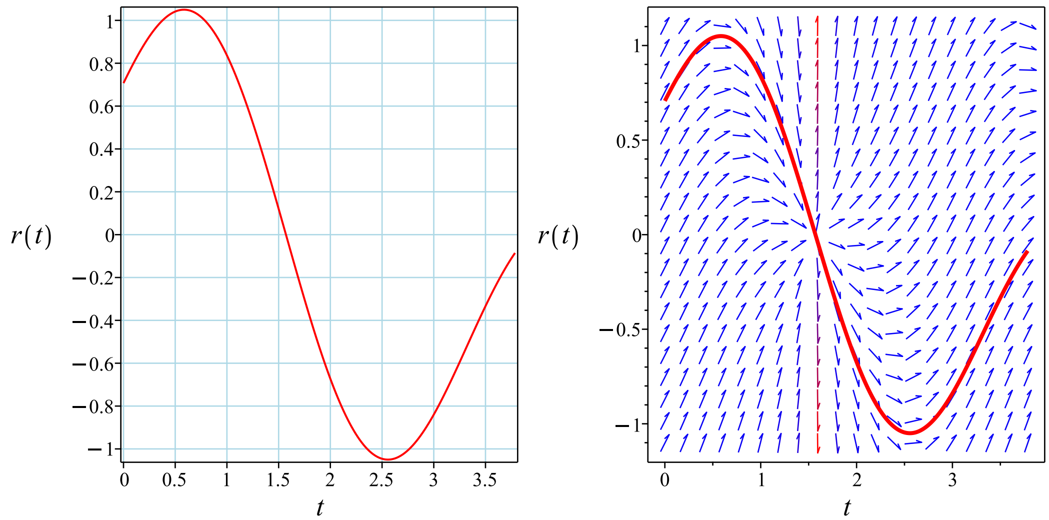
$$c_1 = \frac{\sqrt{2}}{2}$$

Substituting c_1 found above in the general solution gives

$$r = \cos(t) \sin(t) + \frac{\cos(t) \sqrt{2}}{2}$$

Summary
The solution(s) found are the following

$$r = \cos(t) \sin(t) + \frac{\cos(t) \sqrt{2}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$r = \cos(t)\sin(t) + \frac{\cos(t)\sqrt{2}}{2}$$

Verified OK.

5.23.5 Maple step by step solution

Let's solve

$\left[r' + r\tan(t) = \cos(t)^2, r\left(\frac{\pi}{4}\right) = 1\right]$

- Highest derivative means the order of the ODE is 1

  r'

- Isolate the derivative

  $r' = -r\tan(t) + \cos(t)^2$

- Group terms with r on the lhs of the ODE and the rest on the rhs of the ODE

  $r' + r\tan(t) = \cos(t)^2$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(r' + r\tan(t)\right) = \mu(t)\cos(t)^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\, r)$

  $\mu(t)\left(r' + r\tan(t)\right) = \mu'(t)\, r + \mu(t)\, r'$

- Isolate $\mu'(t)$

  $\mu'(t) = \mu(t) \tan(t)$
- Solve to find the integrating factor

  $\mu(t) = \frac{1}{\cos(t)}$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t) r)\right) dt = \int \mu(t) \cos(t)^2 dt + c_1$
- Evaluate the integral on the lhs

  $\mu(t) r = \int \mu(t) \cos(t)^2 dt + c_1$
- Solve for r

  $r = \frac{\int \mu(t) \cos(t)^2 dt + c_1}{\mu(t)}$
- Substitute $\mu(t) = \frac{1}{\cos(t)}$

  $r = \cos(t) \left(\int \cos(t) dt + c_1\right)$
- Evaluate the integrals on the rhs

  $r = \cos(t) (\sin(t) + c_1)$
- Use initial condition $r\left(\frac{\pi}{4}\right) = 1$

  $1 = \frac{\sqrt{2}\left(\frac{\sqrt{2}}{2} + c_1\right)}{2}$
- Solve for c_1

  $c_1 = \frac{\sqrt{2}}{2}$
- Substitute $c_1 = \frac{\sqrt{2}}{2}$ into general solution and simplify

  $r = \frac{\left(2\sin(t) + \sqrt{2}\right)\cos(t)}{2}$
- Solution to the IVP

  $r = \frac{\left(2\sin(t) + \sqrt{2}\right)\cos(t)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([diff(r(t),t)+r(t)*tan(t)=cos(t)^2,r(1/4*Pi) = 1],r(t), singsol=all)
```

$$r(t) = \frac{(2\sin(t) + \sqrt{2})\cos(t)}{2}$$

✓ Solution by Mathematica

Time used: 0.077 (sec). Leaf size: 16

```
DSolve[{r'[t]+r[t]*Tan[t]==Cos[t]^2,{r[Pi/4]==1}},r[t],t,IncludeSingularSolutions -> True]
```

$$r(t) \to \left(\sin(t) + \frac{1}{\sqrt{2}}\right)\cos(t)$$

5.24 problem 24

Internal problem ID [11661]
Internal file name [`OUTPUT/11670_Wednesday_April_10_2024_04_54_05_PM_5880669/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{-x + x' = \sin(2t)}$$

With initial conditions

$$[x(0) = 0]$$

5.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x' + p(t)x = q(t)$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= \sin(2t) \end{aligned}$$

Hence the ode is

$$-x + x' = \sin(2t)$$

The domain of $p(t) = -1$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \sin(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

5.24.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-1)dt} \\ &= e^{-t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu x) &= (\mu)(\sin(2t)) \\ \frac{d}{dt}(e^{-t}x) &= (e^{-t})(\sin(2t)) \\ d(e^{-t}x) &= (e^{-t}\sin(2t))\, dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-t}x &= \int e^{-t}\sin(2t)\, dt \\ e^{-t}x &= -\frac{2e^{-t}\cos(2t)}{5} - \frac{e^{-t}\sin(2t)}{5} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-t}$ results in

$$x = e^{t}\left(-\frac{2e^{-t}\cos(2t)}{5} - \frac{e^{-t}\sin(2t)}{5}\right) + c_1 e^{t}$$

which simplifies to

$$x = c_1 e^{t} - \frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 - \frac{2}{5}$$

$$c_1 = \frac{2}{5}$$

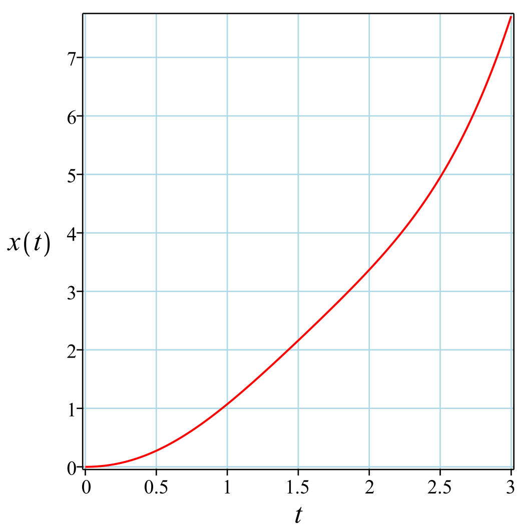
Substituting c_1 found above in the general solution gives

$$x = \frac{2\,\mathrm{e}^{t}}{5} - \frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5}$$

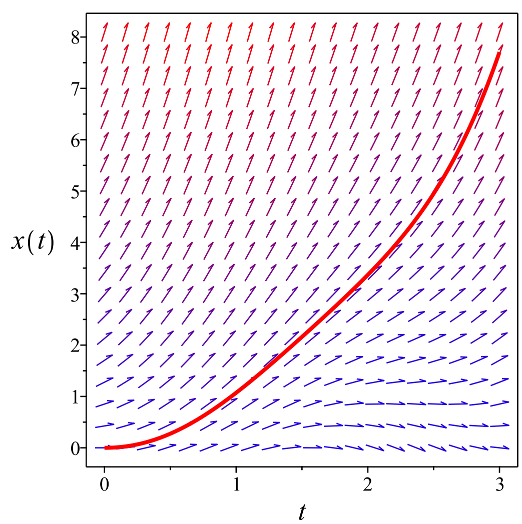
Summary

The solution(s) found are the following

$$x = \frac{2\,\mathrm{e}^{t}}{5} - \frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{2\,\mathrm{e}^{t}}{5} - \frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5}$$

Verified OK.

5.24.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
x' &= x + \sin(2t) \\
x' &= \omega(t, x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 177: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(t,x) = 0$$
$$\eta(t,x) = e^t \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^t} dy \end{aligned}$$

Which results in

$$S = e^{-t} x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x) S_x}{R_t + \omega(t,x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = x + \sin(2t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_x &= 0 \\ S_t &= -e^{-t} x \\ S_x &= e^{-t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{-t} \sin(2t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{-R} \sin(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 - \frac{\mathrm{e}^{-R}(2\cos(2R) + \sin(2R))}{5} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, x coordinates. This results in

$$\mathrm{e}^{-t} x = -\frac{(\sin(2t) + 2\cos(2t))\,\mathrm{e}^{-t}}{5} + c_1$$

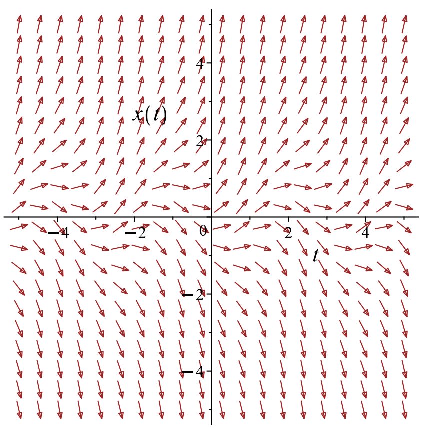
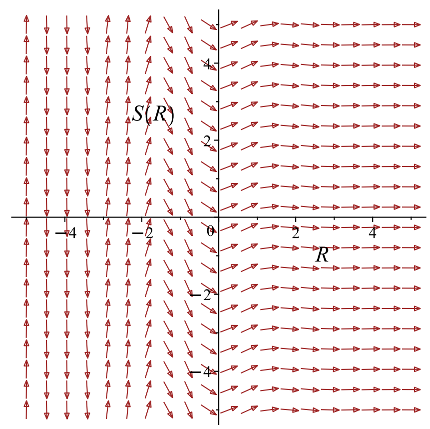
Which simplifies to

$$\mathrm{e}^{-t} x = -\frac{(\sin(2t) + 2\cos(2t))\,\mathrm{e}^{-t}}{5} + c_1$$

Which gives

$$x = -\frac{\mathrm{e}^{t}(\mathrm{e}^{-t}\sin(2t) + 2\,\mathrm{e}^{-t}\cos(2t) - 5c_1)}{5}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in t, x coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dx}{dt} = x + \sin(2t)$
</td><td>$R = t$
$S = e^{-t} x$</td><td>$\frac{dS}{dR} = e^{-R} \sin(2R)$
</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 - \frac{2}{5}$$

$$c_1 = \frac{2}{5}$$

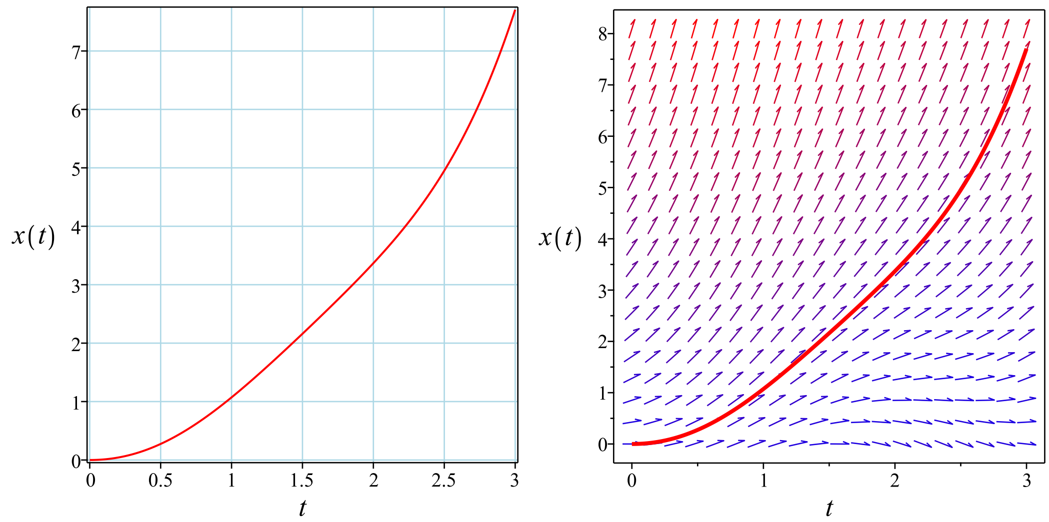
Substituting c_1 found above in the general solution gives

$$x = \frac{2\,e^t}{5} - \frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5}$$

Summary

The solution(s) found are the following

$$x = \frac{2\,e^t}{5} - \frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \frac{2\,e^{t}}{5} - \frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5}$$

Verified OK.

5.24.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}x &= (x + \sin(2t))\,\mathrm{d}t\\ (-x - \sin(2t))\,\mathrm{d}t + \mathrm{d}x &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, x) &= -x - \sin(2t)\\ N(t, x) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}(-x - \sin(2t))\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right)\\ &= 1((-1) - (0))\\ &= -1\end{aligned}$$

Since A does not depend on x, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, dt} \\
&= e^{\int -1 \, dt}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-t} \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-t}(-x - \sin(2t)) \\
&= -\mathrm{e}^{-t}(x + \sin(2t))
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-t}(1) \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\
\left(-\mathrm{e}^{-t}(x + \sin(2t))\right) + \left(\mathrm{e}^{-t}\right)\frac{\mathrm{d}x}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, x)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int \overline{M} \, \mathrm{d}t \\
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t &= \int -\mathrm{e}^{-t}(x + \sin(2t)) \, \mathrm{d}t
\end{aligned}
$$

$$
\phi = \frac{(5x + 2\cos(2t) + \sin(2t))\,\mathrm{e}^{-t}}{5} + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \mathrm{e}^{-t} + f'(x) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \mathrm{e}^{-t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-t} = \mathrm{e}^{-t} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{(5x + 2\cos(2t) + \sin(2t))\,\mathrm{e}^{-t}}{5} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(5x + 2\cos(2t) + \sin(2t))\,\mathrm{e}^{-t}}{5}$$

The solution becomes

$$x = -\frac{\mathrm{e}^{t}(\mathrm{e}^{-t}\sin(2t) + 2\,\mathrm{e}^{-t}\cos(2t) - 5c_1)}{5}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $x = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 - \frac{2}{5}$$

$$c_1 = \frac{2}{5}$$

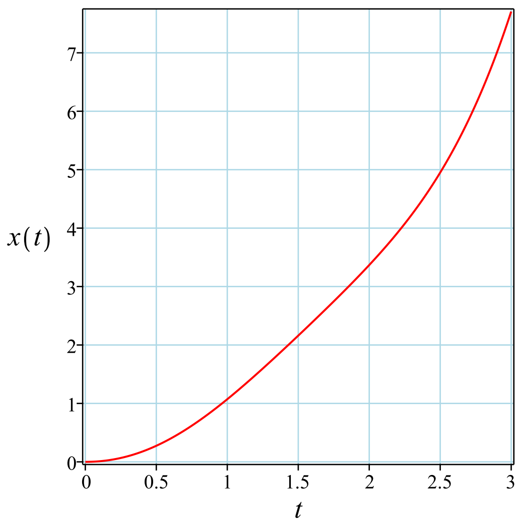
Substituting c_1 found above in the general solution gives

$$x = \frac{2\,e^{t}}{5} - \frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5}$$

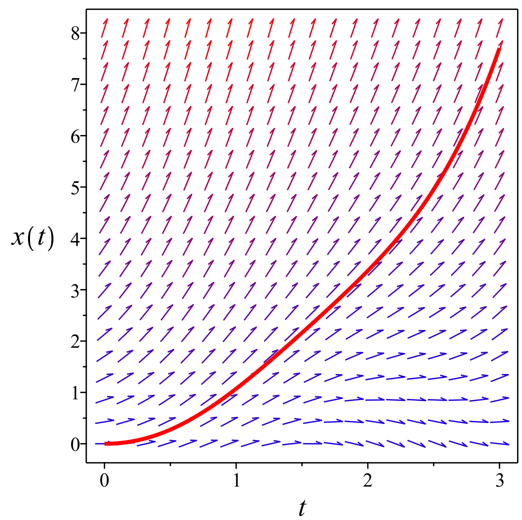
Summary

The solution(s) found are the following

$$x = \frac{2\,e^{t}}{5} - \frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{2\,e^{t}}{5} - \frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5}$$

Verified OK.

5.24.5 Maple step by step solution

Let's solve

$[-x + x' = \sin(2t), x(0) = 0]$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

$x' = x + \sin(2t)$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $-x + x' = \sin(2t)$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)(-x + x') = \mu(t)\sin(2t)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,x)$

  $\mu(t)(-x + x') = \mu'(t)\,x + \mu(t)\,x'$

- Isolate $\mu'(t)$

  $\mu'(t) = -\mu(t)$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{-t}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\,x)\right) dt = \int \mu(t)\sin(2t)\,dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\,x = \int \mu(t)\sin(2t)\,dt + c_1$

- Solve for x

  $x = \frac{\int \mu(t)\sin(2t)dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{-t}$

  $x = \frac{\int \mathrm{e}^{-t}\sin(2t)dt + c_1}{\mathrm{e}^{-t}}$

- Evaluate the integrals on the rhs

  $x = \frac{-\frac{\mathrm{e}^{-t}\sin(2t)}{5} - \frac{2\,\mathrm{e}^{-t}\cos(2t)}{5} + c_1}{\mathrm{e}^{-t}}$

- Simplify

  $x = c_1\mathrm{e}^{t} - \frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5}$

- Use initial condition $x(0) = 0$

  $0 = c_1 - \frac{2}{5}$

- Solve for c_1

  $c_1 = \frac{2}{5}$

- Substitute $c_1 = \frac{2}{5}$ into general solution and simplify

  $x = \frac{2\,\mathrm{e}^{t}}{5} - \frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5}$

- Solution to the IVP

  $x = \frac{2\,e^{t}}{5} - \frac{\sin(2t)}{5} - \frac{2\cos(2t)}{5}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([diff(x(t),t)-x(t)=sin(2*t),x(0) = 0],x(t), singsol=all)
```

$$x(t) = -\frac{2\cos(2t)}{5} - \frac{\sin(2t)}{5} + \frac{2\,e^{t}}{5}$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 27

```
DSolve[{x'[t]-x[t]==Sin[2*t],{x[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{5}\left(2e^{t} - \sin(2t) - 2\cos(2t)\right)$$

5.25 problem 25

Internal problem ID [11662]
Internal file name [`OUTPUT/11671_Wednesday_April_10_2024_04_54_06_PM_54278466/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y' + \frac{y}{2x} - \frac{x}{y^3} = 0}$$

With initial conditions

$$[y(1) = 2]$$

5.25.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{y^4 - 2x^2}{2x\, y^3} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 2$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y^4 - 2x^2}{2x\,y^3}\right)\\ &= -\frac{2}{x} + \frac{\frac{3y^4}{2} - 3x^2}{x\,y^4}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

5.25.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{y^4 - 2x^2}{2x\,y^3}\\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 180: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{y^3x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y^3 x^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^4 x^2}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^4 - 2x^2}{2x\, y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x\, y^4}{2} \\ S_y &= y^3 x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^4}{4} + c_1 \tag{4}$$

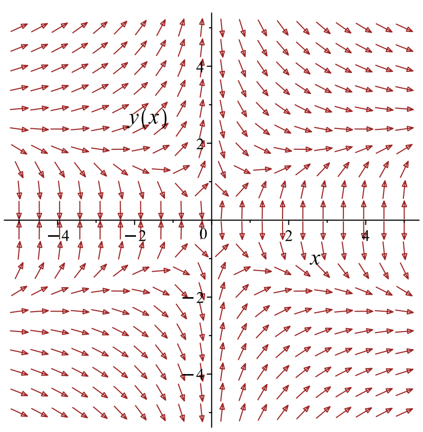
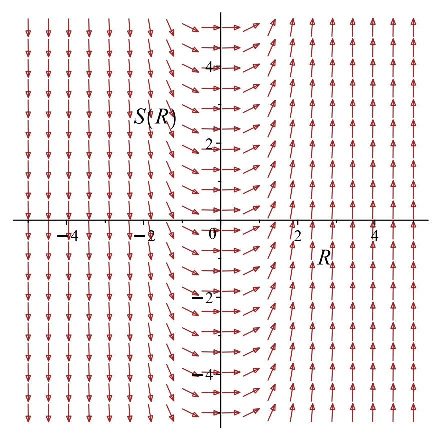
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^4 x^2}{4} = \frac{x^4}{4} + c_1$$

Which simplifies to

$$\frac{y^4 x^2}{4} = \frac{x^4}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^4-2x^2}{2x\,y^3}$ 	$R = x$ $S = \frac{y^4 x^2}{4}$	$\frac{dS}{dR} = R^3$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$4 = \frac{1}{4} + c_1$$

$$c_1 = \frac{15}{4}$$

Substituting c_1 found above in the general solution gives

$$\frac{y^4 x^2}{4} = \frac{x^4}{4} + \frac{15}{4}$$

Summary

The solution(s) found are the following

$$\frac{y^4 x^2}{4} = \frac{x^4}{4} + \frac{15}{4} \tag{1}$$

Verification of solutions

$$\frac{y^4 x^2}{4} = \frac{x^4}{4} + \frac{15}{4}$$

Verified OK.

5.25.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^4 - 2x^2}{2x\, y^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{2x} y + x \frac{1}{y^3} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x) y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{2x} \\ f_1(x) &= x \\ n &= -3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^3}$ gives

$$y'y^3 = -\frac{y^4}{2x} + x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^4 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 4y^3y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{4} &= -\frac{w(x)}{2x} + x \\ w' &= -\frac{2w}{x} + 4x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= 4x \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = 4x$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{2}{x} dx} \\ &= x^2 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(4x) \\ \frac{\mathrm{d}}{\mathrm{d}x}(x^2 w) &= (x^2)(4x) \\ \mathrm{d}(x^2 w) &= (4x^3)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$x^2 w = \int 4x^3 \, dx$$
$$x^2 w = x^4 + c_1$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = x^2 + \frac{c_1}{x^2}$$

Replacing w in the above by y^4 using equation (5) gives the final solution.

$$y^4 = x^2 + \frac{c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$16 = 1 + c_1$$

$$c_1 = 15$$

Substituting c_1 found above in the general solution gives

$$y^4 = \frac{x^4 + 15}{x^2}$$

The above simplifies to

$$y^4 x^2 - x^4 - 15 = 0$$

Summary
The solution(s) found are the following

$$y^4 x^2 - x^4 - 15 = 0 \tag{1}$$

Verification of solutions

$$y^4 x^2 - x^4 - 15 = 0$$

Verified OK.

5.25.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(2x\,y^3\right)\mathrm{d}y&=\left(-y^4+2x^2\right)\mathrm{d}x\\ \left(y^4-2x^2\right)\mathrm{d}x+\left(2x\,y^3\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y^4-2x^2\\ N(x,y)&=2x\,y^3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^4 - 2x^2\right)\\ &= 4y^3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2x\,y^3\right)\\ &= 2y^3\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2x\,y^3}\left(\left(4y^3\right) - \left(2y^3\right)\right)\\ &= \frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int \frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)}\\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= x\left(y^4 - 2x^2\right)\\ &= x\,y^4 - 2x^3\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= x(2x\,y^3)\\ &= 2y^3x^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(x\,y^4 - 2x^3\right) + \left(2y^3x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int x\,y^4 - 2x^3\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{\left(-y^4 + 2x^2\right)^2}{8} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \left(-y^4 + 2x^2\right)y^3 + f'(y) \tag{4}$$

$$= -y^7 + 2y^3x^2 + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y^3x^2$. Therefore equation (4) becomes

$$2y^3x^2 = -y^7 + 2y^3x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^7$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y^7)\,\mathrm{d}y$$

$$f(y) = \frac{y^8}{8} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{(-y^4 + 2x^2)^2}{8} + \frac{y^8}{8} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{(-y^4 + 2x^2)^2}{8} + \frac{y^8}{8}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{15}{2} = c_1$$

$$c_1 = \frac{15}{2}$$

Substituting c_1 found above in the general solution gives

$$-\frac{(-y^4 + 2x^2)^2}{8} + \frac{y^8}{8} = \frac{15}{2}$$

Summary
The solution(s) found are the following

$$\frac{y^4x^2}{2} - \frac{x^4}{2} = \frac{15}{2} \tag{1}$$

Verification of solutions

$$\frac{y^4x^2}{2} - \frac{x^4}{2} = \frac{15}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)+y(x)/(2*x)=x/y(x)^3,y(1) = 2],y(x), singsol=all)
```

$$y(x) = \sqrt{\frac{\sqrt{x^4+15}}{x}}$$

✓ Solution by Mathematica

Time used: 0.277 (sec). Leaf size: 20

```
DSolve[{y'[x]+y[x]/(2*x)==x/y[x]^3,{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[4]{x^4+15}}{\sqrt{x}}$$

5.26 problem 26

Internal problem ID [11663]
Internal file name [`OUTPUT/11672_Wednesday_April_10_2024_04_54_07_PM_3027235/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y'x + y - (yx)^{\frac{3}{2}} = 0}$$

With initial conditions

$$[y(1) = 4]$$

5.26.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{-y + (xy)^{\frac{3}{2}}}{x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 4$ is

$$\{0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\{0 \leq y\}$$

And the point $y_0 = 4$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-y + (xy)^{\frac{3}{2}}}{x}\right) \\ &= \frac{-1 + \frac{3x\sqrt{xy}}{2}}{x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 4$ is

$$\{0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{0 \leq y\}$$

And the point $y_0 = 4$ is inside this domain. Therefore solution exists and is unique.

5.26.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= \frac{-y + (xy)^{\frac{3}{2}}}{x} \\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{\left(-y + (xy)^{\frac{3}{2}}\right)(b_3 - a_2)}{x} - \frac{\left(-y + (xy)^{\frac{3}{2}}\right)^2 a_3}{x^2} \\
&- \left(\frac{3\sqrt{xy}\, y}{2x} - \frac{-y + (xy)^{\frac{3}{2}}}{x^2}\right)(x a_2 + y a_3 + a_1) \\
&- \frac{\left(-1 + \frac{3x\sqrt{xy}}{2}\right)(x b_2 + y b_3 + b_1)}{x} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{-2x^3y^3a_3 + 2(xy)^{\frac{3}{2}} x b_3 + 6(xy)^{\frac{3}{2}} y a_3 - 3\sqrt{xy}\, x^3 b_2 - 3\sqrt{xy}\, x^2 y a_2 - 3\sqrt{xy}\, x^2 y b_3 - 3\sqrt{xy}\, x\, y^2 a_3 + 2(xy)^{\frac{3}{2}} a}{2x^2}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
&-2x^3y^3a_3 + 2(xy)^{\frac{3}{2}} x b_3 + 6(xy)^{\frac{3}{2}} y a_3 - 3\sqrt{xy}\, x^3 b_2 - 3\sqrt{xy}\, x^2 y a_2 \\
&- 3\sqrt{xy}\, x^2 y b_3 - 3\sqrt{xy}\, x\, y^2 a_3 + 2(xy)^{\frac{3}{2}} a_1 - 3\sqrt{xy}\, x^2 b_1 \\
&- 3\sqrt{xy}\, x y a_1 + 4b_2 x^2 - 4y^2 a_3 + 2x b_1 - 2y a_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-2x^3y^3a_3 - 3\sqrt{xy}\, x^3 b_2 - 3\sqrt{xy}\, x^2 y a_2 - \sqrt{xy}\, x^2 y b_3 + 3\sqrt{xy}\, x\, y^2 a_3 \\
&- 3\sqrt{xy}\, x^2 b_1 - \sqrt{xy}\, x y a_1 + 4b_2 x^2 - 4y^2 a_3 + 2x b_1 - 2y a_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y, \sqrt{xy}\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2, \sqrt{xy} = v_3\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2v_1^3 v_2^3 a_3 - 3v_3 v_1^2 v_2 a_2 + 3v_3 v_1 v_2^2 a_3 - 3v_3 v_1^3 b_2 - v_3 v_1^2 v_2 b_3 \\
&- v_3 v_1 v_2 a_1 - 3v_3 v_1^2 b_1 - 4v_2^2 a_3 + 4b_2 v_1^2 - 2v_2 a_1 + 2v_1 b_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}&-2v_1^3v_2^3a_3 - 3v_3v_1^3b_2 + (-3a_2 - b_3)\, v_1^2v_2v_3 - 3v_3v_1^2b_1 + 4b_2v_1^2 \\&+ 3v_3v_1v_2^2a_3 - v_3v_1v_2a_1 + 2v_1b_1 - 4v_2^2a_3 - 2v_2a_1 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-2a_1 &= 0\\-a_1 &= 0\\-4a_3 &= 0\\-2a_3 &= 0\\3a_3 &= 0\\-3b_1 &= 0\\2b_1 &= 0\\-3b_2 &= 0\\4b_2 &= 0\\-3a_2 - b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\a_2 &= a_2\\a_3 &= 0\\b_1 &= 0\\b_2 &= 0\\b_3 &= -3a_2\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\\eta &= -3y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -3y - \left(\frac{-y+(xy)^{\frac{3}{2}}}{x}\right)(x) \\
&= -2y - (xy)^{\frac{3}{2}} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-2y-(xy)^{\frac{3}{2}}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln\left(x\sqrt{xy}+2\right)}{2} - \frac{\ln\left(x\sqrt{xy}-2\right)}{2} + \frac{\ln\left(y\,x^3-4\right)}{2} - \frac{\ln(y)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{-y+(xy)^{\frac{3}{2}}}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{3x^2 y - 6\sqrt{x}\,\sqrt{y}}{2y\,x^3 - 8} \\
S_y &= -\frac{x^{\frac{3}{2}} y - 2\sqrt{y}}{y^{\frac{3}{2}}\,(y\,x^3 - 4)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{3x^3 y^{\frac{3}{2}} - 2x^{\frac{5}{2}}\sqrt{xy}\,y + 4x\sqrt{xy}\,\sqrt{y} - 4x^{\frac{3}{2}} y - 4\sqrt{y}}{2\sqrt{y}\,\left(y\,x^3 - 4\right)x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{2}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -2\ln(R) + c_1 \tag{4}
$$

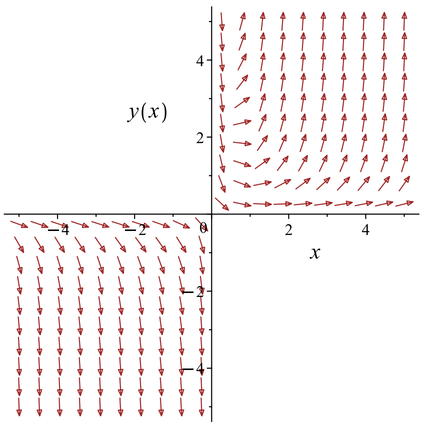
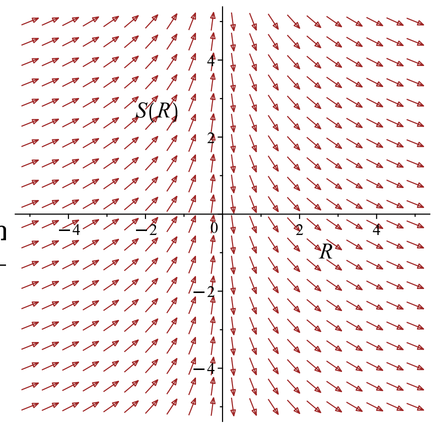
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln\left(\sqrt{y}\,x^{\frac{3}{2}} + 2\right)}{2} - \frac{\ln\left(\sqrt{y}\,x^{\frac{3}{2}} - 2\right)}{2} + \frac{\ln(yx^3 - 4)}{2} - \frac{\ln(y)}{2} = -2\ln(x) + c_1
$$

Which simplifies to

$$
\frac{\ln\left(\sqrt{y}\,x^{\frac{3}{2}} + 2\right)}{2} - \frac{\ln\left(\sqrt{y}\,x^{\frac{3}{2}} - 2\right)}{2} + \frac{\ln(yx^3 - 4)}{2} - \frac{\ln(y)}{2} = -2\ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y+(xy)^{\frac{3}{2}}}{x}$ 	$R = x$ $S = \frac{\ln\left(\sqrt{y}\,x^{\frac{3}{2}}+2\right)}{2} -$	$\frac{dS}{dR} = -\frac{2}{R}$ $-\ln$  $\frac{\ln(y)}{2}$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$\ln(2) = c_1$$

$$c_1 = \ln(2)$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln\left(\sqrt{y}\,x^{\frac{3}{2}}+2\right)}{2} - \frac{\ln\left(\sqrt{y}\,x^{\frac{3}{2}}-2\right)}{2} + \frac{\ln(y\,x^3-4)}{2} - \frac{\ln(y)}{2} = -2\ln(x) + \ln(2)$$

<u>Summary</u>

The solution(s) found are the following

$$\frac{\ln\left(\sqrt{y}\,x^{\frac{3}{2}}+2\right)}{2} - \frac{\ln\left(\sqrt{y}\,x^{\frac{3}{2}}-2\right)}{2} + \frac{\ln(yx^3-4)}{2} - \frac{\ln(y)}{2} = -2\ln(x) + \ln(2) \quad (1)$$

<u>Verification of solutions</u>

$$\frac{\ln\left(\sqrt{y}\,x^{\frac{3}{2}}+2\right)}{2} - \frac{\ln\left(\sqrt{y}\,x^{\frac{3}{2}}-2\right)}{2} + \frac{\ln(yx^3-4)}{2} - \frac{\ln(y)}{2} = -2\ln(x) + \ln(2)$$

Verified OK.

5.26.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(-y+(xy)^{\frac{3}{2}}\right)\mathrm{d}x\\ \left(y-(xy)^{\frac{3}{2}}\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y-(xy)^{\frac{3}{2}}\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(y-(xy)^{\frac{3}{2}}\right)\\&=1-\frac{3x\sqrt{xy}}{2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(x)\\&=1\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\&=\frac{1}{x}\left(\left(1-\frac{3x\sqrt{xy}}{2}\right)-(1)\right)\\&=-\frac{3\sqrt{xy}}{2}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B&=\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\\&=-\frac{1}{y\left(x\sqrt{xy}-1\right)}\left((1)-\left(1-\frac{3x\sqrt{xy}}{2}\right)\right)\\&=-\frac{3x\sqrt{xy}}{2y\left(x\sqrt{xy}-1\right)}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R=\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{xM-yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - \left(1 - \frac{3x\sqrt{xy}}{2}\right)}{x\left(y - (xy)^{\frac{3}{2}}\right) - y\,(x)} \\
&= -\frac{3}{2xy}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{3}{2t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \, \mathrm{d}t} \\
&= e^{\int \left(-\frac{3}{2t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{3\ln(t)}{2}} \\
&= \frac{1}{t^{\frac{3}{2}}}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{(xy)^{\frac{3}{2}}}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{(xy)^{\frac{3}{2}}}\left(y - (xy)^{\frac{3}{2}}\right) \\
&= -\frac{x\sqrt{xy} - 1}{x\sqrt{xy}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{(xy)^{\frac{3}{2}}}(x) \\
&= \frac{1}{y\sqrt{xy}}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-\frac{x\sqrt{xy} - 1}{x\sqrt{xy}}\right) + \left(\frac{1}{y\sqrt{xy}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{x\sqrt{xy} - 1}{x\sqrt{xy}}\,\mathrm{d}x$$
$$\phi = -x - \frac{2}{\sqrt{xy}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{(xy)^{\frac{3}{2}}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y\sqrt{xy}}$. Therefore equation (4) becomes

$$\frac{1}{y\sqrt{xy}} = \frac{1}{y\sqrt{xy}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \frac{2}{\sqrt{xy}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \frac{2}{\sqrt{xy}}$$

The solution becomes

$$y = \frac{4}{x\left(c_1^2 + 2c_1 x + x^2\right)}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = \frac{4}{c_1^2 + 2c_1 + 1}$$

$$c_1 = -2$$

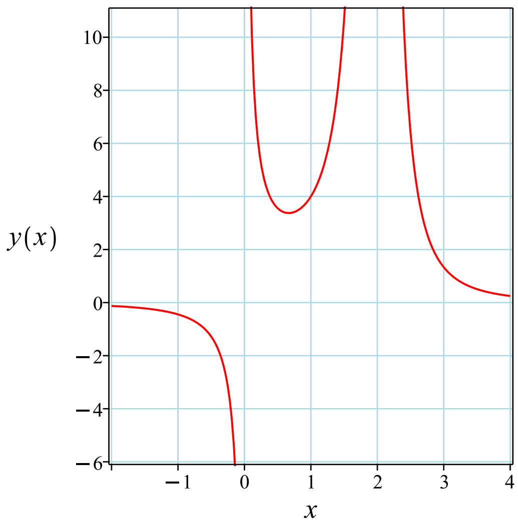
Substituting c_1 found above in the general solution gives

$$y = \frac{4}{x\left(x^2 - 4x + 4\right)}$$

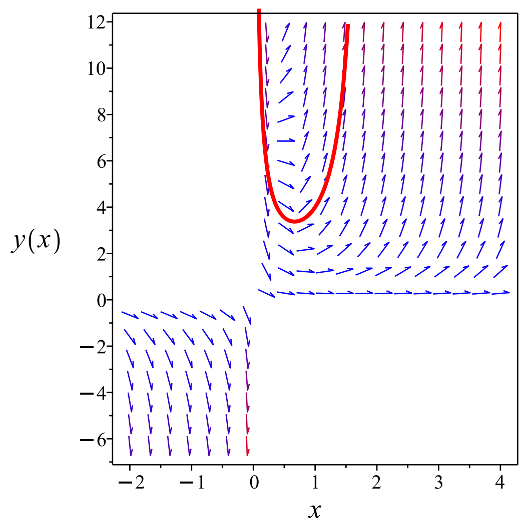
Summary

The solution(s) found are the following

$$y = \frac{4}{x\left(x^2 - 4x + 4\right)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4}{x\,(x^2 - 4x + 4)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 9

```
dsolve([x*diff(y(x),x)+y(x)=(x*y(x))^(3/2),y(1) = 4],y(x), singsol=all)
```

$$y(x) = \frac{4}{x^3}$$

✓ Solution by Mathematica

Time used: 0.258 (sec). Leaf size: 24

```
DSolve[{x*y'[x]+y[x]==(x*y[x])^(3/2),{y[1]==4}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4}{x^3}$$
$$y(x) \to \frac{4}{(x-2)^2 x}$$

5.27 problem 27

Internal problem ID [11664]
Internal file name [`OUTPUT/11673_Wednesday_April_10_2024_04_54_24_PM_95791151/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = \begin{cases} 2 & 0 \leq x < 1 \\ 0 & 1 \leq x \end{cases}}$$

With initial conditions

$$[y(0) = 0]$$

5.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= \begin{cases} 0 & x<0 \\ 2 & x<1 \\ 0 & 1\le x \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y'+y = \begin{cases} 0 & x<0 \\ 2 & x<1 \\ 0 & 1\le x \end{cases}
$$

The domain of $p(x) = 1$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \begin{cases} 0 & x<0 \\ 2 & x<1 \\ 0 & 1\le x \end{cases}$ is

$$
\{0 \le x \le 1, 1 \le x \le \infty, -\infty \le x \le 0\}
$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

5.27.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 1dx} \\
&= e^{x}
\end{aligned}
$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\left\{\begin{array}{cc}0 & x<0 \\ 2 & x<1 \\ 0 & 1\le x\end{array}\right.\right)$$

$$\frac{d}{dx}(e^{x} y) = (e^{x})\left(\left\{\begin{array}{cc}0 & x<0 \\ 2 & x<1 \\ 0 & 1\le x\end{array}\right.\right)$$

$$d(e^{x} y) = \left\{\begin{array}{cc}0 & x<0 \\ 2\,e^{x} & x<1 \\ 0 & 1\le x\end{array}\right. \; dx$$

Integrating gives

$$e^{x} y = \int \left\{\begin{array}{cc}0 & x<0 \\ 2\,e^{x} & x<1 \\ 0 & 1\le x\end{array}\right. \; dx$$

$$e^{x} y = \left\{\begin{array}{cc}0 & x\le 0 \\ 2\,e^{x}-2 & x\le 1 \\ 2\,e-2 & 1<x\end{array}\right. + c_1$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = e^{-x}\left(\left\{\begin{array}{cc}0 & x\le 0 \\ 2\,e^{x}-2 & x\le 1 \\ 2\,e-2 & 1<x\end{array}\right.\right) + c_1 e^{-x}$$

which simplifies to

$$y = e^{-x}\left(-2\left(\left\{\begin{array}{cc}0 & x\le 0 \\ 1-e^{x} & x\le 1 \\ 1-e & 1<x\end{array}\right.\right) + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

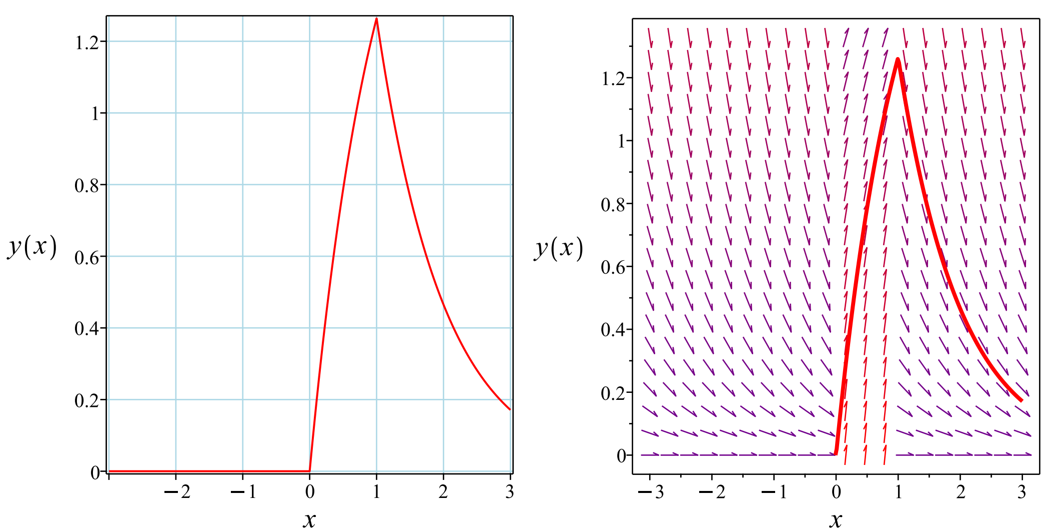
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \begin{cases} 0 & x \leq 0 \\ 2-2\,\mathrm{e}^{-x} & x \leq 1 \\ 2\,\mathrm{e}^{1-x}-2\,\mathrm{e}^{-x} & 1<x \\ 0 & \text{otherwise} \end{cases}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \begin{cases} 0 & x \leq 0 \\ 2-2\,\mathrm{e}^{-x} & x \leq 1 \\ 2\,\mathrm{e}^{1-x}-2\,\mathrm{e}^{-x} & 1<x \\ 0 & \text{otherwise} \end{cases} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \begin{cases} 0 & x \leq 0 \\ 2-2\,\mathrm{e}^{-x} & x \leq 1 \\ 2\,\mathrm{e}^{1-x}-2\,\mathrm{e}^{-x} & 1 < x \\ 0 & \text{otherwise} \end{cases}$$

Verified OK.

5.27.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y + \left(\begin{cases} 2 & 0 \leq x < 1 \\ 0 & 1 \leq x \end{cases}\right)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 182: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy \end{aligned}$$

Which results in

$$S = e^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y + \left(\begin{cases} 2 & 0 \leq x < 1 \\ 0 & 1 \leq x \end{cases} \right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{x} y \\ S_y &= e^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \begin{cases} 0 & x < 0 \\ 2\,e^{x} & x < 1 \\ 0 & 1 \leq x \end{cases} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \begin{cases} 0 & R < 0 \\ 2\,e^{R} & R < 1 \\ 0 & 1 \leq R \end{cases}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \begin{cases} c_1 & R < 0 \\ c_1 + 2\,\mathrm{e}^{R} - 2 & R < 1 \\ c_1 + 2\,\mathrm{e} - 2 & 1 \leq R \end{cases} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{x} = \begin{cases} c_1 & x < 0 \\ c_1 + 2\,\mathrm{e}^{x} - 2 & x < 1 \\ c_1 + 2\,\mathrm{e} - 2 & 1 \leq x \end{cases}$$

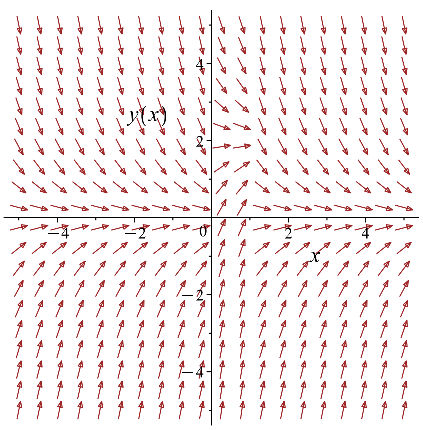
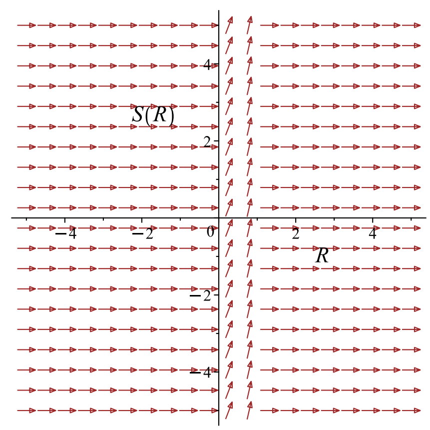
Which simplifies to

$$y\,\mathrm{e}^{x} = \begin{cases} c_1 & x < 0 \\ c_1 + 2\,\mathrm{e}^{x} - 2 & x < 1 \\ c_1 + 2\,\mathrm{e} - 2 & 1 \leq x \end{cases}$$

Which gives

$$y = \begin{cases} [c_1 \mathrm{e}^{-x}] & x < 0 \\ [2 + (-2 + c_1)\,\mathrm{e}^{-x}] & x < 1 \\ [\mathrm{e}^{-x}(c_1 + 2\,\mathrm{e} - 2)] & 1 \leq x \end{cases}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y + \left(\begin{cases} 2 & 0 \leq x < 1 \\ 0 & 1 \leq x \end{cases}\right)$ 	$R = x$ $S = e^{x} y$	$\frac{dS}{dR} = \begin{cases} 0 & R < 0 \\ 2\,e^{R} & R < 1 \\ 0 & 1 \leq R \end{cases}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = [c_1]$$

Unable to solve for constant of integration. Verification of solutions N/A

5.27.4 Maple step by step solution

Let's solve

$$\left[y' + y = \begin{cases} 2 & 0 \leq x < 1 \\ 0 & 1 \leq x \end{cases}, y(0) = 0\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = -y + \left\{\begin{array}{cc} 2 & 0 \le x < 1 \\ 0 & 1 \le x \end{array}\right.$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + y = \left\{\begin{array}{cc} 2 & 0 \le x < 1 \\ 0 & 1 \le x \end{array}\right.$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\,(y' + y) = \mu(x)\left(\left\{\begin{array}{cc} 2 & 0 \le x < 1 \\ 0 & 1 \le x \end{array}\right.\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\,(y' + y) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \mu(x)$$

- Solve to find the integrating factor

$$\mu(x) = \mathrm{e}^{x}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\left(\left\{\begin{array}{cc} 2 & 0 \le x < 1 \\ 0 & 1 \le x \end{array}\right.\right) dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \mu(x)\left(\left\{\begin{array}{cc} 2 & 0 \le x < 1 \\ 0 & 1 \le x \end{array}\right.\right) dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)\left(\left\{\begin{array}{cc} 2 & 0 \le x < 1 \\ 0 & 1 \le x \end{array}\right.\right) dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \mathrm{e}^{x}$

$$y = \frac{\int \mathrm{e}^{x}\left(\left\{\begin{array}{cc} 2 & 0 \le x < 1 \\ 0 & 1 \le x \end{array}\right.\right) dx + c_1}{\mathrm{e}^{x}}$$

- Evaluate the integrals on the rhs

$$y=\frac{\left\{\begin{array}{cc}0 & x\leq 0\\ 2\,\mathrm{e}^{x}-2 & x\leq 1\\ 2\,\mathrm{e}-2 & 1<x\end{array}\right.+c_1}{\mathrm{e}^{x}}$$

- Simplify

$$y=\mathrm{e}^{-x}\left(-2\left(\left\{\begin{array}{cc}0 & x\leq 0\\ 1-\mathrm{e}^{x} & x\leq 1\\ 1-\mathrm{e} & 1<x\end{array}\right.\right)+c_1\right)$$

- Use initial condition $y(0)=0$

$0=c_1$

- Solve for c_1

$c_1=0$

- Substitute $c_1=0$ into general solution and simplify

$$y=-2\,\mathrm{e}^{-x}\left(\left\{\begin{array}{cc}0 & x\leq 0\\ 1-\mathrm{e}^{x} & x\leq 1\\ 1-\mathrm{e} & 1<x\end{array}\right.\right)$$

- Solution to the IVP

$$y=-2\,\mathrm{e}^{-x}\left(\left\{\begin{array}{cc}0 & x\leq 0\\ 1-\mathrm{e}^{x} & x\leq 1\\ 1-\mathrm{e} & 1<x\end{array}\right.\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.328 (sec). Leaf size: 38

```
dsolve([diff(y(x),x)+y(x)=piecewise(0<=x and x<1,2,x>=1,0),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \begin{cases} 0 & x < 0 \\ 2 - 2\,\mathrm{e}^{-x} & x < 1 \\ 2\,\mathrm{e}^{1-x} - 2\,\mathrm{e}^{-x} & 1 \leq x \end{cases}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 38

```
DSolve[{y'[x]+y[x]==Piecewise[{{2,0<=x<1},{0,x>=1}}],{y[0]==0}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \{ \begin{array}{cc} 0 & x \leq 0 \\ 2 - 2e^{-x} & 0 < x \leq 1 \\ 2(-1+e)e^{-x} & \text{True} \end{array}$$

5.28 problem 28

Internal problem ID [11665]
Internal file name [`OUTPUT/11674_Wednesday_April_10_2024_04_54_25_PM_60313912/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = \begin{cases} 5 & 0 \leq x < 10 \\ 1 & 10 \leq x \end{cases}}$$

With initial conditions

$$[y(0) = 6]$$

5.28.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= \begin{cases} 0 & x < 0 \\ 5 & x < 10 \\ 1 & 10 \leq x \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y' + y = \begin{cases} 0 & x < 0 \\ 5 & x < 10 \\ 1 & 10 \leq x \end{cases}
$$

The domain of $p(x) = 1$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \begin{cases} 0 & x < 0 \\ 5 & x < 10 \\ 1 & 10 \leq x \end{cases}$ is

$$
\{0 \leq x \leq 10, 10 \leq x \leq \infty, -\infty \leq x \leq 0\}
$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

5.28.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 1 dx} \\
&= e^{x}
\end{aligned}
$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\begin{cases} 0 & x<0 \\ 5 & x<10 \\ 1 & 10\le x \end{cases}\right)$$

$$\frac{d}{dx}(e^x y) = (e^x)\left(\begin{cases} 0 & x<0 \\ 5 & x<10 \\ 1 & 10\le x \end{cases}\right)$$

$$d(e^x y) = \left(\left(\begin{cases} 0 & x<0 \\ 5 & x<10 \\ 1 & 10\le x \end{cases}\right) e^x\right) dx$$

Integrating gives

$$e^x y = \int \left(\begin{cases} 0 & x<0 \\ 5 & x<10 \\ 1 & 10\le x \end{cases}\right) e^x\, dx$$

$$e^x y = \begin{cases} 0 & x\le 0 \\ 5\,e^x - 5 & x\le 10 \\ e^x + 4\,e^{10} - 5 & 10<x \end{cases} + c_1$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(\begin{cases} 0 & x\le 0 \\ 5\,e^x - 5 & x\le 10 \\ e^x + 4\,e^{10} - 5 & 10<x \end{cases}\right) + c_1 e^{-x}$$

which simplifies to

$$y = e^{-x}\left(\left(\begin{cases} 0 & x\le 0 \\ 5\,e^x - 5 & x\le 10 \\ e^x + 4\,e^{10} - 5 & 10<x \end{cases}\right) + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 6$ in the above solution gives an equation to solve for the constant of integration.

$$6 = c_1$$

$$c_1 = 6$$

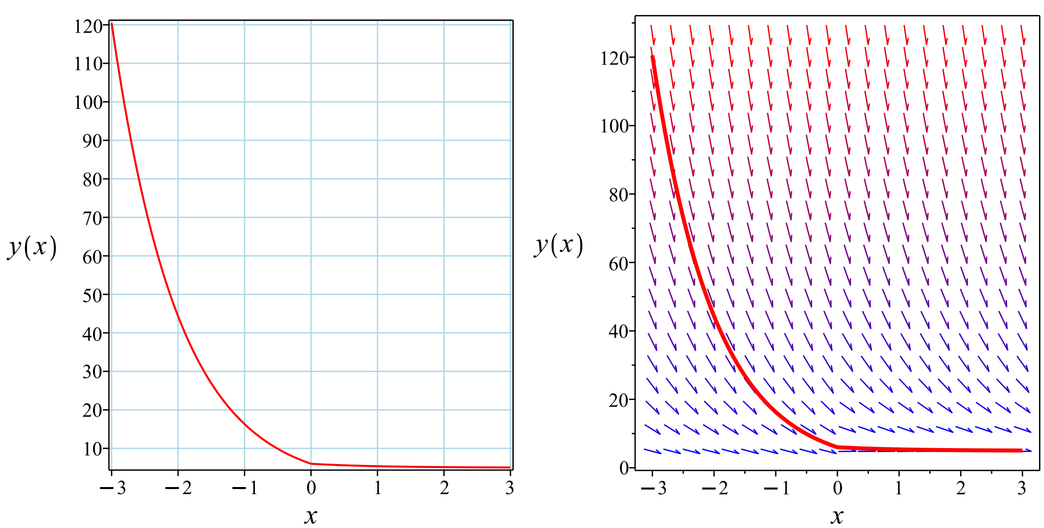
Substituting c_1 found above in the general solution gives

$$y = \begin{cases} 6\,\mathrm{e}^{-x} & x \le 0 \\ 5+\mathrm{e}^{-x} & x \le 10 \\ 4\,\mathrm{e}^{10-x}+1+\mathrm{e}^{-x} & 10 < x \\ 6\,\mathrm{e}^{-x} & \text{otherwise} \end{cases}$$

Summary

The solution(s) found are the following

$$y = \begin{cases} 6\,\mathrm{e}^{-x} & x \le 0 \\ 5+\mathrm{e}^{-x} & x \le 10 \\ 4\,\mathrm{e}^{10-x}+1+\mathrm{e}^{-x} & 10 < x \\ 6\,\mathrm{e}^{-x} & \text{otherwise} \end{cases} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = \begin{cases} 6\,\mathrm{e}^{-x} & x \le 0 \\ 5+\mathrm{e}^{-x} & x \le 10 \\ 4\,\mathrm{e}^{10-x}+1+\mathrm{e}^{-x} & 10 < x \\ 6\,\mathrm{e}^{-x} & \text{otherwise} \end{cases}
$$

Verified OK.

5.28.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -y + \left(\begin{cases} 5 & 0 \le x < 10 \\ 1 & 10 \le x \end{cases}\right) \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 185: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= e^{-x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy \end{aligned}$$

Which results in

$$S = e^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y + \left(\begin{cases} 5 & 0 \leq x < 10 \\ 1 & 10 \leq x \end{cases} \right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{x} y \\ S_y &= e^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \left(\begin{cases} 0 & x < 0 \\ 5 & x < 10 \\ 1 & 10 \leq x \end{cases} \right) e^{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \left(\begin{cases} 0 & R < 0 \\ 5 & R < 10 \\ 1 & 10 \leq R \end{cases} \right) e^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \begin{cases} c_1 & R < 0 \\ c_1 + 5\,\mathrm{e}^{R} - 5 & R < 10 \\ c_1 + \mathrm{e}^{R} + 4\,\mathrm{e}^{10} - 5 & 10 \leq R \end{cases} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{x} = \begin{cases} c_1 & x < 0 \\ c_1 + 5\,\mathrm{e}^{x} - 5 & x < 10 \\ c_1 + \mathrm{e}^{x} + 4\,\mathrm{e}^{10} - 5 & 10 \leq x \end{cases}$$

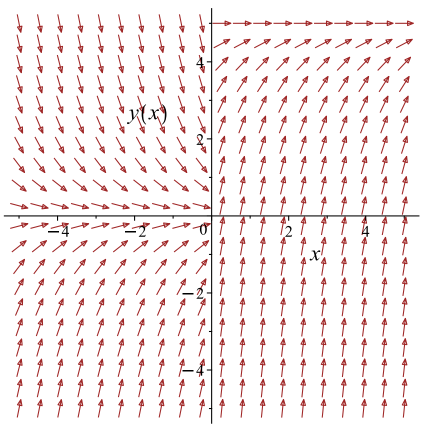
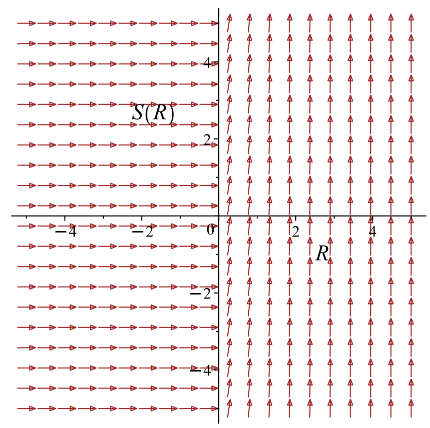
Which simplifies to

$$y\,\mathrm{e}^{x} = \begin{cases} c_1 & x < 0 \\ c_1 + 5\,\mathrm{e}^{x} - 5 & x < 10 \\ c_1 + \mathrm{e}^{x} + 4\,\mathrm{e}^{10} - 5 & 10 \leq x \end{cases}$$

Which gives

$$y = \begin{cases} [c_1 \mathrm{e}^{-x}] & x < 0 \\ [5 + (c_1 - 5)\,\mathrm{e}^{-x}] & x < 10 \\ [4\,\mathrm{e}^{10-x} + 1 + (c_1 - 5)\,\mathrm{e}^{-x}] & 10 \leq x \end{cases}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y + \left(\begin{cases} 5 & 0 \leq x < 10 \\ 1 & 10 \leq x \end{cases}\right)$ 	$R = x$ $S = e^{x} y$	$\frac{dS}{dR} = \left(\begin{cases} 0 & R < 0 \\ 5 & R < 10 \\ 1 & 10 \leq R \end{cases}\right) e^{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 6$ in the above solution gives an equation to solve for the constant of integration.

$$6 = [c_1]$$

Unable to solve for constant of integration. Verification of solutions N/A

5.28.4 Maple step by step solution

Let's solve

$$\left[y' + y = \begin{cases} 5 & 0 \leq x < 10 \\ 1 & 10 \leq x \end{cases}, y(0) = 6\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = -y + \left\{\begin{array}{cc} 5 & 0 \le x < 10 \\ 1 & 10 \le x \end{array}\right.$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + y = \left\{\begin{array}{cc} 5 & 0 \le x < 10 \\ 1 & 10 \le x \end{array}\right.$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + y\right) = \mu(x)\left(\left\{\begin{array}{cc} 5 & 0 \le x < 10 \\ 1 & 10 \le x \end{array}\right.\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x)\left(y' + y\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \mu(x)$$

- Solve to find the integrating factor

$$\mu(x) = \mathrm{e}^{x}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\left(\left\{\begin{array}{cc} 5 & 0 \le x < 10 \\ 1 & 10 \le x \end{array}\right.\right) dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \mu(x)\left(\left\{\begin{array}{cc} 5 & 0 \le x < 10 \\ 1 & 10 \le x \end{array}\right.\right) dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)\left(\left\{\begin{array}{cc} 5 & 0 \le x < 10 \\ 1 & 10 \le x \end{array}\right.\right) dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \mathrm{e}^{x}$

$$y = \frac{\int \mathrm{e}^{x}\left(\left\{\begin{array}{cc} 5 & 0 \le x < 10 \\ 1 & 10 \le x \end{array}\right.\right) dx + c_1}{\mathrm{e}^{x}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\left\{\begin{array}{cc} 0 & x \leq 0 \\ 5\,\mathrm{e}^{x}-5 & x \leq 10 \\ \mathrm{e}^{x}+4\,\mathrm{e}^{10}-5 & 10<x \end{array}\right. + c_1}{\mathrm{e}^{x}}$$

- Simplify

$$y = \mathrm{e}^{-x}\left(\left\{\begin{array}{cc} 0 & x \leq 0 \\ 5\,\mathrm{e}^{x}-5 & x \leq 10 \\ \mathrm{e}^{x}+4\,\mathrm{e}^{10}-5 & 10<x \end{array}\right. + c_1\right)$$

- Use initial condition $y(0) = 6$

$$6 = c_1$$

- Solve for c_1

$$c_1 = 6$$

- Substitute $c_1 = 6$ into general solution and simplify

$$y = \mathrm{e}^{-x}\left(\left\{\begin{array}{cc} 6 & x \leq 0 \\ 1+5\,\mathrm{e}^{x} & x \leq 10 \\ 1+\mathrm{e}^{x}+4\,\mathrm{e}^{10} & 10<x \end{array}\right.\right)$$

- Solution to the IVP

$$y = \mathrm{e}^{-x}\left(\left\{\begin{array}{cc} 6 & x \leq 0 \\ 1+5\,\mathrm{e}^{x} & x \leq 10 \\ 1+\mathrm{e}^{x}+4\,\mathrm{e}^{10} & 10<x \end{array}\right.\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 7.219 (sec). Leaf size: 40

```
dsolve([diff(y(x),x)+y(x)=piecewise(0<=x and x<10,5,x>=10,1),y(0) = 6],y(x), singsol=all)
```

$$y(x)=\begin{cases} 6\,\mathrm{e}^{-x} & x<0 \\ \mathrm{e}^{-x}+5 & x<10 \\ \mathrm{e}^{-x}+1+4\,\mathrm{e}^{10-x} & 10\leq x \end{cases}$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 45

```
DSolve[{y'[x]+y[x]==Piecewise[{{5,0<=x<10},{1,x>=10}}],{y[0]==6}},y[x],x,IncludeSingularSolu
```

$$y(x)\to \left\{ \begin{array}{cc} 6e^{-x} & x\leq 0 \\ e^{-x}(1+4e^{10}+e^{x}) & x>10 \\ 5+e^{-x} & \text{True} \end{array} \right.$$

5.29 problem 29

Internal problem ID [11666]
Internal file name [`OUTPUT/11675_Wednesday_April_10_2024_04_54_26_PM_30453963/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = \begin{cases} \mathrm{e}^{-x} & 0 \leq x < 2 \\ \mathrm{e}^{-2} & 2 \leq x \end{cases}}$$

With initial conditions

$$[y(0) = 1]$$

5.29.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= \begin{cases} 0 & x < 0 \\ \mathrm{e}^{-x} & x < 2 \\ \mathrm{e}^{-2} & 2 \leq x \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y' + y = \begin{cases} 0 & x < 0 \\ \mathrm{e}^{-x} & x < 2 \\ \mathrm{e}^{-2} & 2 \leq x \end{cases}
$$

The domain of $p(x) = 1$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \begin{cases} 0 & x < 0 \\ \mathrm{e}^{-x} & x < 2 \\ \mathrm{e}^{-2} & 2 \leq x \end{cases}$ is

$$
\{x < 0 \lor 0 < x\}
$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

5.29.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1 dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\left(\begin{cases} 0 & x<0 \\ e^{-x} & x<2 \\ e^{-2} & 2\le x \end{cases}\right)\right)$$

$$\frac{d}{dx}(e^{x} y) = (e^{x})\left(\left(\begin{cases} 0 & x<0 \\ e^{-x} & x<2 \\ e^{-2} & 2\le x \end{cases}\right)\right)$$

$$d(e^{x} y) = \begin{cases} 0 & x<0 \\ 1 & x<2 \\ e^{x-2} & 2\le x \end{cases}\, dx$$

Integrating gives

$$e^{x} y = \int \begin{cases} 0 & x<0 \\ 1 & x<2 \\ e^{x-2} & 2\le x \end{cases} dx$$

$$e^{x} y = \begin{cases} 0 & x\le 0 \\ x & x\le 2 \\ e^{x-2}+1 & 2<x \end{cases} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = e^{-x}\left(\begin{cases} 0 & x\le 0 \\ x & x\le 2 \\ e^{x-2}+1 & 2<x \end{cases}\right) + c_1 e^{-x}$$

which simplifies to

$$y = e^{-x}\left(\left(\begin{cases} 0 & x\le 0 \\ x & x\le 2 \\ e^{x-2}+1 & 2<x \end{cases}\right) + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

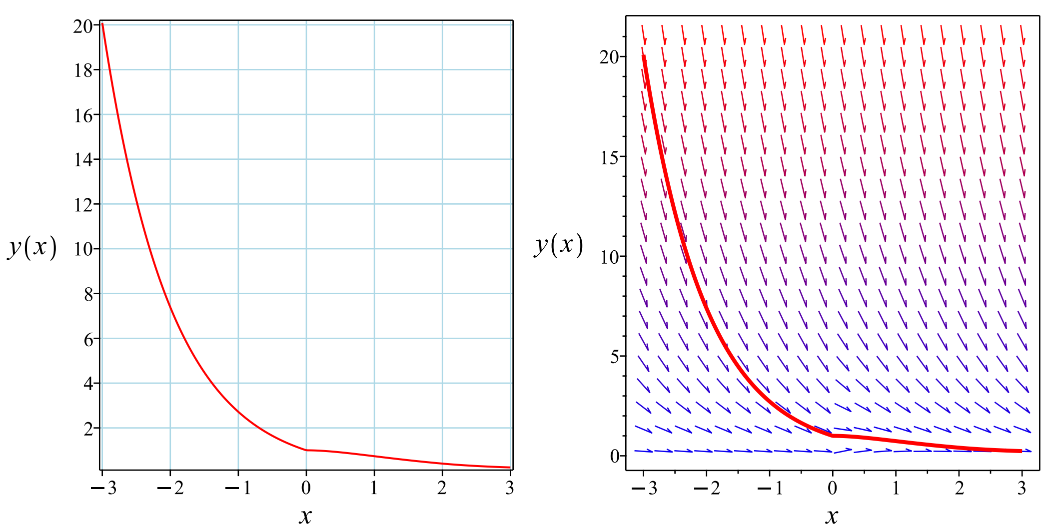
Substituting c_1 found above in the general solution gives

$$y = \begin{cases} e^{-x} & x \le 0 \\ x\,e^{-x} + e^{-x} & x \le 2 \\ e^{-2} + 2\,e^{-x} & 2 < x \\ e^{-x} & \text{otherwise} \end{cases}$$

Summary

The solution(s) found are the following

$$y = \begin{cases} e^{-x} & x \le 0 \\ x\,e^{-x} + e^{-x} & x \le 2 \\ e^{-2} + 2\,e^{-x} & 2 < x \\ e^{-x} & \text{otherwise} \end{cases} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$
y = \begin{cases} e^{-x} & x \le 0 \\ x\,e^{-x} + e^{-x} & x \le 2 \\ e^{-2} + 2\,e^{-x} & 2 < x \\ e^{-x} & \text{otherwise} \end{cases}
$$

Verified OK.

5.29.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -y + \left(\begin{cases} e^{-x} & 0 \le x < 2 \\ e^{-2} & 2 \le x \end{cases}\right) \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 188: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy \end{aligned}$$

Which results in

$$S = e^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y + \left(\begin{cases} e^{-x} & 0 \leq x < 2 \\ e^{-2} & 2 \leq x \end{cases} \right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{x} y \\ S_y &= e^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \begin{cases} 0 & x < 0 \\ 1 & x < 2 \\ e^{x-2} & 2 \leq x \end{cases} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \begin{cases} 0 & R < 0 \\ 1 & R < 2 \\ e^{R-2} & 2 \leq R \end{cases}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \begin{cases} c_1 & R < 0 \\ c_1 + R & R < 2 \\ c_1 + e^{R-2} + 1 & 2 \leq R \end{cases} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,e^{x} = \begin{cases} c_1 & x < 0 \\ x + c_1 & x < 2 \\ c_1 + e^{x-2} + 1 & 2 \leq x \end{cases}$$

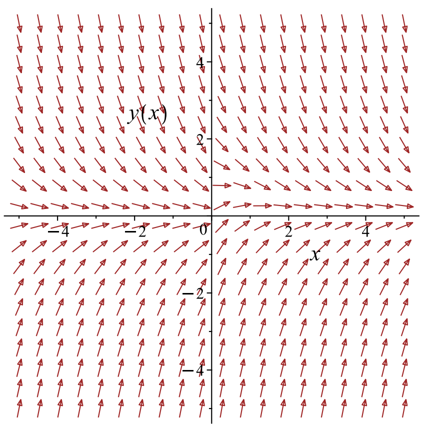
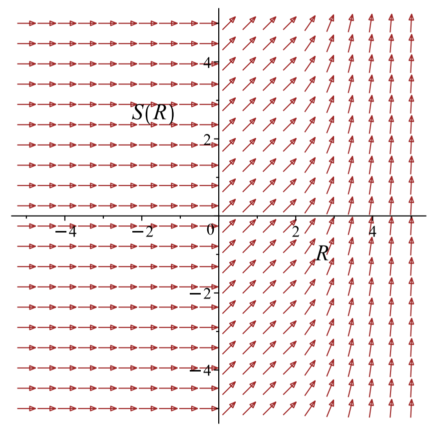
Which simplifies to

$$y\,e^{x} = \begin{cases} c_1 & x < 0 \\ x + c_1 & x < 2 \\ c_1 + e^{x-2} + 1 & 2 \leq x \end{cases}$$

Which gives

$$y = \begin{cases} [c_1 e^{-x}] & x < 0 \\ [e^{-x}(x + c_1)] & x < 2 \\ [e^{-x}(c_1 + e^{x-2} + 1)] & 2 \leq x \end{cases}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y + \left(\begin{cases} e^{-x} & 0 \leq x < 2 \\ e^{-2} & 2 \leq x \end{cases} \right)$ 	$R = x$ $S = e^{x} y$	$\frac{dS}{dR} = \begin{cases} 0 & R < 0 \\ 1 & R < 2 \\ e^{R-2} & 2 \leq R \end{cases}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = [c_1]$$

Unable to solve for constant of integration. Verification of solutions N/A

5.29.4 Maple step by step solution

Let's solve

$$\left[y' + y = \begin{cases} e^{-x} & 0 \leq x < 2 \\ e^{-2} & 2 \leq x \end{cases}, y(0) = 1 \right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = -y + \left\{\begin{array}{cc} \mathrm{e}^{-x} & 0 \le x < 2 \\ \mathrm{e}^{-2} & 2 \le x \end{array}\right.$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + y = \left\{\begin{array}{cc} \mathrm{e}^{-x} & 0 \le x < 2 \\ \mathrm{e}^{-2} & 2 \le x \end{array}\right.$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\,(y' + y) = \mu(x) \left(\left\{\begin{array}{cc} \mathrm{e}^{-x} & 0 \le x < 2 \\ \mathrm{e}^{-2} & 2 \le x \end{array}\right.\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\,(y' + y) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \mu(x)$$

- Solve to find the integrating factor

$$\mu(x) = \mathrm{e}^{x}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x) \left(\left\{\begin{array}{cc} \mathrm{e}^{-x} & 0 \le x < 2 \\ \mathrm{e}^{-2} & 2 \le x \end{array}\right.\right) dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \mu(x) \left(\left\{\begin{array}{cc} \mathrm{e}^{-x} & 0 \le x < 2 \\ \mathrm{e}^{-2} & 2 \le x \end{array}\right.\right) dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x) \left(\left\{\begin{array}{cc} \mathrm{e}^{-x} & 0 \le x < 2 \\ \mathrm{e}^{-2} & 2 \le x \end{array}\right.\right) dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \mathrm{e}^{x}$

$$y = \frac{\int \mathrm{e}^{x} \left(\left\{\begin{array}{cc} \mathrm{e}^{-x} & 0 \le x < 2 \\ \mathrm{e}^{-2} & 2 \le x \end{array}\right.\right) dx + c_1}{\mathrm{e}^{x}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\left\{\begin{array}{cc} 0 & x \leq 0 \\ x & x \leq 2 \\ \mathrm{e}^{x-2}+1 & 2<x \end{array}\right. + c_1}{\mathrm{e}^{x}}$$

- Simplify

$$y = \mathrm{e}^{-x}\left(\left\{\begin{array}{cc} 0 & x \leq 0 \\ x & x \leq 2 \\ \mathrm{e}^{x-2}+1 & 2<x \end{array}\right. + c_1\right)$$

- Use initial condition $y(0) = 1$

$$1 = c_1$$

- Solve for c_1

$$c_1 = 1$$

- Substitute $c_1 = 1$ into general solution and simplify

$$y = \mathrm{e}^{-x}\left(\left\{\begin{array}{cc} 0 & x \leq 0 \\ x & x \leq 2 \\ \mathrm{e}^{x-2}+1 & 2<x \end{array}\right. + 1\right)$$

- Solution to the IVP

$$y = \mathrm{e}^{-x}\left(\left\{\begin{array}{cc} 0 & x \leq 0 \\ x & x \leq 2 \\ \mathrm{e}^{x-2}+1 & 2<x \end{array}\right. + 1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.359 (sec). Leaf size: 35

```
dsolve([diff(y(x),x)+y(x)=piecewise(0<=x and x<2,exp(-x),x>=2,exp(-2)),y(0) = 1],y(x), sings
```

$$y(x)=\begin{cases} e^{-x} & x<0 \\ e^{-x}(1+x) & x<2 \\ 2\,e^{-x}+e^{-2} & 2\leq x \end{cases}$$

✓ Solution by Mathematica

Time used: 0.103 (sec). Leaf size: 40

```
DSolve[{y'[x]+y[x]==Piecewise[{{Exp[-x],0<=x<2},{Exp[-2],x>=2}}],{y[0]==1}},y[x],x,IncludeSi
```

$$y(x)\to \{ \begin{array}{cc} e^{-x} & x\leq 0 \\ \frac{1}{e^2}+2e^{-x} & x>2 \\ e^{-x}(x+1) & \text{True} \end{array}$$

5.30 problem 30

Internal problem ID [11667]
Internal file name [`OUTPUT/11676_Wednesday_April_10_2024_04_54_27_PM_23925784/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$(x+2)\,y'+y = \begin{cases} 2x & 0\le x<2 \\ 4 & 2\le x \end{cases}$$

With initial conditions

$$[y(0)=4]$$

5.30.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'+p(x)y=q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x+2} \\
q(x) &= \frac{2\left(\left\{\begin{array}{cc} 0 & x<0 \\ x & x<2 \\ 2 & 2\leq x \end{array}\right.\right)}{x+2}
\end{aligned}
$$

Hence the ode is

$$
y' + \frac{y}{x+2} = \frac{2\left(\left\{\begin{array}{cc} 0 & x<0 \\ x & x<2 \\ 2 & 2\leq x \end{array}\right.\right)}{x+2}
$$

The domain of $p(x) = \frac{1}{x+2}$ is

$$
\{x < -2 \lor -2 < x\}
$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2\left(\left\{\begin{array}{cc} 0 & x<0 \\ x & x<2 \\ 2 & 2\leq x \end{array}\right.\right)}{x+2}$ is

$$
\{x < -2 \lor -2 < x\}
$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

5.30.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x+2}dx} \\
&= x+2
\end{aligned}
$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{2\left(\left(\begin{cases} 0 & x < 0 \\ x & x < 2 \\ 2 & 2 \leq x \end{cases}\right)\right)}{x+2}\right)$$

$$\frac{d}{dx}((x+2)\,y) = (x+2)\left(\frac{2\left(\left(\begin{cases} 0 & x < 0 \\ x & x < 2 \\ 2 & 2 \leq x \end{cases}\right)\right)}{x+2}\right)$$

$$d((x+2)\,y) = \left(2\left(\left(\begin{cases} 0 & x < 0 \\ x & x < 2 \\ 2 & 2 \leq x \end{cases}\right)\right)\right) dx$$

Integrating gives

$$(x+2)\,y = \int 2\left(\begin{cases} 0 & x < 0 \\ x & x < 2 \\ 2 & 2 \leq x \end{cases}\right) dx$$

$$(x+2)\,y = 2\left(\left(\begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & x \leq 2 \\ 2x-2 & 2 < x \end{cases}\right)\right) + c_1$$

Dividing both sides by the integrating factor $\mu = x + 2$ results in

$$y = \frac{2\left(\left(\begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & x \leq 2 \\ 2x-2 & 2 < x \end{cases}\right)\right)}{x+2} + \frac{c_1}{x+2}$$

which simplifies to

$$y = \frac{\left(\begin{cases} 0 & x \le 0 \\ x^2 & x \le 2 \\ 4x - 4 & 2 < x \end{cases}\right) + c_1}{x + 2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = \frac{c_1}{2}$$

$$c_1 = 8$$

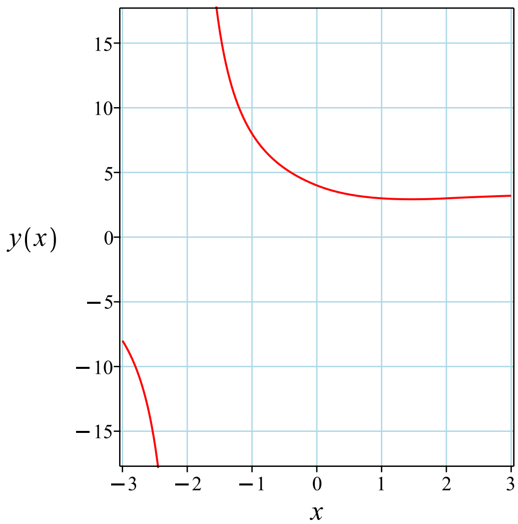
Substituting c_1 found above in the general solution gives

$$y = \begin{cases} \frac{8}{x+2} & x \le 0 \\ \frac{x^2+8}{x+2} & x \le 2 \\ \frac{4+4x}{x+2} & 2 < x \\ \frac{8}{x+2} & \text{otherwise} \end{cases}$$

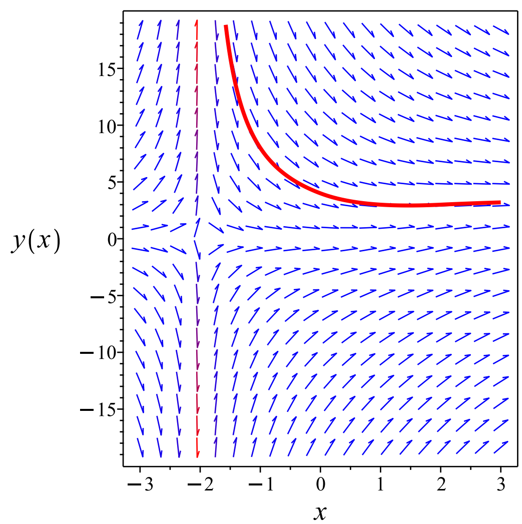
Summary

The solution(s) found are the following

$$y = \begin{cases} \frac{8}{x+2} & x \le 0 \\ \frac{x^2+8}{x+2} & x \le 2 \\ \frac{4+4x}{x+2} & 2 < x \\ \frac{8}{x+2} & \text{otherwise} \end{cases} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \begin{cases} \frac{8}{x+2} & x \leq 0 \\ \frac{x^2+8}{x+2} & x \leq 2 \\ \frac{4+4x}{x+2} & 2 < x \\ \frac{8}{x+2} & \text{otherwise} \end{cases}$$

Verified OK.

5.30.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y - \left(\begin{cases} 2x & 0 \leq x < 2 \\ 4 & 2 \leq x \end{cases}\right)}{x+2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 191: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x + 2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x+2}} dy \end{aligned}$$

Which results in

$$S = (x + 2)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y - \left(\begin{cases} 2x & 0 \le x < 2 \\ 4 & 2 \le x \end{cases}\right)}{x + 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x + 2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2\left(\begin{cases} 0 & x < 0 \\ x & x < 2 \\ 2 & 2 \le x \end{cases}\right) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2\left(\begin{cases} 0 & R < 0 \\ R & R < 2 \\ 2 & 2 \leq R \end{cases}\right)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \begin{cases} c_1 & R < 0 \\ R^2 + c_1 & R < 2 \\ c_1 - 4 + 4R & 2 \leq R \end{cases} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x + 2)\, y = \begin{cases} c_1 & x < 0 \\ x^2 + c_1 & x < 2 \\ c_1 + 4x - 4 & 2 \leq x \end{cases}$$

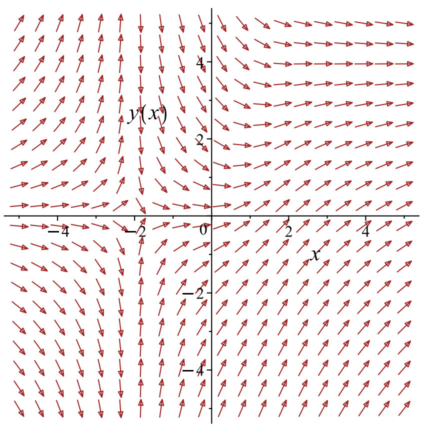
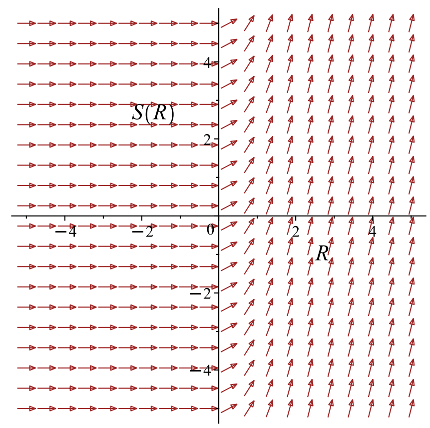
Which simplifies to

$$(x + 2)\, y = \begin{cases} c_1 & x < 0 \\ x^2 + c_1 & x < 2 \\ c_1 + 4x - 4 & 2 \leq x \end{cases}$$

Which gives

$$y = \frac{\begin{cases} c_1 & x < 0 \\ x^2 + c_1 & x < 2 \\ c_1 + 4x - 4 & 2 \leq x \end{cases}}{x + 2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y - \left(\begin{cases} 2x & 0 \le x < 2 \\ 4 & 2 \le x \end{cases}\right)}{x+2}$ 	$R = x$ $S = (x+2)\, y$	$\frac{dS}{dR} = 2\left(\begin{cases} 0 & R < 0 \\ R & R < 2 \\ 2 & 2 \le R \end{cases}\right)$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = \frac{c_1}{2}$$

$$c_1 = 8$$

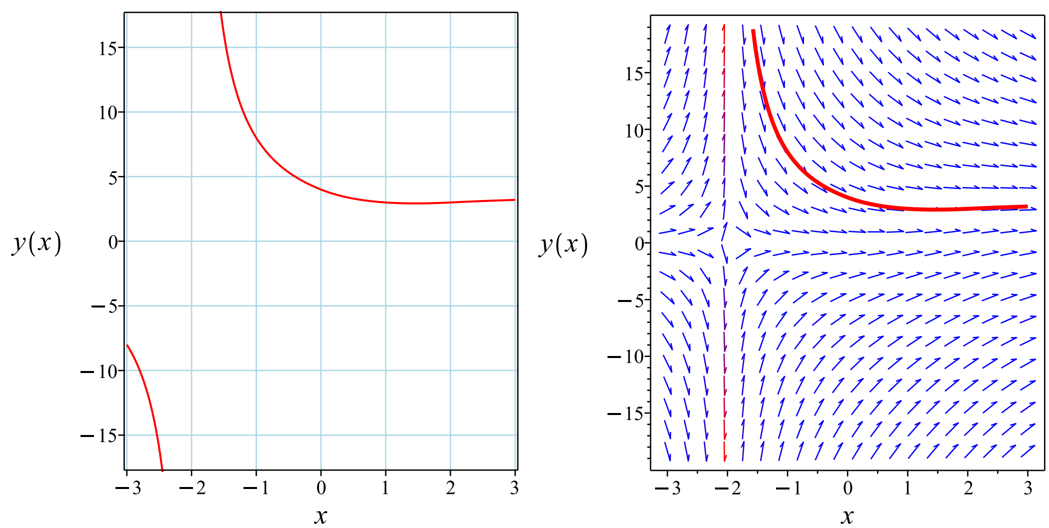
Substituting c_1 found above in the general solution gives

$$y = \begin{cases} \frac{8}{x+2} & x < 0 \\ \frac{x^2+8}{x+2} & x < 2 \\ \frac{4+4x}{x+2} & 2 \le x \\ 0 & \text{otherwise} \end{cases}$$

Summary

The solution(s) found are the following

$$y = \begin{cases} \frac{8}{x+2} & x < 0 \\ \frac{x^2+8}{x+2} & x < 2 \\ \frac{4+4x}{x+2} & 2 \leq x \\ 0 & \text{otherwise} \end{cases} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \begin{cases} \frac{8}{x+2} & x < 0 \\ \frac{x^2+8}{x+2} & x < 2 \\ \frac{4+4x}{x+2} & 2 \leq x \\ 0 & \text{otherwise} \end{cases}$$

Verified OK.

5.30.4 Maple step by step solution

Let's solve

$$\left[(x+2)\,y'+y = \left\{\begin{array}{cc} 2x & 0\le x<2 \\ 4 & 2\le x \end{array}\right., y(0)=4\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $$y' = -\frac{y}{x+2}+\frac{\left\{\begin{array}{cc} 2x & 0\le x<2 \\ 4 & 2\le x \end{array}\right.}{x+2}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $$y'+\frac{y}{x+2} = \frac{\left\{\begin{array}{cc} 2x & 0\le x<2 \\ 4 & 2\le x \end{array}\right.}{x+2}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $$\mu(x)\left(y'+\frac{y}{x+2}\right) = \frac{\mu(x)\left(\left\{\begin{array}{cc} 2x & 0\le x<2 \\ 4 & 2\le x \end{array}\right.\right)}{x+2}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{y}{x+2}\right) = \mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x+2}$

- Solve to find the integrating factor

  $\mu(x) = x+2$

- Integrate both sides with respect to x

  $$\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \frac{\mu(x)\left(\left\{\begin{array}{cc} 2x & 0\le x<2 \\ 4 & 2\le x \end{array}\right.\right)}{x+2}dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \frac{\mu(x) \left(\left\{\begin{array}{cc} 2x & 0\le x<2 \\ 4 & 2\le x \end{array}\right.\right)}{x+2} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x) \left(\left\{\begin{array}{cc} 2x & 0\le x<2 \\ 4 & 2\le x \end{array}\right.\right)}{x+2} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = x+2$

$$y = \frac{\int \left(\left\{\begin{array}{cc} 2x & 0\le x<2 \\ 4 & 2\le x \end{array}\right.\right) dx + c_1}{x+2}$$

- Evaluate the integrals on the rhs

$$y = \frac{\left\{\begin{array}{cc} 0 & x\le 0 \\ x^2 & x\le 2 \\ 4x-4 & 2<x \end{array}\right. + c_1}{x+2}$$

- Use initial condition $y(0) = 4$

$4 = \frac{c_1}{2}$

- Solve for c_1

$c_1 = 8$

- Substitute $c_1 = 8$ into general solution and simplify

$$y = \frac{\left\{\begin{array}{cc} 0 & x\le 0 \\ x^2 & x\le 2 \\ 4x-4 & 2<x \end{array}\right. + 8}{x+2}$$

- Solution to the IVP

$$y = \frac{\left\{\begin{array}{cc} 0 & x\le 0 \\ x^2 & x\le 2 \\ 4x-4 & 2<x \end{array}\right. + 8}{x+2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.265 (sec). Leaf size: 31

```
dsolve([(x+2)*diff(y(x),x)+y(x)=piecewise(0<=x and x<2,2*x,x>=2,4),y(0) = 4],y(x), singsol=a
```

$$y(x) = \frac{\begin{cases} 8 & x < 0 \\ x^2+8 & x < 2 \\ 4+4x & 2 \leq x \end{cases}}{x+2}$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 43

```
DSolve[{(x+2)*y'[x]+y[x]==Piecewise[{{2*x,0<=x<2},{4,x>=2}}],{y[0]==4}},y[x],x,IncludeSingul
```

$$y(x) \to \left\{ \begin{array}{ll} \frac{8}{x+2} & x \leq 0 \\ \frac{4(x+1)}{x+2} & x > 2 \\ \frac{x^2+8}{x+2} & \text{True} \end{array} \right.$$

5.31 problem 31

Internal problem ID [11668]
Internal file name [`OUTPUT/11677_Wednesday_April_10_2024_04_54_28_PM_25898883/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{ay' + yb = k\,\mathrm{e}^{-\lambda x}}$$

5.31.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{b}{a}$$
$$q(x) = \frac{k\,\mathrm{e}^{-\lambda x}}{a}$$

Hence the ode is

$$y' + \frac{by}{a} = \frac{k\,\mathrm{e}^{-\lambda x}}{a}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{b}{a}dx} \\
&= \mathrm{e}^{\frac{bx}{a}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{k\,\mathrm{e}^{-\lambda x}}{a}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{bx}{a}} y\right) &= \left(\mathrm{e}^{\frac{bx}{a}}\right)\left(\frac{k\,\mathrm{e}^{-\lambda x}}{a}\right) \\
\mathrm{d}\left(\mathrm{e}^{\frac{bx}{a}} y\right) &= \left(\frac{k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}}}{a}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{\frac{bx}{a}} y &= \int \frac{k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}}}{a}\,\mathrm{d}x \\
\mathrm{e}^{\frac{bx}{a}} y &= -\frac{k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}}}{a\lambda-b} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{bx}{a}}$ results in

$$
y = -\frac{\mathrm{e}^{-\frac{bx}{a}} k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}}}{a\lambda-b} + c_1 \mathrm{e}^{-\frac{bx}{a}}
$$

which simplifies to

$$
y = \frac{c_1(a\lambda-b)\,\mathrm{e}^{-\frac{bx}{a}} - k\,\mathrm{e}^{-\lambda x}}{a\lambda-b}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{c_1(a\lambda-b)\,\mathrm{e}^{-\frac{bx}{a}} - k\,\mathrm{e}^{-\lambda x}}{a\lambda-b} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1(a\lambda-b)\,\mathrm{e}^{-\frac{bx}{a}} - k\,\mathrm{e}^{-\lambda x}}{a\lambda-b}
$$

Verified OK.

5.31.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{by - k\,e^{-\lambda x}}{a}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 194: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-\frac{bx}{a}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-\frac{bx}{a}}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{\frac{bx}{a}} y
$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{by - k\,\mathrm{e}^{-\lambda x}}{a}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{b\,\mathrm{e}^{\frac{bx}{a}}y}{a}\\
S_y &= \mathrm{e}^{\frac{bx}{a}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}}}{a} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{k\,\mathrm{e}^{-\frac{R(a\lambda-b)}{a}}}{a}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{k\,\mathrm{e}^{-\frac{R(a\lambda-b)}{a}}}{a\lambda-b} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\mathrm{e}^{\frac{bx}{a}}y = -\frac{k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}}}{a\lambda-b} + c_1
$$

Which simplifies to

$$
\mathrm{e}^{\frac{bx}{a}}y = -\frac{k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}}}{a\lambda-b} + c_1
$$

Which gives

$$
y = -\frac{\left(-\lambda c_1 a + k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}} + c_1 b\right)\mathrm{e}^{-\frac{bx}{a}}}{a\lambda-b}
$$

Summary
The solution(s) found are the following

$$y = -\frac{\left(-\lambda c_1 a + k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}} + c_1 b\right)\mathrm{e}^{-\frac{bx}{a}}}{a\lambda - b} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(-\lambda c_1 a + k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}} + c_1 b\right)\mathrm{e}^{-\frac{bx}{a}}}{a\lambda - b}$$

Verified OK.

5.31.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(a)\,\mathrm{d}y &= \left(-by + k\,\mathrm{e}^{-\lambda x}\right)\mathrm{d}x\\ \left(by - k\,\mathrm{e}^{-\lambda x}\right)\mathrm{d}x + (a)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= by - k\,\mathrm{e}^{-\lambda x}\\ N(x,y) &= a \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(by - k\,\mathrm{e}^{-\lambda x}\right)\\ &= b\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(a)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{a}((b) - (0))\\ &= \frac{b}{a}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{b}{a}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\frac{bx}{a}} \\
&= \mathrm{e}^{\frac{bx}{a}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{\frac{bx}{a}} \left(by - k\,\mathrm{e}^{-\lambda x}\right) \\
&= \mathrm{e}^{\frac{bx}{a}} \left(by - k\,\mathrm{e}^{-\lambda x}\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\frac{bx}{a}} (a) \\
&= \mathrm{e}^{\frac{bx}{a}} a
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\mathrm{e}^{\frac{bx}{a}} \left(by - k\,\mathrm{e}^{-\lambda x}\right)\right) + \left(\mathrm{e}^{\frac{bx}{a}} a\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \mathrm{e}^{\frac{bx}{a}} \left(by - k\,\mathrm{e}^{-\lambda x}\right) \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{\left(k\,\mathrm{e}^{-\frac{x(a\lambda - b)}{a}} + y(a\lambda - b)\,\mathrm{e}^{\frac{bx}{a}}\right) a}{a\lambda - b} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{bx}{a}} a + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{\frac{bx}{a}} a$. Therefore equation (4) becomes

$$\mathrm{e}^{\frac{bx}{a}} a = \mathrm{e}^{\frac{bx}{a}} a + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\left(k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}} + y(a\lambda - b)\,\mathrm{e}^{\frac{bx}{a}}\right) a}{a\lambda - b} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\left(k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}} + y(a\lambda - b)\,\mathrm{e}^{\frac{bx}{a}}\right) a}{a\lambda - b}$$

The solution becomes

$$y = -\frac{\left(k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}} a - \lambda c_1 a + c_1 b\right) \mathrm{e}^{-\frac{bx}{a}}}{a\,(a\lambda - b)}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{\left(k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}} a - \lambda c_1 a + c_1 b\right) \mathrm{e}^{-\frac{bx}{a}}}{a\,(a\lambda - b)} \tag{1}$$

Verification of solutions

$$y = -\frac{\left(k\,\mathrm{e}^{-\frac{x(a\lambda-b)}{a}} a - \lambda c_1 a + c_1 b\right) \mathrm{e}^{-\frac{bx}{a}}}{a\,(a\lambda - b)}$$

Verified OK.

5.31.4 Maple step by step solution

Let's solve

$ay' + yb = k\,\mathrm{e}^{-\lambda x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{by}{a} + \frac{k\,\mathrm{e}^{-\lambda x}}{a}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{by}{a} = \frac{k\,\mathrm{e}^{-\lambda x}}{a}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{by}{a}\right) = \frac{\mu(x)k\,\mathrm{e}^{-\lambda x}}{a}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{by}{a}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)b}{a}$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\frac{bx}{a}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)k\,\mathrm{e}^{-\lambda x}}{a} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)k\,\mathrm{e}^{-\lambda x}}{a} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)k\,\mathrm{e}^{-\lambda x}}{a} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{\frac{bx}{a}}$

$$y = \frac{\int \frac{k\,e^{-\lambda x} e^{\frac{bx}{a}}}{a} dx + c_1}{e^{\frac{bx}{a}}}$$

- Evaluate the integrals on the rhs

$$y = \frac{-\frac{k\,e^{-\lambda x + \frac{bx}{a}}}{a\lambda - b} + c_1}{e^{\frac{bx}{a}}}$$

- Simplify

$$y = \frac{e^{-\frac{bx}{a}} \left(-k\,e^{-\frac{x(a\lambda - b)}{a}} + (a\lambda - b) c_1\right)}{a\lambda - b}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
dsolve(a*diff(y(x),x)+b*y(x)=k*exp(-lambda*x),y(x), singsol=all)
```

$$y(x) = \frac{\left(-k\,e^{-\frac{x(a\lambda - b)}{a}} + c_1(a\lambda - b)\right) e^{-\frac{bx}{a}}}{a\lambda - b}$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 44

```
DSolve[a*y'[x]+b*y[x]==k*Exp[\[Lambda]*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-\frac{bx}{a}} \left(k e^{x\left(\frac{b}{a} + \lambda\right)} + c_1(a\lambda + b)\right)}{a\lambda + b}$$

5.32 problem 35 (b)

Internal problem ID [11669]
Internal file name [`OUTPUT/11678_Wednesday_April_10_2024_04_54_29_PM_13645783/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 35 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = 2\sin(x) + 5\sin(2x)}$$

5.32.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 2\sin(x) + 5\sin(2x) \end{aligned}$$

Hence the ode is

$$y' + y = 2\sin(x) + 5\sin(2x)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(2\sin(x)+5\sin(2x)) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} y) &= (\mathrm{e}^{x})\,(2\sin(x)+5\sin(2x)) \\
\mathrm{d}(\mathrm{e}^{x} y) &= ((2\sin(x)+5\sin(2x))\,\mathrm{e}^{x})\ \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{x} y &= \int (2\sin(x)+5\sin(2x))\,\mathrm{e}^{x}\,\mathrm{d}x \\
\mathrm{e}^{x} y &= \mathrm{e}^{x}(\sin(2x)-2\cos(2x)) - \mathrm{e}^{x}\cos(x) + \mathrm{e}^{x}\sin(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$
y = \mathrm{e}^{-x}(\mathrm{e}^{x}(\sin(2x)-2\cos(2x)) - \mathrm{e}^{x}\cos(x) + \mathrm{e}^{x}\sin(x)) + c_1\mathrm{e}^{-x}
$$

which simplifies to

$$
y = c_1\mathrm{e}^{-x} - 4\cos(x)^2 + (2\sin(x)-1)\cos(x) + \sin(x) + 2
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-x} - 4\cos(x)^2 + (2\sin(x)-1)\cos(x) + \sin(x) + 2 \tag{1}
$$

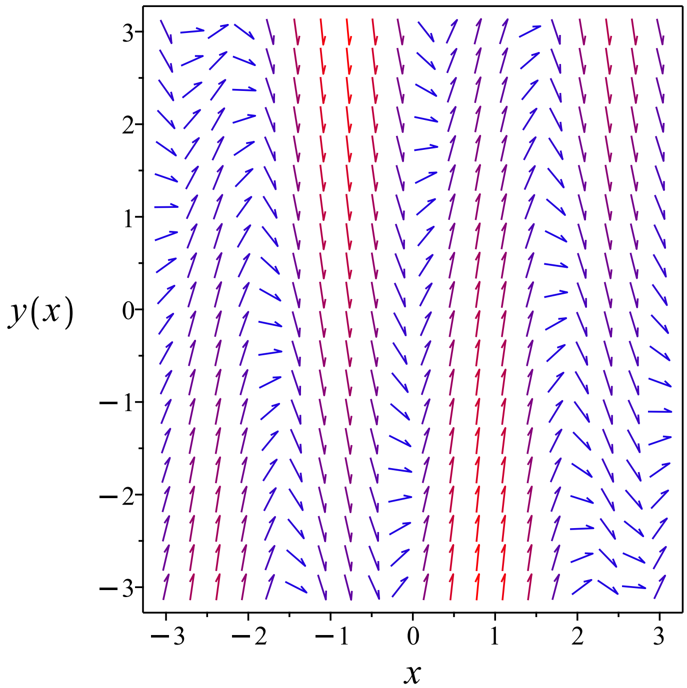


Figure 231: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} - 4\cos(x)^2 + (2\sin(x) - 1)\cos(x) + \sin(x) + 2$$

Verified OK.

5.32.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y + 2\sin(x) + 5\sin(2x)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 197: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y + 2\sin(x) + 5\sin(2x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \mathrm{e}^{x} y \\ S_y &= \mathrm{e}^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (2\sin(x) + 5\sin(2x))\,\mathrm{e}^{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (2\sin(R) + 5\sin(2R))\,\mathrm{e}^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 - e^R(\cos(R) + 2\cos(2R) - \sin(R) - \sin(2R)) \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,e^x = c_1 - e^x(-\sin(2x) + 2\cos(2x) + \cos(x) - \sin(x))$$

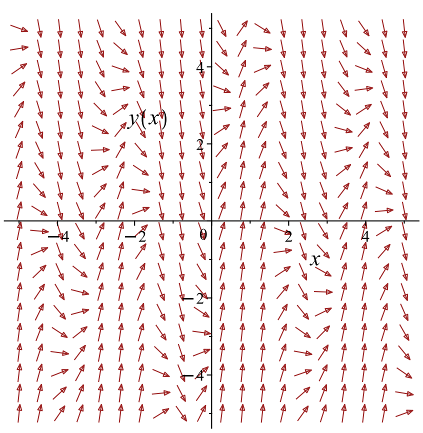
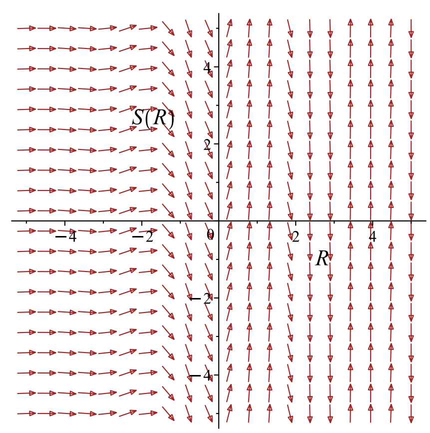
Which simplifies to

$$y\,e^x = c_1 - e^x(-\sin(2x) + 2\cos(2x) + \cos(x) - \sin(x))$$

Which gives

$$y = e^{-x}(e^x \sin(2x) + e^x \sin(x) - e^x \cos(x) - 2e^x \cos(2x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y + 2\sin(x) + 5\sin(2x)$ 	$R = x$ $S = e^x y$	$\frac{dS}{dR} = (2\sin(R) + 5\sin(2R))\,e^R$ 

Summary
The solution(s) found are the following

$$y = e^{-x}(e^x \sin(2x) + e^x \sin(x) - e^x \cos(x) - 2e^x \cos(2x) + c_1) \tag{1}$$

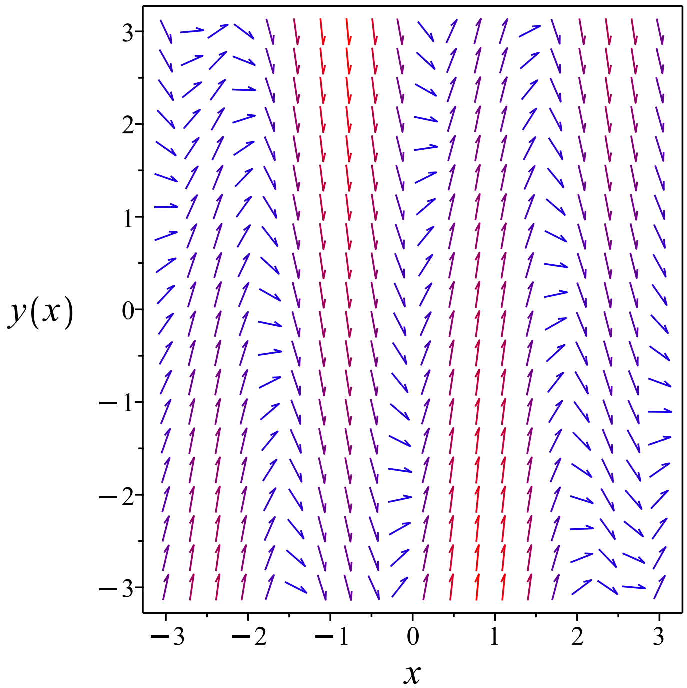


Figure 232: Slope field plot

Verification of solutions

$$y = e^{-x}(e^x \sin(2x) + e^x \sin(x) - e^x \cos(x) - 2 e^x \cos(2x) + c_1)$$

Verified OK.

5.32.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-y + 2\sin(x) + 5\sin(2x))\,\mathrm{d}x$$
$$(y - 2\sin(x) - 5\sin(2x))\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y - 2\sin(x) - 5\sin(2x)$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y - 2\sin(x) - 5\sin(2x))$$
$$= 1$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((1) - (0))\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int 1\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x}\\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{x}(y - 2\sin(x) - 5\sin(2x))\\ &= (y - 2\sin(x) - 5\sin(2x))\,\mathrm{e}^{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{x}(1)\\ &= \mathrm{e}^{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ ((y - 2\sin(x) - 5\sin(2x))\,\mathrm{e}^{x}) + (\mathrm{e}^{x})\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int (y - 2\sin(x) - 5\sin(2x))\,\mathrm{e}^{x} \,\mathrm{d}x$$

$$\phi = \mathrm{e}^{x}(-\sin(2x) + 2\cos(2x) + \cos(x) - \sin(x) + y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{x}(-\sin(2x) + 2\cos(2x) + \cos(x) - \sin(x) + y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{x}(-\sin(2x) + 2\cos(2x) + \cos(x) - \sin(x) + y)$$

The solution becomes

$$y = \mathrm{e}^{-x}(\mathrm{e}^{x}\sin(2x) + \mathrm{e}^{x}\sin(x) - \mathrm{e}^{x}\cos(x) - 2\,\mathrm{e}^{x}\cos(2x) + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{-x}\left(e^{x} \sin \left(2 x\right)+e^{x} \sin \left(x\right)-e^{x} \cos \left(x\right)-2 e^{x} \cos \left(2 x\right)+c_{1}\right) \tag{1}$$

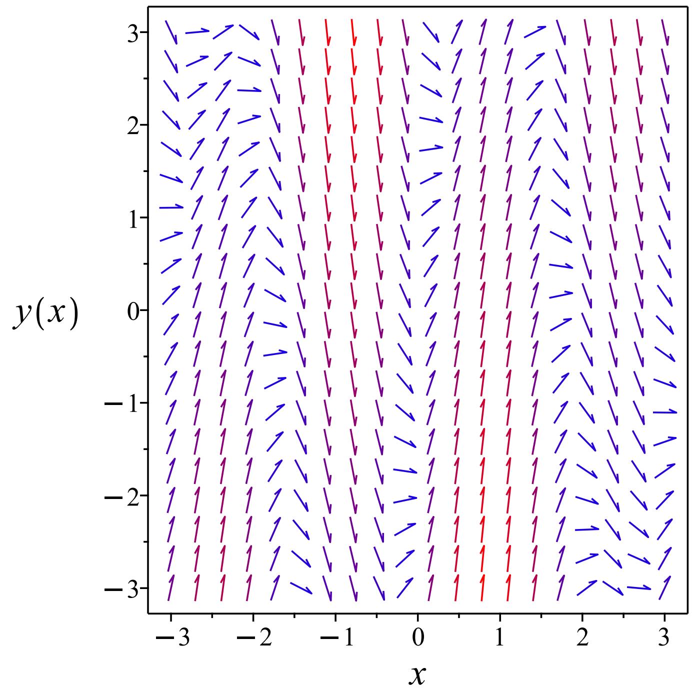


Figure 233: Slope field plot

Verification of solutions

$$y = e^{-x}\left(e^{x} \sin \left(2 x\right)+e^{x} \sin \left(x\right)-e^{x} \cos \left(x\right)-2 e^{x} \cos \left(2 x\right)+c_{1}\right)$$

Verified OK.

5.32.4 Maple step by step solution

Let's solve

$y'+y=2\sin\left(x\right)+5\sin\left(2x\right)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-y+2\sin\left(x\right)+5\sin\left(2x\right)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $$y' + y = 2\sin(x) + 5\sin(2x)$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $$\mu(x)(y' + y) = \mu(x)(2\sin(x) + 5\sin(2x))$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $$\mu(x)(y' + y) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

  $$\mu'(x) = \mu(x)$$

- Solve to find the integrating factor

  $$\mu(x) = \mathrm{e}^{x}$$

- Integrate both sides with respect to x

  $$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)(2\sin(x) + 5\sin(2x))\, dx + c_1$$

- Evaluate the integral on the lhs

  $$\mu(x)\, y = \int \mu(x)(2\sin(x) + 5\sin(2x))\, dx + c_1$$

- Solve for y

  $$y = \frac{\int \mu(x)(2\sin(x)+5\sin(2x))dx+c_1}{\mu(x)}$$

- Substitute $\mu(x) = \mathrm{e}^{x}$

  $$y = \frac{\int (2\sin(x)+5\sin(2x))\mathrm{e}^{x}dx+c_1}{\mathrm{e}^{x}}$$

- Evaluate the integrals on the rhs

  $$y = \frac{\mathrm{e}^{x}(\sin(2x)-2\cos(2x))-\mathrm{e}^{x}\cos(x)+\mathrm{e}^{x}\sin(x)+c_1}{\mathrm{e}^{x}}$$

- Simplify

  $$y = c_1\mathrm{e}^{-x} - 4\cos(x)^2 + (2\sin(x) - 1)\cos(x) + \sin(x) + 2$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x)+y(x)=2*sin(x)+5*sin(2*x),y(x), singsol=all)
```

$$y(x) = -\cos(x) - 2\cos(2x) + \sin(x) + \sin(2x) + c_1 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.131 (sec). Leaf size: 30

```
DSolve[y'[x]+y[x]==2*Sin[x]+5*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x) + \sin(2x) - \cos(x) - 2\cos(2x) + c_1 e^{-x}$$

5.33 problem 37 (a)

Internal problem ID [11670]
Internal file name [`OUTPUT/11679_Wednesday_April_10_2024_04_54_30_PM_22507291/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 37 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{\cos(y)\,y' + \frac{\sin(y)}{x} = 1}$$

5.33.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x\cos(y))\,\mathrm{d}y &= (-\sin(y) + x)\,\mathrm{d}x\\ (\sin(y) - x)\,\mathrm{d}x + (x\cos(y))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \sin(y) - x\\ N(x, y) &= x\cos(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\sin(y) - x)\\ &= \cos(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x\cos(y))\\ &= \cos(y)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \sin(y) - x \, \mathrm{d}x$$

$$\phi = -\frac{x(-2\sin(y) + x)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x\cos(y) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x\cos(y)$. Therefore equation (4) becomes

$$x\cos(y) = x\cos(y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x(-2\sin(y) + x)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x(-2\sin(y) + x)}{2}$$

Summary
The solution(s) found are the following

$$-\frac{x(-2\sin(y) + x)}{2} = c_1 \tag{1}$$

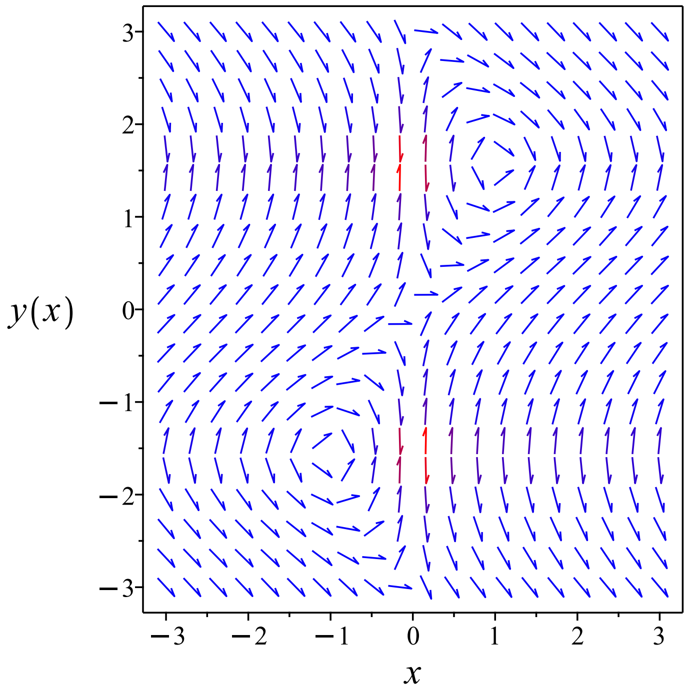


Figure 234: Slope field plot

Verification of solutions

$$-\frac{x(-2\sin(y)+x)}{2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(cos(y(x))*diff(y(x),x)+1/x*sin(y(x))=1,y(x), singsol=all)
```

$$y(x) = -\arcsin\left(\frac{-x^2 + 2c_1}{2x}\right)$$

✓ Solution by Mathematica

Time used: 8.67 (sec). Leaf size: 18

```
DSolve[Cos[y[x]]*y'[x]+1/x*Sin[y[x]]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin\left(\frac{x}{2} + \frac{c_1}{x}\right)$$

5.34 problem 37 (b)

Internal problem ID [11671]
Internal file name [`OUTPUT/11680_Wednesday_April_10_2024_04_54_32_PM_51224347/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 37 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(y+1)\,y'+x\left(y^2+2y\right)=x}$$

5.34.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x(y^2+2y-1)}{y+1} \end{aligned}$$

Where $f(x)=-x$ and $g(y)=\frac{y^2+2y-1}{y+1}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{y^2+2y-1}{y+1}}\,dy &= -x\,dx \\ \int \frac{1}{\frac{y^2+2y-1}{y+1}}\,dy &= \int -x\,dx \end{aligned}$$

$$\frac{\ln \left(y^2+2y-1\right)}{2} = -\frac{x^2}{2}+c_1$$

Raising both side to exponential gives

$$\sqrt{y^2+2y-1} = e^{-\frac{x^2}{2}+c_1}$$

Which simplifies to

$$\sqrt{y^2+2y-1} = c_2 e^{-\frac{x^2}{2}}$$

The solution is

$$\sqrt{y^2+2y-1} = c_2 e^{-\frac{x^2}{2}+c_1}$$

Summary

The solution(s) found are the following

$$\sqrt{y^2+2y-1} = c_2 e^{-\frac{x^2}{2}+c_1} \tag{1}$$

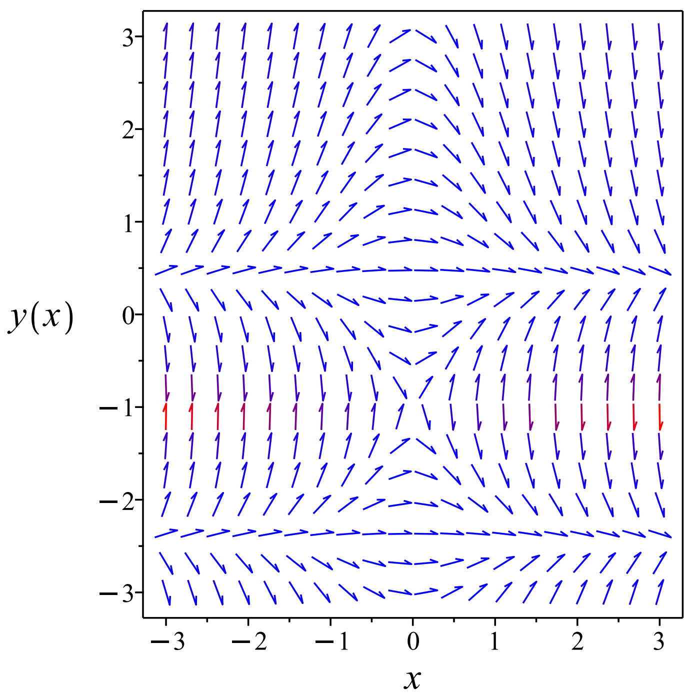


Figure 235: Slope field plot

Verification of solutions

$$\sqrt{y^2+2y-1} = c_2 e^{-\frac{x^2}{2}+c_1}$$

Verified OK.

5.34.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{x(y^2+2y-1)}{y+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 200: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{1}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{1}{x}} dx
\end{aligned}
$$

Which results in

$$
S = -\frac{x^2}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{x(y^2+2y-1)}{y+1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y+1}{y^2+2y-1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R+1}{R^2+2R-1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R^2+2R-1)}{2} + c_1 \tag{4}$$

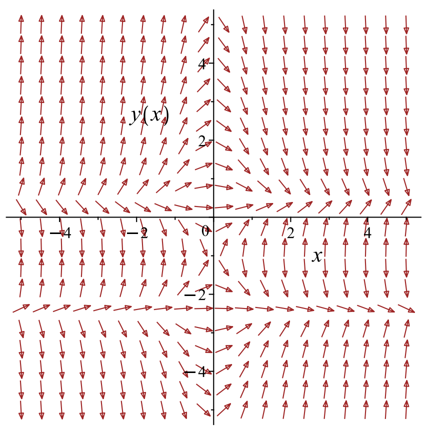
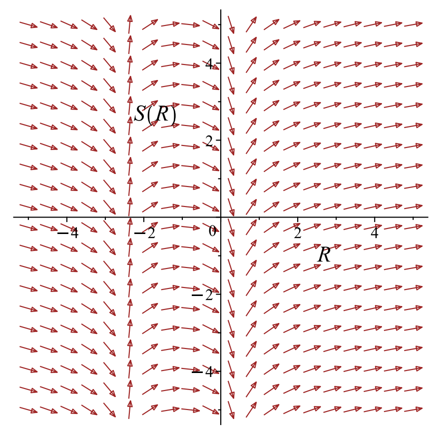
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{2} = \frac{\ln(y^2+2y-1)}{2} + c_1$$

Which simplifies to

$$-\frac{x^2}{2} = \frac{\ln(y^2+2y-1)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x(y^2+2y-1)}{y+1}$ 	$R = y$ $S = -\frac{x^2}{2}$	$\frac{dS}{dR} = \frac{R+1}{R^2+2R-1}$ 

<u>Summary</u>

The solution(s) found are the following

$$-\frac{x^2}{2} = \frac{\ln(y^2+2y-1)}{2} + c_1 \tag{1}$$

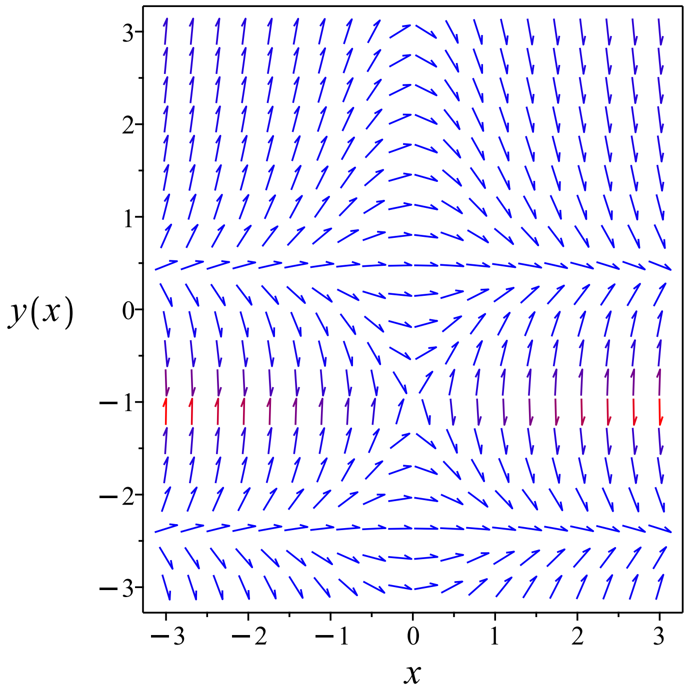


Figure 236: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} = \frac{\ln(y^2 + 2y - 1)}{2} + c_1$$

Verified OK.

5.34.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{y+1}{y^2+2y-1}\right)\mathrm{d}y = (x)\,\mathrm{d}x$$
$$(-x)\,\mathrm{d}x + \left(-\frac{y+1}{y^2+2y-1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = -\frac{y+1}{y^2+2y-1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y+1}{y^2+2y-1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -x\, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y+1}{y^2+2y-1}$. Therefore equation (4) becomes

$$
-\frac{y+1}{y^2+2y-1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{y+1}{y^2+2y-1}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{-y-1}{y^2+2y-1}\right) \mathrm{d}y$$
$$f(y) = -\frac{\ln(y^2+2y-1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{\ln(y^2+2y-1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{\ln(y^2+2y-1)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} - \frac{\ln(y^2+2y-1)}{2} = c_1 \tag{1}$$

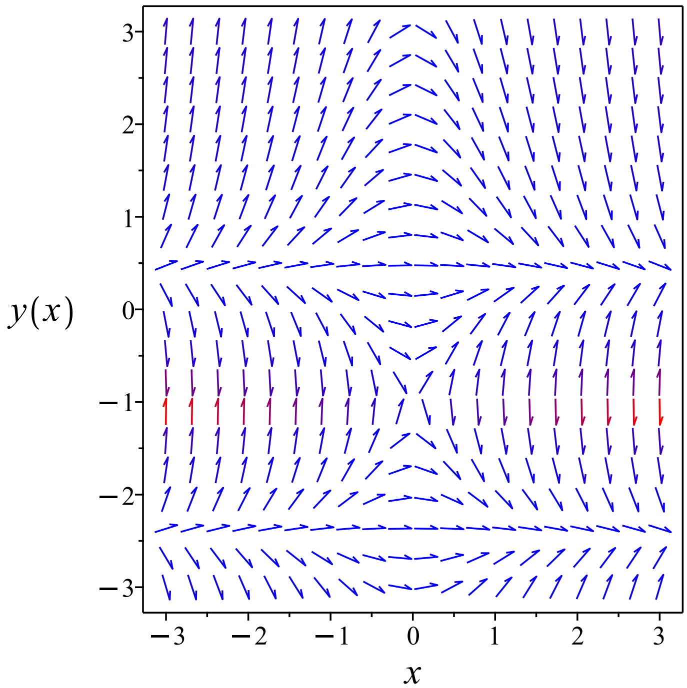


Figure 237: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} - \frac{\ln(y^2 + 2y - 1)}{2} = c_1$$

Verified OK.

5.34.4 Maple step by step solution

Let's solve

$(y + 1)\, y' + x(y^2 + 2y) = x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(y+1)}{y^2+2y-1} = -x$

- Integrate both sides with respect to x

  $\int \frac{y'(y+1)}{y^2+2y-1} dx = \int -x dx + c_1$

- Evaluate integral

$$\frac{\ln(y^2+2y-1)}{2} = -\frac{x^2}{2} + c_1$$

- Solve for y

$$\left\{y = -1 - \sqrt{2 + \mathrm{e}^{-x^2+2c_1}}, y = -1 + \sqrt{2 + \mathrm{e}^{-x^2+2c_1}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 37

```
dsolve((y(x)+1)*diff(y(x),x)+x*(y(x)^2+2*y(x))=x,y(x), singsol=all)
```

$$y(x) = -1 - \sqrt{2 + \mathrm{e}^{-x^2} c_1}$$
$$y(x) = -1 + \sqrt{2 + \mathrm{e}^{-x^2} c_1}$$

✓ Solution by Mathematica

Time used: 29.843 (sec). Leaf size: 163

```
DSolve[(y[x]+1)*y'[x]+x*(y[x]^2+2*y[x])==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -1 - e^{-x^2} \sqrt{e^{x^2} \left(2e^{x^2} + e^{2c_1}\right)}$$
$$y(x) \to -1 + e^{-x^2} \sqrt{e^{x^2} \left(2e^{x^2} + e^{2c_1}\right)}$$
$$y(x) \to -1 - \sqrt{2}$$
$$y(x) \to \sqrt{2} - 1$$
$$y(x) \to \sqrt{2} e^{-x^2} \sqrt{e^{2x^2}} - 1$$
$$y(x) \to -\sqrt{2} e^{-x^2} \sqrt{e^{2x^2}} - 1$$

5.35 problem 39

Internal problem ID [11672]
Internal file name [OUTPUT/11681_Wednesday_April_10_2024_04_54_33_PM_13637091/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' - (1-x)\,y^2 - (2x-1)\,y = -x}$$

5.35.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -x\,y^2 + 2xy + y^2 - x - y \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&2xb_4 + yb_5 + b_2 \\
&+ \left(-x\,y^2 + 2xy + y^2 - x - y\right)\left(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3\right) \\
&- \left(-x\,y^2 + 2xy + y^2 - x - y\right)^2 \left(xa_5 + 2ya_6 + a_3\right) \\
&- \left(-y^2 + 2y - 1\right)\left(x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1\right) \\
&- \left(-2xy + 2x + 2y - 1\right)\left(x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1\right) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned}
&-x^3y^4a_5 - 2x^2y^5a_6 + 4x^3y^3a_5 - x^2y^4a_3 + 2x^2y^4a_5 + 8x^2y^4a_6 + 4x\,y^5a_6 \\
&- 6x^3y^2a_5 + 4x^2y^3a_3 - 6x^2y^3a_5 - 12x^2y^3a_6 + 2x\,y^4a_3 - x\,y^4a_5 \\
&- 12x\,y^4a_6 - 2y^5a_6 + 4x^3ya_5 + 2x^3yb_4 - 6x^2y^2a_3 + 3x^2y^2a_4 + 6x^2y^2a_5 \\
&+ 8x^2y^2a_6 + x^2y^2b_5 - 6x\,y^3a_3 + 4x\,y^3a_5 + 12x\,y^3a_6 - y^4a_3 + 5y^4a_6 \\
&- x^3a_5 - 2x^3b_4 + 4x^2ya_3 - 6x^2ya_4 - 2x^2ya_5 - 2x^2ya_6 + 2x^2yb_2 - 2x^2yb_4 \\
&+ 2x\,y^2a_2 + 6x\,y^2a_3 - 2x\,y^2a_4 - 5x\,y^2a_5 - 4x\,y^2a_6 + x\,y^2b_3 - x\,y^2b_5 \\
&+ 2x\,y^2b_6 + 3y^3a_3 - y^3a_5 - 4y^3a_6 - x^2a_3 + 3x^2a_4 - 2x^2b_2 + x^2b_4 - x^2b_5 \\
&- 4xya_2 - 2xya_3 + 2xya_4 + 2xya_5 + 2xyb_1 - 2xyb_2 - 2xyb_6 + y^2a_1 \\
&- y^2a_2 - 3y^2a_3 + y^2a_5 + y^2a_6 - y^2b_3 - y^2b_6 + 2xa_2 - 2xb_1 + xb_2 \\
&- xb_3 + 2xb_4 - 2ya_1 + ya_2 + ya_3 - 2yb_1 + yb_5 + a_1 + b_1 + b_2 = 0
\end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned}
&-x^3y^4a_5 - 2x^2y^5a_6 + 4x^3y^3a_5 - x^2y^4a_3 + 2x^2y^4a_5 + 8x^2y^4a_6 \\
&+ 4x\,y^5a_6 - 6x^3y^2a_5 + 4x^2y^3a_3 - 6x^2y^3a_5 - 12x^2y^3a_6 + 2x\,y^4a_3 \\
&- x\,y^4a_5 - 12x\,y^4a_6 - 2y^5a_6 + 4x^3ya_5 + 2x^3yb_4 - 6x^2y^2a_3 \\
&+ 3x^2y^2a_4 + 6x^2y^2a_5 + 8x^2y^2a_6 + x^2y^2b_5 - 6x\,y^3a_3 + 4x\,y^3a_5 \\
&+ 12x\,y^3a_6 - y^4a_3 + 5y^4a_6 - x^3a_5 - 2x^3b_4 + 4x^2ya_3 - 6x^2ya_4 \\
&- 2x^2ya_5 - 2x^2ya_6 + 2x^2yb_2 - 2x^2yb_4 + 2x\,y^2a_2 + 6x\,y^2a_3 \\
&- 2x\,y^2a_4 - 5x\,y^2a_5 - 4x\,y^2a_6 + x\,y^2b_3 - x\,y^2b_5 + 2x\,y^2b_6 + 3y^3a_3 \\
&- y^3a_5 - 4y^3a_6 - x^2a_3 + 3x^2a_4 - 2x^2b_2 + x^2b_4 - x^2b_5 - 4xya_2 \\
&- 2xya_3 + 2xya_4 + 2xya_5 + 2xyb_1 - 2xyb_2 - 2xyb_6 + y^2a_1 - y^2a_2 \\
&- 3y^2a_3 + y^2a_5 + y^2a_6 - y^2b_3 - y^2b_6 + 2xa_2 - 2xb_1 + xb_2 - xb_3 \\
&+ 2xb_4 - 2ya_1 + ya_2 + ya_3 - 2yb_1 + yb_5 + a_1 + b_1 + b_2 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a_5v_1^3v_2^4 - 2a_6v_1^2v_2^5 - a_3v_1^2v_2^4 + 4a_5v_1^3v_2^3 + 2a_5v_1^2v_2^4 + 8a_6v_1^2v_2^4 \\
&+ 4a_6v_1v_2^5 + 4a_3v_1^2v_2^3 + 2a_3v_1v_2^4 - 6a_5v_1^3v_2^2 - 6a_5v_1^2v_2^3 - a_5v_1v_2^4 \\
&- 12a_6v_1^2v_2^3 - 12a_6v_1v_2^4 - 2a_6v_2^5 - 6a_3v_1^2v_2^2 - 6a_3v_1v_2^3 - a_3v_2^4 \\
&+ 3a_4v_1^2v_2^2 + 4a_5v_1^3v_2 + 6a_5v_1^2v_2^2 + 4a_5v_1v_2^3 + 8a_6v_1^2v_2^2 + 12a_6v_1v_2^3 \\
&+ 5a_6v_2^4 + 2b_4v_1^3v_2 + b_5v_1^2v_2^2 + 2a_2v_1v_2^2 + 4a_3v_1^2v_2 + 6a_3v_1v_2^2 + 3a_3v_2^3 \\
&- 6a_4v_1^2v_2 - 2a_4v_1v_2^2 - a_5v_1^3 - 2a_5v_1^2v_2 - 5a_5v_1v_2^2 - a_5v_2^3 - 2a_6v_1^2v_2 \\
&- 4a_6v_1v_2^2 - 4a_6v_2^3 + 2b_2v_1^2v_2 + b_3v_1v_2^2 - 2b_4v_1^3 - 2b_4v_1^2v_2 - b_5v_1v_2^2 \\
&+ 2b_6v_1v_2^2 + a_1v_2^2 - 4a_2v_1v_2 - a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - 3a_3v_2^2 + 3a_4v_1^2 \\
&+ 2a_4v_1v_2 + 2a_5v_1v_2 + a_5v_2^2 + a_6v_2^2 + 2b_1v_1v_2 - 2b_2v_1^2 - 2b_2v_1v_2 \\
&- b_3v_2^2 + b_4v_1^2 - b_5v_1^2 - 2b_6v_1v_2 - b_6v_2^2 - 2a_1v_2 + 2a_2v_1 + a_2v_2 + a_3v_2 \\
&- 2b_1v_1 - 2b_1v_2 + b_2v_1 - b_3v_1 + 2b_4v_1 + b_5v_2 + a_1 + b_1 + b_2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(4a_5 + 2b_4)\, v_1^3v_2 + (-a_3 + 2a_5 + 8a_6)\, v_1^2v_2^4 \\
&+ (4a_3 - 6a_5 - 12a_6)\, v_1^2v_2^3 + (-6a_3 + 3a_4 + 6a_5 + 8a_6 + b_5)\, v_1^2v_2^2 \\
&+ (4a_3 - 6a_4 - 2a_5 - 2a_6 + 2b_2 - 2b_4)\, v_1^2v_2 \\
&+ (2a_3 - a_5 - 12a_6)\, v_1v_2^4 + (-6a_3 + 4a_5 + 12a_6)\, v_1v_2^3 \\
&+ (2a_2 + 6a_3 - 2a_4 - 5a_5 - 4a_6 + b_3 - b_5 + 2b_6)\, v_1v_2^2 \\
&+ (-4a_2 - 2a_3 + 2a_4 + 2a_5 + 2b_1 - 2b_2 - 2b_6)\, v_1v_2 + a_1 + b_1 \\
&+ b_2 + (-a_5 - 2b_4)\, v_1^3 - 2a_6v_2^5 + (-a_3 + 3a_4 - 2b_2 + b_4 - b_5)\, v_1^2 \\
&+ (2a_2 - 2b_1 + b_2 - b_3 + 2b_4)\, v_1 + (-a_3 + 5a_6)\, v_2^4 + (3a_3 - a_5 - 4a_6)\, v_2^3 \\
&+ (a_1 - a_2 - 3a_3 + a_5 + a_6 - b_3 - b_6)\, v_2^2 + (-2a_1 + a_2 + a_3 - 2b_1 + b_5)\, v_2 \\
&- a_5v_1^3v_2^4 - 2a_6v_1^2v_2^5 + 4a_5v_1^3v_2^3 + 4a_6v_1v_2^5 - 6a_5v_1^3v_2^2 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-6a_5 &= 0 \\
-a_5 &= 0 \\
4a_5 &= 0 \\
-2a_6 &= 0 \\
4a_6 &= 0 \\
-a_3 + 5a_6 &= 0 \\
-a_5 - 2b_4 &= 0 \\
4a_5 + 2b_4 &= 0 \\
a_1 + b_1 + b_2 &= 0 \\
-6a_3 + 4a_5 + 12a_6 &= 0 \\
-a_3 + 2a_5 + 8a_6 &= 0 \\
2a_3 - a_5 - 12a_6 &= 0 \\
3a_3 - a_5 - 4a_6 &= 0 \\
4a_3 - 6a_5 - 12a_6 &= 0 \\
-2a_1 + a_2 + a_3 - 2b_1 + b_5 &= 0 \\
2a_2 - 2b_1 + b_2 - b_3 + 2b_4 &= 0 \\
-6a_3 + 3a_4 + 6a_5 + 8a_6 + b_5 &= 0 \\
-a_3 + 3a_4 - 2b_2 + b_4 - b_5 &= 0 \\
4a_3 - 6a_4 - 2a_5 - 2a_6 + 2b_2 - 2b_4 &= 0 \\
a_1 - a_2 - 3a_3 + a_5 + a_6 - b_3 - b_6 &= 0 \\
-4a_2 - 2a_3 + 2a_4 + 2a_5 + 2b_1 - 2b_2 - 2b_6 &= 0 \\
2a_2 + 6a_3 - 2a_4 - 5a_5 - 4a_6 + b_3 - b_5 + 2b_6 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -b_6 \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= b_6 \\
b_2 &= 0 \\
b_3 &= -2b_6 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= b_6
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -1 \\
\eta &= y^2 - 2y + 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y^2 - 2y + 1 - \left(-x\, y^2 + 2xy + y^2 - x - y\right)(-1) \\
&= -x\, y^2 + 2xy + 2y^2 - x - 3y + 1 \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-x\,y^2 + 2xy + 2y^2 - x - 3y + 1} dy \end{aligned}$$

Which results in

$$S = \ln(y - 1) + \frac{(-x + 2)\ln(xy - x - 2y + 1)}{x - 2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -x\,y^2 + 2xy + y^2 - x - y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-y + 1}{(x - 2)\,y + 1 - x} \\ S_y &= -\frac{1}{(y - 1)\,((x - 2)\,y + 1 - x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y-1) - \ln((y-1)x - 2y + 1) = x + c_1$$

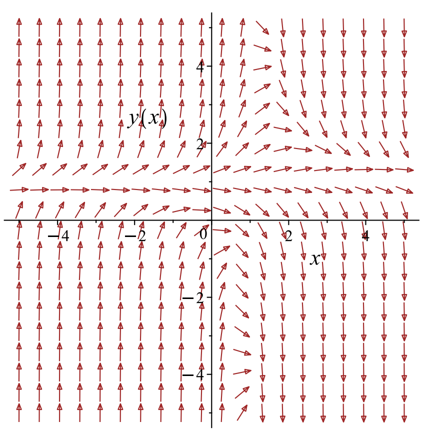
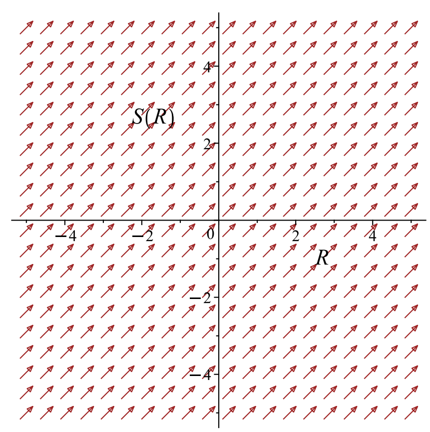
Which simplifies to

$$\ln(y-1) - \ln((y-1)x - 2y + 1) = x + c_1$$

Which gives

$$y = \frac{x\,\mathrm{e}^{x+c_1} - \mathrm{e}^{x+c_1} - 1}{-1 + x\,\mathrm{e}^{x+c_1} - 2\,\mathrm{e}^{x+c_1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -x\,y^2 + 2xy + y^2 - x - y$ 	$R = x$ $S = \ln(y-1) - \ln((y -$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = \frac{x\,\mathrm{e}^{x+c_1} - \mathrm{e}^{x+c_1} - 1}{-1 + x\,\mathrm{e}^{x+c_1} - 2\,\mathrm{e}^{x+c_1}} \tag{1}$$

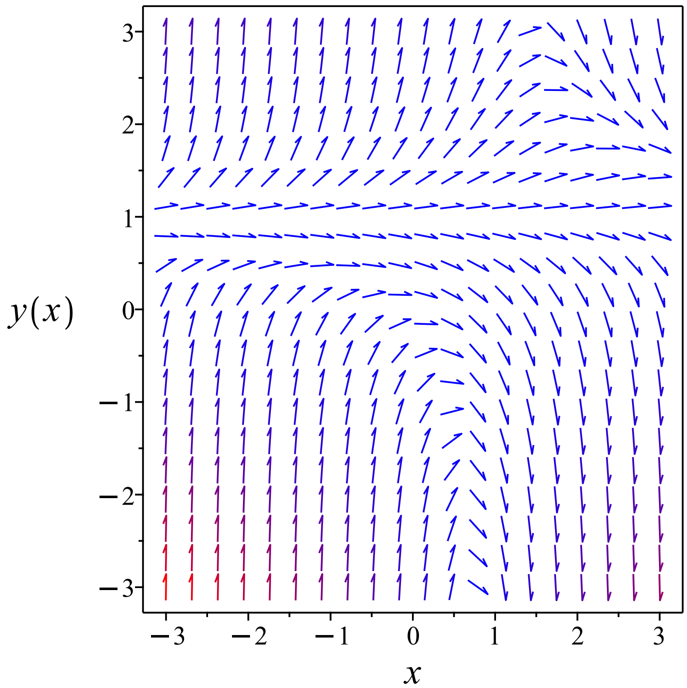


Figure 238: Slope field plot

Verification of solutions

$$y = \frac{x\,e^{x+c_1} - e^{x+c_1} - 1}{-1 + x\,e^{x+c_1} - 2\,e^{x+c_1}}$$

Verified OK.

5.35.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -x\,y^2 + 2xy + y^2 - x - y \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -x\,y^2 + 2xy + y^2 - x - y$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -x$, $f_1(x) = 2x - 1$ and $f_2(x) = 1 - x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(1-x)\,u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -1 \\ f_1 f_2 &= (2x-1)(1-x) \\ f_2^2 f_0 &= -(1-x)^2\, x \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(1-x)\,u''(x) - (-1 + (2x-1)(1-x))\,u'(x) - (1-x)^2\, x u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \mathrm{e}^{\frac{x(x-2)}{2}} + c_2 \mathrm{e}^{\frac{x^2}{2}}(x-2)$$

The above shows that

$$u'(x) = \left((x-1)\, c_2 \mathrm{e}^{\frac{x^2}{2}} + c_1 \mathrm{e}^{\frac{x(x-2)}{2}}\right)(x-1)$$

Using the above in (1) gives the solution

$$y = -\frac{\left((x-1)\, c_2 \mathrm{e}^{\frac{x^2}{2}} + c_1 \mathrm{e}^{\frac{x(x-2)}{2}}\right)(x-1)}{(1-x)\left(c_1 \mathrm{e}^{\frac{x(x-2)}{2}} + c_2 \mathrm{e}^{\frac{x^2}{2}}(x-2)\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(x-1)\,\mathrm{e}^{\frac{x^2}{2}} + c_3 \mathrm{e}^{\frac{x(x-2)}{2}}}{c_3 \mathrm{e}^{\frac{x(x-2)}{2}} + \mathrm{e}^{\frac{x^2}{2}}(x-2)}$$

Summary

The solution(s) found are the following

$$y = \frac{(x-1)\,\mathrm{e}^{\frac{x^2}{2}} + c_3 \mathrm{e}^{\frac{x(x-2)}{2}}}{c_3 \mathrm{e}^{\frac{x(x-2)}{2}} + \mathrm{e}^{\frac{x^2}{2}}\,(x-2)} \tag{1}$$

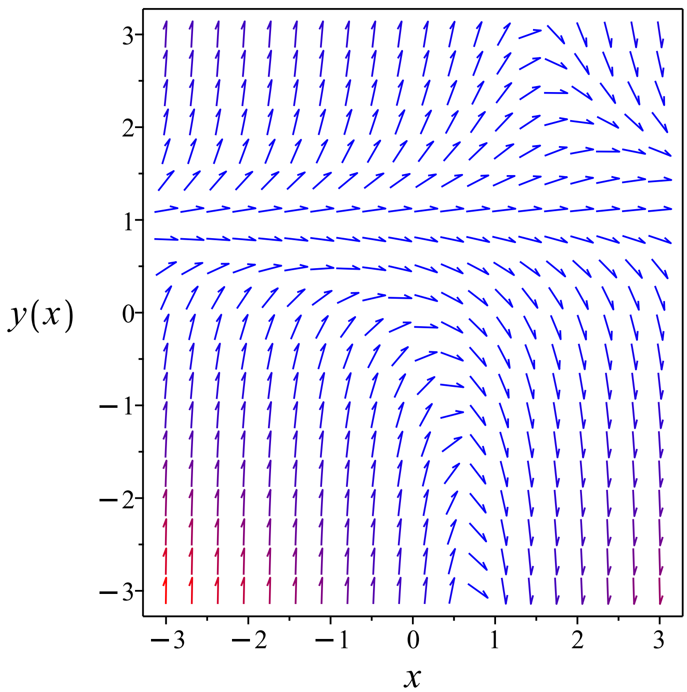


Figure 239: Slope field plot

Verification of solutions

$$y = \frac{(x-1)\,\mathrm{e}^{\frac{x^2}{2}} + c_3 \mathrm{e}^{\frac{x(x-2)}{2}}}{c_3 \mathrm{e}^{\frac{x(x-2)}{2}} + \mathrm{e}^{\frac{x^2}{2}}\,(x-2)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Riccati particular case Kamke (a) successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x)=(1-x)*y(x)^2+(2*x-1)*y(x)-x,y(x), singsol=all)
```

$$y(x) = \frac{(2x-2)\,\mathrm{e}^{x} - c_1}{(2x-4)\,\mathrm{e}^{x} - c_1}$$

✓ Solution by Mathematica
Time used: 0.197 (sec). Leaf size: 28

```
DSolve[y'[x]==(1-x)*y[x]^2+(2*x-1)*y[x]-x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 1 + \frac{e^x}{e^x(x-2)+c_1}$$
$$y(x) \to 1$$

5.36 problem 40

Internal problem ID [11673]
Internal file name [`OUTPUT/11682_Wednesday_April_10_2024_04_54_35_PM_23538320/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

```
[_Riccati]
```

$$\boxed{y' + y^2 - yx = 1}$$

5.36.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= xy - y^2 + 1 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = xy - y^2 + 1$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = -1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= -x \\ f_2^2 f_0 &= 1 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) + xu'(x) + u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{\frac{x^2}{2}} \left(c_1 \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2} \right) + c_2 \right)$$

The above shows that

$$u'(x) = \frac{x\sqrt{\pi}\left(c_1 \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) + c_2\right) \mathrm{e}^{\frac{x^2}{2}} + \sqrt{2}\, c_1}{\sqrt{\pi}}$$

Using the above in (1) gives the solution

$$y = \frac{\left(x\sqrt{\pi}\left(c_1 \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) + c_2\right) \mathrm{e}^{\frac{x^2}{2}} + \sqrt{2}\, c_1\right) \mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{\pi}\left(c_1 \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) + c_2\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\sqrt{2}\, c_3 \mathrm{e}^{-\frac{x^2}{2}} + \sqrt{\pi}\, x\left(c_3 \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) + 1\right)}{\sqrt{\pi}\left(c_3 \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) + 1\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\, c_3 \mathrm{e}^{-\frac{x^2}{2}} + \sqrt{\pi}\, x \left(c_3 \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) + 1\right)}{\sqrt{\pi}\, \left(c_3 \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) + 1\right)} \tag{1}$$

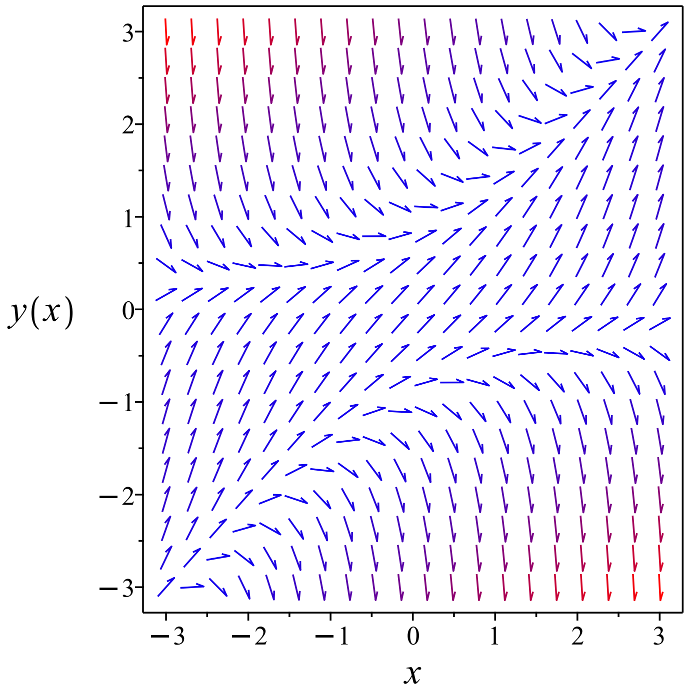


Figure 240: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{2}\, c_3 \mathrm{e}^{-\frac{x^2}{2}} + \sqrt{\pi}\, x \left(c_3 \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) + 1\right)}{\sqrt{\pi}\, \left(c_3 \operatorname{erf}\left(\frac{\sqrt{2}\, x}{2}\right) + 1\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
found: 2 potential symmetries. Proceeding with integration step`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 53

```
dsolve(diff(y(x),x)=-y(x)^2+x*y(x)+1,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{x\sqrt{2}}{2}\right)x + 2c_1x + 2\,\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{x\sqrt{2}}{2}\right) + 2c_1}$$

✓ Solution by Mathematica

Time used: 0.154 (sec). Leaf size: 45

```
DSolve[y'[x]==-y[x]^2+x*y[x]+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + \frac{e^{-\frac{x^2}{2}}}{\sqrt{\frac{\pi}{2}}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) + c_1}$$
$$y(x) \to x$$

5.37 problem 41

Internal problem ID [11674]
Internal file name [`OUTPUT/11683_Wednesday_April_10_2024_04_54_36_PM_3623463/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, section 2.3 (Linear equations). Exercises page 56
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`], _Riccati]
```

$$\boxed{y' + 8y^2x - 4x(4x+1)\,y = -8x^3 - 4x^2 + 1}$$

5.37.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -8x^3 + 16x^2y - 8x\,y^2 - 4x^2 + 4xy + 1 \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&2xb_4 + yb_5 + b_2 \\
&+ \left(-8x^3+16x^2y-8x\,y^2-4x^2+4xy+1\right)\left(-2xa_4+xb_5-ya_5+2yb_6-a_2+b_3\right) \\
&- \left(-8x^3+16x^2y-8x\,y^2-4x^2+4xy+1\right)^2\left(xa_5+2ya_6+a_3\right) \\
&- \left(-24x^2+32xy-8y^2-8x+4y\right)\left(x^2a_4+xya_5+y^2a_6+xa_2+ya_3+a_1\right) \\
&- \left(16x^2-16xy+4x\right)\left(x^2b_4+xyb_5+y^2b_6+xb_2+yb_3+b_1\right) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned}
&-64x^7a_5 + 256x^6ya_5 - 128x^6ya_6 - 384x^5y^2a_5 + 512x^5y^2a_6 \\
&+ 256x^4y^3a_5 - 768x^4y^3a_6 - 64x^3y^4a_5 + 512x^3y^4a_6 - 128x^2y^5a_6 \\
&- 64x^6a_3 - 64x^6a_5 + 256x^5ya_3 + 192x^5ya_5 - 128x^5ya_6 - 384x^4y^2a_3 \\
&- 192x^4y^2a_5 + 384x^4y^2a_6 + 256x^3y^3a_3 + 64x^3y^3a_5 - 384x^3y^3a_6 \\
&- 64x^2y^4a_3 + 128x^2y^4a_6 - 64x^5a_3 - 16x^5a_5 + 192x^4ya_3 + 32x^4ya_5 \\
&- 32x^4ya_6 - 192x^3y^2a_3 - 16x^3y^2a_5 + 64x^3y^2a_6 + 64x^2y^3a_3 \\
&- 32x^2y^3a_6 - 16x^4a_3 + 40x^4a_4 + 16x^4a_5 - 16x^4b_4 - 8x^4b_5 + 32x^3ya_3 \\
&- 64x^3ya_4 + 32x^3ya_6 + 16x^3yb_4 - 16x^3yb_6 - 16x^2y^2a_3 + 24x^2y^2a_4 \\
&- 32x^2y^2a_5 - 40x^2y^2a_6 + 8x^2y^2b_5 + 16x^2y^2b_6 + 16x\,y^3a_5 + 8y^4a_6 \\
&+ 32x^3a_2 + 16x^3a_3 + 16x^3a_4 + 8x^3a_5 - 16x^3b_2 - 8x^3b_3 - 4x^3b_4 - 4x^3b_5 \\
&- 48x^2ya_2 - 8x^2ya_3 - 12x^2ya_4 + 4x^2ya_5 + 16x^2ya_6 + 16x^2yb_2 - 8x^2yb_6 \\
&+ 16x\,y^2a_2 - 16x\,y^2a_3 - 8x\,y^2a_5 - 8x\,y^2a_6 + 8x\,y^2b_3 + 4x\,y^2b_6 + 8y^3a_3 \\
&- 4y^3a_6 + 24x^2a_1 + 12x^2a_2 + 8x^2a_3 - 16x^2b_1 - 4x^2b_2 - 4x^2b_3 - 32xya_1 \\
&- 8xya_2 + 16xyb_1 + 8y^2a_1 - 4y^2a_3 + 8xa_1 - 2xa_4 - xa_5 - 4xb_1 \\
&+ 2xb_4 + xb_5 - 4ya_1 - ya_5 - 2ya_6 + yb_5 + 2yb_6 - a_2 - a_3 + b_2 + b_3 = 0
\end{aligned}$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-64x^7a_5 + 256x^6ya_5 - 128x^6ya_6 - 384x^5y^2a_5 + 512x^5y^2a_6 \\
&+256x^4y^3a_5 - 768x^4y^3a_6 - 64x^3y^4a_5 + 512x^3y^4a_6 - 128x^2y^5a_6 \\
&- 64x^6a_3 - 64x^6a_5 + 256x^5ya_3 + 192x^5ya_5 - 128x^5ya_6 \\
&- 384x^4y^2a_3 - 192x^4y^2a_5 + 384x^4y^2a_6 + 256x^3y^3a_3 \\
&+ 64x^3y^3a_5 - 384x^3y^3a_6 - 64x^2y^4a_3 + 128x^2y^4a_6 - 64x^5a_3 \\
&- 16x^5a_5 + 192x^4ya_3 + 32x^4ya_5 - 32x^4ya_6 - 192x^3y^2a_3 \\
&- 16x^3y^2a_5 + 64x^3y^2a_6 + 64x^2y^3a_3 - 32x^2y^3a_6 - 16x^4a_3 \\
&+ 40x^4a_4 + 16x^4a_5 - 16x^4b_4 - 8x^4b_5 + 32x^3ya_3 - 64x^3ya_4 \\
&+ 32x^3ya_6 + 16x^3yb_4 - 16x^3yb_6 - 16x^2y^2a_3 + 24x^2y^2a_4 \\
&- 32x^2y^2a_5 - 40x^2y^2a_6 + 8x^2y^2b_5 + 16x^2y^2b_6 + 16x\,y^3a_5 \\
&+ 8y^4a_6 + 32x^3a_2 + 16x^3a_3 + 16x^3a_4 + 8x^3a_5 - 16x^3b_2 - 8x^3b_3 \\
&- 4x^3b_4 - 4x^3b_5 - 48x^2ya_2 - 8x^2ya_3 - 12x^2ya_4 + 4x^2ya_5 \\
&+ 16x^2ya_6 + 16x^2yb_2 - 8x^2yb_6 + 16x\,y^2a_2 - 16x\,y^2a_3 - 8x\,y^2a_5 \\
&- 8x\,y^2a_6 + 8x\,y^2b_3 + 4x\,y^2b_6 + 8y^3a_3 - 4y^3a_6 + 24x^2a_1 \\
&+ 12x^2a_2 + 8x^2a_3 - 16x^2b_1 - 4x^2b_2 - 4x^2b_3 - 32xya_1 - 8xya_2 \\
&+ 16xyb_1 + 8y^2a_1 - 4y^2a_3 + 8xa_1 - 2xa_4 - xa_5 - 4xb_1 + 2xb_4 \\
&+ xb_5 - 4ya_1 - ya_5 - 2ya_6 + yb_5 + 2yb_6 - a_2 - a_3 + b_2 + b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-64a_5v_1^7+256a_5v_1^6v_2-384a_5v_1^5v_2^2+256a_5v_1^4v_2^3-64a_5v_1^3v_2^4\\
&-128a_6v_1^6v_2+512a_6v_1^5v_2^2-768a_6v_1^4v_2^3+512a_6v_1^3v_2^4\\
&-128a_6v_1^2v_2^5-64a_3v_1^6+256a_3v_1^5v_2-384a_3v_1^4v_2^2+256a_3v_1^3v_2^3\\
&-64a_3v_1^2v_2^4-64a_5v_1^6+192a_5v_1^5v_2-192a_5v_1^4v_2^2+64a_5v_1^3v_2^3\\
&-128a_6v_1^5v_2+384a_6v_1^4v_2^2-384a_6v_1^3v_2^3+128a_6v_1^2v_2^4\\
&-64a_3v_1^5+192a_3v_1^4v_2-192a_3v_1^3v_2^2+64a_3v_1^2v_2^3-16a_5v_1^5\\
&+32a_5v_1^4v_2-16a_5v_1^3v_2^2-32a_6v_1^4v_2+64a_6v_1^3v_2^2-32a_6v_1^2v_2^3\\
&-16a_3v_1^4+32a_3v_1^3v_2-16a_3v_1^2v_2^2+40a_4v_1^4-64a_4v_1^3v_2\\
&+24a_4v_1^2v_2^2+16a_5v_1^4-32a_5v_1^2v_2^2+16a_5v_1v_2^3+32a_6v_1^3v_2\\
&-40a_6v_1^2v_2^2+8a_6v_2^4-16b_4v_1^4+16b_4v_1^3v_2-8b_5v_1^4+8b_5v_1^2v_2^2\\
&-16b_6v_1^3v_2+16b_6v_1^2v_2^2+32a_2v_1^3-48a_2v_1^2v_2+16a_2v_1v_2^2\\
&+16a_3v_1^3-8a_3v_1^2v_2-16a_3v_1v_2^2+8a_3v_2^3+16a_4v_1^3-12a_4v_1^2v_2\\
&+8a_5v_1^3+4a_5v_1^2v_2-8a_5v_1v_2^2+16a_6v_1^2v_2-8a_6v_1v_2^2-4a_6v_2^3\\
&-16b_2v_1^3+16b_2v_1^2v_2-8b_3v_1^3+8b_3v_1v_2^2-4b_4v_1^3-4b_5v_1^3\\
&-8b_6v_1^2v_2+4b_6v_1v_2^2+24a_1v_1^2-32a_1v_1v_2+8a_1v_2^2+12a_2v_1^2\\
&-8a_2v_1v_2+8a_3v_1^2-4a_3v_2^2-16b_1v_1^2+16b_1v_1v_2-4b_2v_1^2\\
&-4b_3v_1^2+8a_1v_1-4a_1v_2-2a_4v_1-a_5v_1-a_5v_2-2a_6v_2\\
&-4b_1v_1+2b_4v_1+b_5v_1+b_5v_2+2b_6v_2-a_2-a_3+b_2+b_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-a_2 - a_3 + b_2 + b_3 \\
&+ (-16a_3 + 24a_4 - 32a_5 - 40a_6 + 8b_5 + 16b_6)\, v_1^2 v_2^2 \\
&+ (-48a_2 - 8a_3 - 12a_4 + 4a_5 + 16a_6 + 16b_2 - 8b_6)\, v_1^2 v_2 \\
&+ (16a_2 - 16a_3 - 8a_5 - 8a_6 + 8b_3 + 4b_6)\, v_1 v_2^2 \\
&+ (-32a_1 - 8a_2 + 16b_1)\, v_1 v_2 + (256a_5 - 128a_6)\, v_1^6 v_2 \\
&+ (-384a_5 + 512a_6)\, v_1^5 v_2^2 + (256a_3 + 192a_5 - 128a_6)\, v_1^5 v_2 \\
&+ (256a_5 - 768a_6)\, v_1^4 v_2^3 + (-384a_3 - 192a_5 + 384a_6)\, v_1^4 v_2^2 \\
&+ (192a_3 + 32a_5 - 32a_6)\, v_1^4 v_2 + (-64a_5 + 512a_6)\, v_1^3 v_2^4 \\
&+ (256a_3 + 64a_5 - 384a_6)\, v_1^3 v_2^3 + (-192a_3 - 16a_5 + 64a_6)\, v_1^3 v_2^2 \\
&+ (32a_3 - 64a_4 + 32a_6 + 16b_4 - 16b_6)\, v_1^3 v_2 + (-64a_3 + 128a_6)\, v_1^2 v_2^4 \\
&+ (64a_3 - 32a_6)\, v_1^2 v_2^3 - 128a_6 v_1^2 v_2^5 - 64a_5 v_1^7 + 8a_6 v_2^4 \\
&+ 16a_5 v_1 v_2^3 + (24a_1 + 12a_2 + 8a_3 - 16b_1 - 4b_2 - 4b_3)\, v_1^2 \\
&+ (8a_1 - 2a_4 - a_5 - 4b_1 + 2b_4 + b_5)\, v_1 + (8a_3 - 4a_6)\, v_2^3 \\
&+ (8a_1 - 4a_3)\, v_2^2 + (-4a_1 - a_5 - 2a_6 + b_5 + 2b_6)\, v_2 \\
&+ (-64a_3 - 64a_5)\, v_1^6 + (-64a_3 - 16a_5)\, v_1^5 \\
&+ (-16a_3 + 40a_4 + 16a_5 - 16b_4 - 8b_5)\, v_1^4 \\
&+ (32a_2 + 16a_3 + 16a_4 + 8a_5 - 16b_2 - 8b_3 - 4b_4 - 4b_5)\, v_1^3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-64a_5 &= 0 \\
16a_5 &= 0 \\
-128a_6 &= 0 \\
8a_6 &= 0 \\
8a_1 - 4a_3 &= 0 \\
-64a_3 - 64a_5 &= 0 \\
-64a_3 - 16a_5 &= 0 \\
-64a_3 + 128a_6 &= 0 \\
8a_3 - 4a_6 &= 0 \\
64a_3 - 32a_6 &= 0 \\
-384a_5 + 512a_6 &= 0 \\
-64a_5 + 512a_6 &= 0 \\
256a_5 - 768a_6 &= 0 \\
256a_5 - 128a_6 &= 0 \\
-32a_1 - 8a_2 + 16b_1 &= 0 \\
-384a_3 - 192a_5 + 384a_6 &= 0 \\
-192a_3 - 16a_5 + 64a_6 &= 0 \\
192a_3 + 32a_5 - 32a_6 &= 0 \\
256a_3 + 64a_5 - 384a_6 &= 0 \\
256a_3 + 192a_5 - 128a_6 &= 0 \\
-a_2 - a_3 + b_2 + b_3 &= 0 \\
-4a_1 - a_5 - 2a_6 + b_5 + 2b_6 &= 0 \\
-16a_3 + 40a_4 + 16a_5 - 16b_4 - 8b_5 &= 0 \\
32a_3 - 64a_4 + 32a_6 + 16b_4 - 16b_6 &= 0 \\
8a_1 - 2a_4 - a_5 - 4b_1 + 2b_4 + b_5 &= 0 \\
24a_1 + 12a_2 + 8a_3 - 16b_1 - 4b_2 - 4b_3 &= 0 \\
16a_2 - 16a_3 - 8a_5 - 8a_6 + 8b_3 + 4b_6 &= 0 \\
-16a_3 + 24a_4 - 32a_5 - 40a_6 + 8b_5 + 16b_6 &= 0 \\
-48a_2 - 8a_3 - 12a_4 + 4a_5 + 16a_6 + 16b_2 - 8b_6 &= 0 \\
32a_2 + 16a_3 + 16a_4 + 8a_5 - 16b_2 - 8b_3 - 4b_4 - 4b_5 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= 0 \\
b_2 &= \frac{b_6}{2} \\
b_3 &= -\frac{b_6}{2} \\
b_4 &= b_6 \\
b_5 &= -2b_6 \\
b_6 &= b_6
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= x^2 - 2xy + y^2 + \frac{1}{2}x - \frac{1}{2}y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{x^2 - 2xy + y^2 + \frac{1}{2}x - \frac{1}{2}y} dy
\end{aligned}
$$

Which results in

$$
S = 2\ln(2y - 2x - 1) - 2\ln(y - x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -8x^3 + 16x^2y - 8x\,y^2 - 4x^2 + 4xy + 1
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2}{(2x - 2y + 1)(-y + x)} \\
S_y &= \frac{2}{(2x - 2y + 1)(-y + x)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -8x \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -8R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -4R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2\ln(2y - 2x - 1) - 2\ln(y - x) = -4x^2 + c_1$$

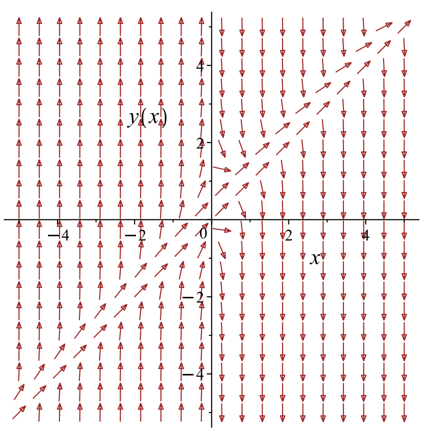
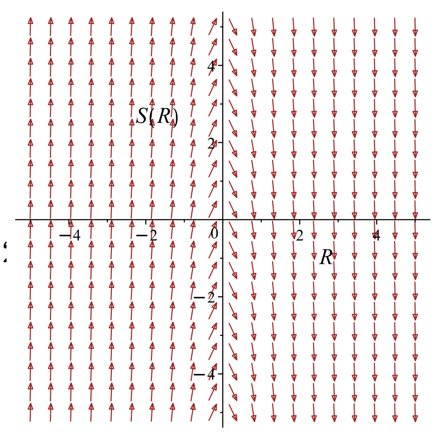
Which simplifies to

$$2\ln(2y - 2x - 1) - 2\ln(y - x) = -4x^2 + c_1$$

Which gives

$$y = \frac{x\,\mathrm{e}^{-2x^2+\frac{c_1}{2}} - 2x - 1}{\mathrm{e}^{-2x^2+\frac{c_1}{2}} - 2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -8x^3+16x^2y-8x\,y^2-4x^2+4xy+1$ 	$R = x$ $S = 2\ln(2y-2x-1) -$	$\frac{dS}{dR} = -8R$ 

Summary
The solution(s) found are the following

$$y = \frac{x\,\mathrm{e}^{-2x^2+\frac{c_1}{2}} - 2x - 1}{\mathrm{e}^{-2x^2+\frac{c_1}{2}} - 2} \tag{1}$$

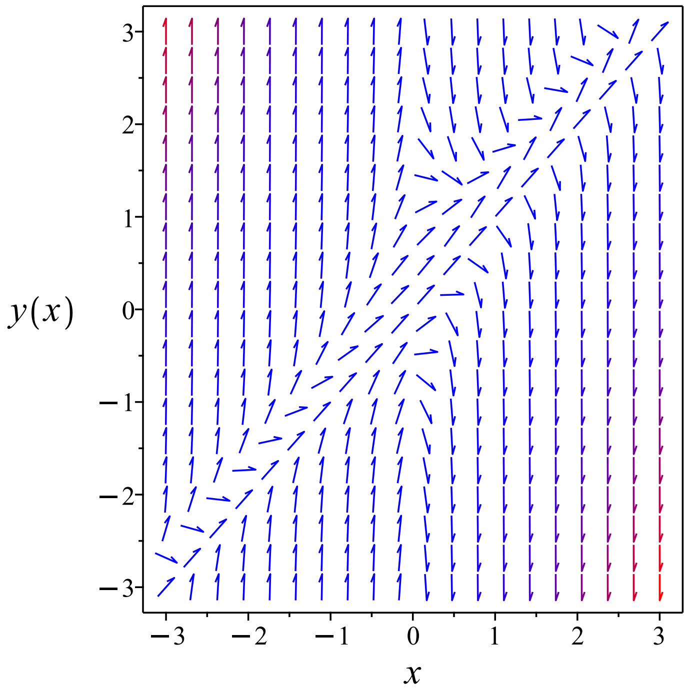


Figure 241: Slope field plot

Verification of solutions

$$y = \frac{x\,\mathrm{e}^{-2x^2+\frac{c_1}{2}} - 2x - 1}{\mathrm{e}^{-2x^2+\frac{c_1}{2}} - 2}$$

Verified OK.

5.37.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -8x^3 + 16x^2y - 8x\,y^2 - 4x^2 + 4xy + 1 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -8x^3 + 16x^2y - 8x\,y^2 - 4x^2 + 4xy + 1$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -8x^3 - 4x^2 + 1$, $f_1(x) = 16x^2 + 4x$ and $f_2(x) = -8x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-8xu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -8 \\ f_1 f_2 &= -8\left(16x^2 + 4x\right) x \\ f_2^2 f_0 &= 64x^2\left(-8x^3 - 4x^2 + 1\right) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-8xu''(x) - \left(-8 - 8\left(16x^2 + 4x\right) x\right) u'(x) + 64x^2\left(-8x^3 - 4x^2 + 1\right) u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \mathrm{e}^{\frac{8x^3}{3}} + c_2 \mathrm{e}^{\frac{8}{3}x^3 + 2x^2}$$

The above shows that

$$u'(x) = 8x\left(c_2\left(x + \frac{1}{2}\right) \mathrm{e}^{\frac{8}{3}x^3 + 2x^2} + \mathrm{e}^{\frac{8x^3}{3}} c_1 x\right)$$

Using the above in (1) gives the solution

$$y = \frac{c_2\left(x + \frac{1}{2}\right) \mathrm{e}^{\frac{8}{3}x^3 + 2x^2} + \mathrm{e}^{\frac{8x^3}{3}} c_1 x}{c_1 \mathrm{e}^{\frac{8x^3}{3}} + c_2 \mathrm{e}^{\frac{8}{3}x^3 + 2x^2}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{2\,\mathrm{e}^{\frac{8x^3}{3}} c_3 x + 2\,\mathrm{e}^{\frac{8}{3}x^3 + 2x^2} x + \mathrm{e}^{\frac{8}{3}x^3 + 2x^2}}{2c_3 \mathrm{e}^{\frac{8x^3}{3}} + 2\,\mathrm{e}^{\frac{8}{3}x^3 + 2x^2}}$$

Summary

The solution(s) found are the following

$$y = \frac{2\,\mathrm{e}^{\frac{8x^3}{3}} c_3 x + 2\,\mathrm{e}^{\frac{8}{3}x^3+2x^2} x + \mathrm{e}^{\frac{8}{3}x^3+2x^2}}{2c_3\mathrm{e}^{\frac{8x^3}{3}} + 2\,\mathrm{e}^{\frac{8}{3}x^3+2x^2}} \tag{1}$$

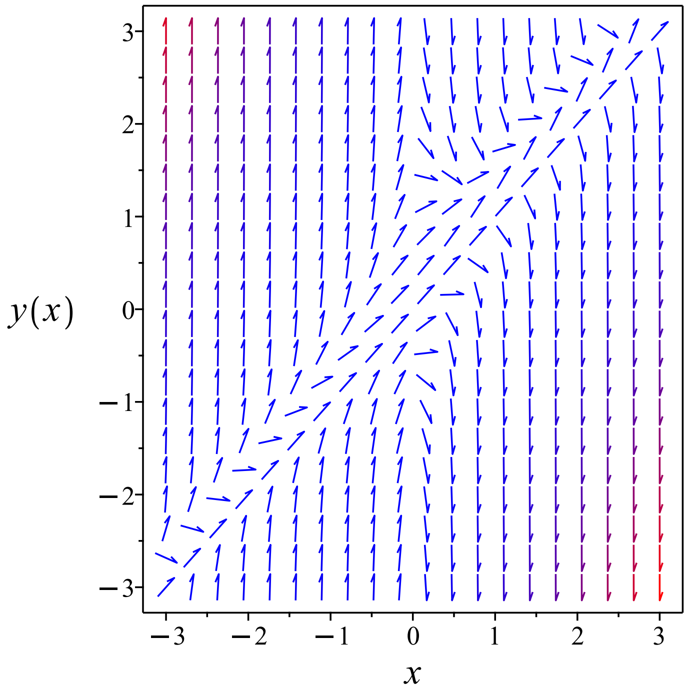


Figure 242: Slope field plot

Verification of solutions

$$y = \frac{2\,\mathrm{e}^{\frac{8x^3}{3}} c_3 x + 2\,\mathrm{e}^{\frac{8}{3}x^3+2x^2} x + \mathrm{e}^{\frac{8}{3}x^3+2x^2}}{2c_3\mathrm{e}^{\frac{8x^3}{3}} + 2\,\mathrm{e}^{\frac{8}{3}x^3+2x^2}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (16*x^3+4*x^2+1)*(diff(y(x), x
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Reducible group (found an exponential solution)
         Reducible group (found another exponential solution)
      <- Kovacics algorithm successful
   <- Riccati to 2nd Order successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 60

```
dsolve(diff(y(x),x)=-8*x*y(x)^2+4*x*(4*x+1)*y(x)-(8*x^3+4*x^2-1),y(x), singsol=all)
```

$$y(x) = \frac{c_1(2x+1)\,\mathrm{e}^{\frac{8}{3}x^3+2x^2} + 2\,\mathrm{e}^{\frac{8x^3}{3}}x}{2c_1\mathrm{e}^{\frac{8}{3}x^3+2x^2} + 2\,\mathrm{e}^{\frac{8x^3}{3}}}$$

✓ Solution by Mathematica

Time used: 0.196 (sec). Leaf size: 30

```
DSolve[y'[x]==-8*x*y[x]^2+4*x*(4*x+1)*y[x]-(8*x^3+4*x^2-1),y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{1}{4}\left(\tanh\left(x^2 + ic_1\right) + 4x + 1\right)$$
$$y(x) \to \text{Indeterminate}$$

6 Chapter 2, Miscellaneous Review. Exercises page 60

6.1 problem 1

Internal problem ID [11675]
Internal file name [`OUTPUT/11684_Wednesday_April_10_2024_04_54_38_PM_81155065/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{6x^2y - \left(x^3+1\right)y' = 0}$$

6.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{6x^2y}{x^3+1} \end{aligned}$$

Where $f(x) = \frac{6x^2}{x^3+1}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\, dy &= \frac{6x^2}{x^3+1}\, dx \\
\int \frac{1}{y}\, dy &= \int \frac{6x^2}{x^3+1}\, dx \\
\ln(y) &= 2\ln\left(x^3+1\right) + c_1 \\
y &= \mathrm{e}^{2\ln(x^3+1)+c_1} \\
&= c_1\left(x^3+1\right)^2
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\left(x^3+1\right)^2 \tag{1}$$

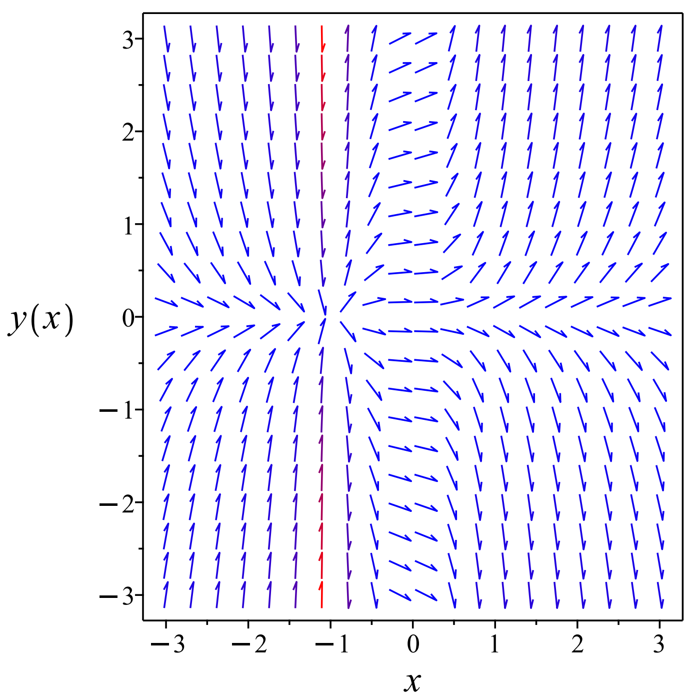


Figure 243: Slope field plot

Verification of solutions

$$y = c_1\left(x^3+1\right)^2$$

Verified OK.

6.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{6x^2}{x^3+1}\\ q(x) &= 0\end{aligned}$$

Hence the ode is

$$y' - \frac{6x^2 y}{x^3+1} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{6x^2}{x^3+1}dx}\\ &= \frac{1}{(x^3+1)^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{(x^3+1)^2}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{y}{(x^3+1)^2} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(x^3+1)^2}$ results in

$$y = c_1(x^3+1)^2$$

Summary
The solution(s) found are the following

$$y = c_1(x^3+1)^2 \tag{1}$$

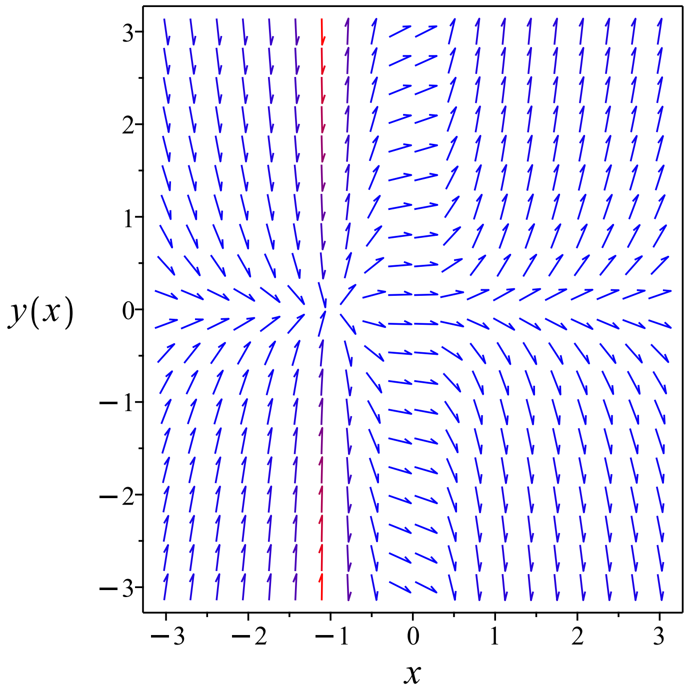


Figure 244: Slope field plot

Verification of solutions

$$y = c_1 (x^3 + 1)^2$$

Verified OK.

6.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$6x^3u(x) - (x^3 + 1)\,(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(5x^3 - 1)}{x\,(x^3 + 1)} \end{aligned}$$

Where $f(x) = \frac{5x^3-1}{x(x^3+1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{5x^3-1}{x\,(x^3+1)}\,dx\\
\int \frac{1}{u}\,du &= \int \frac{5x^3-1}{x\,(x^3+1)}\,dx\\
\ln(u) &= 2\ln(1+x) - \ln(x) + 2\ln\left(x^2-x+1\right) + c_2\\
u &= \mathrm{e}^{2\ln(1+x)-\ln(x)+2\ln\left(x^2-x+1\right)+c_2}\\
&= c_2\mathrm{e}^{2\ln(1+x)-\ln(x)+2\ln\left(x^2-x+1\right)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_2\left(x^5 + 2x^2 + \frac{1}{x}\right)
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu\\
&= xc_2\left(x^5 + 2x^2 + \frac{1}{x}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = xc_2\left(x^5 + 2x^2 + \frac{1}{x}\right) \tag{1}
$$

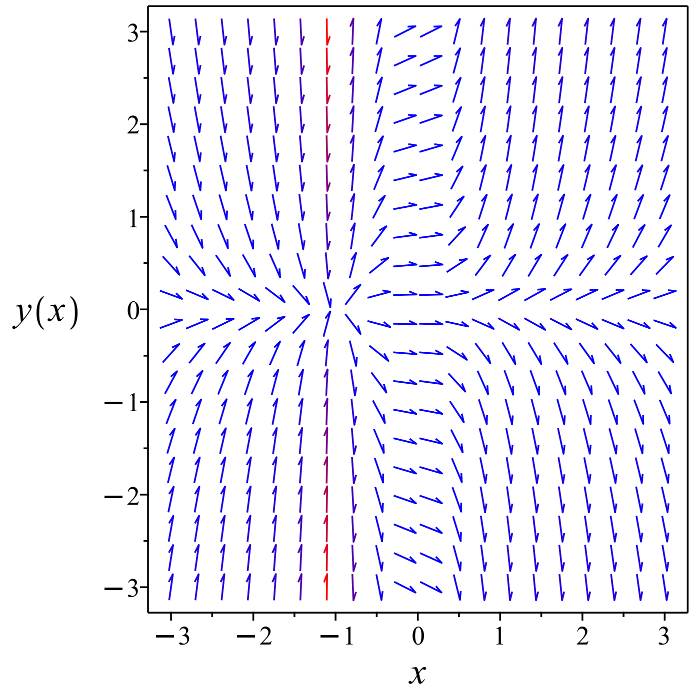


Figure 245: Slope field plot

Verification of solutions

$$y = xc_2\left(x^5 + 2x^2 + \frac{1}{x}\right)$$

Verified OK.

6.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{6x^2y}{x^3+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 203: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \left(x^3+1\right)^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{(x^3+1)^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{(x^3+1)^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{6x^2 y}{x^3+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{6y\,x^2}{(x^3+1)^3} \\ S_y &= \frac{1}{(x^3+1)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{(x^3+1)^2} = c_1$$

Which simplifies to

$$\frac{y}{(x^3+1)^2} = c_1$$

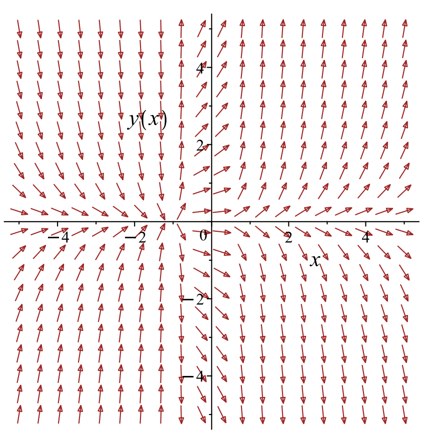
Which gives

$$y = c_1 (x^3+1)^2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>

$\frac{dy}{dx} = \frac{6x^2y}{x^3+1}$



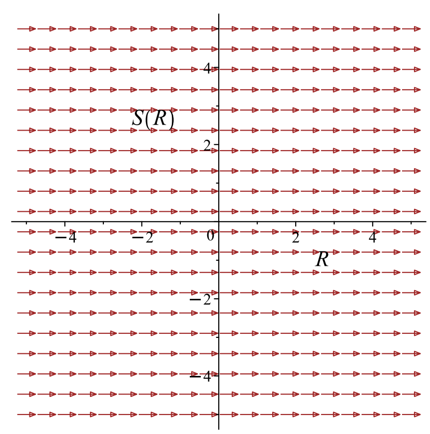
</td><td>

$R = x$

$S = \frac{y}{(x^3+1)^2}$

</td><td>

$\frac{dS}{dR} = 0$



</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = c_1 (x^3+1)^2 \tag{1}$$

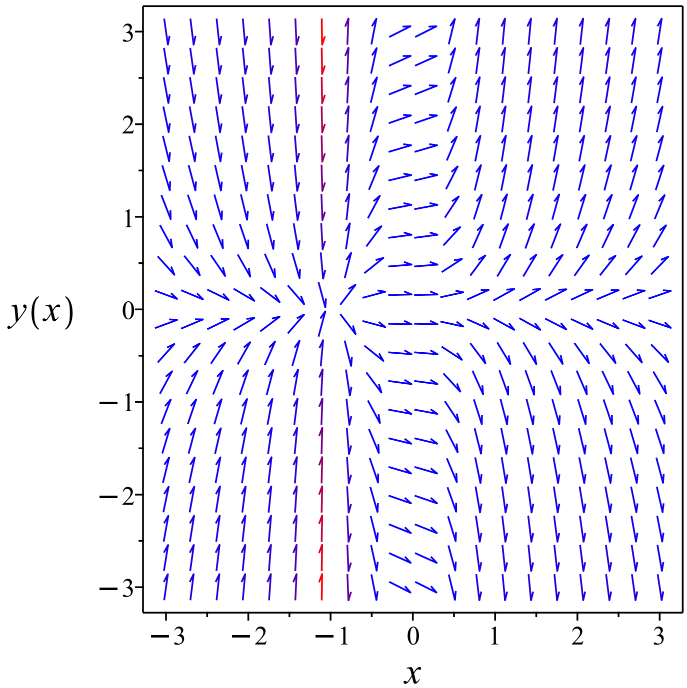


Figure 246: Slope field plot

<u>Verification of solutions</u>

$$y = c_1(x^3+1)^2$$

Verified OK.

6.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{6y}\right) \mathrm{d}y = \left(\frac{x^2}{x^3 + 1}\right) \mathrm{d}x$$
$$\left(-\frac{x^2}{x^3 + 1}\right) \mathrm{d}x + \left(\frac{1}{6y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x^2}{x^3 + 1}$$
$$N(x, y) = \frac{1}{6y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{x^2}{x^3 + 1}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{6y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{x^2}{x^3+1} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln(x^3+1)}{3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{6y}$. Therefore equation (4) becomes

$$
\frac{1}{6y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{6y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{6y}\right) \mathrm{d}y$$

$$f(y) = \frac{\ln(y)}{6} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^3+1)}{3} + \frac{\ln(y)}{6} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^3+1)}{3} + \frac{\ln(y)}{6}$$

The solution becomes

$$y = \mathrm{e}^{6c_1}\left(x^3+1\right)^2$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{6c_1}\left(x^3+1\right)^2 \tag{1}$$

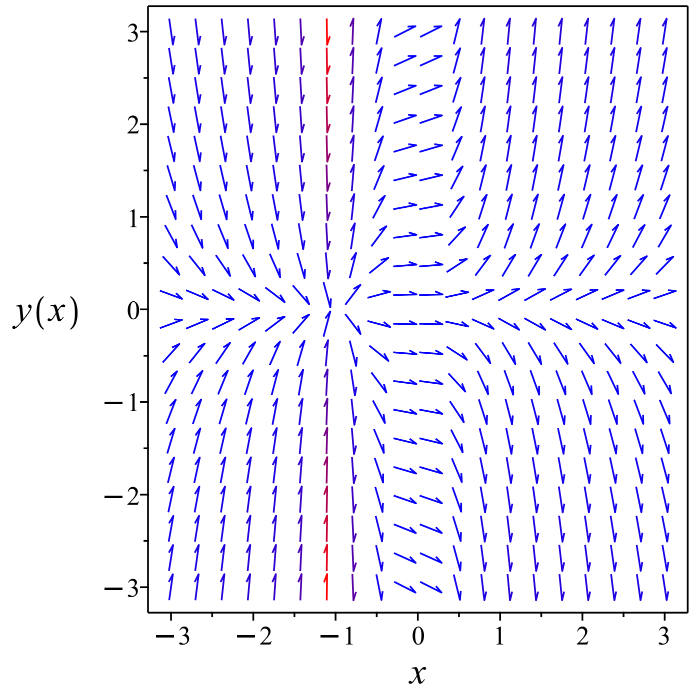


Figure 247: Slope field plot

Verification of solutions

$$y = e^{6c_1}(x^3+1)^2$$

Verified OK.

6.1.6 Maple step by step solution

Let's solve

$6x^2y - (x^3+1)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{6x^2}{x^3+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{6x^2}{x^3+1}dx + c_1$

- Evaluate integral

$$\ln(y) = 2\ln(x^3+1) + c_1$$

- Solve for y

$$y = e^{c_1}(x^3+1)^2$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(6*x^2*y(x)-(x^3+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1(x^3+1)^2$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 20

```
DSolve[6*x^2*y[x]-(x^3+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x^3+1)^2$$
$$y(x) \to 0$$

6.2 problem 2

Internal problem ID [11676]
Internal file name [`OUTPUT/11685_Wednesday_April_10_2024_04_54_39_PM_80429987/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _exact, _rational]
```

$$\boxed{(3x^2y^2 - x)\,y' + 2y^3x - y = 0}$$

6.2.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(2x\,y^2 - 1)}{x\,(3x\,y^2 - 1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{y(2x\,y^2-1)\,(b_3-a_2)}{x\,(3x\,y^2-1)} - \frac{y^2(2x\,y^2-1)^2\,a_3}{x^2\,(3x\,y^2-1)^2} \\ &- \left(-\frac{2y^3}{x\,(3x\,y^2-1)} + \frac{y(2x\,y^2-1)}{x^2\,(3x\,y^2-1)} + \frac{3y^3(2x\,y^2-1)}{x\,(3x\,y^2-1)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{2x\,y^2-1}{x\,(3x\,y^2-1)} - \frac{4y^2}{3x\,y^2-1} + \frac{6y^2(2x\,y^2-1)}{(3x\,y^2-1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{15x^4y^4b_2 - 10x^2y^6a_3 + 6x^3y^4b_1 - 6x^2y^5a_1 - 9x^3y^2b_2 + x^2y^3a_2 + 2x^2y^3b_3 + 10x\,y^4a_3 - 3x^2y^2b_1 + 6x\,y^3a_1}{x^2\,(3x\,y^2-1)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &15x^4y^4b_2 - 10x^2y^6a_3 + 6x^3y^4b_1 - 6x^2y^5a_1 - 9x^3y^2b_2 + x^2y^3a_2 + 2x^2y^3b_3 \\ &+ 10x\,y^4a_3 - 3x^2y^2b_1 + 6x\,y^3a_1 + 2b_2x^2 - 2y^2a_3 + xb_1 - ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-10a_3v_1^2v_2^6 + 15b_2v_1^4v_2^4 - 6a_1v_1^2v_2^5 + 6b_1v_1^3v_2^4 + a_2v_1^2v_2^3 + 10a_3v_1v_2^4 - 9b_2v_1^3v_2^2 \\ &+ 2b_3v_1^2v_2^3 + 6a_1v_1v_2^3 - 3b_1v_1^2v_2^2 - 2a_3v_2^2 + 2b_2v_1^2 - a_1v_2 + b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$15b_2v_1^4v_2^4 + 6b_1v_1^3v_2^4 - 9b_2v_1^3v_2^2 - 10a_3v_1^2v_2^6 - 6a_1v_1^2v_2^5 + (a_2 + 2b_3)\, v_1^2v_2^3 - 3b_1v_1^2v_2^2 + 2b_2v_1^2 + 10a_3v_1v_2^4 + 6a_1v_1v_2^3 + b_1v_1 - 2a_3v_2^2 - a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -6a_1 &= 0 \\ -a_1 &= 0 \\ 6a_1 &= 0 \\ -10a_3 &= 0 \\ -2a_3 &= 0 \\ 10a_3 &= 0 \\ -3b_1 &= 0 \\ 6b_1 &= 0 \\ -9b_2 &= 0 \\ 2b_2 &= 0 \\ 15b_2 &= 0 \\ a_2 + 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -2b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -2x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{y(2x\,y^2 - 1)}{x\,(3x\,y^2 - 1)}\right)(-2x) \\
&= \frac{-x\,y^3 + y}{3x\,y^2 - 1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x\,y^3 + y}{3x\,y^2 - 1}} dy
\end{aligned}$$

Which results in

$$S = -\ln\left(y\left(x\,y^2 - 1\right)\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(2x\,y^2 - 1)}{x\,(3x\,y^2 - 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y^2}{x\,y^2-1} \\
S_y &= -\frac{1}{y}-\frac{2xy}{x\,y^2-1}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R)=\ln (R)+c_1 \tag{4}$$

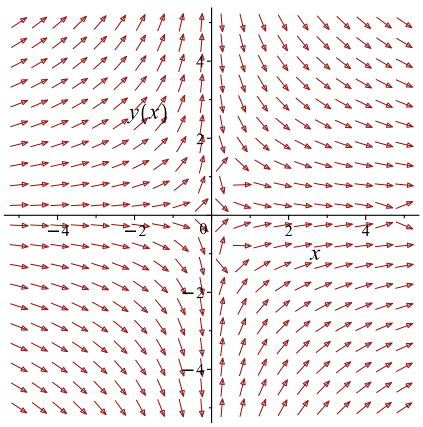
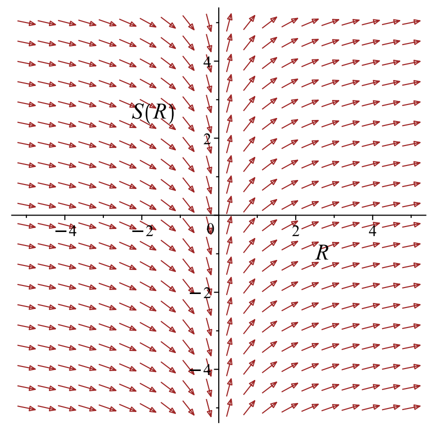
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln (y)-\ln \left(y^2 x-1\right)=\ln (x)+c_1$$

Which simplifies to

$$-\ln (y)-\ln \left(y^2 x-1\right)=\ln (x)+c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(2x\,y^2-1)}{x(3x\,y^2-1)}$ 	$R = x$ $S = -\ln(y) - \ln(x\,y^2 -$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$-\ln(y) - \ln\left(y^2 x - 1\right) = \ln(x) + c_1 \tag{1}$$

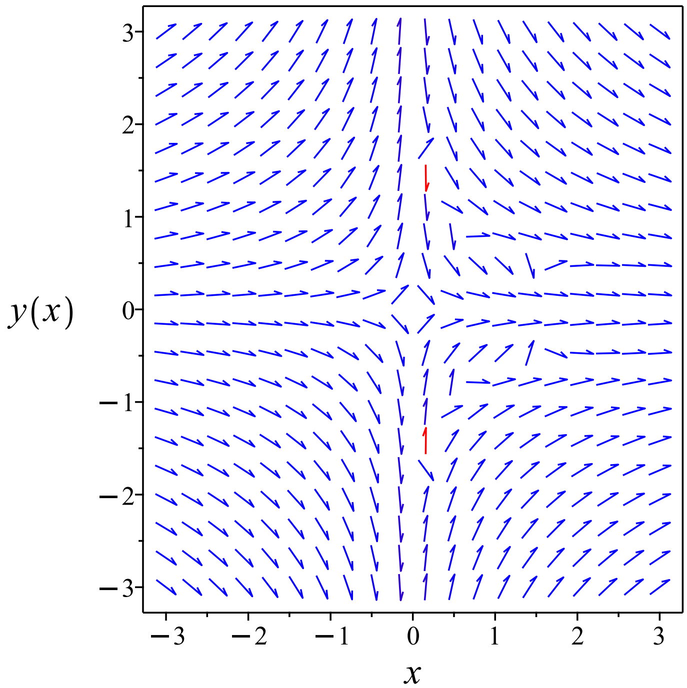


Figure 248: Slope field plot

Verification of solutions

$$-\ln(y) - \ln\left(y^2 x - 1\right) = \ln(x) + c_1$$

Verified OK.

6.2.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(3x^2 y^2 - x\right) \mathrm{d}y &= \left(-2x \, y^3 + y\right) \mathrm{d}x \\
\left(2x \, y^3 - y\right) \mathrm{d}x + \left(3x^2 y^2 - x\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 2x \, y^3 - y \\
N(x, y) &= 3x^2 y^2 - x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x \, y^3 - y\right) \\
&= 6x \, y^2 - 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(3x^2 y^2 - x\right) \\
&= 6x \, y^2 - 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$
$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int 2x\,y^3-y\,\mathrm{d}x$$
$$\phi=xy\left(x\,y^2-1\right)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=x\left(x\,y^2-1\right)+2x^2y^2+f'(y) \tag{4}$$
$$=3x^2y^2-x+f'(y)$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=3x^2y^2-x$. Therefore equation (4) becomes

$$3x^2y^2-x=3x^2y^2-x+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=xy\left(x\,y^2-1\right)+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=xy\left(x\,y^2-1\right)$$

Summary

The solution(s) found are the following

$$y(y^2x - 1)\,x = c_1 \tag{1}$$

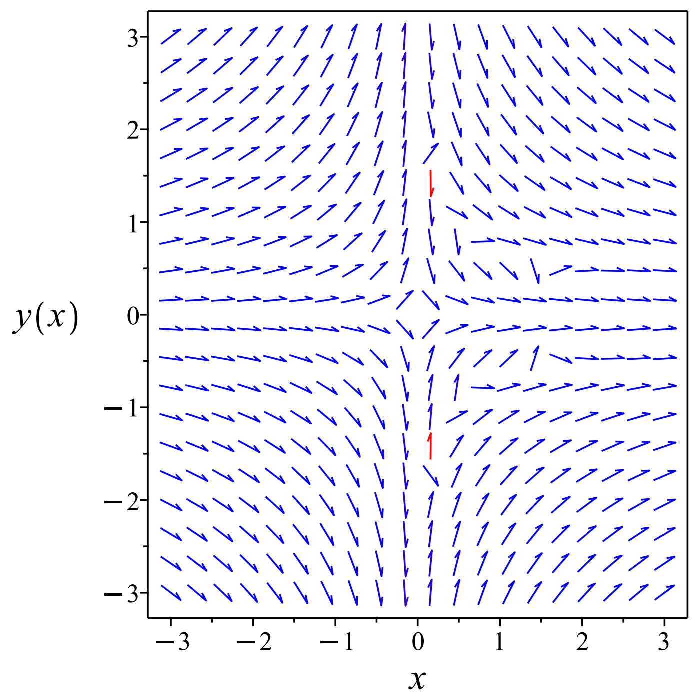


Figure 249: Slope field plot

Verification of solutions

$$y(y^2x - 1)\,x = c_1$$

Verified OK.

6.2.3 Maple step by step solution

Let's solve

$(3x^2y^2 - x)\,y' + 2y^3x - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ○ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $6x\,y^2 - 1 = 6x\,y^2 - 1$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (2x\,y^3 - y)\,dx + f_1(y)$

- Evaluate integral

  $F(x, y) = y(x^2y^2 - x) + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $3x^2y^2 - x = 3x^2y^2 - x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = y(x^2y^2 - x)$

- Substitute $F(x, y)$ into the solution of the ODE

  $y(x^2y^2 - x) = c_1$

- Solve for y

$$\left\{y = \frac{\left(\left(12\sqrt{3}\,\sqrt{27c_1^2-4x}+108c_1\right)x\right)^{\frac{1}{3}}}{6x} + \frac{2}{\left(\left(12\sqrt{3}\,\sqrt{27c_1^2-4x}+108c_1\right)x\right)^{\frac{1}{3}}}, y = -\frac{\left(\left(12\sqrt{3}\,\sqrt{27c_1^2-4x}+108c_1\right)x\right)^{\frac{1}{3}}}{12x} -\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 1127

```
dsolve((3*x^2*y(x)^2-x)*diff(y(x),x)+(2*x*y(x)^3-y(x))=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{3}\,\sqrt{2}\,\sqrt{\left(\left(3\sqrt{3}\,\sqrt{27c_1^2-4c_1x}+27c_1-2x\right)x^2\right)^{\frac{1}{3}}\left(2x^2 2^{\frac{1}{3}}+2^{\frac{2}{3}}\left(\left(3\sqrt{3}\,\sqrt{27c_1^2-4c_1x}+27c_1-2x\right)\right.\right.}}{6\left(\left(3\sqrt{3}\,\sqrt{27c_1^2-4c_1x}+27c_1-2x\right)x^2\right)^{\frac{1}{3}}x}$$

$$y(x) = \frac{\sqrt{3}\,\sqrt{2}\,\sqrt{\left(\left(3\sqrt{3}\,\sqrt{27c_1^2-4c_1x}+27c_1-2x\right)x^2\right)^{\frac{1}{3}}\left(2x^2 2^{\frac{1}{3}}+2^{\frac{2}{3}}\left(\left(3\sqrt{3}\,\sqrt{27c_1^2-4c_1x}+27c_1-2x\right)^2 x\right.\right.}}{6\left(\left(3\sqrt{3}\,\sqrt{27c_1^2-4c_1x}+27c_1-2x\right)x^2\right)^{\frac{1}{3}}x}$$

$$y(x) = -\frac{\sqrt{3}\,\sqrt{\left(i\left(-2x^2 2^{\frac{1}{3}}+2^{\frac{2}{3}}\left(\left(3\sqrt{3}\,\sqrt{27c_1^2-4c_1x}+27c_1-2x\right)^2 x^4\right)^{\frac{1}{3}}\right)\sqrt{3}-2x^2 2^{\frac{1}{3}}+8x\left(\left(3\sqrt{3}\,\sqrt{27c_1^2\right.\right.\right.}}}{6\left(\left(3\sqrt{3}\,\sqrt{2\right.\right.}}$$

$$y(x) = \frac{\sqrt{3}\,\sqrt{\left(i\left(-2x^2 2^{\frac{1}{3}}+2^{\frac{2}{3}}\left(\left(3\sqrt{3}\,\sqrt{27c_1^2-4c_1x}+27c_1-2x\right)^2 x^4\right)^{\frac{1}{3}}\right)\sqrt{3}-2x^2 2^{\frac{1}{3}}+8x\left(\left(3\sqrt{3}\,\sqrt{27c_1^2-\right.\right.\right.}}}{6\left(\left(3\sqrt{3}\,\sqrt{27c_1^2\right.\right.}}$$

$$y(x) = -\frac{\sqrt{3}\,\sqrt{\left(-i\left(-2x^2 2^{\frac{1}{3}}+2^{\frac{2}{3}}\left(\left(3\sqrt{3}\,\sqrt{27c_1^2-4c_1x}+27c_1-2x\right)^2 x^4\right)^{\frac{1}{3}}\right)\sqrt{3}-2x^2 2^{\frac{1}{3}}+8x\left(\left(3\sqrt{3}\,\sqrt{27\right.\right.\right.}}}{6\left(\left(3\sqrt{3}\,\sqrt{2\right.\right.}}$$

$$y(x) = \frac{\sqrt{3}\,\sqrt{\left(-i\left(-2x^2 2^{\frac{1}{3}}+2^{\frac{2}{3}}\left(\left(3\sqrt{3}\,\sqrt{27c_1^2-4c_1x}+27c_1-2x\right)^2 x^4\right)^{\frac{1}{3}}\right)\sqrt{3}-2x^2 2^{\frac{1}{3}}+8x\left(\left(3\sqrt{3}\,\sqrt{27c_1^2\right.\right.\right.}}}{6\left(\left(3\sqrt{3}\,\sqrt{27c\right.\right.}}$$

✓ Solution by Mathematica

Time used: 30.566 (sec). Leaf size: 356

```
DSolve[(3*x^2*y[x]^2-x)*y'[x]+(2*x*y[x]^3-y[x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2\sqrt[3]{3}x^3 + \sqrt[3]{2}\left(\sqrt{3}\sqrt{x^8\left(-4x+27c_1^2\right)}+9c_1x^4\right)^{2/3}}{6^{2/3}x^2\sqrt[3]{\sqrt{3}\sqrt{x^8\left(-4x+27c_1^2\right)}+9c_1x^4}}$$

$$y(x) \to \frac{i\sqrt[3]{3}\left(\sqrt{3}+i\right)\left(2\sqrt{3}\sqrt{x^8\left(-4x+27c_1^2\right)}+18c_1x^4\right)^{2/3} - 2\sqrt[3]{2}\sqrt[6]{3}\left(\sqrt{3}+3i\right)x^3}{12x^2\sqrt[3]{\sqrt{3}\sqrt{x^8\left(-4x+27c_1^2\right)}+9c_1x^4}}$$

$$y(x) \to \frac{\sqrt[3]{3}\left(-1-i\sqrt{3}\right)\left(2\sqrt{3}\sqrt{x^8\left(-4x+27c_1^2\right)}+18c_1x^4\right)^{2/3} - 2\sqrt[3]{2}\sqrt[6]{3}\left(\sqrt{3}-3i\right)x^3}{12x^2\sqrt[3]{\sqrt{3}\sqrt{x^8\left(-4x+27c_1^2\right)}+9c_1x^4}}$$

6.3 problem 3

Internal problem ID [11677]
Internal file name [`OUTPUT/11686_Wednesday_April_10_2024_04_54_41_PM_24706420/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y + x(1+x)\, y' = 1}$$

6.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{1-y}{x\,(1+x)} \end{aligned}$$

Where $f(x) = \frac{1}{x(1+x)}$ and $g(y) = 1-y$. Integrating both sides gives

$$\frac{1}{1-y}\, dy = \frac{1}{x\,(1+x)}\, dx$$

$$\int \frac{1}{1-y}\,dy = \int \frac{1}{x(1+x)}\,dx$$
$$-\ln(y-1) = -\ln(1+x) + \ln(x) + c_1$$

Raising both side to exponential gives

$$\frac{1}{y-1} = e^{-\ln(1+x)+\ln(x)+c_1}$$

Which simplifies to

$$\frac{1}{y-1} = c_2 e^{-\ln(1+x)+\ln(x)}$$

Which simplifies to

$$y = \frac{\left(\frac{c_2 e^{c_1} x}{1+x} + 1\right)(1+x)\, e^{-c_1}}{c_2 x}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\frac{c_2 e^{c_1} x}{1+x} + 1\right)(1+x)\, e^{-c_1}}{c_2 x} \tag{1}$$

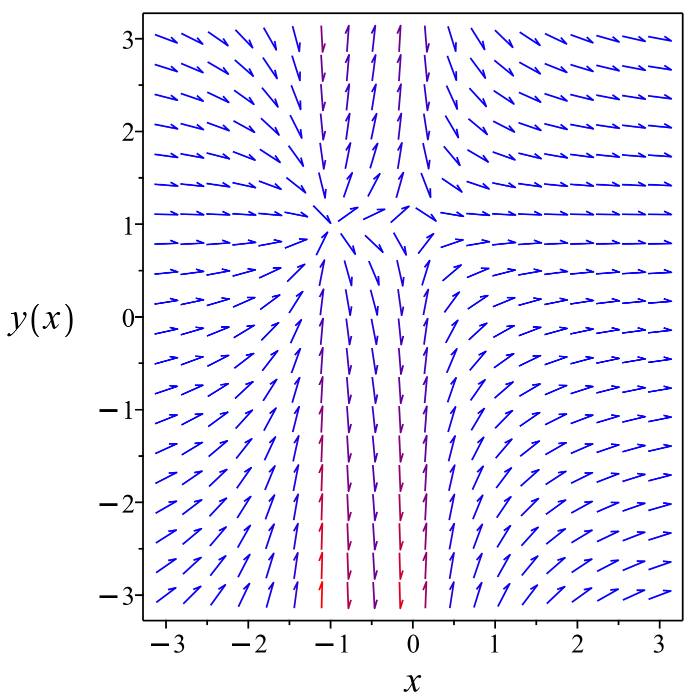


Figure 250: Slope field plot

Verification of solutions

$$y = \frac{\left(\frac{c_2 e^{c_1 x}}{1+x} + 1\right)(1+x)\,e^{-c_1}}{c_2 x}$$

Verified OK.

6.3.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= \frac{1}{x\,(1+x)} \\
q(x) &= \frac{1}{x\,(1+x)}
\end{aligned}$$

Hence the ode is

$$y' + \frac{y}{x\,(1+x)} = \frac{1}{x\,(1+x)}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{1}{x(1+x)}dx} \\
&= e^{-\ln(1+x)+\ln(x)}
\end{aligned}$$

Which simplifies to

$$\mu = \frac{x}{1+x}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{1}{x\,(1+x)}\right) \\
\frac{d}{dx}\left(\frac{xy}{1+x}\right) &= \left(\frac{x}{1+x}\right)\left(\frac{1}{x\,(1+x)}\right) \\
d\left(\frac{xy}{1+x}\right) &= \frac{1}{(1+x)^2}\,dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{xy}{1+x} &= \int \frac{1}{(1+x)^2}\,dx \\
\frac{xy}{1+x} &= -\frac{1}{1+x} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{x}{1+x}$ results in

$$y = -\frac{1}{x} + \frac{c_1(1+x)}{x}$$

which simplifies to

$$y = \frac{c_1x + c_1 - 1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1x + c_1 - 1}{x} \tag{1}$$

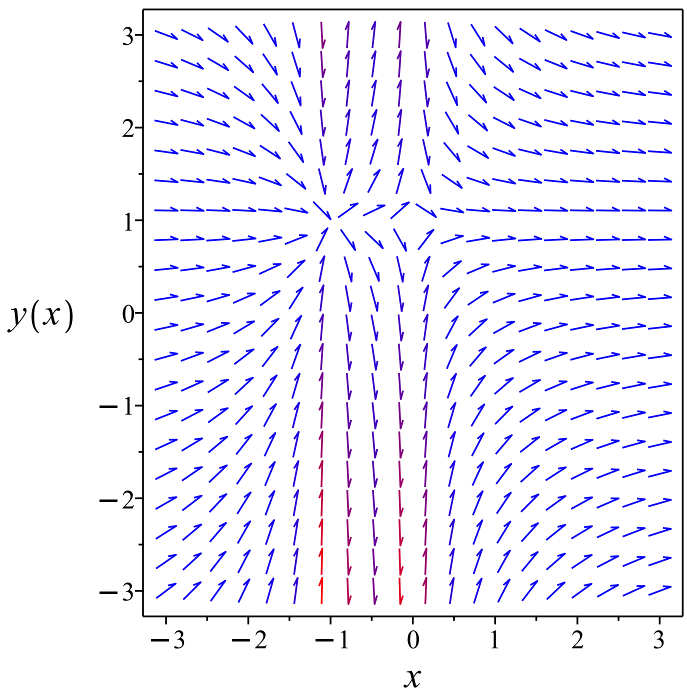


Figure 251: Slope field plot

Verification of solutions

$$y = \frac{c_1x + c_1 - 1}{x}$$

Verified OK.

6.3.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y-1}{x\,(1+x)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 207: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{\ln(1+x)-\ln(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{\ln(1+x)-\ln(x)}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{xy}{1+x}
$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y-1}{x\,(1+x)}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{(1+x)^2} \\ S_y &= \frac{x}{1+x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(1+x)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(1+R)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{1+R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{xy}{1+x} = -\frac{1}{1+x} + c_1$$

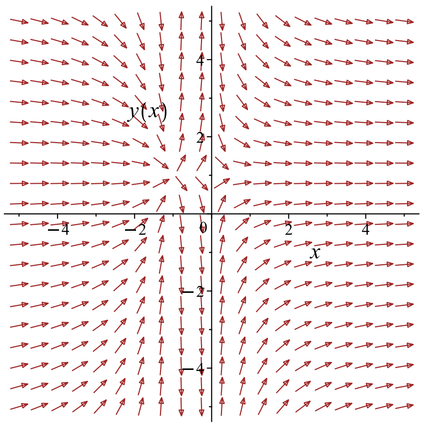
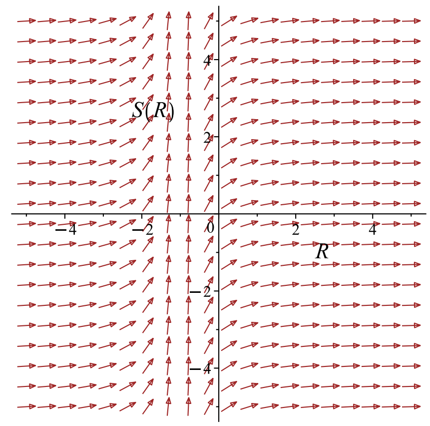
Which simplifies to

$$\frac{xy}{1+x} = -\frac{1}{1+x} + c_1$$

Which gives

$$y = \frac{c_1 x + c_1 - 1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y-1}{x(1+x)}$ 	$R = x$ $S = \frac{xy}{1+x}$	$\frac{dS}{dR} = \frac{1}{(1+R)^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1 x + c_1 - 1}{x} \tag{1}$$

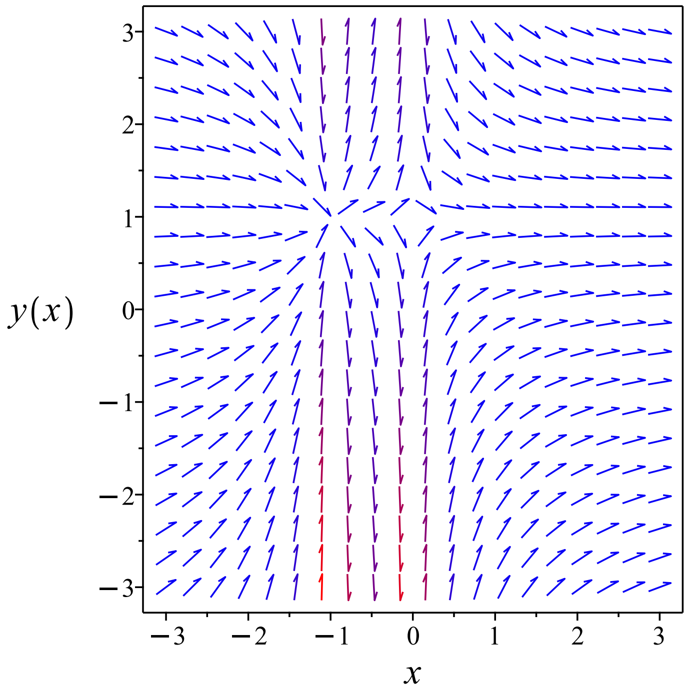


Figure 252: Slope field plot

Verification of solutions

$$y = \frac{c_1 x + c_1 - 1}{x}$$

Verified OK.

6.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{1-y}\right) \mathrm{d}y = \left(\frac{1}{x(1+x)}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x(1+x)}\right) \mathrm{d}x + \left(\frac{1}{1-y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x(1+x)}$$
$$N(x, y) = \frac{1}{1-y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x(1+x)}\right)\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{1-y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{x\,(1+x)} \, \mathrm{d}x$$
$$\phi = \ln(1+x) - \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1-y}$. Therefore equation (4) becomes

$$\frac{1}{1-y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y-1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{1}{y-1}\right) \mathrm{d}y$$
$$f(y) = -\ln(y-1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(1+x) - \ln(x) - \ln(y-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(1+x) - \ln(x) - \ln(y-1)$$

The solution becomes

$$y = \frac{(e^{c_1}x + x + 1)\, e^{-c_1}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{(e^{c_1}x + x + 1)\, e^{-c_1}}{x} \tag{1}$$

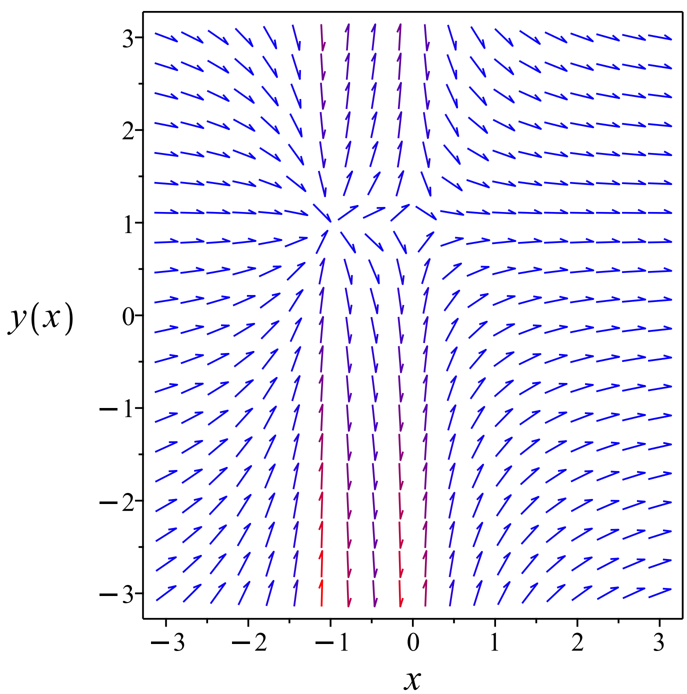


Figure 253: Slope field plot

Verification of solutions

$$y = \frac{(e^{c_1}x + x + 1)\, e^{-c_1}}{x}$$

Verified OK.

6.3.5 Maple step by step solution

Let's solve

$y + x(1 + x)\, y' = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1-y} = \frac{1}{x(1+x)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{1-y}dx = \int \frac{1}{x(1+x)}dx + c_1$

- Evaluate integral

  $-\ln(1 - y) = -\ln(1 + x) + \ln(x) + c_1$

- Solve for y

  $y = \frac{e^{c_1}x - x - 1}{e^{c_1}x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve((y(x)-1)+(x*(x+1))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x + c_1 - 1}{x}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 22

```
DSolve[(y[x]-1)+(x*(x+1))*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-1 + c_1(x+1)}{x}$$
$$y(x) \to 1$$

6.4 problem 4

Internal problem ID [11678]
Internal file name [`OUTPUT/11687_Wednesday_April_10_2024_04_54_41_PM_36212687/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-2y + y'x = -x^2}$$

6.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = -x$$

Hence the ode is

$$y' - \frac{2y}{x} = -x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{2}{x}dx} \\
&= \frac{1}{x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(-x) \\
\frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)(-x) \\
d\left(\frac{y}{x^2}\right) &= \left(-\frac{1}{x}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x^2} &= \int -\frac{1}{x}\, dx \\
\frac{y}{x^2} &= -\ln(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$
y = -x^2 \ln(x) + c_1 x^2
$$

which simplifies to

$$
y = x^2(-\ln(x) + c_1)
$$

Summary

The solution(s) found are the following

$$
y = x^2(-\ln(x) + c_1) \tag{1}
$$

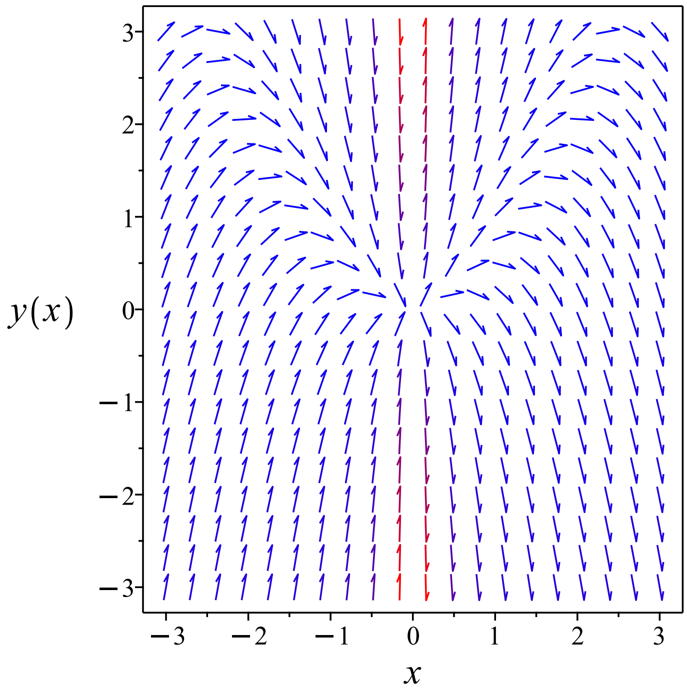


Figure 254: Slope field plot

Verification of solutions

$$y = x^2(-\ln(x) + c_1)$$

Verified OK.

6.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-x^2 + 2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 210: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= x^2 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x^2 + 2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{x^3} \\ S_y &= \frac{1}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2} = -\ln(x) + c_1$$

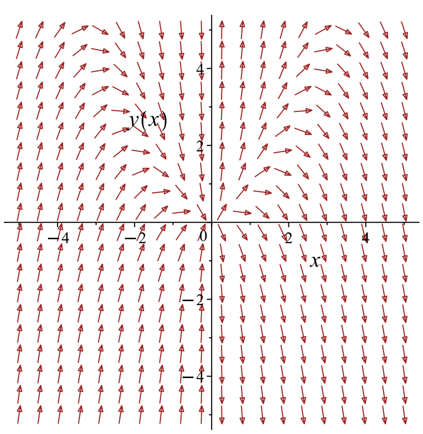
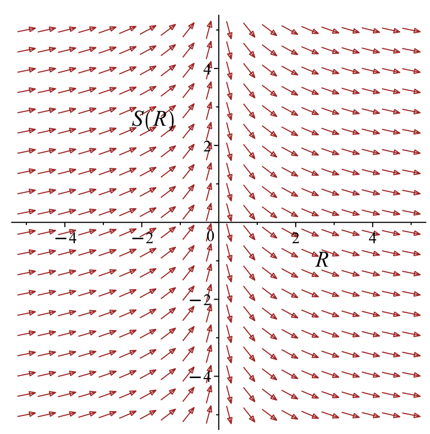
Which simplifies to

$$\frac{y}{x^2} = -\ln(x) + c_1$$

Which gives

$$y = -x^2(\ln(x) - c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x^2+2y}{x}$ 	$R = x$ $S = \frac{y}{x^2}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = -x^2(\ln(x) - c_1) \tag{1}$$

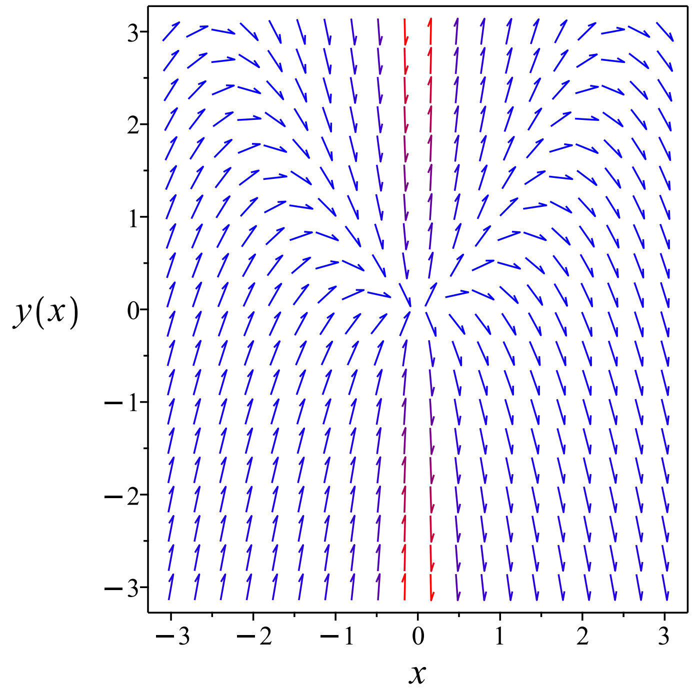


Figure 255: Slope field plot

Verification of solutions

$$y = -x^2(\ln(x) - c_1)$$

Verified OK.

6.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(-x^2+2y\right)\mathrm{d}x\\ \left(x^2-2y\right)\mathrm{d}x+(x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x^2-2y\\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2-2y\right)\\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((-2) - (1)) \\ &= -\frac{3}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int -\frac{3}{x}\,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-3\ln(x)} \\ &= \frac{1}{x^3} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^3}(x^2 - 2y) \\ &= \frac{x^2 - 2y}{x^3} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^3}(x) \\ &= \frac{1}{x^2} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{x^2 - 2y}{x^3}\right) + \left(\frac{1}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \overline{M} \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \frac{x^2 - 2y}{x^3} \, dx$$

$$\phi = \frac{y}{x^2} + \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2}$. Therefore equation (4) becomes

$$\frac{1}{x^2} = \frac{1}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y}{x^2} + \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y}{x^2} + \ln(x)$$

The solution becomes

$$y = -x^2(\ln(x) - c_1)$$

Summary

The solution(s) found are the following

$$y = -x^2(\ln(x) - c_1) \tag{1}$$

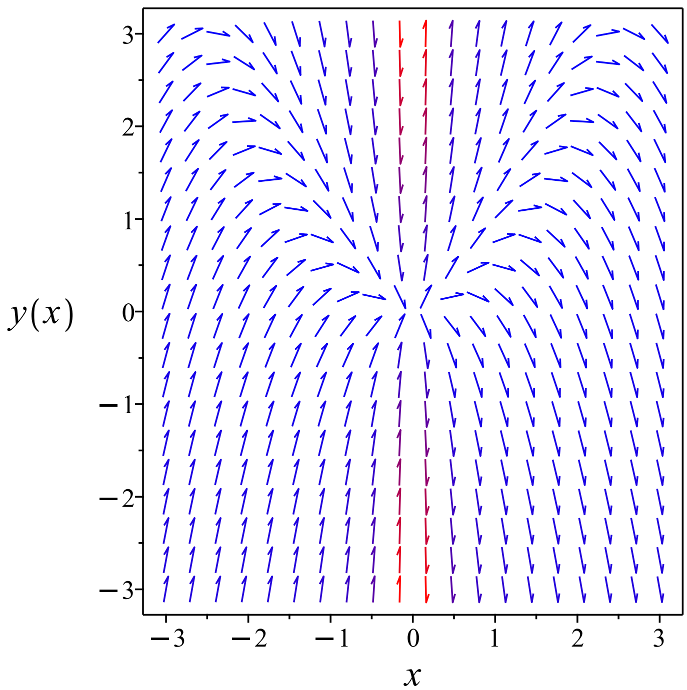


Figure 256: Slope field plot

Verification of solutions

$$y = -x^2(\ln(x) - c_1)$$

Verified OK.

6.4.4 Maple step by step solution

Let's solve

$-2y + y'x = -x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{x} - x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{x} = -x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = -\mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\mu(x)\,x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\mu(x)\,x dx + c_1$

- Solve for y

  $y = \frac{\int -\mu(x)x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^2}$

  $y = x^2\left(\int -\frac{1}{x} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^2(-\ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve((x^2-2*y(x))+(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (-\ln(x) + c_1)\, x^2$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 16

```
DSolve[(x^2-2*y[x])+(x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(-\log(x) + c_1)$$

6.5 problem 5

Internal problem ID [11679]
Internal file name [`OUTPUT/11688_Wednesday_April_10_2024_04_54_42_PM_93828384/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-5y + (y + x)\, y' = -3x}$$

6.5.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$-5u(x)\, x + (u(x)\, x + x)\, (u'(x)\, x + u(x)) = -3x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u^2 - 4u + 3}{x\,(u + 1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-4u+3}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-4u+3}{u+1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^2-4u+3}{u+1}}\,du = \int -\frac{1}{x}\,dx$$

$$2\ln(u-3) - \ln(u-1) = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{2\ln(u-3)-\ln(u-1)} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{(u-3)^2}{u-1} = \frac{c_3}{x}$$

The solution is

$$\frac{(u(x)-3)^2}{u(x)-1} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\left(\frac{y}{x}-3\right)^2}{\frac{y}{x}-1} = \frac{c_3}{x}$$

$$\frac{(-3x+y)^2}{x(y-x)} = \frac{c_3}{x}$$

Which simplifies to

$$-\frac{(3x-y)^2}{-y+x} = c_3$$

Summary

The solution(s) found are the following

$$-\frac{(3x-y)^2}{-y+x} = c_3 \tag{1}$$

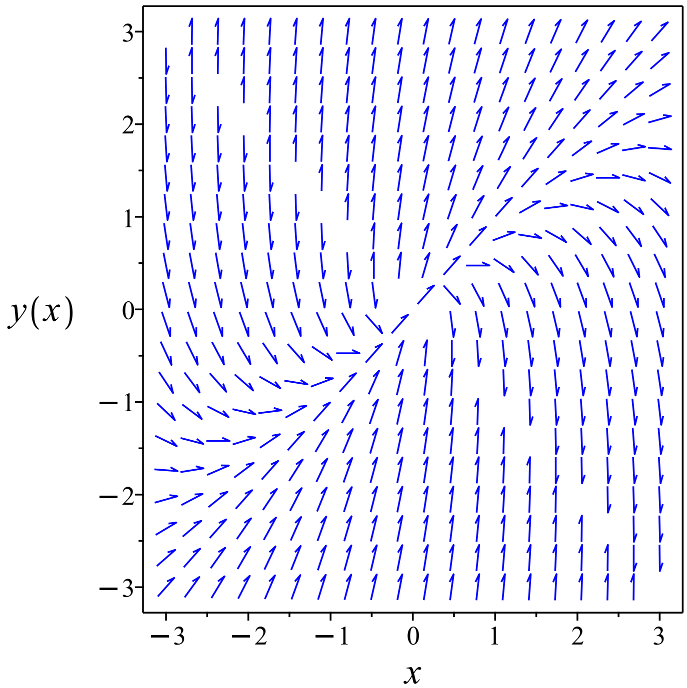


Figure 257: Slope field plot

Verification of solutions

$$-\frac{(3x-y)^2}{-y+x} = c_3$$

Verified OK.

6.5.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-3x+5y}{y+x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(-3x+5y)(b_3-a_2)}{y+x} - \frac{(-3x+5y)^2 a_3}{(y+x)^2} - \left(-\frac{3}{y+x} - \frac{-3x+5y}{(y+x)^2}\right)(xa_2+ya_3+a_1) - \left(\frac{5}{y+x} - \frac{-3x+5y}{(y+x)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^2a_2 - 9x^2a_3 - 7x^2b_2 - 3x^2b_3 + 6xya_2 + 30xya_3 + 2xyb_2 - 6xyb_3 - 5y^2a_2 - 17y^2a_3 + y^2b_2 + 5y^2b_3 - 8x}{(y+x)^2} = 0$$

Setting the numerator to zero gives

$$3x^2a_2 - 9x^2a_3 - 7x^2b_2 - 3x^2b_3 + 6xya_2 + 30xya_3 + 2xyb_2 - 6xyb_3 - 5y^2a_2 - 17y^2a_3 + y^2b_2 + 5y^2b_3 - 8xb_1 + 8ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$3a_2v_1^2 + 6a_2v_1v_2 - 5a_2v_2^2 - 9a_3v_1^2 + 30a_3v_1v_2 - 17a_3v_2^2 - 7b_2v_1^2 + 2b_2v_1v_2 + b_2v_2^2 - 3b_3v_1^2 - 6b_3v_1v_2 + 5b_3v_2^2 + 8a_1v_2 - 8b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(3a_2 - 9a_3 - 7b_2 - 3b_3) v_1^2 + (6a_2 + 30a_3 + 2b_2 - 6b_3) v_1 v_2 \\ - 8b_1 v_1 + (-5a_2 - 17a_3 + b_2 + 5b_3) v_2^2 + 8a_1 v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 8a_1 &= 0 \\ -8b_1 &= 0 \\ -5a_2 - 17a_3 + b_2 + 5b_3 &= 0 \\ 3a_2 - 9a_3 - 7b_2 - 3b_3 &= 0 \\ 6a_2 + 30a_3 + 2b_2 - 6b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -4a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 0 \\ b_2 &= -3a_3 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(\frac{-3x + 5y}{y + x}\right)(x) \\ &= \frac{3x^2 - 4xy + y^2}{y + x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{3x^2 - 4xy + y^2}{y + x}} dy \end{aligned}$$

Which results in

$$S = -\ln(y - x) + 2\ln(-3x + y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-3x + 5y}{y + x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y - x} - \frac{6}{-3x + y} \\ S_y &= \frac{y + x}{(3x - y)(-y + x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

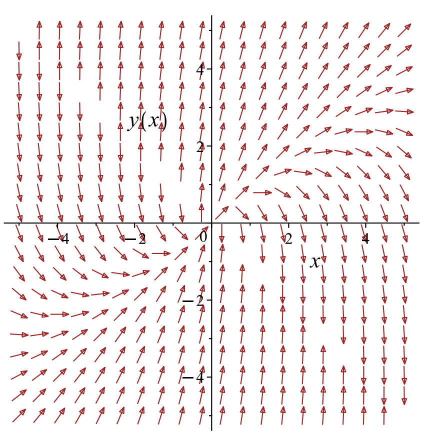
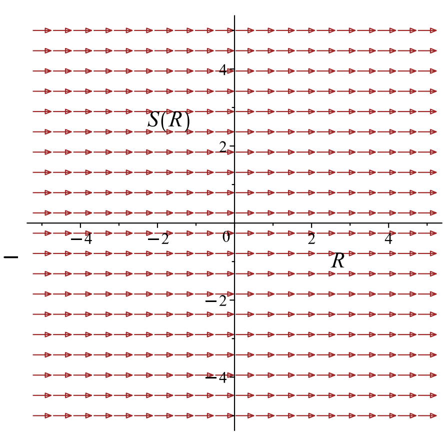
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(y - x) + 2\ln(-3x + y) = c_1$$

Which simplifies to

$$-\ln(y - x) + 2\ln(-3x + y) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-3x+5y}{y+x}$ 	$R = x$ $S = -\ln(y - x) + 2\ln(-$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$-\ln(y-x)+2\ln(-3x+y)=c_1 \tag{1}$$

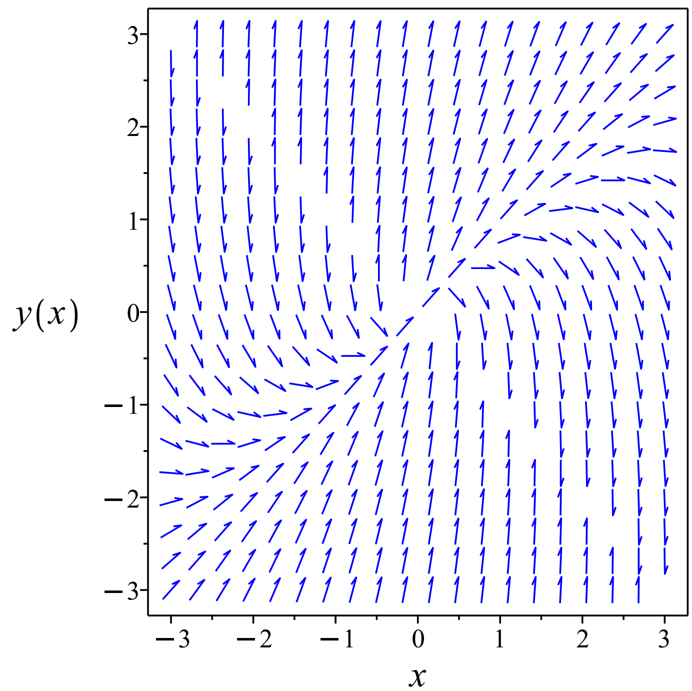


Figure 258: Slope field plot

Verification of solutions

$$-\ln(y-x)+2\ln(-3x+y)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
dsolve((3*x-5*y(x))+(x+y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{6c_1x - \sqrt{8c_1x + 1} + 1}{2c_1}$$
$$y(x) = \frac{6c_1x + 1 + \sqrt{8c_1x + 1}}{2c_1}$$

✓ Solution by Mathematica

Time used: 1.033 (sec). Leaf size: 80

```
DSolve[(3*x-5*y[x])+(x+y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(6x - e^{\frac{c_1}{2}}\sqrt{-8x + e^{c_1}} - e^{c_1}\right)$$
$$y(x) \to \frac{1}{2}\left(6x + e^{\frac{c_1}{2}}\sqrt{-8x + e^{c_1}} - e^{c_1}\right)$$

6.6 problem 6

Internal problem ID [11680]
Internal file name [`OUTPUT/11689_Wednesday_April_10_2024_04_54_44_PM_77178628/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{e^{2x}y^2 + \left(y\,e^{2x} - 2y\right) y' = 0}$$

6.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{e^{2x}y}{e^{2x}-2} \end{aligned}$$

Where $f(x) = -\frac{e^{2x}}{e^{2x}-2}$ and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y}\,dy = -\frac{e^{2x}}{e^{2x}-2}\,dx$$

$$\int \frac{1}{y}\,dy = \int -\frac{e^{2x}}{e^{2x}-2}\,dx$$

$$\ln(y) = -\frac{\ln(e^{2x}-2)}{2} + c_1$$

$$y = e^{-\frac{\ln(e^{2x}-2)}{2}+c_1}$$

$$= \frac{c_1}{\sqrt{e^{2x}-2}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{e^{2x}-2}} \tag{1}$$

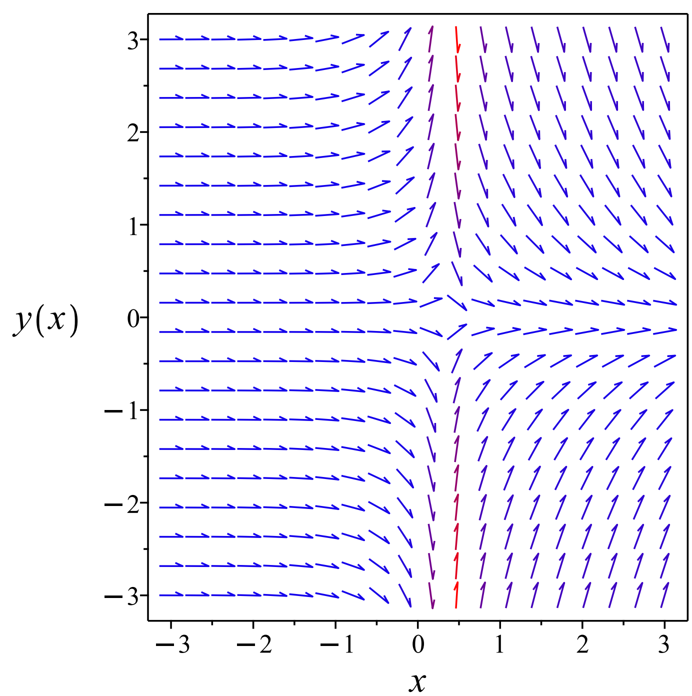


Figure 259: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\sqrt{e^{2x}-2}}$$

Verified OK.

6.6.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{e^{2x}}{e^{2x} - 2} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{e^{2x} y}{e^{2x} - 2} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{e^{2x}}{e^{2x}-2} dx} \\ &= \sqrt{e^{2x} - 2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx} \mu y &= 0 \\ \frac{d}{dx} \left(\sqrt{e^{2x} - 2}\, y \right) &= 0 \end{aligned}$$

Integrating gives

$$\sqrt{e^{2x} - 2}\, y = c_1$$

Dividing both sides by the integrating factor $\mu = \sqrt{e^{2x} - 2}$ results in

$$y = \frac{c_1}{\sqrt{e^{2x} - 2}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{e^{2x} - 2}} \tag{1}$$

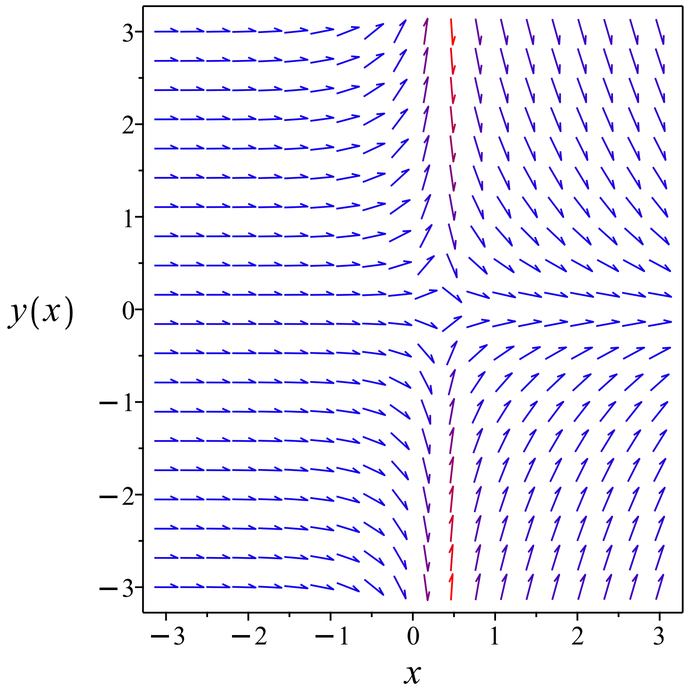


Figure 260: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\sqrt{e^{2x} - 2}}$$

Verified OK.

6.6.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$e^{2x}u(x)^2\,x^2 + \left(u(x)\,x\,e^{2x} - 2u(x)\,x\right)\left(u'(x)\,x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(e^{2x}x + e^{2x} - 2)}{x\,(e^{2x} - 2)} \end{aligned}$$

Where $f(x) = -\frac{e^{2x}x+e^{2x}-2}{x(e^{2x}-2)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{e^{2x}x + e^{2x} - 2}{x\left(e^{2x} - 2\right)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{e^{2x}x + e^{2x} - 2}{x\left(e^{2x} - 2\right)}\,dx \\
\ln\left(u\right) &= -\ln\left(x\right) - \frac{\ln\left(e^{2x} - 2\right)}{2} + c_2 \\
u &= e^{-\ln(x) - \frac{\ln\left(e^{2x}-2\right)}{2} + c_2} \\
&= c_2 e^{-\ln(x) - \frac{\ln\left(e^{2x}-2\right)}{2}}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2}{x\sqrt{e^{2x} - 2}}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= \frac{c_2}{\sqrt{e^{2x} - 2}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2}{\sqrt{e^{2x} - 2}} \tag{1}$$

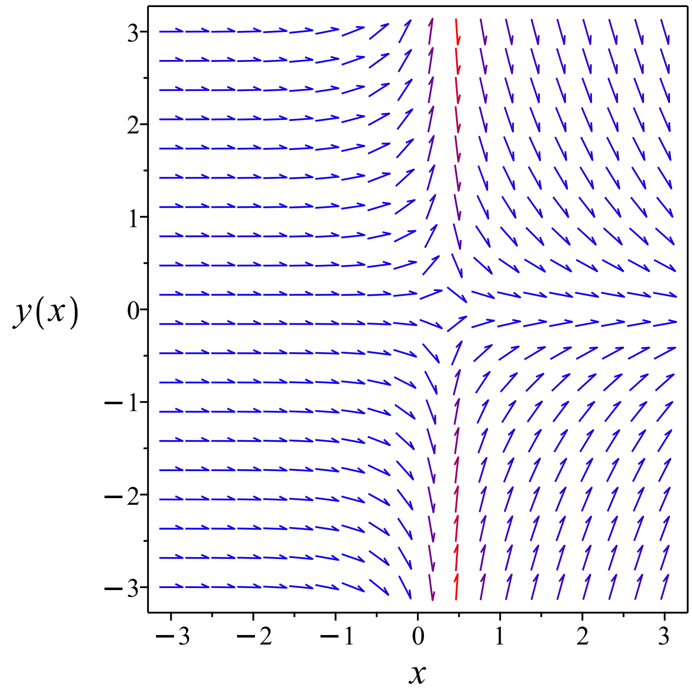


Figure 261: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{c_2}{\sqrt{e^{2x} - 2}}$$

Verified OK.

6.6.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{e^{2x} y}{e^{2x} - 2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 213: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \frac{1}{\sqrt{e^{2x} - 2}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{\sqrt{e^{2x}-2}}} dy
\end{aligned}$$

Which results in

$$S = \sqrt{e^{2x} - 2}\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{e^{2x} y}{e^{2x} - 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y\, e^{2x}}{\sqrt{e^{2x} - 2}} \\
S_y &= \sqrt{e^{2x} - 2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\sqrt{e^{2x}-2} = c_1$$

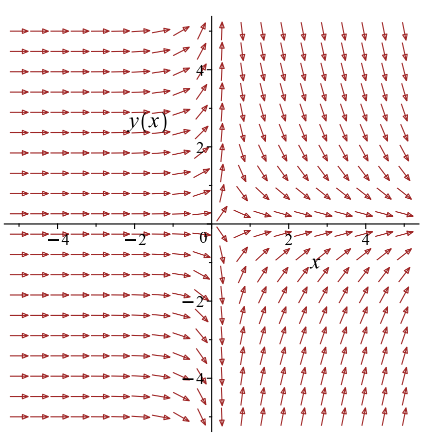
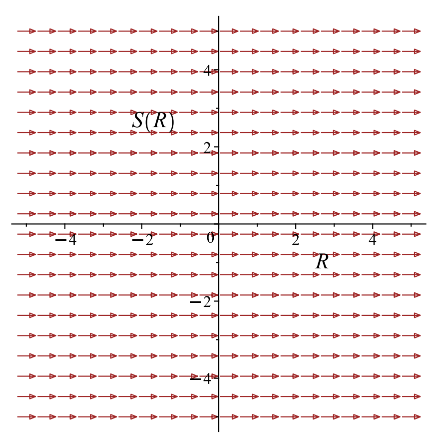
Which simplifies to

$$y\sqrt{e^{2x}-2} = c_1$$

Which gives

$$y = \frac{c_1}{\sqrt{e^{2x}-2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{e^{2x}y}{e^{2x}-2}$ 	$R = x$ $S = \sqrt{e^{2x}-2}\,y$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{e^{2x}-2}} \tag{1}$$

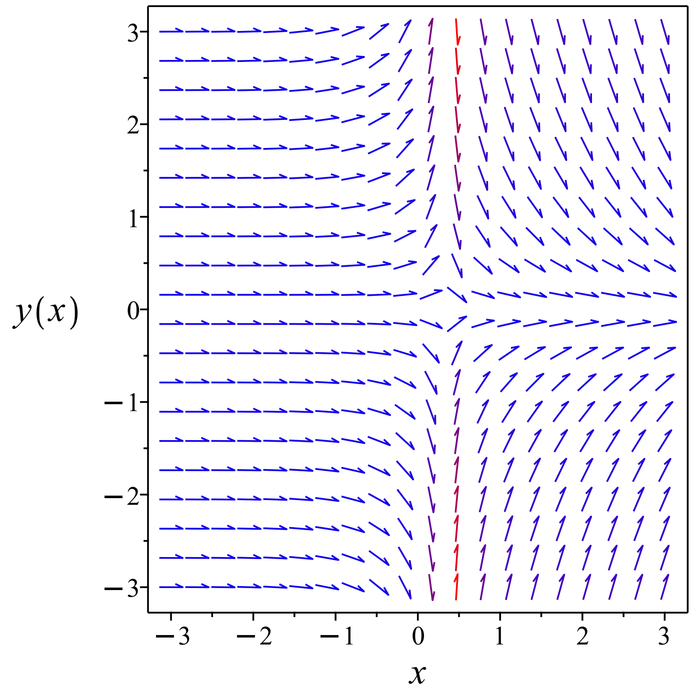


Figure 262: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\sqrt{e^{2x} - 2}}$$

Verified OK.

6.6.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y}\right)\mathrm{d}y = \left(\frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x} - 2}\right)\mathrm{d}x$$
$$\left(-\frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x} - 2}\right)\mathrm{d}x + \left(-\frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x} - 2}$$
$$N(x, y) = -\frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x} - 2}\right)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x}-2} \, \mathrm{d}x$$
$$\phi = -\frac{\ln\left(\mathrm{e}^{2x}-2\right)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(e^{2x}-2)}{2} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(e^{2x}-2)}{2} - \ln(y)$$

The solution becomes

$$y = e^{-\frac{\ln(e^{2x}-2)}{2} - c_1}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{\ln(e^{2x}-2)}{2} - c_1} \tag{1}$$

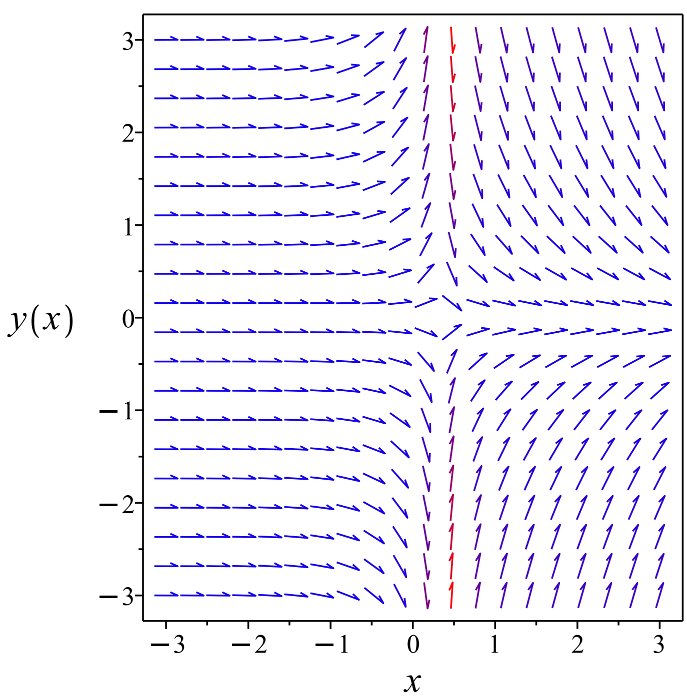


Figure 263: Slope field plot

Verification of solutions

$$y = e^{-\frac{\ln\left(e^{2x}-2\right)}{2}-c_1}$$

Verified OK.

6.6.6 Maple step by step solution

Let's solve

$e^{2x}y^2 + (y\,e^{2x} - 2y)\,y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(e^{2x}y^2 + (y\,e^{2x} - 2y)\,y'\right)dx = \int 0dx + c_1$

- Evaluate integral

  $\frac{(e^{2x}-2)y^2}{2} = c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{2}\,\sqrt{(e^{2x}-2)c_1}}{e^{2x}-2}, y = -\frac{\sqrt{2}\,\sqrt{(e^{2x}-2)c_1}}{e^{2x}-2}\right\}$

Maple trace

```
`Classification methods on request
Methods to be used are: [exact]
----------------------------
* Tackling ODE using method: exact
--- Trying classification methods ---
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 58

```
dsolve((exp(2*x)*y(x)^2)+(exp(2*x)*y(x)-2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= 0 \\
y(x) &= \frac{\sqrt{2}\,\sqrt{-\left(\mathrm{e}^{2x}-2\right)c_1}}{\mathrm{e}^{2x}-2} \\
y(x) &= -\frac{\sqrt{2}\,\sqrt{-\left(\mathrm{e}^{2x}-2\right)c_1}}{\mathrm{e}^{2x}-2}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.071 (sec). Leaf size: 29

```
DSolve[(Exp[2*x]*y[x]^2)+(Exp[2*x]*y[x]-2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$
\begin{aligned}
y(x) &\to 0 \\
y(x) &\to \frac{c_1}{\sqrt{e^{2x}-2}} \\
y(x) &\to 0
\end{aligned}
$$

6.7 problem 7

Internal problem ID [11681]
Internal file name [`OUTPUT/11690_Wednesday_April_10_2024_04_54_45_PM_15215251/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{8yx^3 + \left(x^4+1\right) y' = 12x^3}$$

6.7.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x^3(-8y+12)}{x^4+1} \end{aligned}$$

Where $f(x) = \frac{x^3}{x^4+1}$ and $g(y) = -8y+12$. Integrating both sides gives

$$\frac{1}{-8y+12}\, dy = \frac{x^3}{x^4+1}\, dx$$

$$\int \frac{1}{-8y+12}\,dy = \int \frac{x^3}{x^4+1}\,dx$$

$$-\frac{\ln(-2y+3)}{8} = \frac{\ln(x^4+1)}{4} + c_1$$

Raising both side to exponential gives

$$\frac{1}{(-2y+3)^{\frac{1}{8}}} = \mathrm{e}^{\frac{\ln\left(x^4+1\right)}{4}+c_1}$$

Which simplifies to

$$\frac{1}{(-2y+3)^{\frac{1}{8}}} = c_2\left(x^4+1\right)^{\frac{1}{4}}$$

Which simplifies to

$$y = \frac{\left(3c_2^8\mathrm{e}^{8c_1}(x^4+1)^2 - 1\right)\mathrm{e}^{-8c_1}}{2c_2^8\,(x^4+1)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(3c_2^8\mathrm{e}^{8c_1}(x^4+1)^2 - 1\right)\mathrm{e}^{-8c_1}}{2c_2^8\,(x^4+1)^2} \tag{1}$$

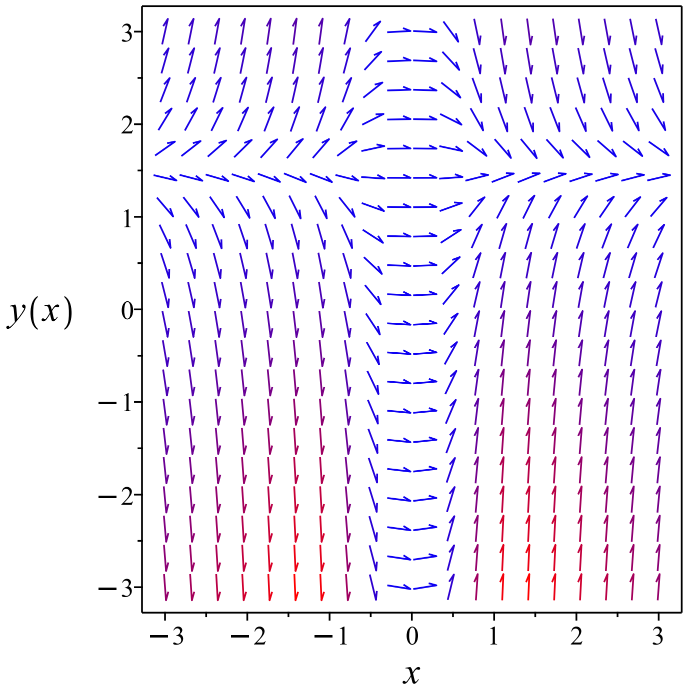


Figure 264: Slope field plot

Verification of solutions

$$y = \frac{\left(3c_2^8 \mathrm{e}^{8c_1} \left(x^4+1\right)^2 - 1\right) \mathrm{e}^{-8c_1}}{2c_2^8 \left(x^4+1\right)^2}$$

Verified OK.

6.7.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{8x^3}{x^4+1}$$
$$q(x) = \frac{12x^3}{x^4+1}$$

Hence the ode is

$$y' + \frac{8x^3 y}{x^4+1} = \frac{12x^3}{x^4+1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{8x^3}{x^4+1}dx} \\ &= \left(x^4+1\right)^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{12x^3}{x^4+1}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^4+1\right)^2 y\right) &= \left(\left(x^4+1\right)^2\right)\left(\frac{12x^3}{x^4+1}\right) \\ \mathrm{d}\left(\left(x^4+1\right)^2 y\right) &= \left(12x^3\left(x^4+1\right)\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\left(x^4+1\right)^2 y &= \int 12x^3\left(x^4+1\right)\,\mathrm{d}x \\ \left(x^4+1\right)^2 y &= \frac{3\left(x^4+1\right)^2}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \left(x^4+1\right)^2$ results in

$$y = \frac{3}{2} + \frac{c_1}{\left(x^4+1\right)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{3}{2} + \frac{c_1}{\left(x^4+1\right)^2} \tag{1}$$

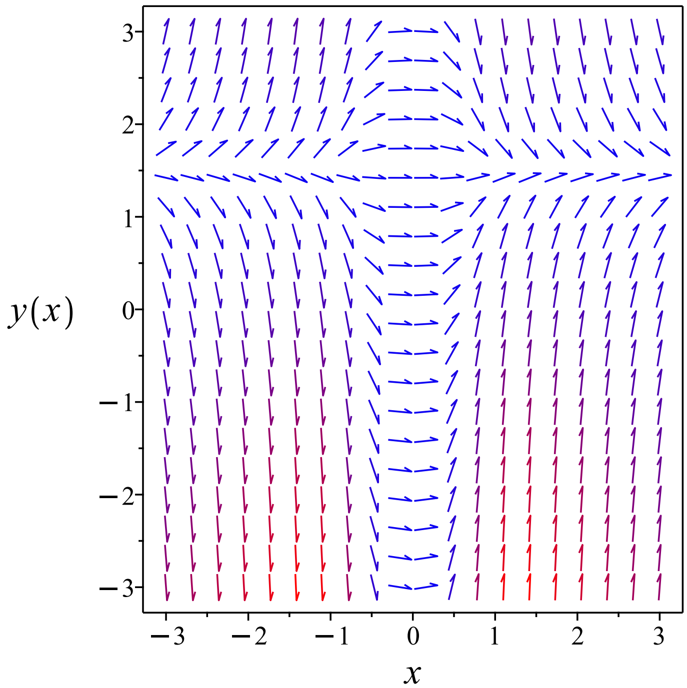


Figure 265: Slope field plot

Verification of solutions

$$y = \frac{3}{2} + \frac{c_1}{(x^4 + 1)^2}$$

Verified OK.

6.7.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{4x^3(2y - 3)}{x^4 + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 216: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{\left(x^4+1\right)^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(x^4+1)^2}} dy \end{aligned}$$

Which results in

$$S = (x^4 + 1)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{4x^3(2y - 3)}{x^4 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 8(x^4 + 1) y x^3 \\ S_y &= (x^4 + 1)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 12x^7 + 12x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 12R^7 + 12R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{3(R^4+1)^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x^4+1)^2 y = \frac{3(x^4+1)^2}{2} + c_1$$

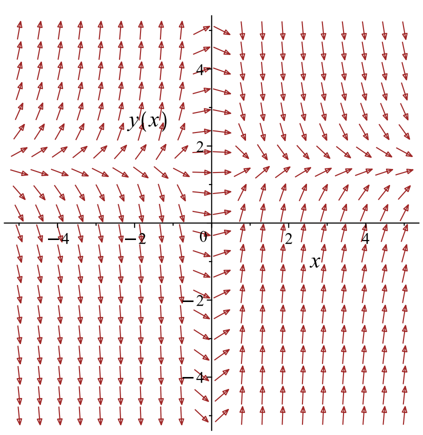
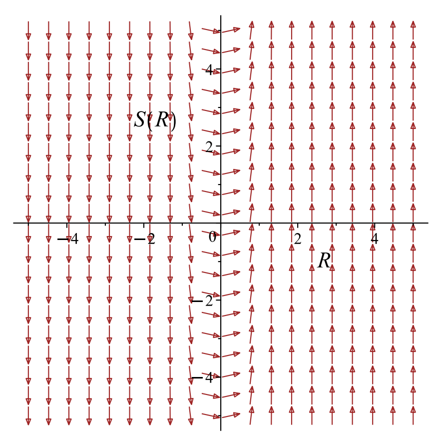
Which simplifies to

$$(x^4+1)^2 y = \frac{3(x^4+1)^2}{2} + c_1$$

Which gives

$$y = \frac{3x^8 + 6x^4 + 2c_1 + 3}{2(x^4+1)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4x^3(2y-3)}{x^4+1}$ 	$R = x$ $S = (x^4+1)^2 y$	$\frac{dS}{dR} = 12R^7 + 12R^3$ 

Summary
The solution(s) found are the following

$$y = \frac{3x^8 + 6x^4 + 2c_1 + 3}{2\left(x^4 + 1\right)^2} \tag{1}$$

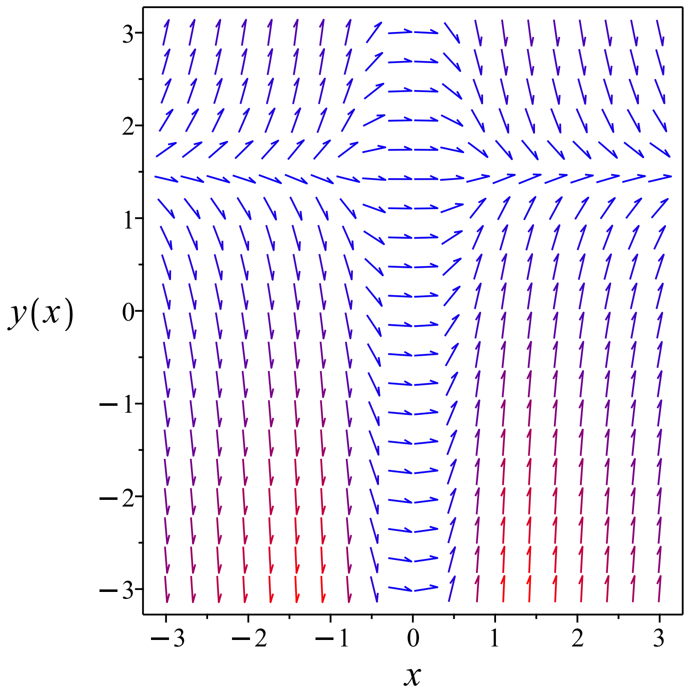


Figure 266: Slope field plot

Verification of solutions

$$y = \frac{3x^8 + 6x^4 + 2c_1 + 3}{2\left(x^4 + 1\right)^2}$$

Verified OK.

6.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-8y+12}\right)\mathrm{d}y&=\left(\frac{x^3}{x^4+1}\right)\mathrm{d}x\\ \left(-\frac{x^3}{x^4+1}\right)\mathrm{d}x+\left(\frac{1}{-8y+12}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{x^3}{x^4+1}\\ N(x,y)&=\frac{1}{-8y+12}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^3}{x^4+1}\right) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-8y+12}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x^3}{x^4+1} \, \mathrm{d}x
\end{aligned}$$

$$\phi = -\frac{\ln(x^4+1)}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-8y+12}$. Therefore equation (4) becomes

$$\frac{1}{-8y+12} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{4(2y-3)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{8y-12}\right) \mathrm{d}y$$

$$f(y) = -\frac{\ln(2y-3)}{8} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^4+1)}{4} - \frac{\ln(2y-3)}{8} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^4+1)}{4} - \frac{\ln(2y-3)}{8}$$

The solution becomes

$$y = \frac{3x^8 + 6x^4 + \mathrm{e}^{-8c_1} + 3}{2x^8 + 4x^4 + 2}$$

Summary

The solution(s) found are the following

$$y = \frac{3x^8 + 6x^4 + \mathrm{e}^{-8c_1} + 3}{2x^8 + 4x^4 + 2} \tag{1}$$

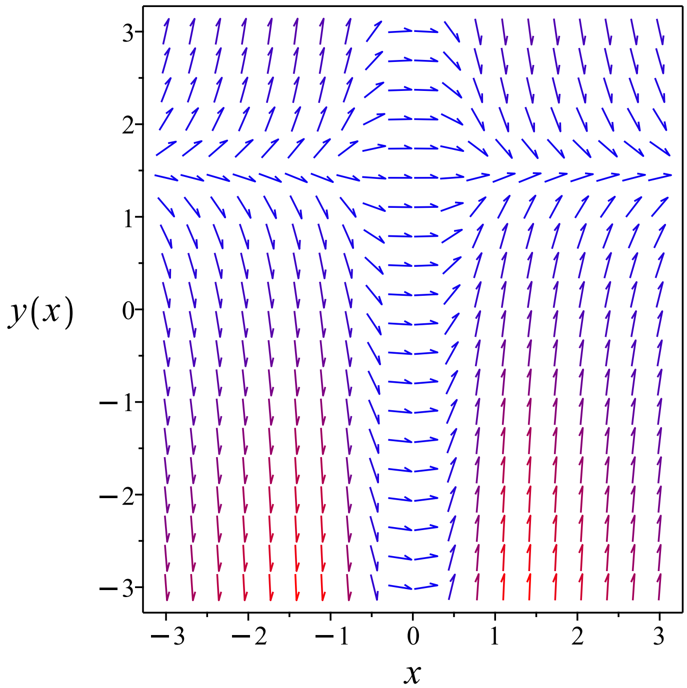


Figure 267: Slope field plot

Verification of solutions

$$y = \frac{3x^8 + 6x^4 + e^{-8c_1} + 3}{2x^8 + 4x^4 + 2}$$

Verified OK.

6.7.5 Maple step by step solution

Let's solve

$8yx^3 + (x^4 + 1)\, y' = 12x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{2y-3} = -\frac{4x^3}{x^4+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{2y-3}dx = \int -\frac{4x^3}{x^4+1}dx + c_1$

- Evaluate integral

$\frac{\ln(2y-3)}{2} = -\ln(x^4+1) + c_1$

- Solve for y

$y = \frac{3x^8+6x^4+e^{2c_1}+3}{2(x^8+2x^4+1)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((8*x^3*y(x)-12*x^3)+(1+x^4)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{3}{2} + \frac{c_1}{(x^4+1)^2}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 38

```
DSolve[(8*x^3*y[x]-12*x^3)+(1+x^4)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3x^8+6x^4+2c_1}{2(x^4+1)^2}$$
$$y(x) \to \frac{3}{2}$$

6.8 problem 8

Internal problem ID [11682]
Internal file name [`OUTPUT/11691_Wednesday_April_10_2024_04_54_46_PM_66245698/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Riccati]
```

$$\boxed{yx + y^2 + 2x^2y' = -2x^2}$$

6.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x^2 + u(x)^2\,x^2 + 2x^2(u'(x)\,x + u(x)) = -2x^2$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{-\frac{1}{2}u^2 - \frac{3}{2}u - 1}{x}
\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -\frac{1}{2}u^2 - \frac{3}{2}u - 1$. Integrating both sides gives

$$\frac{1}{-\frac{1}{2}u^2 - \frac{3}{2}u - 1}\, du = \frac{1}{x}\, dx$$

$$\int \frac{1}{-\frac{1}{2}u^2 - \frac{3}{2}u - 1}\, du = \int \frac{1}{x}\, dx$$

$$-2\ln(u+1) + 2\ln(u+2) = \ln(x) + c_2$$

The above can be written as

$$\begin{aligned}(-2)\left(\ln(u+1) - \ln(u+2)\right) &= \ln(x) + 2c_2\\ \ln(u+1) - \ln(u+2) &= \left(-\frac{1}{2}\right)(\ln(x) + 2c_2)\\ &= -\frac{\ln(x)}{2} - c_2\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u+1)-\ln(u+2)} = e^{-\frac{\ln(x)}{2} - \frac{c_2}{2}}$$

Which simplifies to

$$\begin{aligned}\frac{u+1}{u+2} &= -\frac{c_2}{2\sqrt{x}}\\ &= \frac{c_3}{\sqrt{x}}\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}y &= xu\\ &= -\frac{x\left(\sqrt{x} - 2c_3\right)}{\sqrt{x} - c_3}\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{x\left(\sqrt{x} - 2c_3\right)}{\sqrt{x} - c_3} \tag{1}$$

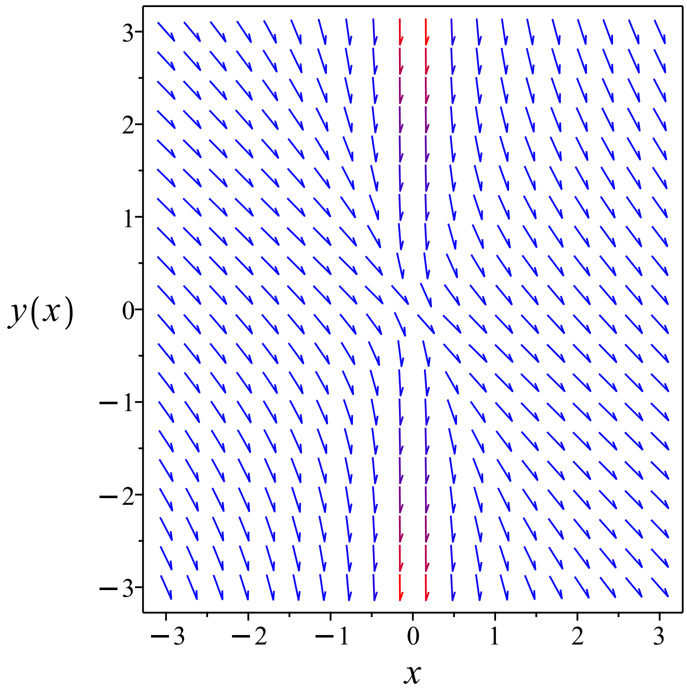


Figure 268: Slope field plot

Verification of solutions

$$y = -\frac{x(\sqrt{x} - 2c_3)}{\sqrt{x} - c_3}$$

Verified OK.

6.8.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2x^2 + xy + y^2}{2x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{(2x^2 + xy + y^2)(b_3 - a_2)}{2x^2} - \frac{(2x^2 + xy + y^2)^2 a_3}{4x^4} \\
& - \left(-\frac{4x + y}{2x^2} + \frac{2x^2 + xy + y^2}{x^3}\right)(xa_2 + ya_3 + a_1) \\
& + \frac{(x + 2y)(xb_2 + yb_3 + b_1)}{2x^2} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{4x^4a_2 - 4x^4a_3 + 6b_2x^4 - 4x^4b_3 - 4x^3ya_3 + 4x^3yb_2 - 2x^2y^2a_2 - 7x^2y^2a_3 + 2x^2y^2b_3 - 6x\,y^3a_3 - y^4a_3 + 2x}{4x^4} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& 4x^4a_2 - 4x^4a_3 + 6b_2x^4 - 4x^4b_3 - 4x^3ya_3 + 4x^3yb_2 - 2x^2y^2a_2 - 7x^2y^2a_3 \\
& + 2x^2y^2b_3 - 6x\,y^3a_3 - y^4a_3 + 2x^3b_1 - 2x^2ya_1 + 4x^2yb_1 - 4x\,y^2a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& 4a_2v_1^4 - 2a_2v_1^2v_2^2 - 4a_3v_1^4 - 4a_3v_1^3v_2 - 7a_3v_1^2v_2^2 - 6a_3v_1v_2^3 - a_3v_2^4 + 6b_2v_1^4 \\
& + 4b_2v_1^3v_2 - 4b_3v_1^4 + 2b_3v_1^2v_2^2 - 2a_1v_1^2v_2 - 4a_1v_1v_2^2 + 2b_1v_1^3 + 4b_1v_1^2v_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(4a_2 - 4a_3 + 6b_2 - 4b_3) v_1^4 + (-4a_3 + 4b_2) v_1^3 v_2 + 2b_1 v_1^3 + (-2a_2 - 7a_3 + 2b_3) v_1^2 v_2^2 + (-2a_1 + 4b_1) v_1^2 v_2 - 6a_3 v_1 v_2^3 - 4a_1 v_1 v_2^2 - a_3 v_2^4 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-4a_1 &= 0 \\
-6a_3 &= 0 \\
-a_3 &= 0 \\
2b_1 &= 0 \\
-2a_1 + 4b_1 &= 0 \\
-4a_3 + 4b_2 &= 0 \\
-2a_2 - 7a_3 + 2b_3 &= 0 \\
4a_2 - 4a_3 + 6b_2 - 4b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{2x^2 + xy + y^2}{2x^2}\right)(x) \\
&= \frac{2x^2 + 3xy + y^2}{2x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2x^2+3xy+y^2}{2x}} dy
\end{aligned}$$

Which results in

$$S = 2\ln(y + x) - 2\ln(2x + y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2x^2 + xy + y^2}{2x^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2y}{(y+x)(2x+y)} \\
S_y &= \frac{2x}{(y+x)(2x+y)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(R) + c_1 \tag{4}
$$

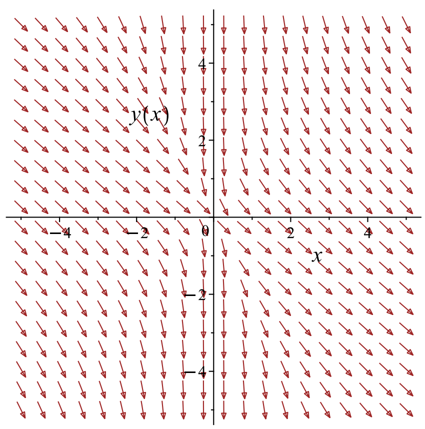
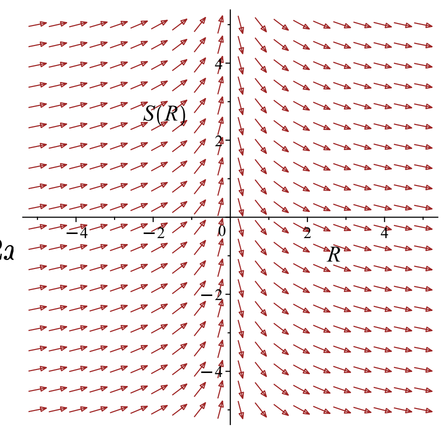
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
2\ln(y+x) - 2\ln(2x+y) = -\ln(x) + c_1
$$

Which simplifies to

$$
2\ln(y+x) - 2\ln(2x+y) = -\ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x^2+xy+y^2}{2x^2}$ 	$R = x$ $S = 2\ln(y+x) - 2\ln(2x$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$2\ln(y+x) - 2\ln(2x+y) = -\ln(x) + c_1 \tag{1}$$

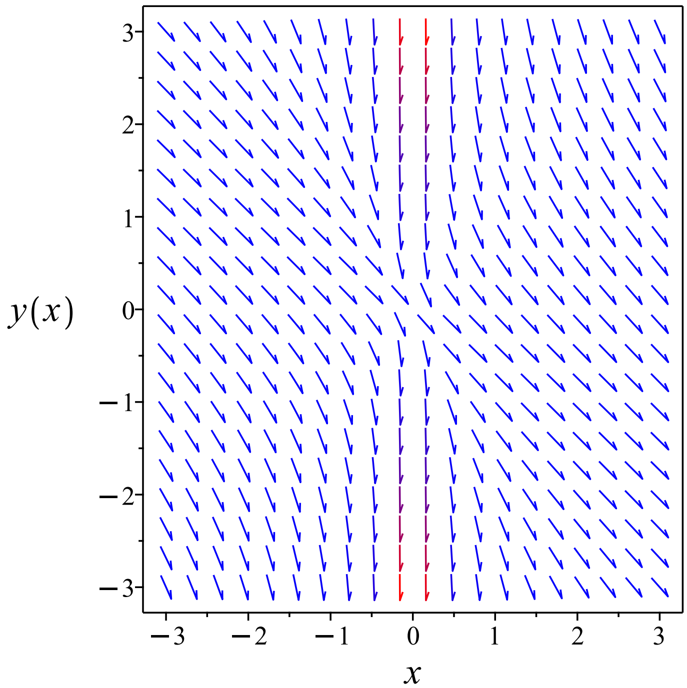


Figure 269: Slope field plot

Verification of solutions

$$2\ln(y+x) - 2\ln(2x+y) = -\ln(x) + c_1$$

Verified OK.

6.8.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{2x^2+xy+y^2}{2x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -1 - \frac{y}{2x} - \frac{y^2}{2x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -1$, $f_1(x) = -\frac{1}{2x}$ and $f_2(x) = -\frac{1}{2x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{2x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{1}{x^3} \\ f_1 f_2 &= \frac{1}{4x^3} \\ f_2^2 f_0 &= -\frac{1}{4x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{2x^2} - \frac{5u'(x)}{4x^3} - \frac{u(x)}{4x^4} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2}{x}$$

The above shows that

$$u'(x) = -\frac{c_1}{2x^{\frac{3}{2}}} - \frac{c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = \frac{2\left(-\frac{c_1}{2x^{\frac{3}{2}}} - \frac{c_2}{x^2}\right) x^2}{\frac{c_1}{\sqrt{x}} + \frac{c_2}{x}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-c_3 x - 2\sqrt{x}\right) x}{c_3 x + \sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-c_3 x - 2\sqrt{x}\right) x}{c_3 x + \sqrt{x}} \tag{1}$$

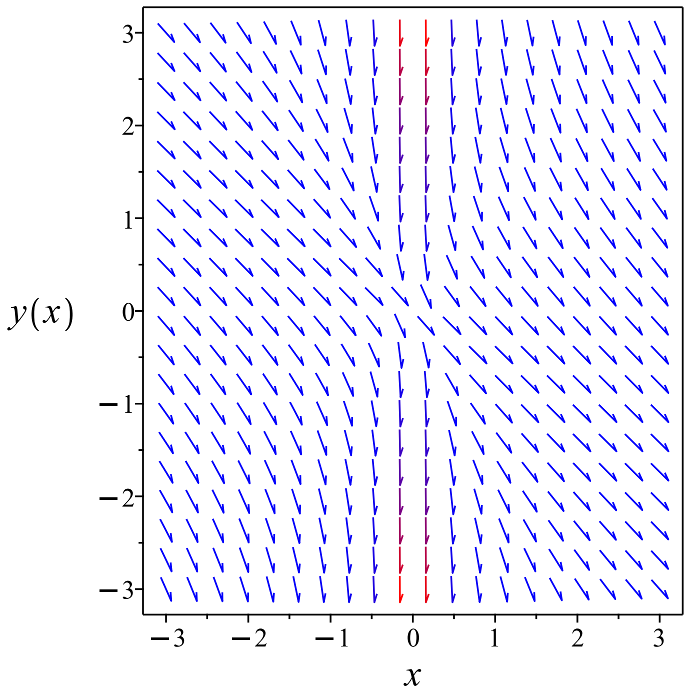


Figure 270: Slope field plot

Verification of solutions

$$y = \frac{\left(-c_3 x - 2\sqrt{x}\right) x}{c_3 x + \sqrt{x}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
dsolve((2*x^2+x*y(x)+y(x)^2)+(2*x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\left(c_1 x - \sqrt{c_1 x} - 2\right) x}{c_1 x - 1}$$
$$y(x) = -\frac{\left(c_1 x + \sqrt{c_1 x} - 2\right) x}{c_1 x - 1}$$

✓ Solution by Mathematica

Time used: 2.203 (sec). Leaf size: 47

```
DSolve[(2*x^2+x*y[x]+y[x]^2)+(2*x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x\left(\sqrt{x} - 2e^{c_1}\right)}{-\sqrt{x} + e^{c_1}}$$
$$y(x) \to -2x$$
$$y(x) \to -x$$

6.9 problem 9

Internal problem ID [11683]
Internal file name [`OUTPUT/11692_Wednesday_April_10_2024_04_54_48_PM_84959607/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y' - \frac{4y^2x^3 - 3x^2y}{x^3 - 2yx^4} = 0}$$

6.9.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(4xy - 3)}{x(2xy - 1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2-\frac{y(4xy-3)\left(b_3-a_2\right)}{x\left(2xy-1\right)}-\frac{y^2(4xy-3)^2 a_3}{x^2\left(2xy-1\right)^2} \\ & -\left(\frac{y(4xy-3)}{x^2\left(2xy-1\right)}-\frac{4y^2}{x\left(2xy-1\right)}+\frac{2y^2(4xy-3)}{x\left(2xy-1\right)^2}\right)\left(xa_2+ya_3+a_1\right) \\ & -\left(-\frac{4xy-3}{x\left(2xy-1\right)}-\frac{4y}{2xy-1}+\frac{2y(4xy-3)}{\left(2xy-1\right)^2}\right)\left(xb_2+yb_3+b_1\right)=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{12x^4y^2b_2-24x^2y^4a_3+8x^3y^2b_1-8x^2y^3a_1-12x^3yb_2+2x^2y^2a_2+2x^2y^2b_3+36x\,y^3a_3-8x^2yb_1+12x\,y^2a}{x^2\left(2xy-1\right)^2}$$
$$=0$$

Setting the numerator to zero gives

$$\begin{aligned} & 12x^4y^2b_2-24x^2y^4a_3+8x^3y^2b_1-8x^2y^3a_1-12x^3yb_2+2x^2y^2a_2+2x^2y^2b_3 \\ & +36x\,y^3a_3-8x^2yb_1+12x\,y^2a_1+4b_2x^2-12y^2a_3+3xb_1-3ya_1=0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x=v_1, y=v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -24a_3v_1^2v_2^4+12b_2v_1^4v_2^2-8a_1v_1^2v_2^3+8b_1v_1^3v_2^2+2a_2v_1^2v_2^2+36a_3v_1v_2^3-12b_2v_1^3v_2 \\ & +2b_3v_1^2v_2^2+12a_1v_1v_2^2-8b_1v_1^2v_2-12a_3v_2^2+4b_2v_1^2-3a_1v_2+3b_1v_1=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &12b_2v_1^4v_2^2 + 8b_1v_1^3v_2^2 - 12b_2v_1^3v_2 - 24a_3v_1^2v_2^4 - 8a_1v_1^2v_2^3 + (2a_2 + 2b_3)\,v_1^2v_2^2 \\ &- 8b_1v_1^2v_2 + 4b_2v_1^2 + 36a_3v_1v_2^3 + 12a_1v_1v_2^2 + 3b_1v_1 - 12a_3v_2^2 - 3a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -8a_1 &= 0 \\ -3a_1 &= 0 \\ 12a_1 &= 0 \\ -24a_3 &= 0 \\ -12a_3 &= 0 \\ 36a_3 &= 0 \\ -8b_1 &= 0 \\ 3b_1 &= 0 \\ 8b_1 &= 0 \\ -12b_2 &= 0 \\ 4b_2 &= 0 \\ 12b_2 &= 0 \\ 2a_2 + 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{y(4xy-3)}{x\,(2xy-1)}\right)(-x) \\
&= \frac{-2x\,y^2 + 2y}{2xy-1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2x\,y^2+2y}{2xy-1}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{\ln\left(y(xy-1)\right)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{y(4xy-3)}{x\,(2xy-1)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y}{2xy - 2} \\
S_y &= \frac{-2xy + 1}{2y\,(xy - 1)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{3}{2x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{3}{2R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{3\ln(R)}{2} + c_1 \tag{4}
$$

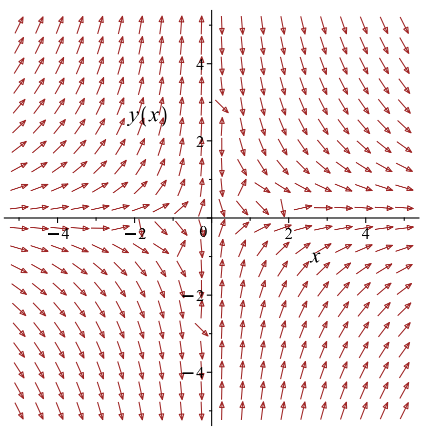
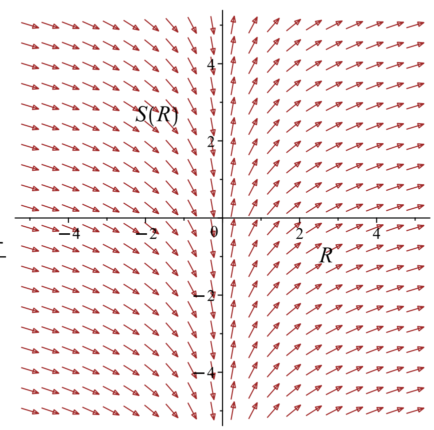
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{\ln(y)}{2} - \frac{\ln(yx - 1)}{2} = \frac{3\ln(x)}{2} + c_1
$$

Which simplifies to

$$
-\frac{\ln(y)}{2} - \frac{\ln(yx - 1)}{2} = \frac{3\ln(x)}{2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(4xy-3)}{x(2xy-1)}$ 	$R = x$ $S = -\frac{\ln(y)}{2} - \frac{\ln(xy-1)}{2}$	$\frac{dS}{dR} = \frac{3}{2R}$ 

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} - \frac{\ln(yx-1)}{2} = \frac{3\ln(x)}{2} + c_1 \tag{1}$$

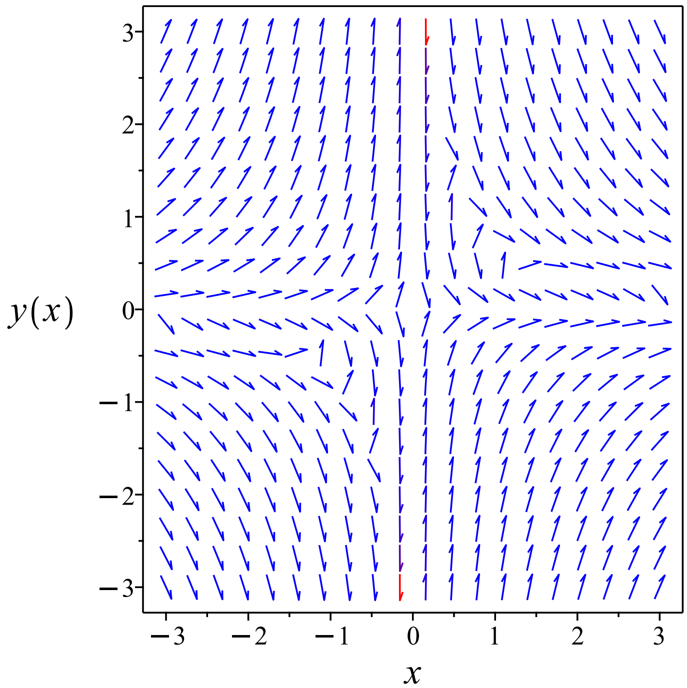


Figure 271: Slope field plot

Verification of solutions

$$-\frac{\ln(y)}{2} - \frac{\ln(yx-1)}{2} = \frac{3\ln(x)}{2} + c_1$$

Verified OK.

6.9.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{4y^2x^3 - 3x^2y}{-2x^4y + x^3} \right) \mathrm{d}x$$
$$\left(-\frac{4y^2x^3 - 3x^2y}{-2x^4y + x^3} \right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{4y^2x^3 - 3x^2y}{-2x^4y + x^3}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{4y^2x^3 - 3x^2y}{-2x^4y + x^3} \right) \\ &= \frac{8x^2y^2 - 8xy + 3}{x (2xy - 1)^2} \end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{8y\,x^3 - 3x^2}{-2x^4y + x^3} - \frac{2(4y^2x^3 - 3x^2y)\,x^4}{(-2x^4y + x^3)^2}\right) - (0)\right) \\
&= \frac{8x^2y^2 - 8xy + 3}{x\,(2xy - 1)^2}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{2x^2y - x}{4x\,y^2 - 3y}\left((0) - \left(-\frac{8y\,x^3 - 3x^2}{-2x^4y + x^3} - \frac{2(4y^2x^3 - 3x^2y)\,x^4}{(-2x^4y + x^3)^2}\right)\right) \\
&= \frac{-8x^2y^2 + 8xy - 3}{8y^3x^2 - 10x\,y^2 + 3y}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(0) - \left(-\frac{8y\,x^3 - 3x^2}{-2x^4y + x^3} - \frac{2\left(4y^2x^3 - 3x^2y\right)x^4}{\left(-2x^4y + x^3\right)^2}\right)}{x\left(-\frac{4y^2x^3 - 3x^2y}{-2x^4y + x^3}\right) - y\,(1)} \\
&= \frac{-8x^2y^2 + 8xy - 3}{4x\,(xy - 1)\,y\left(xy - \frac{1}{2}\right)}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$R = \frac{-8t^2 + 8t - 3}{4t(t-1)\left(t - \frac{1}{2}\right)}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}\mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int \left(\frac{-8t^2+8t-3}{4t(t-1)\left(t-\frac{1}{2}\right)}\right) \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{3\ln(t-1)}{2} - \frac{3\ln(t)}{2} + \ln(2t-1)} \\ &= \frac{2t-1}{(t-1)^{\frac{3}{2}} t^{\frac{3}{2}}}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{2xy-1}{(xy-1)^{\frac{3}{2}} (xy)^{\frac{3}{2}}}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{2xy-1}{(xy-1)^{\frac{3}{2}} (xy)^{\frac{3}{2}}} \left(-\frac{4y^2x^3 - 3x^2y}{-2x^4y + x^3}\right) \\ &= \frac{4xy-3}{x^2\sqrt{xy}\,(xy-1)^{\frac{3}{2}}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{2xy-1}{(xy-1)^{\frac{3}{2}} (xy)^{\frac{3}{2}}}(1) \\ &= \frac{2xy-1}{(xy-1)^{\frac{3}{2}}\, xy\sqrt{xy}}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{4xy-3}{x^2\sqrt{xy}\,(xy-1)^{\frac{3}{2}}}\right) + \left(\frac{2xy-1}{(xy-1)^{\frac{3}{2}}\, xy\sqrt{xy}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{4xy-3}{x^2\sqrt{xy}\,(xy-1)^{\frac{3}{2}}}\,\mathrm{d}x$$
$$\phi = -\frac{2}{\sqrt{xy-1}\,x\sqrt{xy}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{(xy-1)^{\frac{3}{2}}\sqrt{xy}} + \frac{1}{\sqrt{xy-1}\,(xy)^{\frac{3}{2}}} + f'(y) \tag{4}$$
$$= \frac{2xy-1}{(xy-1)^{\frac{3}{2}}\,xy\sqrt{xy}} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2xy-1}{(xy-1)^{\frac{3}{2}}\,xy\sqrt{xy}}$. Therefore equation (4) becomes

$$\frac{2xy-1}{(xy-1)^{\frac{3}{2}}\,xy\sqrt{xy}} = \frac{2xy-1}{(xy-1)^{\frac{3}{2}}\,xy\sqrt{xy}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{2}{\sqrt{xy-1}\,x\sqrt{xy}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{2}{\sqrt{xy-1}\, x\sqrt{xy}}$$

Summary

The solution(s) found are the following

$$-\frac{2}{\sqrt{yx-1}\, x\sqrt{yx}} = c_1 \tag{1}$$

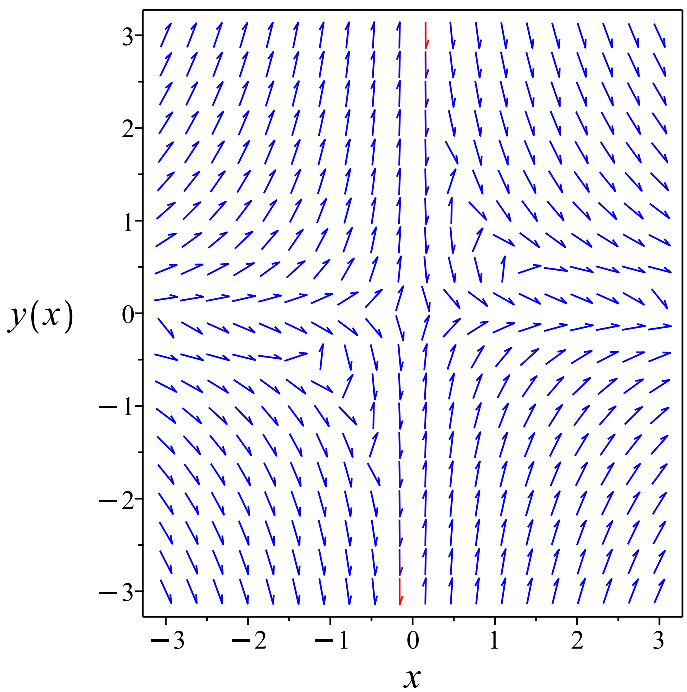


Figure 272: Slope field plot

Verification of solutions

$$-\frac{2}{\sqrt{yx-1}\, x\sqrt{yx}} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 41

```
dsolve(diff(y(x),x)=(4*x^3*y(x)^2-3*x^2*y(x))/(x^3-2*x^4*y(x)),y(x), singsol=all)
```

$$y(x) = \frac{x - \sqrt{x^2 + 4c_1}}{2x^2}$$
$$y(x) = \frac{x + \sqrt{x^2 + 4c_1}}{2x^2}$$

✓ Solution by Mathematica

Time used: 0.575 (sec). Leaf size: 78

```
DSolve[y'[x]==(4*x^3*y[x]^2-3*x^2*y[x])/(x^3-2*x^4*y[x]),y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{x^3 - \sqrt{x^2}\sqrt{x^4 + 4c_1x^2}}{2x^4}$$
$$y(x) \to \frac{x^3 + \sqrt{x^2}\sqrt{x^4 + 4c_1x^2}}{2x^4}$$

6.10 problem 10

Internal problem ID [11684]
Internal file name [`OUTPUT/11693_Wednesday_April_10_2024_04_54_49_PM_86942080/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(1+x)\, y' + yx = \mathrm{e}^{-x}}$$

6.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{x}{1+x}$$
$$q(x) = \frac{\mathrm{e}^{-x}}{1+x}$$

Hence the ode is

$$y' + \frac{xy}{1+x} = \frac{\mathrm{e}^{-x}}{1+x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{x}{1+x}dx} \\
&= \mathrm{e}^{x-\ln(1+x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{\mathrm{e}^{x}}{1+x}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{\mathrm{e}^{-x}}{1+x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{x} y}{1+x}\right) &= \left(\frac{\mathrm{e}^{x}}{1+x}\right)\left(\frac{\mathrm{e}^{-x}}{1+x}\right) \\
\mathrm{d}\left(\frac{\mathrm{e}^{x} y}{1+x}\right) &= \frac{1}{(1+x)^{2}}\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\mathrm{e}^{x} y}{1+x} &= \int \frac{1}{(1+x)^{2}}\,\mathrm{d}x \\
\frac{\mathrm{e}^{x} y}{1+x} &= -\frac{1}{1+x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{x}}{1+x}$ results in

$$
y = -\mathrm{e}^{-x} + c_1(1+x)\,\mathrm{e}^{-x}
$$

which simplifies to

$$
y = \mathrm{e}^{-x}(c_1 x + c_1 - 1)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-x}(c_1 x + c_1 - 1) \tag{1}
$$

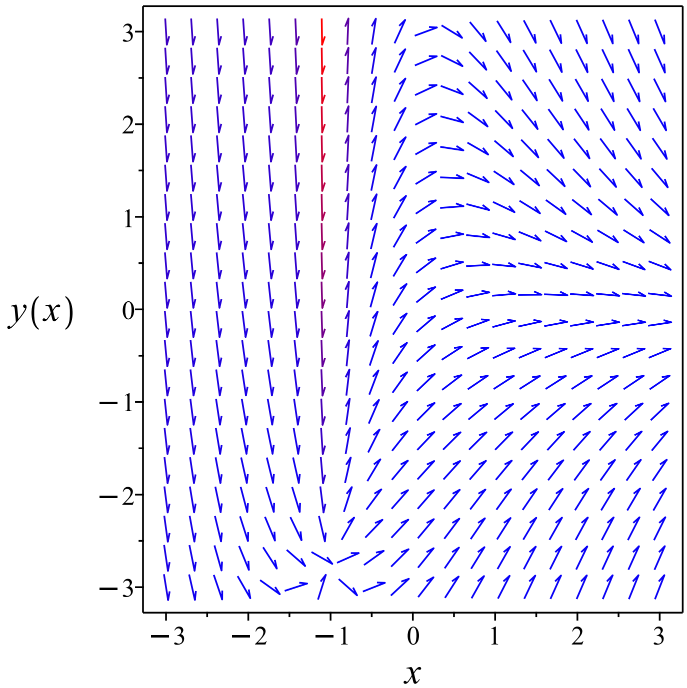


Figure 273: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 x + c_1 - 1)$$

Verified OK.

6.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-xy + e^{-x}}{1 + x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 219: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x+\ln(1+x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x+\ln(1+x)}} dy \end{aligned}$$

Which results in

$$S = \frac{e^x y}{1+x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-xy + e^{-x}}{1+x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{e^x y x}{(1+x)^2} \\ S_y &= \frac{e^x}{1+x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(1+x)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(1+R)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{1+R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{e^x y}{1+x} = -\frac{1}{1+x} + c_1$$

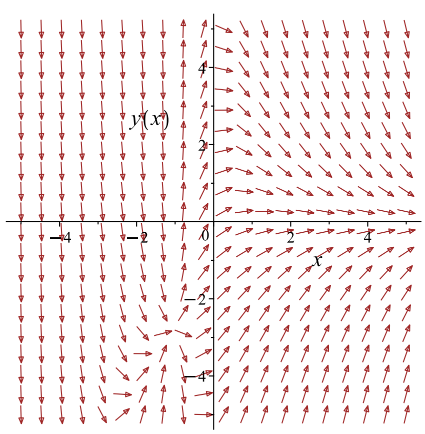
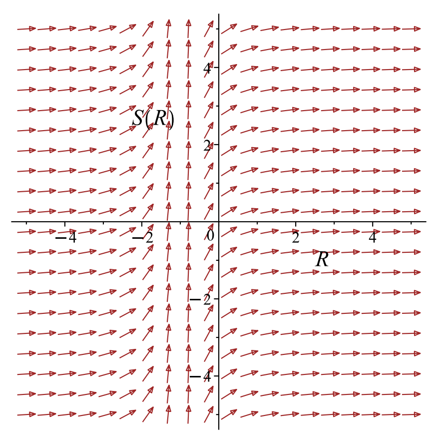
Which simplifies to

$$\frac{e^x y}{1+x} = -\frac{1}{1+x} + c_1$$

Which gives

$$y = e^{-x}(c_1 x + c_1 - 1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-xy+e^{-x}}{1+x}$ 	$R = x$ $S = \frac{e^x y}{1+x}$	$\frac{dS}{dR} = \frac{1}{(1+R)^2}$ 

Summary
The solution(s) found are the following

$$y = e^{-x}(c_1 x + c_1 - 1) \tag{1}$$

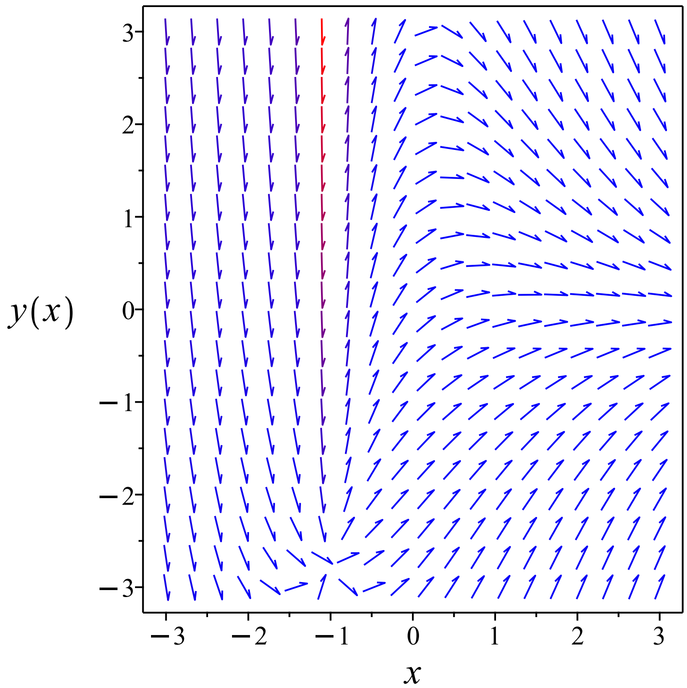


Figure 274: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 x + c_1 - 1)$$

Verified OK.

6.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(1 + x)\,\mathrm{d}y = \left(-xy + \mathrm{e}^{-x}\right)\mathrm{d}x$$
$$\left(xy - \mathrm{e}^{-x}\right)\mathrm{d}x + (1 + x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = xy - \mathrm{e}^{-x}$$
$$N(x, y) = 1 + x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(xy - \mathrm{e}^{-x}\right) \\ &= x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1 + x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{1+x}((x) - (1)) \\ &= \frac{x-1}{1+x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{x-1}{1+x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{x-2\ln(1+x)} \\ &= \frac{\mathrm{e}^{x}}{(1+x)^{2}} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{\mathrm{e}^{x}}{(1+x)^{2}}\left(xy - \mathrm{e}^{-x}\right) \\ &= \frac{x\,\mathrm{e}^{x} y - 1}{(1+x)^{2}} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{\mathrm{e}^{x}}{(1+x)^{2}}(1+x) \\ &= \frac{\mathrm{e}^{x}}{1+x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{x\,\mathrm{e}^{x} y - 1}{(1+x)^{2}}\right) + \left(\frac{\mathrm{e}^{x}}{1+x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{x \, \mathrm{e}^x y - 1}{(1 + x)^2} \, \mathrm{d}x$$

$$\phi = \frac{\mathrm{e}^x y + 1}{1 + x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^x}{1 + x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^x}{1+x}$. Therefore equation (4) becomes

$$\frac{\mathrm{e}^x}{1 + x} = \frac{\mathrm{e}^x}{1 + x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\mathrm{e}^x y + 1}{1 + x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\mathrm{e}^x y + 1}{1 + x}$$

The solution becomes

$$y = e^{-x}(c_1 x + c_1 - 1)$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_1 x + c_1 - 1) \tag{1}$$

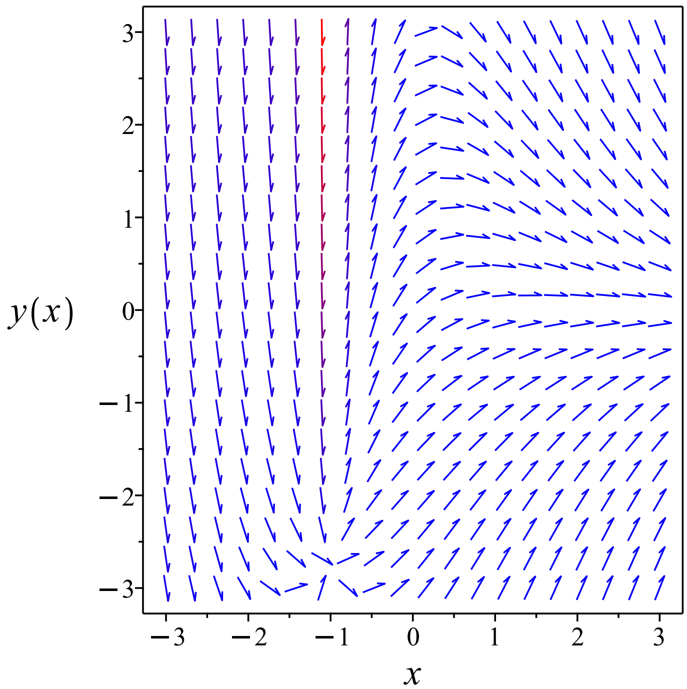


Figure 275: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 x + c_1 - 1)$$

Verified OK.

6.10.4 Maple step by step solution

Let's solve

$(1+x)\,y' + yx = \mathrm{e}^{-x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{xy}{1+x} + \frac{\mathrm{e}^{-x}}{1+x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{xy}{1+x} = \frac{\mathrm{e}^{-x}}{1+x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{xy}{1+x}\right) = \frac{\mu(x)\mathrm{e}^{-x}}{1+x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{xy}{1+x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)x}{1+x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{\mathrm{e}^{x}}{1+x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)\mathrm{e}^{-x}}{1+x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)\mathrm{e}^{-x}}{1+x} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\mathrm{e}^{-x}}{1+x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{\mathrm{e}^{x}}{1+x}$

  $y = \frac{(1+x)\left(\int \frac{\mathrm{e}^{-x}\mathrm{e}^{x}}{(1+x)^2} dx + c_1\right)}{\mathrm{e}^{x}}$

- Evaluate the integrals on the rhs

  $y = \frac{(1+x)\left(-\frac{1}{1+x} + c_1\right)}{\mathrm{e}^{x}}$

- Simplify

  $y = e^{-x}(c_1 x + c_1 - 1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x+1)*diff(y(x),x)+x*y(x)=exp(-x),y(x), singsol=all)
```

$$y(x) = e^{-x}(c_1 x + c_1 - 1)$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 19

```
DSolve[(x+1)*y'[x]+x*y[x]==Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(-1 + c_1(x + 1))$$

6.11 problem 11

Internal problem ID [11685]
Internal file name [`OUTPUT/11694_Wednesday_April_10_2024_04_54_50_PM_17922417/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{2x - 7y}{3y - 8x} = 0}$$

6.11.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - \frac{2x - 7u(x)\, x}{3u\,(x)\, x - 8x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u^2 - u - 2}{x\,(3u - 8)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{3u^2-u-2}{3u-8}$. Integrating both sides gives

$$\frac{1}{\frac{3u^2-u-2}{3u-8}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{3u^2-u-2}{3u-8}}\, du = \int -\frac{1}{x}\, dx$$

$$-\ln(u-1) + 2\ln(3u+2) = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{-\ln(u-1)+2\ln(3u+2)} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{(3u+2)^2}{u-1} = \frac{c_3}{x}$$

The solution is

$$\frac{(3u(x)+2)^2}{u(x)-1} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\left(\frac{3y}{x}+2\right)^2}{\frac{y}{x}-1} = \frac{c_3}{x}$$

$$\frac{(2x+3y)^2}{(y-x)\,x} = \frac{c_3}{x}$$

Which simplifies to

$$-\frac{(2x+3y)^2}{-y+x} = c_3$$

Summary

The solution(s) found are the following

$$-\frac{(2x+3y)^2}{-y+x} = c_3 \tag{1}$$

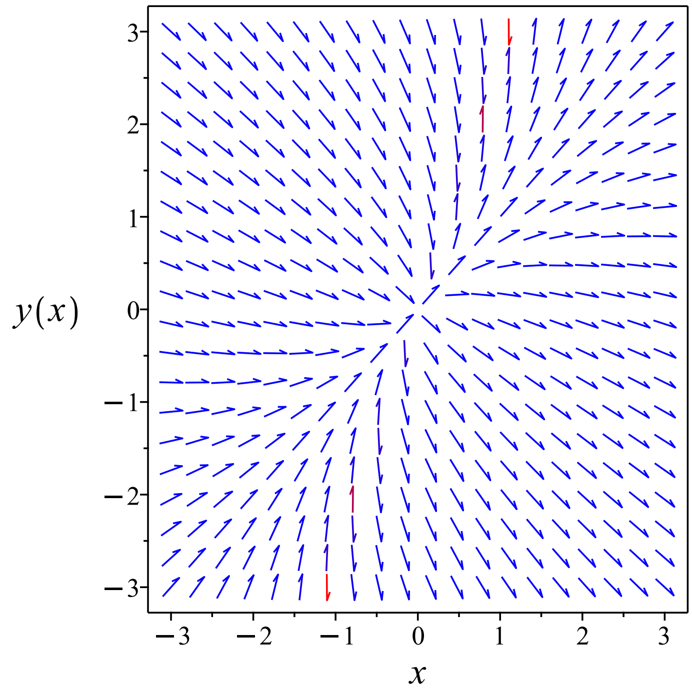


Figure 276: Slope field plot

Verification of solutions

$$-\frac{(2x+3y)^2}{-y+x} = c_3$$

Verified OK.

6.11.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{-2x+7y}{3y-8x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(-2x+7y)(b_3-a_2)}{3y-8x} - \frac{(-2x+7y)^2 a_3}{(3y-8x)^2} \\ & - \left(\frac{2}{3y-8x} - \frac{8(-2x+7y)}{(3y-8x)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{7}{3y-8x} + \frac{-6x+21y}{(3y-8x)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{16x^2a_2 - 4x^2a_3 + 14x^2b_2 - 16x^2b_3 - 12xya_2 + 28xya_3 - 48xyb_2 + 12xyb_3 + 21y^2a_2 + y^2a_3 + 9y^2b_2 - 21y}{(-3y+8x)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 16x^2a_2 - 4x^2a_3 + 14x^2b_2 - 16x^2b_3 - 12xya_2 + 28xya_3 - 48xyb_2 \\ & + 12xyb_3 + 21y^2a_2 + y^2a_3 + 9y^2b_2 - 21y^2b_3 - 50xb_1 + 50ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 16a_2v_1^2 - 12a_2v_1v_2 + 21a_2v_2^2 - 4a_3v_1^2 + 28a_3v_1v_2 + a_3v_2^2 + 14b_2v_1^2 \\ & - 48b_2v_1v_2 + 9b_2v_2^2 - 16b_3v_1^2 + 12b_3v_1v_2 - 21b_3v_2^2 + 50a_1v_2 - 50b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(16a_2 - 4a_3 + 14b_2 - 16b_3)\, v_1^2 + (-12a_2 + 28a_3 - 48b_2 + 12b_3)\, v_1 v_2 - 50b_1 v_1 + (21a_2 + a_3 + 9b_2 - 21b_3)\, v_2^2 + 50a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 50a_1 &= 0 \\ -50b_1 &= 0 \\ -12a_2 + 28a_3 - 48b_2 + 12b_3 &= 0 \\ 16a_2 - 4a_3 + 14b_2 - 16b_3 &= 0 \\ 21a_2 + a_3 + 9b_2 - 21b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= -3a_2 + 3b_3 \\ b_1 &= 0 \\ b_2 &= -2a_2 + 2b_3 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 3y \\ \eta &= 2x + y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= 2x + y - \left(-\frac{-2x + 7y}{3y - 8x}\right)(3y) \\ &= \frac{16x^2 + 8xy - 24y^2}{-3y + 8x} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{16x^2+8xy-24y^2}{-3y+8x}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(2x+3y)}{4} - \frac{\ln(y-x)}{8}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-2x+7y}{3y-8x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x-7y}{16x^2+8xy-24y^2} \\ S_y &= \frac{-3y+8x}{16x^2+8xy-24y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

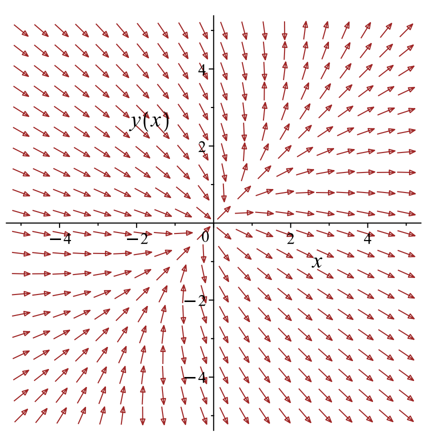
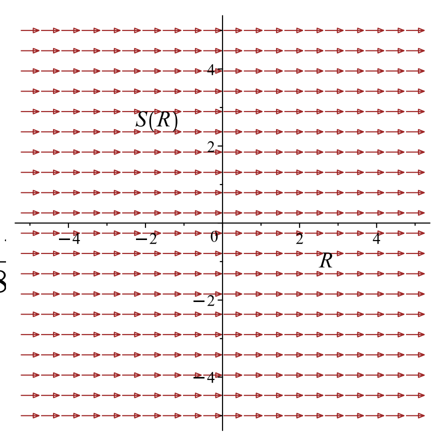
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(2x+3y)}{4} - \frac{\ln(y-x)}{8} = c_1$$

Which simplifies to

$$\frac{\ln(2x+3y)}{4} - \frac{\ln(y-x)}{8} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-2x+7y}{3y-8x}$ 	$R = x$ $S = \frac{\ln(2x+3y)}{4} - \frac{\ln(y}{8}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(2x+3y)}{4}-\frac{\ln(y-x)}{8}=c_1 \tag{1}$$

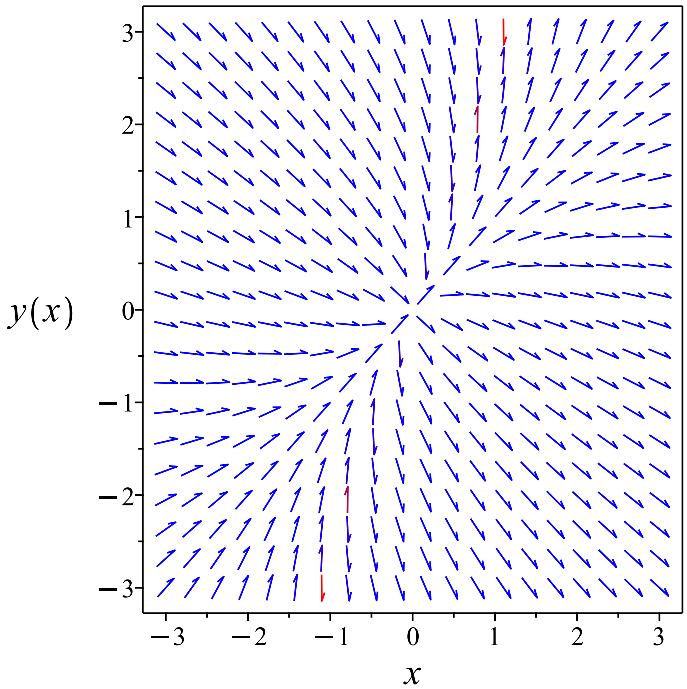


Figure 277: Slope field plot

Verification of solutions

$$\frac{\ln(2x+3y)}{4}-\frac{\ln(y-x)}{8}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 47

```
dsolve(diff(y(x),x)=(2*x-7*y(x))/(3*y(x)-8*x),y(x), singsol=all)
```

$$y(x) = \frac{-12c_1x - \sqrt{-60c_1x+1}+1}{18c_1}$$
$$y(x) = \frac{-12c_1x + 1 + \sqrt{-60c_1x+1}}{18c_1}$$

✓ Solution by Mathematica

Time used: 0.969 (sec). Leaf size: 80

```
DSolve[y'[x]==(2*x-7*y[x])/(3*y[x]-8*x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{18}\left(-12x - e^{\frac{c_1}{2}}\sqrt{60x+e^{c_1}} - e^{c_1}\right)$$
$$y(x) \to \frac{1}{18}\left(-12x + e^{\frac{c_1}{2}}\sqrt{60x+e^{c_1}} - e^{c_1}\right)$$

6.12 problem 12

Internal problem ID [11686]
Internal file name [`OUTPUT/11695_Wednesday_April_10_2024_04_54_52_PM_34915414/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x^2y' + yx - y^3x = 0}$$

6.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^3 - y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y^3 - y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y^3 - y}\, dy &= \frac{1}{x}\, dx \\ \int \frac{1}{y^3 - y}\, dy &= \int \frac{1}{x}\, dx \end{aligned}$$

$$\frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \ln(y) = \ln(x) + c_1$$

Raising both side to exponential gives

$$e^{\frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \ln(y)} = e^{\ln(x) + c_1}$$

Which simplifies to

$$\frac{\sqrt{y-1}\,\sqrt{y+1}}{y} = c_2 x$$

The solution is

$$\frac{\sqrt{y-1}\,\sqrt{y+1}}{y} = c_2 x$$

Summary

The solution(s) found are the following

$$\frac{\sqrt{y-1}\,\sqrt{y+1}}{y} = c_2 x \tag{1}$$

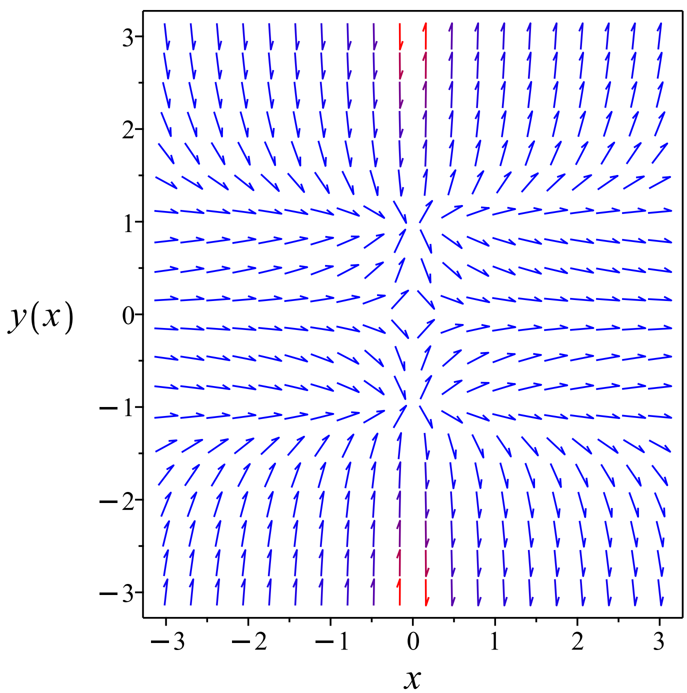


Figure 278: Slope field plot

Verification of solutions

$$\frac{\sqrt{y-1}\,\sqrt{y+1}}{y} = c_2 x$$

Verified OK.

6.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(y^2-1)}{x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 222: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x} dx \end{aligned}$$

Which results in

$$S = \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(y^2-1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^3 - y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3 - R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R-1)}{2} + \frac{\ln(R+1)}{2} - \ln(R) + c_1 \tag{4}$$

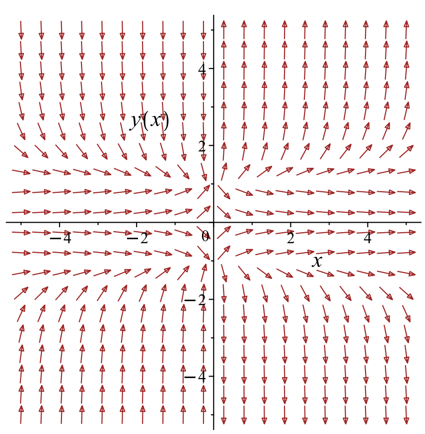
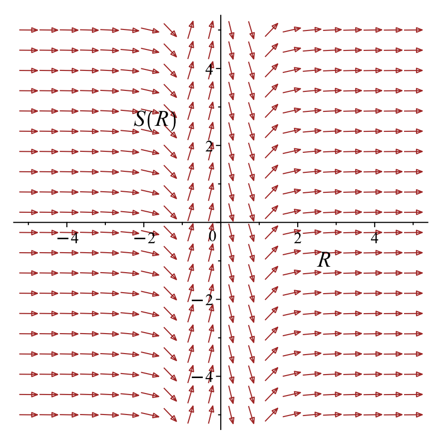
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \ln(y) + c_1$$

Which simplifies to

$$\ln(x) = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \ln(y) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(y^2-1)}{x}$ 	$R = y$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{1}{R^3-R}$ 

Summary

The solution(s) found are the following

$$\ln(x) = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \ln(y) + c_1 \tag{1}$$

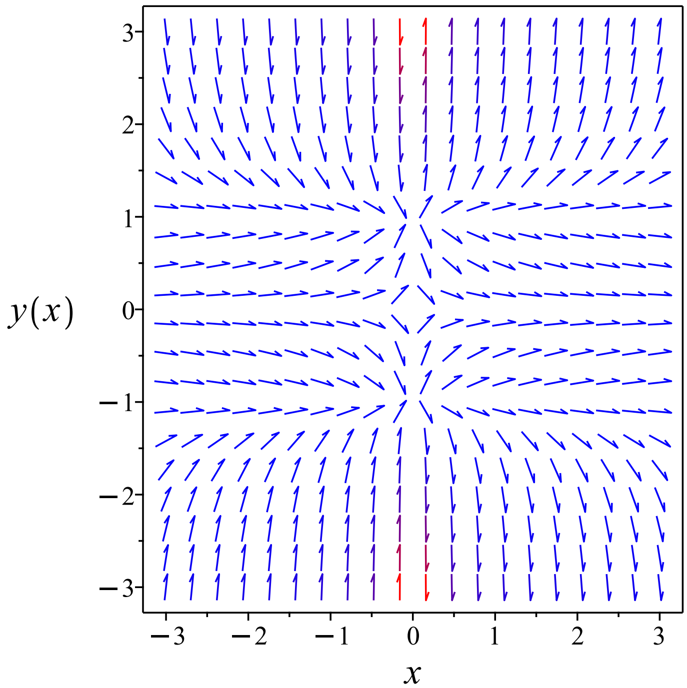


Figure 279: Slope field plot

Verification of solutions

$$\ln(x) = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \ln(y) + c_1$$

Verified OK.

6.12.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(y^2-1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + \frac{1}{x}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \frac{1}{x} \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y' \frac{1}{y^3} = -\frac{1}{x\,y^2} + \frac{1}{x} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{2} &= -\frac{w(x)}{x} + \frac{1}{x} \\ w' &= \frac{2w}{x} - \frac{2}{x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= -\frac{2}{x} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = -\frac{2}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{2}{x}\right) \\ \frac{d}{dx}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(-\frac{2}{x}\right) \\ d\left(\frac{w}{x^2}\right) &= \left(-\frac{2}{x^3}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^2} &= \int -\frac{2}{x^3} dx \\ \frac{w}{x^2} &= \frac{1}{x^2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = c_1 x^2 + 1$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = c_1 x^2 + 1$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{1}{\sqrt{c_1 x^2 + 1}} \\ y(x) &= -\frac{1}{\sqrt{c_1 x^2 + 1}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{c_1 x^2 + 1}} \tag{1}$$

$$y = -\frac{1}{\sqrt{c_1 x^2 + 1}} \tag{2}$$

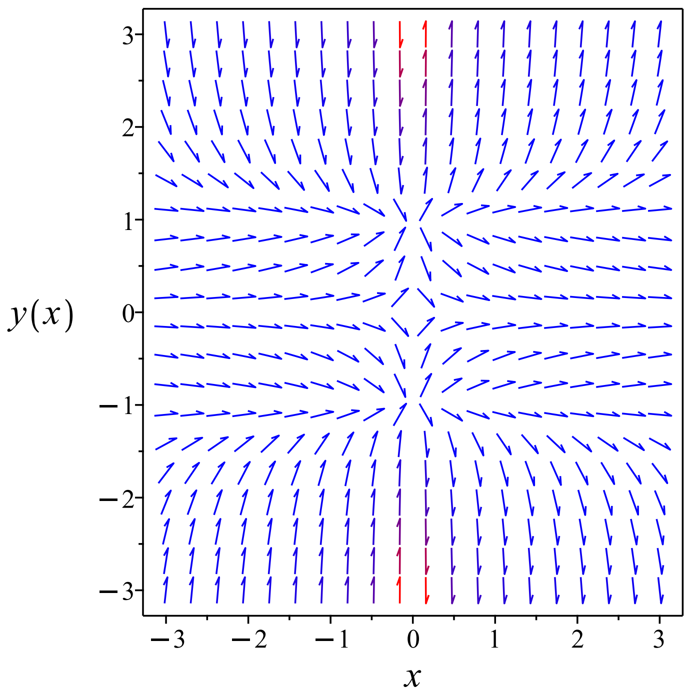


Figure 280: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{c_1 x^2 + 1}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{c_1 x^2 + 1}}$$

Verified OK.

6.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y\,(y^2-1)}\right)\mathrm{d}y&=\left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x+\left(\frac{1}{y\,(y^2-1)}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= -\frac{1}{x} \\
N(x,y) &= \frac{1}{y\,(y^2-1)}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y\,(y^2-1)}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x \\
\phi &= -\ln(x) + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{y(y^2-1)}$. Therefore equation (4) becomes

$$\frac{1}{y\left(y^2-1\right)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= \frac{1}{y\left(y^2-1\right)} \\ &= \frac{1}{y^3-y} \end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned} \int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y^3-y}\right)\mathrm{d}y \\ f(y) &= \frac{\ln\left(y-1\right)}{2} + \frac{\ln\left(y+1\right)}{2} - \ln\left(y\right) + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln\left(x\right) + \frac{\ln\left(y-1\right)}{2} + \frac{\ln\left(y+1\right)}{2} - \ln\left(y\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln\left(x\right) + \frac{\ln\left(y-1\right)}{2} + \frac{\ln\left(y+1\right)}{2} - \ln\left(y\right)$$

Summary

The solution(s) found are the following

$$-\ln\left(x\right) + \frac{\ln\left(y-1\right)}{2} + \frac{\ln\left(y+1\right)}{2} - \ln\left(y\right) = c_1 \tag{1}$$

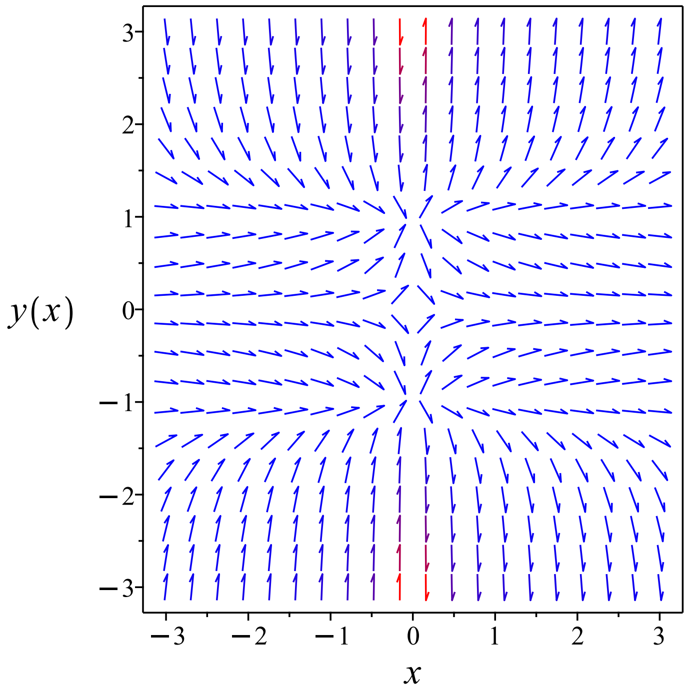


Figure 281: Slope field plot

Verification of solutions

$$-\ln(x) + \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} - \ln(y) = c_1$$

Verified OK.

6.12.5 Maple step by step solution

Let's solve

$x^2y' + yx - y^3x = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(y-1)(y+1)} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(y-1)(y+1)}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

$$\frac{\ln(y-1)}{2}+\frac{\ln(y+1)}{2}-\ln(y)=\ln(x)+c_1$$

- Solve for y

$$\left\{y=-\frac{\sqrt{-(e^{c_1})^2x^2+1}-1}{\sqrt{-(e^{c_1})^2x^2+1}}+1, y=-\frac{\sqrt{-(e^{c_1})^2x^2+1}+1}{\sqrt{-(e^{c_1})^2x^2+1}}+1\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x)+x*y(x)=x*y(x)^3,y(x), singsol=all)
```

$$y(x)=\frac{1}{\sqrt{c_1x^2+1}}$$
$$y(x)=-\frac{1}{\sqrt{c_1x^2+1}}$$

✓ Solution by Mathematica

Time used: 0.242 (sec). Leaf size: 58

```
DSolve[x^2*y'[x]+x*y[x]==x*y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\frac{1}{\sqrt{1+e^{2c_1}x^2}}$$
$$y(x)\to \frac{1}{\sqrt{1+e^{2c_1}x^2}}$$
$$y(x)\to -1$$
$$y(x)\to 0$$
$$y(x)\to 1$$

6.13 problem 13

Internal problem ID [11687]
Internal file name [`OUTPUT/11696_Wednesday_April_10_2024_04_54_53_PM_72324803/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x^3+1)\,y'+6x^2y=6x^2}$$

6.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{x^2(-6y+6)}{x^3+1} \end{aligned}$$

Where $f(x)=\frac{x^2}{x^3+1}$ and $g(y)=-6y+6$. Integrating both sides gives

$$\frac{1}{-6y+6}\,dy=\frac{x^2}{x^3+1}\,dx$$

$$\int \frac{1}{-6y+6}\, dy = \int \frac{x^2}{x^3+1}\, dx$$

$$-\frac{\ln (y-1)}{6} = \frac{\ln (x^3+1)}{3} + c_1$$

Raising both side to exponential gives

$$\frac{1}{(y-1)^{\frac{1}{6}}} = \mathrm{e}^{\frac{\ln\left(x^3+1\right)}{3}+c_1}$$

Which simplifies to

$$\frac{1}{(y-1)^{\frac{1}{6}}} = c_2\left(x^3+1\right)^{\frac{1}{3}}$$

Which simplifies to

$$y = \frac{\left(c_2^6 \mathrm{e}^{6c_1} (x^3+1)^2 + 1\right) \mathrm{e}^{-6c_1}}{c_2^6 \,(x^3+1)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_2^6 \mathrm{e}^{6c_1} (x^3+1)^2 + 1\right) \mathrm{e}^{-6c_1}}{c_2^6 \,(x^3+1)^2} \tag{1}$$

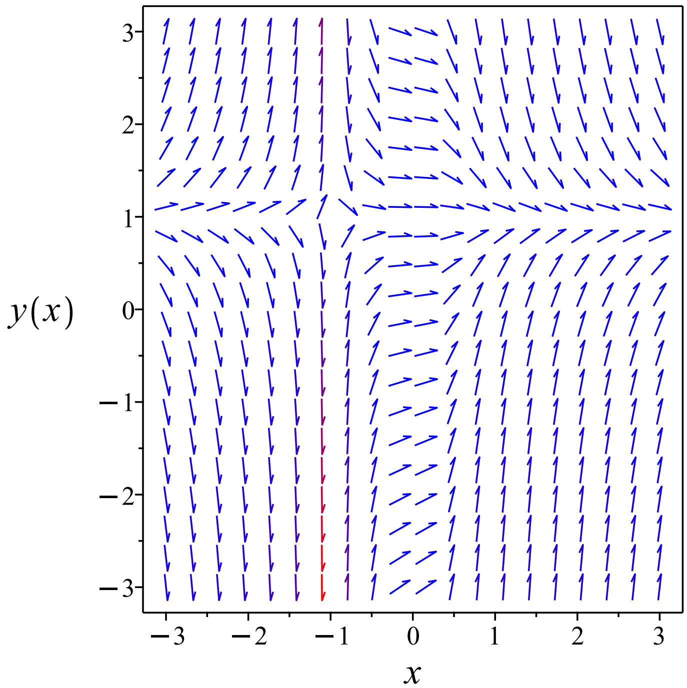


Figure 282: Slope field plot

Verification of solutions

$$y = \frac{\left(c_2^6 e^{6c_1} (x^3+1)^2 + 1\right) e^{-6c_1}}{c_2^6 (x^3+1)^2}$$

Verified OK.

6.13.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{6x^2}{x^3+1}$$
$$q(x) = \frac{6x^2}{x^3+1}$$

Hence the ode is

$$y' + \frac{6x^2 y}{x^3+1} = \frac{6x^2}{x^3+1}$$

The integrating factor μ is

$$\mu = e^{\int \frac{6x^2}{x^3+1}dx} \\ = (x^3+1)^2$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{6x^2}{x^3+1}\right) \\ \frac{d}{dx}\left((x^3+1)^2 y\right) = \left((x^3+1)^2\right)\left(\frac{6x^2}{x^3+1}\right) \\ d\left((x^3+1)^2 y\right) = (6x^5+6x^2)\, dx$$

Integrating gives

$$(x^3+1)^2 y = \int 6x^5 + 6x^2 \, dx \\ (x^3+1)^2 y = x^6 + 2x^3 + c_1$$

Dividing both sides by the integrating factor $\mu = (x^3+1)^2$ results in

$$y = \frac{x^6+2x^3}{(x^3+1)^2} + \frac{c_1}{(x^3+1)^2}$$

which simplifies to

$$y = \frac{x^6+2x^3+c_1}{(x^3+1)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^6+2x^3+c_1}{(x^3+1)^2} \tag{1}$$

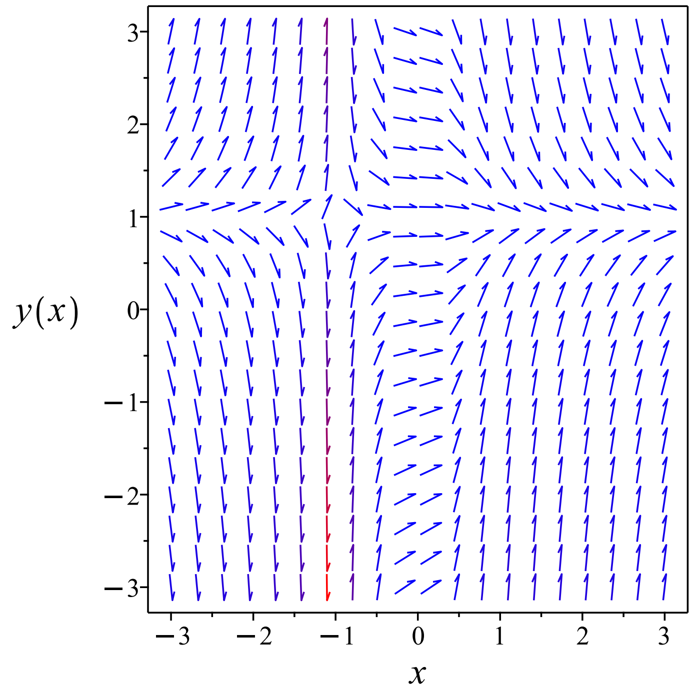


Figure 283: Slope field plot

Verification of solutions

$$y = \frac{x^6 + 2x^3 + c_1}{(x^3 + 1)^2}$$

Verified OK.

6.13.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{6(y-1)\,x^2}{x^3+1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 225: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{(x^3+1)^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(x^3+1)^2}} dy \end{aligned}$$

Which results in

$$S = (x^3 + 1)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{6(y-1)\,x^2}{x^3+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 6x^5y + 6x^2y \\ S_y &= (x^3+1)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 6x^5 + 6x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 6R^5 + 6R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R^6 + 2R^3 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x^3+1)^2 y = x^6 + 2x^3 + c_1$$

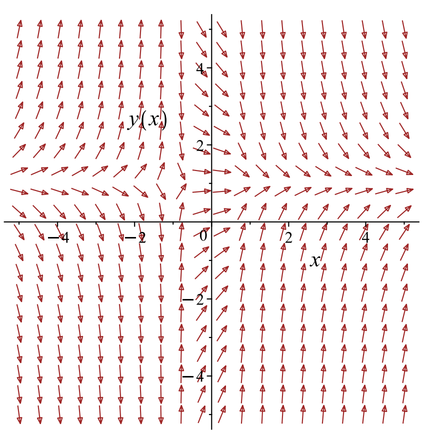
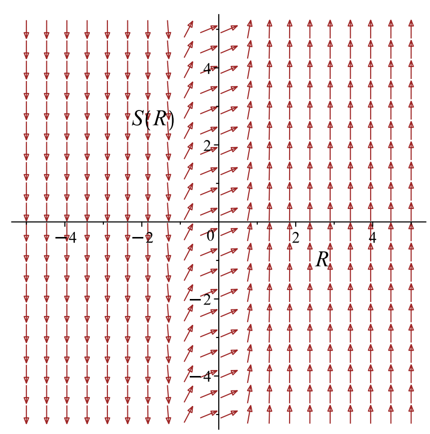
Which simplifies to

$$(x^3+1)^2 y = x^6 + 2x^3 + c_1$$

Which gives

$$y = \frac{x^6 + 2x^3 + c_1}{(x^3+1)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{6(y-1)x^2}{x^3+1}$ 	$R = x$ $S = (x^3+1)^2 y$	$\frac{dS}{dR} = 6R^5 + 6R^2$ 

Summary
The solution(s) found are the following

$$y = \frac{x^6 + 2x^3 + c_1}{(x^3 + 1)^2} \tag{1}$$

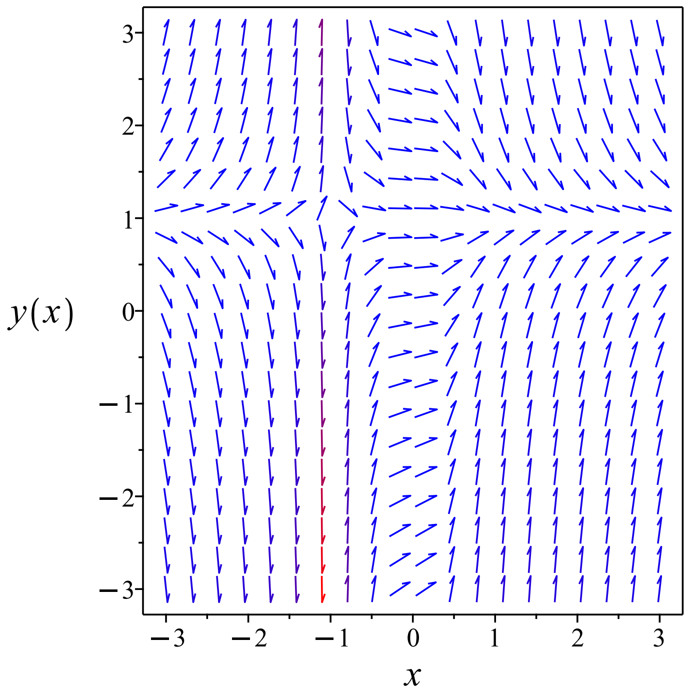


Figure 284: Slope field plot

Verification of solutions

$$y = \frac{x^6 + 2x^3 + c_1}{(x^3 + 1)^2}$$

Verified OK.

6.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-6y+6}\right)\mathrm{d}y &= \left(\frac{x^2}{x^3+1}\right)\mathrm{d}x \\ \left(-\frac{x^2}{x^3+1}\right)\mathrm{d}x + \left(\frac{1}{-6y+6}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{x^2}{x^3+1} \\ N(x, y) &= \frac{1}{-6y+6}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^2}{x^3+1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-6y+6}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x^2}{x^3+1} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln(x^3+1)}{3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-6y+6}$. Therefore equation (4) becomes

$$
\frac{1}{-6y+6} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{6(y-1)}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{6y-6}\right) \mathrm{d}y$$

$$f(y) = -\frac{\ln(y-1)}{6} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^3+1)}{3} - \frac{\ln(y-1)}{6} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^3+1)}{3} - \frac{\ln(y-1)}{6}$$

The solution becomes

$$y = \frac{x^6 + 2x^3 + \mathrm{e}^{-6c_1} + 1}{x^6 + 2x^3 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{x^6 + 2x^3 + \mathrm{e}^{-6c_1} + 1}{x^6 + 2x^3 + 1} \tag{1}$$

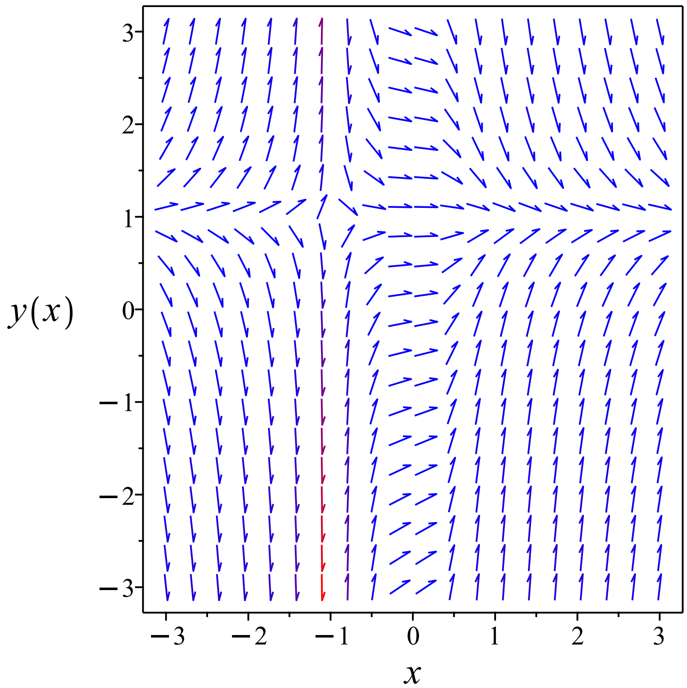


Figure 285: Slope field plot

Verification of solutions

$$y = \frac{x^6 + 2x^3 + e^{-6c_1} + 1}{x^6 + 2x^3 + 1}$$

Verified OK.

6.13.5 Maple step by step solution

Let's solve

$(x^3 + 1)\, y' + 6x^2 y = 6x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y-1} = -\frac{6x^2}{(1+x)(x^2-x+1)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y-1} dx = \int -\frac{6x^2}{(1+x)(x^2-x+1)} dx + c_1$

- Evaluate integral

$$\ln(y-1) = -2\ln((1+x)(x^2-x+1)) + c_1$$

- Solve for y

$$y = \frac{x^6+2x^3+e^{c_1}+1}{x^6+2x^3+1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve((x^3+1)*diff(y(x),x)+6*x^2*y(x)=6*x^2,y(x), singsol=all)
```

$$y(x) = \frac{x^6+2x^3+c_1}{(x^3+1)^2}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 29

```
DSolve[(x^3+1)*y'[x]+6*x^2*y[x]==6*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^6+2x^3+c_1}{(x^3+1)^2}$$

$$y(x) \to 1$$

6.14 problem 14

Internal problem ID [11688]
Internal file name [`OUTPUT/11697_Wednesday_April_10_2024_04_54_54_PM_9116467/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y' - \frac{2x^2 + y^2}{2yx - x^2} = 0}$$

6.14.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - \frac{2x^2 + u(x)^2\, x^2}{2u\,(x)\, x^2 - x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{u^2 - u - 2}{x\,(2u - 1)}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-u-2}{2u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-u-2}{2u-1}} \, du = -\frac{1}{x} \, dx$$

$$\int \frac{1}{\frac{u^2-u-2}{2u-1}} \, du = \int -\frac{1}{x} \, dx$$

$$\ln\left(u^2 - u - 2\right) = -\ln(x) + c_2$$

Raising both side to exponential gives

$$u^2 - u - 2 = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$u^2 - u - 2 = \frac{c_3}{x}$$

Which simplifies to

$$u(x)^2 - u(x) - 2 = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$u(x)^2 - u(x) - 2 = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2} - \frac{y}{x} - 2 = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

$$\frac{y^2}{x^2} - \frac{y}{x} - 2 = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Which simplifies to

$$\frac{y^2 - yx - 2x^2}{x} = c_3 \mathrm{e}^{c_2}$$

Summary

The solution(s) found are the following

$$\frac{y^2 - yx - 2x^2}{x} = c_3 \mathrm{e}^{c_2} \tag{1}$$

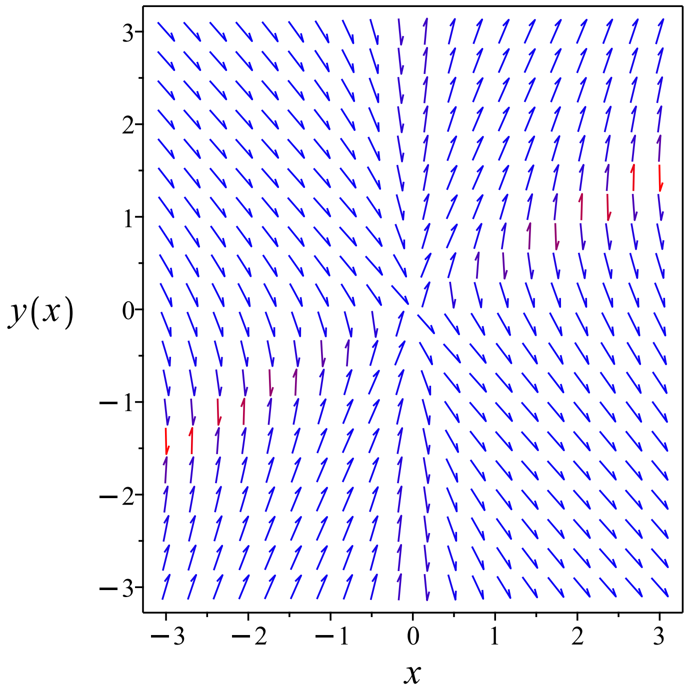


Figure 286: Slope field plot

Verification of solutions

$$\frac{y^2 - yx - 2x^2}{x} = c_3 \mathrm{e}^{c_2}$$

Verified OK.

6.14.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{2x^2 + y^2}{x\left(-x + 2y\right)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(2x^2+y^2)(b_3-a_2)}{x(-x+2y)} - \frac{(2x^2+y^2)^2 a_3}{x^2(-x+2y)^2} \\ & - \left(\frac{4}{-x+2y} - \frac{2x^2+y^2}{x^2(-x+2y)} + \frac{2x^2+y^2}{x(-x+2y)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(\frac{2y}{x(-x+2y)} - \frac{2(2x^2+y^2)}{x(-x+2y)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^4a_2 - 4x^4a_3 + 5x^4b_2 - 2x^4b_3 - 8x^3ya_2 - 2x^3yb_2 + 8x^3yb_3 - x^2y^2a_2 - 8x^2y^2a_3 + 2x^2y^2b_2 + x^2y^2b_3 - 2x}{(x-2y)^2 x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 2x^4a_2 - 4x^4a_3 + 5x^4b_2 - 2x^4b_3 - 8x^3ya_2 - 2x^3yb_2 + 8x^3yb_3 \\ & - x^2y^2a_2 - 8x^2y^2a_3 + 2x^2y^2b_2 + x^2y^2b_3 - 2x\,y^3a_3 + y^4a_3 \\ & + 4x^3b_1 - 4x^2ya_1 + 2x^2yb_1 - 2x\,y^2a_1 - 2x\,y^2b_1 + 2y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 2a_2v_1^4 - 8a_2v_1^3v_2 - a_2v_1^2v_2^2 - 4a_3v_1^4 - 8a_3v_1^2v_2^2 - 2a_3v_1v_2^3 + a_3v_2^4 \\ & + 5b_2v_1^4 - 2b_2v_1^3v_2 + 2b_2v_1^2v_2^2 - 2b_3v_1^4 + 8b_3v_1^3v_2 + b_3v_1^2v_2^2 \\ & - 4a_1v_1^2v_2 - 2a_1v_1v_2^2 + 2a_1v_2^3 + 4b_1v_1^3 + 2b_1v_1^2v_2 - 2b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(2a_2 - 4a_3 + 5b_2 - 2b_3) v_1^4 + (-8a_2 - 2b_2 + 8b_3) v_1^3 v_2 \\&+ 4b_1 v_1^3 + (-a_2 - 8a_3 + 2b_2 + b_3) v_1^2 v_2^2 + (-4a_1 + 2b_1) v_1^2 v_2 \\&- 2a_3 v_1 v_2^3 + (-2a_1 - 2b_1) v_1 v_2^2 + a_3 v_2^4 + 2a_1 v_2^3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}a_3 &= 0 \\ 2a_1 &= 0 \\ -2a_3 &= 0 \\ 4b_1 &= 0 \\ -4a_1 + 2b_1 &= 0 \\ -2a_1 - 2b_1 &= 0 \\ -8a_2 - 2b_2 + 8b_3 &= 0 \\ -a_2 - 8a_3 + 2b_2 + b_3 &= 0 \\ 2a_2 - 4a_3 + 5b_2 - 2b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x \\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{2x^2 + y^2}{x\,(-x + 2y)}\right)(x) \\
&= \frac{2x^2 + xy - y^2}{x - 2y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2x^2 + xy - y^2}{x - 2y}} dy
\end{aligned}$$

Which results in

$$S = \ln\left(-2x^2 - xy + y^2\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2x^2 + y^2}{x\,(-x + 2y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{4x+y}{(y+x)(2x-y)} \\
S_y &= \frac{x-2y}{(y+x)(2x-y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

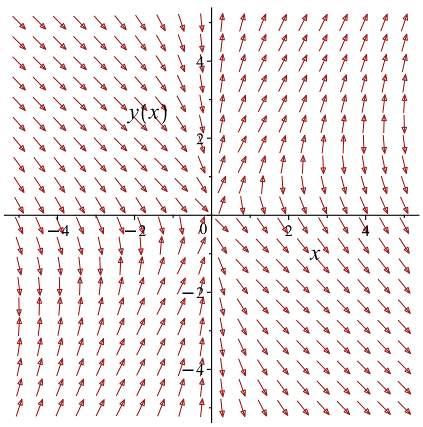
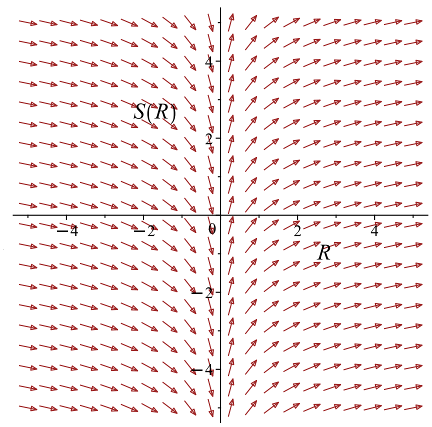
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(-y-x) + \ln(2x-y) = \ln(x) + c_1$$

Which simplifies to

$$\ln(-y-x) + \ln(2x-y) = \ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x^2+y^2}{x(-x+2y)}$ 	$R = x$ $S = \ln(-y-x) + \ln(2x$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$\ln(-y-x) + \ln(2x-y) = \ln(x) + c_1 \tag{1}$$

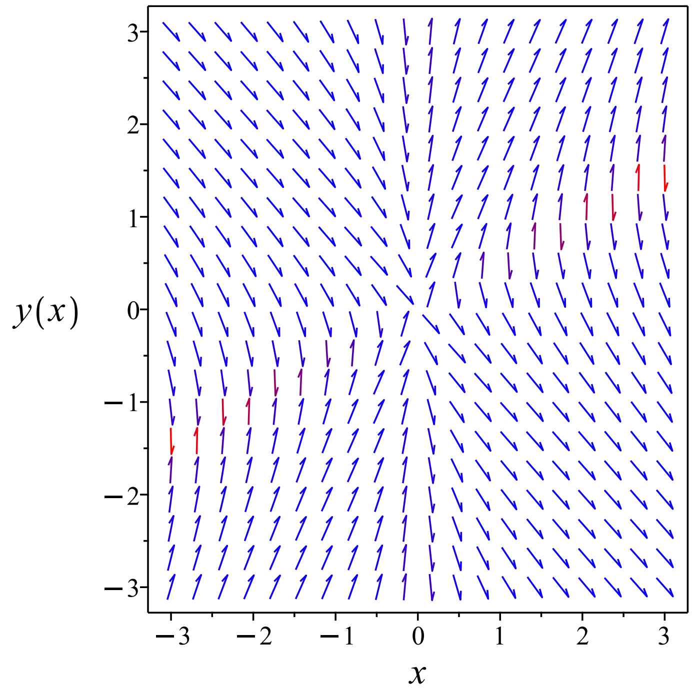


Figure 287: Slope field plot

Verification of solutions

$$\ln(-y-x)+\ln(2x-y)=\ln(x)+c_1$$

Verified OK.

6.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x(-x + 2y))\,\mathrm{d}y = \left(2x^2 + y^2\right)\mathrm{d}x$$
$$\left(-2x^2 - y^2\right)\mathrm{d}x + (x(-x + 2y))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -2x^2 - y^2$$
$$N(x, y) = x(-x + 2y)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2x^2 - y^2\right) \\ &= -2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x(-x + 2y)) \\ &= -2x + 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x(x-2y)}((-2y) - (-2x+2y)) \\
&= -\frac{2}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}(-2x^2 - y^2) \\
&= \frac{-2x^2 - y^2}{x^2}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(x(-x+2y)) \\
&= \frac{-x+2y}{x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-2x^2 - y^2}{x^2}\right) + \left(\frac{-x+2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{-2x^2 - y^2}{x^2} \,\mathrm{d}x$$

$$\phi = -2x + \frac{y^2}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2y}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+2y}{x}$. Therefore equation (4) becomes

$$\frac{-x+2y}{x} = \frac{2y}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -1$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (-1) \,\mathrm{d}y$$

$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2x + \frac{y^2}{x} - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2x + \frac{y^2}{x} - y$$

Summary

The solution(s) found are the following

$$-2x + \frac{y^2}{x} - y = c_1 \tag{1}$$

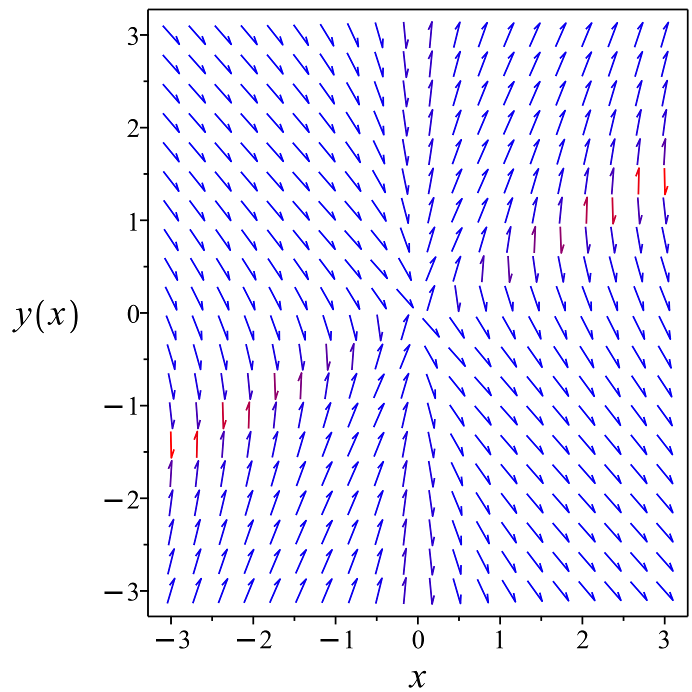


Figure 288: Slope field plot

Verification of solutions

$$-2x + \frac{y^2}{x} - y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 57

```
dsolve(diff(y(x),x)=(2*x^2+y(x)^2)/(2*x*y(x)-x^2),y(x), singsol=all)
```

$$y(x) = \frac{c_1 x - \sqrt{9c_1^2x^2 + 4c_1 x}}{2c_1}$$
$$y(x) = \frac{c_1 x + \sqrt{9c_1^2x^2 + 4c_1 x}}{2c_1}$$

✓ Solution by Mathematica

Time used: 2.748 (sec). Leaf size: 93

```
DSolve[y'[x]==(2*x^2+y[x]^2)/(2*x*y[x]-x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(x - \sqrt{x(9x - 4e^{c_1})}\right)$$
$$y(x) \to \frac{1}{2}\left(x + \sqrt{x(9x - 4e^{c_1})}\right)$$
$$y(x) \to \frac{1}{2}\left(x - 3\sqrt{x^2}\right)$$
$$y(x) \to \frac{1}{2}\left(3\sqrt{x^2} + x\right)$$

6.15 problem 15

Internal problem ID [11689]
Internal file name [`OUTPUT/11698_Wednesday_April_10_2024_04_54_58_PM_46377333/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 - 2xyy' = -x^2}$$

With initial conditions

$$[y(1) = 2]$$

6.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{x^2 + y^2}{2yx} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 2$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x^2+y^2}{2yx}\right)\\ &= \frac{1}{x} - \frac{x^2+y^2}{2y^2x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

6.15.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 - 2x^2u(x)\,(u'(x)\,x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{u^2-1}{2ux}\end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{u}}\,du = -\frac{1}{2x}\,dx$$

$$\int \frac{1}{\frac{u^2-1}{u}}\,du = \int -\frac{1}{2x}\,dx$$

$$\frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} = -\frac{\ln(x)}{2} + c_2$$

The above can be written as

$$\begin{aligned}\left(\frac{1}{2}\right)(\ln (u-1)+\ln (u+1)) &= -\frac{\ln (x)}{2}+2c_2\\ \ln (u-1)+\ln (u+1) &= (2)\left(-\frac{\ln (x)}{2}+2c_2\right)\\ &= -\ln (x)+4c_2\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(u-1)+\ln(u+1)} = \mathrm{e}^{-\ln(x)+2c_2}$$

Which simplifies to

$$\begin{aligned}u^2-1 &= \frac{2c_2}{x}\\ &= \frac{c_3}{x}\end{aligned}$$

The solution is

$$u(x)^2-1 = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

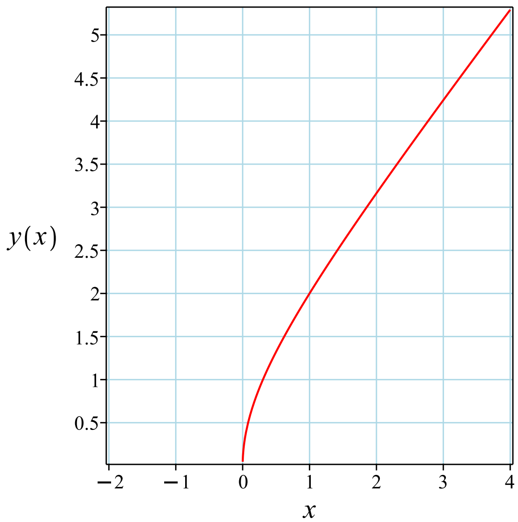
$$\begin{aligned}\frac{y^2}{x^2}-1 &= \frac{c_3}{x}\\ \frac{y^2}{x^2}-1 &= \frac{c_3}{x}\end{aligned}$$

Substituting initial conditions and solving for c_3 gives $c_3 = 3$. Hence the solution becomes Solving for y from the above gives

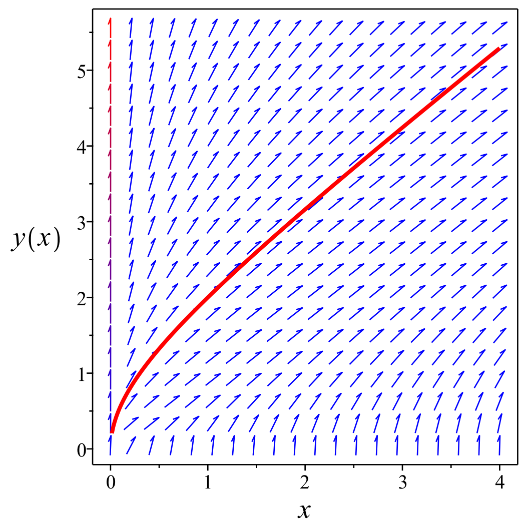
$$y = \sqrt{x\,(x+3)}$$

Summary
The solution(s) found are the following

$$y = \sqrt{x\,(x+3)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{x\,(x+3)}$$

Verified OK.

6.15.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + y^2}{2yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 228: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x}{y}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2 + y^2}{2yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2}{2x^2} \\ S_y &= \frac{y}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \tag{4}$$

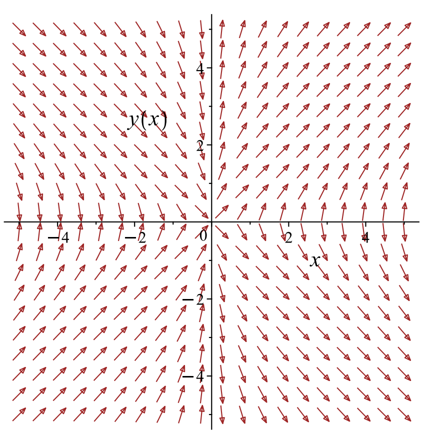
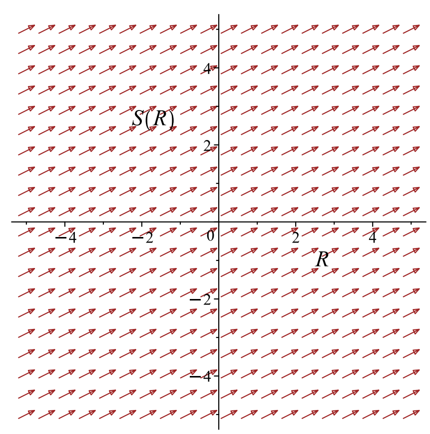
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x} = \frac{x}{2} + c_1$$

Which simplifies to

$$\frac{y^2}{2x} = \frac{x}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y^2}{2yx}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{1}{2} + c_1$$

$$c_1 = \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{y^2}{2x} = \frac{x}{2} + \frac{3}{2}$$

The above simplifies to

$$-x^2 + y^2 - 3x = 0$$

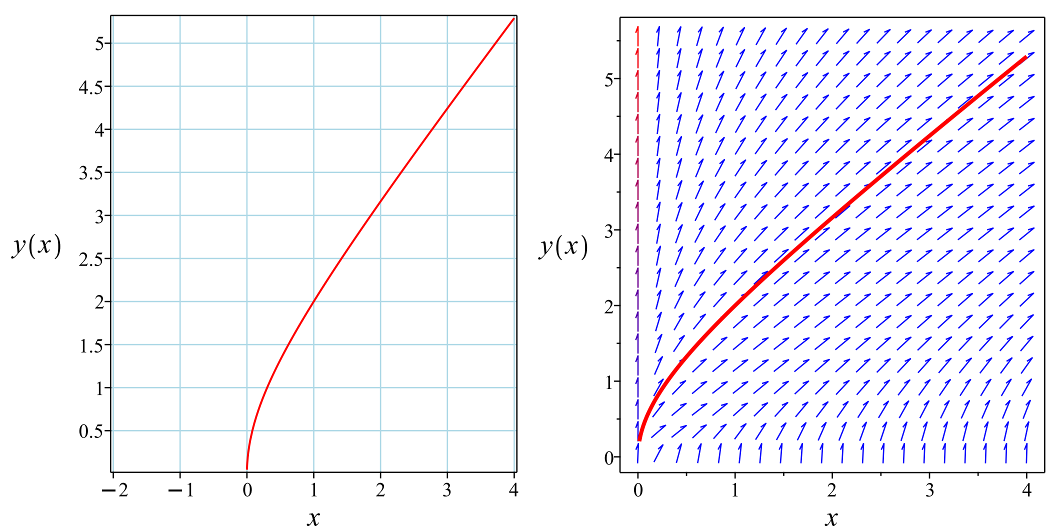
Solving for y from the above gives

$$y = \sqrt{x(x+3)}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x(x+3)} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \sqrt{x(x+3)}$$

Verified OK.

6.15.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^2+y^2}{2yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y + \frac{x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{2x} \\ f_1(x) &= \frac{x}{2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{y^2}{2x} + \frac{x}{2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}\frac{w'(x)}{2} &= \frac{w(x)}{2x} + \frac{x}{2} \\ w' &= \frac{w}{x} + x\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{1}{x} \\ q(x) &= x\end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = x$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(x) \\ \mathrm{d}\left(\frac{w}{x}\right) &= \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int \mathrm{d}x \\ \frac{w}{x} &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = c_1 x + x^2$$

which simplifies to

$$w(x) = x(x + c_1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x(x + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$4 = 1 + c_1$$

$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

$$y^2 = x(x + 3)$$

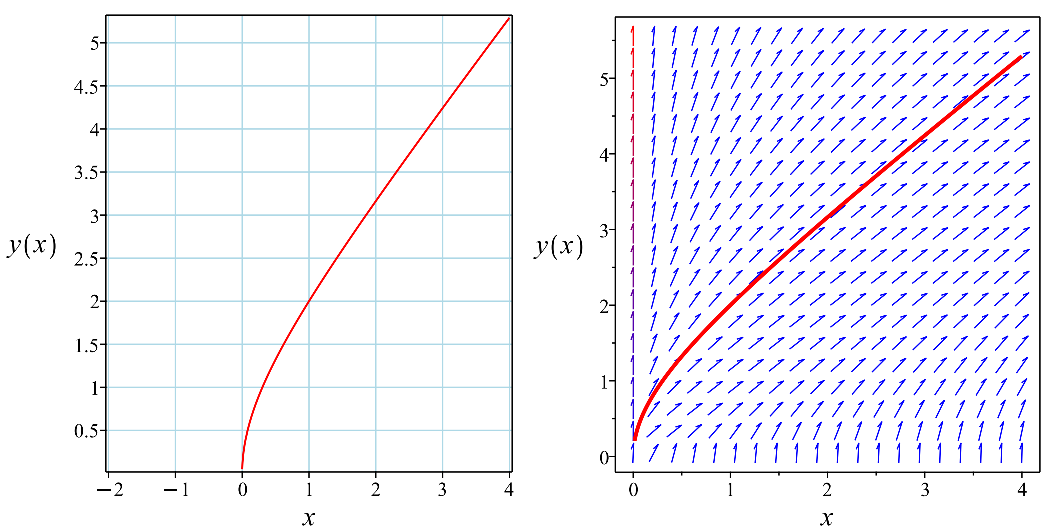
Solving for y from the above gives

$$y = \sqrt{x (x + 3)}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x (x + 3)} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = \sqrt{x(x+3)}
$$

Verified OK.

6.15.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial\phi}{\partial x} &= M \\
\frac{\partial\phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(-2xy)\,\mathrm{d}y &= \left(-x^2 - y^2\right)\mathrm{d}x \\
\left(x^2 + y^2\right)\mathrm{d}x + (-2xy)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= x^2 + y^2 \\
N(x,y) &= -2xy
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2 + y^2) \\
&= 2y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2xy) \\
&= -2y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{2xy}((2y) - (-2y)) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}\left(x^2+y^2\right) \\
&= \frac{x^2+y^2}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(-2xy) \\
&= -\frac{2y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^2+y^2}{x^2}\right) + \left(-\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{x^2+y^2}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x - \frac{y^2}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{2y}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{x}$. Therefore equation (4) becomes

$$-\frac{2y}{x} = -\frac{2y}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x - \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x - \frac{y^2}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = c_1$$

$$c_1 = -3$$

Substituting c_1 found above in the general solution gives

$$x - \frac{y^2}{x} = -3$$

The above simplifies to

$$x^2 - y^2 + 3x = 0$$

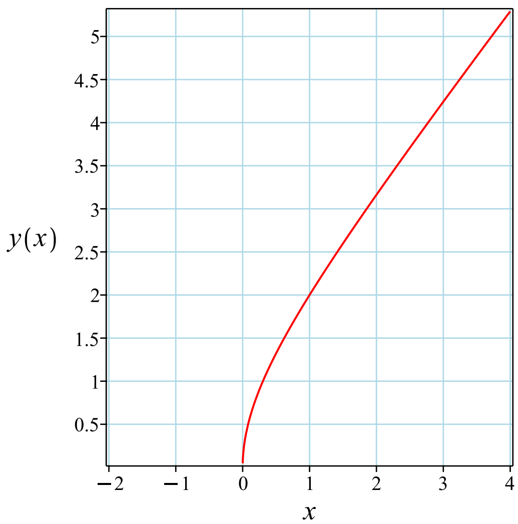
Solving for y from the above gives

$$y = \sqrt{x(x+3)}$$

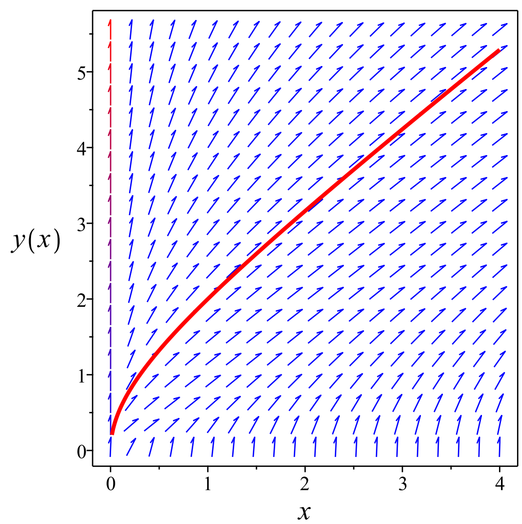
Summary

The solution(s) found are the following

$$y = \sqrt{x(x+3)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{x(x+3)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 13

```
dsolve([(x^2+y(x)^2)-2*x*y(x)*diff(y(x),x)=0,y(1) = 2],y(x), singsol=all)
```

$$y(x) = \sqrt{(x+3)\,x}$$

✓ Solution by Mathematica

Time used: 0.175 (sec). Leaf size: 18

```
DSolve[{(x^2+y[x]^2)-2*x*y[x]*y'[x]==0,{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x}\sqrt{x+3}$$

6.16 problem 16

Internal problem ID [11690]
Internal file name [`OUTPUT/11699_Wednesday_April_10_2024_04_55_03_PM_17615973/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2y^2 + \left(-x^2 + 1\right) y y' = -8}$$

With initial conditions

$$[y(3) = 0]$$

6.16.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{2y^2 + 8}{(x^2 - 1)\, y} \end{aligned}$$

$f(x, y)$ is not defined at $y = 0$ therefore existence and uniqueness theorem do not apply.

6.16.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{2y^2+8}{(x^2-1)\,y}
\end{aligned}$$

Where $f(x)=\frac{2}{x^2-1}$ and $g(y)=\frac{y^2+4}{y}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{y^2+4}{y}}\,dy &= \frac{2}{x^2-1}\,dx \\
\int \frac{1}{\frac{y^2+4}{y}}\,dy &= \int \frac{2}{x^2-1}\,dx \\
\frac{\ln(y^2+4)}{2} &= -2\ \mathrm{arctanh}(x)+c_1
\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{y^2+4} = \mathrm{e}^{-2\ \mathrm{arctanh}(x)+c_1}$$

Which simplifies to

$$\sqrt{y^2+4} = \frac{c_2(-x^2+1)}{(1+x)^2}$$

The solution is

$$\sqrt{y^2+4} = c_2\mathrm{e}^{-2\ \mathrm{arctanh}(x)+c_1}$$

Initial conditions are used to solve for c_1. Substituting $x=3$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\frac{\mathrm{e}^{c_1}c_2}{2}$$

$$c_1 = \ln\left(-\frac{4}{c_2}\right)$$

Substituting c_1 found above in the general solution gives

$$\sqrt{y^2+4} = \frac{4x-4}{1+x}$$

The above simplifies to

$$\sqrt{y^2+4}\,x+\sqrt{y^2+4}-4x+4=0$$

Summary

The solution(s) found are the following

$$\sqrt{y^2+4}\,x+\sqrt{y^2+4}-4x+4=0 \tag{1}$$

Verification of solutions

$$\sqrt{y^2+4}\,x+\sqrt{y^2+4}-4x+4=0$$

Verified OK.

6.16.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{2y^2+8}{(x^2-1)\,y} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 230: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{x^2}{2} - \frac{1}{2} \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x^2}{2} - \frac{1}{2}} dx \end{aligned}$$

Which results in

$$S = -2 \operatorname{arctanh}(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2y^2 + 8}{(x^2 - 1)\, y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{2}{x^2 - 1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2 + 4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2 + 4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R^2+4)}{2} + c_1 \tag{4}$$

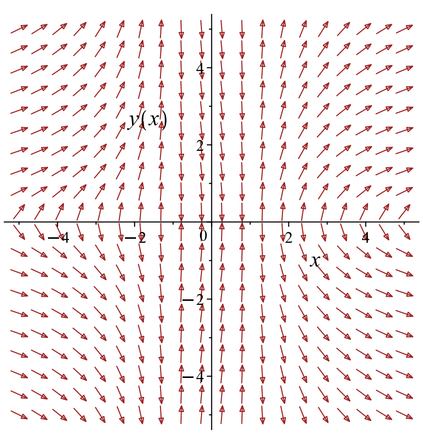
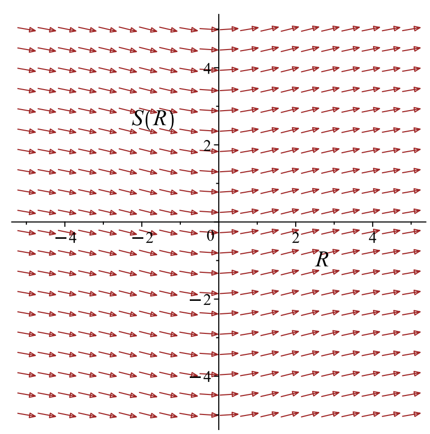
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-2\ \operatorname{arctanh}(x) = \frac{\ln(y^2+4)}{2} + c_1$$

Which simplifies to

$$-2\ \operatorname{arctanh}(x) = \frac{\ln(y^2+4)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y^2+8}{(x^2-1)y}$ 	$R = y$ $S = -2\ \operatorname{arctanh}(x)$	$\frac{dS}{dR} = \frac{R}{R^2+4}$ 

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-2\ \operatorname{arccoth}(3) + i\pi = \ln(2) + c_1$$

$$c_1 = i\pi - \ln(2) - 2\,\text{arccoth}(3)$$

Substituting c_1 found above in the general solution gives

$$-2\,\text{arctanh}(x) = \frac{\ln(y^2+4)}{2} + i\pi - \ln(2) - 2\,\text{arccoth}(3)$$

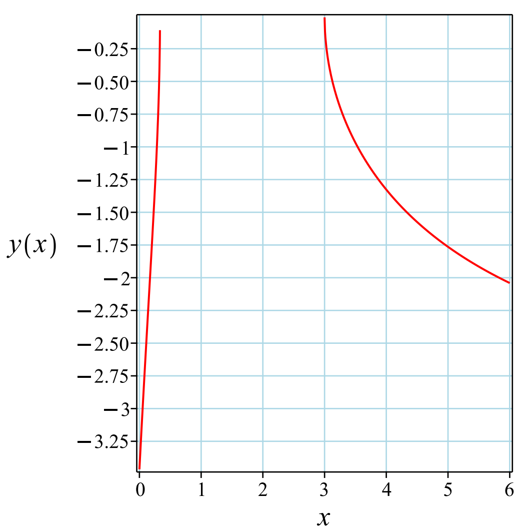
Solving for y from the above gives

$$y = 2\sqrt{-1+\mathrm{e}^{-4\,\text{arctanh}(x)+4\,\text{arccoth}(3)}}$$
$$y = -2\sqrt{-1+\mathrm{e}^{-4\,\text{arctanh}(x)+4\,\text{arccoth}(3)}}$$

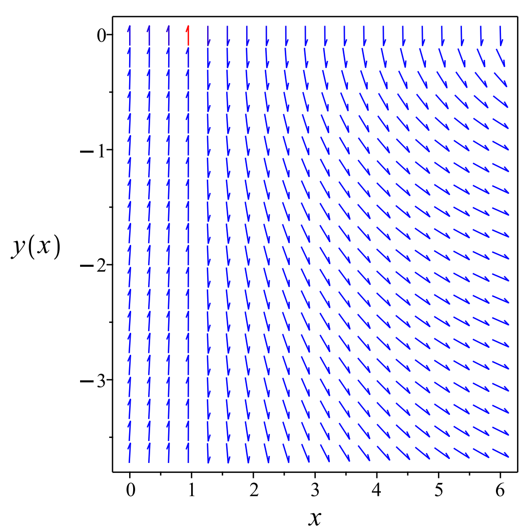
Summary

The solution(s) found are the following

$$y = 2\sqrt{-1+\mathrm{e}^{-4\,\text{arctanh}(x)+4\,\text{arccoth}(3)}} \tag{1}$$
$$y = -2\sqrt{-1+\mathrm{e}^{-4\,\text{arctanh}(x)+4\,\text{arccoth}(3)}} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\sqrt{-1+\mathrm{e}^{-4\,\text{arctanh}(x)+4\,\text{arccoth}(3)}}$$

Verified OK. `{positive}`

$$y = -2\sqrt{-1+\mathrm{e}^{-4\,\text{arctanh}(x)+4\,\text{arccoth}(3)}}$$

Verified OK. `{positive}`

6.16.4 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x,y)\\
&= \frac{2y^2+8}{(x^2-1)\,y}
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = \frac{2}{x^2-1}y + \frac{8}{x^2-1}\frac{1}{y} \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(x)y + f_1(x)y^n \tag{2}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{2}{x^2-1}\\
f_1(x) &= \frac{8}{x^2-1}\\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = \frac{2y^2}{x^2-1} + \frac{8}{x^2-1} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n}\\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{2w(x)}{x^2-1} + \frac{8}{x^2-1} \\
w' &= \frac{4w}{x^2-1} + \frac{16}{x^2-1}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{4}{x^2-1} \\
q(x) &= \frac{16}{x^2-1}
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{4w(x)}{x^2-1} = \frac{16}{x^2-1}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{4}{x^2-1}dx} \\
&= \frac{(1+x)^4}{(-x^2+1)^2}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{(1+x)^2}{(x-1)^2}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\frac{16}{x^2-1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(1+x)^2 w}{(x-1)^2}\right) &= \left(\frac{(1+x)^2}{(x-1)^2}\right)\left(\frac{16}{x^2-1}\right) \\
\mathrm{d}\left(\frac{(1+x)^2 w}{(x-1)^2}\right) &= \left(\frac{16+16x}{(x-1)^3}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$\frac{(1+x)^2 w}{(x-1)^2} = \int \frac{16+16x}{(x-1)^3}\, dx$$
$$\frac{(1+x)^2 w}{(x-1)^2} = -\frac{16}{x-1} - \frac{16}{(x-1)^2} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{(1+x)^2}{(x-1)^2}$ results in

$$w(x) = \frac{(x-1)^2 \left(-\frac{16}{x-1} - \frac{16}{(x-1)^2}\right)}{(1+x)^2} + \frac{c_1 (x-1)^2}{(1+x)^2}$$

which simplifies to

$$w(x) = \frac{c_1 x^2 + (-2c_1 - 16)\, x + c_1}{(1+x)^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{c_1 x^2 + (-2c_1 - 16)\, x + c_1}{(1+x)^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{c_1}{4} - 3$$

$$c_1 = 12$$

Substituting c_1 found above in the general solution gives

$$y^2 = \frac{12x^2 - 40x + 12}{(1+x)^2}$$

The above simplifies to

$$x^2y^2 + 2x\, y^2 - 12x^2 + y^2 + 40x - 12 = 0$$

Summary

The solution(s) found are the following

$$x^2y^2 + 2y^2x - 12x^2 + y^2 + 40x - 12 = 0 \tag{1}$$

Verification of solutions

$$x^2y^2 + 2y^2x - 12x^2 + y^2 + 40x - 12 = 0$$

Verified OK.

6.16.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{2y^2+8}\right)\mathrm{d}y&=\left(\frac{1}{x^2-1}\right)\mathrm{d}x\\ \left(-\frac{1}{x^2-1}\right)\mathrm{d}x+\left(\frac{y}{2y^2+8}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x^2-1} \\
N(x,y) &= \frac{y}{2y^2+8}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2-1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{2y^2+8}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{x^2-1} \,\mathrm{d}x \\
\phi &= \operatorname{arctanh}(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{2y^2+8}$. Therefore equation (4) becomes

$$\frac{y}{2y^2+8} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{2y^2+8}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y}{2y^2+8}\right) \mathrm{d}y$$

$$f(y) = \frac{\ln(y^2+4)}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \operatorname{arctanh}(x) + \frac{\ln(y^2+4)}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \operatorname{arctanh}(x) + \frac{\ln(y^2+4)}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\operatorname{arccoth}(3) - \frac{i\pi}{2} + \frac{\ln(2)}{2} = c_1$$

$$c_1 = \operatorname{arccoth}(3) - \frac{i\pi}{2} + \frac{\ln(2)}{2}$$

Substituting c_1 found above in the general solution gives

$$\operatorname{arctanh}(x) + \frac{\ln(y^2+4)}{4} = \operatorname{arccoth}(3) - \frac{i\pi}{2} + \frac{\ln(2)}{2}$$

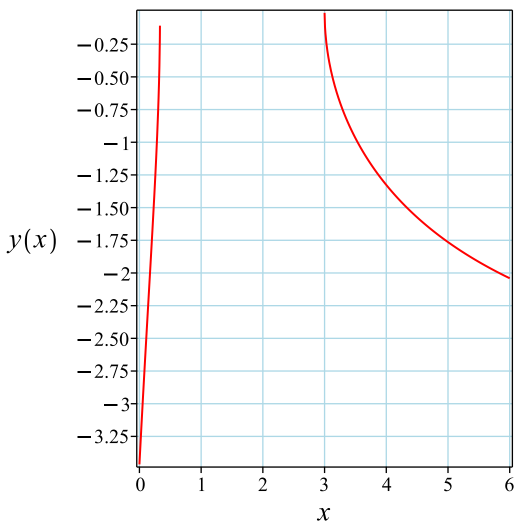
Solving for y from the above gives

$$y = 2\sqrt{-1+\mathrm{e}^{-4 \operatorname{arctanh}(x)+4 \operatorname{arccoth}(3)}}$$
$$y = -2\sqrt{-1+\mathrm{e}^{-4 \operatorname{arctanh}(x)+4 \operatorname{arccoth}(3)}}$$

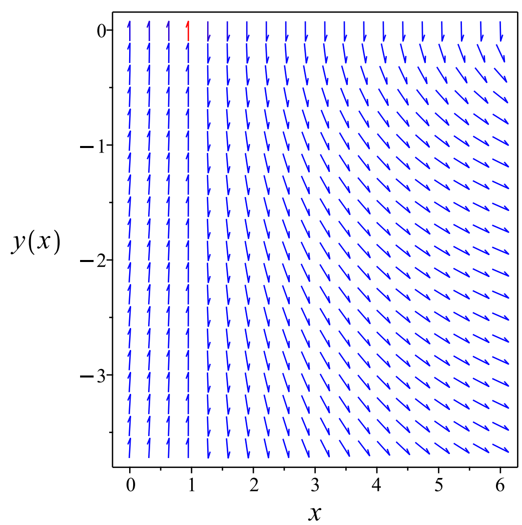
Summary

The solution(s) found are the following

$$y = 2\sqrt{-1+\mathrm{e}^{-4 \operatorname{arctanh}(x)+4 \operatorname{arccoth}(3)}} \tag{1}$$
$$y = -2\sqrt{-1+\mathrm{e}^{-4 \operatorname{arctanh}(x)+4 \operatorname{arccoth}(3)}} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\sqrt{-1+\mathrm{e}^{-4 \operatorname{arctanh}(x)+4 \operatorname{arccoth}(3)}}$$

Verified OK. `{positive}`

$$y = -2\sqrt{-1+\mathrm{e}^{-4 \operatorname{arctanh}(x)+4 \operatorname{arccoth}(3)}}$$

Verified OK. `{positive}`

6.16.6 Maple step by step solution

Let's solve

$\left[2 y^{2}+\left(-x^{2}+1\right) y y^{\prime}=-8, y(3)=0\right]$

- Highest derivative means the order of the ODE is 1

  $y^{\prime}$

- Separate variables

  $\frac{y^{\prime} y}{-2 y^{2}-8}=\frac{1}{-x^{2}+1}$

- Integrate both sides with respect to x

  $\int \frac{y^{\prime} y}{-2 y^{2}-8} d x=\int \frac{1}{-x^{2}+1} d x+c_{1}$

- Evaluate integral

  $-\frac{\ln \left(y^{2}+4\right)}{4}=\operatorname{arctanh}(x)+c_{1}$

- Solve for y

  $\left\{y=\sqrt{-4+\mathrm{e}^{-4 \operatorname{arctanh}(x)-4 c_{1}}}, y=-\sqrt{-4+\mathrm{e}^{-4 \operatorname{arctanh}(x)-4 c_{1}}}\right\}$

- Use initial condition $y(3)=0$

  $0=\sqrt{-4+\mathrm{e}^{-4 \operatorname{arctanh}\left(\frac{1}{3}\right)-4 c_{1}}}$

- Solve for c_{1}

  $c_{1}=-\operatorname{arctanh}\left(\frac{1}{3}\right)-\frac{\ln (2)}{2}$

- Substitute $c_{1}=-\operatorname{arctanh}\left(\frac{1}{3}\right)-\frac{\ln (2)}{2}$ into general solution and simplify

  $y=2 \sqrt{-1+\mathrm{e}^{-4 \operatorname{arctanh}(x)+4 \operatorname{arctanh}\left(\frac{1}{3}\right)}}$

- Use initial condition $y(3)=0$

  $0=-\sqrt{-4+\mathrm{e}^{-4 \operatorname{arctanh}\left(\frac{1}{3}\right)-4 c_{1}}}$

- Solve for c_{1}

  $c_{1}=-\operatorname{arctanh}\left(\frac{1}{3}\right)-\frac{\ln (2)}{2}$

- Substitute $c_{1}=-\operatorname{arctanh}\left(\frac{1}{3}\right)-\frac{\ln (2)}{2}$ into general solution and simplify

  $y=-2 \sqrt{-1+\mathrm{e}^{-4 \operatorname{arctanh}(x)+4 \operatorname{arctanh}\left(\frac{1}{3}\right)}}$

- Solutions to the IVP

  $\left\{y=-2 \sqrt{-1+\mathrm{e}^{-4 \operatorname{arctanh}(x)+4 \operatorname{arctanh}\left(\frac{1}{3}\right)}}, y=2 \sqrt{-1+\mathrm{e}^{-4 \operatorname{arctanh}(x)+4 \operatorname{arctanh}\left(\frac{1}{3}\right)}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 45

```
dsolve([2*(y(x)^2+4)+(1-x^2)*y(x)*diff(y(x),x)=0,y(3) = 0],y(x), singsol=all)
```

$$y(x) = -\frac{2\sqrt{3x^2-10x+3}}{1+x}$$
$$y(x) = \frac{2\sqrt{3x^2-10x+3}}{1+x}$$

✓ Solution by Mathematica

Time used: 0.886 (sec). Leaf size: 51

```
DSolve[{2*(y[x]^2+4)+(1-x^2)*y[x]*y'[x]==0,{y[3]==0}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to -\frac{2\sqrt{3x^2-10x+3}}{x+1}$$
$$y(x) \to \frac{2\sqrt{3x^2-10x+3}}{x+1}$$

6.17 problem 17

Internal problem ID [11691]
Internal file name [`OUTPUT/11700_Wednesday_April_10_2024_04_55_07_PM_1436126/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_exact, _Bernoulli]
```

$$\boxed{e^{2x}y^2 + e^{2x}yy' = 2x}$$

With initial conditions

$$[y(0) = 2]$$

6.17.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{(e^{2x}y^2 - 2x)\,e^{-2x}}{y} \end{aligned}$$

The x domain of $f(x,y)$ when $y = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{(e^{2x}y^2 - 2x)\,e^{-2x}}{y}\right) \\ &= -2 + \frac{(e^{2x}y^2 - 2x)\,e^{-2x}}{y^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

6.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{(e^{2x}y^2 - 2x)\,e^{-2x}}{y} \\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 233: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{e^{-2x}}{y}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{e^{-2x}}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{e^{2x} y^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{(e^{2x} y^2 - 2x) e^{-2x}}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{2x} y^2 \\ S_y &= y e^{2x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

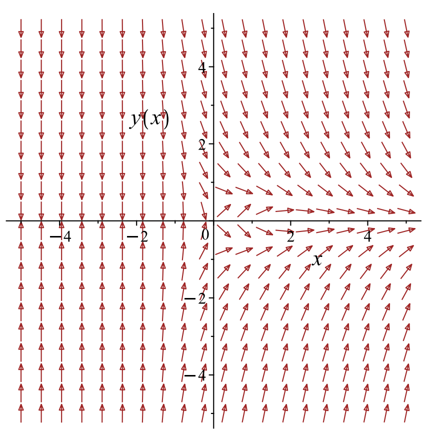
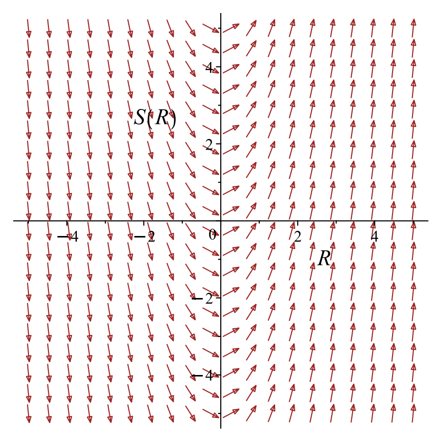
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{e^{2x} y^2}{2} = x^2 + c_1$$

Which simplifies to

$$\frac{e^{2x} y^2}{2} = x^2 + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{(e^{2x} y^2 - 2x) e^{-2x}}{y}$ 	$R = x$ $S = \frac{e^{2x} y^2}{2}$	$\frac{dS}{dR} = 2R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$\frac{e^{2x}y^2}{2} = x^2 + 2$$

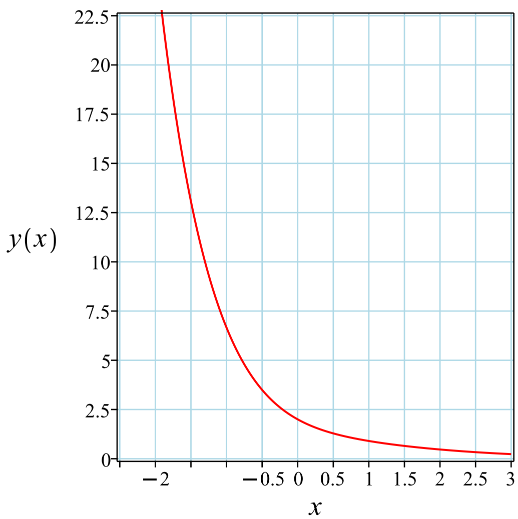
Solving for y from the above gives

$$y = e^{-2x}\sqrt{2}\,\sqrt{e^{2x}(x^2 + 2)}$$

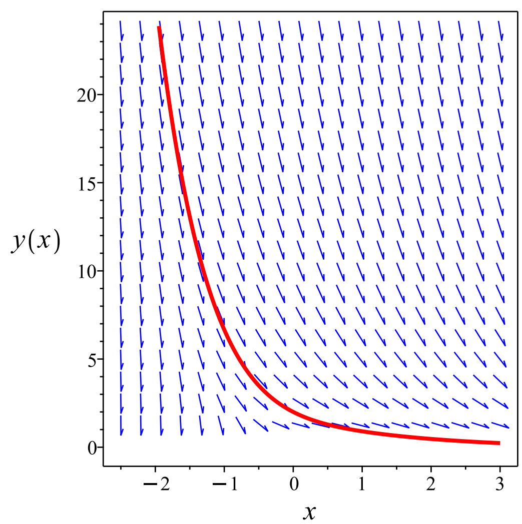
Summary

The solution(s) found are the following

$$y = e^{-2x}\sqrt{2}\,\sqrt{e^{2x}(x^2 + 2)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-2x}\sqrt{2}\,\sqrt{e^{2x}(x^2 + 2)}$$

Verified OK.

6.17.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= -\frac{(e^{2x}y^2-2x)\,e^{-2x}}{y}
\end{aligned}$$

This is a Bernoulli ODE.

$$y' = -y + 2x\,e^{-2x}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= -1\\
f_1(x) &= 2x\,e^{-2x}\\
n &= -1
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -y^2 + 2x\,e^{-2x} \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n}\\
&= y^2
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
\frac{w'(x)}{2} &= -w(x) + 2x\,\mathrm{e}^{-2x} \\
w' &= -2w + 4x\,\mathrm{e}^{-2x} \qquad (7)
\end{aligned}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= 2 \\
q(x) &= 4x\,\mathrm{e}^{-2x}
\end{aligned}$$

Hence the ode is

$$w'(x) + 2w(x) = 4x\,\mathrm{e}^{-2x}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int 2dx} \\
&= \mathrm{e}^{2x}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(4x\,\mathrm{e}^{-2x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x} w\right) &= \left(\mathrm{e}^{2x}\right)\left(4x\,\mathrm{e}^{-2x}\right) \\
\mathrm{d}\left(\mathrm{e}^{2x} w\right) &= (4x)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\mathrm{e}^{2x} w &= \int 4x\,\mathrm{d}x \\
\mathrm{e}^{2x} w &= 2x^2 + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2x}$ results in

$$w(x) = 2x^2 \mathrm{e}^{-2x} + c_1 \mathrm{e}^{-2x}$$

which simplifies to

$$w(x) = \mathrm{e}^{-2x}\left(2x^2 + c_1\right)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = e^{-2x}\left(2x^2 + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$4 = c_1$$

$$c_1 = 4$$

Substituting c_1 found above in the general solution gives

$$y^2 = 2\,e^{-2x}\left(x^2 + 2\right)$$

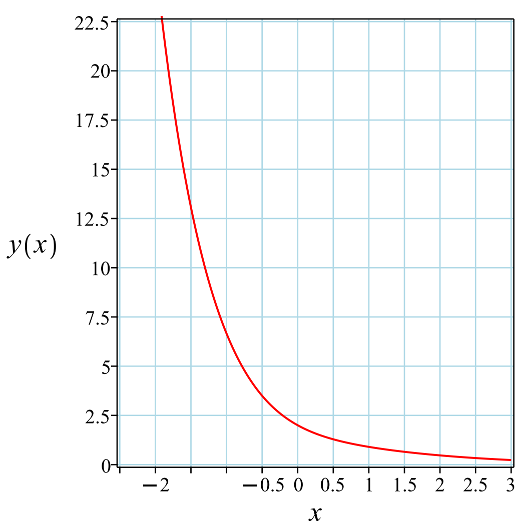
Solving for y from the above gives

$$y = \sqrt{2}\,\sqrt{e^{-2x}\left(x^2 + 2\right)}$$

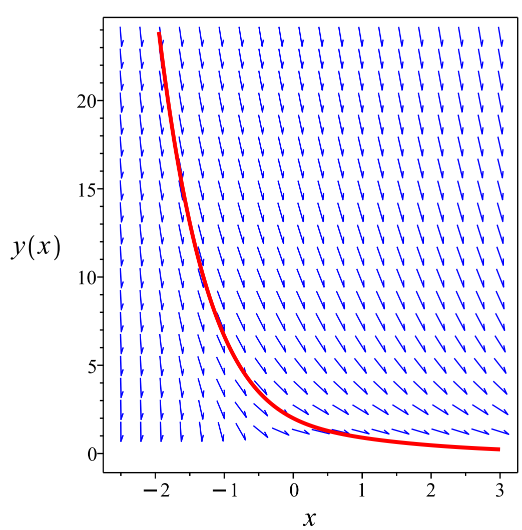
Summary

The solution(s) found are the following

$$y = \sqrt{2}\,\sqrt{e^{-2x}\left(x^2 + 2\right)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{2}\,\sqrt{e^{-2x}\left(x^2 + 2\right)}$$

Verified OK.

6.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(y\,\mathrm{e}^{2x}\right)\mathrm{d}y=\left(-\mathrm{e}^{2x}y^2+2x\right)\mathrm{d}x$$
$$\left(\mathrm{e}^{2x}y^2-2x\right)\mathrm{d}x+\left(y\,\mathrm{e}^{2x}\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y)=\mathrm{e}^{2x}y^2-2x$$
$$N(x,y)=y\,\mathrm{e}^{2x}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(e^{2x}y^2 - 2x\right)\\ &= 2y\,e^{2x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(y\,e^{2x}\right)\\ &= 2y\,e^{2x}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,dx &= \int M\,dx\\ \int \frac{\partial \phi}{\partial x}\,dx &= \int e^{2x}y^2 - 2x\,dx\\ \phi &= \frac{e^{2x}y^2}{2} - x^2 + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = y\,e^{2x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y\,e^{2x}$. Therefore equation (4) becomes

$$y\,e^{2x} = y\,e^{2x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{e^{2x} y^2}{2} - x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{e^{2x} y^2}{2} - x^2$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$\frac{e^{2x} y^2}{2} - x^2 = 2$$

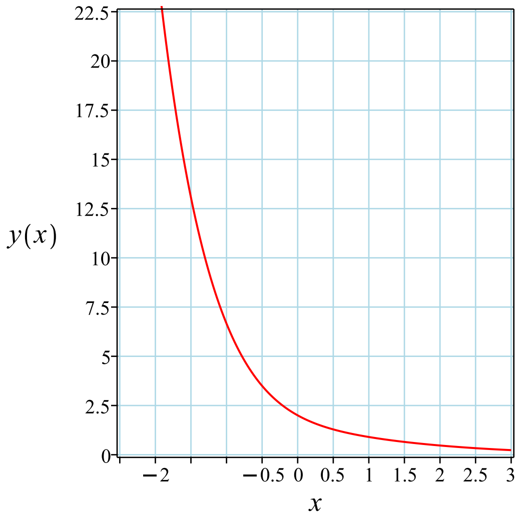
Solving for y from the above gives

$$y = e^{-2x} \sqrt{2} \sqrt{e^{2x} (x^2 + 2)}$$

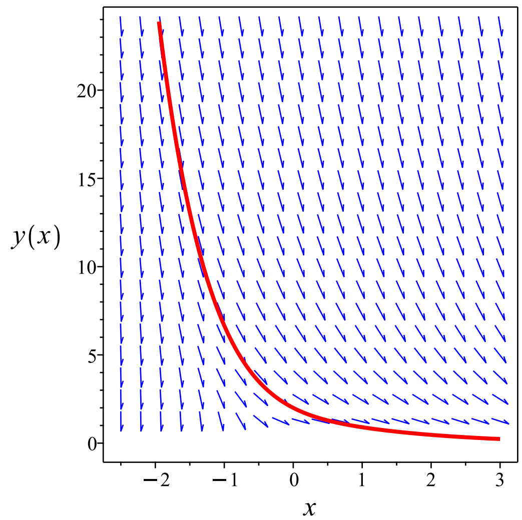
Summary

The solution(s) found are the following

$$y = e^{-2x} \sqrt{2} \sqrt{e^{2x} (x^2 + 2)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-2x}\sqrt{2}\,\sqrt{e^{2x}\left(x^{2}+2\right)}$$

Verified OK.

6.17.5 Maple step by step solution

Let's solve

$[e^{2x}y^{2} + e^{2x}yy' = 2x, y(0) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$
  - ◦ Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$
  - ◦ Evaluate derivatives

    $2y\,e^{2x} = 2y\,e^{2x}$
  - ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $$\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $$F(x,y)=\int (\mathrm{e}^{2x}y^2-2x)\,dx+f_1(y)$$

- Evaluate integral

  $$F(x,y)=\frac{\mathrm{e}^{2x}y^2}{2}-x^2+f_1(y)$$

- Take derivative of $F(x,y)$ with respect to y

  $$N(x,y)=\frac{\partial}{\partial y}F(x,y)$$

- Compute derivative

  $$y\,\mathrm{e}^{2x}=y\,\mathrm{e}^{2x}+\frac{d}{dy}f_1(y)$$

- Isolate for $\frac{d}{dy}f_1(y)$

  $$\frac{d}{dy}f_1(y)=0$$

- Solve for $f_1(y)$

  $$f_1(y)=0$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $$F(x,y)=\frac{\mathrm{e}^{2x}y^2}{2}-x^2$$

- Substitute $F(x,y)$ into the solution of the ODE

  $$\frac{\mathrm{e}^{2x}y^2}{2}-x^2=c_1$$

- Solve for y

  $$\left\{y=\frac{\sqrt{2}\,\sqrt{\mathrm{e}^{2x}(x^2+c_1)}}{\mathrm{e}^{2x}}, y=-\frac{\sqrt{2}\,\sqrt{\mathrm{e}^{2x}(x^2+c_1)}}{\mathrm{e}^{2x}}\right\}$$

- Use initial condition $y(0)=2$

  $$2=\sqrt{c_1}\,\sqrt{2}$$

- Solve for c_1

  $$c_1=2$$

- Substitute $c_1=2$ into general solution and simplify

  $$y=\mathrm{e}^{-2x}\sqrt{2}\,\sqrt{(\mathrm{e}^{x})^2\,(x^2+2)}$$

- Use initial condition $y(0)=2$

  $$2=-\sqrt{c_1}\,\sqrt{2}$$

- Solution does not satisfy initial condition
- Solution to the IVP

$$y = e^{-2x}\sqrt{2}\,\sqrt{\left(e^{x}\right)^{2}\left(x^{2}+2\right)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 24

```
dsolve([(exp(2*x)*y(x)^2-2*x)+(exp(2*x)*y(x))*diff(y(x),x)=0,y(0) = 2],y(x), singsol=all)
```

$$y(x) = e^{-2x}\sqrt{2}\,\sqrt{e^{2x}\left(x^{2}+2\right)}$$

✓ Solution by Mathematica

Time used: 0.337 (sec). Leaf size: 25

```
DSolve[{(Exp[2*x]*y[x]^2-2*x)+(Exp[2*x]*y[x])*y'[x]==0,{y[0]==2}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \sqrt{2}e^{-x}\sqrt{x^{2}+2}$$

6.18 problem 18

Internal problem ID [11692]
Internal file name [`OUTPUT/11701_Wednesday_April_10_2024_04_55_08_PM_24887235/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{2y^2x+(2x^2y+6y^2)\,y'=-3x^2}$$

With initial conditions

$$[y(1)=2]$$

6.18.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned}y'&=f(x,y)\\&=-\frac{x(2y^2+3x)}{2y\,(x^2+3y)}\end{aligned}$$

The x domain of $f(x,y)$ when $y=2$ is

$$\{-\infty<x<\infty\}$$

And the point $x_0=1$ is inside this domain. The y domain of $f(x,y)$ when $x=1$ is

$$\left\{-\infty\leq y<0,0<y<-\frac{1}{3},-\frac{1}{3}<y\leq\infty\right\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x(2y^2+3x)}{2y\,(x^2+3y)}\right) \\ &= -\frac{2x}{x^2+3y} + \frac{x(2y^2+3x)}{2y^2\,(x^2+3y)} + \frac{3x(2y^2+3x)}{2y\,(x^2+3y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\left\{-\infty \leq y < 0, 0 < y < -\frac{1}{3}, -\frac{1}{3} < y \leq \infty\right\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

6.18.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x^2y + 6y^2)\,\mathrm{d}y &= (-2x\,y^2 - 3x^2)\,\mathrm{d}x\\ (2x\,y^2 + 3x^2)\,\mathrm{d}x + (2x^2y + 6y^2)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 2x\,y^2 + 3x^2\\ N(x, y) &= 2x^2y + 6y^2 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x\,y^2 + 3x^2)\\ &= 4xy\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x^2y + 6y^2)\\ &= 4xy\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int 2x\, y^2 + 3x^2 \mathrm{d}x \\
\phi &= x^2 \left(y^2 + x\right) + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x^2 y + f'(y) \qquad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x^2 y + 6y^2$. Therefore equation (4) becomes

$$2x^2 y + 6y^2 = 2x^2 y + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 6y^2$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(6y^2\right) \mathrm{d}y \\
f(y) &= 2y^3 + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2 \left(y^2 + x\right) + 2y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2 \left(y^2 + x\right) + 2y^3$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$21 = c_1$$

$$c_1 = 21$$

Substituting c_1 found above in the general solution gives

$$x^2(y^2 + x) + 2y^3 = 21$$

Summary
The solution(s) found are the following

$$x^2y^2 + x^3 + 2y^3 = 21 \tag{1}$$

Verification of solutions

$$x^2y^2 + x^3 + 2y^3 = 21$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 87

```
dsolve([(3*x^2+2*x*y(x)^2)+(2*x^2*y(x)+6*y(x)^2)*diff(y(x),x)=0,y(1) = 2],y(x), singsol=all)
```

$$y(x) = \frac{\left(1134-54x^3-x^6+6\sqrt{3x^9+18x^6-3402x^3+35721}\right)^{\frac{1}{3}}}{6} + \frac{x^4}{6\left(1134-54x^3-x^6+6\sqrt{3x^9+18x^6-3402x^3+35721}\right)^{\frac{1}{3}}} - \frac{x^2}{6}$$

✓ Solution by Mathematica

Time used: 4.797 (sec). Leaf size: 103

```
DSolve[{(3*x^2+2*x*y[x]^2)+(2*x^2*y[x]+6*y[x]^2)*y'[x]==0,{y[1]==2}},y[x],x,IncludeSingularS
```

$$y(x) \to \frac{1}{6}\left(-x^2+\sqrt[3]{-x^6-54x^3+6\sqrt{3}\sqrt{x^9+6x^6-1134x^3+11907}+1134} + \frac{x^4}{\sqrt[3]{-x^6-54x^3+6\sqrt{3}\sqrt{x^9+6x^6-1134x^3+11907}+1134}}\right)$$

6.19 problem 19

Internal problem ID [11693]
Internal file name [`OUTPUT/11702_Wednesday_April_10_2024_04_55_10_PM_49928285/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{4xyy' - y^2 = 1}$$

With initial conditions

$$[y(2) = 1]$$

6.19.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y^2+1}{4yx} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x,y)$ when $x = 2$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y^2+1}{4yx}\right)\\ &= \frac{1}{2x} - \frac{y^2+1}{4y^2x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 2$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

6.19.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{y^2+1}{4yx}\end{aligned}$$

Where $f(x) = \frac{1}{4x}$ and $g(y) = \frac{y^2+1}{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{y^2+1}{y}}\,dy &= \frac{1}{4x}\,dx\\ \int \frac{1}{\frac{y^2+1}{y}}\,dy &= \int \frac{1}{4x}\,dx\\ \frac{\ln(y^2+1)}{2} &= \frac{\ln(x)}{4} + c_1\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{y^2+1} = e^{\frac{\ln(x)}{4}+c_1}$$

Which simplifies to

$$\sqrt{y^2+1} = c_2 x^{\frac{1}{4}}$$

Which can be simplified to become

$$\sqrt{1+y^2} = c_2 x^{\frac{1}{4}} e^{c_1}$$

The solution is

$$\sqrt{1+y^2} = c_2 x^{\frac{1}{4}} e^{c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\sqrt{2} = 2^{\frac{1}{4}} e^{c_1} c_2$$

$$c_1 = \frac{\ln\left(\frac{2}{c_2^4}\right)}{4}$$

Substituting c_1 found above in the general solution gives

$$\sqrt{y^2+1} = c_2 x^{\frac{1}{4}} \sqrt{\sqrt{2}\,\sqrt{\frac{1}{c_2^4}}}$$

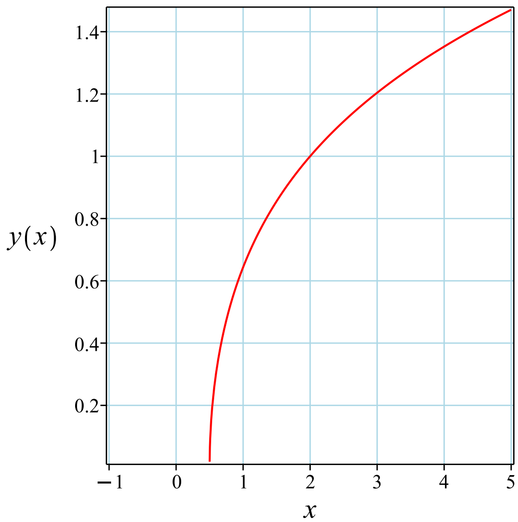
Solving for y from the above gives

$$y = \sqrt{\sqrt{x}\,\sqrt{2}-1}$$

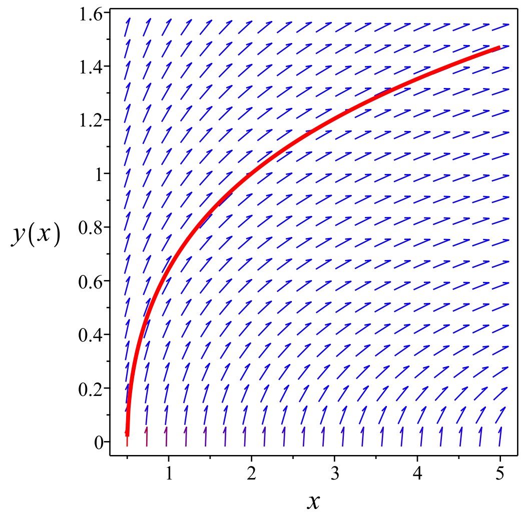
Summary

The solution(s) found are the following

$$y = \sqrt{\sqrt{x}\,\sqrt{2}-1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \sqrt{\sqrt{x}\,\sqrt{2} - 1}$$

Verified OK. `{positive}`

6.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2 + 1}{4yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 236: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 4x \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{4x} dx \end{aligned}$$

Which results in

$$S = \frac{\ln(x)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2 + 1}{4yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{4x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R^2+1)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x)}{4} = \frac{\ln(1+y^2)}{2} + c_1$$

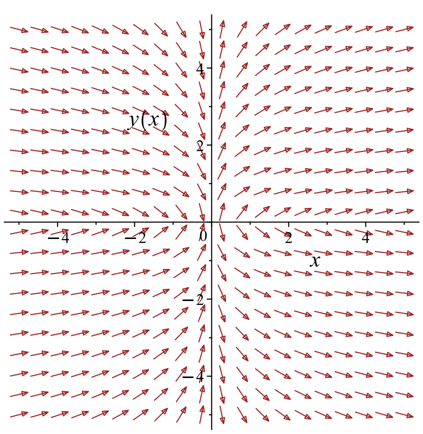
Which simplifies to

$$\frac{\ln(x)}{4} = \frac{\ln(1+y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = \frac{y^2+1}{4yx}$



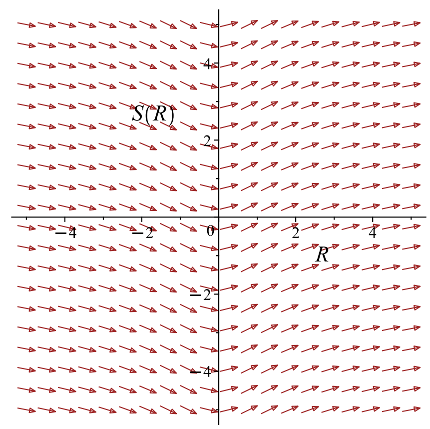
</td><td>

$R = y$

$S = \frac{\ln(x)}{4}$

</td><td>

$\frac{dS}{dR} = \frac{R}{R^2+1}$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\ln(2)}{4} = \frac{\ln(2)}{2} + c_1$$

$$
c_1 = -\frac{\ln (2)}{4}
$$

Substituting c_1 found above in the general solution gives

$$
\frac{\ln (x)}{4} = \frac{\ln (y^2+1)}{2} - \frac{\ln (2)}{4}
$$

Summary

The solution(s) found are the following

$$
\frac{\ln (x)}{4} = \frac{\ln (1+y^2)}{2} - \frac{\ln (2)}{4} \tag{1}
$$

Verification of solutions

$$
\frac{\ln (x)}{4} = \frac{\ln (1+y^2)}{2} - \frac{\ln (2)}{4}
$$

Verified OK.

6.19.4 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= \frac{y^2+1}{4yx}
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = \frac{1}{4x}y + \frac{1}{4x}\frac{1}{y} \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(x)y + f_1(x)y^n \tag{2}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{4x} \\
f_1(x) &= \frac{1}{4x} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = \frac{y^2}{4x} + \frac{1}{4x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{w(x)}{4x} + \frac{1}{4x} \\
w' &= \frac{w}{2x} + \frac{1}{2x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{2x} \\
q(x) &= \frac{1}{2x}
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{w(x)}{2x} = \frac{1}{2x}
$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{2x}dx}$$
$$= \frac{1}{\sqrt{x}}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)\left(\frac{1}{2x}\right)$$
$$\frac{d}{dx}\left(\frac{w}{\sqrt{x}}\right) = \left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{2x}\right)$$
$$d\left(\frac{w}{\sqrt{x}}\right) = \left(\frac{1}{2x^{\frac{3}{2}}}\right) dx$$

Integrating gives

$$\frac{w}{\sqrt{x}} = \int \frac{1}{2x^{\frac{3}{2}}} dx$$
$$\frac{w}{\sqrt{x}} = -\frac{1}{\sqrt{x}} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x}}$ results in

$$w(x) = -1 + c_1\sqrt{x}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -1 + c_1\sqrt{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -1 + \sqrt{2}\, c_1$$

$$c_1 = \sqrt{2}$$

Substituting c_1 found above in the general solution gives

$$y^2 = \sqrt{x}\,\sqrt{2} - 1$$

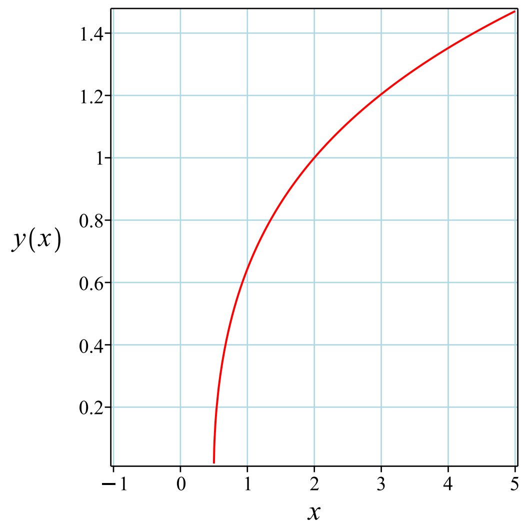
Solving for y from the above gives

$$y = \sqrt{\sqrt{x}\,\sqrt{2} - 1}$$

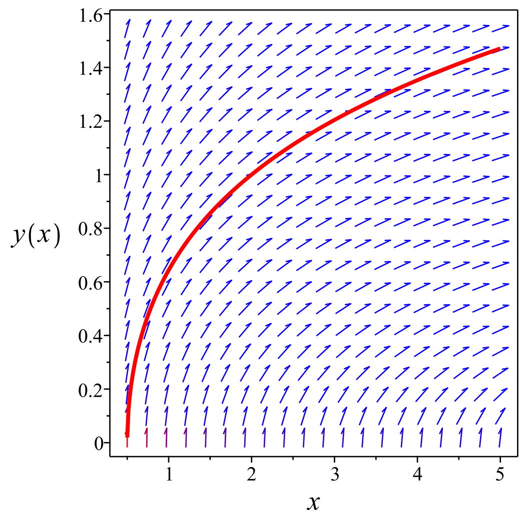
Summary

The solution(s) found are the following

$$y = \sqrt{\sqrt{x}\,\sqrt{2} - 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{\sqrt{x}\,\sqrt{2} - 1}$$

Verified OK. `{positive}`

6.19.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{4y}{y^2+1}\right) \mathrm{d}y &= \left(\frac{1}{x}\right) \mathrm{d}x \\
\left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{4y}{y^2+1}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= \frac{4y}{y^2+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{4y}{y^2+1}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{1}{x}\,\mathrm{d}x$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{4y}{y^2+1}$. Therefore equation (4) becomes

$$\frac{4y}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{4y}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{4y}{y^2+1}\right)\mathrm{d}y$$
$$f(y) = 2\ln\left(y^2+1\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + 2\ln(y^2+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + 2\ln(y^2+1)$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\ln(2) = c_1$$

$$c_1 = \ln(2)$$

Substituting c_1 found above in the general solution gives

$$-\ln(x) + 2\ln(y^2+1) = \ln(2)$$

Summary
The solution(s) found are the following

$$-\ln(x) + 2\ln(1+y^2) = \ln(2) \tag{1}$$

Verification of solutions

$$-\ln(x) + 2\ln(1+y^2) = \ln(2)$$

Verified OK.

6.19.6 Maple step by step solution

Let's solve

$[4xyy' - y^2 = 1, y(2) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{1+y^2} = \frac{1}{4x}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{1+y^2}dx = \int \frac{1}{4x}dx + c_1$

- Evaluate integral

  $\frac{\ln(1+y^2)}{2} = \frac{\ln(x)}{4} + c_1$

- Use initial condition $y(2) = 1$

  $\frac{\ln(2)}{2} = \frac{\ln(2)}{4} + c_1$

- Solve for c_1

  $c_1 = \frac{\ln(2)}{4}$

- Substitute $c_1 = \frac{\ln(2)}{4}$ into general solution and simplify

  $\frac{\ln(1+y^2)}{2} = \frac{\ln(x)}{4} + \frac{\ln(2)}{4}$

- Solution to the IVP

  $\frac{\ln(1+y^2)}{2} = \frac{\ln(x)}{4} + \frac{\ln(2)}{4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 15

```
dsolve([4*x*y(x)*diff(y(x),x)=y(x)^2+1,y(2) = 1],y(x), singsol=all)
```

$$y(x) = \sqrt{\sqrt{2}\,\sqrt{x} - 1}$$

✓ Solution by Mathematica

Time used: 3.741 (sec). Leaf size: 22

```
DSolve[{4*x*y[x]*y'[x]==y[x]^2+1,{y[2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{\sqrt{2}\sqrt{x} - 1}$$

6.20 problem 20

Internal problem ID [11694]
Internal file name [`OUTPUT/11703_Wednesday_April_10_2024_04_55_13_PM_71646592/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{2x + 7y}{-2y + 2x} = 0}$$

With initial conditions

$$[y(1) = 2]$$

6.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{2x + 7y}{2(y - x)} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 2$ is

$$\{x < 2 \lor 2 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x+7y}{2(y-x)}\right)\\ &= -\frac{7}{2(y-x)} + \frac{2x+7y}{2(y-x)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{x < 2 \lor 2 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 1 \lor 1 < y\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

6.20.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{2x + 7u(x)\,x}{-2u(x)\,x + 2x} = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{2u^2+5u+2}{2x(u-1)}\end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{2u^2+5u+2}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+5u+2}{u-1}}\,du = -\frac{1}{2x}\,dx$$

$$\int \frac{1}{\frac{2u^2+5u+2}{u-1}}\,du = \int -\frac{1}{2x}\,dx$$

$$-\frac{\ln(2u+1)}{2} + \ln(u+2) = -\frac{\ln(x)}{2} + c_2$$

Raising both side to exponential gives

$$e^{-\frac{\ln(2u+1)}{2}+\ln(u+2)} = e^{-\frac{\ln(x)}{2}+c_2}$$

Which simplifies to

$$\frac{u+2}{\sqrt{2u+1}} = \frac{c_3}{\sqrt{x}}$$

The solution is

$$\frac{u(x)+2}{\sqrt{2u(x)+1}} = \frac{c_3}{\sqrt{x}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\frac{y}{x}+2}{\sqrt{\frac{2y}{x}+1}} = \frac{c_3}{\sqrt{x}}$$
$$\frac{2x+y}{\sqrt{\frac{x+2y}{x}}\,x} = \frac{c_3}{\sqrt{x}}$$

Which simplifies to

$$\frac{2x+y}{\sqrt{\frac{x+2y}{x}}\,\sqrt{x}} = c_3$$

Substituting initial conditions and solving for c_3 gives $c_3 = \frac{4\sqrt{5}}{5}$. Hence the solution be-

Summary

The solution(s) found are the following

comes

$$\frac{2x+y}{\sqrt{\frac{x+2y}{x}}\,\sqrt{x}} = \frac{4\sqrt{5}}{5} \tag{1}$$

Verification of solutions

$$\frac{2x+y}{\sqrt{\frac{x+2y}{x}}\,\sqrt{x}} = \frac{4\sqrt{5}}{5}$$

Verified OK.

6.20.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2x+7y}{2(y-x)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(2x+7y)(b_3-a_2)}{2(y-x)} - \frac{(2x+7y)^2 a_3}{4(y-x)^2} \\
&- \left(-\frac{1}{y-x} - \frac{2x+7y}{2(y-x)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(-\frac{7}{2(y-x)} + \frac{2x+7y}{2(y-x)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{4x^2a_2 + 4x^2a_3 + 14x^2b_2 - 4x^2b_3 - 8xya_2 + 28xya_3 + 8xyb_2 + 8xyb_3 - 14y^2a_2 + 31y^2a_3 - 4y^2b_2 + 14y^2b}{4(-y+x)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-4x^2a_2 - 4x^2a_3 - 14x^2b_2 + 4x^2b_3 + 8xya_2 - 28xya_3 - 8xyb_2 \\
&- 8xyb_3 + 14y^2a_2 - 31y^2a_3 + 4y^2b_2 - 14y^2b_3 - 18xb_1 + 18ya_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-4a_2v_1^2 + 8a_2v_1v_2 + 14a_2v_2^2 - 4a_3v_1^2 - 28a_3v_1v_2 - 31a_3v_2^2 - 14b_2v_1^2 - 8b_2v_1v_2 + 4b_2v_2^2 + 4b_3v_1^2 - 8b_3v_1v_2 - 14b_3v_2^2 + 18a_1v_2 - 18b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-4a_2 - 4a_3 - 14b_2 + 4b_3)\, v_1^2 + (8a_2 - 28a_3 - 8b_2 - 8b_3)\, v_1v_2 - 18b_1v_1 + (14a_2 - 31a_3 + 4b_2 - 14b_3)\, v_2^2 + 18a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
18a_1 &= 0 \\
-18b_1 &= 0 \\
-4a_2 - 4a_3 - 14b_2 + 4b_3 &= 0 \\
8a_2 - 28a_3 - 8b_2 - 8b_3 &= 0 \\
14a_2 - 31a_3 + 4b_2 - 14b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= -\frac{5b_2}{2} + b_3 \\
a_3 &= -b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{2x + 7y}{2\,(y - x)}\right)(x) \\
&= \frac{-2x^2 - 5xy - 2y^2}{-2y + 2x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2x^2 - 5xy - 2y^2}{-2y + 2x}} dy
\end{aligned}
$$

Which results in

$$
S = 2\ln(2x + y) - \ln(x + 2y)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2x + 7y}{2(y - x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{4}{2x + y} - \frac{1}{x + 2y} \\
S_y &= \frac{2y - 2x}{(x + 2y)(2x + y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

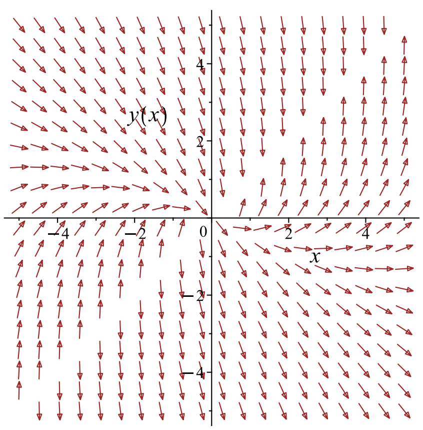
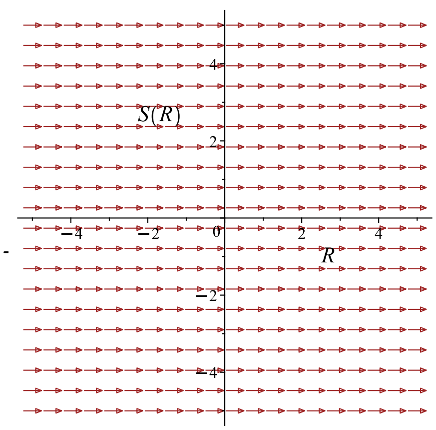
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2\ln(2x + y) - \ln(x + 2y) = c_1$$

Which simplifies to

$$2\ln(2x + y) - \ln(x + 2y) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+7y}{2(y-x)}$ 	$R = x$ $S = 2\ln(2x+y) - \ln(x$ -	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$4\ln(2) - \ln(5) = c_1$$

$$c_1 = 4\ln(2) - \ln(5)$$

Substituting c_1 found above in the general solution gives

$$2\ln(2x+y) - \ln(x+2y) = 4\ln(2) - \ln(5)$$

Summary

The solution(s) found are the following

$$2\ln(2x+y) - \ln(x+2y) = 4\ln(2) - \ln(5) \tag{1}$$

Verification of solutions

$$2\ln(2x+y) - \ln(x+2y) = 4\ln(2) - \ln(5)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 18

```
dsolve([diff(y(x),x)=(2*x+7*y(x))/(2*x-2*y(x)),y(1) = 2],y(x), singsol=all)
```

$$y(x) = \frac{4\sqrt{16-15x}}{5} - 2x + \frac{16}{5}$$

✓ Solution by Mathematica

Time used: 1.383 (sec). Leaf size: 25

```
DSolve[{y'[x]==(2*x+7*y[x])/(2*x-2*y[x]),{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{5}\left(-5x + 2\sqrt{16-15x} + 8\right)$$

6.21 problem 21

Internal problem ID [11695]
Internal file name [`OUTPUT/11704_Wednesday_April_10_2024_04_55_15_PM_67684932/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{xy}{x^2+1} = 0}$$

With initial conditions

$$\left[y\left(\sqrt{15}\right) = 2\right]$$

6.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{x}{x^2+1} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{xy}{x^2+1} = 0$$

The domain of $p(x) = -\frac{x}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \sqrt{15}$ is inside this domain. Hence solution exists and is unique.

6.21.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{xy}{x^2+1} \end{aligned}$$

Where $f(x) = \frac{x}{x^2+1}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= \frac{x}{x^2+1}\,dx \\ \int \frac{1}{y}\,dy &= \int \frac{x}{x^2+1}\,dx \\ \ln(y) &= \frac{\ln(x^2+1)}{2} + c_1 \\ y &= e^{\frac{\ln(x^2+1)}{2}+c_1} \\ &= c_1\sqrt{x^2+1} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = \sqrt{15}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 4c_1$$

$$c_1 = \frac{1}{2}$$

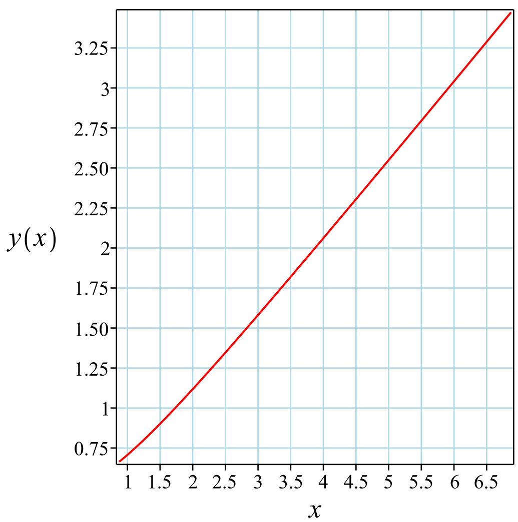
Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{x^2 + 1}}{2}$$

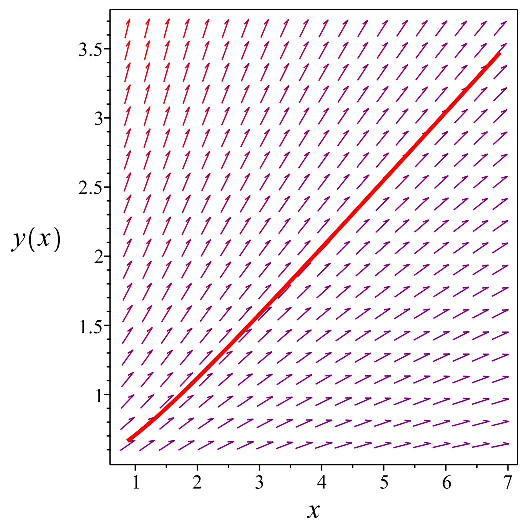
Summary

The solution(s) found are the following

$$y = \frac{\sqrt{x^2 + 1}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x^2 + 1}}{2}$$

Verified OK.

6.21.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int -\frac{x}{x^2+1}dx}$$
$$= \frac{1}{\sqrt{x^2+1}}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(\frac{y}{\sqrt{x^2+1}}\right) = 0$$

Integrating gives

$$\frac{y}{\sqrt{x^2+1}} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x^2+1}}$ results in

$$y = c_1\sqrt{x^2+1}$$

Initial conditions are used to solve for c_1. Substituting $x = \sqrt{15}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 4c_1$$

$$c_1 = \frac{1}{2}$$

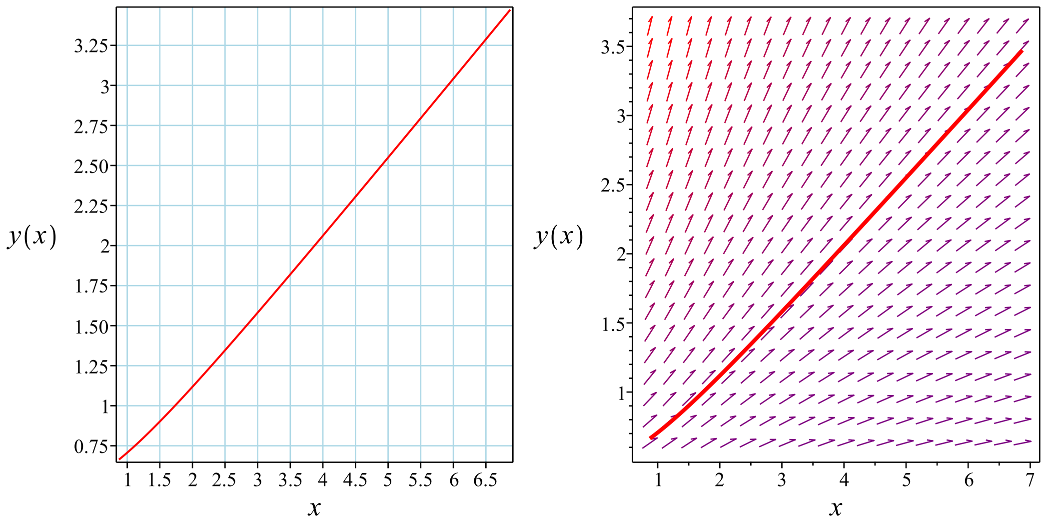
Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{x^2+1}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{x^2+1}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x^2+1}}{2}$$

Verified OK.

6.21.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x^2 u(x)}{x^2+1} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{x\,(x^2+1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{x\left(x^2+1\right)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{x\left(x^2+1\right)}\,dx \\
\ln\left(u\right) &= -\ln\left(x\right) + \frac{\ln\left(x^2+1\right)}{2} + c_2 \\
u &= \mathrm{e}^{-\ln(x)+\frac{\ln\left(x^2+1\right)}{2}+c_2} \\
&= c_2\mathrm{e}^{-\ln(x)+\frac{\ln\left(x^2+1\right)}{2}}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2\sqrt{x^2+1}}{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= c_2\sqrt{x^2+1}
\end{aligned}
$$

Initial conditions are used to solve for c_2. Substituting $x = \sqrt{15}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$
2 = 4c_2
$$

$$
c_2 = \frac{1}{2}
$$

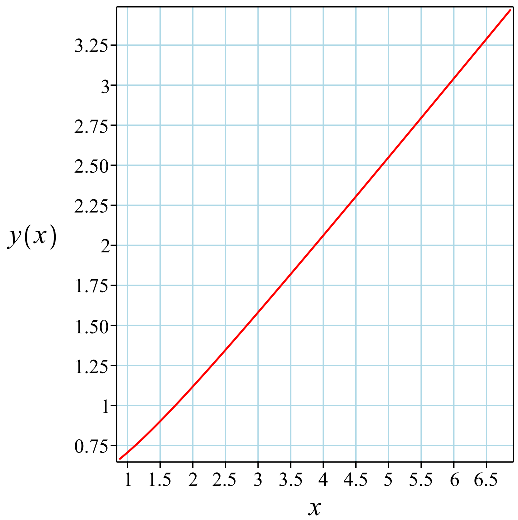
Substituting c_2 found above in the general solution gives

$$
y = \frac{\sqrt{x^2+1}}{2}
$$

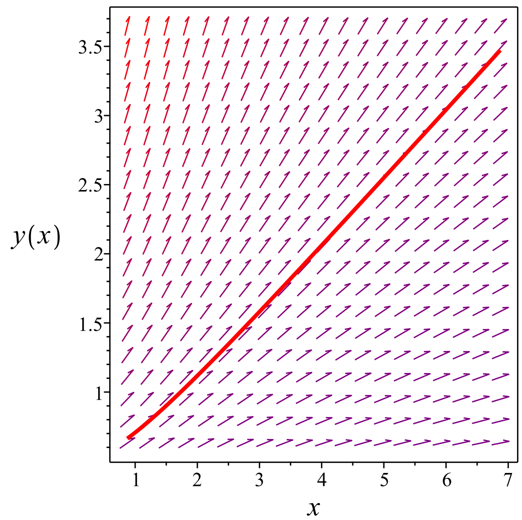
Summary

The solution(s) found are the following

$$
y = \frac{\sqrt{x^2+1}}{2} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x^2 + 1}}{2}$$

Verified OK.

6.21.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{xy}{x^2 + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 239: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \sqrt{x^2 + 1}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{x^2+1}} dy\end{aligned}$$

Which results in

$$S = \frac{y}{\sqrt{x^2+1}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{xy}{x^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{yx}{(x^2+1)^{\frac{3}{2}}} \\ S_y &= \frac{1}{\sqrt{x^2+1}}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\sqrt{x^2+1}} = c_1$$

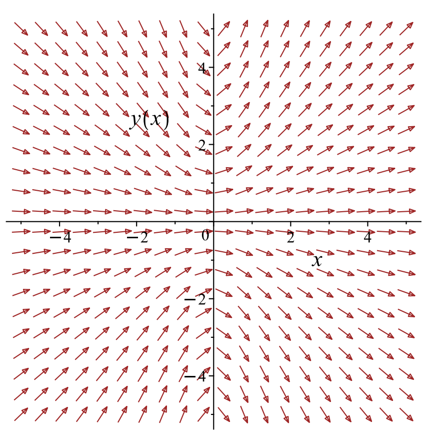
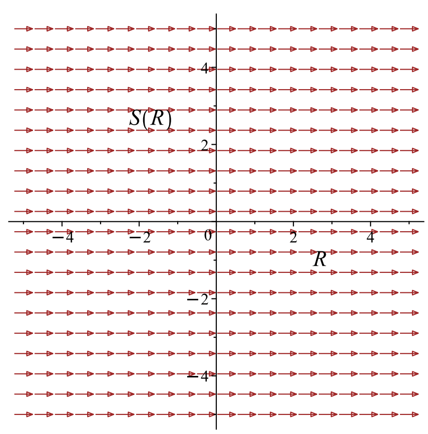
Which simplifies to

$$\frac{y}{\sqrt{x^2+1}} = c_1$$

Which gives

$$y = c_1\sqrt{x^2+1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{xy}{x^2+1}$ 	$R = x$ $S = \frac{y}{\sqrt{x^2+1}}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = \sqrt{15}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 4c_1$$

$$
c_1 = \frac{1}{2}
$$

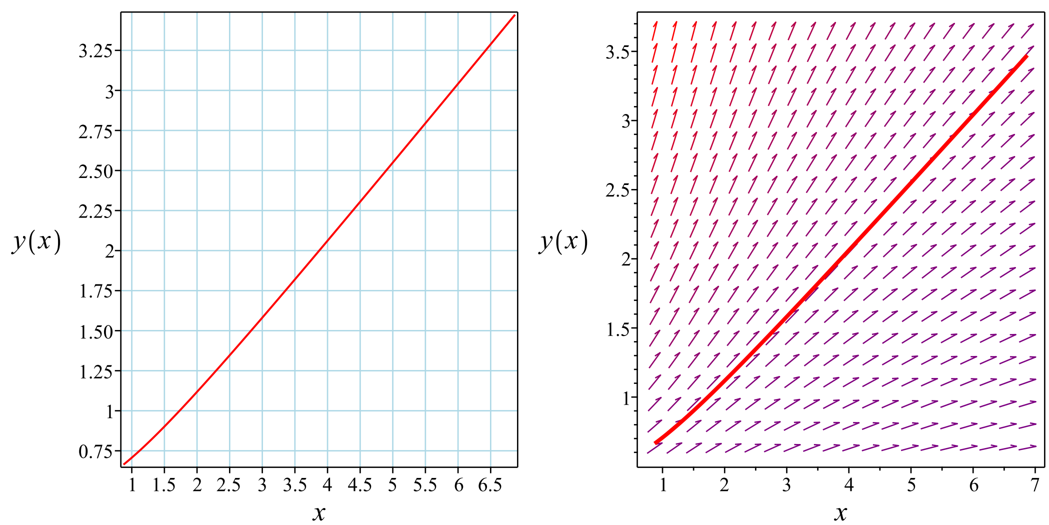
Substituting c_1 found above in the general solution gives

$$
y = \frac{\sqrt{x^2+1}}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\sqrt{x^2+1}}{2} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = \frac{\sqrt{x^2+1}}{2}
$$

Verified OK.

6.21.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y &= \left(\frac{x}{x^2+1}\right)\mathrm{d}x\\ \left(-\frac{x}{x^2+1}\right)\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{x}{x^2+1} \\
N(x, y) &= \frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{x}{x^2+1} \mathrm{d}x \\
\phi &= -\frac{\ln(x^2+1)}{2} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$

$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} + \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\frac{\ln(x^2+1)}{2} + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = \sqrt{15}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 4\,\mathrm{e}^{c_1}$$

$$c_1 = -\ln(2)$$

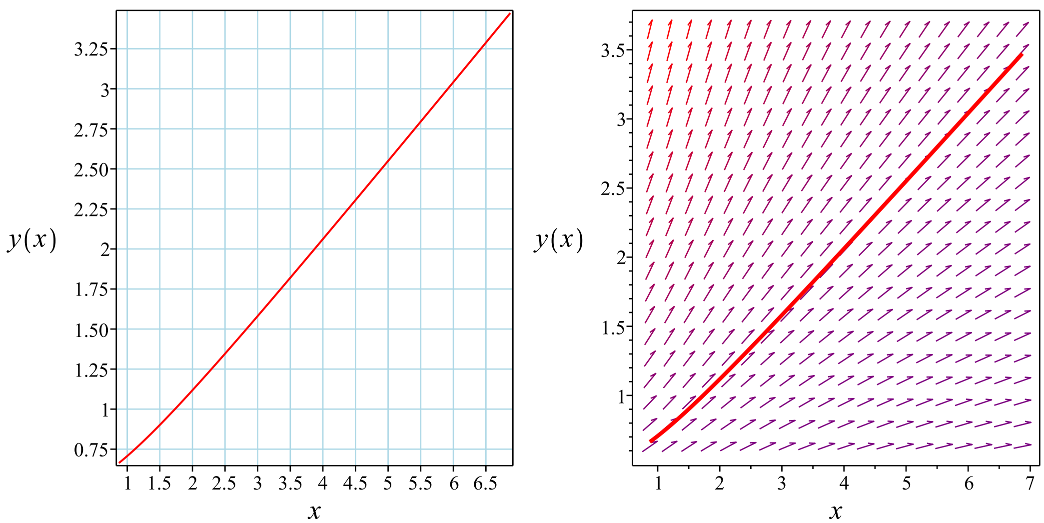
Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{x^2+1}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{x^2+1}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x^2+1}}{2}$$

Verified OK.

6.21.7 Maple step by step solution

Let's solve

$\left[y' - \frac{xy}{x^2+1} = 0, y\left(\sqrt{15}\right) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{x}{x^2+1}$
- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{x}{x^2+1}dx + c_1$
- Evaluate integral

  $\ln(y) = \frac{\ln(x^2+1)}{2} + c_1$
- Solve for y

  $y = e^{\frac{\ln(x^2+1)}{2}+c_1}$
- Use initial condition $y(\sqrt{15}) = 2$

  $2 = e^{2\ln(2)+c_1}$
- Solve for c_1

  $c_1 = -\ln(2)$
- Substitute $c_1 = -\ln(2)$ into general solution and simplify

  $y = \frac{\sqrt{x^2+1}}{2}$
- Solution to the IVP

  $y = \frac{\sqrt{x^2+1}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 13

```
dsolve([diff(y(x),x)=(x*y(x))/(x^2+1),y(sqrt(15)) = 2],y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{x^2+1}}{2}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 18

```
DSolve[{y'[x]==(x*y[x])/(x^2+1),{y[Sqrt[15]]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{x^2+1}}{2}$$

6.22 problem 22

Internal problem ID [11696]
Internal file name [`OUTPUT/11705_Wednesday_April_10_2024_04_55_16_PM_71593933/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$y' + y = \begin{cases} 1 & 0 \leq x < 2 \\ 0 & 0 < x \end{cases}$$

With initial conditions

$$[y(0) = 0]$$

6.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= \begin{cases} 0 & x < 0 \\ 1 & x < 2 \\ 0 & 2 \leq x \end{cases}
\end{aligned}
$$

Hence the ode is

$$
y' + y = \begin{cases} 0 & x < 0 \\ 1 & x < 2 \\ 0 & 2 \leq x \end{cases}
$$

The domain of $p(x) = 1$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \begin{cases} 0 & x < 0 \\ 1 & x < 2 \\ 0 & 2 \leq x \end{cases}$ is

$$
\{0 \leq x \leq 2, 2 \leq x \leq \infty, -\infty \leq x \leq 0\}
$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.22.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 1 dx} \\
&= e^{x}
\end{aligned}
$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\left(\begin{cases} 0 & x<0 \\ 1 & x<2 \\ 0 & 2\le x \end{cases}\right)\right)$$

$$\frac{d}{dx}(e^x y) = (e^x)\left(\left(\begin{cases} 0 & x<0 \\ 1 & x<2 \\ 0 & 2\le x \end{cases}\right)\right)$$

$$d(e^x y) = \begin{cases} 0 & x<0 \\ e^x & x<2 \\ 0 & 2\le x \end{cases}\, dx$$

Integrating gives

$$e^x y = \int \begin{cases} 0 & x<0 \\ e^x & x<2 \\ 0 & 2\le x \end{cases}\, dx$$

$$e^x y = \begin{cases} 0 & x\le 0 \\ -1+e^x & x\le 2 \\ e^2-1 & 2<x \end{cases} + c_1$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(\left(\begin{cases} 0 & x\le 0 \\ -1+e^x & x\le 2 \\ e^2-1 & 2<x \end{cases}\right)\right) + c_1 e^{-x}$$

which simplifies to

$$y = e^{-x}\left(\left(\left(\begin{cases} 0 & x\le 0 \\ -1+e^x & x\le 2 \\ e^2-1 & 2<x \end{cases}\right)\right) + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

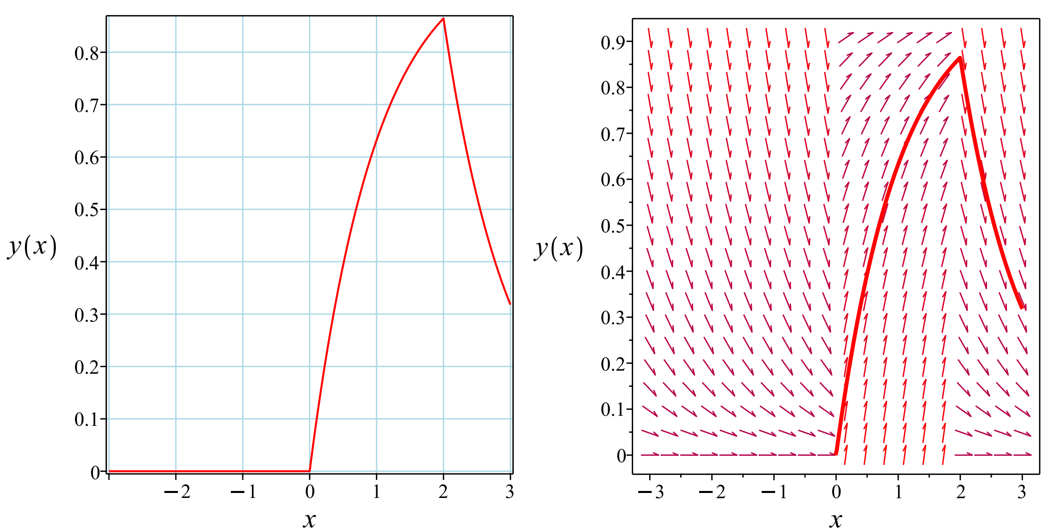
Substituting c_1 found above in the general solution gives

$$y = \begin{cases} 0 & x \le 0 \\ 1 - e^{-x} & x \le 2 \\ e^{-x+2} - e^{-x} & 2 < x \\ 0 & \text{otherwise} \end{cases}$$

Summary

The solution(s) found are the following

$$y = \begin{cases} 0 & x \le 0 \\ 1 - e^{-x} & x \le 2 \\ e^{-x+2} - e^{-x} & 2 < x \\ 0 & \text{otherwise} \end{cases} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$
y = \begin{cases} 0 & x \le 0 \\ 1 - \mathrm{e}^{-x} & x \le 2 \\ \mathrm{e}^{-x+2} - \mathrm{e}^{-x} & 2 < x \\ 0 & \text{otherwise} \end{cases}
$$

Verified OK.

6.22.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -y + \left(\begin{cases} 1 & 0 \le x < 2 \\ 0 & 0 < x \end{cases} \right) \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 242: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= e^{-x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy \end{aligned}$$

Which results in

$$S = e^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y + \left(\begin{cases} 1 & 0 \leq x < 2 \\ 0 & 0 < x \end{cases} \right)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{x} y \\ S_y &= e^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \begin{cases} 0 & x < 0 \\ e^{x} & x < 2 \\ 0 & 2 \leq x \end{cases} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \begin{cases} 0 & R < 0 \\ e^{R} & R < 2 \\ 0 & 2 \leq R \end{cases}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \begin{cases} c_1 & R < 0 \\ c_1 + e^R - 1 & R < 2 \\ c_1 + e^2 - 1 & 2 \leq R \end{cases} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,e^x = \begin{cases} c_1 & x < 0 \\ c_1 - 1 + e^x & x < 2 \\ c_1 + e^2 - 1 & 2 \leq x \end{cases}$$

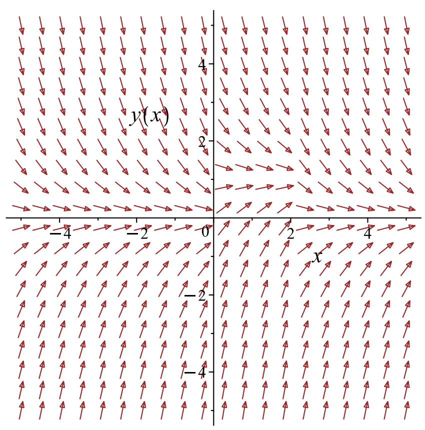
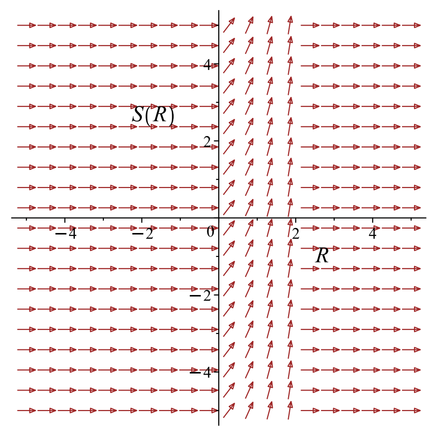
Which simplifies to

$$y\,e^x = \begin{cases} c_1 & x < 0 \\ c_1 - 1 + e^x & x < 2 \\ c_1 + e^2 - 1 & 2 \leq x \end{cases}$$

Which gives

$$y = \begin{cases} [c_1 e^{-x}] & x < 0 \\ [1 + (-1 + c_1)\,e^{-x}] & x < 2 \\ [e^{-x}(c_1 + e^2 - 1)] & 2 \leq x \end{cases}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y + \left(\begin{cases} 1 & 0 \leq x < 2 \\ 0 & 0 < x \end{cases}\right)$ 	$R = x$ $S = e^x y$	$\frac{dS}{dR} = \begin{cases} 0 & R < 0 \\ e^R & R < 2 \\ 0 & 2 \leq R \end{cases}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = [c_1]$$

Unable to solve for constant of integration. Verification of solutions N/A

6.22.4 Maple step by step solution

Let's solve

$$\left[y' + y = \begin{cases} 1 & 0 \leq x < 2 \\ 0 & 0 < x \end{cases}, y(0) = 0\right]$$

- Highest derivative means the order of the ODE is 1
  y'
- Isolate the derivative

$$y' = -y + \left\{\begin{array}{cc} 1 & 0 \le x < 2 \\ 0 & 0 < x \end{array}\right.$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + y = \left\{\begin{array}{cc} 1 & 0 \le x < 2 \\ 0 & 0 < x \end{array}\right.$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + y\right) = \mu(x)\left(\left\{\begin{array}{cc} 1 & 0 \le x < 2 \\ 0 & 0 < x \end{array}\right.\right)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x)\left(y' + y\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \mu(x)$$

- Solve to find the integrating factor

$$\mu(x) = e^{x}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\left(\left\{\begin{array}{cc} 1 & 0 \le x < 2 \\ 0 & 0 < x \end{array}\right.\right) dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \mu(x)\left(\left\{\begin{array}{cc} 1 & 0 \le x < 2 \\ 0 & 0 < x \end{array}\right.\right) dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)\left(\left\{\begin{array}{cc} 1 & 0 \le x < 2 \\ 0 & 0 < x \end{array}\right.\right) dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = e^{x}$

$$y = \frac{\int e^{x}\left(\left\{\begin{array}{cc} 1 & 0 \le x < 2 \\ 0 & 0 < x \end{array}\right.\right) dx + c_1}{e^{x}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\left\{\begin{array}{cc} 0 & x \leq 0 \\ -1+\mathrm{e}^{x} & x \leq 2 \\ \mathrm{e}^{2}-1 & 2<x \end{array}\right. + c_1}{\mathrm{e}^{x}}$$

- Simplify

$$y = \mathrm{e}^{-x}\left(\left\{\begin{array}{cc} 0 & x \leq 0 \\ -1+\mathrm{e}^{x} & x \leq 2 \\ \mathrm{e}^{2}-1 & 2<x \end{array}\right. + c_1\right)$$

- Use initial condition $y(0) = 0$

$$0 = c_1$$

- Solve for c_1

$$c_1 = 0$$

- Substitute $c_1 = 0$ into general solution and simplify

$$y = \mathrm{e}^{-x}\left(\left\{\begin{array}{cc} 0 & x \leq 0 \\ -1+\mathrm{e}^{x} & x \leq 2 \\ \mathrm{e}^{2}-1 & 2<x \end{array}\right.\right)$$

- Solution to the IVP

$$y = \mathrm{e}^{-x}\left(\left\{\begin{array}{cc} 0 & x \leq 0 \\ -1+\mathrm{e}^{x} & x \leq 2 \\ \mathrm{e}^{2}-1 & 2<x \end{array}\right.\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.313 (sec). Leaf size: 36

```
dsolve([diff(y(x),x)+y(x)=piecewise(0<=x and x<2,1,x>0,0),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x < 2 \\ e^{2-x} - e^{-x} & 2 \leq x \end{cases}$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 39

```
DSolve[{y'[x]+y[x]==Piecewise[{{1,0<=x<2},{0,x>2}}],{y[0]==0}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x} & 0 < x \leq 2 \\ e^{-x}(-1 + e^2) & \text{True} \end{cases}$$

6.23 problem 23

Internal problem ID [11697]
Internal file name [`OUTPUT/11706_Wednesday_April_10_2024_04_55_17_PM_81399686/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x+2)\, y'+y = \begin{cases} 2x & 0\le x\le 2 \\ 4 & 2<x \end{cases}}$$

With initial conditions

$$[y(0) = 4]$$

6.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'+p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x+2} \\
q(x) &= \frac{2\left(\left\{\begin{array}{cc} 0 & x<0 \\ x & x\leq 2 \\ 2 & 2<x \end{array}\right.\right)}{x+2}
\end{aligned}
$$

Hence the ode is

$$
y' + \frac{y}{x+2} = \frac{2\left(\left\{\begin{array}{cc} 0 & x<0 \\ x & x\leq 2 \\ 2 & 2<x \end{array}\right.\right)}{x+2}
$$

The domain of $p(x) = \frac{1}{x+2}$ is

$$
\{x < -2 \vee -2 < x\}
$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2\left(\left\{\begin{array}{cc} 0 & x<0 \\ x & x\leq 2 \\ 2 & 2<x \end{array}\right.\right)}{x+2}$ is

$$
\{x < -2 \vee -2 < x\}
$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

6.23.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x+2}dx} \\
&= x + 2
\end{aligned}
$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{2\left(\left\{\begin{array}{cc}0 & x<0 \\ x & x\le 2 \\ 2 & 2<x\end{array}\right.\right)}{x+2}\right)$$

$$\frac{d}{dx}((x+2)\,y) = (x+2)\left(\frac{2\left(\left\{\begin{array}{cc}0 & x<0 \\ x & x\le 2 \\ 2 & 2<x\end{array}\right.\right)}{x+2}\right)$$

$$d((x+2)\,y) = \left(2\left(\left\{\begin{array}{cc}0 & x<0 \\ x & x\le 2 \\ 2 & 2<x\end{array}\right.\right)\right) dx$$

Integrating gives

$$(x+2)\,y = \int 2\left(\left\{\begin{array}{cc}0 & x<0 \\ x & x\le 2 \\ 2 & 2<x\end{array}\right.\right) dx$$

$$(x+2)\,y = 2\left(\left\{\begin{array}{cc}0 & x\le 0 \\ \frac{x^2}{2} & x\le 2 \\ 2x-2 & 2<x\end{array}\right.\right) + c_1$$

Dividing both sides by the integrating factor $\mu = x+2$ results in

$$y = \frac{2\left(\left\{\begin{array}{cc}0 & x\le 0 \\ \frac{x^2}{2} & x\le 2 \\ 2x-2 & 2<x\end{array}\right.\right)}{x+2} + \frac{c_1}{x+2}$$

which simplifies to

$$y = \frac{\left(\begin{cases} 0 & x \le 0 \\ x^2 & x \le 2 \\ 4x - 4 & 2 < x \end{cases}\right) + c_1}{x + 2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = \frac{c_1}{2}$$

$$c_1 = 8$$

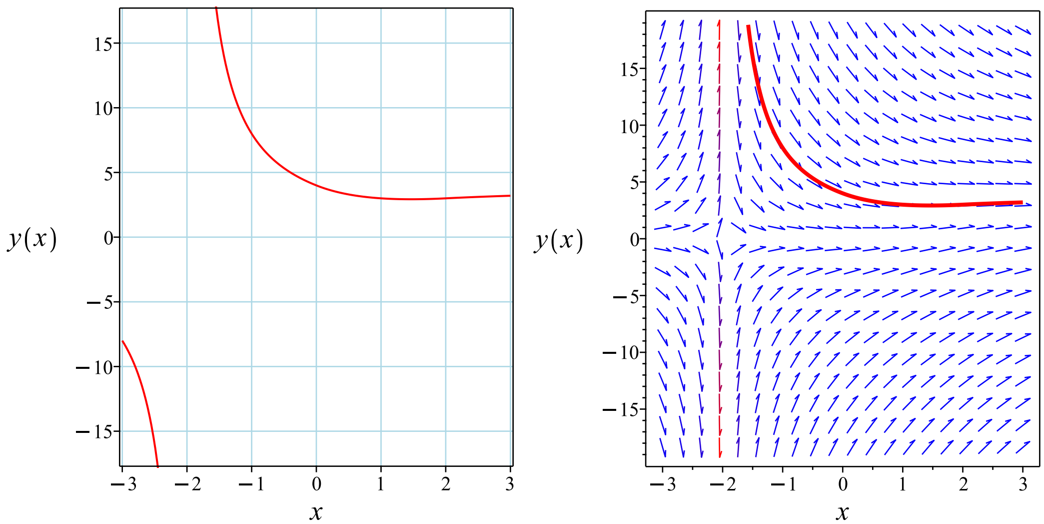
Substituting c_1 found above in the general solution gives

$$y = \begin{cases} \frac{8}{x+2} & x \le 0 \\ \frac{x^2+8}{x+2} & x \le 2 \\ \frac{4+4x}{x+2} & 2 < x \\ \frac{8}{x+2} & \text{otherwise} \end{cases}$$

Summary

The solution(s) found are the following

$$y = \begin{cases} \frac{8}{x+2} & x \le 0 \\ \frac{x^2+8}{x+2} & x \le 2 \\ \frac{4+4x}{x+2} & 2 < x \\ \frac{8}{x+2} & \text{otherwise} \end{cases} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \begin{cases} \frac{8}{x+2} & x \le 0 \\ \frac{x^2+8}{x+2} & x \le 2 \\ \frac{4+4x}{x+2} & 2 < x \\ \frac{8}{x+2} & \text{otherwise} \end{cases}$$

Verified OK.

6.23.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y - \left(\begin{cases} 2x & 0 \le x \le 2 \\ 4 & 2 < x \end{cases}\right)}{x+2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 245: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x + 2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x+2}} dy \end{aligned}$$

Which results in

$$S = (x+2)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y - \left(\begin{cases} 2x & 0 \leq x \leq 2 \\ 4 & 2 < x \end{cases}\right)}{x+2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x+2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2\left(\begin{cases} 0 & x < 0 \\ x & x \leq 2 \\ 2 & 2 < x \end{cases}\right) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2\left(\begin{cases} 0 & R < 0 \\ R & R \leq 2 \\ 2 & 2 < R \end{cases}\right)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \begin{cases} c_1 & R < 0 \\ R^2 + c_1 & R < 2 \\ c_1 - 4 + 4R & 2 \leq R \end{cases} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x+2)\, y = \begin{cases} c_1 & x < 0 \\ x^2 + c_1 & x < 2 \\ c_1 + 4x - 4 & 2 \leq x \end{cases}$$

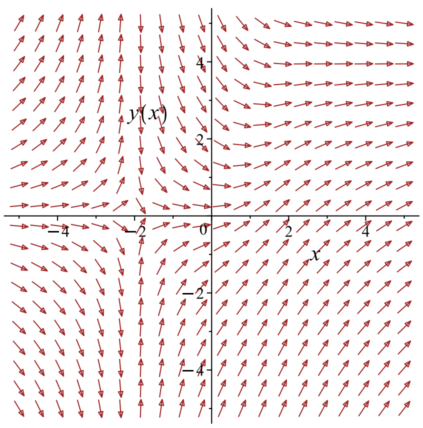
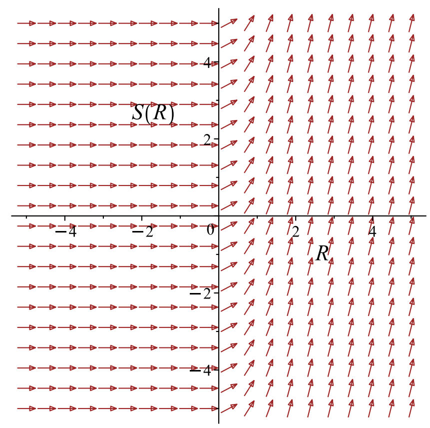
Which simplifies to

$$(x+2)\, y = \begin{cases} c_1 & x < 0 \\ x^2 + c_1 & x < 2 \\ c_1 + 4x - 4 & 2 \leq x \end{cases}$$

Which gives

$$y = \frac{\begin{cases} c_1 & x < 0 \\ x^2 + c_1 & x < 2 \\ c_1 + 4x - 4 & 2 \leq x \end{cases}}{x+2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y-\left(\left\{\begin{array}{cc} 2x & 0\le x\le 2 \\ 4 & 2<x \end{array}\right.\right)}{x+2}$ 	$R = x$ $S = (x+2)\, y$	$\frac{dS}{dR} = 2\left(\left\{\begin{array}{cc} 0 & R<0 \\ R & R\le 2 \\ 2 & 2<R \end{array}\right.\right)$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = \frac{c_1}{2}$$

$$c_1 = 8$$

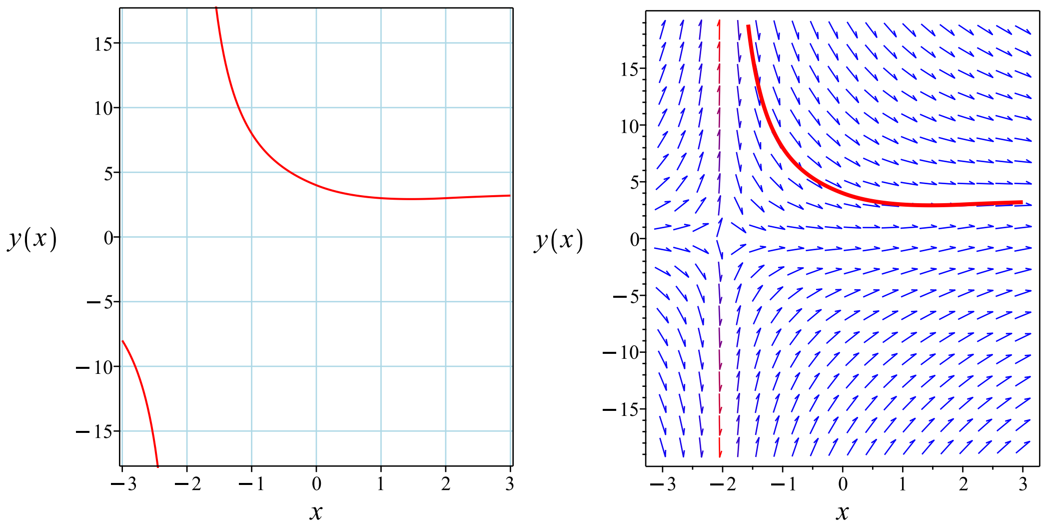
Substituting c_1 found above in the general solution gives

$$y = \left\{\begin{array}{cc} \frac{8}{x+2} & x<0 \\ \frac{x^2+8}{x+2} & x<2 \\ \frac{4+4x}{x+2} & 2\le x \\ 0 & \text{otherwise} \end{array}\right.$$

Summary

The solution(s) found are the following

$$y = \begin{cases} \frac{8}{x+2} & x < 0 \\ \frac{x^2+8}{x+2} & x < 2 \\ \frac{4+4x}{x+2} & 2 \leq x \\ 0 & \text{otherwise} \end{cases} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \begin{cases} \frac{8}{x+2} & x < 0 \\ \frac{x^2+8}{x+2} & x < 2 \\ \frac{4+4x}{x+2} & 2 \leq x \\ 0 & \text{otherwise} \end{cases}$$

Verified OK.

6.23.4 Maple step by step solution

Let's solve

$$\left[(x+2)\, y' + y = \left\{\begin{array}{cc} 2x & 0\le x\le 2 \\ 4 & 2<x \end{array}\right., y(0)=4\right]$$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $$y' = -\frac{y}{x+2} + \frac{\left\{\begin{array}{cc} 2x & 0\le x\le 2 \\ 4 & 2<x \end{array}\right.}{x+2}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $$y' + \frac{y}{x+2} = \frac{\left\{\begin{array}{cc} 2x & 0\le x\le 2 \\ 4 & 2<x \end{array}\right.}{x+2}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $$\mu(x)\left(y' + \frac{y}{x+2}\right) = \frac{\mu(x)\left(\left\{\begin{array}{cc} 2x & 0\le x\le 2 \\ 4 & 2<x \end{array}\right.\right)}{x+2}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $$\mu(x)\left(y' + \frac{y}{x+2}\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

  $$\mu'(x) = \frac{\mu(x)}{x+2}$$

- Solve to find the integrating factor

  $$\mu(x) = x+2$$

- Integrate both sides with respect to x

  $$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)\left(\left\{\begin{array}{cc} 2x & 0\le x\le 2 \\ 4 & 2<x \end{array}\right.\right)}{x+2} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \frac{\mu(x) \left(\begin{cases} 2x & 0 \leq x \leq 2 \\ 4 & 2 < x \end{cases}\right)}{x+2} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x) \left(\begin{cases} 2x & 0 \leq x \leq 2 \\ 4 & 2 < x \end{cases}\right)}{x+2} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = x + 2$

$$y = \frac{\int \left(\begin{cases} 2x & 0 \leq x \leq 2 \\ 4 & 2 < x \end{cases}\right) dx + c_1}{x+2}$$

- Evaluate the integrals on the rhs

$$y = \frac{\begin{cases} 0 & x \leq 0 \\ x^2 & x \leq 2 \\ 4x - 4 & 2 < x \end{cases} + c_1}{x+2}$$

- Use initial condition $y(0) = 4$

$4 = \frac{c_1}{2}$

- Solve for c_1

$c_1 = 8$

- Substitute $c_1 = 8$ into general solution and simplify

$$y = \frac{\begin{cases} 0 & x \leq 0 \\ x^2 & x \leq 2 \\ 4x - 4 & 2 < x \end{cases} + 8}{x+2}$$

- Solution to the IVP

$$y = \frac{\begin{cases} 0 & x \leq 0 \\ x^2 & x \leq 2 \\ 4x - 4 & 2 < x \end{cases} + 8}{x+2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 31

```
dsolve([(x+2)*diff(y(x),x)+y(x)=piecewise(0<=x and x<=2,2*x,x>2,4),y(0) = 4],y(x), singsol=a
```

$$y(x) = \frac{\begin{cases} 8 & x < 0 \\ x^2 + 8 & x < 2 \\ 4 + 4x & 2 \leq x \end{cases}}{x + 2}$$

✓ Solution by Mathematica

Time used: 0.248 (sec). Leaf size: 43

```
DSolve[{(x+2)*y'[x]+y[x]==Piecewise[{{2*x,0<=x<=2},{4,x>2}}],{y[0]==4}},y[x],x,IncludeSingul
```

$$y(x) \to \begin{cases} \frac{8}{x+2} & x \leq 0 \\ \frac{4(x+1)}{x+2} & x > 2 \\ \frac{x^2+8}{x+2} & \text{True} \end{cases}$$

6.24 problem 24

Internal problem ID [11698]
Internal file name [`OUTPUT/11707_Wednesday_April_10_2024_04_55_18_PM_22551060/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Miscellaneous Review. Exercises page 60
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{x^2y' + yx - \frac{y^3}{x} = 0}$$

With initial conditions

$$[y(1) = 1]$$

6.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y(-x^2 + y^2)}{x^3} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y(-x^2+y^2)}{x^3}\right) \\ &= \frac{-x^2+y^2}{x^3} + \frac{2y^2}{x^3}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

6.24.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^2(u'(x)\,x + u(x)) + u(x)\,x^2 - u(x)^3\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(u^2-2)}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u(u^2 - 2)$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u\,(u^2-2)}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{u\,(u^2-2)}\,du &= \int \frac{1}{x}\,dx \\ -\frac{\ln(u)}{2} + \frac{\ln(u^2-2)}{4} &= \ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$e^{-\frac{\ln(u)}{2}+\frac{\ln\left(u^2-2\right)}{4}} = e^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{(u^2-2)^{\frac{1}{4}}}{\sqrt{u}} = c_3 x$$

The solution is

$$\frac{\left(u(x)^2-2\right)^{\frac{1}{4}}}{\sqrt{u\left(x\right)}} = c_3 x$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\left(\frac{y^2}{x^2}-2\right)^{\frac{1}{4}}}{\sqrt{\frac{y}{x}}} = c_3 x$$

$$\frac{\left(\frac{y^2-2x^2}{x^2}\right)^{\frac{1}{4}}}{\sqrt{\frac{y}{x}}} = c_3 x$$

Substituting initial conditions and solving for c_3 gives $c_3 = (-1)^{\frac{1}{4}}$. Hence the solution be-

Summary

The solution(s) found are the following

comes

$$\frac{\left(\frac{y^2-2x^2}{x^2}\right)^{\frac{1}{4}}}{\sqrt{\frac{y}{x}}} = (-1)^{\frac{1}{4}} x \tag{1}$$

Verification of solutions

$$\frac{\left(\frac{y^2-2x^2}{x^2}\right)^{\frac{1}{4}}}{\sqrt{\frac{y}{x}}} = (-1)^{\frac{1}{4}} x$$

Verified OK.

6.24.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y(-x^2+y^2)}{x^3} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 248: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= y^3x^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{y^3x^2} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{1}{2x^2y^2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{y(-x^2+y^2)}{x^3}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{x^3 y^2} \\
S_y &= \frac{1}{y^3 x^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{x^5} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^5}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{1}{4R^4} + c_1 \tag{4}
$$

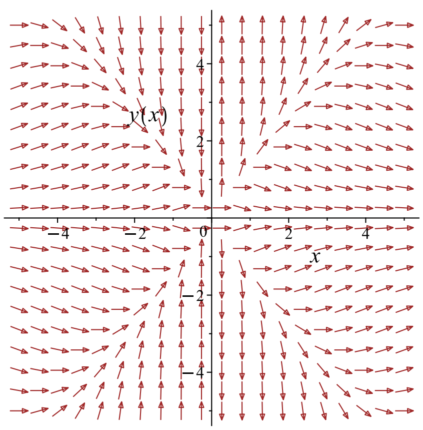
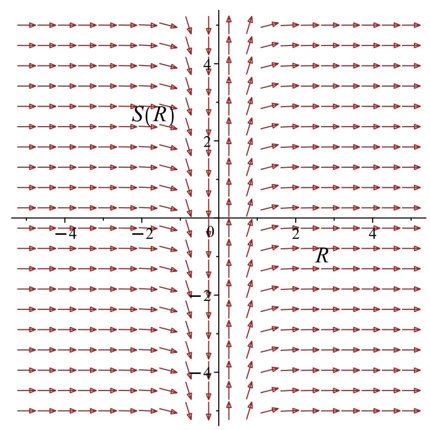
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{1}{2x^2y^2} = -\frac{1}{4x^4} + c_1
$$

Which simplifies to

$$
-\frac{1}{2x^2y^2} = -\frac{1}{4x^4} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(-x^2+y^2)}{x^3}$ 	$R = x$ $S = -\frac{1}{2x^2y^2}$	$\frac{dS}{dR} = \frac{1}{R^5}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = c_1 - \frac{1}{4}$$

$$c_1 = -\frac{1}{4}$$

Substituting c_1 found above in the general solution gives

$$-\frac{1}{2x^2y^2} = -\frac{x^4+1}{4x^4}$$

The above simplifies to

$$x^4y^2 - 2x^2 + y^2 = 0$$

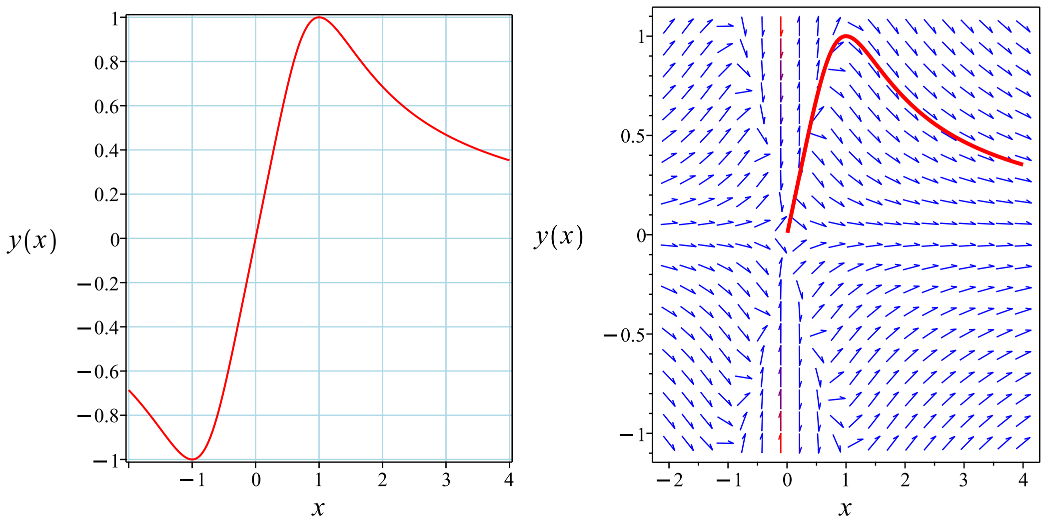
Solving for y from the above gives

$$y = \frac{2x}{\sqrt{2x^4+2}}$$

Summary

The solution(s) found are the following

$$y = \frac{2x}{\sqrt{2x^4 + 2}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{2x}{\sqrt{2x^4 + 2}}$$

Verified OK.

6.24.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(-x^2 + y^2)}{x^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + \frac{1}{x^3}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \frac{1}{x^3} \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y'\frac{1}{y^3} = -\frac{1}{x\,y^2} + \frac{1}{x^3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{2} &= -\frac{w(x)}{x} + \frac{1}{x^3} \\ w' &= \frac{2w}{x} - \frac{2}{x^3} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= -\frac{2}{x^3}
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{2w(x)}{x} = -\frac{2}{x^3}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2}{x} dx} \\
&= \frac{1}{x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-\frac{2}{x^3}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(-\frac{2}{x^3}\right) \\
\mathrm{d}\left(\frac{w}{x^2}\right) &= \left(-\frac{2}{x^5}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x^2} &= \int -\frac{2}{x^5} \,\mathrm{d}x \\
\frac{w}{x^2} &= \frac{1}{2x^4} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$
w(x) = \frac{1}{2x^2} + c_1 x^2
$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$
\frac{1}{y^2} = \frac{1}{2x^2} + c_1 x^2
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = \frac{1}{2} + c_1
$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{y^2} = \frac{x^4+1}{2x^2}$$

The above simplifies to

$$-x^4y^2 + 2x^2 - y^2 = 0$$

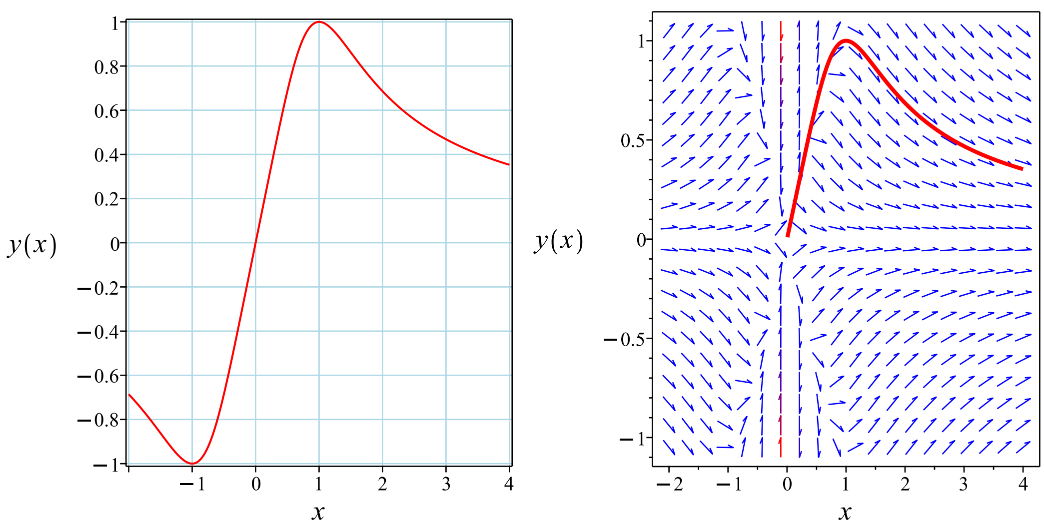
Solving for y from the above gives

$$y = \frac{2x}{\sqrt{2x^4+2}}$$

Summary

The solution(s) found are the following

$$y = \frac{2x}{\sqrt{2x^4+2}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{2x}{\sqrt{2x^4+2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 16

```
dsolve([x^2*diff(y(x),x)+x*y(x)=y(x)^3/x,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{2x}{\sqrt{2x^4 + 2}}$$

✓ Solution by Mathematica

Time used: 0.355 (sec). Leaf size: 21

```
DSolve[{x^2*y'[x]+x*y[x]==y[x]^3/x,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{2}x}{\sqrt{x^4 + 1}}$$

7 Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67

7.1 problem 1

Internal problem ID [11699]
Internal file name [`OUTPUT/11708_Wednesday_April_10_2024_04_55_32_PM_43910952/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{5yx + 4y^2 + (2yx + x^2)\, y' = -1}$$

7.1.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2+2xy\right)\mathrm{d}y &= \left(-5xy-4y^2-1\right)\mathrm{d}x\\ \left(5xy+4y^2+1\right)\mathrm{d}x + \left(x^2+2xy\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 5xy+4y^2+1\\ N(x,y) &= x^2+2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(5xy+4y^2+1\right)\\ &= 5x+8y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2+2xy\right)\\ &= 2y+2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x\left(x+2y\right)}\left(\left(5x+8y\right)-\left(2y+2x\right)\right)\\ &= \frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\, \mathrm{d}x} \\ &= e^{\int \frac{3}{x}\, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{3\ln(x)} \\ &= x^3\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x^3\left(5xy + 4y^2 + 1\right) \\ &= \left(5xy + 4y^2 + 1\right)x^3\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x^3\left(x^2 + 2xy\right) \\ &= x^4(x + 2y)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\left(5xy + 4y^2 + 1\right)x^3\right) + \left(x^4(x + 2y)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int \overline{M}\, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int \left(5xy + 4y^2 + 1\right)x^3\, \mathrm{d}x\end{aligned}$$

$$\phi = x^5y + x^4y^2 + \frac{1}{4}x^4 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= x^5 + 2x^4 y + f'(y) \\ &= x^4(x+2y) + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^4(x+2y)$. Therefore equation (4) becomes

$$x^4(x+2y) = x^4(x+2y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x^5 y + x^4 y^2 + \frac{1}{4}x^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^5 y + x^4 y^2 + \frac{1}{4}x^4$$

Summary

The solution(s) found are the following

$$y x^5 + y^2 x^4 + \frac{x^4}{4} = c_1 \tag{1}$$

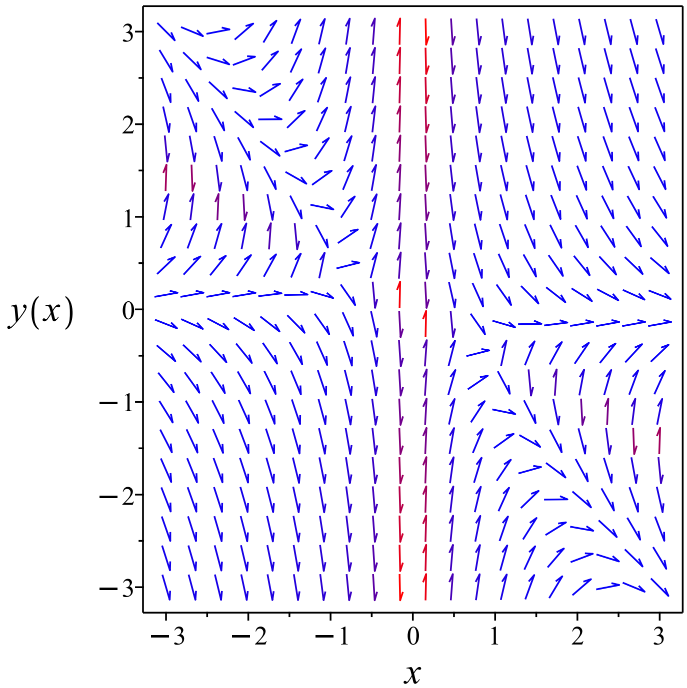


Figure 310: Slope field plot

Verification of solutions

$$yx^5 + y^2x^4 + \frac{x^4}{4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 59

```
dsolve((5*x*y(x)+4*y(x)^2+1)+(x^2+2*x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-x^3 - \sqrt{x^6 - x^4 - 4c_1}}{2x^2}$$
$$y(x) = \frac{-x^3 + \sqrt{x^6 - x^4 - 4c_1}}{2x^2}$$

✓ Solution by Mathematica

Time used: 0.558 (sec). Leaf size: 84

```
DSolve[(5*x*y[x]+4*y[x]^2+1)+(x^2+2*x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\frac{x^5 + \sqrt{x^3}\sqrt{x^7 - x^5 + 4c_1 x}}{2x^4}$$
$$y(x) \to -\frac{x}{2} + \frac{\sqrt{x^3}\sqrt{x^7 - x^5 + 4c_1 x}}{2x^4}$$

7.2 problem 2

Internal problem ID [11700]
Internal file name [`OUTPUT/11709_Thursday_April_11_2024_08_48_33_PM_19754612/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`]]
```

$$\boxed{\tan (y)+\left(x-x^2 \tan (y)\right) y'=-2x}$$

7.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x - x^2\tan(y)\right)\mathrm{d}y &= (-2x - \tan(y))\,\mathrm{d}x\\ (2x + \tan(y))\,\mathrm{d}x + \left(x - x^2\tan(y)\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2x + \tan(y)\\ N(x,y) &= x - x^2\tan(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x + \tan(y))\\ &= \sec(y)^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x - x^2\tan(y)\right)\\ &= 1 - 2\tan(y)\,x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x\left(\tan\left(y\right)x - 1\right)}\left(\left(1 + \tan\left(y\right)^2\right) - \left(1 - 2\tan\left(y\right)x\right)\right) \\
&= -\frac{\left(2x + \tan\left(y\right)\right)\tan\left(y\right)}{x\left(\tan\left(y\right)x - 1\right)}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{2x + \tan\left(y\right)}\left(\left(1 - 2\tan\left(y\right)x\right) - \left(1 + \tan\left(y\right)^2\right)\right) \\
&= -\tan\left(y\right)
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\tan(y) \, \mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{\ln(\cos(y))} \\
&= \cos\left(y\right)
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \cos\left(y\right)\left(2x + \tan\left(y\right)\right) \\
&= 2x\cos\left(y\right) + \sin\left(y\right)
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \cos\left(y\right)\left(x - x^2\tan\left(y\right)\right) \\
&= -x(\sin\left(y\right)x - \cos\left(y\right))
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$(2x\cos(y) + \sin(y)) + (-x(\sin(y)x - \cos(y)))\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int 2x\cos(y) + \sin(y)\,\mathrm{d}x$$
$$\phi = (x\cos(y) + \sin(y))\,x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = (-\sin(y)\,x + \cos(y))\,x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -x(\sin(y)\,x - \cos(y))$. Therefore equation (4) becomes

$$-x(\sin(y)\,x - \cos(y)) = (-\sin(y)\,x + \cos(y))\,x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (x\cos(y) + \sin(y))\,x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (x\cos(y) + \sin(y))\,x$$

Summary

The solution(s) found are the following

$$(\cos(y)\,x + \sin(y))\,x = c_1 \tag{1}$$

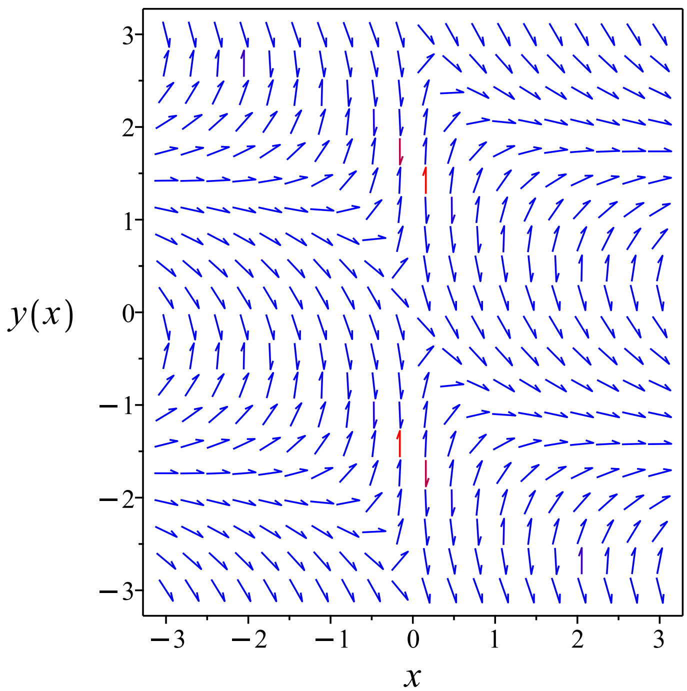


Figure 311: Slope field plot

Verification of solutions

$$(\cos(y)\,x + \sin(y))\,x = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 134

```
dsolve((2*x+tan(y(x)))+(x-x^2*tan(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arctan\left(\frac{-\sqrt{x^4-c_1^2+x^2}\,x-c_1}{(x^2+1)\,x}, \frac{-c_1x+\sqrt{x^4-c_1^2+x^2}}{(x^2+1)\,x}\right)$$
$$y(x) = \arctan\left(\frac{\sqrt{x^4-c_1^2+x^2}\,x-c_1}{(x^2+1)\,x}, \frac{-c_1x-\sqrt{x^4-c_1^2+x^2}}{(x^2+1)\,x}\right)$$

✓ Solution by Mathematica

Time used: 38.283 (sec). Leaf size: 177

```
DSolve[(2*x+Tan[y[x]])+(x-x^2*Tan[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\arccos\left(-\frac{c_1x^2+\sqrt{x^6+x^4-c_1{}^2x^2}}{x^4+x^2}\right)$$
$$y(x) \to \arccos\left(-\frac{c_1x^2+\sqrt{x^6+x^4-c_1{}^2x^2}}{x^4+x^2}\right)$$
$$y(x) \to -\arccos\left(\frac{\sqrt{x^6+x^4-c_1{}^2x^2}-c_1x^2}{x^4+x^2}\right)$$
$$y(x) \to \arccos\left(\frac{\sqrt{x^6+x^4-c_1{}^2x^2}-c_1x^2}{x^4+x^2}\right)$$

7.3 problem 3

Internal problem ID [11701]
Internal file name [`OUTPUT/11710_Thursday_April_11_2024_08_48_36_PM_81880701/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class B`]]
```

$$\boxed{(1+x)\,y^2+y+(2yx+1)\,y'=0}$$

7.3.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy + 1)\,\mathrm{d}y &= \left(-(1 + x)\,y^2 - y\right)\mathrm{d}x \\ \left((1 + x)\,y^2 + y\right)\mathrm{d}x + (2xy + 1)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= (1 + x)\,y^2 + y \\ N(x, y) &= 2xy + 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left((1 + x)\,y^2 + y\right) \\ &= 1 + (2 + 2x)\,y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy + 1) \\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{2xy + 1}((2y(1 + x) + 1) - (2y)) \\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x} \\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{x}\left((1+x)\, y^{2}+y\right) \\ &= y(xy+y+1)\,\mathrm{e}^{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{x}(2xy+1) \\ &= (2xy+1)\,\mathrm{e}^{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(y(xy+y+1)\,\mathrm{e}^{x}\right)+\left((2xy+1)\,\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int y(xy+y+1)\,\mathrm{e}^{x}\,\mathrm{d}x\end{aligned}$$

$$\phi = (xy+1)\,y\,\mathrm{e}^{x}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= x\,\mathrm{e}^{x} y + (xy+1)\,\mathrm{e}^{x} + f'(y) \qquad (4)\\ &= (2xy+1)\,\mathrm{e}^{x} + f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (2xy+1)\,\mathrm{e}^{x}$. Therefore equation (4) becomes

$$(2xy+1)\,\mathrm{e}^{x} = (2xy+1)\,\mathrm{e}^{x} + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (xy+1)\,y\,\mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (xy+1)\,y\,\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$(yx+1)\,y\,\mathrm{e}^{x} = c_1 \qquad (1)$$

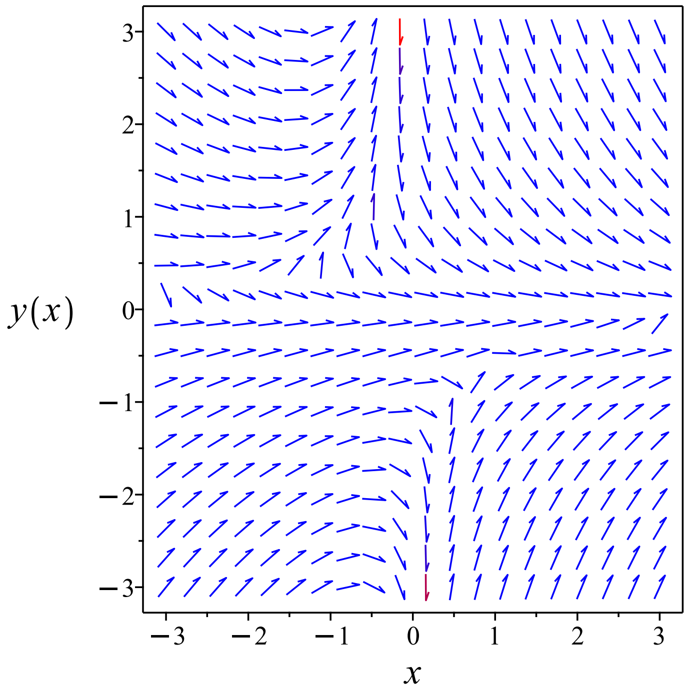


Figure 312: Slope field plot

Verification of solutions

$$(yx+1)\,y\,\mathrm{e}^{x}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 56

```
dsolve((y(x)^2*(x+1)+y(x))+(2*x*y(x)+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-1 + \sqrt{\mathrm{e}^{x} \left(-4c_1 x + \mathrm{e}^{x}\right)}\, \mathrm{e}^{-x}}{2x}$$
$$y(x) = \frac{-\sqrt{\mathrm{e}^{x} \left(-4c_1 x + \mathrm{e}^{x}\right)}\, \mathrm{e}^{-x} - 1}{2x}$$

✓ Solution by Mathematica

Time used: 2.638 (sec). Leaf size: 69

```
DSolve[(y[x]^2*(x+1)+y[x])+(2*x*y[x]+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1 + \frac{\sqrt{e^x + 4c_1 x}}{\sqrt{e^x}}}{2x}$$
$$y(x) \to \frac{-1 + \frac{\sqrt{e^x + 4c_1 x}}{\sqrt{e^x}}}{2x}$$

7.4 problem 4

Internal problem ID [11702]
Internal file name [`OUTPUT/11711_Thursday_April_11_2024_08_48_37_PM_11428769/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{2y^2x + y + (2y^3 - x)\, y' = 0}$$

7.4.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(2xy+1)}{2y^3 - x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 3 to use as anstaz gives

$$\xi = x^3a_7 + x^2ya_8 + x\,y^2a_9 + y^3a_{10} + x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = x^3b_7 + x^2yb_8 + x\,y^2b_9 + y^3b_{10} + x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$3x^2b_7 + 2xyb_8 + y^2b_9 + 2xb_4 + yb_5 + b_2 - \frac{y(2xy+1)\left(-3x^2a_7 + x^2b_8 - 2xya_8 + 2xyb_9 - y^2a_9 + 3y^2b_{10} - 2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3\right)}{2y^3 - x} - \frac{y^2(2xy+1)^2\left(x^2a_8 + 2xya_9 + 3y^2a_{10} + xa_5 + 2ya_6 + a_3\right)}{(2y^3-x)^2} - \left(-\frac{2y^2}{2y^3-x} - \frac{y(2xy+1)}{(2y^3-x)^2}\right)\left(x^3a_7 + x^2ya_8 + x\,y^2a_9 + y^3a_{10} + x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1\right) - \left(-\frac{2xy+1}{2y^3-x} - \frac{2yx}{2y^3-x} + \frac{6y^3(2xy+1)}{(2y^3-x)^2}\right)\left(x^3b_7 + x^2yb_8 + x\,y^2b_9 + y^3b_{10} + x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1\right) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-2x^4b_7 - 4y^8b_9 - 2y^6a_9 + 10y^6b_{10} + 2y^4a_{10} - 4y^8a_{10} - 12x^2y^6b_7 + 16x^3y^3b_7 - 8x\,y^7b_8 + 14x^2y^4b_8 - 2x}{} = 0$$

Setting the numerator to zero gives

$$2x^4b_7 + 4y^8b_9 + 2y^6a_9 - 10y^6b_{10} - 2y^4a_{10} + 4y^8a_{10} + 12x^2y^6b_7 - 16x^3y^3b_7 + 8x\,y^7b_8 - 14x^2y^4b_8 + 2x^3yb_8 - 12x\,y^5b_9 + 2x^2y^2b_9 + 16x^3y^5a_7 - 8x^3y^5b_8 + 12x^2y^6a_8 - 12x^2y^6b_9 + 8x\,y^7a_9 - 16x\,y^7b_{10} - 6x^4y^2a_7 - 2x^4y^2b_8 - 8x^3y^3a_8 + 6x^2y^4a_7 - 10x^2y^4a_9 + 2x^2y^4b_{10} + 4x\,y^5a_8 - 4x^3y^4a_5 - 4x^3y^4b_4 + 12x^2y^5a_4 - 8x^2y^5a_6 - 8x^2y^5b_5 + 8x\,y^6a_5 + 8x\,y^6b_4 - 12x\,y^6b_6 - 4x^4yb_4 - 4x^3y^2a_4 - 2x^3y^2b_5 - 6x^2y^3a_5 - 12x^2y^3b_4 + 4x\,y^4a_4 - 8x\,y^4a_6 - 10x\,y^4b_5 - x^2ya_4 + x^2yb_5 - x\,y^2a_5 + x\,y^2b_6 - 4x^2y^4a_3 - 4x^2y^4b_2 + 8x\,y^5a_2 - 8x\,y^5b_3 - 4x\,y^4b_1 - 4x^3yb_2 - 2x^2y^2a_2 - 2x^2y^2b_3 - 4x\,y^3a_3 - 8x\,y^3b_2 - 4x^2yb_1 + 4y^6a_3 + 4y^6b_2 + 4y^5a_1 + 2y^4a_2 - 6y^4b_3 - 4y^3b_1 - xb_1 + ya_1 + 4y^7a_6 + 4y^7b_5 + 2y^5a_5 - 8y^5b_6 + x^3b_4 - y^3a_6 - 2x^3ya_7 - 2x^2y^2a_8 - 2x\,y^3a_9 + 2x\,y^3b_{10} - 4x^4y^4a_8 - 8x^3y^5a_9 - 12x^2y^6a_{10} - 12x\,y^5a_{10} - 4x^4y^4b_7 - 4x^5yb_7 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 2v_1^4b_7 + 4v_2^8b_9 + 2v_2^6a_9 - 10v_2^6b_{10} - 2v_2^4a_{10} + 4v_2^8a_{10} + 4v_2^6a_3 \\ & + 4v_2^6b_2 + 4v_2^5a_1 + 2v_2^4a_2 - 6v_2^4b_3 - 4v_2^3b_1 - v_1b_1 + v_2a_1 + 4v_2^7a_6 \\ & + 4v_2^7b_5 + 2v_2^5a_5 - 8v_2^5b_6 + v_1^3b_4 - v_2^3a_6 + 12v_1^2v_2^6b_7 - 16v_1^3v_2^3b_7 \\ & + 8v_1v_2^7b_8 - 14v_1^2v_2^4b_8 + 2v_1^3v_2b_8 - 12v_1v_2^5b_9 + 2v_1^2v_2^2b_9 + 16v_1^3v_2^5a_7 \\ & - 8v_1^3v_2^5b_8 + 12v_1^2v_2^6a_8 - 12v_1^2v_2^6b_9 + 8v_1v_2^7a_9 - 16v_1v_2^7b_{10} - 6v_1^4v_2^2a_7 \\ & - 2v_1^4v_2^2b_8 - 8v_1^3v_2^3a_8 + 6v_1^2v_2^4a_7 - 10v_1^2v_2^4a_9 + 2v_1^2v_2^4b_{10} + 4v_1v_2^5a_8 \\ & - 4v_1^3v_2^4a_5 - 4v_1^3v_2^4b_4 + 12v_1^2v_2^5a_4 - 8v_1^2v_2^5a_6 - 8v_1^2v_2^5b_5 + 8v_1v_2^6a_5 \\ & + 8v_1v_2^6b_4 - 12v_1v_2^6b_6 - 4v_1^4v_2b_4 - 4v_1^3v_2^2a_4 - 2v_1^3v_2^2b_5 - 6v_1^2v_2^3a_5 \\ & - 12v_1^2v_2^3b_4 + 4v_1v_2^4a_4 - 8v_1v_2^4a_6 - 10v_1v_2^4b_5 - v_1^2v_2a_4 + v_1^2v_2b_5 \\ & - v_1v_2^2a_5 + v_1v_2^2b_6 - 4v_1^2v_2^4a_3 - 4v_1^2v_2^4b_2 + 8v_1v_2^5a_2 - 8v_1v_2^5b_3 \\ & - 4v_1v_2^4b_1 - 4v_1^3v_2b_2 - 2v_1^2v_2^2a_2 - 2v_1^2v_2^2b_3 - 4v_1v_2^3a_3 - 8v_1v_2^3b_2 \\ & - 4v_1^2v_2b_1 - 2v_1^3v_2a_7 - 2v_1^2v_2^2a_8 - 2v_1v_2^3a_9 + 2v_1v_2^3b_{10} - 4v_1^4v_2^4a_8 \\ & - 8v_1^3v_2^5a_9 - 12v_1^2v_2^6a_{10} - 12v_1v_2^5a_{10} - 4v_1^4v_2^4b_7 - 4v_1^5v_2b_7 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&2v_1^4 b_7 - v_1 b_1 + v_2 a_1 + v_1^3 b_4 + (4b_9 + 4a_{10})\, v_2^8 \\
&+ (2a_9 - 10b_{10} + 4a_3 + 4b_2)\, v_2^6 + (-2a_{10} + 2a_2 - 6b_3)\, v_2^4 \\
&+ (4a_1 + 2a_5 - 8b_6)\, v_2^5 + (-a_6 - 4b_1)\, v_2^3 + (4a_6 + 4b_5)\, v_2^7 \\
&+ (-4a_8 - 4b_7)\, v_2^4 v_1^4 + (12b_7 + 12a_8 - 12b_9 - 12a_{10})\, v_2^6 v_1^2 \\
&+ (-16b_7 - 8a_8)\, v_2^3 v_1^3 + (8b_8 + 8a_9 - 16b_{10})\, v_2^7 v_1 \\
&+ (-14b_8 + 6a_7 - 10a_9 + 2b_{10} - 4a_3 - 4b_2)\, v_2^4 v_1^2 \\
&+ (2b_8 - 4b_2 - 2a_7)\, v_2 v_1^3 + (-12b_9 + 4a_8 + 8a_2 - 8b_3 - 12a_{10})\, v_2^5 v_1 \\
&+ (2b_9 - 2a_2 - 2b_3 - 2a_8)\, v_2^2 v_1^2 + (16a_7 - 8b_8 - 8a_9)\, v_2^5 v_1^3 \\
&+ (-6a_7 - 2b_8)\, v_2^2 v_1^4 + (-4a_5 - 4b_4)\, v_2^4 v_1^3 + (-6a_5 - 12b_4)\, v_2^3 v_1^2 \\
&+ (4a_4 - 8a_6 - 4b_1 - 10b_5)\, v_2^4 v_1 + (-a_4 - 4b_1 + b_5)\, v_2 v_1^2 \\
&+ (-a_5 + b_6)\, v_2^2 v_1 + (-4a_3 - 8b_2 - 2a_9 + 2b_{10})\, v_2^3 v_1 \\
&- 4v_1^4 v_2 b_4 + (12a_4 - 8a_6 - 8b_5)\, v_2^5 v_1^2 \\
&+ (8a_5 + 8b_4 - 12b_6)\, v_2^6 v_1 + (-4a_4 - 2b_5)\, v_2^2 v_1^3 - 4v_1^5 v_2 b_7 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
b_4 &= 0 \\
-b_1 &= 0 \\
-4b_4 &= 0 \\
-4b_7 &= 0 \\
2b_7 &= 0 \\
-4a_4 - 2b_5 &= 0 \\
-6a_5 - 12b_4 &= 0 \\
-4a_5 - 4b_4 &= 0 \\
-a_5 + b_6 &= 0 \\
-a_6 - 4b_1 &= 0 \\
4a_6 + 4b_5 &= 0 \\
-6a_7 - 2b_8 &= 0 \\
-4a_8 - 4b_7 &= 0 \\
-16b_7 - 8a_8 &= 0 \\
4b_9 + 4a_{10} &= 0 \\
4a_1 + 2a_5 - 8b_6 &= 0 \\
-a_4 - 4b_1 + b_5 &= 0 \\
12a_4 - 8a_6 - 8b_5 &= 0 \\
8a_5 + 8b_4 - 12b_6 &= 0 \\
16a_7 - 8b_8 - 8a_9 &= 0 \\
-2a_{10} + 2a_2 - 6b_3 &= 0 \\
2b_8 - 4b_2 - 2a_7 &= 0 \\
8b_8 + 8a_9 - 16b_{10} &= 0 \\
-4a_3 - 8b_2 - 2a_9 + 2b_{10} &= 0 \\
4a_4 - 8a_6 - 4b_1 - 10b_5 &= 0 \\
2a_9 - 10b_{10} + 4a_3 + 4b_2 &= 0 \\
12b_7 + 12a_8 - 12b_9 - 12a_{10} &= 0 \\
2b_9 - 2a_2 - 2b_3 - 2a_8 &= 0 \\
-12b_9 + 4a_8 + 8a_2 - 8b_3 - 12a_{10} &= 0 \\
-14b_8 + 6a_7 - 10a_9 + 2b_{10} - 4a_3 - 4b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= \frac{b_9}{2}\\
a_3 &= 2b_{10}\\
a_4 &= 0\\
a_5 &= 0\\
a_6 &= 0\\
a_7 &= b_{10}\\
a_8 &= 0\\
a_9 &= 5b_{10}\\
a_{10} &= -b_9\\
b_1 &= 0\\
b_2 &= -2b_{10}\\
b_3 &= \frac{b_9}{2}\\
b_4 &= 0\\
b_5 &= 0\\
b_6 &= 0\\
b_7 &= 0\\
b_8 &= -3b_{10}\\
b_9 &= b_9\\
b_{10} &= b_{10}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x^3 + 5x\,y^2 + 2y\\
\eta &= -3x^2y + y^3 - 2x
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

7.4.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(2y^3-x\right)\mathrm{d}y&=\left(-2x\,y^2-y\right)\mathrm{d}x\\ \left(2x\,y^2+y\right)\mathrm{d}x+\left(2y^3-x\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= 2x\,y^2 + y \\
N(x,y) &= 2y^3 - x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x\,y^2 + y\right) \\
&= 4xy + 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2y^3 - x\right) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{2y^3 - x}((4xy + 1) - (-1)) \\
&= \frac{-4xy - 2}{-2y^3 + x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{2x\,y^2 + y}((-1) - (4xy + 1)) \\
&= -\frac{2}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int -\frac{2}{y} \,\mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(y)} \\ &= \frac{1}{y^2}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{y^2}(2x\,y^2 + y) \\ &= \frac{2xy + 1}{y}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{y^2}(2y^3 - x) \\ &= \frac{2y^3 - x}{y^2}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{2xy + 1}{y}\right) + \left(\frac{2y^3 - x}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{2xy + 1}{y}\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{x(xy + 1)}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= \frac{x^2}{y} - \frac{x(xy+1)}{y^2} + f'(y) \\ &= -\frac{x}{y^2} + f'(y) \end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y^3-x}{y^2}$. Therefore equation (4) becomes

$$\frac{2y^3-x}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\begin{aligned} \int f'(y)\,\mathrm{d}y &= \int (2y)\,\mathrm{d}y \\ f(y) &= y^2 + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(xy+1)}{y} + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(xy+1)}{y} + y^2$$

Summary
The solution(s) found are the following

$$\frac{x(yx+1)}{y} + y^2 = c_1 \tag{1}$$

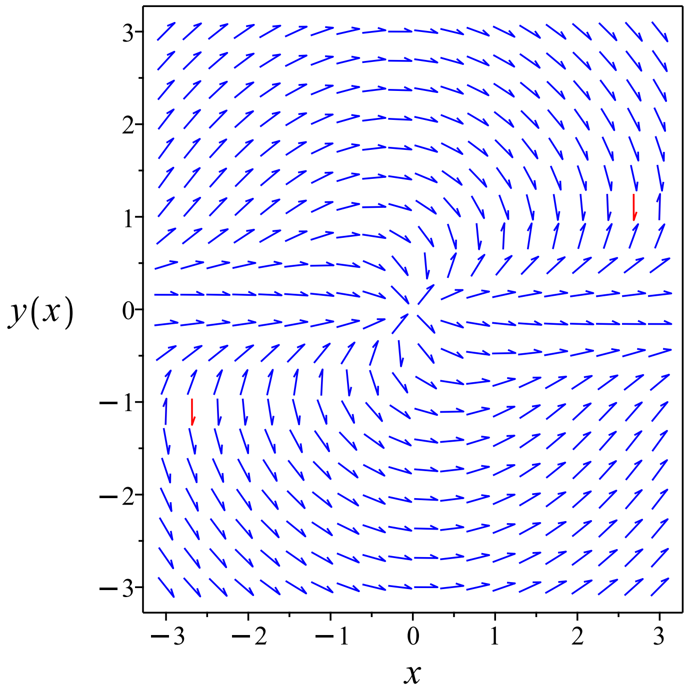


Figure 313: Slope field plot

Verification of solutions

$$\frac{x(yx+1)}{y}+y^2=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 301

```
dsolve((2*x*y(x)^2+y(x))+(2*y(x)^3-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-12x^2 - 12c_1 + \left(-108x + 12\sqrt{12x^6 + 36x^4c_1 + (36c_1^2 + 81)\,x^2 + 12c_1^3}\right)^{\frac{2}{3}}}{6\left(-108x + 12\sqrt{12x^6 + 36x^4c_1 + (36c_1^2 + 81)\,x^2 + 12c_1^3}\right)^{\frac{1}{3}}}$$

$$y(x) = -\frac{\left(\frac{i\sqrt{3}}{12} + \frac{1}{12}\right)\left(-108x + 12\sqrt{12x^6 + 36x^4c_1 + (36c_1^2 + 81)\,x^2 + 12c_1^3}\right)^{\frac{2}{3}} + (x^2 + c_1)\left(i\sqrt{3} - 1\right)}{\left(-108x + 12\sqrt{12x^6 + 36x^4c_1 + (36c_1^2 + 81)\,x^2 + 12c_1^3}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{\frac{\left(i\sqrt{3}-1\right)\left(-108x+12\sqrt{12x^6+36x^4c_1+(36c_1^2+81)x^2+12c_1^3}\right)^{\frac{2}{3}}}{12} + (x^2 + c_1)\left(1 + i\sqrt{3}\right)}{\left(-108x + 12\sqrt{12x^6 + 36x^4c_1 + (36c_1^2 + 81)\,x^2 + 12c_1^3}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 6.163 (sec). Leaf size: 316

```
DSolve[(2*x*y[x]^2+y[x])+(2*y[x]^3-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2^{2/3}\left(-27x + \sqrt{729x^2 + 108\,(x^2 - c_1)^3}\right)^{2/3} - 6\sqrt[3]{2}(x^2 - c_1)}{6\sqrt[3]{-27x + \sqrt{729x^2 + 108\,(x^2 - c_1)^3}}}$$

$$y(x) \to \frac{\left(1 - i\sqrt{3}\right)(x^2 - c_1)}{2^{2/3}\sqrt[3]{-27x + \sqrt{729x^2 + 108\,(x^2 - c_1)^3}}} - \frac{\left(1 + i\sqrt{3}\right)\sqrt[3]{-27x + \sqrt{729x^2 + 108\,(x^2 - c_1)^3}}}{6\sqrt[3]{2}}$$

$$y(x) \to \frac{\left(1 + i\sqrt{3}\right)(x^2 - c_1)}{2^{2/3}\sqrt[3]{-27x + \sqrt{729x^2 + 108\,(x^2 - c_1)^3}}} + \frac{\left(-1 + i\sqrt{3}\right)\sqrt[3]{-27x + \sqrt{729x^2 + 108\,(x^2 - c_1)^3}}}{6\sqrt[3]{2}}$$

$$y(x) \to 0$$

7.5 problem 5

Internal problem ID [11703]
Internal file name [`OUTPUT/11712_Thursday_April_11_2024_08_48_39_PM_50875666/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{4y^2x + 6y + (5x^2y + 8x)\, y' = 0}$$

7.5.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2y(2xy + 3)}{x\,(5xy + 8)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{2y(2xy+3)\,(b_3-a_2)}{x\,(5xy+8)} - \frac{4y^2(2xy+3)^2\,a_3}{x^2\,(5xy+8)^2} \\
&- \left(-\frac{4y^2}{x\,(5xy+8)} + \frac{2y(2xy+3)}{x^2\,(5xy+8)} + \frac{10y^2(2xy+3)}{x\,(5xy+8)^2}\right)(xa_2 + ya_3 + a_1) \\
&- \left(-\frac{2(2xy+3)}{x\,(5xy+8)} - \frac{4y}{5xy+8} + \frac{10y(2xy+3)}{(5xy+8)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{45x^4y^2b_2 - 36x^2y^4a_3 + 20x^3y^2b_1 - 20x^2y^3a_1 + 144x^3yb_2 + 2x^2y^2a_2 + 2x^2y^2b_3 - 108x\,y^3a_3 + 64x^2yb_1 - 6}{x^2\,(5xy+8)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&45x^4y^2b_2 - 36x^2y^4a_3 + 20x^3y^2b_1 - 20x^2y^3a_1 + 144x^3yb_2 + 2x^2y^2a_2 + 2x^2y^2b_3 \\
&- 108x\,y^3a_3 + 64x^2yb_1 - 60x\,y^2a_1 + 112b_2x^2 - 84y^2a_3 + 48xb_1 - 48ya_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-36a_3v_1^2v_2^4 + 45b_2v_1^4v_2^2 - 20a_1v_1^2v_2^3 + 20b_1v_1^3v_2^2 + 2a_2v_1^2v_2^2 - 108a_3v_1v_2^3 + 144b_2v_1^3v_2 \\
&+ 2b_3v_1^2v_2^2 - 60a_1v_1v_2^2 + 64b_1v_1^2v_2 - 84a_3v_2^2 + 112b_2v_1^2 - 48a_1v_2 + 48b_1v_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$45b_2v_1^4v_2^2 + 20b_1v_1^3v_2^2 + 144b_2v_1^3v_2 - 36a_3v_1^2v_2^4 - 20a_1v_1^2v_2^3 + (2a_2 + 2b_3)v_1^2v_2^2 + 64b_1v_1^2v_2 + 112b_2v_1^2 - 108a_3v_1v_2^3 - 60a_1v_1v_2^2 + 48b_1v_1 - 84a_3v_2^2 - 48a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -60a_1 &= 0 \\ -48a_1 &= 0 \\ -20a_1 &= 0 \\ -108a_3 &= 0 \\ -84a_3 &= 0 \\ -36a_3 &= 0 \\ 20b_1 &= 0 \\ 48b_1 &= 0 \\ 64b_1 &= 0 \\ 45b_2 &= 0 \\ 112b_2 &= 0 \\ 144b_2 &= 0 \\ 2a_2 + 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{2y(2xy+3)}{x\,(5xy+8)}\right)(-x) \\
&= \frac{x\,y^2 + 2y}{5xy+8} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x\,y^2+2y}{5xy+8}} dy
\end{aligned}
$$

Which results in

$$
S = 4\ln(y) + \ln(xy+2)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{2y(2xy+3)}{x\,(5xy+8)}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{xy+2} \\
S_y &= \frac{4}{y} + \frac{x}{xy+2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{3}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{3}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -3\ln(R) + c_1 \tag{4}$$

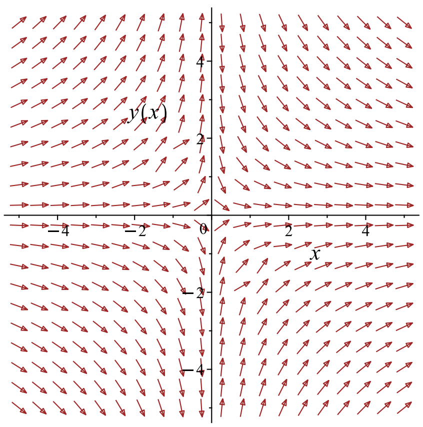
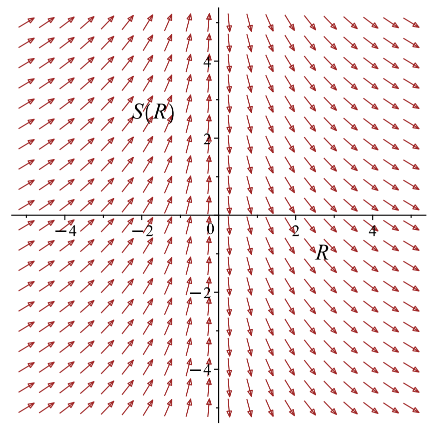
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$4\ln(y) + \ln(yx+2) = -3\ln(x) + c_1$$

Which simplifies to

$$4\ln(y) + \ln(yx+2) = -3\ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y(2xy+3)}{x(5xy+8)}$ 	$R = x$ $S = 4\ln(y) + \ln(xy + 2)$	$\frac{dS}{dR} = -\frac{3}{R}$ 

Summary

The solution(s) found are the following

$$4\ln(y) + \ln(yx + 2) = -3\ln(x) + c_1 \tag{1}$$

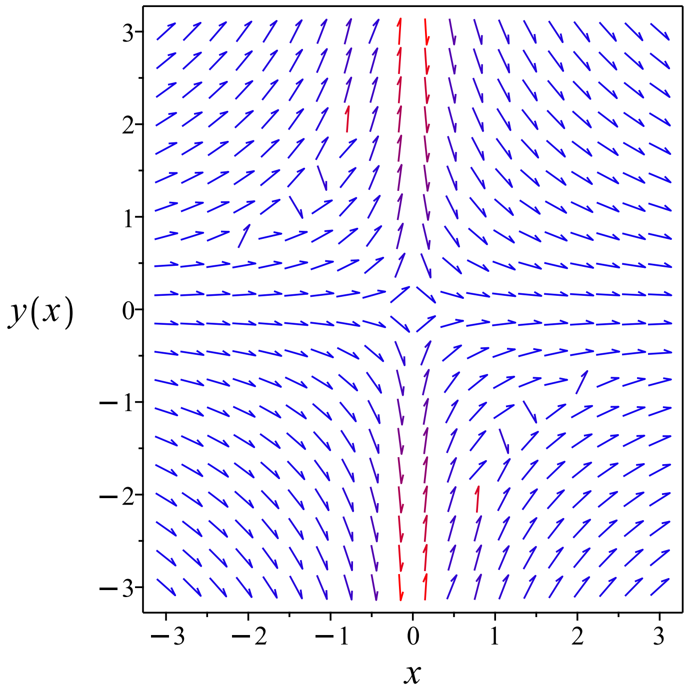


Figure 314: Slope field plot

Verification of solutions

$$4\ln(y) + \ln(yx + 2) = -3\ln(x) + c_1$$

Verified OK.

7.5.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(5x^2 y + 8x)\,\mathrm{d}y &= (-4x\,y^2 - 6y)\,\mathrm{d}x\\ (4x\,y^2 + 6y)\,\mathrm{d}x + (5x^2 y + 8x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 4x\,y^2 + 6y$$
$$N(x,y) = 5x^2 y + 8x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(4x\,y^2 + 6y)\\ &= 8xy + 6\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(5x^2 y + 8x)\\ &= 10xy + 8\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{5x^2y+8x}((8xy+6)-(10xy+8)) \\ &= \frac{-2xy-2}{5x^2y+8x} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{4x\,y^2+6y}((10xy+8)-(8xy+6)) \\ &= \frac{xy+1}{2x\,y^2+3y} \end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(10xy+8)-(8xy+6)}{x\,(4x\,y^2+6y) - y\,(5x^2y+8x)} \\ &= \frac{-2xy-2}{xy\,(xy+2)} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{-2t-2}{t\,(t+2)}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int \left(\frac{-2t-2}{t(t+2)}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(t(t+2))}\\ &= \frac{1}{t\,(t+2)}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{xy\,(xy+2)}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{xy\,(xy+2)}\left(4x\,y^2+6y\right)\\ &= \frac{4xy+6}{(xy+2)\,x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{xy\,(xy+2)}\left(5x^2y+8x\right)\\ &= \frac{5xy+8}{y\,(xy+2)}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{4xy+6}{(xy+2)\,x}\right)+\left(\frac{5xy+8}{y\,(xy+2)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{4xy + 6}{(xy + 2)\, x} \mathrm{d}x$$
$$\phi = \ln(xy + 2) + 3\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{xy + 2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{5xy+8}{y(xy+2)}$. Therefore equation (4) becomes

$$\frac{5xy + 8}{y\,(xy + 2)} = \frac{x}{xy + 2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{4}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{4}{y}\right) \mathrm{d}y$$
$$f(y) = 4\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(xy + 2) + 3\ln(x) + 4\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(xy + 2) + 3\ln(x) + 4\ln(y)$$

Summary

The solution(s) found are the following

$$4\ln(y) + \ln(yx + 2) + 3\ln(x) = c_1 \tag{1}$$

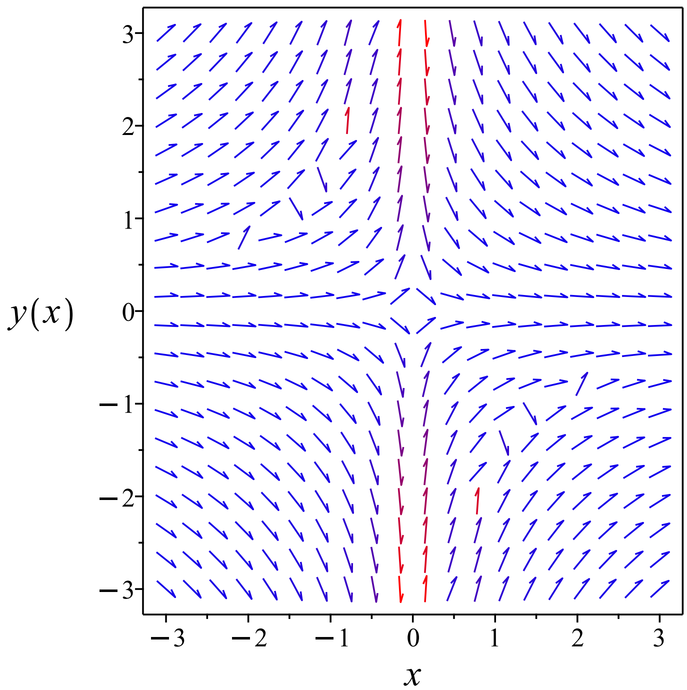


Figure 315: Slope field plot

Verification of solutions

$$4\ln(y) + \ln(yx + 2) + 3\ln(x) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 23

```
dsolve((4*x*y(x)^2+6*y(x))+(5*x^2*y(x)+8*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\operatorname{RootOf}\left(-\ln\left(x\right) + c_1 + \ln\left(\_Z + 2\right) + 4\ln\left(\_Z\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 1.767 (sec). Leaf size: 156

```
DSolve[(4*x*y[x]^2+6*y[x])+(5*x^2*y[x]+8*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \operatorname{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 1\right]$$
$$y(x) \to \operatorname{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 2\right]$$
$$y(x) \to \operatorname{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 3\right]$$
$$y(x) \to \operatorname{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 4\right]$$
$$y(x) \to \operatorname{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 5\right]$$

7.6 problem 6

Internal problem ID [11704]
Internal file name [`OUTPUT/11713_Thursday_April_11_2024_08_48_41_PM_99209226/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{8y^3x^2 - 2y^4 + \left(5y^2x^3 - 8y^3x\right)y' = 0}$$

7.6.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2y(-4x^2 + y)}{x\,(-5x^2 + 8y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{2y(-4x^2+y)\,(b_3-a_2)}{x\,(-5x^2+8y)} - \frac{4y^2(-4x^2+y)^2\,a_3}{x^2\,(-5x^2+8y)^2} \\ &- \left(\frac{16y}{-5x^2+8y} + \frac{2y(-4x^2+y)}{x^2\,(-5x^2+8y)} - \frac{20y(-4x^2+y)}{(-5x^2+8y)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{2(-4x^2+y)}{x\,(-5x^2+8y)} - \frac{2y}{x\,(-5x^2+8y)} \right. \\ &\left. + \frac{16y(-4x^2+y)}{x\,(-5x^2+8y)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{65x^6b_2 - 104x^4y^2a_3 + 40x^5b_1 - 40x^4ya_1 - 100x^4yb_2 - 108x^3y^2a_2 + 54x^3y^2b_3 - 2x^2y^3a_3 - 20x^3yb_1 - 34x}{(5x^2-8y)^2\,x^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &65x^6b_2 - 104x^4y^2a_3 + 40x^5b_1 - 40x^4ya_1 - 100x^4yb_2 - 108x^3y^2a_2 + 54x^3y^2b_3 \\ &- 2x^2y^3a_3 - 20x^3yb_1 - 34x^2y^2a_1 + 80x^2y^2b_2 - 20y^4a_3 + 16x\,y^2b_1 - 16y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-104a_3v_1^4v_2^2 + 65b_2v_1^6 - 40a_1v_1^4v_2 - 108a_2v_1^3v_2^2 - 2a_3v_1^2v_2^3 + 40b_1v_1^5 - 100b_2v_1^4v_2 \\ &+ 54b_3v_1^3v_2^2 - 34a_1v_1^2v_2^2 - 20a_3v_2^4 - 20b_1v_1^3v_2 + 80b_2v_1^2v_2^2 - 16a_1v_2^3 + 16b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$65b_2v_1^6 + 40b_1v_1^5 - 104a_3v_1^4v_2^2 + (-40a_1 - 100b_2)\, v_1^4v_2 + (-108a_2 + 54b_3)\, v_1^3v_2^2 \\ - 20b_1v_1^3v_2 - 2a_3v_1^2v_2^3 + (-34a_1 + 80b_2)\, v_1^2v_2^2 + 16b_1v_1v_2^2 - 20a_3v_2^4 - 16a_1v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-16a_1 &= 0 \\
-104a_3 &= 0 \\
-20a_3 &= 0 \\
-2a_3 &= 0 \\
-20b_1 &= 0 \\
16b_1 &= 0 \\
40b_1 &= 0 \\
65b_2 &= 0 \\
-40a_1 - 100b_2 &= 0 \\
-34a_1 + 80b_2 &= 0 \\
-108a_2 + 54b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= 2y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 2y - \left(-\frac{2y(-4x^2+y)}{x\,(-5x^2+8y)}\right)(x) \\
&= \frac{18x^2y - 18y^2}{5x^2 - 8y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{18x^2y-18y^2}{5x^2-8y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(-x^2+y)}{6} + \frac{5\ln(y)}{18}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y(-4x^2+y)}{x\,(-5x^2+8y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x}{3x^2 - 3y} \\
S_y &= -\frac{1}{6x^2 - 6y} + \frac{5}{18y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{9x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{9R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln(R)}{9} + c_1 \tag{4}
$$

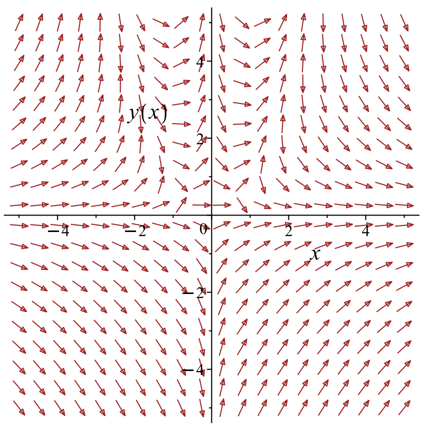
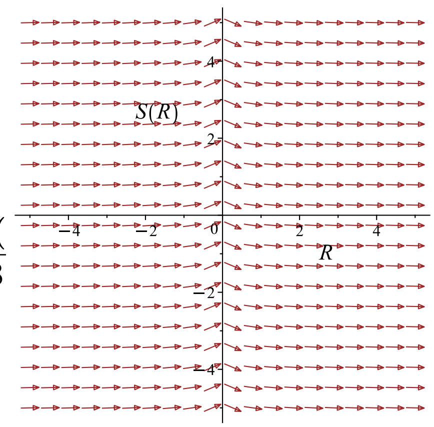
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(-x^2 + y)}{6} + \frac{5\ln(y)}{18} = -\frac{\ln(x)}{9} + c_1
$$

Which simplifies to

$$
\frac{\ln(-x^2 + y)}{6} + \frac{5\ln(y)}{18} = -\frac{\ln(x)}{9} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y(-4x^2+y)}{x(-5x^2+8y)}$ 	$R = x$ $S = \frac{\ln(-x^2+y)}{6} + \frac{5\ln(}{18}$	$\frac{dS}{dR} = -\frac{1}{9R}$ 

Summary

The solution(s) found are the following

$$\frac{\ln(-x^2+y)}{6} + \frac{5\ln(y)}{18} = -\frac{\ln(x)}{9} + c_1 \tag{1}$$

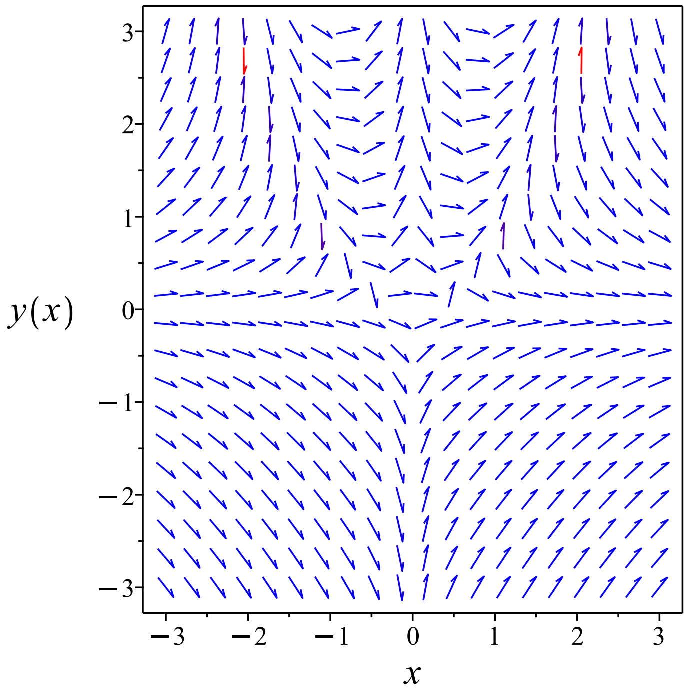


Figure 316: Slope field plot

Verification of solutions

$$\frac{\ln(-x^2+y)}{6}+\frac{5\ln(y)}{18}=-\frac{\ln(x)}{9}+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.703 (sec). Leaf size: 34

```
dsolve((8*x^2*y(x)^3-2*y(x)^4)+(5*x^3*y(x)^2-8*x*y(x)^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \text{RootOf}\left(x^6\_Z^{48} - x^6\_Z^{30} - c_1\right)^{18} x^2$$

✓ Solution by Mathematica

Time used: 3.924 (sec). Leaf size: 411

```
DSolve[(8*x^2*y[x]^3-2*y[x]^4)+(5*x^3*y[x]^2-8*x*y[x]^3)*y'[x]==0,y[x],x,IncludeSingularSolu
```

$$y(x) \to 0$$
$$y(x) \to \text{Root}\left[-\#1^8 + 3\#1^7x^2 - 3\#1^6x^4 + \#1^5x^6 + \frac{e^{18c_1}}{x^2}\&, 1\right]$$
$$y(x) \to \text{Root}\left[-\#1^8 + 3\#1^7x^2 - 3\#1^6x^4 + \#1^5x^6 + \frac{e^{18c_1}}{x^2}\&, 2\right]$$
$$y(x) \to \text{Root}\left[-\#1^8 + 3\#1^7x^2 - 3\#1^6x^4 + \#1^5x^6 + \frac{e^{18c_1}}{x^2}\&, 3\right]$$
$$y(x) \to \text{Root}\left[-\#1^8 + 3\#1^7x^2 - 3\#1^6x^4 + \#1^5x^6 + \frac{e^{18c_1}}{x^2}\&, 4\right]$$
$$y(x) \to \text{Root}\left[-\#1^8 + 3\#1^7x^2 - 3\#1^6x^4 + \#1^5x^6 + \frac{e^{18c_1}}{x^2}\&, 5\right]$$
$$y(x) \to \text{Root}\left[-\#1^8 + 3\#1^7x^2 - 3\#1^6x^4 + \#1^5x^6 + \frac{e^{18c_1}}{x^2}\&, 6\right]$$
$$y(x) \to \text{Root}\left[-\#1^8 + 3\#1^7x^2 - 3\#1^6x^4 + \#1^5x^6 + \frac{e^{18c_1}}{x^2}\&, 7\right]$$
$$y(x) \to \text{Root}\left[-\#1^8 + 3\#1^7x^2 - 3\#1^6x^4 + \#1^5x^6 + \frac{e^{18c_1}}{x^2}\&, 8\right]$$
$$y(x) \to 0$$

7.7 problem 7

Internal problem ID [11705]
Internal file name [`OUTPUT/11714_Thursday_April_11_2024_08_48_42_PM_29529941/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{2y + (2x + y + 1)\, y' = -5x - 1}$$

7.7.1 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-5x - 2y - 1}{2x + y + 1} \tag{1}$$

Which becomes

$$(y + 1)\, dy = (-2x)\, dy + (-5x - 2y - 1)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-2x)\, dy + (-5x - 2y - 1)\, dx = d\left(-\frac{5}{2}x^2 - 2xy - x\right)$$

Hence (2) becomes

$$(y+1)\,dy = d\left(-\frac{5}{2}x^2 - 2xy - x\right)$$

Integrating both sides gives gives these solutions

$$y = -2x - 1 + \sqrt{-x^2 + 2c_1 + 2x + 1} + c_1$$
$$y = -2x - 1 - \sqrt{-x^2 + 2c_1 + 2x + 1} + c_1$$

Summary

The solution(s) found are the following

$$y = -2x - 1 + \sqrt{-x^2 + 2c_1 + 2x + 1} + c_1 \tag{1}$$
$$y = -2x - 1 - \sqrt{-x^2 + 2c_1 + 2x + 1} + c_1 \tag{2}$$

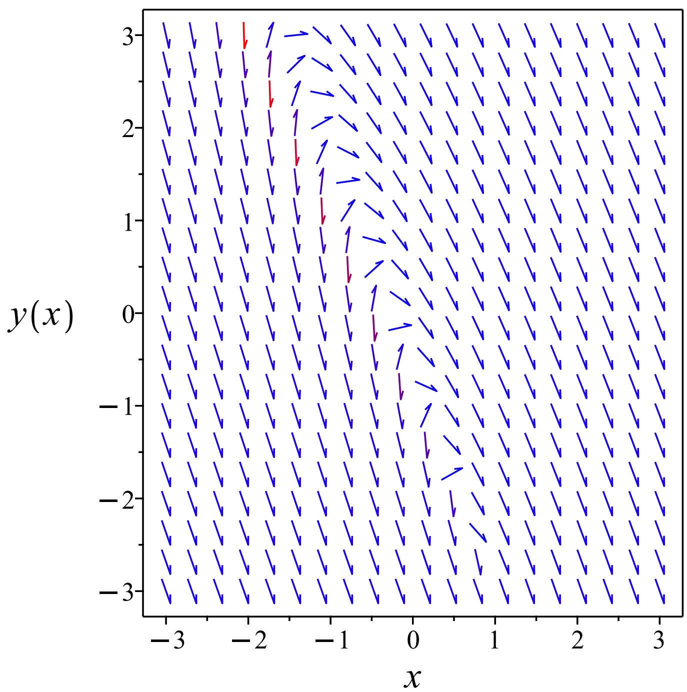


Figure 317: Slope field plot

Verification of solutions

$$y = -2x - 1 + \sqrt{-x^2 + 2c_1 + 2x + 1} + c_1$$

Verified OK.

$$y = -2x - 1 - \sqrt{-x^2 + 2c_1 + 2x + 1} + c_1$$

Verified OK.

7.7.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{5X + 5x_0 + 2Y(X) + 2y_0 + 1}{2X + 2x_0 + Y(X) + y_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned} x_0 &= 1 \\ y_0 &= -3 \end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{5X + 2Y(X)}{2X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{5X + 2Y}{2X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -5X - 2Y$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dX}X + u &= \frac{-2u - 5}{u + 2} \\ \frac{du}{dX} &= \frac{\frac{-2u(X)-5}{u(X)+2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-2u(X)-5}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) Xu(X) + 2\left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 4u(X) + 5 = 0$$

Or

$$X(u(X) + 2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 4u(X) + 5 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X)g(u) \\ &= -\frac{u^2 + 4u + 5}{X(u + 2)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+4u+5}{u+2}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+4u+5}{u+2}}\, du = -\frac{1}{X}\, dX$$

$$\int \frac{1}{\frac{u^2+4u+5}{u+2}}\, du = \int -\frac{1}{X}\, dX$$

$$\frac{\ln(u^2 + 4u + 5)}{2} = -\ln(X) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2 + 4u + 5} = e^{-\ln(X)+c_2}$$

Which simplifies to

$$\sqrt{u^2 + 4u + 5} = \frac{c_3}{X}$$

Which simplifies to

$$\sqrt{u(X)^2 + 4u(X) + 5} = \frac{c_3 e^{c_2}}{X}$$

The solution is

$$\sqrt{u(X)^2 + 4u(X) + 5} = \frac{c_3 e^{c_2}}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\sqrt{\frac{Y(X)^2}{X^2}+\frac{4Y(X)}{X}+5} = \frac{c_3 e^{c_2}}{X}$$

Using the solution for $Y(X)$

$$\sqrt{\frac{Y(X)^2+4Y(X)X+5X^2}{X^2}} = \frac{c_3 e^{c_2}}{X}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 3$$
$$X = 1 + x$$

Then the solution in y becomes

$$\sqrt{\frac{(y+3)^2+4(y+3)(x-1)+5(x-1)^2}{(x-1)^2}} = \frac{c_3 e^{c_2}}{x-1}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{(y+3)^2+4(y+3)(x-1)+5(x-1)^2}{(x-1)^2}} = \frac{c_3 e^{c_2}}{x-1} \tag{1}$$

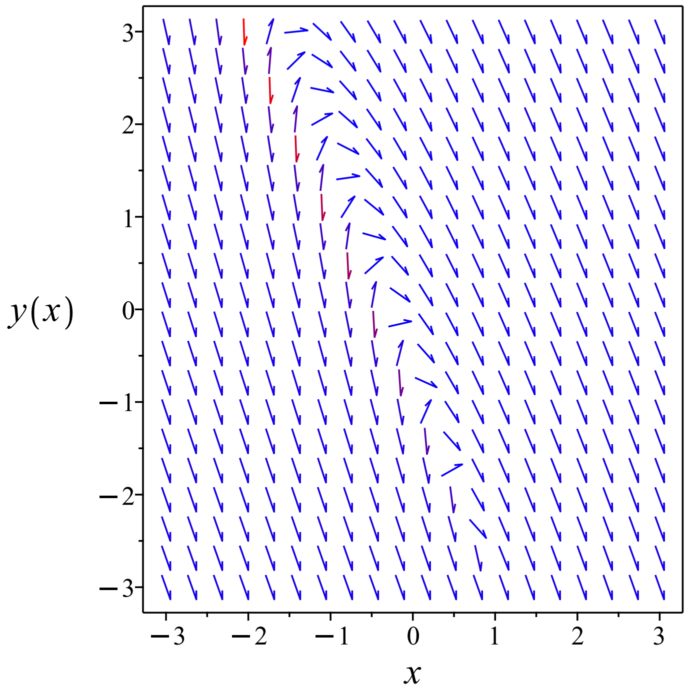


Figure 318: Slope field plot

Verification of solutions

$$\sqrt{\frac{(y+3)^2+4(y+3)(x-1)+5(x-1)^2}{(x-1)^2}} = \frac{c_3 e^{c_2}}{x-1}$$

Verified OK.

7.7.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{5x+2y+1}{2x+y+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(5x+2y+1)(b_3-a_2)}{2x+y+1} - \frac{(5x+2y+1)^2 a_3}{(2x+y+1)^2} - \left(-\frac{5}{2x+y+1} + \frac{10x+4y+2}{(2x+y+1)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{2}{2x+y+1} + \frac{5x+2y+1}{(2x+y+1)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{10x^2a_2 - 25x^2a_3 + 3x^2b_2 - 10x^2b_3 + 10xya_2 - 20xya_3 + 4xyb_2 - 10xyb_3 + 2y^2a_2 - 3y^2a_3 + y^2b_2 - 2y^2b_3}{(2x+y} = 0$$

Setting the numerator to zero gives

$$10x^2a_2 - 25x^2a_3 + 3x^2b_2 - 10x^2b_3 + 10xya_2 - 20xya_3 + 4xyb_2 - 10xyb_3 + 2y^2a_2 - 3y^2a_3 + y^2b_2 - 2y^2b_3 + 10xa_2 - 10xa_3 - xb_1 + 5xb_2 - 7xb_3 + ya_1 + 3ya_2 - ya_3 + 2yb_2 - 2yb_3 + 3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$10a_2v_1^2 + 10a_2v_1v_2 + 2a_2v_2^2 - 25a_3v_1^2 - 20a_3v_1v_2 - 3a_3v_2^2 + 3b_2v_1^2 + 4b_2v_1v_2 + b_2v_2^2 - 10b_3v_1^2 - 10b_3v_1v_2 - 2b_3v_2^2 + a_1v_2 + 10a_2v_1 + 3a_2v_2 - 10a_3v_1 - a_3v_2 - b_1v_1 + 5b_2v_1 + 2b_2v_2 - 7b_3v_1 - 2b_3v_2 + 3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(10a_2 - 25a_3 + 3b_2 - 10b_3)\, v_1^2 + (10a_2 - 20a_3 + 4b_2 - 10b_3)\, v_1 v_2 \\ &+ (10a_2 - 10a_3 - b_1 + 5b_2 - 7b_3)\, v_1 + (2a_2 - 3a_3 + b_2 - 2b_3)\, v_2^2 \\ &+ (a_1 + 3a_2 - a_3 + 2b_2 - 2b_3)\, v_2 + 3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}2a_2 - 3a_3 + b_2 - 2b_3 &= 0\\ 10a_2 - 25a_3 + 3b_2 - 10b_3 &= 0\\ 10a_2 - 20a_3 + 4b_2 - 10b_3 &= 0\\ a_1 + 3a_2 - a_3 + 2b_2 - 2b_3 &= 0\\ 10a_2 - 10a_3 - b_1 + 5b_2 - 7b_3 &= 0\\ 3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -a_3 - b_3\\ a_2 &= 4a_3 + b_3\\ a_3 &= a_3\\ b_1 &= 5a_3 + 3b_3\\ b_2 &= -5a_3\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x - 1\\ \eta &= 3 + y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi\\ &= 3 + y - \left(-\frac{5x + 2y + 1}{2x + y + 1}\right)(x - 1)\\ &= \frac{5x^2 + 4xy + y^2 + 2x + 2y + 2}{2x + y + 1}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{5x^2+4xy+y^2+2x+2y+2}{2x+y+1}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln\left(5x^2 + 4xy + y^2 + 2x + 2y + 2\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{5x + 2y + 1}{2x + y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{5x + 2y + 1}{5x^2 + (4y + 2)x + y^2 + 2y + 2} \\ S_y &= \frac{2x + y + 1}{y^2 + (4x + 2)y + 5x^2 + 2x + 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

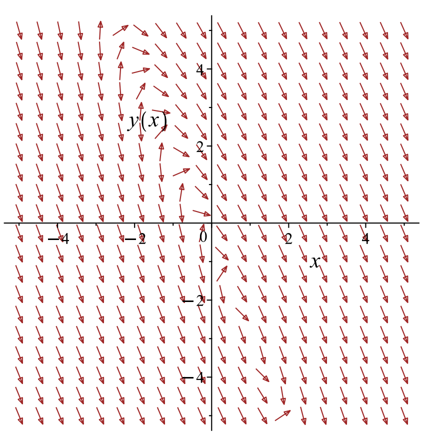
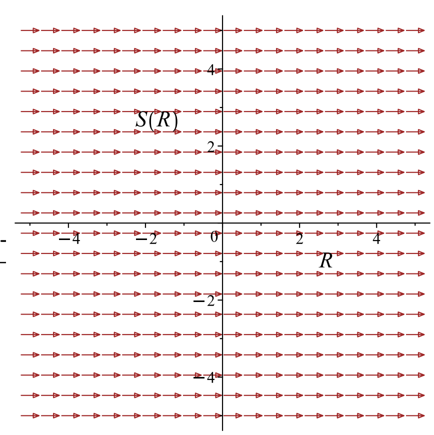
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(5x^2 + (4y+2)x + y^2 + 2y + 2)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(5x^2 + (4y+2)x + y^2 + 2y + 2)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{5x+2y+1}{2x+y+1}$ 	$R = x$ $S = \frac{\ln(5x^2 + (4y+2)x -}{2}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln \left(5x^2 + (4y+2)\,x + y^2 + 2y + 2\right)}{2} = c_1 \tag{1}$$

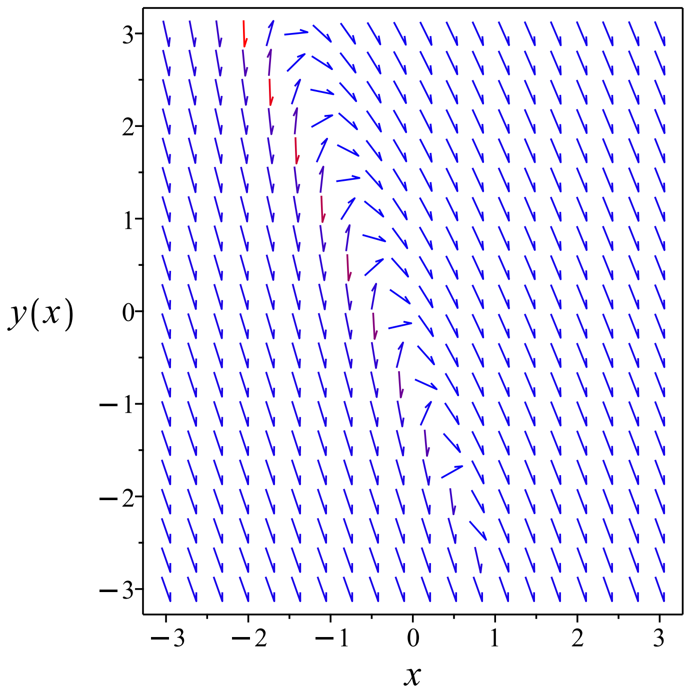


Figure 319: Slope field plot

Verification of solutions

$$\frac{\ln \left(5x^2 + (4y+2)\,x + y^2 + 2y + 2\right)}{2} = c_1$$

Verified OK.

7.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x + y + 1)\,\mathrm{d}y &= (-5x - 2y - 1)\,\mathrm{d}x\\ (5x + 2y + 1)\,\mathrm{d}x + (2x + y + 1)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 5x + 2y + 1\\ N(x,y) &= 2x + y + 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(5x + 2y + 1)\\ &= 2\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x + y + 1) \\
&= 2
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 5x + 2y + 1 \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x(5x + 4y + 2)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 2x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x + y + 1$. Therefore equation (4) becomes

$$
2x + y + 1 = 2x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y + 1
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int (y + 1) \, \mathrm{d}y \\
f(y) &= \frac{1}{2}y^2 + y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(5x + 4y + 2)}{2} + \frac{y^2}{2} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(5x + 4y + 2)}{2} + \frac{y^2}{2} + y$$

Summary

The solution(s) found are the following

$$\frac{x(5x + 4y + 2)}{2} + \frac{y^2}{2} + y = c_1 \tag{1}$$

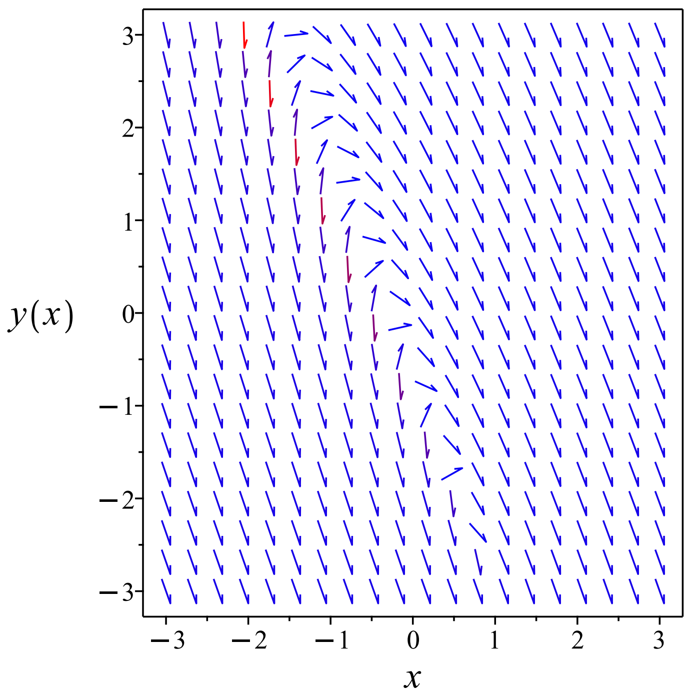


Figure 320: Slope field plot

Verification of solutions

$$\frac{x(5x + 4y + 2)}{2} + \frac{y^2}{2} + y = c_1$$

Verified OK.

7.7.5 Maple step by step solution

Let's solve

$2y + (2x + y + 1) y' = -5x - 1$

- Highest derivative means the order of the ODE is 1

y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

  ◦ Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  ◦ Evaluate derivatives

  $2 = 2$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

$\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

$F(x, y) = \int (5x + 2y + 1) dx + f_1(y)$

- Evaluate integral

$F(x, y) = \frac{5x^2}{2} + 2xy + x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

$N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

$2x + y + 1 = 2x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

$\frac{d}{dy} f_1(y) = y + 1$

- Solve for $f_1(y)$

$f_1(y) = \frac{1}{2} y^2 + y$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$F(x,y) = \frac{5}{2}x^2 + 2xy + x + \frac{1}{2}y^2 + y$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{5}{2}x^2 + 2xy + x + \frac{1}{2}y^2 + y = c_1$

- Solve for y

  $\left\{y = -2x - 1 - \sqrt{-x^2 + 2c_1 + 2x + 1}, y = -2x - 1 + \sqrt{-x^2 + 2c_1 + 2x + 1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.562 (sec). Leaf size: 32

```
dsolve((5*x+2*y(x)+1)+(2*x+y(x)+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-\sqrt{-\left(-1+x\right)^2 c_1^2 + 1} + \left(-2x - 1\right) c_1}{c_1}$$

✓ Solution by Mathematica

Time used: 0.134 (sec). Leaf size: 53

```
DSolve[(5*x+2*y[x]+1)+(2*x+y[x]+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x^2+2x+1+c_1}-2x-1$$
$$y(x) \to \sqrt{-x^2+2x+1+c_1}-2x-1$$

7.8 problem 8

Internal problem ID [11706]
Internal file name [`OUTPUT/11715_Thursday_April_11_2024_08_48_44_PM_93206069/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y-(6x-2y-3)\,y'=-3x-1}$$

7.8.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{-3x+y-1}{-6x+2y+3}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(-3x + y - 1)(b_3 - a_2)}{-6x + 2y + 3} - \frac{(-3x + y - 1)^2 a_3}{(-6x + 2y + 3)^2} - \left(-\frac{3}{-6x + 2y + 3} + \frac{-18x + 6y - 6}{(-6x + 2y + 3)^2}\right)(xa_2 + ya_3 + a_1) - \left(\frac{1}{-6x + 2y + 3} - \frac{2(-3x + y - 1)}{(-6x + 2y + 3)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{18x^2a_2 + 9x^2a_3 - 36x^2b_2 - 18x^2b_3 - 12xya_2 - 6xya_3 + 24xyb_2 + 12xyb_3 + 2y^2a_2 + y^2a_3 - 4y^2b_2 - 2y^2b}{(6x - 2y}$$
$$= 0$$

Setting the numerator to zero gives

$$-18x^2a_2 - 9x^2a_3 + 36x^2b_2 + 18x^2b_3 + 12xya_2 + 6xya_3 - 24xyb_2 - 12xyb_3 - 2y^2a_2 - y^2a_3 + 4y^2b_2 + 2y^2b_3 + 18xa_2 - 6xa_3 - 41xb_2 - 3xb_3 - ya_2 + 17ya_3 + 12yb_2 - 4yb_3 + 15a_1 + 3a_2 - a_3 - 5b_1 + 9b_2 - 3b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-18a_2v_1^2 + 12a_2v_1v_2 - 2a_2v_2^2 - 9a_3v_1^2 + 6a_3v_1v_2 - a_3v_2^2 + 36b_2v_1^2 - 24b_2v_1v_2 + 4b_2v_2^2 + 18b_3v_1^2 - 12b_3v_1v_2 + 2b_3v_2^2 + 18a_2v_1 - a_2v_2 - 6a_3v_1 + 17a_3v_2 - 41b_2v_1 + 12b_2v_2 - 3b_3v_1 - 4b_3v_2 + 15a_1 + 3a_2 - a_3 - 5b_1 + 9b_2 - 3b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(-18a_2 - 9a_3 + 36b_2 + 18b_3)\, v_1^2 + (12a_2 + 6a_3 - 24b_2 - 12b_3)\, v_1 v_2 \\ &+ (18a_2 - 6a_3 - 41b_2 - 3b_3)\, v_1 + (-2a_2 - a_3 + 4b_2 + 2b_3)\, v_2^2 \\ &+ (-a_2 + 17a_3 + 12b_2 - 4b_3)\, v_2 + 15a_1 + 3a_2 - a_3 - 5b_1 + 9b_2 - 3b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-18a_2 - 9a_3 + 36b_2 + 18b_3 &= 0\\ -2a_2 - a_3 + 4b_2 + 2b_3 &= 0\\ -a_2 + 17a_3 + 12b_2 - 4b_3 &= 0\\ 12a_2 + 6a_3 - 24b_2 - 12b_3 &= 0\\ 18a_2 - 6a_3 - 41b_2 - 3b_3 &= 0\\ 15a_1 + 3a_2 - a_3 - 5b_1 + 9b_2 - 3b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= a_1\\ a_2 &= 2b_2\\ a_3 &= -\frac{2b_2}{3}\\ b_1 &= 3a_1 + \frac{10b_2}{3}\\ b_2 &= b_2\\ b_3 &= -\frac{b_2}{3}\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 1\\ \eta &= 3\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= 3 - \left(\frac{-3x+y-1}{-6x+2y+3}\right)(1) \\ &= \frac{15x-5y-10}{6x-2y-3} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{15x-5y-10}{6x-2y-3}} dy\end{aligned}$$

Which results in

$$S = \frac{2y}{5} - \frac{\ln(-3x+y+2)}{5}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-3x+y-1}{-6x+2y+3}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= -\frac{3}{15x-5y-10}\\
S_y &= \frac{2}{5}+\frac{1}{15x-5y-10}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{5} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{5}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R}{5} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{2y}{5} - \frac{\ln(-3x+y+2)}{5} = \frac{x}{5} + c_1
$$

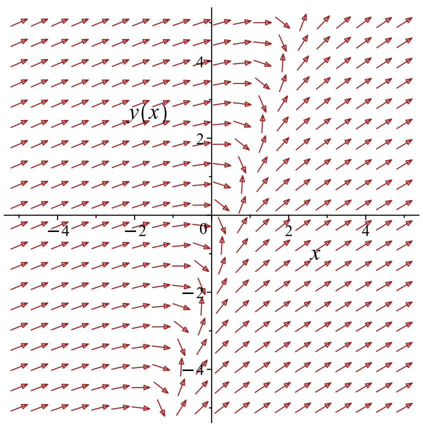
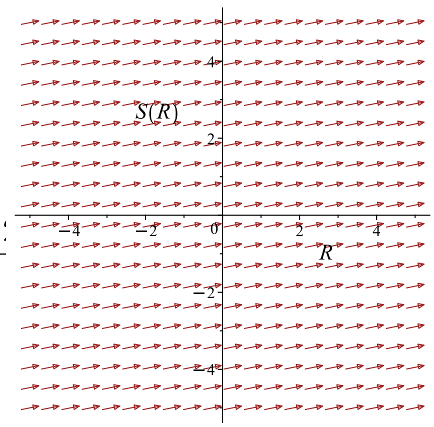
Which simplifies to

$$
\frac{2y}{5} - \frac{\ln(-3x+y+2)}{5} = \frac{x}{5} + c_1
$$

Which gives

$$
y = -\frac{\operatorname{LambertW}\left(-2\,\mathrm{e}^{5x-4-5c_1}\right)}{2} + 3x - 2
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-3x+y-1}{-6x+2y+3}$ 	$R = x$ $S = \frac{2y}{5} - \frac{\ln(-3x+y+}{5}$	$\frac{dS}{dR} = \frac{1}{5}$ 

Summary

The solution(s) found are the following

$$y = -\frac{\text{LambertW}\left(-2\,e^{5x-4-5c_1}\right)}{2} + 3x - 2 \tag{1}$$

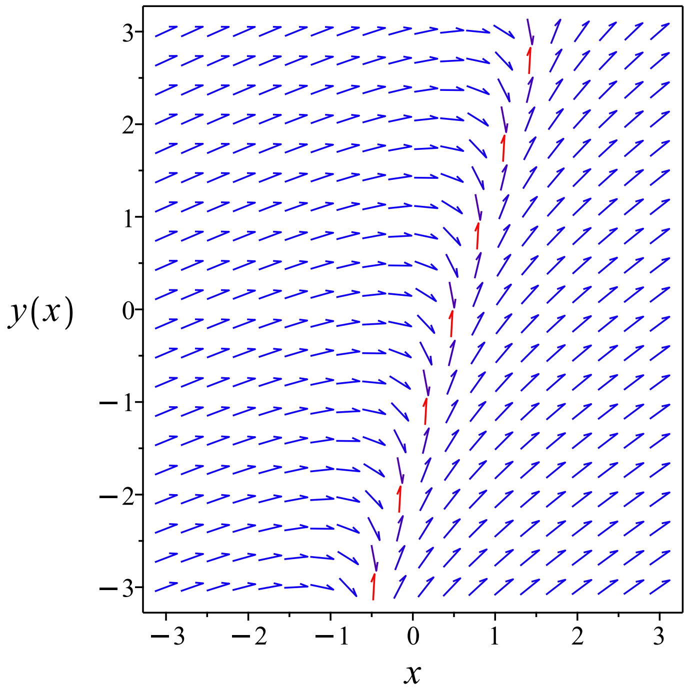


Figure 321: Slope field plot

Verification of solutions

$$y = -\frac{\text{LambertW}\left(-2\,e^{5x-4-5c_1}\right)}{2} + 3x - 2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = 3, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 23

```
dsolve((3*x-y(x)+1)-(6*x-2*y(x)-3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\mathrm{LambertW}\left(-2\,\mathrm{e}^{5x-4-5c_1}\right)}{2} + 3x - 2$$

✓ Solution by Mathematica

Time used: 3.097 (sec). Leaf size: 35

```
DSolve[(3*x-y[x]+1)-(6*x-2*y[x]-3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}W\left(-e^{5x-1+c_1}\right) + 3x - 2$$
$$y(x) \to 3x - 2$$

7.9 problem 9

Internal problem ID [11707]
Internal file name [`OUTPUT/11716_Thursday_April_11_2024_08_48_45_PM_49666120/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-2y + (2x + y - 1)\, y' = -x + 3}$$

7.9.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{-X - x_0 + 2Y(X) + 2y_0 + 3}{2X + 2x_0 + Y(X) + y_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = -1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-X + 2Y(X)}{2X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{-X+2Y}{2X+Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -X+2Y$ and $N = 2X+Y$ are both homogeneous and of the same order $n=1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{2u-1}{u+2} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{2u(X)-1}{u(X)+2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{2u(X)-1}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) + 2\left(\frac{d}{dX}u(X)\right)X + u(X)^2 + 1 = 0$$

Or

$$X(u(X)+2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{u^2+1}{X(u+2)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+1}{u+2}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u+2}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+1}{u+2}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(u^2+1)}{2} + 2\arctan(u) = -\ln(X) + c_2$$

The solution is

$$\frac{\ln\left(u(X)^2+1\right)}{2} + 2\arctan(u(X)) + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} + 2\arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} + 2\arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 1$$
$$X = 1 + x$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{(y+1)^2}{(x-1)^2}+1\right)}{2} + 2\arctan\left(\frac{y+1}{x-1}\right) + \ln(x-1) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{(y+1)^2}{(x-1)^2}+1\right)}{2}+2\arctan\left(\frac{y+1}{x-1}\right)+\ln(x-1)-c_2=0 \tag{1}$$

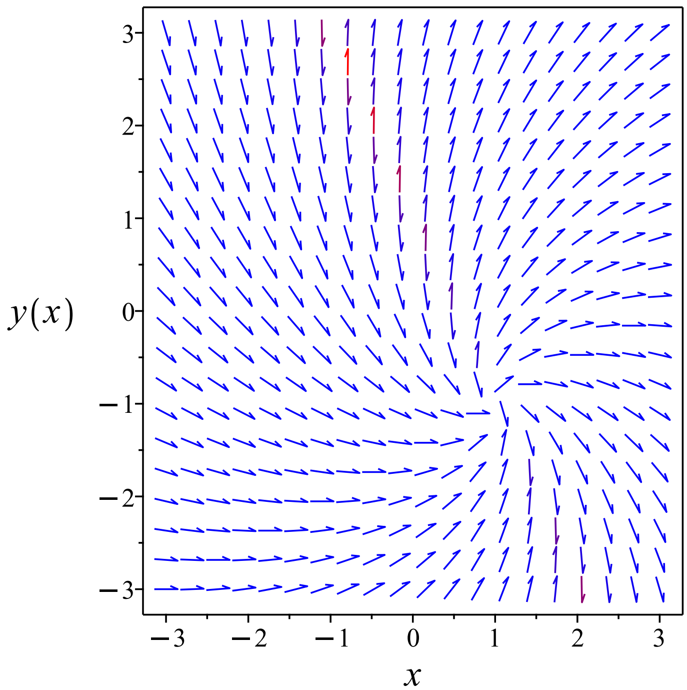


Figure 322: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{(y+1)^2}{(x-1)^2}+1\right)}{2}+2\arctan\left(\frac{y+1}{x-1}\right)+\ln(x-1)-c_2=0$$

Verified OK.

7.9.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{-x+2y+3}{2x+y-1} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(-x+2y+3)(b_3-a_2)}{2x+y-1} - \frac{(-x+2y+3)^2 a_3}{(2x+y-1)^2} \\ & - \left(-\frac{1}{2x+y-1} - \frac{2(-x+2y+3)}{(2x+y-1)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(\frac{2}{2x+y-1} - \frac{-x+2y+3}{(2x+y-1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2a_2 - x^2a_3 - x^2b_2 - 2x^2b_3 + 2xya_2 + 4xya_3 + 4xyb_2 - 2xyb_3 - 2y^2a_2 + y^2a_3 + y^2b_2 + 2y^2b_3 - 2xa_2 + 6}{(2x + y - 1} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 2x^2a_2 - x^2a_3 - x^2b_2 - 2x^2b_3 + 2xya_2 + 4xya_3 + 4xyb_2 - 2xyb_3 - 2y^2a_2 \\ & + y^2a_3 + y^2b_2 + 2y^2b_3 - 2xa_2 + 6xa_3 - 5xb_1 + xb_2 + 7xb_3 + 5ya_1 \\ & - ya_2 - 7ya_3 - 2yb_2 + 6yb_3 + 5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&2a_2v_1^2 + 2a_2v_1v_2 - 2a_2v_2^2 - a_3v_1^2 + 4a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 + 4b_2v_1v_2 + b_2v_2^2 \\
&- 2b_3v_1^2 - 2b_3v_1v_2 + 2b_3v_2^2 + 5a_1v_2 - 2a_2v_1 - a_2v_2 + 6a_3v_1 - 7a_3v_2 - 5b_1v_1 \\
&+ b_2v_1 - 2b_2v_2 + 7b_3v_1 + 6b_3v_2 + 5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(2a_2 - a_3 - b_2 - 2b_3)\, v_1^2 + (2a_2 + 4a_3 + 4b_2 - 2b_3)\, v_1v_2 \\
&+ (-2a_2 + 6a_3 - 5b_1 + b_2 + 7b_3)\, v_1 + (-2a_2 + a_3 + b_2 + 2b_3)\, v_2^2 \\
&+ (5a_1 - a_2 - 7a_3 - 2b_2 + 6b_3)\, v_2 + 5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2a_2 + a_3 + b_2 + 2b_3 &= 0 \\
2a_2 - a_3 - b_2 - 2b_3 &= 0 \\
2a_2 + 4a_3 + 4b_2 - 2b_3 &= 0 \\
5a_1 - a_2 - 7a_3 - 2b_2 + 6b_3 &= 0 \\
-2a_2 + 6a_3 - 5b_1 + b_2 + 7b_3 &= 0 \\
5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -b_2 - b_3 \\
a_2 &= b_3 \\
a_3 &= -b_2 \\
b_1 &= -b_2 + b_3 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x - 1$$
$$\eta = y + 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y + 1 - \left(\frac{-x + 2y + 3}{2x + y - 1}\right)(x - 1) \\
&= \frac{x^2 + y^2 - 2x + 2y + 2}{2x + y - 1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2 + y^2 - 2x + 2y + 2}{2x + y - 1}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + y^2 - 2x + 2y + 2)}{2} + 2\arctan\left(\frac{2y + 2}{2x - 2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-x+2y+3}{2x+y-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x-2y-3}{x^2+y^2-2x+2y+2} \\
S_y &= \frac{2x+y-1}{x^2+y^2-2x+2y+2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

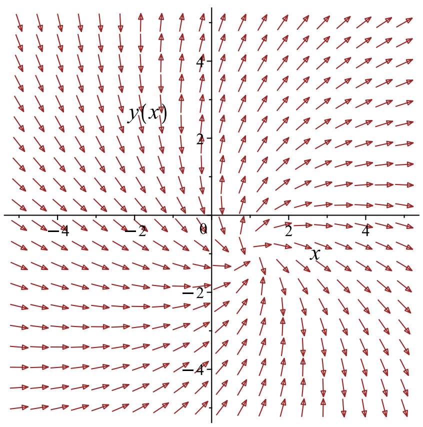
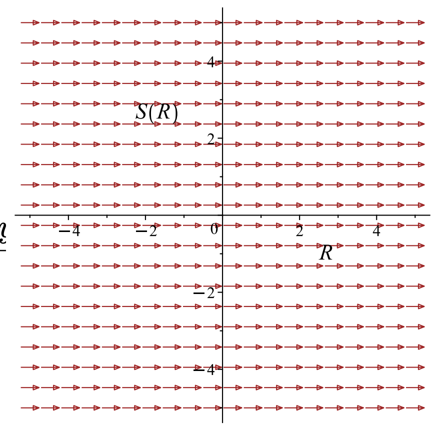
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2+x^2+2y-2x+2)}{2} + 2\arctan\left(\frac{y+1}{x-1}\right) = c_1$$

Which simplifies to

$$\frac{\ln(y^2+x^2+2y-2x+2)}{2} + 2\arctan\left(\frac{y+1}{x-1}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x+2y+3}{2x+y-1}$ 	$R = x$ $S = \frac{\ln(x^2+y^2-2x+2y}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y^2+x^2+2y-2x+2)}{2} + 2\arctan\left(\frac{y+1}{x-1}\right) = c_1 \tag{1}$$

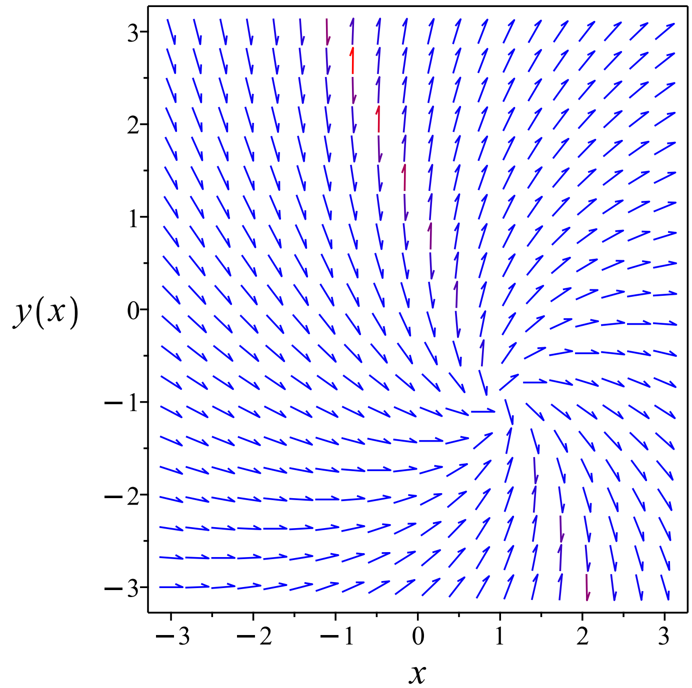


Figure 323: Slope field plot

Verification of solutions

$$\frac{\ln\left(y^2+x^2+2y-2x+2\right)}{2}+2\arctan\left(\frac{y+1}{x-1}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 31

```
dsolve((x-2*y(x)-3)+(2*x+y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -1 - \tan\left(\operatorname{RootOf}\left(-4\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(-1+x\right) + 2c_1\right)\right)\left(-1+x\right)$$

✓ Solution by Mathematica

Time used: 0.061 (sec). Leaf size: 66

```
DSolve[(x-2*y[x]-3)+(2*x+y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[32\arctan\left(\frac{2y(x)-x+3}{y(x)+2x-1}\right) + 8\log\left(\frac{x^2+y(x)^2+2y(x)-2x+2}{5(x-1)^2}\right) + 16\log(x-1) + 5c_1 = 0, y(x)\right]$$

7.10 problem 10

Internal problem ID [11708]
Internal file name [OUTPUT/11717_Thursday_April_11_2024_08_48_47_PM_9399003/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-4y-(x+5y+3)\,y'=-10x-12}$$

7.10.1 Solving as homogeneousTypeMapleC ode

Let $Y=y+y_0$ and $X=x+x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X)=-\frac{2(2Y(X)+2y_0-5X-5x_0-6)}{X+x_0+5Y(X)+5y_0+3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0=-\frac{4}{3}$$
$$y_0=-\frac{1}{3}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X)=-\frac{2(2Y(X)-5X)}{X+5Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= -\frac{2(2Y-5X)}{X+5Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -4Y + 10X$ and $N = X + 5Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-4u+10}{5u+1} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-4u(X)+10}{5u(X)+1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-4u(X)+10}{5u(X)+1} - u(X)}{X} = 0$$

Or

$$5\left(\frac{d}{dX}u(X)\right)Xu(X) + \left(\frac{d}{dX}u(X)\right)X + 5u(X)^2 + 5u(X) - 10 = 0$$

Or

$$-10 + (5u(X)+1)X\left(\frac{d}{dX}u(X)\right) + 5u(X)^2 + 5u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{5(u^2+u-2)}{(5u+1)X} \end{aligned}$$

Where $f(X) = -\frac{5}{X}$ and $g(u) = \frac{u^2+u-2}{5u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+u-2}{5u+1}}\,du = -\frac{5}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+u-2}{5u+1}}\,du = \int -\frac{5}{X}\,dX$$

$$2\ln(u-1) + 3\ln(u+2) = -5\ln(X) + c_2$$

Raising both side to exponential gives

$$e^{2\ln(u-1)+3\ln(u+2)} = e^{-5\ln(X)+c_2}$$

Which simplifies to

$$(u-1)^2(u+2)^3 = \frac{c_3}{X^5}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \operatorname{RootOf}\left(8X^5 - 4X^4\_Z - 10X^3\_Z^2 + X^2\_Z^3 + 4X\_Z^4 + \_Z^5 - c_3\right)$$

Using the solution for $Y(X)$

$$Y(X) = \operatorname{RootOf}\left(8X^5 - 4X^4\_Z - 10X^3\_Z^2 + X^2\_Z^3 + 4X\_Z^4 + \_Z^5 - c_3\right)$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - \frac{1}{3}$$
$$X = x - \frac{4}{3}$$

Then the solution in y becomes

$$y + \frac{1}{3} = \operatorname{RootOf}\left(243\_Z^5 + (972x + 1296)\_Z^4 + \left(243x^2 + 648x + 432\right)\_Z^3 + \left(-2430x^3 - 9720x^2 - 12\right.$$

Summary

The solution(s) found are the following

$$y+\frac{1}{3}=\operatorname{RootOf}\left(243\_Z^5+(972x+1296)\_Z^4+\left(243x^2+648x+432\right)\_Z^3+\left(-2430x^3-9720x^2-12960x-5760\right)\_Z^2+\left(-972x^4-5184x^3-10368x^2-9216x-3072\right)\_Z+1944x^5+12960x^4+34560x^3+46080x^2-243c_3+30720x+8192\right)$$

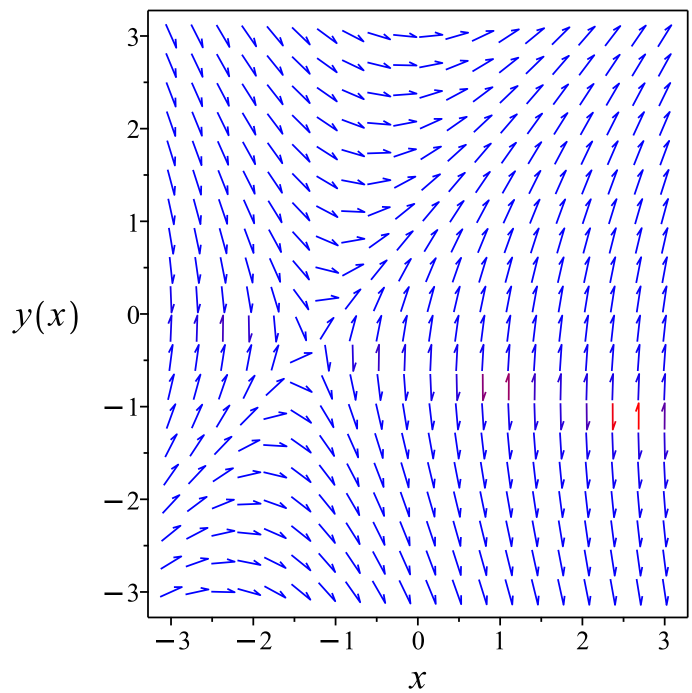


Figure 324: Slope field plot

Verification of solutions

$$y+\frac{1}{3}=\operatorname{RootOf}\left(243\_Z^5+(972x+1296)\_Z^4+\left(243x^2+648x+432\right)\_Z^3+\left(-2430x^3-9720x^2-12960x-5760\right)\_Z^2+\left(-972x^4-5184x^3-10368x^2-9216x-3072\right)\_Z+1944x^5+12960x^4+34560x^3+46080x^2-243c_3+30720x+8192\right)$$

Verified OK.

7.10.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2(-5x+2y-6)}{x+5y+3} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{2(-5x+2y-6)(b_3-a_2)}{x+5y+3} - \frac{4(-5x+2y-6)^2 a_3}{(x+5y+3)^2} \\
&- \left(\frac{10}{x+5y+3} + \frac{-10x+4y-12}{(x+5y+3)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(-\frac{4}{x+5y+3} + \frac{-50x+20y-60}{(x+5y+3)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{10x^2a_2 + 100x^2a_3 - 55x^2b_2 - 10x^2b_3 + 100xya_2 - 80xya_3 - 10xyb_2 - 100xyb_3 - 20y^2a_2 + 70y^2a_3 - 25}{}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-10x^2a_2 - 100x^2a_3 + 55x^2b_2 + 10x^2b_3 - 100xya_2 + 80xya_3 \\
&+ 10xyb_2 + 100xyb_3 + 20y^2a_2 - 70y^2a_3 + 25y^2b_2 - 20y^2b_3 - 60xa_2 \\
&- 240xa_3 + 54xb_1 + 78xb_2 + 42xb_3 - 54ya_1 - 48ya_2 + 78ya_3 \\
&+ 30yb_2 + 120yb_3 - 18a_1 - 36a_2 - 144a_3 + 72b_1 + 9b_2 + 36b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-10a_2v_1^2 - 100a_2v_1v_2 + 20a_2v_2^2 - 100a_3v_1^2 + 80a_3v_1v_2 - 70a_3v_2^2 \\ &+ 55b_2v_1^2 + 10b_2v_1v_2 + 25b_2v_2^2 + 10b_3v_1^2 + 100b_3v_1v_2 - 20b_3v_2^2 - 54a_1v_2 \\ &- 60a_2v_1 - 48a_2v_2 - 240a_3v_1 + 78a_3v_2 + 54b_1v_1 + 78b_2v_1 + 30b_2v_2 \\ &+ 42b_3v_1 + 120b_3v_2 - 18a_1 - 36a_2 - 144a_3 + 72b_1 + 9b_2 + 36b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-10a_2 - 100a_3 + 55b_2 + 10b_3)\, v_1^2 + (-100a_2 + 80a_3 + 10b_2 + 100b_3)\, v_1v_2 \\ &+ (-60a_2 - 240a_3 + 54b_1 + 78b_2 + 42b_3)\, v_1 + (20a_2 - 70a_3 + 25b_2 - 20b_3)\, v_2^2 \\ &+ (-54a_1 - 48a_2 + 78a_3 + 30b_2 + 120b_3)\, v_2 - 18a_1 \\ &- 36a_2 - 144a_3 + 72b_1 + 9b_2 + 36b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -100a_2 + 80a_3 + 10b_2 + 100b_3 &= 0 \\ -10a_2 - 100a_3 + 55b_2 + 10b_3 &= 0 \\ 20a_2 - 70a_3 + 25b_2 - 20b_3 &= 0 \\ -54a_1 - 48a_2 + 78a_3 + 30b_2 + 120b_3 &= 0 \\ -60a_2 - 240a_3 + 54b_1 + 78b_2 + 42b_3 &= 0 \\ -18a_1 - 36a_2 - 144a_3 + 72b_1 + 9b_2 + 36b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -9a_3 + 4b_1 \\
a_2 &= -7a_3 + 3b_1 \\
a_3 &= a_3 \\
b_1 &= b_1 \\
b_2 &= 2a_3 \\
b_3 &= -8a_3 + 3b_1
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 4 + 3x \\
\eta &= 3y + 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 3y + 1 - \left(-\frac{2(-5x + 2y - 6)}{x + 5y + 3}\right)(4 + 3x) \\
&= \frac{-30x^2 + 15xy + 15y^2 - 75x + 30y - 45}{x + 5y + 3} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-30x^2 + 15xy + 15y^2 - 75x + 30y - 45}{x + 5y + 3}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln(2x+y+3)}{5} + \frac{2\ln(y-1-x)}{15}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2(-5x+2y-6)}{x+5y+3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2}{10x+5y+15} + \frac{2}{15x-15y+15} \\
S_y &= \frac{-x-5y-3}{15(2x+y+3)(-y+1+x)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

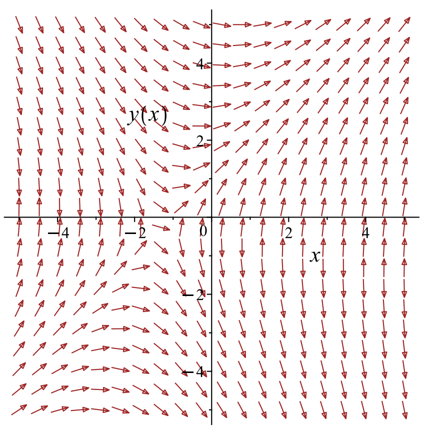
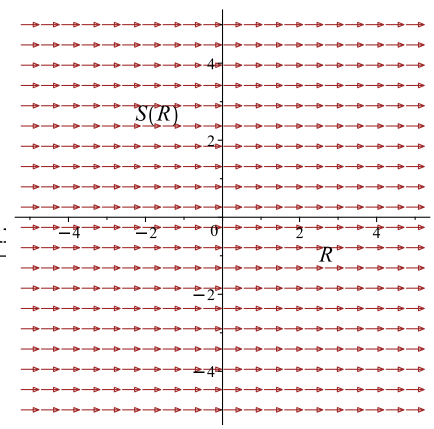
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(2x+y+3)}{5} + \frac{2\ln(y-1-x)}{15} = c_1$$

Which simplifies to

$$\frac{\ln(2x+y+3)}{5} + \frac{2\ln(y-1-x)}{15} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2(-5x+2y-6)}{x+5y+3}$ 	$R = x$ $S = \frac{\ln(2x+y+3)}{5} + 2l$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(2x+y+3)}{5} + \frac{2\ln(y-1-x)}{15} = c_1 \tag{1}$$

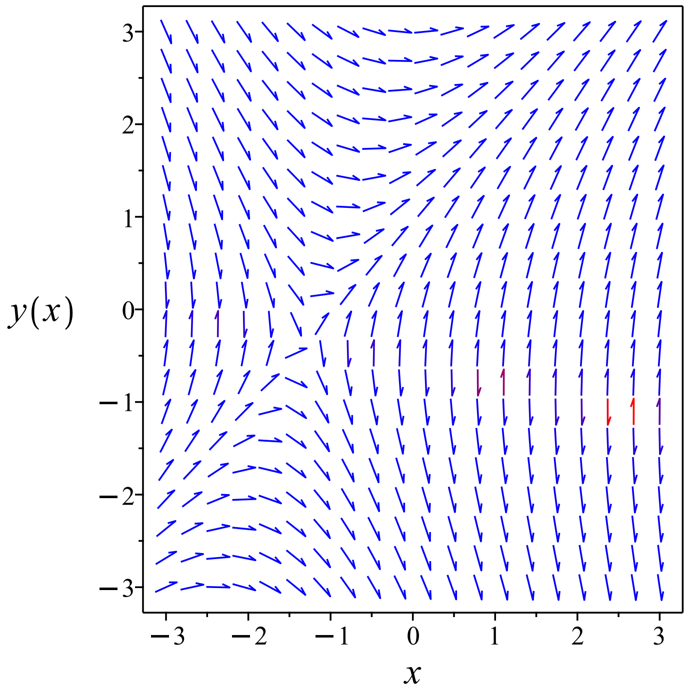


Figure 325: Slope field plot

Verification of solutions

$$\frac{\ln(2x+y+3)}{5}+\frac{2\ln(y-1-x)}{15}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.859 (sec). Leaf size: 129

```
dsolve((10*x-4*y(x)+12)-(x+5*y(x)+3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(-3x-4)\operatorname{RootOf}\left(-1+\left(243c_1x^5+1620x^4c_1+4320c_1x^3+5760c_1x^2+3840c_1x+1024c_1\right)\_Z^{25}+(14\ldots\right.}{\ldots}$$
$$-2x-3$$

✓ Solution by Mathematica

Time used: 60.443 (sec). Leaf size: 3061

```
DSolve[(10*x-4*y[x]+12)-(x+5*y[x]+3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

7.11 problem 11

Internal problem ID [11709]
Internal file name [`OUTPUT/11718_Thursday_April_11_2024_08_48_50_PM_63010928/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{4y + (4x + 2y + 2)\, y' = -6x - 1}$$

With initial conditions

$$\left[y\left(\frac{1}{2}\right) = 3\right]$$

7.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(x,y) \\
&= -\frac{6x+4y+1}{2\left(2x+y+1\right)}
\end{aligned}
$$

The x domain of $f(x,y)$ when $y=3$ is

$$
\{x<-2 \lor -2<x\}
$$

And the point $x_0=\frac{1}{2}$ is inside this domain. The y domain of $f(x,y)$ when $x=\frac{1}{2}$ is

$$
\{y<-2 \lor -2<y\}
$$

And the point $y_0=3$ is inside this domain. Now we will look at the continuity of

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{6x+4y+1}{2\left(2x+y+1\right)}\right) \\
&= -\frac{2}{2x+y+1}+\frac{6x+4y+1}{2\left(2x+y+1\right)^{2}}
\end{aligned}
$$

The x domain of $\frac{\partial f}{\partial y}$ when $y=3$ is

$$
\{x<-2 \lor -2<x\}
$$

And the point $x_0=\frac{1}{2}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x=\frac{1}{2}$ is

$$
\{y<-2 \lor -2<y\}
$$

And the point $y_0=3$ is inside this domain. Therefore solution exists and is unique.

7.11.2 Solving as differentialType ode

Writing the ode as

$$
y'=\frac{-6x-4y-1}{4x+2y+2} \tag{1}
$$

Which becomes

$$
(2y+2)\,dy=(-4x)\,dy+(-6x-4y-1)\,dx \tag{2}
$$

But the RHS is complete differential because

$$(-4x)\,dy + (-6x - 4y - 1)\,dx = d\big(-3x^2 - 4xy - x\big)$$

Hence (2) becomes

$$(2y + 2)\,dy = d\big(-3x^2 - 4xy - x\big)$$

Integrating both sides gives gives these solutions

$$y = -2x - 1 + \sqrt{x^2 + c_1 + 3x + 1} + c_1$$
$$y = -2x - 1 - \sqrt{x^2 + c_1 + 3x + 1} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{2}$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = -2 - \frac{\sqrt{11 + 4c_1}}{2} + c_1$$

$$c_1 = \frac{11}{2} + 2\sqrt{2}$$

Substituting c_1 found above in the general solution gives

$$y = -2x + \frac{9}{2} - \frac{\sqrt{4x^2 + 26 + 8\sqrt{2} + 12x}}{2} + 2\sqrt{2}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{2}$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = -2 + \frac{\sqrt{11 + 4c_1}}{2} + c_1$$

$$c_1 = \frac{11}{2} - 2\sqrt{2}$$

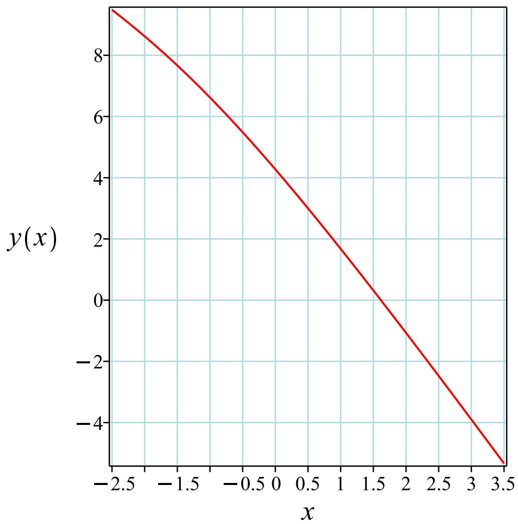
Substituting c_1 found above in the general solution gives

$$y = -2x + \frac{9}{2} + \frac{\sqrt{4x^2 + 26 - 8\sqrt{2} + 12x}}{2} - 2\sqrt{2}$$

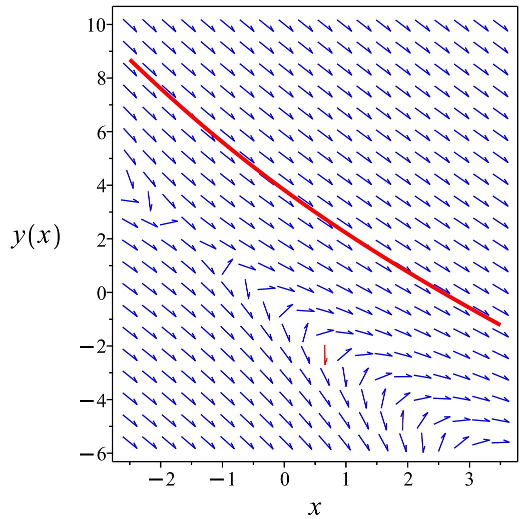
Summary

The solution(s) found are the following

$$y = -2x + \frac{9}{2} + \frac{\sqrt{4x^2 + 26 - 8\sqrt{2} + 12x}}{2} - 2\sqrt{2} \tag{1}$$
$$y = -2x + \frac{9}{2} - \frac{\sqrt{4x^2 + 26 + 8\sqrt{2} + 12x}}{2} + 2\sqrt{2} \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2x + \frac{9}{2} + \frac{\sqrt{4x^2 + 26 - 8\sqrt{2} + 12x}}{2} - 2\sqrt{2}$$

Verified OK.

$$y = -2x + \frac{9}{2} - \frac{\sqrt{4x^2 + 26 + 8\sqrt{2} + 12x}}{2} + 2\sqrt{2}$$

Verified OK.

7.11.3 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{6X + 6x_0 + 4Y(X) + 4y_0 + 1}{2\,(2X + 2x_0 + Y\,(X) + y_0 + 1)}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -\frac{3}{2}$$
$$y_0 = 2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{6X + 4Y(X)}{2(2X + Y(X))}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{3X + 2Y}{2X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -3X - 2Y$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-2u - 3}{u + 2} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-2u(X)-3}{u(X)+2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-2u(X)-3}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) + 2\left(\frac{d}{dX}u(X)\right)X + u(X)^2 + 4u(X) + 3 = 0$$

Or

$$X(u(X) + 2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 4u(X) + 3 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X)g(u) \\ &= -\frac{u^2 + 4u + 3}{X(u + 2)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+4u+3}{u+2}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+4u+3}{u+2}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+4u+3}{u+2}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(u^2+4u+3)}{2} = -\ln(X) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+4u+3} = \mathrm{e}^{-\ln(X)+c_2}$$

Which simplifies to

$$\sqrt{u^2+4u+3} = \frac{c_3}{X}$$

Which simplifies to

$$\sqrt{u(X)^2+4u(X)+3} = \frac{c_3\mathrm{e}^{c_2}}{X}$$

The solution is

$$\sqrt{u(X)^2+4u(X)+3} = \frac{c_3\mathrm{e}^{c_2}}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\sqrt{\frac{Y(X)^2}{X^2} + \frac{4Y(X)}{X} + 3} = \frac{c_3\mathrm{e}^{c_2}}{X}$$

Which simplifies to

$$\sqrt{\frac{(Y(X)+3X)(Y(X)+X)}{X^2}} = \frac{c_3\mathrm{e}^{c_2}}{X}$$

Using the solution for $Y(X)$

$$\sqrt{\frac{(Y(X)+3X)(Y(X)+X)}{X^2}} = \frac{c_3\mathrm{e}^{c_2}}{X}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 2$$
$$X = x - \frac{3}{2}$$

Then the solution in y becomes

$$\sqrt{\frac{\left(y+\frac{5}{2}+3x\right)\left(y-\frac{1}{2}+x\right)}{\left(x+\frac{3}{2}\right)^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x+\frac{3}{2}}$$

Initial conditions are used to solve for c_2. Substituting $x = \frac{1}{2}$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\sqrt{21}}{2} = \frac{c_3 \mathrm{e}^{c_2}}{2}$$

$$c_2 = \frac{\ln\left(\frac{21}{c_3^2}\right)}{2}$$

Substituting c_2 found above in the general solution gives

$$\sqrt{\frac{\left(y+\frac{5}{2}+3x\right)\left(x+y-\frac{1}{2}\right)}{\left(x+\frac{3}{2}\right)^2}} = \frac{2c_3\sqrt{21}\,\sqrt{\frac{1}{c_3^2}}}{3+2x}$$

The above simplifies to

$$-2c_3\sqrt{21}\,\sqrt{\frac{1}{c_3^2}} + 2\sqrt{\frac{(2y+5+6x)(2x+2y-1)}{(3+2x)^2}}\,x + 3\sqrt{\frac{(2y+5+6x)(2x+2y-1)}{(3+2x)^2}} = 0$$

<u>Summary</u>
The solution(s) found are the following

$$\sqrt{\frac{(2y+5+6x)(2x+2y-1)}{(3+2x)^2}}\,(3+2x) - 2\sqrt{21}\,\mathrm{csgn}\left(\frac{1}{c_3}\right) = 0 \tag{1}$$

<u>Verification of solutions</u>

$$\sqrt{\frac{(2y+5+6x)(2x+2y-1)}{(3+2x)^2}}\,(3+2x) - 2\sqrt{21}\,\mathrm{csgn}\left(\frac{1}{c_3}\right) = 0$$

Verified OK.

7.11.4 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{6x+4y+1}{2\left(2x+y+1\right)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{(6x+4y+1)(b_3-a_2)}{2\left(2x+y+1\right)} - \frac{(6x+4y+1)^2 a_3}{4\left(2x+y+1\right)^2} \\
& - \left(-\frac{3}{2x+y+1} + \frac{6x+4y+1}{(2x+y+1)^2}\right)(x a_2 + y a_3 + a_1) \\
& - \left(-\frac{2}{2x+y+1} + \frac{6x+4y+1}{2\left(2x+y+1\right)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{24x^2a_2 - 36x^2a_3 + 20x^2b_2 - 24x^2b_3 + 24xya_2 - 48xya_3 + 16xyb_2 - 24xyb_3 + 8y^2a_2 - 20y^2a_3 + 4y^2b_2 - 8}{4\left(2\right.}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& 24x^2a_2 - 36x^2a_3 + 20x^2b_2 - 24x^2b_3 + 24xya_2 - 48xya_3 + 16xyb_2 - 24xyb_3 \\
& + 8y^2a_2 - 20y^2a_3 + 4y^2b_2 - 8y^2b_3 + 24xa_2 - 12xa_3 + 4xb_1 + 22xb_2 - 16xb_3 \\
& - 4ya_1 + 10ya_2 + 8yb_2 - 4yb_3 + 8a_1 + 2a_2 - a_3 + 6b_1 + 4b_2 - 2b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&24a_2v_1^2 + 24a_2v_1v_2 + 8a_2v_2^2 - 36a_3v_1^2 - 48a_3v_1v_2 - 20a_3v_2^2 + 20b_2v_1^2 + 16b_2v_1v_2 \\
&+ 4b_2v_2^2 - 24b_3v_1^2 - 24b_3v_1v_2 - 8b_3v_2^2 - 4a_1v_2 + 24a_2v_1 + 10a_2v_2 - 12a_3v_1 + 4b_1v_1 \\
&+ 22b_2v_1 + 8b_2v_2 - 16b_3v_1 - 4b_3v_2 + 8a_1 + 2a_2 - a_3 + 6b_1 + 4b_2 - 2b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(24a_2 - 36a_3 + 20b_2 - 24b_3)\, v_1^2 + (24a_2 - 48a_3 + 16b_2 - 24b_3)\, v_1v_2 \\
&+ (24a_2 - 12a_3 + 4b_1 + 22b_2 - 16b_3)\, v_1 + (8a_2 - 20a_3 + 4b_2 - 8b_3)\, v_2^2 \\
&+ (-4a_1 + 10a_2 + 8b_2 - 4b_3)\, v_2 + 8a_1 + 2a_2 - a_3 + 6b_1 + 4b_2 - 2b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-4a_1 + 10a_2 + 8b_2 - 4b_3 &= 0 \\
8a_2 - 20a_3 + 4b_2 - 8b_3 &= 0 \\
24a_2 - 48a_3 + 16b_2 - 24b_3 &= 0 \\
24a_2 - 36a_3 + 20b_2 - 24b_3 &= 0 \\
24a_2 - 12a_3 + 4b_1 + 22b_2 - 16b_3 &= 0 \\
8a_1 + 2a_2 - a_3 + 6b_1 + 4b_2 - 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 4a_3 + \frac{3b_3}{2} \\
a_2 &= 4a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= -\frac{9a_3}{2} - 2b_3 \\
b_2 &= -3a_3 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x + \frac{3}{2} \\
\eta &= y - 2
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - 2 - \left(-\frac{6x + 4y + 1}{2\,(2x + y + 1)}\right)\left(x + \frac{3}{2}\right) \\
&= \frac{12x^2 + 16xy + 4y^2 + 4x + 8y - 5}{8x + 4y + 4} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{12x^2+16xy+4y^2+4x+8y-5}{8x+4y+4}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln \left(12x^2+16xy+4y^2+4x+8y-5\right)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{6x+4y+1}{2\left(2x+y+1\right)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{3}{2y+5+6x} + \frac{1}{2x+2y-1} \\
S_y &= \frac{1}{2y+5+6x} + \frac{1}{2x+2y-1}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

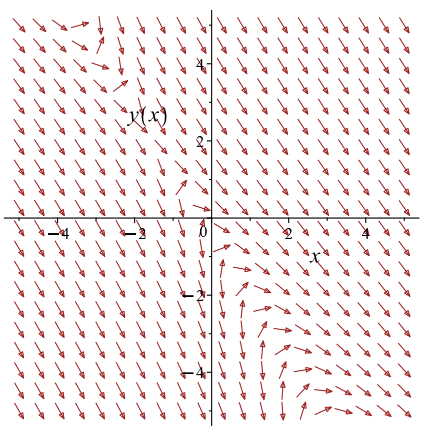
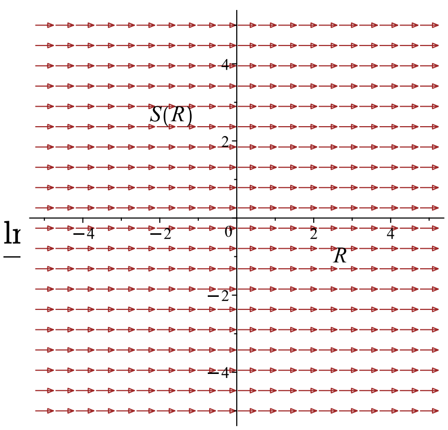
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(2y+5+6x)}{2} + \frac{\ln(2x+2y-1)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(2y+5+6x)}{2} + \frac{\ln(2x+2y-1)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{6x+4y+1}{2(2x+y+1)}$ 	$R = x$ $S = \frac{\ln(2y+5+6x)}{2} + \ln$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{2}$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$\ln(2) + \frac{\ln(7)}{2} + \frac{\ln(3)}{2} = c_1$$

$$c_1 = \ln(2) + \frac{\ln(7)}{2} + \frac{\ln(3)}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{\ln(2y+5+6x)}{2}+\frac{\ln(2x+2y-1)}{2}=\ln(2)+\frac{\ln(7)}{2}+\frac{\ln(3)}{2}$$

Summary

The solution(s) found are the following

$$\frac{\ln(2y+5+6x)}{2}+\frac{\ln(2x+2y-1)}{2}=\ln(2)+\frac{\ln(7)}{2}+\frac{\ln(3)}{2} \tag{1}$$

Verification of solutions

$$\frac{\ln(2y+5+6x)}{2}+\frac{\ln(2x+2y-1)}{2}=\ln(2)+\frac{\ln(7)}{2}+\frac{\ln(3)}{2}$$

Verified OK.

7.11.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(4x+2y+2)\,\mathrm{d}y &= (-6x-4y-1)\,\mathrm{d}x\\ (6x+4y+1)\,\mathrm{d}x + (4x+2y+2)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 6x+4y+1\\ N(x,y) &= 4x+2y+2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(6x+4y+1)\\ &= 4\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(4x+2y+2)\\ &= 4\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int 6x+4y+1\,\mathrm{d}x\\ \phi &= x(3x+4y+1)+f(y)\end{aligned}\tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 4x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 4x + 2y + 2$. Therefore equation (4) becomes

$$4x + 2y + 2 = 4x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y + 2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (2y + 2)\,\mathrm{d}y$$
$$f(y) = y^2 + 2y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(3x + 4y + 1) + y^2 + 2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(3x + 4y + 1) + y^2 + 2y$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{1}{2}$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{89}{4} = c_1$$

$$c_1 = \frac{89}{4}$$

Substituting c_1 found above in the general solution gives

$$x(3x+4y+1)+y^2+2y=\frac{89}{4}$$

Summary
The solution(s) found are the following

$$y^2+(4x+2)\,y+3x^2+x=\frac{89}{4} \tag{1}$$

Verification of solutions

$$y^2+(4x+2)\,y+3x^2+x=\frac{89}{4}$$

Verified OK.

7.11.6 Maple step by step solution

Let's solve

$\left[4y+(4x+2y+2)\,y'=-6x-1, y\left(\frac{1}{2}\right)=3\right]$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$
  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
  - Evaluate derivatives

    $4=4$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(6x+4y+1)\,dx+f_1(y)$
- Evaluate integral

$F(x,y) = 3x^2 + 4xy + x + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y} F(x,y)$

- Compute derivative

  $4x + 2y + 2 = 4x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 2y + 2$

- Solve for $f_1(y)$

  $f_1(y) = y^2 + 2y$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = 3x^2 + 4xy + y^2 + x + 2y$

- Substitute $F(x,y)$ into the solution of the ODE

  $3x^2 + 4xy + y^2 + x + 2y = c_1$

- Solve for y

  $\left\{y = -2x - 1 - \sqrt{x^2 + c_1 + 3x + 1}, y = -2x - 1 + \sqrt{x^2 + c_1 + 3x + 1}\right\}$

- Use initial condition $y\left(\frac{1}{2}\right) = 3$

  $3 = -2 - \sqrt{\frac{11}{4} + c_1}$

- Solution does not satisfy initial condition
- Use initial condition $y\left(\frac{1}{2}\right) = 3$

  $3 = -2 + \sqrt{\frac{11}{4} + c_1}$

- Solve for c_1

  $c_1 = \frac{89}{4}$

- Substitute $c_1 = \frac{89}{4}$ into general solution and simplify

  $y = -2x - 1 + \frac{\sqrt{4x^2+12x+93}}{2}$

- Solution to the IVP

  $y = -2x - 1 + \frac{\sqrt{4x^2+12x+93}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.297 (sec). Leaf size: 23

```
dsolve([(6*x+4*y(x)+1)+(4*x+2*y(x)+2)*diff(y(x),x)=0,y(1/2) = 3],y(x), singsol=all)
```

$$y(x) = -2x - 1 + \frac{\sqrt{4x^2 + 12x + 93}}{2}$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 28

```
DSolve[{(6*x+4*y[x]+1)+(4*x+2*y[x]+2)*y'[x]==0,{y[1/2]==3}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{2}\left(\sqrt{4x^2 + 12x + 93} - 4x - 2\right)$$

7.12 problem 12

Internal problem ID [11710]
Internal file name [`OUTPUT/11719_Thursday_April_11_2024_08_48_53_PM_62273029/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y + (x + y + 2)\, y' = -3x + 6}$$

With initial conditions

$$[y(2) = -2]$$

7.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{-3x + y + 6}{x + y + 2} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -2$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x,y)$ when $x = 2$ is

$$\{y < -4 \lor -4 < y\}$$

And the point $y_0 = -2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-3x+y+6}{x+y+2}\right)\\ &= \frac{1}{x+y+2} - \frac{-3x+y+6}{(x+y+2)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -2$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 2$ is

$$\{y < -4 \lor -4 < y\}$$

And the point $y_0 = -2$ is inside this domain. Therefore solution exists and is unique.

7.12.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{-3X - 3x_0 + Y(X) + y_0 + 6}{X + x_0 + Y(X) + y_0 + 2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= 1\\ y_0 &= -3\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-3X + Y(X)}{X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned}Y' &= F(X,Y)\\ &= \frac{-3X+Y}{X+Y} \qquad (1)\end{aligned}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -3X + Y$ and $N = X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u-3}{u+1}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)-3}{u(X)+1} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{u(X)-3}{u(X)+1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) + \left(\frac{d}{dX}u(X)\right)X + u(X)^2 + 3 = 0$$

Or

$$X(u(X)+1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 3 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X,u)\\ &= f(X)g(u)\\ &= -\frac{u^2+3}{X(u+1)}\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+3}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+3}{u+1}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+3}{u+1}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln \left(u^{2}+3\right)}{2}+\frac{\sqrt{3}\, \arctan \left(\frac{\sqrt{3}\, u}{3}\right)}{3} = -\ln \left(X \right)+c_{2}$$

The solution is

$$\frac{\ln \left(u \left(X \right)^{2}+3\right)}{2}+\frac{\sqrt{3}\, \arctan \left(\frac{\sqrt{3}\, u \left(X \right)}{3}\right)}{3}+\ln \left(X \right)-c_{2} = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln \left(\frac{Y \left(X \right)^{2}}{X^{2}}+3\right)}{2}+\frac{\sqrt{3}\, \arctan \left(\frac{\sqrt{3}\, Y \left(X \right)}{3 X}\right)}{3}+\ln \left(X \right)-c_{2} = 0$$

Using the solution for $Y(X)$

$$\frac{\ln \left(\frac{Y \left(X \right)^{2}}{X^{2}}+3\right)}{2}+\frac{\sqrt{3}\, \arctan \left(\frac{\sqrt{3}\, Y \left(X \right)}{3 X}\right)}{3}+\ln \left(X \right)-c_{2} = 0$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y + y_0 \\ X &= x + x_0 \end{aligned}$$

Or

$$\begin{aligned} Y &= y - 3 \\ X &= 1 + x \end{aligned}$$

Then the solution in y becomes

$$\frac{\ln \left(\frac{\left(y +3\right)^{2}}{\left(x -1\right)^{2}}+3\right)}{2}+\frac{\sqrt{3}\, \arctan \left(\frac{\sqrt{3}\, \left(y +3\right)}{-3+3 x}\right)}{3}+\ln \left(x -1\right)-c_{2} = 0$$

Initial conditions are used to solve for c_2. Substituting $x = 2$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$\ln \left(2\right)+\frac{\pi \sqrt{3}}{18}-c_{2} = 0$$

$$c_{2} = \ln \left(2\right)+\frac{\pi \sqrt{3}}{18}$$

Substituting c_2 found above in the general solution gives

$$\frac{\ln\left(\frac{3x^2+y^2-6x+6y+12}{(x-1)^2}\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,(3+y)}{-3+3x}\right)}{3}+\ln(x-1)-\ln(2)-\frac{\pi\sqrt{3}}{18}=0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{3x^2+y^2-6x+6y+12}{(x-1)^2}\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,(y+3)}{-3+3x}\right)}{3}+\ln(x-1)-\ln(2)-\frac{\pi\sqrt{3}}{18}=0 \quad (1)$$

Verification of solutions

$$\frac{\ln\left(\frac{3x^2+y^2-6x+6y+12}{(x-1)^2}\right)}{2}+\frac{\sqrt{3}\,\arctan\left(\frac{\sqrt{3}\,(y+3)}{-3+3x}\right)}{3}+\ln(x-1)-\ln(2)-\frac{\pi\sqrt{3}}{18}=0$$

Verified OK.

7.12.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= \frac{-3x+y+6}{x+y+2}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2+\frac{(-3x+y+6)(b_3-a_2)}{x+y+2}-\frac{(-3x+y+6)^2 a_3}{(x+y+2)^2} \\
& -\left(-\frac{3}{x+y+2}-\frac{-3x+y+6}{(x+y+2)^2}\right)(xa_2+ya_3+a_1) \\
& -\left(\frac{1}{x+y+2}-\frac{-3x+y+6}{(x+y+2)^2}\right)(xb_2+yb_3+b_1)=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{3x^2a_2-9x^2a_3-3x^2b_2-3x^2b_3+6xya_2+6xya_3+2xyb_2-6xyb_3-y^2a_2+3y^2a_3+y^2b_2+y^2b_3+12xa_2}{(x+y+2)}
$$

$=0$

Setting the numerator to zero gives

$$
\begin{aligned}
& 3x^2a_2-9x^2a_3-3x^2b_2-3x^2b_3+6xya_2+6xya_3+2xyb_2-6xyb_3 \\
& -y^2a_2+3y^2a_3+y^2b_2+y^2b_3+12xa_2+36xa_3-4xb_1+8xb_2+4ya_1 \\
& -8ya_2+4yb_2+12yb_3+12a_1-12a_2-36a_3+4b_1+4b_2+12b_3=0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x=v_1, y=v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& 3a_2v_1^2+6a_2v_1v_2-a_2v_2^2-9a_3v_1^2+6a_3v_1v_2+3a_3v_2^2-3b_2v_1^2+2b_2v_1v_2+b_2v_2^2 \\
& -3b_3v_1^2-6b_3v_1v_2+b_3v_2^2+4a_1v_2+12a_2v_1-8a_2v_2+36a_3v_1-4b_1v_1 \\
& +8b_2v_1+4b_2v_2+12b_3v_2+12a_1-12a_2-36a_3+4b_1+4b_2+12b_3=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$\begin{aligned}(3a_2 - 9a_3 - 3b_2 - 3b_3)\, v_1^2 + (6a_2 + 6a_3 + 2b_2 - 6b_3)\, v_1 v_2 \\ + (12a_2 + 36a_3 - 4b_1 + 8b_2)\, v_1 + (-a_2 + 3a_3 + b_2 + b_3)\, v_2^2 \\ + (4a_1 - 8a_2 + 4b_2 + 12b_3)\, v_2 + 12a_1 - 12a_2 - 36a_3 + 4b_1 + 4b_2 + 12b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}4a_1 - 8a_2 + 4b_2 + 12b_3 &= 0 \\ -a_2 + 3a_3 + b_2 + b_3 &= 0 \\ 3a_2 - 9a_3 - 3b_2 - 3b_3 &= 0 \\ 6a_2 + 6a_3 + 2b_2 - 6b_3 &= 0 \\ 12a_2 + 36a_3 - 4b_1 + 8b_2 &= 0 \\ 12a_1 - 12a_2 - 36a_3 + 4b_1 + 4b_2 + 12b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -b_3 + 3a_3 \\ a_2 &= b_3 \\ a_3 &= a_3 \\ b_1 &= 3a_3 + 3b_3 \\ b_2 &= -3a_3 \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= 3 + y \\ \eta &= -3x + 3\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi \\ &= -3x + 3 - \left(\frac{-3x + y + 6}{x + y + 2}\right)(3 + y) \\ &= \frac{-3x^2 - y^2 + 6x - 6y - 12}{x + y + 2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-3x^2-y^2+6x-6y-12}{x+y+2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(3x^2 + y^2 - 6x + 6y + 12)}{2} + \frac{(1-x)\sqrt{3}\arctan\left(\frac{(2y+6)\sqrt{3}}{6x-6}\right)}{-3+3x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-3x + y + 6}{x + y + 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-3x + y + 6}{3x^2 + y^2 - 6x + 6y + 12} \\ S_y &= \frac{-x - y - 2}{3x^2 + y^2 - 6x + 6y + 12} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

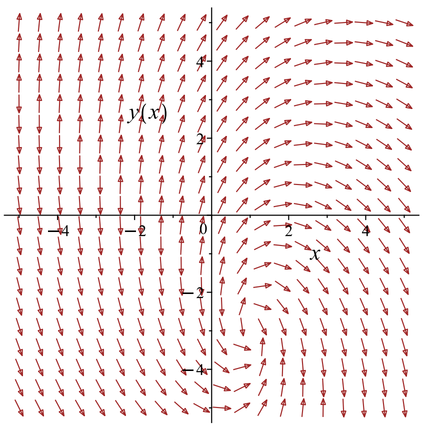
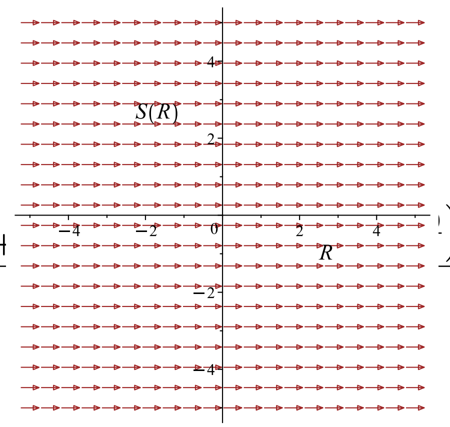
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(3x^2 + y^2 - 6x + 6y + 12)}{2} - \frac{\sqrt{3}\, \arctan\left(\frac{\sqrt{3}\,(y+3)}{-3+3x}\right)}{3} = c_1$$

Which simplifies to

$$-\frac{\ln(3x^2 + y^2 - 6x + 6y + 12)}{2} - \frac{\sqrt{3}\, \arctan\left(\frac{\sqrt{3}\,(y+3)}{-3+3x}\right)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-3x+y+6}{x+y+2}$ 	$R = x$ $S = -\frac{\ln(3x^2+y^2-6x+}{2}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-\ln(2) - \frac{\pi\sqrt{3}}{18} = c_1$$

$$c_1 = -\ln(2) - \frac{\pi\sqrt{3}}{18}$$

Substituting c_1 found above in the general solution gives

$$-\frac{\ln(3x^2+y^2-6x+6y+12)}{2} - \frac{\sqrt{3}\arctan\left(\frac{\sqrt{3}(3+y)}{-3+3x}\right)}{3} = -\ln(2) - \frac{\pi\sqrt{3}}{18}$$

Summary

The solution(s) found are the following

$$-\frac{\ln(3x^2+y^2-6x+6y+12)}{2} - \frac{\sqrt{3}\arctan\left(\frac{\sqrt{3}(y+3)}{-3+3x}\right)}{3} = -\ln(2) - \frac{\pi\sqrt{3}}{18} \quad (1)$$

Verification of solutions

$$-\frac{\ln \left(3x^{2}+y^{2}-6x+6y+12\right)}{2}-\frac{\sqrt{3}\, \arctan \left(\frac{\sqrt{3}\, (y+3)}{-3+3x}\right)}{3} = -\ln \left(2\right)-\frac{\pi \sqrt{3}}{18}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 9.453 (sec). Leaf size: 120

```
dsolve([(3*x-y(x)-6)+(x+y(x)+2)*diff(y(x),x)=0,y(2) = -2],y(x), singsol=all)
```

$$y(x) = -3-\sqrt{3}\, \tan \left(\operatorname{RootOf}\left(-3\sqrt{3}\, \ln \left(3\right)+6\sqrt{3}\, \ln \left(2\right) - 3\sqrt{3}\, \ln \left(\sec \left(\_Z\right)^{2} \left(-1+x\right)^{2}\right)+\pi +6\_Z\right)\right) \left(-1+x\right)$$

✓ Solution by Mathematica

Time used: 0.141 (sec). Leaf size: 90

```
DSolve[{(3*x-y[x]-6)+(x+y[x]+2)*y'[x]==0,{y[2]==-2}},y[x],x,IncludeSingularSolutions -> True
```

$$\text{Solve}\left[\frac{\arctan\left(\frac{-y(x)+3x-6}{\sqrt{3}(y(x)+x+2)}\right)}{\sqrt{3}}+\log(2)=\frac{1}{2}\log\left(\frac{3x^2+y(x)^2+6y(x)-6x+12}{(x-1)^2}\right)+\log(x-1)+\frac{1}{18}\left(\sqrt{3}\pi+18\log(2)-9\log(4)\right),y(x)\right]$$

7.13 problem 13

Internal problem ID [11711]
Internal file name [`OUTPUT/11720_Thursday_April_11_2024_08_48_58_PM_23035239/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{3y + (4x + 6y + 1)\, y' = -2x - 1}$$

With initial conditions

$$[y(-2) = 2]$$

7.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{2x + 3y + 1}{4x + 6y + 1} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 2$ is

$$\left\{x < -\frac{13}{4} \vee -\frac{13}{4} < x\right\}$$

And the point $x_0 = -2$ is inside this domain. The y domain of $f(x, y)$ when $x = -2$ is

$$\left\{y < \frac{7}{6} \vee \frac{7}{6} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2x + 3y + 1}{4x + 6y + 1}\right) \\ &= -\frac{3}{4x + 6y + 1} + \frac{12x + 18y + 6}{(4x + 6y + 1)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\left\{x < -\frac{13}{4} \vee -\frac{13}{4} < x\right\}$$

And the point $x_0 = -2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -2$ is

$$\left\{y < \frac{7}{6} \vee \frac{7}{6} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

7.13.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{2x + 3y + 1}{4x + 6y + 1} \\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{(2x+3y+1)(b_3-a_2)}{4x+6y+1} - \frac{(2x+3y+1)^2 a_3}{(4x+6y+1)^2} \\
&- \left(-\frac{2}{4x+6y+1} + \frac{8x+12y+4}{(4x+6y+1)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(-\frac{3}{4x+6y+1} + \frac{12x+18y+6}{(4x+6y+1)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{8x^2a_2 - 4x^2a_3 + 16x^2b_2 - 8x^2b_3 + 24xya_2 - 12xya_3 + 48xyb_2 - 24xyb_3 + 18y^2a_2 - 9y^2a_3 + 36y^2b_2 - 18y}{(4x + 6y +}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&8x^2a_2 - 4x^2a_3 + 16x^2b_2 - 8x^2b_3 + 24xya_2 - 12xya_3 + 48xyb_2 - 24xyb_3 \\
&+ 18y^2a_2 - 9y^2a_3 + 36y^2b_2 - 18y^2b_3 + 4xa_2 - 4xa_3 + 5xb_2 - 6xb_3 \\
&+ 9ya_2 - 8ya_3 + 12yb_2 - 12yb_3 - 2a_1 + a_2 - a_3 - 3b_1 + b_2 - b_3 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&8a_2v_1^2 + 24a_2v_1v_2 + 18a_2v_2^2 - 4a_3v_1^2 - 12a_3v_1v_2 - 9a_3v_2^2 + 16b_2v_1^2 + 48b_2v_1v_2 \\
&+ 36b_2v_2^2 - 8b_3v_1^2 - 24b_3v_1v_2 - 18b_3v_2^2 + 4a_2v_1 + 9a_2v_2 - 4a_3v_1 - 8a_3v_2 \\
&+ 5b_2v_1 + 12b_2v_2 - 6b_3v_1 - 12b_3v_2 - 2a_1 + a_2 - a_3 - 3b_1 + b_2 - b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(8a_2 - 4a_3 + 16b_2 - 8b_3)\, v_1^2 + (24a_2 - 12a_3 + 48b_2 - 24b_3)\, v_1 v_2 \\
&+ (4a_2 - 4a_3 + 5b_2 - 6b_3)\, v_1 + (18a_2 - 9a_3 + 36b_2 - 18b_3)\, v_2^2 \\
&+ (9a_2 - 8a_3 + 12b_2 - 12b_3)\, v_2 - 2a_1 + a_2 - a_3 - 3b_1 + b_2 - b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
4a_2 - 4a_3 + 5b_2 - 6b_3 &= 0 \\
8a_2 - 4a_3 + 16b_2 - 8b_3 &= 0 \\
9a_2 - 8a_3 + 12b_2 - 12b_3 &= 0 \\
18a_2 - 9a_3 + 36b_2 - 18b_3 &= 0 \\
24a_2 - 12a_3 + 48b_2 - 24b_3 &= 0 \\
-2a_1 + a_2 - a_3 - 3b_1 + b_2 - b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= a_1 \\
a_2 &= -8a_1 - 12b_1 \\
a_3 &= -12a_1 - 18b_1 \\
b_1 &= b_1 \\
b_2 &= 4a_1 + 6b_1 \\
b_3 &= 6a_1 + 9b_1
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -12x - 18y \\
\eta &= 6x + 9y + 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 6x + 9y + 1 - \left(-\frac{2x+3y+1}{4x+6y+1}\right)(-12x - 18y) \\
&= \frac{-2x-3y+1}{4x+6y+1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2x-3y+1}{4x+6y+1}} dy
\end{aligned}$$

Which results in

$$S = -2y - \ln(2x + 3y - 1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2x+3y+1}{4x+6y+1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2}{2x+3y-1} \\
S_y &= -2-\frac{3}{2x+3y-1}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 1 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 1
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = R + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-2y - \ln(2x+3y-1) = x + c_1
$$

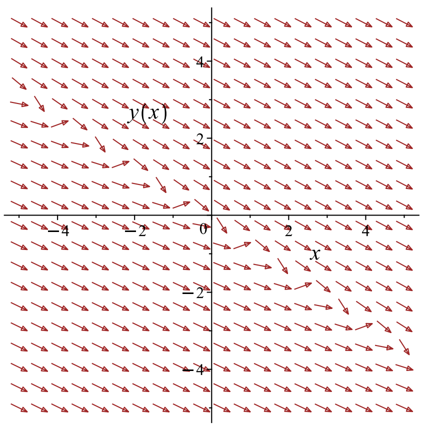
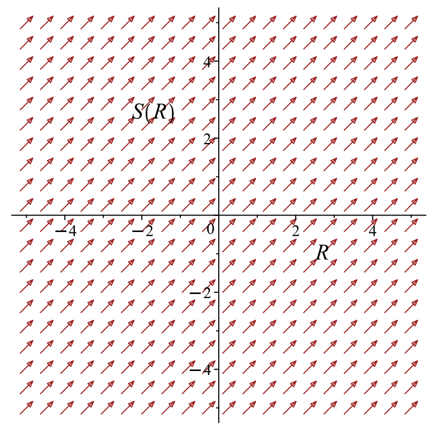
Which simplifies to

$$
-2y - \ln(2x+3y-1) = x + c_1
$$

Which gives

$$
y = -\frac{2x}{3} + \frac{\operatorname{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{x}{3}-\frac{2}{3}-c_1}}{3}\right)}{2} + \frac{1}{3}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x+3y+1}{4x+6y+1}$ 	$R = x$ $S = -2y - \ln(2x + 3y -$	$\frac{dS}{dR} = 1$ 

Initial conditions are used to solve for c_1. Substituting $x = -2$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{5}{3} + \frac{\text{LambertW}\left(\frac{2\,e^{-\frac{4}{3}-c_1}}{3}\right)}{2}$$

$$c_1 = -2$$

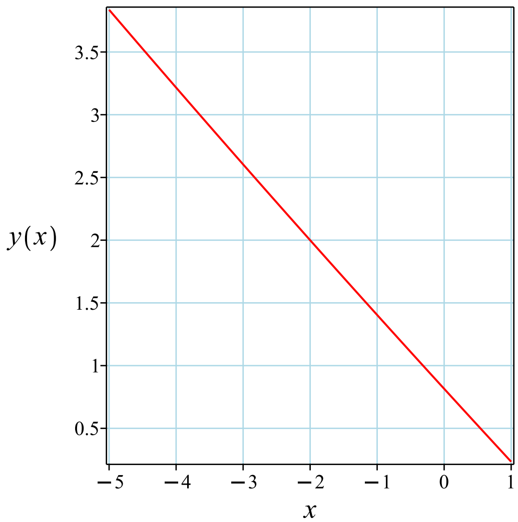
Substituting c_1 found above in the general solution gives

$$y = -\frac{2x}{3} + \frac{\text{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2} + \frac{1}{3}$$

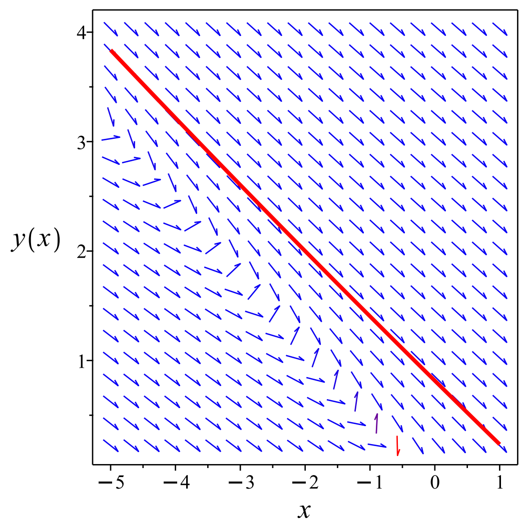
Summary

The solution(s) found are the following

$$y = -\frac{2x}{3} + \frac{\text{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2} + \frac{1}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2x}{3} + \frac{\text{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2} + \frac{1}{3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.156 (sec). Leaf size: 20

```
dsolve([(2*x+3*y(x)+1)+(4*x+6*y(x)+1)*diff(y(x),x)=0,y(-2) = 2],y(x), singsol=all)
```

$$y(x) = \frac{1}{3} - \frac{2x}{3} + \frac{\text{LambertW}\left(\frac{2\,e^{\frac{4}{3}+\frac{x}{3}}}{3}\right)}{2}$$

✓ Solution by Mathematica

Time used: 4.146 (sec). Leaf size: 30

```
DSolve[{(2*x+3*y[x]+1)+(4*x+6*y[x]+1)*y'[x]==0,{y[-2]==2}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{1}{6}\left(3W\left(\frac{2}{3}e^{\frac{x+4}{3}}\right) - 4x + 2\right)$$

7.14 problem 14

Internal problem ID [11712]
Internal file name [`OUTPUT/11721_Thursday_April_11_2024_08_49_00_PM_57163841/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 2, Section 2.4. Special integrating factors and transformations. Exercises page 67
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{3y + (x + y + 1)\, y' = -4x - 1}$$

With initial conditions

$$[y(3) = -4]$$

7.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{4x + 3y + 1}{x + y + 1} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -4$ is

$$\{x < 3 \lor 3 < x\}$$

But the point $x_0 = 3$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

7.14.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{4X + 4x_0 + 3Y(X) + 3y_0 + 1}{X + x_0 + Y(X) + y_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 2$$
$$y_0 = -3$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{4X + 3Y(X)}{X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{4X + 3Y}{X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -4X - 3Y$ and $N = X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dX}X + u &= \frac{-3u - 4}{u + 1} \\ \frac{du}{dX} &= \frac{\frac{-3u(X)-4}{u(X)+1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-3u(X)-4}{u(X)+1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) Xu(X) + \left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 4u(X) + 4 = 0$$

Or

$$X(u(X)+1)\left(\frac{d}{dX}u(X)\right) + (u(X)+2)^2 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{(u+2)^2}{X(u+1)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{(u+2)^2}{u+1}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{(u+2)^2}{u+1}} du &= -\frac{1}{X} dX \\ \int \frac{1}{\frac{(u+2)^2}{u+1}} du &= \int -\frac{1}{X} dX \\ \ln(u+2) + \frac{1}{u+2} &= -\ln(X) + c_2 \end{aligned}$$

The solution is

$$\ln(u(X)+2) + \frac{1}{u(X)+2} + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\ln\left(\frac{Y(X)}{X} + 2\right) + \frac{1}{\frac{Y(X)}{X} + 2} + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\ln\left(\frac{Y(X)+2X}{X}\right) + \frac{X}{Y(X)+2X} + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 3$$
$$X = x + 2$$

Then the solution in y becomes

$$\ln\left(\frac{2x+y-1}{x-2}\right) + \frac{x-2}{2x+y-1} + \ln(x-2) - c_2 = 0$$

Initial conditions are used to solve for c_2. Substituting $x = 3$ and $y = -4$ in the above solution gives an equation to solve for the constant of integration.

$$-c_2 + 1 = 0$$

$$c_2 = 1$$

Substituting c_2 found above in the general solution gives

$$\frac{2\ln\left(\frac{2x+y-1}{x-2}\right)x + \ln\left(\frac{2x+y-1}{x-2}\right)y + 2\ln(x-2)x + \ln(x-2)y - \ln\left(\frac{2x+y-1}{x-2}\right) - \ln(x-2) - x - y - 1}{2x+y-1} = 0$$

The above simplifies to

$$2\ln\left(\frac{2x+y-1}{x-2}\right)x + \ln\left(\frac{2x+y-1}{x-2}\right)y + 2\ln(x-2)x + \ln(x-2)y - \ln\left(\frac{2x+y-1}{x-2}\right) - \ln(x-2)$$

Summary
The solution(s) found are the following

$$(2x+y-1)\ln\left(\frac{2x+y-1}{x-2}\right) + (2x+y-1)\ln(x-2) - x - y - 1 = 0 \tag{1}$$

Verification of solutions

$$(2x+y-1)\ln\left(\frac{2x+y-1}{x-2}\right) + (2x+y-1)\ln(x-2) - x - y - 1 = 0$$

Verified OK.

7.14.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{4x+3y+1}{x+y+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2 + ya_3 + a_1 \tag{1E}
$$

$$
\eta = xb_2 + yb_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(4x+3y+1)(b_3-a_2)}{x+y+1} - \frac{(4x+3y+1)^2 a_3}{(x+y+1)^2} \\
&- \left(-\frac{4}{x+y+1} + \frac{4x+3y+1}{(x+y+1)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{3}{x+y+1} + \frac{4x+3y+1}{(x+y+1)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{4x^2a_2 - 16x^2a_3 - 4x^2b_3 + 8xya_2 - 24xya_3 + 2xyb_2 - 8xyb_3 + 3y^2a_2 - 8y^2a_3 + y^2b_2 - 3y^2b_3 + 8xa_2 - 8x}{(x+y+1)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&4x^2a_2 - 16x^2a_3 - 4x^2b_3 + 8xya_2 - 24xya_3 + 2xyb_2 - 8xyb_3 + 3y^2a_2 \\
&- 8y^2a_3 + y^2b_2 - 3y^2b_3 + 8xa_2 - 8xa_3 - xb_1 + 4xb_2 - 5xb_3 + ya_1 \\
&+ 4ya_2 - 3ya_3 + 2yb_2 - 2yb_3 + 3a_1 + a_2 - a_3 + 2b_1 + b_2 - b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&4a_2v_1^2 + 8a_2v_1v_2 + 3a_2v_2^2 - 16a_3v_1^2 - 24a_3v_1v_2 - 8a_3v_2^2 + 2b_2v_1v_2 + b_2v_2^2 \\
&- 4b_3v_1^2 - 8b_3v_1v_2 - 3b_3v_2^2 + a_1v_2 + 8a_2v_1 + 4a_2v_2 - 8a_3v_1 - 3a_3v_2 - b_1v_1 \\
&+ 4b_2v_1 + 2b_2v_2 - 5b_3v_1 - 2b_3v_2 + 3a_1 + a_2 - a_3 + 2b_1 + b_2 - b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(4a_2 - 16a_3 - 4b_3)\, v_1^2 + (8a_2 - 24a_3 + 2b_2 - 8b_3)\, v_1v_2 \\
&+ (8a_2 - 8a_3 - b_1 + 4b_2 - 5b_3)\, v_1 + (3a_2 - 8a_3 + b_2 - 3b_3)\, v_2^2 \\
&+ (a_1 + 4a_2 - 3a_3 + 2b_2 - 2b_3)\, v_2 + 3a_1 + a_2 - a_3 + 2b_1 + b_2 - b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
4a_2 - 16a_3 - 4b_3 &= 0 \\
3a_2 - 8a_3 + b_2 - 3b_3 &= 0 \\
8a_2 - 24a_3 + 2b_2 - 8b_3 &= 0 \\
a_1 + 4a_2 - 3a_3 + 2b_2 - 2b_3 &= 0 \\
8a_2 - 8a_3 - b_1 + 4b_2 - 5b_3 &= 0 \\
3a_1 + a_2 - a_3 + 2b_1 + b_2 - b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -5a_3 - 2b_3 \\
a_2 &= 4a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= 8a_3 + 3b_3 \\
b_2 &= -4a_3 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x-2 \\
\eta &= 3+y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 3+y-\left(-\frac{4x+3y+1}{x+y+1}\right)(x-2) \\
&= \frac{4x^2+4xy+y^2-4x-2y+1}{x+y+1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{4x^2+4xy+y^2-4x-2y+1}{x+y+1}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{-x+2}{2x+y-1} + \ln(2x+y-1)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{4x+3y+1}{x+y+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{4x+3y+1}{(2x+y-1)^2} \\ S_y &= \frac{x+y+1}{(2x+y-1)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(2x+y-1)\ln(2x+y-1)+x-2}{2x+y-1} = c_1$$

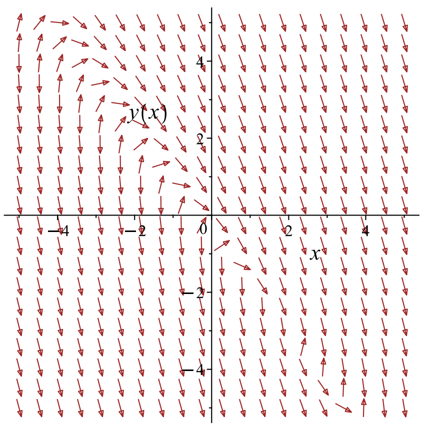
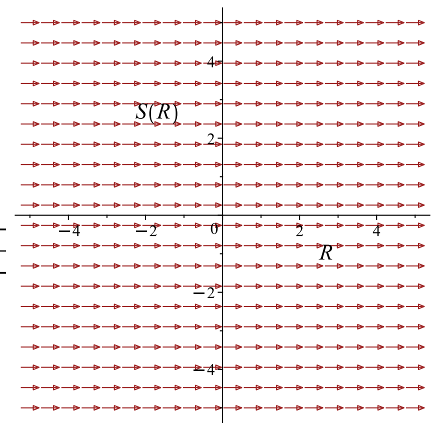
Which simplifies to

$$\frac{(2x+y-1)\ln(2x+y-1)+x-2}{2x+y-1} = c_1$$

Which gives

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-(x-2)\mathrm{e}^{-c_1}\right)+c_1} - 2x + 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4x+3y+1}{x+y+1}$ 	$R = x$ $S = \frac{(2x+y-1)\ln(2x+}{2x+y-}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 3$ and $y = -4$ in the above solution gives an equation to solve for the constant of integration.

$$-4 = \frac{-1 - 5\,\mathrm{LambertW}\left(-\mathrm{e}^{-c_1}\right)}{\mathrm{LambertW}\left(-\mathrm{e}^{-c_1}\right)}$$

$$c_1 = 1$$

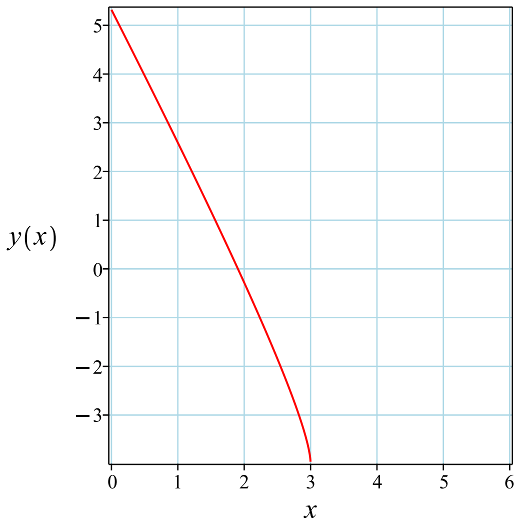
Substituting c_1 found above in the general solution gives

$$y = \frac{-2x\,\mathrm{LambertW}\left(-(x-2)\,\mathrm{e}^{-1}\right) + \mathrm{LambertW}\left(-(x-2)\,\mathrm{e}^{-1}\right) - x + 2}{\mathrm{LambertW}\left(-(x-2)\,\mathrm{e}^{-1}\right)}$$

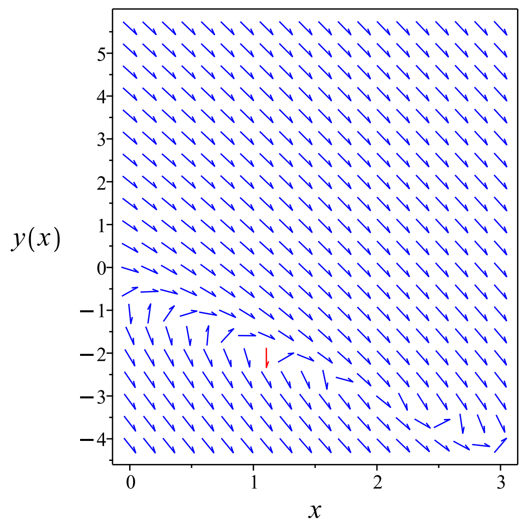
Summary

The solution(s) found are the following

$$y = \frac{-2x\,\mathrm{LambertW}\left(-(x-2)\,\mathrm{e}^{-1}\right) + \mathrm{LambertW}\left(-(x-2)\,\mathrm{e}^{-1}\right) - x + 2}{\mathrm{LambertW}\left(-(x-2)\,\mathrm{e}^{-1}\right)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-2x \,\mathrm{LambertW}\left(-(x-2)\,\mathrm{e}^{-1}\right) + \mathrm{LambertW}\left(-(x-2)\,\mathrm{e}^{-1}\right) - x + 2}{\mathrm{LambertW}\left(-\left(x-2\right)\mathrm{e}^{-1}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.265 (sec). Leaf size: 39

```
dsolve([(4*x+3*y(x)+1)+(x+y(x)+1)*diff(y(x),x)=0,y(3) = -4],y(x), singsol=all)
```

$$y(x) = \frac{-2x \operatorname{LambertW}\left(-(x-2)\,\mathrm{e}^{-1}\right) + \operatorname{LambertW}\left(-(x-2)\,\mathrm{e}^{-1}\right) - x + 2}{\operatorname{LambertW}\left(-(x-2)\,\mathrm{e}^{-1}\right)}$$

✓ Solution by Mathematica

Time used: 65.902 (sec). Leaf size: 197

```
DSolve[{(4*x+3*y[x]+1)+(x+y[x]+1)*y'[x]==0,{y[-2]==2}},y[x],x,IncludeSingularSolutions -> Tr
```

$$\text{Solve}\left[\frac{(-2)^{2/3}\left(-2x\log\left(\frac{3(-2)^{2/3}(y(x)+2x-1)}{y(x)+x+1}\right) + (2x-1)\log\left(-\frac{3(-2)^{2/3}(x-2)}{y(x)+x+1}\right) + \log\left(\frac{3(-2)^{2/3}(y(x)+2x-1)}{y(x)+x+1}\right) + \cdots}{9(y(x)+2x-1)}\right.$$

8 Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 113

8.1 problem 1 (a)

Internal problem ID [11713]
Internal file name [`OUTPUT/11722_Thursday_April_11_2024_08_49_05_PM_81323681/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 113
Problem number: 1 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 5y' + 6y = e^{x}}$$

With initial conditions

$$[y(0) = 5, y'(0) = 7]$$

8.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 5 \\ q(x) &= 6 \\ F &= e^{x} \end{aligned}$$

Hence the ode is

$$y'' + 5y' + 6y = e^x$$

The domain of $p(x) = 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.1.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 5, C = 6, f(x) = e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 5\lambda e^{\lambda x} + 6 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{5^2 - (4)(1)(6)} \\ &= -\frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{5}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{5}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 \\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(-2)x} + c_2 e^{(-3)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3x}, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{-3x}\right) + \left(\frac{e^x}{12}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^x}{12} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 0$ in the above gives

$$5 = c_1 + c_2 + \frac{1}{12} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2x} - 3c_2 e^{-3x} + \frac{e^x}{12}$$

substituting $y' = 7$ and $x = 0$ in the above gives

$$7 = -2c_1 - 3c_2 + \frac{1}{12} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{65}{3}$$
$$c_2 = -\frac{67}{4}$$

Substituting these values back in above solution results in

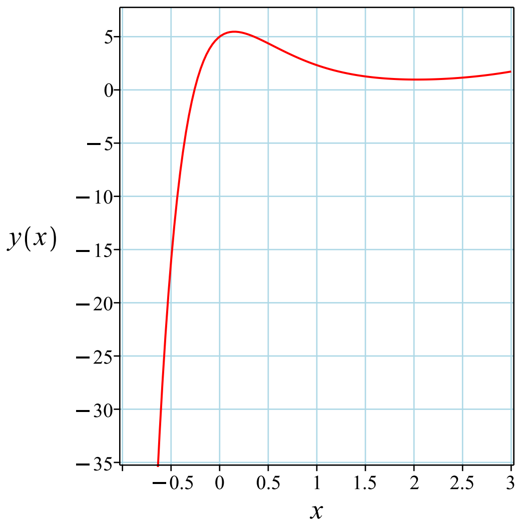
$$y = \frac{65 e^{-2x}}{3} - \frac{67 e^{-3x}}{4} + \frac{e^x}{12}$$

Which simplifies to

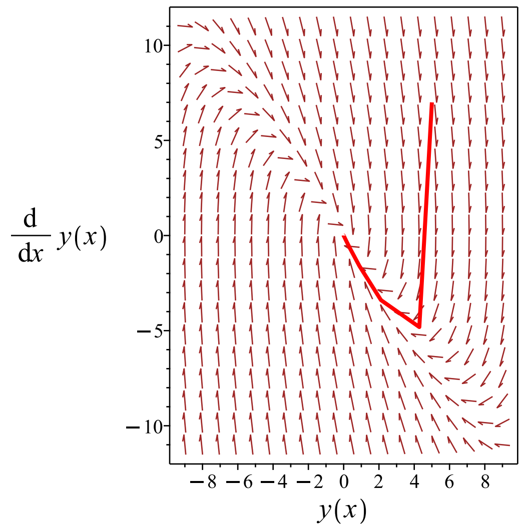
$$y = \frac{(e^{4x} + 260 e^x - 201) e^{-3x}}{12}$$

Summary
The solution(s) found are the following

$$y = \frac{(e^{4x} + 260 e^x - 201) e^{-3x}}{12} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(e^{4x} + 260\,e^{x} - 201)\,e^{-3x}}{12}$$

Verified OK.

8.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 252: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dx} \\ &= z_1 e^{-\frac{5x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{5x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-5x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} (\mathrm{e}^{x})\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 5y' + 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-3x}, \mathrm{e}^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12 A_1 \mathrm{e}^{x} = \mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + c_2 e^{-2x}\right) + \left(\frac{e^x}{12}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-3x} + c_2 e^{-2x} + \frac{e^x}{12} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 0$ in the above gives

$$5 = c_1 + c_2 + \frac{1}{12} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 e^{-3x} - 2c_2 e^{-2x} + \frac{e^x}{12}$$

substituting $y' = 7$ and $x = 0$ in the above gives

$$7 = -3c_1 - 2c_2 + \frac{1}{12} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{67}{4} \\ c_2 &= \frac{65}{3} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{65\,\mathrm{e}^{-2x}}{3} - \frac{67\,\mathrm{e}^{-3x}}{4} + \frac{\mathrm{e}^{x}}{12}$$

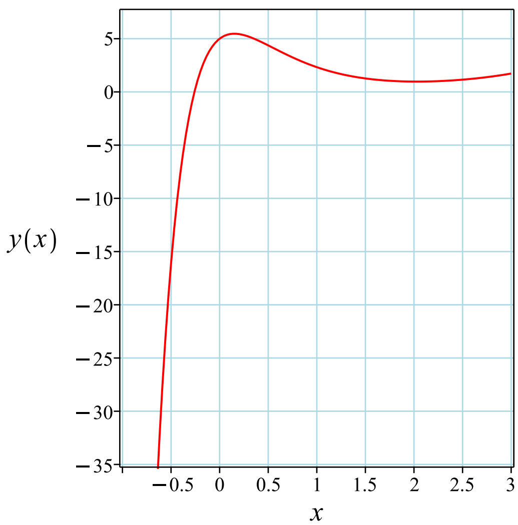
Which simplifies to

$$y = \frac{\left(\mathrm{e}^{4x} + 260\,\mathrm{e}^{x} - 201\right)\mathrm{e}^{-3x}}{12}$$

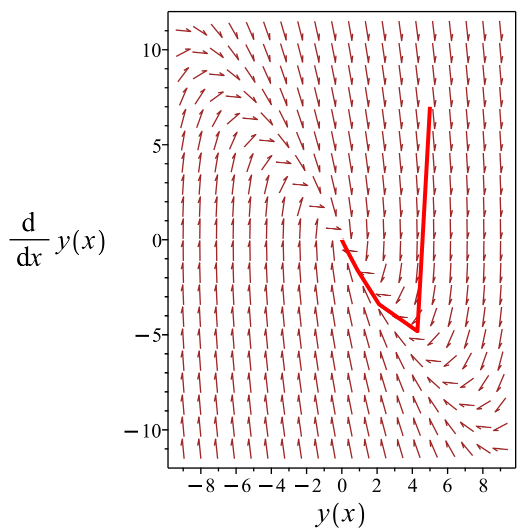
<u>Summary</u>

The solution(s) found are the following

$$y = \frac{\left(\mathrm{e}^{4x} + 260\,\mathrm{e}^{x} - 201\right)\mathrm{e}^{-3x}}{12} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\left(\mathrm{e}^{4x} + 260\,\mathrm{e}^{x} - 201\right)\mathrm{e}^{-3x}}{12}$$

Verified OK.

8.1.4 Maple step by step solution

Let's solve

$\left[y'' + 5y' + 6y = e^{x}, y(0) = 5, y'\Big|_{\{x=0\}} = 7\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r+3)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-3, -2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3x} + c_2 e^{-2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-3x} & e^{-2x} \\ -3e^{-3x} & -2e^{-2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{-5x}$

  - Substitute functions into equation for $y_p(x)$

$y_p(x) = -\mathrm{e}^{-3x}\left(\int \mathrm{e}^{4x}dx\right) + \mathrm{e}^{-2x}\left(\int \mathrm{e}^{3x}dx\right)$

- Compute integrals

  $y_p(x) = \frac{\mathrm{e}^x}{12}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-3x} + c_2\mathrm{e}^{-2x} + \frac{\mathrm{e}^x}{12}$

☐ Check validity of solution $y = c_1\mathrm{e}^{-3x} + c_2\mathrm{e}^{-2x} + \frac{\mathrm{e}^x}{12}$

- Use initial condition $y(0) = 5$

  $5 = c_1 + c_2 + \frac{1}{12}$

- Compute derivative of the solution

  $y' = -3c_1\mathrm{e}^{-3x} - 2c_2\mathrm{e}^{-2x} + \frac{\mathrm{e}^x}{12}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 7$

  $7 = -3c_1 - 2c_2 + \frac{1}{12}$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{67}{4}, c_2 = \frac{65}{3}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{(\mathrm{e}^{4x}+260\,\mathrm{e}^x-201)\,\mathrm{e}^{-3x}}{12}$

- Solution to the IVP

  $y = \frac{(\mathrm{e}^{4x}+260\,\mathrm{e}^x-201)\,\mathrm{e}^{-3x}}{12}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve([diff(y(x),x$2)+5*diff(y(x),x)+6*y(x)=exp(x),y(0) = 5, D(y)(0) = 7],y(x), singsol=all
```

$$y(x) = \frac{(e^{4x} + 260 e^{x} - 201) e^{-3x}}{12}$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 26

```
DSolve[{y''[x]+5*y'[x]+6*y[x]==Exp[x],{y[0]==5,y'[0]==7}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{12} e^{-3x} \left(260 e^{x} + e^{4x} - 201\right)$$

8.2 problem 1 (b)

Internal problem ID [11714]
Internal file name [`OUTPUT/11723_Thursday_April_11_2024_08_49_06_PM_48965104/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 113
Problem number: 1 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 5y' + 6y = e^{x}}$$

With initial conditions

$$[y(0) = 5, y'(1) = 7]$$

8.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 5, C = 6, f(x) = e^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 5\lambda\, e^{\lambda x} + 6\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 6$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{5^2 - (4)(1)(6)} \\
&= -\frac{5}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{5}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{5}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -2 \\
\lambda_2 &= -3
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(-2)x} + c_2 e^{(-3)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-2x} + c_2 e^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3x}, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{-3x}\right) + \left(\frac{e^x}{12}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^x}{12} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 0$ in the above gives

$$5 = c_1 + c_2 + \frac{1}{12} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2x} - 3c_2 e^{-3x} + \frac{e^x}{12}$$

substituting $y' = 7$ and $x = 1$ in the above gives

$$7 = \frac{(e^4 - 24\, e c_1 - 36 c_2)\, e^{-3}}{12} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{e^4 - 84\, e^3 - 177}{24\, e - 36}$$
$$c_2 = \frac{(-e^3 + 84\, e^2 + 118)\, e}{24\, e - 36}$$

Substituting these values back in above solution results in

$$y = \frac{171\, e^{4-3x} - 252\, e^{3-3x} - 2\, e^{-3x+5} + 236\, e^{-3x+2} + 2\, e^{-2x+5} - 171\, e^{-2x+4} + 4\, e^{x+2} - 354\, e^{-2x+1} - 354\, e^{-}}{48\, e^2 - 144\, e + 108}$$

Which simplifies to

$$y = -\frac{(354\, e^{1-x} + 171\, e^4 - 252\, e^3 + 12\, e^{3x+1} - 9\, e^{3x} - 171\, e^{4-x} + 252\, e^{-x+3} + 2\, e^{-x+5} - 236\, e^{-x+2} - 2\, e^5 -}{48\, e^2 - 144\, e + 108}$$

Summary

The solution(s) found are the following

$$y = -\frac{(354\, e^{1-x} + 171\, e^4 - 252\, e^3 + 12\, e^{3x+1} - 9\, e^{3x} - 171\, e^{4-x} + 252\, e^{-x+3} + 2\, e^{-x+5} - 236\, e^{-x+2} - 2\, e^5 -}{48\, e^2 - 144\, e + 108} \tag{1}$$

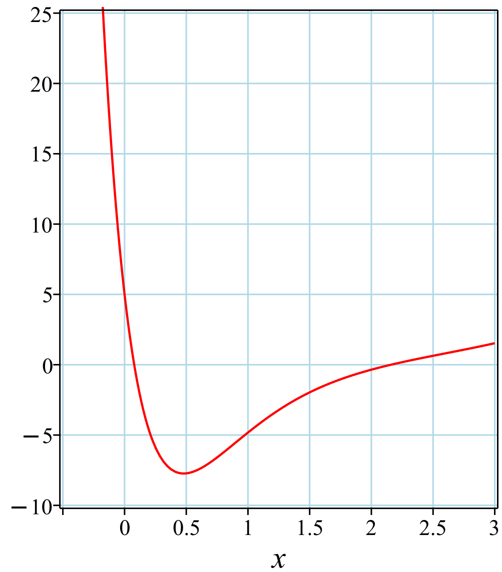


Figure 331: Solution plot

Verification of solutions

$$y = -\frac{(354\,\mathrm{e}^{1-x} + 171\,\mathrm{e}^{4} - 252\,\mathrm{e}^{3} + 12\,\mathrm{e}^{3x+1} - 9\,\mathrm{e}^{3x} - 171\,\mathrm{e}^{4-x} + 252\,\mathrm{e}^{-x+3} + 2\,\mathrm{e}^{-x+5} - 236\,\mathrm{e}^{-x+2} - 2\,\mathrm{e}^{5} -}{48\,\mathrm{e}^{2} - 144\,\mathrm{e} + 108}$$

Verified OK.

8.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 254: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dx} \\ &= z_1 e^{-\frac{5x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{5x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-5x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}\left(\mathrm{e}^{x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 5y' + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-2x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-3x}, \mathrm{e}^{-2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
12A_1 \mathrm{e}^{x} = \mathrm{e}^{x}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + c_2 e^{-2x}\right) + \left(\frac{e^x}{12}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-3x} + c_2 e^{-2x} + \frac{e^x}{12} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 0$ in the above gives

$$5 = c_1 + c_2 + \frac{1}{12} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 e^{-3x} - 2c_2 e^{-2x} + \frac{e^x}{12}$$

substituting $y' = 7$ and $x = 1$ in the above gives

$$7 = \frac{\left(e^4 - 24c_2 e - 36c_1\right) e^{-3}}{12} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{\left(-e^3 + 84\,e^2 + 118\right) e}{24\,e - 36} \\ c_2 &= \frac{e^4 - 84\,e^3 - 177}{24\,e - 36} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{171\,e^{4-3x} - 252\,e^{3-3x} - 2\,e^{-3x+5} + 236\,e^{-3x+2} + 2\,e^{-2x+5} - 171\,e^{-2x+4} + 4\,e^{x+2} - 354\,e^{-2x+1} - 354\,e^{-}}{48\,e^{2} - 144\,e + 108}$$

Which simplifies to

$$y = -\frac{(354\,e^{1-x} + 171\,e^{4} - 252\,e^{3} + 12\,e^{3x+1} - 9\,e^{3x} - 171\,e^{4-x} + 252\,e^{-x+3} + 2\,e^{-x+5} - 236\,e^{-x+2} - 2\,e^{5} -}{48\,e^{2} - 144\,e + 108}$$

Summary

The solution(s) found are the following

$$y = -\frac{(354\,e^{1-x} + 171\,e^{4} - 252\,e^{3} + 12\,e^{3x+1} - 9\,e^{3x} - 171\,e^{4-x} + 252\,e^{-x+3} + 2\,e^{-x+5} - 236\,e^{-x+2} - 2\,e^{5} -}{48\,e^{2} - 144\,e + 108} \quad (1)$$

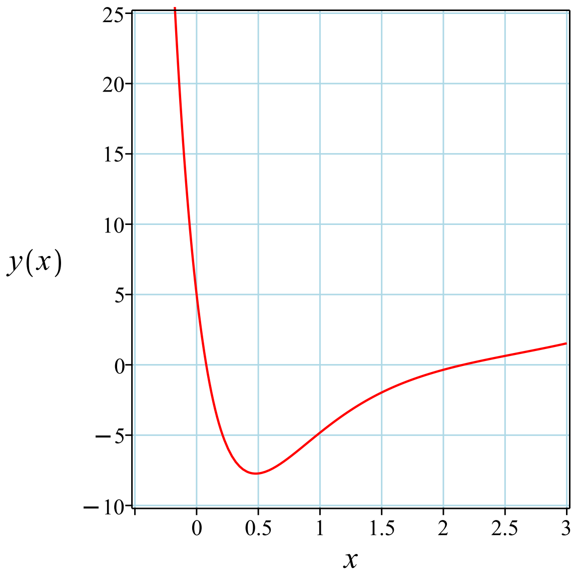


Figure 332: Solution plot

Verification of solutions

$$y = -\frac{(354\,e^{1-x} + 171\,e^{4} - 252\,e^{3} + 12\,e^{3x+1} - 9\,e^{3x} - 171\,e^{4-x} + 252\,e^{-x+3} + 2\,e^{-x+5} - 236\,e^{-x+2} - 2\,e^{5} -}{48\,e^{2} - 144\,e + 108}$$

Verified OK.

8.2.3 Maple step by step solution

Let's solve

$$\left[y''+5y'+6y=e^{x}, y(0)=5, y'\Big|_{\{x=1\}}=7\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+5r+6=0$

- Factor the characteristic polynomial

  $(r+3)(r+2)=0$

- Roots of the characteristic polynomial

  $r=(-3,-2)$

- 1st solution of the homogeneous ODE

  $y_1(x)=e^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x)=e^{-2x}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1e^{-3x}+c_2e^{-2x}+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x)=-y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=e^{x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x),y_2(x))=\begin{bmatrix} e^{-3x} & e^{-2x} \\ -3e^{-3x} & -2e^{-2x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x),y_2(x))=e^{-5x}$

- ○ Substitute functions into equation for $y_p(x)$

$y_p(x) = -\mathrm{e}^{-3x}\left(\int \mathrm{e}^{4x}dx\right) + \mathrm{e}^{-2x}\left(\int \mathrm{e}^{3x}dx\right)$

- Compute integrals

  $y_p(x) = \frac{\mathrm{e}^x}{12}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-3x} + c_2\mathrm{e}^{-2x} + \frac{\mathrm{e}^x}{12}$

- ☐ Check validity of solution $y = c_1\mathrm{e}^{-3x} + c_2\mathrm{e}^{-2x} + \frac{\mathrm{e}^x}{12}$
  - Use initial condition $y(0) = 5$

    $5 = c_1 + c_2 + \frac{1}{12}$
  - Compute derivative of the solution

    $y' = -3c_1\mathrm{e}^{-3x} - 2c_2\mathrm{e}^{-2x} + \frac{\mathrm{e}^x}{12}$
  - Use the initial condition $y'\Big|_{\{x=1\}} = 7$

    $7 = -3c_1\mathrm{e}^{-3} - 2c_2\mathrm{e}^{-2} + \frac{\mathrm{e}}{12}$
  - Solve for c_1 and c_2

    $\left\{c_1 = \frac{118\,\mathrm{e}^{-2} - \mathrm{e} + 84}{12(2\,\mathrm{e}^{-2} - 3\,\mathrm{e}^{-3})}, c_2 = -\frac{-\mathrm{e} + 177\,\mathrm{e}^{-3} + 84}{12(2\,\mathrm{e}^{-2} - 3\,\mathrm{e}^{-3})}\right\}$
  - Substitute constant values into general solution and simplify

    $y = \frac{(\mathrm{e}^{x+4} + 2\,\mathrm{e}^{4x+1} - \mathrm{e}^4 - 84\,\mathrm{e}^{x+3} - 3\,\mathrm{e}^{4x} + 84\,\mathrm{e}^3 + 118\,\mathrm{e} - 177\,\mathrm{e}^x)\,\mathrm{e}^{-3x}}{24\,\mathrm{e} - 36}$

- Solution to the IVP

  $y = \frac{(\mathrm{e}^{x+4} + 2\,\mathrm{e}^{4x+1} - \mathrm{e}^4 - 84\,\mathrm{e}^{x+3} - 3\,\mathrm{e}^{4x} + 84\,\mathrm{e}^3 + 118\,\mathrm{e} - 177\,\mathrm{e}^x)\,\mathrm{e}^{-3x}}{24\,\mathrm{e} - 36}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 55

```
dsolve([diff(y(x),x$2)+5*diff(y(x),x)+6*y(x)=exp(x),y(0) = 5, D(y)(1) = 7],y(x), singsol=all
```

$$y(x) = \frac{(-e^{4-x} + 84\,e^{3-x} + e^{4} + 2\,e^{3x+1} - 84\,e^{3} + 118\,e^{1-x} - 3\,e^{3x} - 177)\,e^{-2x}}{24\,e - 36}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 68

```
DSolve[{y''[x]+5*y'[x]+6*y[x]==Exp[x],{y[0]==5,y'[1]==7}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{e^{-3x}(-177e^{x} - 3e^{4x} - 84e^{x+3} + e^{x+4} + 2e^{4x+1} + 118e + 84e^{3} - e^{4})}{12(2e-3)}$$

8.3 problem 2

Internal problem ID [11715]
Internal file name [`OUTPUT/11724_Thursday_April_11_2024_08_49_07_PM_26398801/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 113
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{y'' + y'x + x^2y = 0}$$

With initial conditions

$$[y(1) = 0, y'(1) = 0]$$

8.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= x \\ q(x) &= x^2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y'x + x^2 y = 0$$

The domain of $p(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

8.3.2 Maple step by step solution

Let's solve

$$\left[y'' + y'x + x^2 y = 0, y(1) = 0, y'\Big|_{\{x=1\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- ☐ Rewrite ODE with series expansions
  - Convert $x^2 \cdot y$ to series expansion

    $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$

  - Shift index using $k- > k - 2$

    $x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  - Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 + (6a_3 + a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_k k + a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 + a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = 0, a_3 = -\frac{a_1}{6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + a_k k + a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + a_{k+2}(k+2) + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{k a_{k+2} + a_k + 2a_{k+2}}{k^2 + 7k + 12}, a_2 = 0, a_3 = -\frac{a_1}{6}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.641 (sec). Leaf size: 5

```
dsolve([diff(y(x),x$2)+x*diff(y(x),x)+x^2*y(x)=0,y(1) = 0, D(y)(1) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 6

```
DSolve[{y''[x]+x*y'[x]+x^2*y[x]==0,{y[1]==0,y'[1]==0}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to 0$$

8.4 problem 4 (a)

Internal problem ID [11716]
Internal file name [`OUTPUT/11725_Thursday_April_11_2024_08_49_08_PM_4293718/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 113
Problem number: 4 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' + 3y = 0}$$

8.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 3 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 3$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-4^2 - (4)(1)(3)} \\
&= 2 \pm 1
\end{aligned}
$$

Hence

$$
\lambda_1 = 2 + 1
$$
$$
\lambda_2 = 2 - 1
$$

Which simplifies to

$$
\lambda_1 = 3
$$
$$
\lambda_2 = 1
$$

Since roots are real and distinct, then the solution is

$$
y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}
$$
$$
y = c_1 e^{(3)x} + c_2 e^{(1)x}
$$

Or

$$
y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{x}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{x} \tag{1}
$$

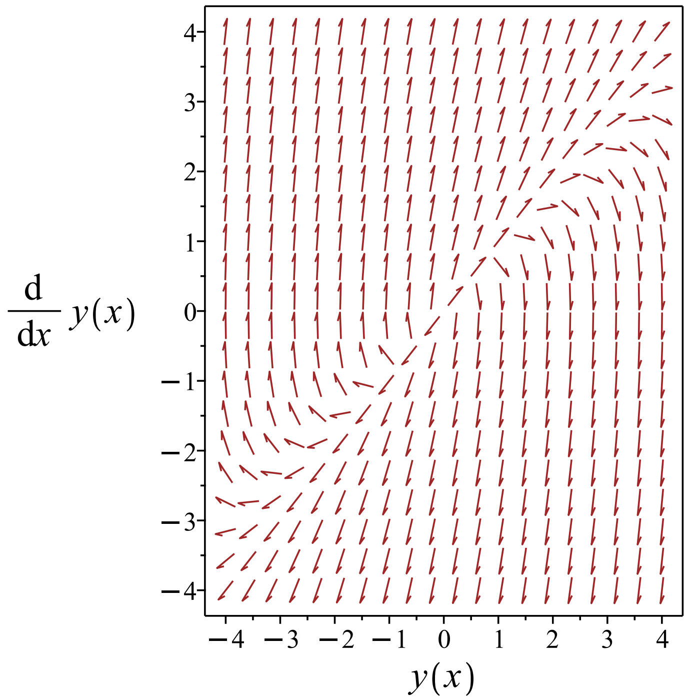


Figure 333: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{x}$$

Verified OK.

8.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 257: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx}\\ &= z_1 e^{2x}\\ &= z_1\left(\mathrm{e}^{2x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^x c_1 + \frac{\mathrm{e}^{3x} c_2}{2} \tag{1}
$$

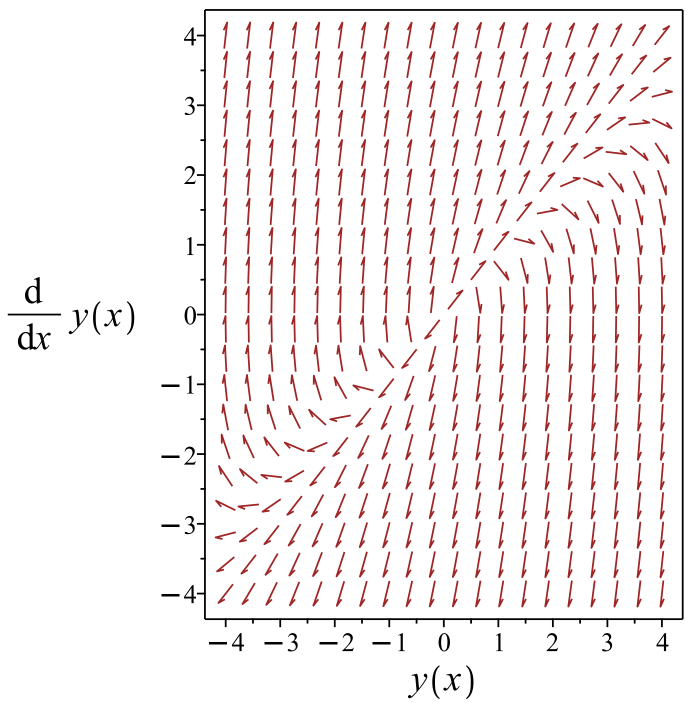


Figure 334: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^x c_1 + \frac{\mathrm{e}^{3x} c_2}{2}
$$

Verified OK.

8.4.3 Maple step by step solution

Let's solve

$y'' - 4y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (1, 3)$

- 1st solution of the ODE

  $y_1(x) = e^x$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = e^x c_1 + e^{3x} c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + c_2 e^{3x}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 20

```
DSolve[y''[x]-4*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x \left(c_2 e^{2x} + c_1\right)$$

8.5 problem 8

Internal problem ID [11717]
Internal file name [`OUTPUT/11726_Thursday_April_11_2024_08_49_08_PM_4170906/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 113
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 4]$$

8.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + y = 0$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.5.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\lambda_{1,2} = \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)}$$
$$= 1$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^x + c_2 x\, e^x \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^x c_1 + c_2 x\, e^x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^x c_1 + c_2 e^x + c_2 x\, e^x$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

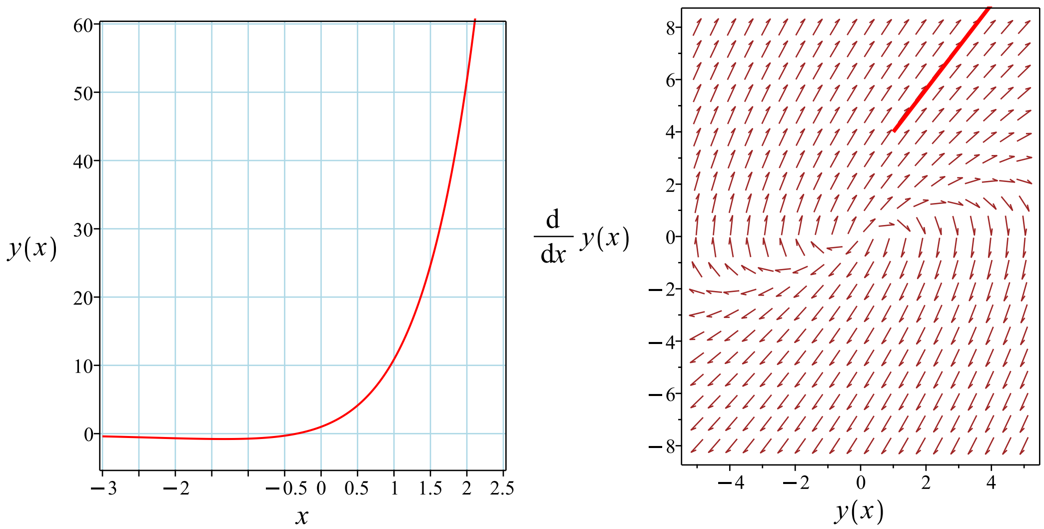
$$c_1 = 1$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = 3x\, e^x + e^x$$

Summary
The solution(s) found are the following

$$y = 3x\, e^x + e^x \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = 3x\,e^{x} + e^{x}$$

Verified OK.

8.5.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^{2} + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(e^{-x}y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(e^{-x} y\right)' = c_1$$

Integrating again gives

$$\left(e^{-x} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{e^{-x}}$$

Or

$$y = c_1 x\, e^{x} + c_2 e^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x\, e^{x} + c_2 e^{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x} c_1 + c_1 x\, e^{x} + c_2 e^{x}$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 1$$

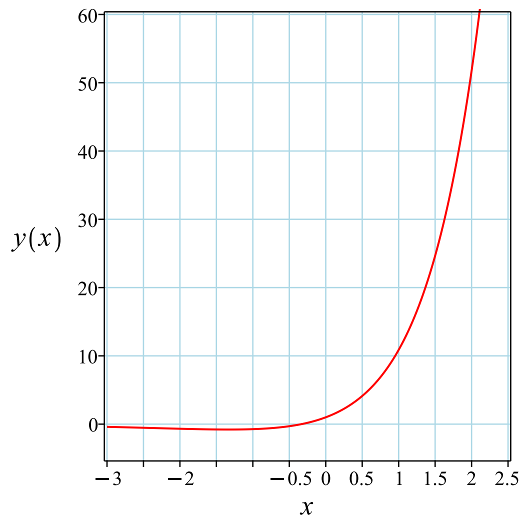
Substituting these values back in above solution results in

$$y = 3x\, e^{x} + e^{x}$$

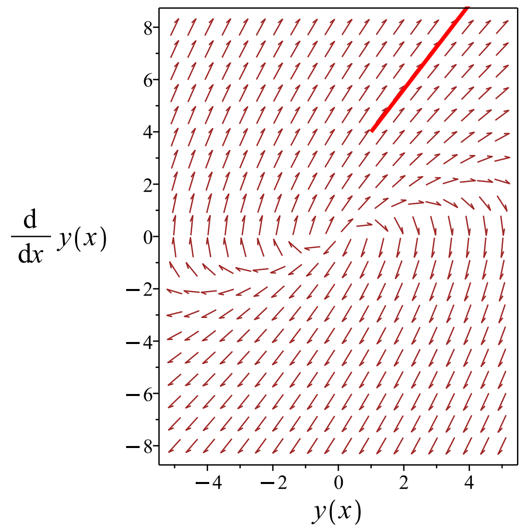
Summary

The solution(s) found are the following

$$y = 3x\, e^{x} + e^{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 3x\,\mathrm{e}^{x} + \mathrm{e}^{x}$$

Verified OK.

8.5.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 259: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(x))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \mathrm{e}^{x} c_1 + c_2 x\,\mathrm{e}^{x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \mathrm{e}^{x} c_1 + c_2 \mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x}
$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$
4 = c_1 + c_2 \tag{2A}
$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= 3
\end{aligned}
$$

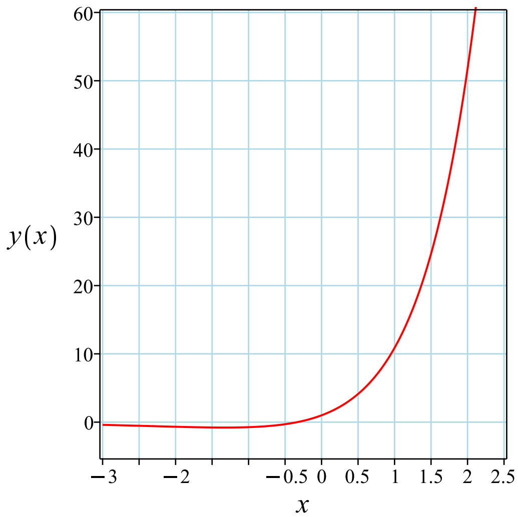
Substituting these values back in above solution results in

$$
y = 3x\,\mathrm{e}^{x} + \mathrm{e}^{x}
$$

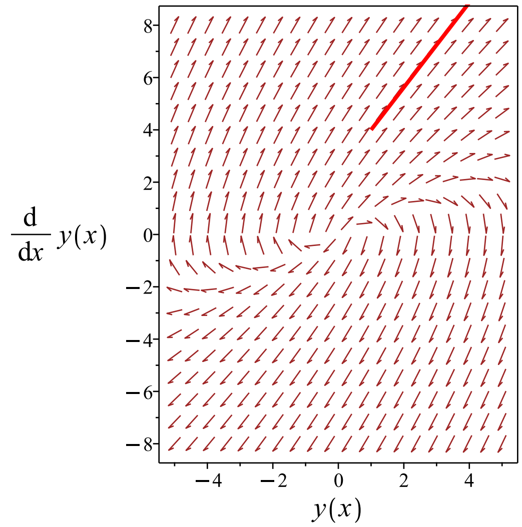
Summary

The solution(s) found are the following

$$
y = 3x\,\mathrm{e}^{x} + \mathrm{e}^{x} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3x\,\mathrm{e}^{x} + \mathrm{e}^{x}$$

Verified OK.

8.5.5 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + y = 0, y(0) = 1, y'\Big|_{\{x=0\}} = 4\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the ODE

$y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = e^x c_1 + c_2 x\,e^x$

- ☐ Check validity of solution $y = e^x c_1 + c_2 x e^x$
  - Use initial condition $y(0) = 1$

    $1 = c_1$
  - Compute derivative of the solution

    $y' = e^x c_1 + c_2 e^x + c_2 x\,e^x$
  - Use the initial condition $y'\Big|_{\{x=0\}} = 4$

    $4 = c_1 + c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = 3\}$
  - Substitute constant values into general solution and simplify

    $y = e^x(3x + 1)$

- Solution to the IVP

  $y = e^x(3x + 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+y(x)=0,y(0) = 1, D(y)(0) = 4],y(x), singsol=all)
```

$$y(x) = e^{x}(3x + 1)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 14

```
DSolve[{y''[x]-2*y'[x]+y[x]==0,{y[0]==1,y'[0]==4}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}(3x + 1)$$

8.6 problem 9

Internal problem ID [11718]
Internal file name [`OUTPUT/11727_Thursday_April_11_2024_08_49_09_PM_74917571/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 113
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 2y'x + 2y = 0}$$

With initial conditions

$$[y(1) = 3, y'(1) = 2]$$

8.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{2}{x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{2y'}{x} + \frac{2y}{x^2} = 0$$

The domain of $p(x) = -\frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{2}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

8.6.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 2xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 2r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 2r + 2 = 0$$

Or

$$r^2 - 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2 x^2 + c_1 x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x^2 + c_1 x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_2 x + c_1$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$2 = 2c_2 + c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = -x^2 + 4x$$

Summary

The solution(s) found are the following

$$y = -x^2 + 4x \tag{1}$$

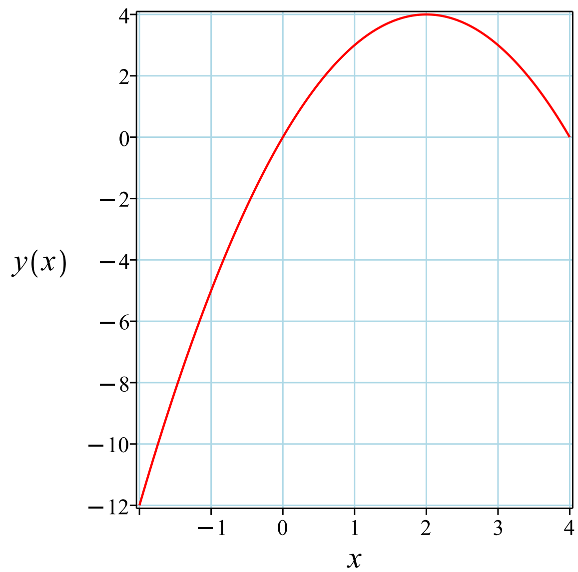


Figure 338: Solution plot

Verification of solutions

$$y = -x^2 + 4x$$

Verified OK.

8.6.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -\frac{2}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -\frac{2}{x}\, dx} \\ &= \frac{1}{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 0$$
$$\left(\frac{y}{x}\right)'' = 0$$

Integrating once gives

$$\left(\frac{y}{x}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{x}\right) = c_1x + c_2$$

Hence the solution is

$$y = \frac{c_1x + c_2}{\frac{1}{x}}$$

Or

$$y = c_1x^2 + c_2x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1x^2 + c_2x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1x + c_2$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$2 = 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 4$$

Substituting these values back in above solution results in

$$y = -x^2 + 4x$$

Summary

The solution(s) found are the following

$$y = -x^2 + 4x \tag{1}$$

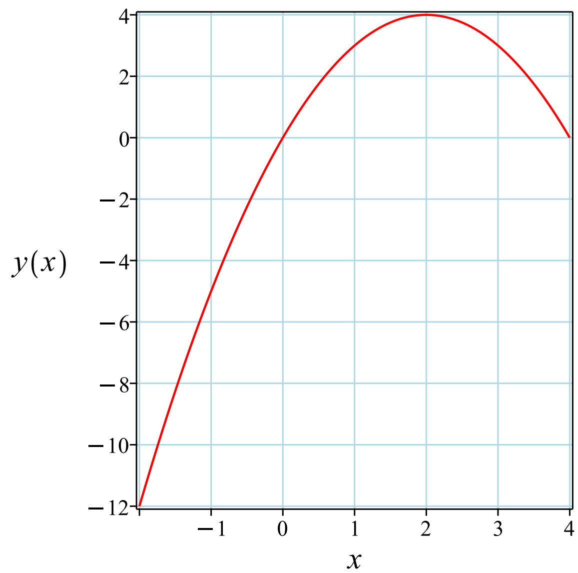


Figure 339: Solution plot

Verification of solutions

$$y = -x^2 + 4x$$

Verified OK.

8.6.4 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - 2y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{2}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int -\frac{2}{x}dx\right)} dx \\
&= \int e^{2\ln(x)}\, dx \\
&= \int x^2 dx \\
&= \frac{x^3}{3}
\end{aligned}
\tag{6}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{2}{x^2}}{x^4} \\
&= \frac{2}{x^6}
\end{aligned}
\tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{x^6} = 0$$

But in terms of τ

$$\frac{2}{x^6} = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{1}{3}} + c_2 \tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = \frac{3^{\frac{1}{3}} \left(c_1 3^{\frac{1}{3}} + c_2\right)}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 3^{\frac{2}{3}} x^2}{3 (x^3)^{\frac{2}{3}}} + \frac{2 c_2 3^{\frac{1}{3}} x^2}{3 (x^3)^{\frac{1}{3}}}$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$2 = \frac{3^{\frac{1}{3}} \left(c_1 3^{\frac{1}{3}} + 2 c_2\right)}{3} \tag{2A}$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 4\, 3^{\frac{1}{3}} \\ c_2 &= -3^{\frac{2}{3}} \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\left(x^3\right)^{\frac{2}{3}} + 4\left(x^3\right)^{\frac{1}{3}}$$

Summary

The solution(s) found are the following

$$y = -\left(x^3\right)^{\frac{2}{3}} + 4\left(x^3\right)^{\frac{1}{3}} \tag{1}$$

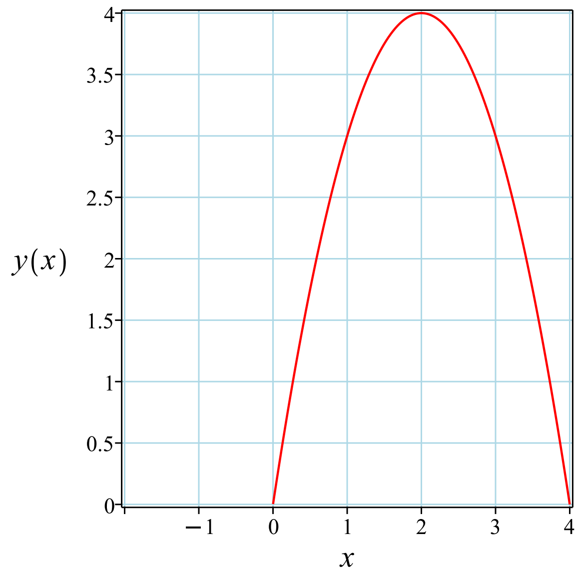


Figure 340: Solution plot

Verification of solutions

$$y = -\left(x^3\right)^{\frac{2}{3}} + 4\left(x^3\right)^{\frac{1}{3}}$$

Verified OK.

8.6.5 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' - 2y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \\ &= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{2}{x}\frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{3c\sqrt{2}}{2}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{3c\sqrt{2}\,\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1\cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2\sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{2}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{3}{2}}\left(c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = x^{\frac{3}{2}}\left(c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)\right) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$
3 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{3\sqrt{x}\left(c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)\right)}{2} + x^{\frac{3}{2}}\left(\frac{c_1\sinh\left(\frac{\ln(x)}{2}\right)}{2x} + \frac{ic_2\cosh\left(\frac{\ln(x)}{2}\right)}{2x}\right)
$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$
2 = \frac{3c_1}{2} + \frac{ic_2}{2} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= 5i
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = -5x^{\frac{3}{2}} \sinh\left(\frac{\ln(x)}{2}\right) + 3\cosh\left(\frac{\ln(x)}{2}\right) x^{\frac{3}{2}}$$

Which simplifies to

$$y = \left(-5\sinh\left(\frac{\ln(x)}{2}\right) + 3\cosh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{3}{2}}$$

Summary

The solution(s) found are the following

$$y = \left(-5\sinh\left(\frac{\ln(x)}{2}\right) + 3\cosh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{3}{2}} \tag{1}$$

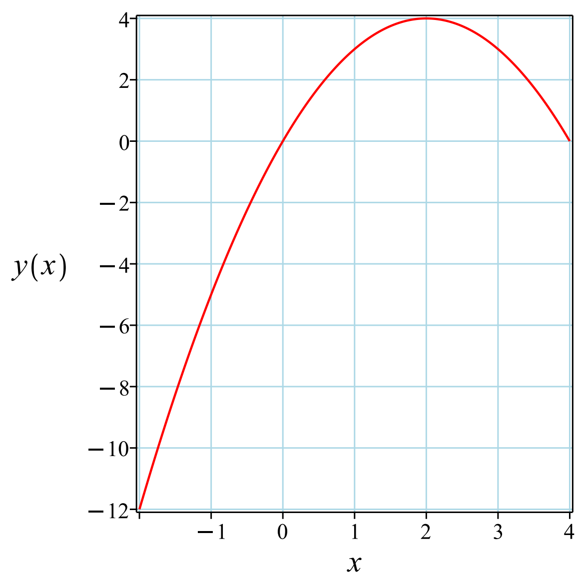


Figure 341: Solution plot

Verification of solutions

$$y = \left(-5\sinh\left(\frac{\ln(x)}{2}\right) + 3\cosh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{3}{2}}$$

Verified OK.

8.6.6 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\ &= \frac{2}{x^2} - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{-\frac{2}{x}}{2}} \\ &= x \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x \tag{4}$$

Applying this change of variable to the original ode results in

$$x^3 v''(x) = 0$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$v(x) = c_1x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1x + c_2)\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x$$

Hence (7) becomes

$$y = (c_1x + c_2)\, x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (c_1x + c_2)\, x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1x + c_2$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$2 = 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -1 \\ c_2 &= 4 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -x(-4 + x)$$

Summary
The solution(s) found are the following

$$y = -x(-4 + x) \tag{1}$$

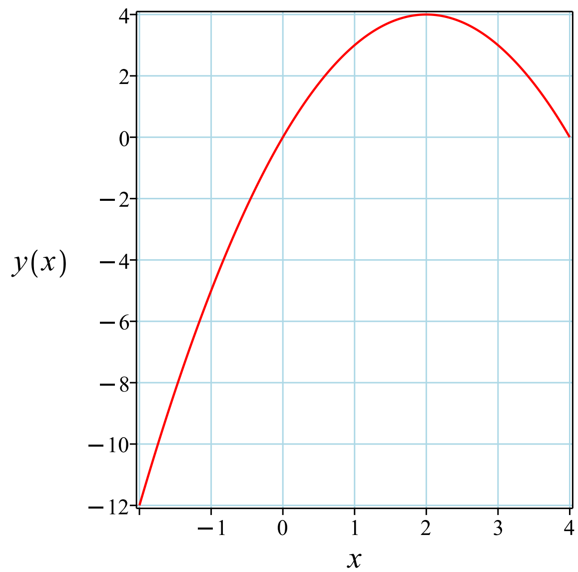


Figure 342: Solution plot

Verification of solutions

$$y = -x(-4 + x)$$

Verified OK.

8.6.7 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' - 2y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{2n}{x^2} + \frac{2}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\
\ln(u) &= -2\ln(x) + c_1 \\
u &= \mathrm{e}^{-2\ln(x)+c_1} \\
&= \frac{c_1}{x^2}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{x} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{x} + c_2\right)x^2 \\
&= (c_2 x - c_1)\,x
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \left(-\frac{c_1}{x} + c_2\right)x^2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$
3 = -c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 + 2\left(-\frac{c_1}{x} + c_2\right)x
$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$2 = -c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -4$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = -x(-4 + x)$$

Summary
The solution(s) found are the following

$$y = -x(-4 + x) \tag{1}$$

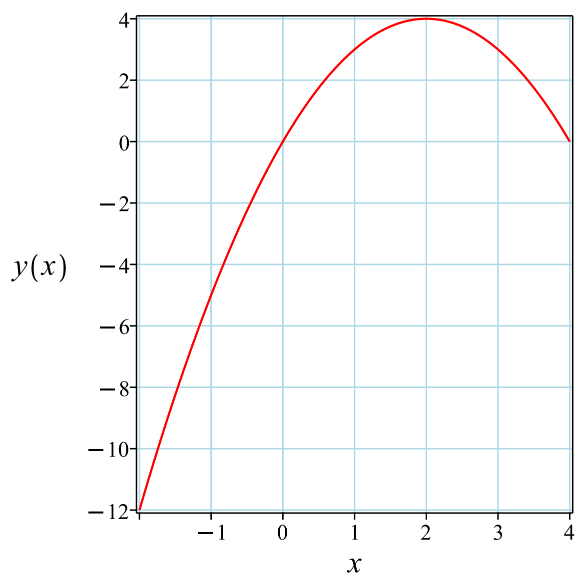


Figure 343: Solution plot

Verification of solutions

$$y = -x(-4 + x)$$

Verified OK.

8.6.8 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= 2 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2\right)(0) + (-2x)(-2) + (2)(-2x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x^3v'' + (0)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$-2x^3u'(x) = 0$$

Which is now solved for u. Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0 \,\mathrm{d}x \\ &= c_1 \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= c_1 \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int c_1 \,\mathrm{d}x \\ &= c_1x + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (-2x)\,(c_1x + c_2) \\ &= -2(c_1x + c_2)\, x \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -2(c_1x + c_2)\, x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = -2c_1 - 2c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1x - 2c_2$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$2 = -4c_1 - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{2}$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$y = -x(-4 + x)$$

Summary

The solution(s) found are the following

$$y = -x(-4 + x) \tag{1}$$

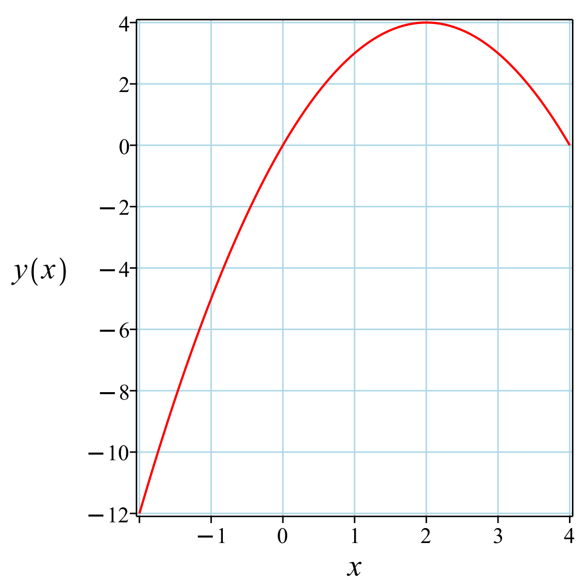


Figure 344: Solution plot

Verification of solutions

$$y = -x(-4 + x)$$

Verified OK.

8.6.9 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 2y'x + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 261: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2(x(x))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_2 x^2 + c_1 x \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$
3 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 2c_2 x + c_1
$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$2 = 2c_2 + c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = -x^2 + 4x$$

Summary

The solution(s) found are the following

$$y = -x^2 + 4x \tag{1}$$

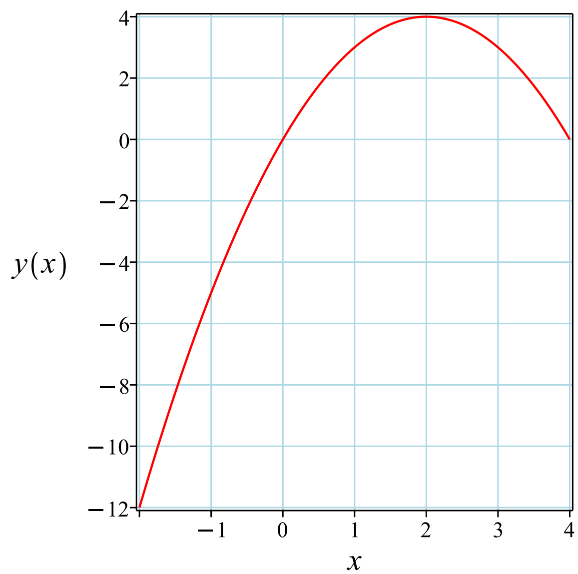


Figure 345: Solution plot

Verification of solutions

$$y = -x^2 + 4x$$

Verified OK.

8.6.10 Maple step by step solution

Let's solve

$$\left[y''x^2-2y'x+2y=0, y(1)=3, y'\Big|_{\{x=1\}}=2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2y'}{x}-\frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2y'}{x}+\frac{2y}{x^2}=0$

- Multiply by denominators of the ODE

  $y''x^2-2y'x+2y=0$

- Make a change of variables

  $t=\ln(x)$

- □ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y'=\left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y'=\frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y''=\left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2+t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y''=\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)x^2-2\frac{d}{dt}y(t)+2y(t)=0$

- Simplify

  $\frac{d^2}{dt^2}y(t)-3\frac{d}{dt}y(t)+2y(t)=0$

- Characteristic polynomial of ODE

$r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the ODE

  $y_1(t) = e^t$

- 2nd solution of the ODE

  $y_2(t) = e^{2t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^t + c_2 e^{2t}$

- Change variables back using $t = \ln(x)$

  $y = c_2 x^2 + c_1 x$

- Simplify

  $y = x(c_2 x + c_1)$

□ Check validity of solution $y = x(c_2 x + c_1)$

  - Use initial condition $y(1) = 3$

    $3 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = 2c_2 x + c_1$

  - Use the initial condition $y'\Big|_{\{x=1\}} = 2$

    $2 = 2c_2 + c_1$

  - Solve for c_1 and c_2

    $\{c_1 = 4, c_2 = -1\}$

  - Substitute constant values into general solution and simplify

    $y = -x(-4 + x)$

- Solution to the IVP

$$y = -x(-4 + x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve([x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(1) = 3, D(y)(1) = 2],y(x), singsol=al
```

$$y(x) = -x^2 + 4x$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 11

```
DSolve[{x^2*y''[x]-2*x*y'[x]+2*y[x]==0,{y[1]==3,y'[1]==2}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \rightarrow -((x - 4)x)$$

8.7 problem 10

Internal problem ID [11719]
Internal file name [`OUTPUT/11728_Thursday_April_11_2024_08_49_11_PM_90841023/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 113
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' + y'x - 4y = 0}$$

With initial conditions

$$[y(2) = 3, y'(2) = -1]$$

8.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{y'}{x} - \frac{4y}{x^2} = 0$$

The domain of $p(x) = \frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = -\frac{4}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

8.7.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 4 = 0$$

Or

$$r^2 - 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -2$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + c_2 x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x^2} + c_2 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 2$ in the above gives

$$3 = \frac{c_1}{4} + 4c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{2c_1}{x^3} + 2c_2 x$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = -\frac{c_1}{4} + 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 8$$
$$c_2 = \frac{1}{4}$$

Substituting these values back in above solution results in

$$y = \frac{x^4 + 32}{4x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^4 + 32}{4x^2} \tag{1}$$

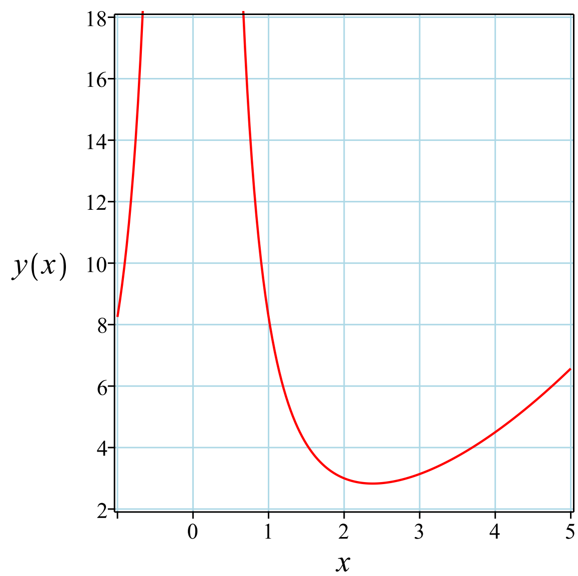


Figure 346: Solution plot

Verification of solutions

$$y = \frac{x^4 + 32}{4x^2}$$

Verified OK.

8.7.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' + y' x - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\
&= \int e^{-\ln(x)}\,dx \\
&= \int \frac{1}{x}dx \\
&= \ln(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{4}{x^2}}{\frac{1}{x^2}} \\
&= -4
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 4y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} - 4\,\mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)} \\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2 \\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\ y(\tau) &= c_1 e^{(2)\tau} + c_2 e^{(-2)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 \mathrm{e}^{2\tau} + c_2 \mathrm{e}^{-2\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^4 + c_2}{x^2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 x^4 + c_2}{x^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 2$ in the above gives

$$3 = 4c_1 + \frac{c_2}{4} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 4c_1 x - \frac{2(c_1 x^4 + c_2)}{x^3}$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = 4c_1 - \frac{c_2}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{4}$$
$$c_2 = 8$$

Substituting these values back in above solution results in

$$y = \frac{x^4 + 32}{4x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^4 + 32}{4x^2} \tag{1}$$

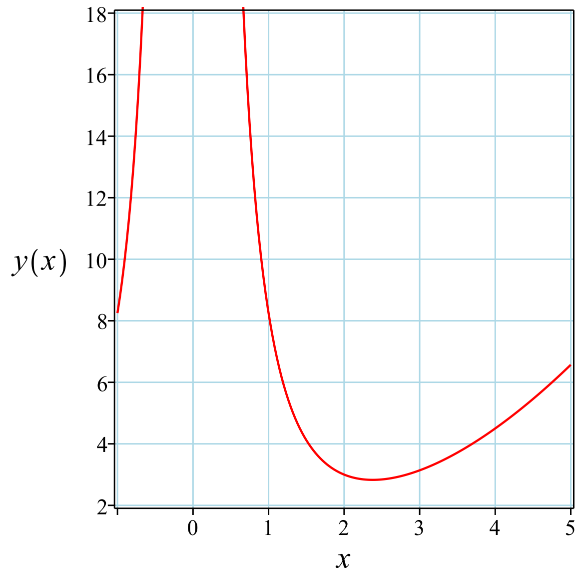


Figure 347: Solution plot

Verification of solutions

$$y = \frac{x^4 + 32}{4x^2}$$

Verified OK.

8.7.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' + y'x - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{2\sqrt{-\frac{1}{x^2}}}{c} \\ \tau'' &= \frac{2}{c\sqrt{-\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2}\\ &= \frac{\frac{2}{c\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{2\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{-\frac{1}{x^2}}}{c}\right)^2}\\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx\\ &= \frac{\int 2\sqrt{-\frac{1}{x^2}}dx}{c}\\ &= \frac{2\sqrt{-\frac{1}{x^2}}\,x\ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cosh(2\ln(x)) + i c_2 \sinh(2\ln(x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cosh(2\ln(x)) + i c_2 \sinh(2\ln(x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 2$ in the above gives

$$3 = \frac{15 i c_2}{8} + \frac{17 c_1}{8} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2 c_1 \sinh(2\ln(x))}{x} + \frac{2 i c_2 \cosh(2\ln(x))}{x}$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = \frac{17 i c_2}{8} + \frac{15 c_1}{8} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{33}{4}$$
$$c_2 = \frac{31 i}{4}$$

Substituting these values back in above solution results in

$$y = \frac{33\cosh(2\ln(x))}{4} - \frac{31\sinh(2\ln(x))}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{33\cosh(2\ln(x))}{4} - \frac{31\sinh(2\ln(x))}{4} \tag{1}$$

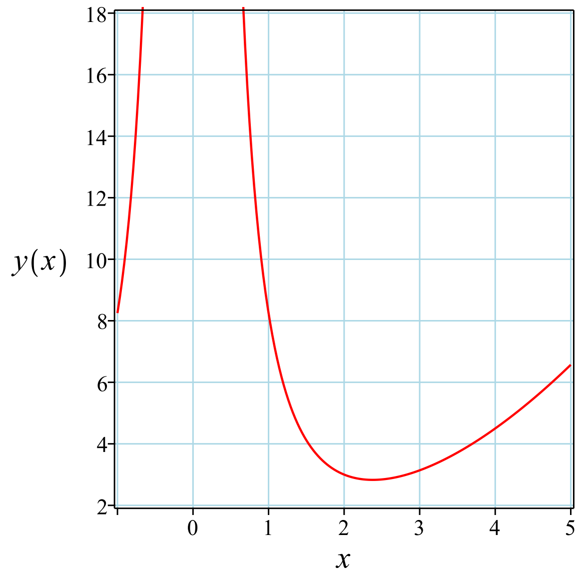


Figure 348: Solution plot

Verification of solutions

$$y = \frac{33 \cosh(2 \ln(x))}{4} - \frac{31 \sinh(2 \ln(x))}{4}$$

Verified OK.

8.7.5 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' + y' x - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{4}{x^2}$$

Applying change of variables on the depndent variable $y = v(x) x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{5v'(x)}{x} &= 0 \\ v''(x) + \frac{5v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{5u}{x} \end{aligned}$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{5}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{5}{x}\,dx \\ \ln(u) &= -5\ln(x) + c_1 \\ u &= \mathrm{e}^{-5\ln(x)+c_1} \\ &= \frac{c_1}{x^5} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{4x^4} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{4x^4} + c_2\right) x^2 \\
&= \frac{4c_2x^4 - c_1}{4x^2}
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \left(-\frac{c_1}{4x^4} + c_2\right) x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 2$ in the above gives

$$3 = -\frac{c_1}{16} + 4c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1}{x^3} + 2\left(-\frac{c_1}{4x^4} + c_2\right) x$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = \frac{c_1}{16} + 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}
c_1 &= -32 \\
c_2 &= \frac{1}{4}
\end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{x^4 + 32}{4x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^4 + 32}{4x^2} \tag{1}$$

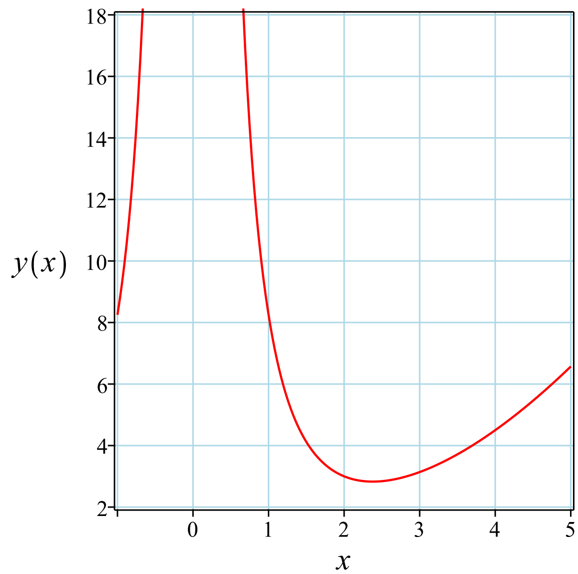


Figure 349: Solution plot

Verification of solutions

$$y = \frac{x^4 + 32}{4x^2}$$

Verified OK.

8.7.6 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + y'x - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 263: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition

of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2x} + (-)\,(0) \\ &= -\frac{3}{2x} \\ &= -\frac{3}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{3}{2x}\right)(0) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x}\right)^2 - \left(\frac{15}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{3}{2x} dx} \\ &= \frac{1}{x^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^4}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^2}\right) + c_2\left(\frac{1}{x^2}\left(\frac{x^4}{4}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_1}{x^2} + \frac{c_2 x^2}{4} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 2$ in the above gives

$$3 = \frac{c_1}{4} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{2c_1}{x^3} + \frac{c_2 x}{2}$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = -\frac{c_1}{4} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 8$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = \frac{x^4 + 32}{4x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^4 + 32}{4x^2} \tag{1}$$

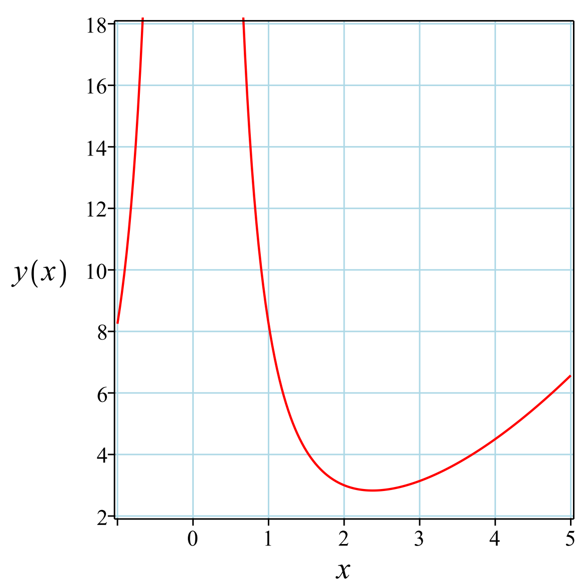


Figure 350: Solution plot

Verification of solutions

$$y = \frac{x^4 + 32}{4x^2}$$

Verified OK.

8.7.7 Maple step by step solution

Let's solve

$$\left[y''x^2 + y'x - 4y = 0, y(2) = 3, y'\Big|_{\{x=2\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + \frac{4y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{4y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 + y'x - 4y = 0$

- Make a change of variables

  $t = \ln(x)$

- □ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)x^2+\frac{d}{dt}y(t)-4y(t)=0$$

- Simplify

  $\frac{d^2}{dt^2}y(t)-4y(t)=0$
- Characteristic polynomial of ODE

  $r^2-4=0$
- Factor the characteristic polynomial

  $(r-2)(r+2)=0$
- Roots of the characteristic polynomial

  $r=(-2,2)$
- 1st solution of the ODE

  $y_1(t)=e^{-2t}$
- 2nd solution of the ODE

  $y_2(t)=e^{2t}$
- General solution of the ODE

  $y(t)=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y(t)=c_1e^{-2t}+c_2e^{2t}$
- Change variables back using $t=\ln(x)$

  $y=\frac{c_1}{x^2}+c_2x^2$
- Simplify

  $y=\frac{c_1}{x^2}+c_2x^2$

☐ Check validity of solution $y=\frac{c_1}{x^2}+c_2x^2$

  - Use initial condition $y(2)=3$

    $3=\frac{c_1}{4}+4c_2$
  - Compute derivative of the solution

    $y'=-\frac{2c_1}{x^3}+2c_2x$
  - Use the initial condition $y'\Big|_{\{x=2\}}=-1$

    $-1=-\frac{c_1}{4}+4c_2$

- Solve for c_1 and c_2

  $\{c_1 = 8, c_2 = \frac{1}{4}\}$

- Substitute constant values into general solution and simplify

  $y = \frac{x^4+32}{4x^2}$

- Solution to the IVP

  $y = \frac{x^4+32}{4x^2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([x^2*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=0,y(2) = 3, D(y)(2) = -1],y(x), singsol=all
```

$$y(x) = \frac{x^4 + 32}{4x^2}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 17

```
DSolve[{x^2*y''[x]+x*y'[x]-4*y[x]==0,{y[2]==3,y'[2]==-1}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{x^4 + 32}{4x^2}$$

8.8 problem 11

Internal problem ID [11720]
Internal file name [`OUTPUT/11729_Thursday_April_11_2024_08_49_12_PM_60684601/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 113
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 5y' + 4y = 0}$$

8.8.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -5, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 5\lambda \,\mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 5\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 4$ into the above gives

$$\lambda_{1,2} = \frac{5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-5^2 - (4)(1)(4)}$$
$$= \frac{5}{2} \pm \frac{3}{2}$$

Hence

$$\lambda_1 = \frac{5}{2} + \frac{3}{2}$$
$$\lambda_2 = \frac{5}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = 4$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(4)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{x} \tag{1}$$

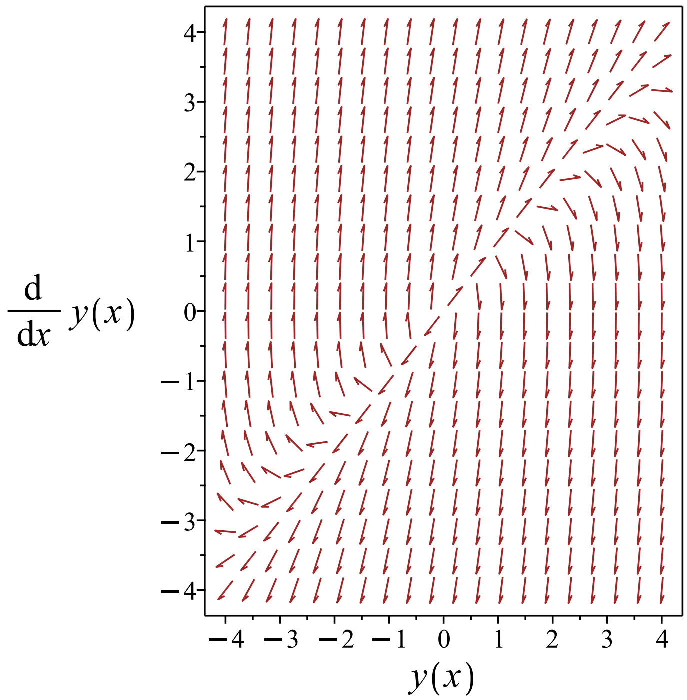


Figure 351: Slope field plot

Verification of solutions

$$y = c_1 e^{4x} + c_2 e^x$$

Verified OK.

8.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 265: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dx}\\ &= z_1 e^{\frac{5x}{2}}\\ &= z_1\left(\mathrm{e}^{\frac{5x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{5x}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\mathrm{e}^{3x}}{3}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^x c_1 + \frac{c_2 \mathrm{e}^{4x}}{3} \tag{1}
$$

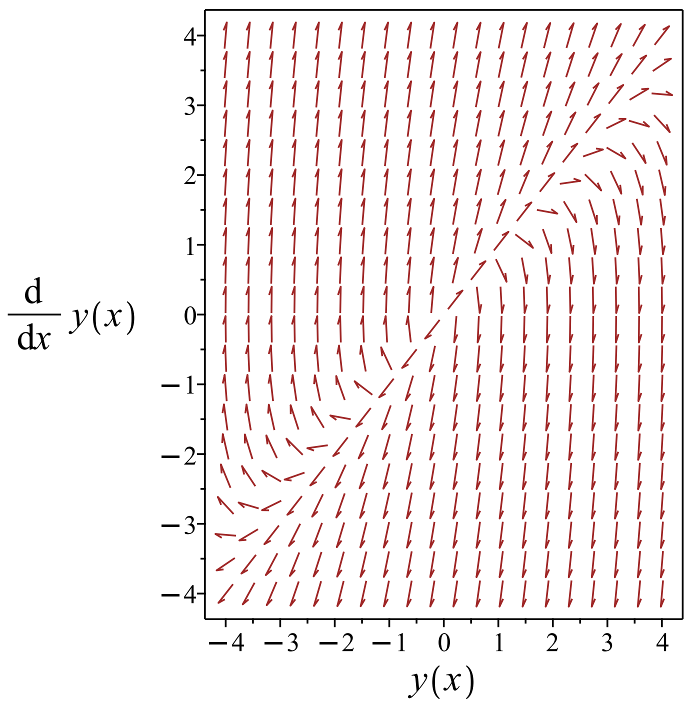


Figure 352: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^x c_1 + \frac{c_2 \mathrm{e}^{4x}}{3}
$$

Verified OK.

8.8.3 Maple step by step solution

Let's solve

$y'' - 5y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 5r + 4 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r - 4) = 0$

- Roots of the characteristic polynomial

  $r = (1, 4)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^x$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = \mathrm{e}^x c_1 + c_2 \mathrm{e}^{4x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-5*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{4x} + c_2 e^{x}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 20

```
DSolve[y''[x]-5*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x \left(c_2 e^{3x} + c_1\right)$$

8.9 problem 12

Internal problem ID [11721]
Internal file name [`OUTPUT/11730_Thursday_April_11_2024_08_49_13_PM_89762227/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 113
Problem number: 12.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 6y'' + 5y' + 12y = 0}$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 5\lambda + 12 = 0$$

The roots of the above equation are

$$\lambda_1 = 3$$
$$\lambda_2 = 4$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{3x} c_2 + \mathrm{e}^{4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{3x}$$
$$y_3 = \mathrm{e}^{4x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + e^{3x} c_2 + e^{4x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^{3x} c_2 + e^{4x} c_3$$

Verified OK.

8.9.1 Maple step by step solution

Let's solve

$y''' - 6y'' + 5y' + 12y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 6y_3(x) - 5y_2(x) - 12y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 6y_3(x) - 5y_2(x) - 12y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -5 & 6 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -5 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{4x} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + e^{3x} c_2 \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + e^{4x} c_3 \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1 e^{-x} + \frac{e^{3x} c_2}{9} + \frac{e^{4x} c_3}{16}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+5*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{4x} + c_2 e^{-x} + c_3 e^{3x}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 29

```
DSolve[y'''[x]-6*y''[x]+5*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(e^{4x}(c_3 e^x + c_2) + c_1\right)$$

8.10 problem 13

Internal problem ID [11722]
Internal file name [OUTPUT/11731_Thursday_April_11_2024_08_49_13_PM_65327505/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 113
Problem number: 13.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y''' - 4x^2y'' + 8y'x - 8y = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' - 4x^2y'' + 8y'x - 8y = 0$$

gives

$$8x\lambda\, x^{\lambda-1} - 4x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} - 8x^\lambda = 0$$

Which simplifies to

$$8\lambda\, x^\lambda - 4\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda - 8x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$8\lambda - 4\lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) - 8 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 4$$

This table summarises the result

root	multiplicity	type of root
1	1	real root
2	1	real root
4	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2 x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_3 x^4 + c_2 x^2 + c_1 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = x$$
$$y_2 = x^2$$
$$y_3 = x^4$$

Summary

The solution(s) found are the following

$$y = c_3 x^4 + c_2 x^2 + c_1 x \tag{1}$$

Verification of solutions

$$y = c_3x^4 + c_2x^2 + c_1x$$

Verified OK.

8.10.1 Maple step by step solution

Let's solve

$x^3y''' - 4y''x^2 + 8y'x - 8y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{8y}{x^3} + \frac{4(y''x - 2y')}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{4y''}{x} + \frac{8y'}{x^2} - \frac{8y}{x^3} = 0$

- Multiply by denominators of the ODE

  $x^3y''' - 4y''x^2 + 8y'x - 8y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

- ○ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

- ○ Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- ○ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- ○ Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

- ○ Calculate the 3rd derivative of y with respect to x , using the chain rule

  $y''' = \left(\frac{d^3}{dt^3}y(t)\right) t'(x)^3 + 3t'(x)\, t''(x) \left(\frac{d^2}{dt^2}y(t)\right) + t'''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

$$y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$$

Substitute the change of variables back into the ODE

$$x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) - 4\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 + 8\frac{d}{dt}y(t) - 8y(t) = 0$$

- Simplify

$$\frac{d^3}{dt^3}y(t) - 7\frac{d^2}{dt^2}y(t) + 14\frac{d}{dt}y(t) - 8y(t) = 0$$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

$$y_1(t) = y(t)$$

- Define new variable $y_2(t)$

$$y_2(t) = \frac{d}{dt}y(t)$$

- Define new variable $y_3(t)$

$$y_3(t) = \frac{d^2}{dt^2}y(t)$$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

$$\frac{d}{dt}y_3(t) = 7y_3(t) - 14y_2(t) + 8y_1(t)$$

Convert linear ODE into a system of first order ODEs

$$\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 7y_3(t) - 14y_2(t) + 8y_1(t)\right]$$

- Define vector

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -14 & 7 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -14 & 7 \end{bmatrix}$$

- Rewrite the system as

  $\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{y}_1 = e^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{y}_2 = e^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \left[\begin{array}{c}\frac{1}{16} \\ \frac{1}{4} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = e^{4t} \cdot \left[\begin{array}{c}\frac{1}{16} \\ \frac{1}{4} \\ 1\end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 e^{t} \cdot \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right] + c_2 e^{2t} \cdot \left[\begin{array}{c}\frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right] + c_3 e^{4t} \cdot \left[\begin{array}{c}\frac{1}{16} \\ \frac{1}{4} \\ 1\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y(t) = c_1 e^{t} + \frac{c_2 e^{2t}}{4} + \frac{c_3 e^{4t}}{16}$$

- Change variables back using $t = \ln(x)$

$$y = c_1 x + \frac{1}{4} c_2 x^2 + \frac{1}{16} c_3 x^4$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve(x^3*diff(y(x),x$3)-4*x^2*diff(y(x),x$2)+8*x*diff(y(x),x)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = x(c_1x^3 + c_3x + c_2)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 20

```
DSolve[x^3*y'''[x]-4*x^2*y''[x]+8*x*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(c_3x^3 + c_2x + c_1)$$

9 Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 124

9.1 problem 1

Internal problem ID [11723]
Internal file name [`OUTPUT/11732_Thursday_April_11_2024_08_49_13_PM_10371206/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 124
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{x^2y'' - 4y'x + 4y = 0}$$

Given that one solution of the ode is

$$y_1 = x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{4}{x}$$

Therefore

$$y_2(x) = x\left(\int \frac{e^{-\left(\int -\frac{4}{x}dx\right)}}{x^2}dx\right)$$

$$y_2(x) = x \int \frac{x^4}{x^2}, dx$$

$$y_2(x) = x\left(\int x^2 dx\right)$$

$$y_2(x) = \frac{x^4}{3}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x + \frac{1}{3} c_2 x^4 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + \frac{1}{3} c_2 x^4 \tag{1}$$

Verification of solutions

$$y = c_1 x + \frac{1}{3} c_2 x^4$$

Verified OK.

9.1.1 Maple step by step solution

Let's solve

$y''x^2 - 4y'x + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{x} - \frac{4y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x} + \frac{4y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 - 4y'x + 4y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^2 - 4\frac{d}{dt}y(t) + 4y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 5\frac{d}{dt}y(t) + 4y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 5r + 4 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-4)=0$
- Roots of the characteristic polynomial

  $r=(1,4)$
- 1st solution of the ODE

  $y_1(t)=e^t$
- 2nd solution of the ODE

  $y_2(t)=e^{4t}$
- General solution of the ODE

  $y(t)=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y(t)=c_1e^t+c_2e^{4t}$
- Change variables back using $t=\ln(x)$

  $y=c_2x^4+c_1x$
- Simplify

  $y=x(c_2x^3+c_1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+4*y(x)=0,x],singsol=all)
```

$$y(x)=x(c_1x^3+c_2)$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 16

```
DSolve[x^2*y''[x]-4*x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(c_2x^3 + c_1)$$

9.2 problem 2

Internal problem ID [11724]
Internal file name [`OUTPUT/11733_Thursday_April_11_2024_08_49_14_PM_11953513/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 124
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1+x)^2 y''-3(1+x) y'+3y=0}$$

Given that one solution of the ode is

$$y_1=1+x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x)=y_1\left(\int \frac{e^{-(\int p dx)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y''+p(x) y'+q(x) y=f(x)$$

Looking at the ode to solve shows that

$$p(x)=\frac{-3x-3}{x^2+2x+1}$$

Therefore

$$y_2(x) = (1+x)\left(\int \frac{e^{-\left(\int \frac{-3x-3}{x^2+2x+1}dx\right)}}{(1+x)^2}dx\right)$$

$$y_2(x) = 1+x\int \frac{(1+x)^3}{(1+x)^2}, dx$$

$$y_2(x) = (1+x)\left(\int (1+x)\,dx\right)$$

$$y_2(x) = (1+x)\left(x+\frac{1}{2}x^2\right)$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x)+c_2y_2(x) \\ &= (1+x)\,c_1+c_2(1+x)\left(x+\frac{1}{2}x^2\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = (1+x)\,c_1+c_2(1+x)\left(x+\frac{1}{2}x^2\right) \tag{1}$$

<u>Verification of solutions</u>

$$y = (1+x)\,c_1+c_2(1+x)\left(x+\frac{1}{2}x^2\right)$$

Verified OK.

9.2.1 Maple step by step solution

Let's solve

$(1+x)^2\,y''+(-3x-3)\,y'+3y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{3y'}{1+x}-\frac{3y}{(1+x)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{1+x} + \frac{3y}{(1+x)^2} = 0$

☐ Check to see if $x_0 = -1$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{3}{1+x}, P_3(x) = \frac{3}{(1+x)^2}\right]$

  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = -3$

  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 3$

  - $x = -1$is a regular singular point

    Check to see if $x_0 = -1$ is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $(1+x)^2 y'' + (-3x - 3) y' + 3y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $u^2 \left(\frac{d^2}{du^2} y(u)\right) - 3u\left(\frac{d}{du} y(u)\right) + 3y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite DE with series expansions

  - Convert $u \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion

    $u \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r) u^{k+r}$

  - Convert $u^2 \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion

    $u^2 \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) u^{k+r}$

    Rewrite DE with series expansions

    $\sum_{k=0}^{\infty} a_k (k+r-1)(k+r-3) u^{k+r} = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k-1)(k-3) = 0$
- Recursion relation that defines series solution to ODE

  $a_k = 0$
- Recursion relation for $r = 0$

  $a_k = 0$
- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_k = 0\right]$$
- Revert the change of variables $u = 1 + x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_k = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve([(x+1)^2*diff(y(x),x$2)-3*(x+1)*diff(y(x),x)+3*y(x)=0,x+1],singsol=all)
```

$$y(x) = (1+x)\left(c_1 + c_2(1+x)^2\right)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 20

```
DSolve[(x+1)^2*y''[x]-3*(x+1)*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2(x+1)^3 + c_1(x+1)$$

9.3 problem 3

Internal problem ID [11725]
Internal file name [`OUTPUT/11734_Thursday_April_11_2024_08_49_14_PM_67310780/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 124
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{(x^2-1)\,y''-2y'x+2y=0}$$

Given that one solution of the ode is

$$y_1=x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x)=y_1\left(\int \frac{e^{-\left(\int p dx\right)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y''+p(x)\,y'+q(x)\,y=f(x)$$

Looking at the ode to solve shows that

$$p(x)=-\frac{2x}{x^2-1}$$

Therefore

$$y_2(x) = x\left(\int \frac{e^{-\left(\int -\frac{2x}{x^2-1}dx\right)}}{x^2}dx\right)$$

$$y_2(x) = x\int \frac{e^{\ln(x-1)+\ln(1+x)}}{x^2}, dx$$

$$y_2(x) = x\left(\int \frac{x^2-1}{x^2}dx\right)$$

$$y_2(x) = x\left(x+\frac{1}{x}\right)$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x + c_2x\left(x+\frac{1}{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x + c_2x\left(x+\frac{1}{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1x + c_2x\left(x+\frac{1}{x}\right)$$

Verified OK.

9.3.1 Maple step by step solution

Let's solve

$y''(x^2-1) - 2y'x + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2xy'}{x^2-1} - \frac{2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{2xy'}{x^2-1} + \frac{2y}{x^2-1} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{2x}{x^2-1}, P_3(x) = \frac{2}{x^2-1}\right]$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = -1$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$

- $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $y''(x^2-1) - 2y'x + 2y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (-2u+2)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

$u^{m}\cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =-2+m}^{\infty}a_{k +2-m} \left(k +2-m +r \right) \left(k +1-m +r \right) u^{k +r}$

Rewrite ODE with series expansions

$-2 a_{0} r \left(-2+r \right) u^{-1+r}+\left(\sum_{k =0}^{\infty}\left(-2 a_{k +1} \left(k +1+r \right) \left(k +r -1\right)+a_{k} \left(k +r -1\right) \left(k +r -2\right)\right) u^{k +r}\right)$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-2 r \left(-2+r \right)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, 2\right\}$

- Each term in the series must be 0, giving the recursion relation

  $\left(k +r -1\right) \left(\left(-2 k -2 r -2\right) a_{k +1}+a_{k} \left(k +r -2\right)\right)=0$

- Recursion relation that defines series solution to ODE

  $a_{k +1}=\frac{a_{k} \left(k +r -2\right)}{2 \left(k +1+r \right)}$

- Recursion relation for $r =0$; series terminates at $k =2$

  $a_{k +1}=\frac{a_{k} \left(k -2\right)}{2 \left(k +1\right)}$

- Apply recursion relation for $k =0$

  $a_{1}=-a_{0}$

- Apply recursion relation for $k =1$

  $a_{2}=-\frac{a_{1}}{4}$

- Express in terms of a_{0}

  $a_{2}=\frac{a_{0}}{4}$

- Terminating series solution of the ODE for $r =0$. Use reduction of order to find the second li

  $y \left(u \right)=a_{0}\cdot \left(1-u +\frac{1}{4} u^{2}\right)$

- Revert the change of variables $u =1+x$

  $\left[y =\frac{a_{0} \left(x -1\right)^{2}}{4}\right]$

- Recursion relation for $r =2$

  $a_{k +1}=\frac{a_{k} k}{2 \left(k +3\right)}$

- Solution for $r =2$

  $\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k +2}, a_{k +1}=\frac{a_{k} k}{2 \left(k +3\right)}\right]$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \frac{a_0 (x-1)^2}{4} + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+2}\right), b_{1+k} = \frac{b_k k}{2(k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([(x^2-1)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,x],singsol=all)
```

$$y(x) = c_2 x^2 + c_1 x + c_2$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 39

```
DSolve[(x^2-1)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{x^2-1}(c_1 (x-1)^2 + c_2 x)}{\sqrt{1-x^2}}$$

9.4 problem 4

Internal problem ID [11726]
Internal file name [`OUTPUT/11735_Thursday_April_11_2024_08_49_14_PM_91859302/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 124
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2 - x + 1)\, y'' - (x^2 + x)\, y' + y(1 + x) = 0}$$

Given that one solution of the ode is

$$y_1 = x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{e^{-(\int p dx)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-x^2 - x}{x^2 - x + 1}$$

Therefore

$$y_2(x) = x\left(\int \frac{e^{-\left(\int \frac{-x^2-x}{x^2-x+1}dx\right)}}{x^2}dx\right)$$

$$y_2(x) = x\int \frac{e^{x+\ln(x^2-x+1)}}{x^2}, dx$$

$$y_2(x) = x\left(\int \frac{e^x(x^2-x+1)}{x^2}dx\right)$$

$$y_2(x) = e^x(x-1)$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x + c_2e^x(x-1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x + c_2e^x(x-1) \tag{1}$$

Verification of solutions

$$y = c_1x + c_2e^x(x-1)$$

Verified OK.

9.4.1 Maple step by step solution

Let's solve

$(x^2-x+1)\,y'' + (-x^2-x)\,y' + y(1+x) = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(1+x)y}{x^2-x+1} + \frac{x(1+x)y'}{x^2-x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{x(1+x)y'}{x^2-x+1} + \frac{(1+x)y}{x^2-x+1} = 0$

☐ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{(1+x)x}{x^2-x+1}, P_3(x) = \frac{1+x}{x^2-x+1}\right]$

- $\left(x-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)\cdot P_2(x)$ is analytic at $x=\frac{1}{2}-\frac{I\sqrt{3}}{2}$

  $$\left(\left(x-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)\cdot P_2(x)\right)\bigg|_{x=\frac{1}{2}-\frac{I\sqrt{3}}{2}} = 0$$

- $\left(x-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)^2\cdot P_3(x)$ is analytic at $x=\frac{1}{2}-\frac{I\sqrt{3}}{2}$

  $$\left(\left(x-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)^2\cdot P_3(x)\right)\bigg|_{x=\frac{1}{2}-\frac{I\sqrt{3}}{2}} = 0$$

- $x=\frac{1}{2}-\frac{I\sqrt{3}}{2}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = \frac{1}{2}-\frac{I\sqrt{3}}{2}$

- Multiply by denominators

  $(x^2-x+1)\,y''-x(1+x)\,y'+y(1+x)=0$

- Change variables using $x=u+\frac{1}{2}-\frac{I\sqrt{3}}{2}$ so that the regular singular point is at $u=0$

  $(u^2-Iu\sqrt{3})\left(\frac{d^2}{du^2}y(u)\right)+(-2u-u^2+Iu\sqrt{3}+I\sqrt{3})\left(\frac{d}{du}y(u)\right)+\left(\frac{3}{2}+u-\frac{I\sqrt{3}}{2}\right)y(u)=0$

- Assume series solution for $y(u)$

  $$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

  $$u^m\cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

  $$u^m\cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

  $$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$-\mathrm{I}\sqrt{3}\, r(r-2)\, a_0 u^{r-1} + \left(-\mathrm{I}\sqrt{3}\,(1+r)\,(r-1)\, a_1 + \frac{\left(3+2\,\mathrm{I}r\sqrt{3}-\mathrm{I}\sqrt{3}+2r^2-6r\right)a_0}{2}\right) u^r + \left(\sum_{k=1}^{\infty} \left(-\mathrm{I}\sqrt{3}\,(\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-\mathrm{I}\sqrt{3}\, r(r-2) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$-\mathrm{I}\sqrt{3}\,(1+r)\,(r-1)\, a_1 + \frac{\left(3+2\,\mathrm{I}r\sqrt{3}-\mathrm{I}\sqrt{3}+2r^2-6r\right)a_0}{2} = 0$$

- Each term in the series must be 0, giving the recursion relation

$$\frac{\mathrm{I}((2k+2r-1)a_k-2(k+1+r)a_{k+1}(k+r-1))\sqrt{3}}{2} + \frac{(3+2k^2+2(-3+2r)k+2r^2-6r)a_k}{2} - a_{k-1}(k-2+r) = 0$$

- Shift index using $k- > k+1$

$$\frac{\mathrm{I}((2k+1+2r)a_{k+1}-2(k+2+r)a_{k+2}(k+r))\sqrt{3}}{2} + \frac{\left(3+2(k+1)^2+2(-3+2r)(k+1)+2r^2-6r\right)a_{k+1}}{2} - a_k(k+r-1) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{-\frac{\mathrm{I}}{6}\left(2\,\mathrm{I}\sqrt{3}\, r a_{k+1}+\mathrm{I}\sqrt{3}\, a_{k+1}+2\,\mathrm{I}\sqrt{3}\, k a_{k+1}+2k^2a_{k+1}+4kra_{k+1}+2r^2a_{k+1}-2ka_k-2a_{k+1}k-2ra_k-2a_{k+1}r+2a_k-a_{k+1}\right)\sqrt{3}}{k^2+2kr+r^2+2k+2r}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{-\frac{\mathrm{I}}{6}\left(\mathrm{I}\sqrt{3}\, a_{k+1}+2\,\mathrm{I}\sqrt{3}\, k a_{k+1}+2k^2a_{k+1}-2ka_k-2a_{k+1}k+2a_k-a_{k+1}\right)\sqrt{3}}{k^2+2k}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = \frac{-\frac{\mathrm{I}}{6}\left(\mathrm{I}\sqrt{3}\, a_{k+1}+2\,\mathrm{I}\sqrt{3}\, k a_{k+1}+2k^2a_{k+1}-2ka_k-2a_{k+1}k+2a_k-a_{k+1}\right)\sqrt{3}}{k^2+2k}$$

- Recursion relation for $r = 2$

$$a_{k+2} = \frac{-\frac{I}{6}\left(5 I\sqrt{3}\, a_{k+1}+2 I\sqrt{3}\, k a_{k+1}+2k^{2} a_{k+1}+6 a_{k+1} k+3 a_{k+1}-2 k a_{k}-2 a_{k}\right)\sqrt{3}}{k^{2}+6k+8}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{-\frac{I}{6}\left(5 I\sqrt{3}\, a_{k+1}+2 I\sqrt{3}\, k a_{k+1}+2k^{2} a_{k+1}+6 a_{k+1} k+3 a_{k+1}-2 k a_{k}-2 a_{k}\right)\sqrt{3}}{k^{2}+6k+8}, -3 I\sqrt{3}\, a_1 +\right.$$

- Revert the change of variables $u = x - \frac{1}{2} + \frac{I\sqrt{3}}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{1}{2} + \frac{I\sqrt{3}}{2}\right)^{k+2}, a_{k+2} = \frac{-\frac{I}{6}\left(5 I\sqrt{3}\, a_{k+1}+2 I\sqrt{3}\, k a_{k+1}+2k^{2} a_{k+1}+6 a_{k+1} k+3 a_{k+1}-2 k a_{k}-2 a_{k}\right)\sqrt{3}}{k^{2}+6k+8}, -3\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([(x^2-x+1)*diff(y(x),x$2)-(x^2+x)*diff(y(x),x)+(x+1)*y(x)=0,x],singsol=all)
```

$$y(x) = c_1 x + c_2 e^{x}(-1 + x)$$

✓ Solution by Mathematica

Time used: 0.093 (sec). Leaf size: 19

```
DSolve[(x^2-x+1)*y''[x]-(x^2+x)*y'[x]+(x+1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x + c_2 e^{x}(x - 1)$$

9.5 problem 5

Internal problem ID [11727]
Internal file name [`OUTPUT/11736_Thursday_April_11_2024_08_49_15_PM_62434432/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 124
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x+1)\,y''-4(1+x)\,y'+4y=0}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{2x}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-4x-4}{2x+1}$$

Therefore

$$y_2(x) = e^{2x}\left(\int e^{-\left(\int \frac{-4x-4}{2x+1}dx\right)} e^{-4x} dx\right)$$

$$y_2(x) = e^{2x}\int \frac{e^{2x+\ln(2x+1)}}{e^{4x}}, dx$$

$$y_2(x) = e^{2x}\left(\int (2x+1)\,e^{-2x} dx\right)$$

$$y_2(x) = -e^{2x}(1+x)\,e^{-2x}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^{2x} - c_2 e^{2x}(1+x)\,e^{-2x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} - c_2 e^{2x}(1+x)\,e^{-2x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{2x} - c_2 e^{2x}(1+x)\,e^{-2x}$$

Verified OK.

9.5.1 Maple step by step solution

Let's solve

$(2x+1)\,y'' + (-4x-4)\,y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{2x+1} + \frac{4(1+x)y'}{2x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4(1+x)y'}{2x+1} + \frac{4y}{2x+1} = 0$

□ Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{4(1+x)}{2x+1}, P_3(x) = \frac{4}{2x+1}\right]$

- $\left(x+\frac{1}{2}\right) \cdot P_2(x)$ is analytic at $x = -\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right) \cdot P_2(x)\right)\bigg|_{x=-\frac{1}{2}} = -1$

- $\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{1}{2}$

  $\left(\left(x+\frac{1}{2}\right)^2 \cdot P_3(x)\right)\bigg|_{x=-\frac{1}{2}} = 0$

- $x = -\frac{1}{2}$is a regular singular point

  Check to see if $x_0 = -\frac{1}{2}$ is a regular singular point

  $x_0 = -\frac{1}{2}$

- Multiply by denominators

  $(2x+1)\, y'' + (-4x-4)\, y' + 4y = 0$

- Change variables using $x = u - \frac{1}{2}$ so that the regular singular point is at $u = 0$

  $2u\left(\frac{d^2}{du^2}y(u)\right) + (-4u-2)\left(\frac{d}{du}y(u)\right) + 4y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, u^{k+r-1}$

- Shift index using $k->k+1$

  $u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$

  Rewrite ODE with series expansions

$$2a_0r(-2+r)\,u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(2a_{k+1}(k+1+r)\,(k+r-1)-4a_k(k+r-1)\right)u^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r(-2+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{0,2\}$$

- Each term in the series must be 0, giving the recursion relation

$$2(a_{k+1}(k+1+r)-2a_k)\,(k+r-1)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1}=\frac{2a_k}{k+1+r}$$

- Recursion relation for $r=0$

$$a_{k+1}=\frac{2a_k}{k+1}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^k, a_{k+1}=\frac{2a_k}{k+1}\right]$$

- Revert the change of variables $u=x+\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{2}\right)^k, a_{k+1}=\frac{2a_k}{k+1}\right]$$

- Recursion relation for $r=2$

$$a_{k+1}=\frac{2a_k}{k+3}$$

- Solution for $r=2$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k+2}, a_{k+1}=\frac{2a_k}{k+3}\right]$$

- Revert the change of variables $u=x+\frac{1}{2}$

$$\left[y=\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{2}\right)^{k+2}, a_{k+1}=\frac{2a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k\left(x+\frac{1}{2}\right)^k\right)+\left(\sum_{k=0}^{\infty}b_k\left(x+\frac{1}{2}\right)^{k+2}\right), a_{1+k}=\frac{2a_k}{1+k}, b_{1+k}=\frac{2b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([(2*x+1)*diff(y(x),x$2)-4*(x+1)*diff(y(x),x)+4*y(x)=0,exp(2*x)],singsol=all)
```

$$y(x) = c_2 e^{2x} + c_1 x + c_1$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 23

```
DSolve[(2*x+1)*y''[x]-4*(x+1)*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{2x+1} - c_2(x+1)$$

9.6 problem 6

Internal problem ID [11728]
Internal file name [`OUTPUT/11737_Thursday_April_11_2024_08_49_15_PM_96516632/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 124
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^{3}-x^{2}\right) y''-\left(x^{3}+2 x^{2}-2 x\right) y'+\left(2 x^{2}+2 x -2\right) y=0}$$

Given that one solution of the ode is

$$y_1 = x^2$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-x^3 - 2x^2 + 2x}{x^3 - x^2}$$

Therefore

$$y_2(x) = x^2 \left(\int \frac{\mathrm{e}^{-\left(\int \frac{-x^3-2x^2+2x}{x^3-x^2}dx\right)}}{x^4}dx\right)$$

$$y_2(x) = x^2 \int \frac{\mathrm{e}^{x+\ln(x-1)+2\ln(x)}}{x^4}, dx$$

$$y_2(x) = x^2 \left(\int \frac{(x-1)\,\mathrm{e}^x}{x^2}dx\right)$$

$$y_2(x) = x\,\mathrm{e}^x$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^2 + c_2 x\,\mathrm{e}^x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_2 x\,\mathrm{e}^x \tag{1}$$

Verification of solutions

$$y = c_1 x^2 + c_2 x\,\mathrm{e}^x$$

Verified OK.

9.6.1 Maple step by step solution

Let's solve

$y''x^2(x-1) + (-x^3 - 2x^2 + 2x)\,y' + (2x^2 + 2x - 2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2(x^2+x-1)y}{x^2(x-1)} + \frac{(x^2+2x-2)y'}{x(x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2+2x-2)y'}{x(x-1)} + \frac{2(x^2+x-1)y}{x^2(x-1)} = 0$

□ Check to see if x_0 is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{x^2+2x-2}{x(x-1)}, P_3(x) = \frac{2(x^2+x-1)}{x^2(x-1)}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -2$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 2$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2(x-1) - x(x^2+2x-2)\,y' + (2x^2+2x-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $x^{m}\cdot y''=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_{0}(-1+r)\,(-2+r)\,x^{r}+\left(-a_{1}r(-1+r)+a_{0}(-1+r)\,(-2+r)\right)x^{1+r}+\left(\sum_{k=2}^{\infty}\left(-a_{k}(k+r-1)\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $-(-1+r)\,(-2+r)=0$
- Values of r that satisfy the indicial equation

  $r\in\{1,2\}$
- Each term must be 0

  $-a_{1}r(-1+r)+a_{0}(-1+r)\,(-2+r)=0$
- Solve for the dependent coefficient(s)

  $a_{1}=\frac{a_{0}(-2+r)}{r}$
- Each term in the series must be 0, giving the recursion relation

  $-a_{k}(k+r-1)\,(k+r-2)+a_{k-1}(k+r-2)\,(k-3+r)-a_{k-2}(k-4+r)=0$
- Shift index using $k->k+2$

  $-a_{k+2}(k+1+r)\,(k+r)+a_{k+1}(k+r)\,(k+r-1)-a_{k}(k+r-2)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{k^{2}a_{k+1}+2kra_{k+1}+r^{2}a_{k+1}-ka_{k}-ka_{k+1}-ra_{k}-ra_{k+1}+2a_{k}}{(k+1+r)(k+r)}$
- Recursion relation for $r=1$

  $a_{k+2}=\frac{k^{2}a_{k+1}-ka_{k}+ka_{k+1}+a_{k}}{(k+2)(k+1)}$
- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty}a_{k}x^{k+1},a_{k+2}=\frac{k^{2}a_{k+1}-ka_{k}+ka_{k+1}+a_{k}}{(k+2)(k+1)},a_{1}=-a_{0}\right]$
- Recursion relation for $r=2$

  $a_{k+2}=\frac{k^{2}a_{k+1}-ka_{k}+3ka_{k+1}+2a_{k+1}}{(k+3)(k+2)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty}a_{k}x^{k+2},a_{k+2}=\frac{k^{2}a_{k+1}-ka_{k}+3ka_{k+1}+2a_{k+1}}{(k+3)(k+2)},a_{1}=0\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{1+k}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+2} = \frac{k^2 a_{1+k} - k a_k + k a_{1+k} + a_k}{(k+2)(1+k)}, a_1 = -a_0, b_{k+2} = \frac{k^2 b_{1+k} - k b_k + 3k b_{}}{(k+3)(k+}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([(x^3-x^2)*diff(y(x),x$2)-(x^3+2*x^2-2*x)*diff(y(x),x)+(2*x^2+2*x-2)*y(x)=0,x^2],sing
```

$$y(x) = x(c_2 e^x + c_1 x)$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 36

```
DSolve[(x^3-x^2)*y''[x]-(x^3+2*x^2-2*x)*y'[x]+(2*x^2+2*x-2)*y[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to -\frac{\sqrt{1-x}x(c_2 x - c_1 e^x)}{\sqrt{x-1}}$$

9.7 problem 8

Internal problem ID [11729]
Internal file name [`OUTPUT/11738_Thursday_April_11_2024_08_49_16_PM_36339295/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 124
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 3y' + 2y = 4x^2}$$

9.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = 4x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda \, \mathrm{e}^{\lambda x} + 2 \, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)} \\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= \frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= 1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_3 x^2 + 2A_2 x - 6x A_3 + 2A_1 - 3A_2 + 2A_3 = 4x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 7, A_2 = 6, A_3 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x^2 + 6x + 7$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^x\right) + \left(2x^2 + 6x + 7\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^x + 2x^2 + 6x + 7 \tag{1}$$

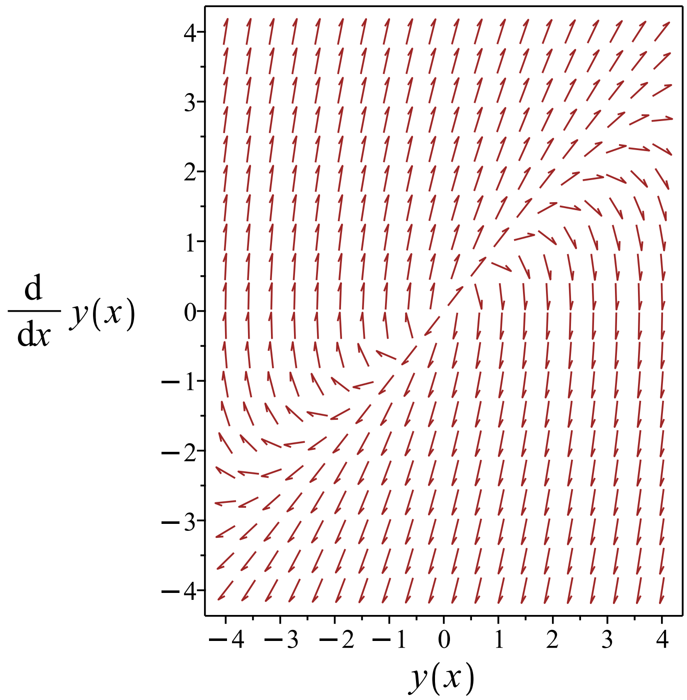


Figure 353: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{x} + 2x^2 + 6x + 7$$

Verified OK.

9.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 275: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(e^x) + c_2(e^x(e^x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = e^x c_1 + c_2 e^{2x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{e^x, e^{2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_3x^2 + A_2x + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_3x^2 + 2A_2x - 6xA_3 + 2A_1 - 3A_2 + 2A_3 = 4x^2
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 7, A_2 = 6, A_3 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x^2 + 6x + 7$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + c_2 e^{2x}\right) + \left(2x^2 + 6x + 7\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x c_1 + c_2 e^{2x} + 2x^2 + 6x + 7 \tag{1}$$

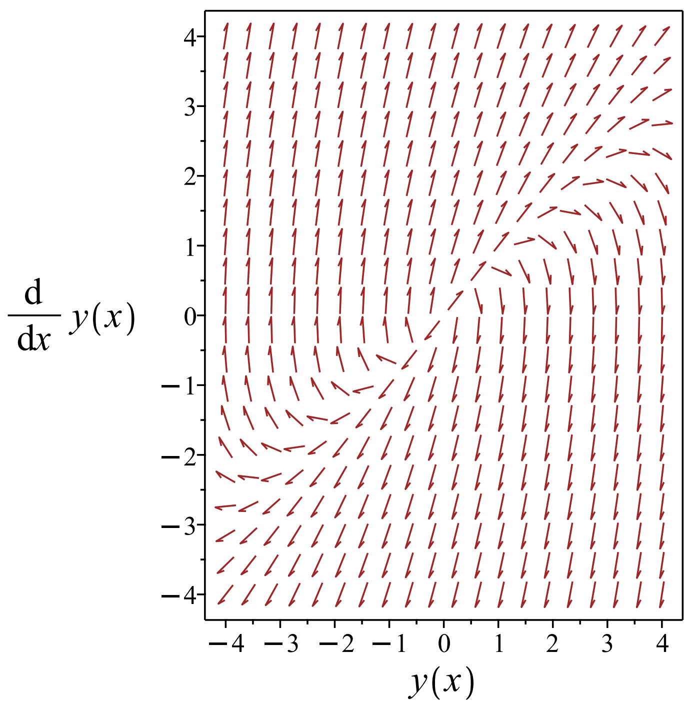


Figure 354: Slope field plot

Verification of solutions

$$y = e^x c_1 + c_2 e^{2x} + 2x^2 + 6x + 7$$

Verified OK.

9.7.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = 4x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x c_1 + c_2 e^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4x^2\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2\,e^{2x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{3x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -4\,e^x\left(\int x^2 e^{-x} dx\right) + 4\,e^{2x}\left(\int x^2 e^{-2x} dx\right)$

- ◦ Compute integrals

  $y_p(x) = 2x^2 + 6x + 7$

- Substitute particular solution into general solution to ODE

  $y = e^x c_1 + c_2 e^{2x} + 2x^2 + 6x + 7$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=4*x^2,y(x), singsol=all)
```

$$y(x) = e^{2x} c_1 + c_2 e^x + 2x^2 + 6x + 7$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 29

```
DSolve[y''[x]-3*y'[x]+2*y[x]==4*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2x^2 + 6x + c_1 e^x + c_2 e^{2x} + 7$$

9.8 problem 9

Internal problem ID [11730]
Internal file name [`OUTPUT/11739_Thursday_April_11_2024_08_49_16_PM_66646944/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.1. Basic theory of linear differential equations. Exercises page 124
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 5y' + 6y = 2 - 12x + 6\,\mathrm{e}^{x}}$$

9.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -5, C = 6, f(x) = 2 - 12x + 6\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 5\lambda \,\mathrm{e}^{\lambda x} + 6\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 6$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-5^2 - (4)\,(1)\,(6)} \\
&= \frac{5}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{5}{2} + \frac{1}{2} \\
\lambda_2 &= \frac{5}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 3 \\
\lambda_2 &= 2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(3)x} + c_2 e^{(2)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2 - 12x + 6\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{2x}, \mathrm{e}^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{x} + A_2 + A_3x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{x} - 5A_3 + 6A_2 + 6A_3x = 2 - 12x + 6\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 3, A_2 = -\frac{4}{3}, A_3 = -2\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3\,\mathrm{e}^{x} - \frac{4}{3} - 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{3x} + c_2\mathrm{e}^{2x}\right) + \left(3\,\mathrm{e}^{x} - \frac{4}{3} - 2x\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1\mathrm{e}^{3x} + c_2\mathrm{e}^{2x} + 3\,\mathrm{e}^{x} - \frac{4}{3} - 2x \tag{1}$$

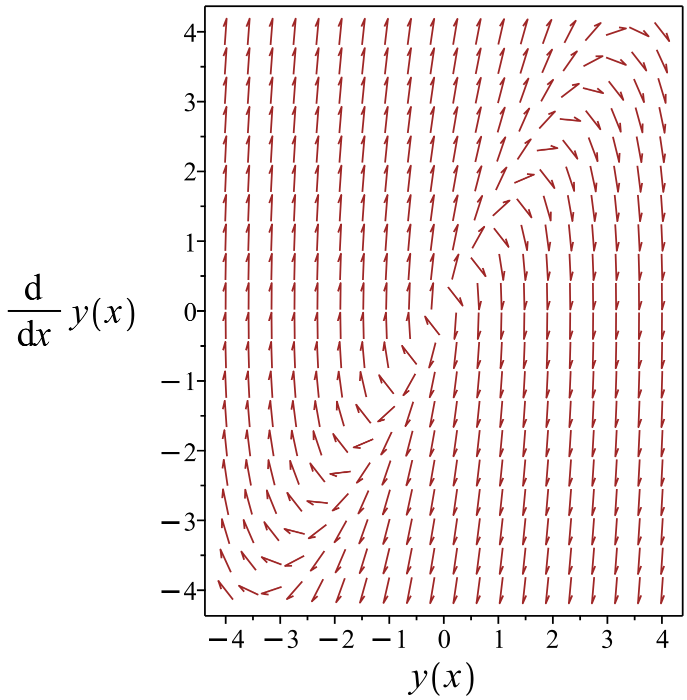


Figure 355: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{2x} + 3 e^{x} - \frac{4}{3} - 2x$$

Verified OK.

9.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 277: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\, dx} \\
&= z_1 e^{\frac{5x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{5x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{5x}}{(y_1)^2}\, dx \\
&= y_1 (\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{2x}\right) + c_2 \left(e^{2x} \left(e^{x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - 5y' + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 e^{2x} + e^{3x} c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2 - 12x + 6\,e^{x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^{x}\}, \{1, x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{e^{2x}, e^{3x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 e^{x} + A_2 + A_3 x
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_1 e^{x} - 5A_3 + 6A_2 + 6A_3 x = 2 - 12x + 6\,e^{x}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 3, A_2 = -\frac{4}{3}, A_3 = -2\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3\,\mathrm{e}^{x} - \frac{4}{3} - 2x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{2x} + \mathrm{e}^{3x} c_2\right) + \left(3\,\mathrm{e}^{x} - \frac{4}{3} - 2x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + \mathrm{e}^{3x} c_2 + 3\,\mathrm{e}^{x} - \frac{4}{3} - 2x \tag{1}$$

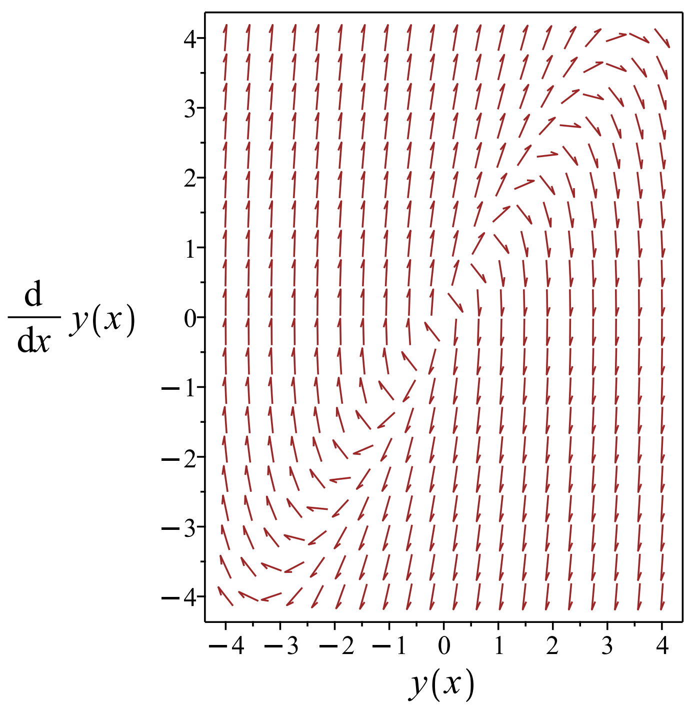


Figure 356: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + e^{3x} c_2 + 3\,e^{x} - \frac{4}{3} - 2x$$

Verified OK.

9.8.3 Maple step by step solution

Let's solve

$y'' - 5y' + 6y = 2 - 12x + 6\,e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{2x} + e^{3x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2 - 12x + 6\,e^{x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & e^{3x} \\ 2\,e^{2x} & 3\,e^{3x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{5x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -2\,e^{2x}\left(\int (1-6x+3\,e^x)\,e^{-2x}dx\right) + 2\,e^{3x}\left(\int (1-6x+3\,e^x)\,e^{-3x}dx\right)$
- Compute integrals

  $y_p(x) = 3\,e^x - \frac{4}{3} - 2x$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{2x} + e^{3x} c_2 + 3\,e^x - \frac{4}{3} - 2x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)-5*diff(y(x),x)+6*y(x)=2-12*x+6*exp(x),y(x), singsol=all)
```

$$y(x) = c_2 e^{2x} + c_1 e^{3x} + 3\,e^x - 2x - \frac{4}{3}$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 33

```
DSolve[y''[x]-5*y'[x]+6*y[x]==2-12*x+6*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x + 3e^x + c_1 e^{2x} + c_2 e^{3x} - \frac{4}{3}$$

10 Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135

10.1 problem 1

Internal problem ID [11731]
Internal file name [`OUTPUT/11740_Thursday_April_11_2024_08_49_17_PM_75053001/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 5y' + 6y = 0}$$

10.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 5\lambda e^{\lambda x} + 6 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-5^2 - (4)(1)(6)} \\ &= \frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{5}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{5}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = 2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(2)x}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{2x} \tag{1}$$

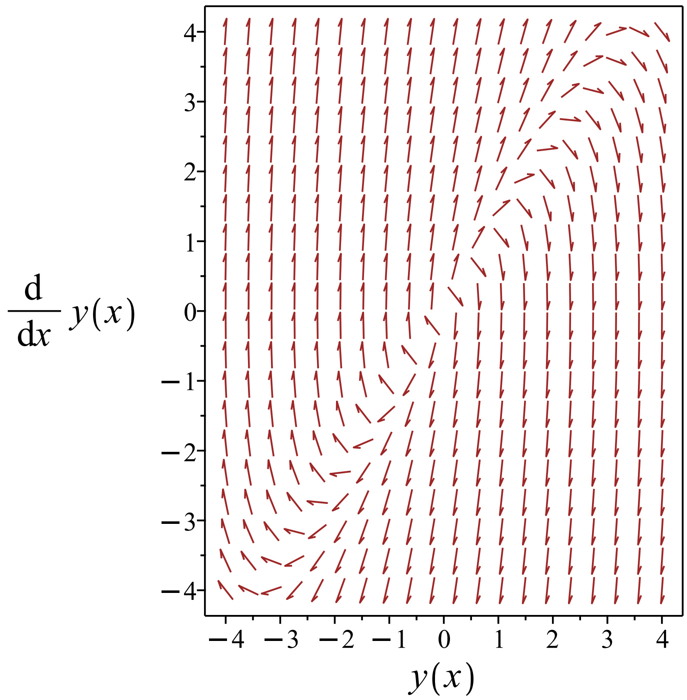


Figure 357: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{2x}$$

Verified OK.

10.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 279: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\, dx} \\ &= z_1 e^{\frac{5x}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{5x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{5x}}{(y_1)^2}\, dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}(\mathrm{e}^{x})\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{2x} + \mathrm{e}^{3x} c_2 \tag{1}
$$

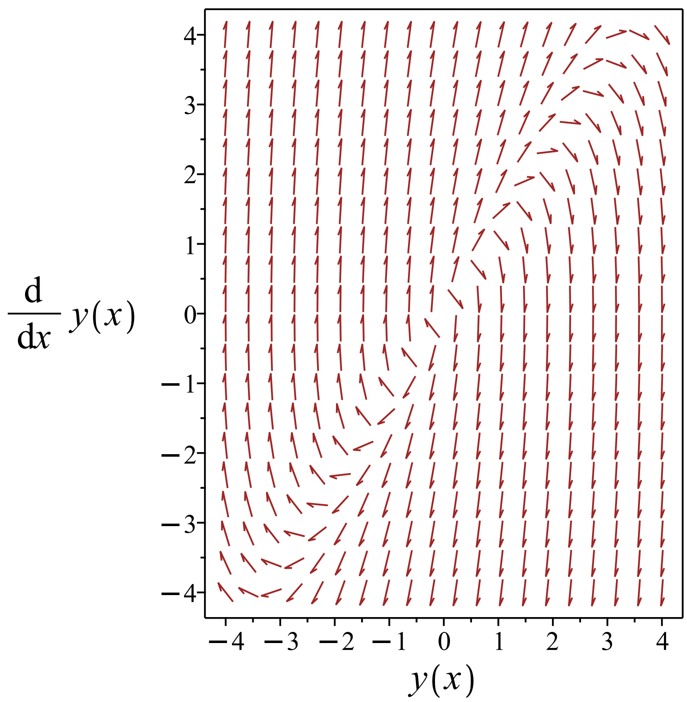


Figure 358: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{2x} + \mathrm{e}^{3x} c_2
$$

Verified OK.

10.1.3 Maple step by step solution

Let's solve

$y'' - 5y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the ODE

  $y_1(x) = e^{2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{2x} + e^{3x} c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-5*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{2x}c_1 + c_2e^{3x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[y''[x]-5*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(c_2e^x + c_1)$$

10.2 problem 2

Internal problem ID [11732]
Internal file name [`OUTPUT/11741_Thursday_April_11_2024_08_49_17_PM_36620420/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' - 3y = 0}$$

10.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} - 3 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -3$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(-3)} \\
&= 1 \pm 2
\end{aligned}$$

Hence

$$\lambda_1 = 1 + 2$$
$$\lambda_2 = 1 - 2$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-x} \tag{1}$$

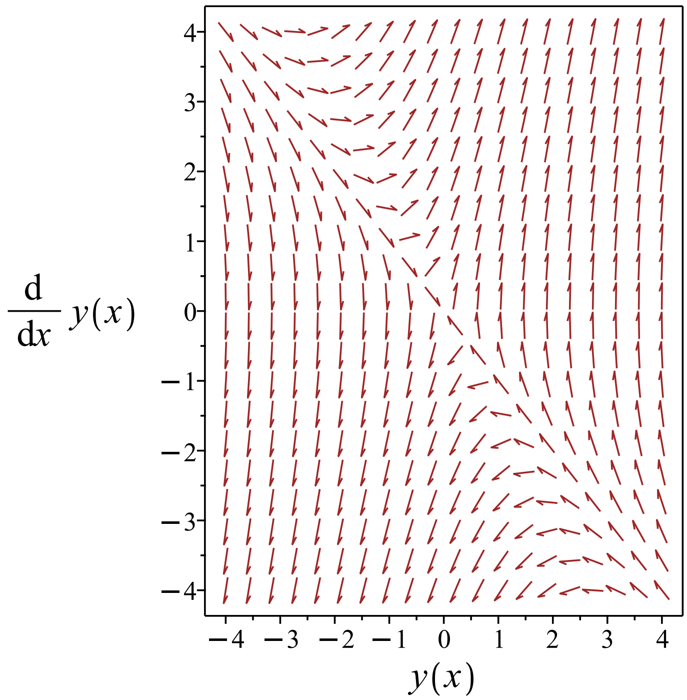


Figure 359: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{-x}$$

Verified OK.

10.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 281: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1(\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x} \left(\frac{e^{4x}}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-x} + \frac{e^{3x} c_2}{4} \tag{1}
$$

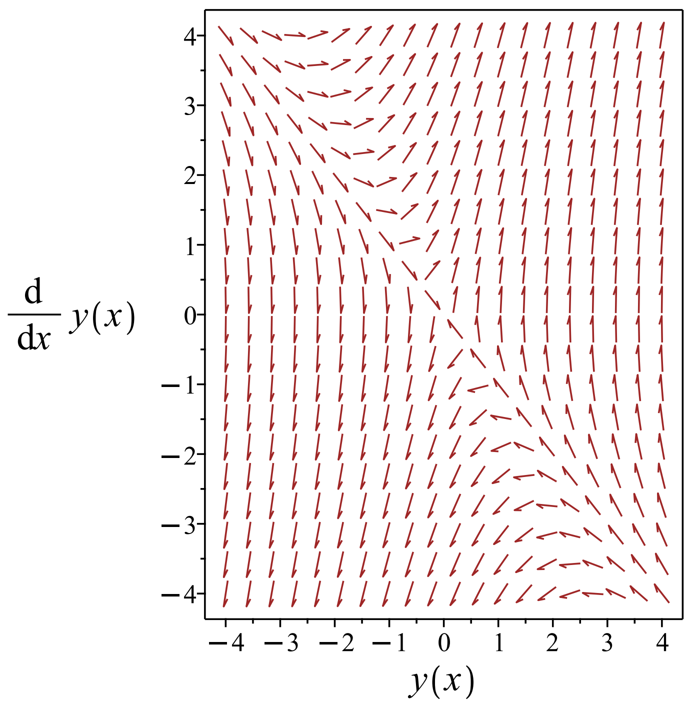


Figure 360: Slope field plot

Verification of solutions

$$
y = c_1 e^{-x} + \frac{e^{3x} c_2}{4}
$$

Verified OK.

10.2.3 Maple step by step solution

Let's solve

$y'' - 2y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 3)$

- 1st solution of the ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-x} + e^{3x} c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x} + c_2 e^{3x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 22

```
DSolve[y''[x]-2*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_2 e^{4x} + c_1\right)$$

10.3 problem 3

Internal problem ID [11733]
Internal file name [`OUTPUT/11742_Thursday_April_11_2024_08_49_18_PM_66750643/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' - 12y' + 5y = 0}$$

10.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = -12, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} - 12\lambda e^{\lambda x} + 5 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 - 12\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = -12, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{12}{(2)\,(4)} \pm \frac{1}{(2)\,(4)}\sqrt{-12^2 - (4)\,(4)\,(5)} \\ &= \frac{3}{2} \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + 1 \\ \lambda_2 &= \frac{3}{2} - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{5}{2} \\ \lambda_2 &= \frac{1}{2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(\frac{5}{2}\right)x} + c_2 e^{\left(\frac{1}{2}\right)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\frac{5x}{2}} + \mathrm{e}^{\frac{x}{2}} c_2$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{5x}{2}} + \mathrm{e}^{\frac{x}{2}} c_2 \tag{1}$$

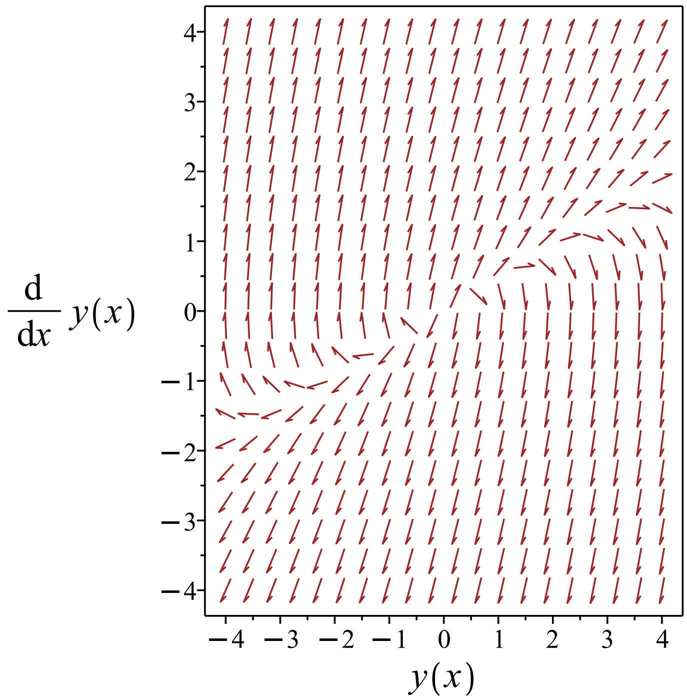


Figure 361: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{5x}{2}} + e^{\frac{x}{2}} c_2$$

Verified OK.

10.3.2 Solving using Kovacic algorithm

Writing the ode as

$$4y'' - 12y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= -12 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 283: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-12}{4}\,dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-12}{4}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{\frac{x}{2}}\right) + c_2\left(e^{\frac{x}{2}}\left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{\frac{x}{2}} + \frac{c_2 e^{\frac{5x}{2}}}{2} \tag{1}
$$

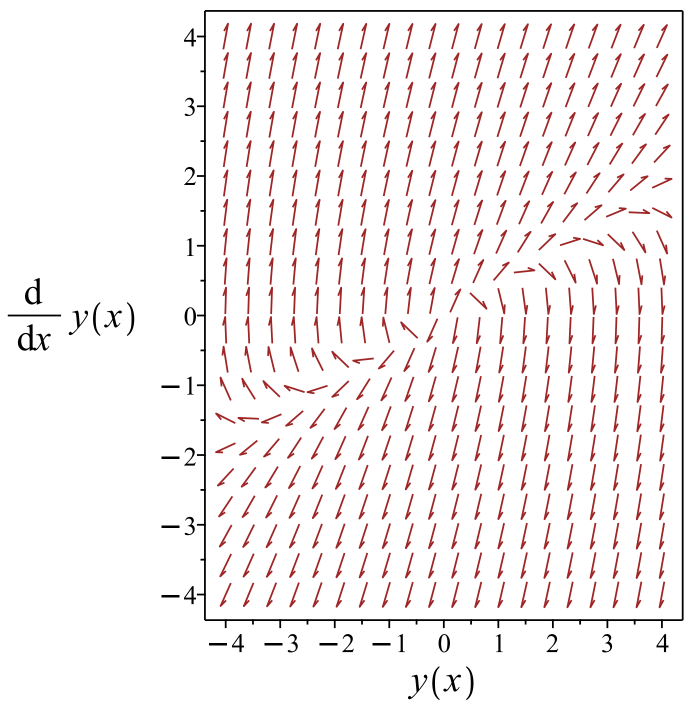


Figure 362: Slope field plot

Verification of solutions

$$
y = c_1 e^{\frac{x}{2}} + \frac{c_2 e^{\frac{5x}{2}}}{2}
$$

Verified OK.

10.3.3 Maple step by step solution

Let's solve

$4y'' - 12y' + 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = 3y' - \frac{5y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 3y' + \frac{5y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - 3r + \frac{5}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)(2r-5)}{4} = 0$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2}, \frac{5}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{\frac{x}{2}}$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{5x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{5x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(4*diff(y(x),x$2)-12*diff(y(x),x)+5*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{x}{2}} + c_2 e^{\frac{5x}{2}}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 24

```
DSolve[4*y''[x]-12*y'[x]+5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x/2} \left(c_2 e^{2x} + c_1\right)$$

10.4 problem 4

Internal problem ID [11734]
Internal file name [`OUTPUT/11743_Thursday_April_11_2024_08_49_18_PM_78335090/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{3y'' - 14y' - 5y = 0}$$

10.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 3, B = -14, C = -5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$3\lambda^2 e^{\lambda x} - 14\lambda e^{\lambda x} - 5 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$3\lambda^2 - 14\lambda - 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 3, B = -14, C = -5$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{14}{(2)(3)} \pm \frac{1}{(2)(3)}\sqrt{-14^2 - (4)(3)(-5)} \\
&= \frac{7}{3} \pm \frac{8}{3}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{7}{3} + \frac{8}{3} \\
\lambda_2 &= \frac{7}{3} - \frac{8}{3}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 5 \\
\lambda_2 &= -\frac{1}{3}
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(5)x} + c_2 e^{\left(-\frac{1}{3}\right)x}
\end{aligned}
$$

Or

$$
y = c_1 \mathrm{e}^{5x} + c_2 \mathrm{e}^{-\frac{x}{3}}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{5x} + c_2 \mathrm{e}^{-\frac{x}{3}} \tag{1}
$$

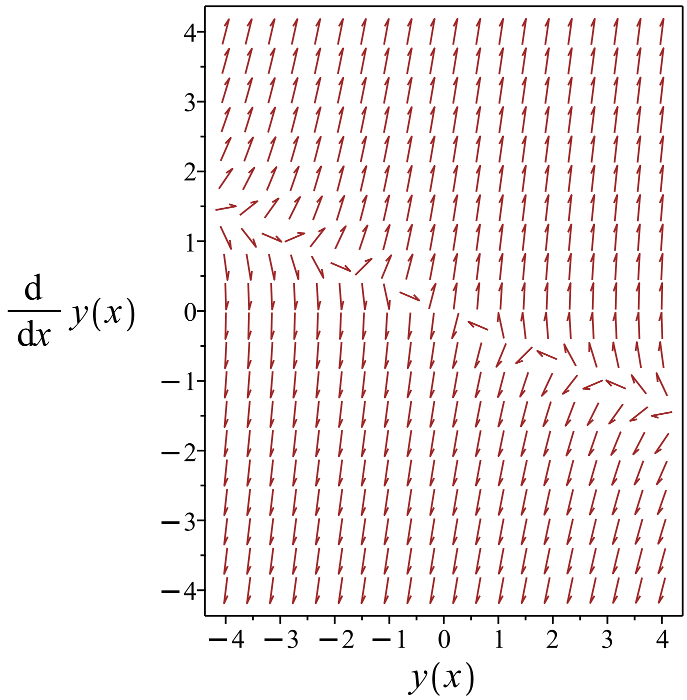


Figure 363: Slope field plot

Verification of solutions

$$y = c_1 e^{5x} + c_2 e^{-\frac{x}{3}}$$

Verified OK.

10.4.2 Solving using Kovacic algorithm

Writing the ode as

$$3y'' - 14y' - 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3 \\ B &= -14 \\ C &= -5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{64}{9} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 64 \\ t &= 9 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{64z(x)}{9} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 285: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{64}{9}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{8x}{3}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-14}{3}\,dx} \\ &= z_1 e^{\frac{7x}{3}} \\ &= z_1 \left(\mathrm{e}^{\frac{7x}{3}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-14}{3}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{14x}{3}}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{3\,\mathrm{e}^{\frac{16x}{3}}}{16}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{3}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{3}}\left(\frac{3\,\mathrm{e}^{\frac{16x}{3}}}{16}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{3}} + \frac{3c_2 \mathrm{e}^{5x}}{16} \tag{1}
$$

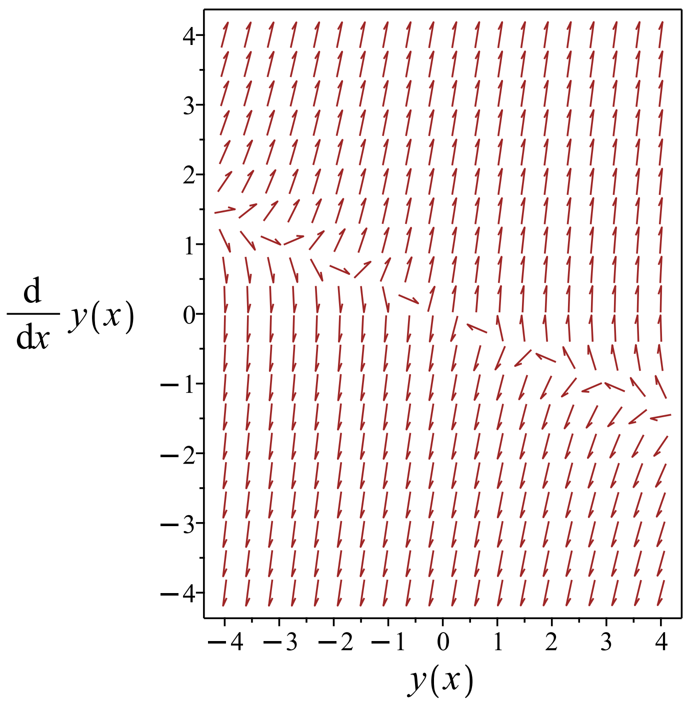


Figure 364: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x}{3}} + \frac{3c_2 \mathrm{e}^{5x}}{16}
$$

Verified OK.

10.4.3 Maple step by step solution

Let's solve

$3y'' - 14y' - 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{14y'}{3} + \frac{5y}{3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{14y'}{3} - \frac{5y}{3} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{14}{3}r - \frac{5}{3} = 0$

- Factor the characteristic polynomial

  $\frac{(3r+1)(r-5)}{3} = 0$

- Roots of the characteristic polynomial

  $r = \left(5, -\frac{1}{3}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{5x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{3}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{5x} + c_2 \mathrm{e}^{-\frac{x}{3}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(3*diff(y(x),x$2)-14*diff(y(x),x)-5*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{5x} + c_2 e^{-\frac{x}{3}}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 24

```
DSolve[3*y''[x]-14*y'[x]-5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x/3} + c_2 e^{5x}$$

10.5 problem 5

Internal problem ID [11735]
Internal file name [`OUTPUT/11744_Thursday_April_11_2024_08_49_19_PM_4021622/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 5.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 3y'' - y' + 3y = 0}$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + c_3 \mathrm{e}^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{x}$$
$$y_3 = \mathrm{e}^{3x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + c_3 \mathrm{e}^{3x} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + c_3 \mathrm{e}^{3x}$$

Verified OK.

10.5.1 Maple step by step solution

Let's solve

$y''' - 3y'' - y' + 3y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ∘ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ∘ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ∘ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ∘ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 3y_3(x) + y_2(x) - 3y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3y_3(x) + y_2(x) - 3y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 1 & 3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 1 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1 e^{-x} + c_2 e^{x} + \frac{c_3 e^{3x}}{9}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)-diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + c_3 \mathrm{e}^{3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 28

```
DSolve[y'''[x]-3*y''[x]-y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x} + c_2 e^{x} + c_3 e^{3x}$$

10.6 problem 6

Internal problem ID [11736]
Internal file name [`OUTPUT/11745_Thursday_April_11_2024_08_49_19_PM_28454319/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 6.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 6y'' + 5y' + 12y = 0}$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 5\lambda + 12 = 0$$

The roots of the above equation are

$$\lambda_1 = 3$$
$$\lambda_2 = 4$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + e^{3x} c_2 + e^{4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{3x}$$
$$y_3 = e^{4x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + e^{3x} c_2 + e^{4x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^{3x} c_2 + e^{4x} c_3$$

Verified OK.

10.6.1 Maple step by step solution

Let's solve

$y''' - 6y'' + 5y' + 12y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 6y_3(x) - 5y_2(x) - 12y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 6y_3(x) - 5y_2(x) - 12y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -5 & 6 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -5 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{4x} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + e^{3x}c_2 \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + e^{4x}c_3 \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1 e^{-x} + \frac{e^{3x}c_2}{9} + \frac{e^{4x}c_3}{16}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+5*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{4x} + c_2 e^{-x} + c_3 e^{3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 29

```
DSolve[y'''[x]-6*y''[x]+5*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(e^{4x}(c_3 e^x + c_2) + c_1\right)$$

10.7 problem 7

Internal problem ID [11737]
Internal file name [`OUTPUT/11746_Thursday_April_11_2024_08_49_19_PM_20027788/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 8y' + 16y = 0}$$

10.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -8, C = 16$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 8\lambda e^{\lambda x} + 16 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 8\lambda + 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -8, C = 16$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{8}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-8)^2 - (4)\,(1)\,(16)} \\ &= 4\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -4$. Therefore the solution is

$$y = c_1 e^{4x} + c_2 x\, e^{4x} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{4x} + c_2 x\, e^{4x} \tag{1}$$

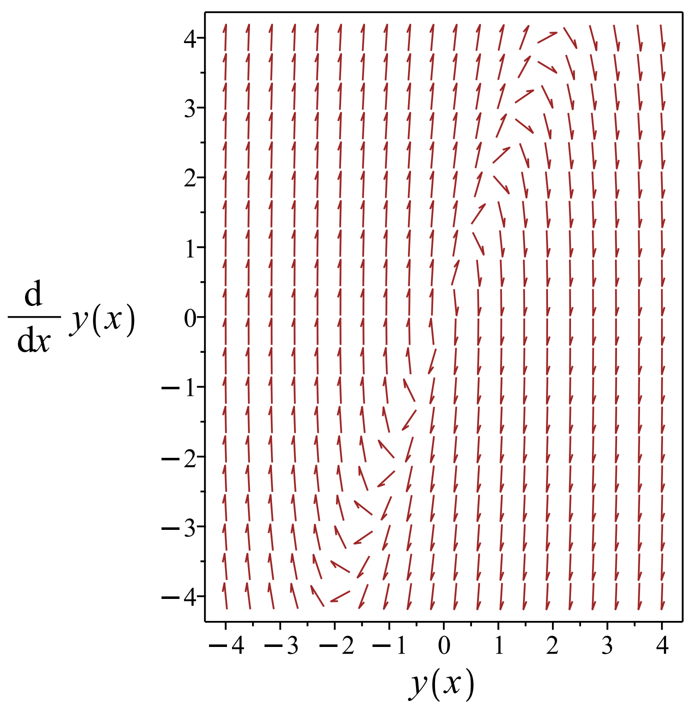


Figure 365: Slope field plot

Verification of solutions

$$y = c_1 e^{4x} + c_2 x\, e^{4x}$$

Verified OK.

10.7.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -8$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x) &= e^{\frac{1}{2}\int p\,dx}\\&= e^{\int -8\,dx}\\&= \mathrm{e}^{-4x}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}(M(x)y)'' &= 0\\\left(\mathrm{e}^{-4x}y\right)'' &= 0\end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-4x}y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-4x}y\right) = c_1x + c_2$$

Hence the solution is

$$y = \frac{c_1x + c_2}{\mathrm{e}^{-4x}}$$

Or

$$y = c_1x\,\mathrm{e}^{4x} + c_2\mathrm{e}^{4x}$$

Summary

The solution(s) found are the following

$$y = c_1x\,\mathrm{e}^{4x} + c_2\mathrm{e}^{4x} \tag{1}$$

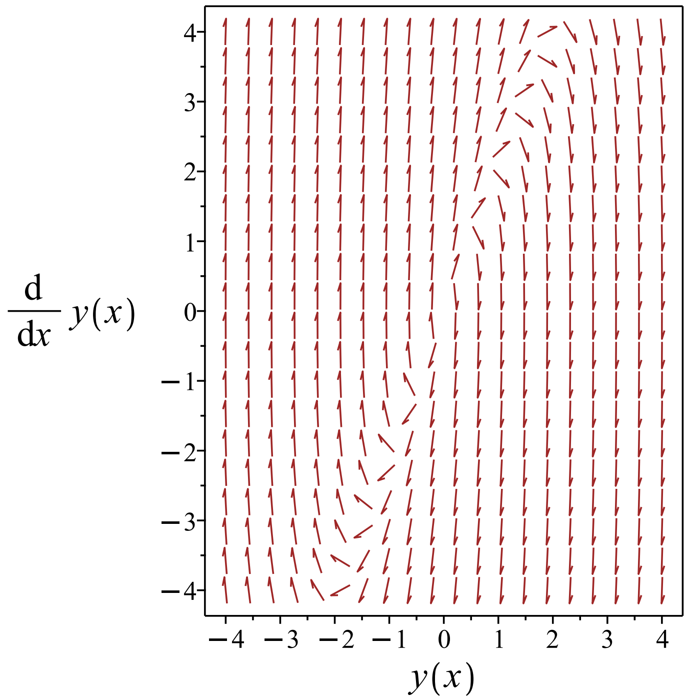


Figure 366: Slope field plot

Verification of solutions

$$y = c_1 x\, e^{4x} + c_2 e^{4x}$$

Verified OK.

10.7.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 8y' + 16y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -8 \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 289: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-8}{1}\,dx} \\ &= z_1 e^{4x} \\ &= z_1\left(\mathrm{e}^{4x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{4x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-8}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{8x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{4x}\right) + c_2\left(\mathrm{e}^{4x}(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{4x} + c_2 x\, \mathrm{e}^{4x} \tag{1}$$

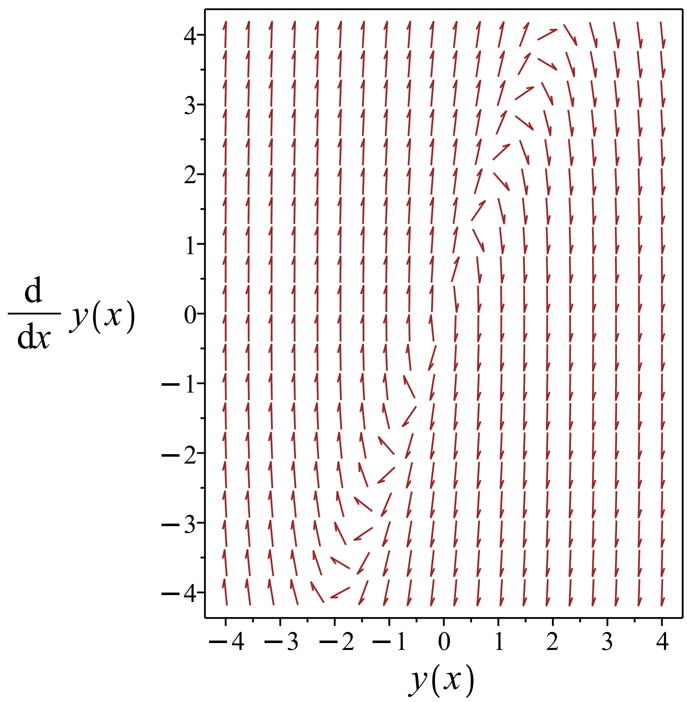


Figure 367: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{4x} + c_2 x\, \mathrm{e}^{4x}$$

Verified OK.

10.7.4 Maple step by step solution

Let's solve

$y'' - 8y' + 16y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 8r + 16 = 0$

- Factor the characteristic polynomial

  $(r-4)^2 = 0$

- Root of the characteristic polynomial

  $r = 4$

- 1st solution of the ODE

  $y_1(x) = e^{4x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{4x} + c_2 x\,e^{4x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)-8*diff(y(x),x)+16*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{4x}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 18

```
DSolve[y''[x]-8*y'[x]+16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{4x}(c_2x + c_1)$$

10.8 problem 8

Internal problem ID [11738]
Internal file name [`OUTPUT/11747_Thursday_April_11_2024_08_49_20_PM_5835726/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' + 4y' + y = 0}$$

10.8.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = 4, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 + 4\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 4, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{(4)^2 - (4)(4)(1)} \\ &= -\frac{1}{2}\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = \frac{1}{2}$. Therefore the solution is

$$y = c_1 e^{-\frac{x}{2}} + c_2 x\, e^{-\frac{x}{2}} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-\frac{x}{2}} + c_2 x\, e^{-\frac{x}{2}} \tag{1}$$

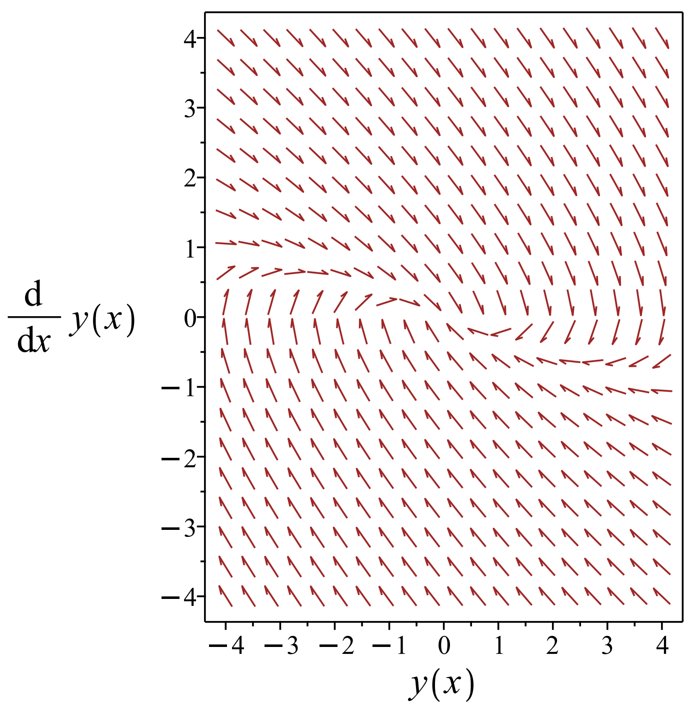


Figure 368: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x}{2}} + c_2 x\, e^{-\frac{x}{2}}$$

Verified OK.

10.8.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^{2} + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 1$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2} \int p \, dx} \\ &= e^{\int 1 \, dx} \\ &= \mathrm{e}^{\frac{x}{2}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{\frac{x}{2}} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{\frac{x}{2}} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{\frac{x}{2}} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{\frac{x}{2}}}$$

Or

$$y = c_1 x \,\mathrm{e}^{-\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 x \,\mathrm{e}^{-\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}} \tag{1}$$

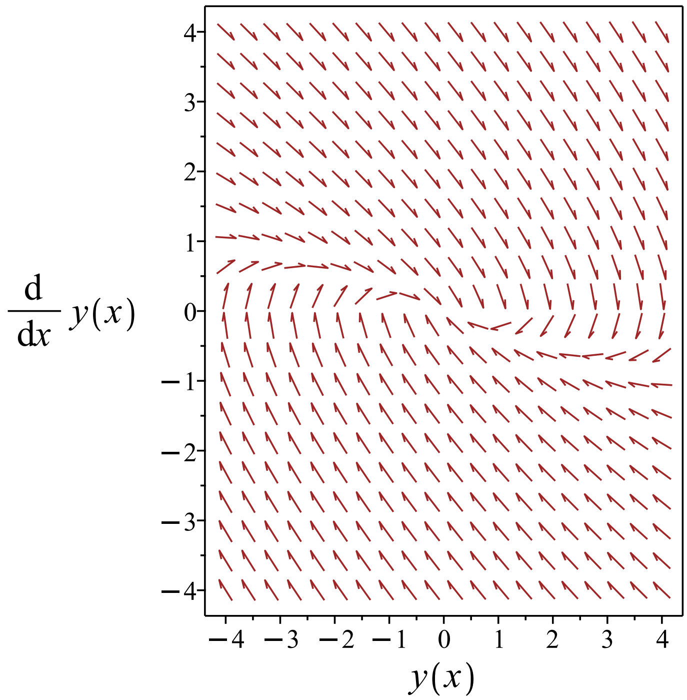


Figure 369: Slope field plot

Verification of solutions

$$y = c_1 x\, \mathrm{e}^{-\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Verified OK.

10.8.3 Solving using Kovacic algorithm

Writing the ode as

$$4y'' + 4y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 4 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 291: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{4}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{4}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2 x \,\mathrm{e}^{-\frac{x}{2}} \tag{1}
$$

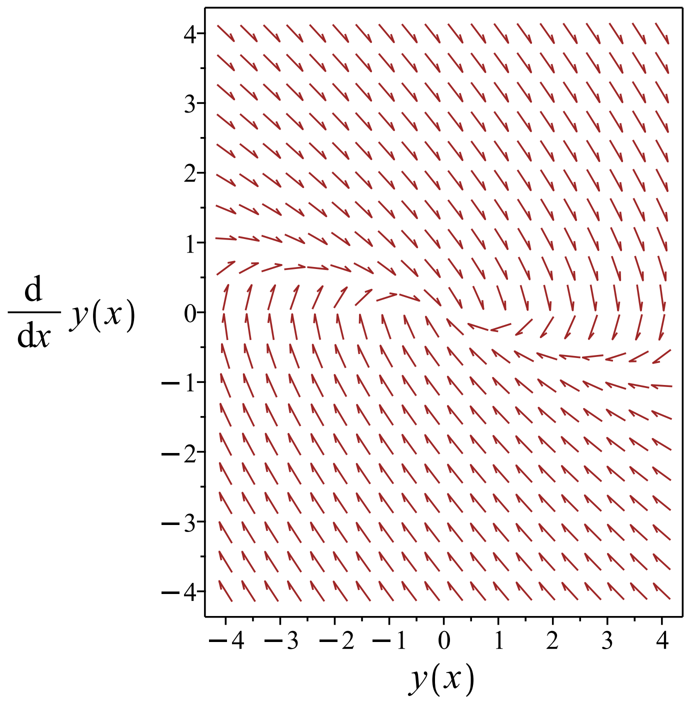


Figure 370: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2 x \,\mathrm{e}^{-\frac{x}{2}}
$$

Verified OK.

10.8.4 Maple step by step solution

Let's solve

$4y'' + 4y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' - \frac{y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' + \frac{y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 + r + \frac{1}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r+1)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = -\frac{1}{2}$

- 1st solution of the ODE

  $y_1(x) = e^{-\frac{x}{2}}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^{-\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-\frac{x}{2}} + c_2 x\, e^{-\frac{x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(4*diff(y(x),x$2)+4*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 20

```
DSolve[4*y''[x]+4*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}(c_2x + c_1)$$

10.9 problem 9

Internal problem ID [11739]
Internal file name [`OUTPUT/11748_Thursday_April_11_2024_08_49_20_PM_31209479/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' + 13y = 0}$$

10.9.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 13$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 13\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 13$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-4^2 - (4)(1)(13)} \\
&= 2 \pm 3i
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= 2 + 3i \\
\lambda_2 &= 2 - 3i
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 2 + 3i \\
\lambda_2 &= 2 - 3i
\end{aligned}
$$

Since roots are complex conjugate of each others, then let the roots be

$$
\lambda_{1,2} = \alpha \pm i\beta
$$

Where $\alpha = 2$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$
y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))
$$

Which becomes

$$
y = e^{2x}(c_1 \cos(3x) + c_2 \sin(3x))
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \mathrm{e}^{2x}(c_1 \cos(3x) + c_2 \sin(3x)) \tag{1}
$$

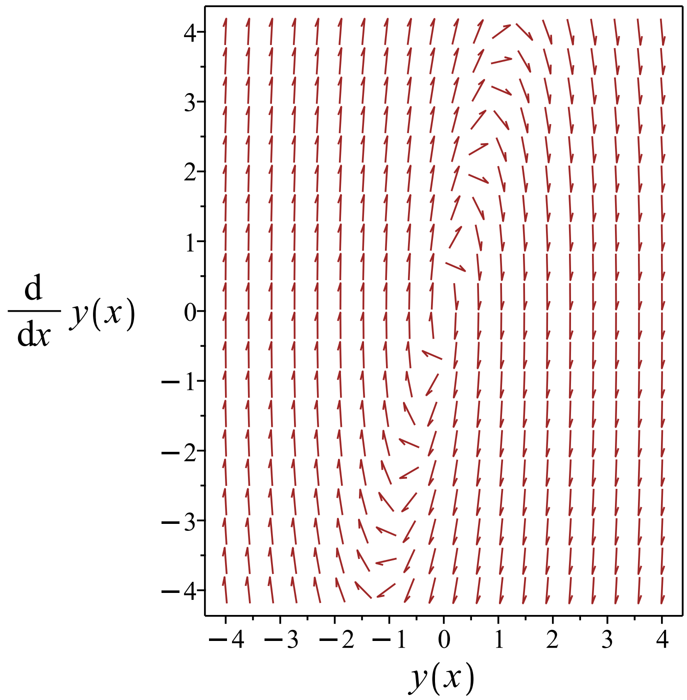


Figure 371: Slope field plot

<u>Verification of solutions</u>

$$y = e^{2x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Verified OK.

10.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 13y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 293: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(3x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4}{1} \, dx} \\
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x} \cos(3x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\tan(3x)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{2x} \cos(3x)\right) + c_2\left(e^{2x} \cos(3x) \left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{2x} \cos(3x) + \frac{c_2 e^{2x} \sin(3x)}{3} \tag{1}
$$

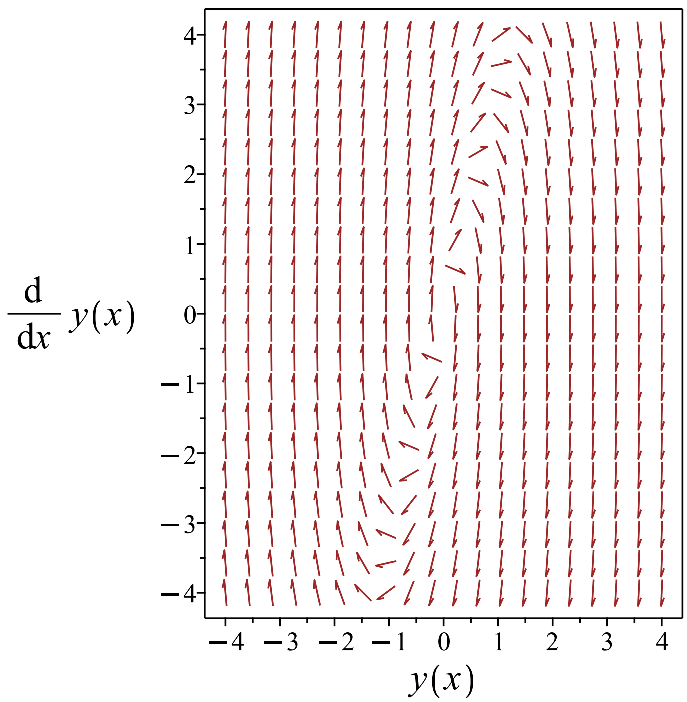


Figure 372: Slope field plot

Verification of solutions

$$
y = c_1 e^{2x} \cos(3x) + \frac{c_2 e^{2x} \sin(3x)}{3}
$$

Verified OK.

10.9.3 Maple step by step solution

Let's solve

$y'' - 4y' + 13y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 4r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - 3\,\mathrm{I}, 2 + 3\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{2x} \cos(3x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{2x} \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{2x} \cos(3x) + c_2 \mathrm{e}^{2x} \sin(3x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+13*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{2x}(c_1 \sin(3x) + c_2 \cos(3x))$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 26

```
DSolve[y''[x]-4*y'[x]+13*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(c_2 \cos(3x) + c_1 \sin(3x))$$

10.10 problem 10

Internal problem ID [11740]
Internal file name [`OUTPUT/11749_Thursday_April_11_2024_08_49_21_PM_75173537/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 6y' + 25y = 0}$$

10.10.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 25$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda\, e^{\lambda x} + 25\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 25$ into the above gives

$$\lambda_{1,2} = \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(25)}$$
$$= -3 \pm 4i$$

Hence

$$\lambda_1 = -3 + 4i$$
$$\lambda_2 = -3 - 4i$$

Which simplifies to

$$\lambda_1 = -3 + 4i$$
$$\lambda_2 = -3 - 4i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-3x}(c_1 \cos(4x) + c_2 \sin(4x))$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-3x}(c_1 \cos(4x) + c_2 \sin(4x)) \tag{1}$$

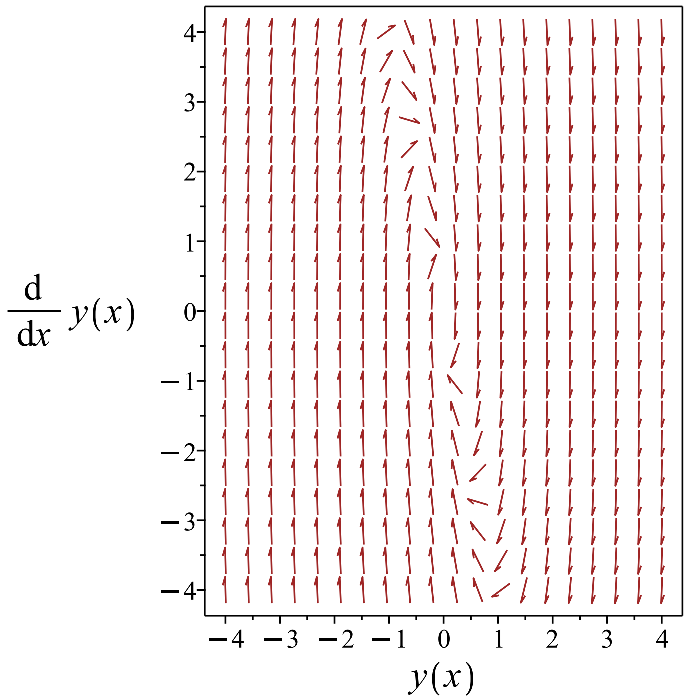


Figure 373: Slope field plot

Verification of solutions

$$y = e^{-3x}(c_1 \cos(4x) + c_2 \sin(4x))$$

Verified OK.

10.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 25y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 295: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(4x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx}\\ &= z_1 e^{-3x}\\ &= z_1\left(\mathrm{e}^{-3x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}\cos(4x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\tan(4x)}{4}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-3x} \cos(4x)\right) + c_2 \left(e^{-3x} \cos(4x) \left(\frac{\tan(4x)}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-3x} \cos(4x) + \frac{c_2 e^{-3x} \sin(4x)}{4} \tag{1}
$$

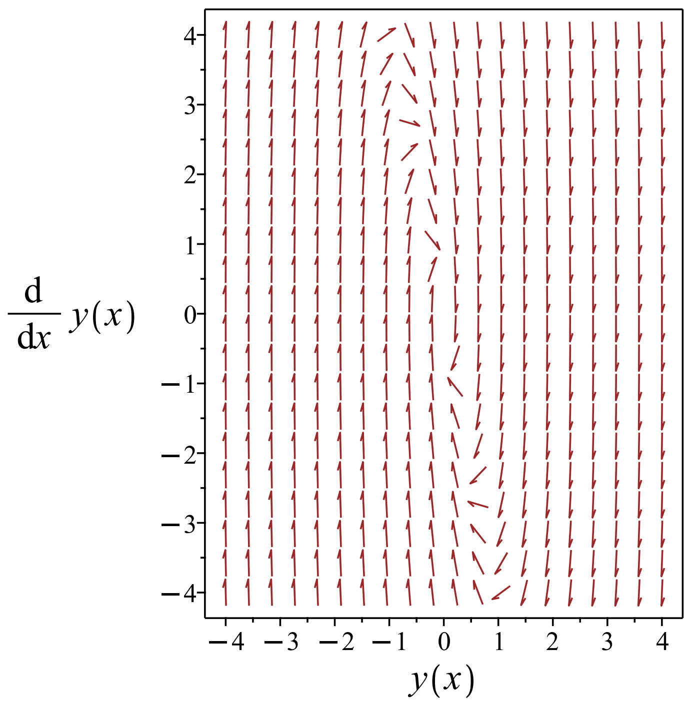


Figure 374: Slope field plot

Verification of solutions

$$
y = c_1 e^{-3x} \cos(4x) + \frac{c_2 e^{-3x} \sin(4x)}{4}
$$

Verified OK.

10.10.3 Maple step by step solution

Let's solve

$y'' + 6y' + 25y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 6r + 25 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6) \pm (\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - 4\,\mathrm{I}, -3 + 4\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-3x} \cos(4x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-3x} \sin(4x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-3x} \cos(4x) + c_2 \mathrm{e}^{-3x} \sin(4x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+25*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-3x}(c_1 \sin(4x) + c_2 \cos(4x))$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 26

```
DSolve[y''[x]+6*y'[x]+25*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}(c_2 \cos(4x) + c_1 \sin(4x))$$

10.11 problem 11

Internal problem ID [11741]
Internal file name [`OUTPUT/11750_Thursday_April_11_2024_08_49_21_PM_92995520/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 9y = 0}$$

10.11.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3x) + c_2 \sin(3x) \tag{1}$$

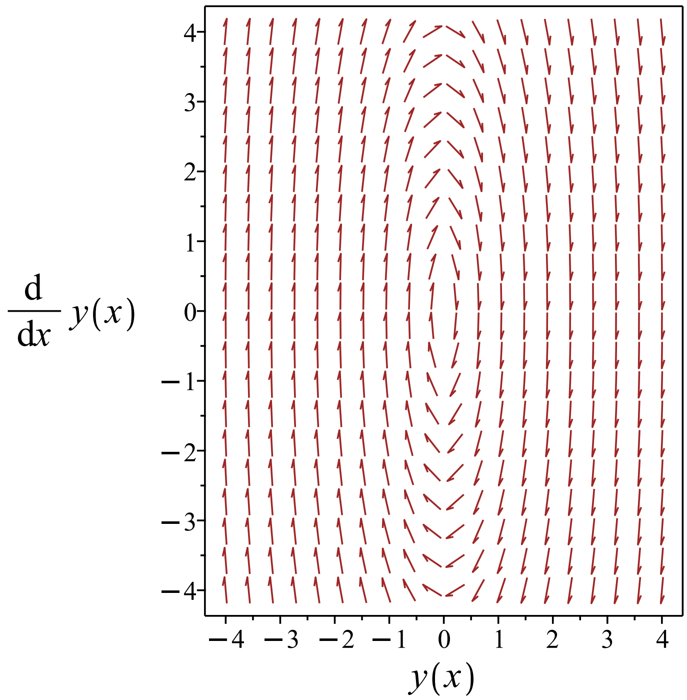


Figure 375: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Verified OK.

10.11.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + 9yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + 9yy')\, dx = 0$$

$$\frac{y'^2}{2} + \frac{9y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-9y^2 + 2c_1} \tag{1}$$

$$y' = -\sqrt{-9y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-9y^2 + 2c_1}} dy = \int dx$$
$$\frac{\arctan\left(\frac{3y}{\sqrt{-9y^2+2c_1}}\right)}{3} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-9y^2 + 2c_1}} dy = \int dx$$
$$-\frac{\arctan\left(\frac{3y}{\sqrt{-9y^2+2c_1}}\right)}{3} = c_3 + x$$

Summary

The solution(s) found are the following

$$\frac{\arctan\left(\frac{3y}{\sqrt{-9y^2+2c_1}}\right)}{3} = x + c_2 \tag{1}$$
$$-\frac{\arctan\left(\frac{3y}{\sqrt{-9y^2+2c_1}}\right)}{3} = c_3 + x \tag{2}$$

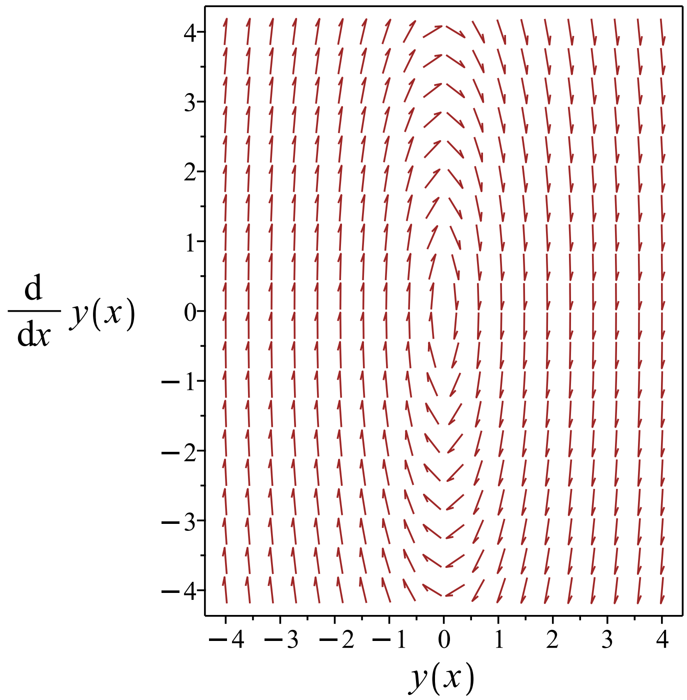


Figure 376: Slope field plot

Verification of solutions

$$\frac{\arctan\left(\frac{3y}{\sqrt{-9y^2+2c_1}}\right)}{3} = x + c_2$$

Verified OK.

$$-\frac{\arctan\left(\frac{3y}{\sqrt{-9y^2+2c_1}}\right)}{3} = c_3 + x$$

Verified OK.

10.11.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 297: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2}\,dx \\ &= \cos(3x)\left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} \tag{1}$$

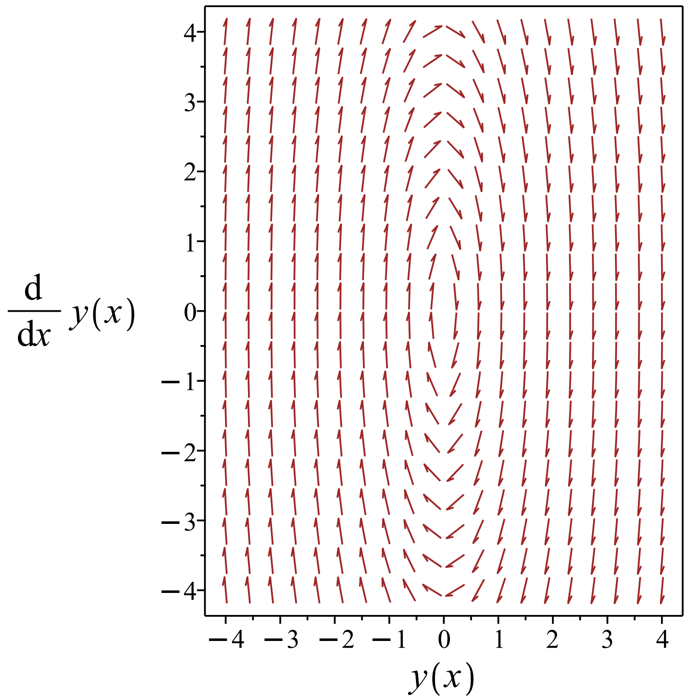


Figure 377: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

Verified OK.

10.11.4 Maple step by step solution

Let's solve

$y'' + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$$r = (-3\,\mathrm{I}, 3\,\mathrm{I})$$

- 1st solution of the ODE

$$y_1(x) = \cos(3x)$$

- 2nd solution of the ODE

$$y_2(x) = \sin(3x)$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+9*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(3x) + c_2 \cos(3x)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 20

```
DSolve[y''[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(3x) + c_2 \sin(3x)$$

10.12 problem 12

Internal problem ID [11742]
Internal file name [`OUTPUT/11751_Thursday_April_11_2024_08_49_22_PM_63064774/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' + y = 0}$$

10.12.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{0^2 - (4)(4)(1)} \\ &= \pm\frac{i}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\frac{i}{2} \\ \lambda_2 &= -\frac{i}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{i}{2} \\ \lambda_2 &= -\frac{i}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \frac{1}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0\left(c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)\right)$$

Or

$$y = c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right) \tag{1}$$

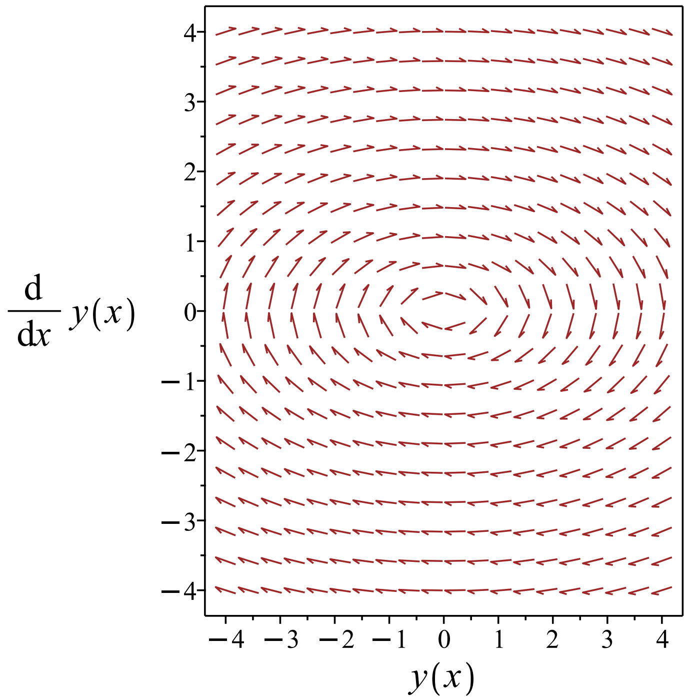


Figure 378: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)$$

Verified OK.

10.12.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$4y'y'' + yy' = 0$$

Integrating the above w.r.t x gives

$$\int (4y'y'' + yy')\, dx = 0$$
$$\frac{y^2}{2} + 2y'^2 = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{-y^2+2c_1}}{2} \tag{1}$$
$$y' = -\frac{\sqrt{-y^2+2c_1}}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2}{\sqrt{-y^2+2c_1}} dy = \int dx$$
$$2\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2}{\sqrt{-y^2+2c_1}} dy = \int dx$$
$$-2\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = c_3 + x$$

Summary

The solution(s) found are the following

$$2\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = x + c_2 \tag{1}$$
$$-2\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right) = c_3 + x \tag{2}$$

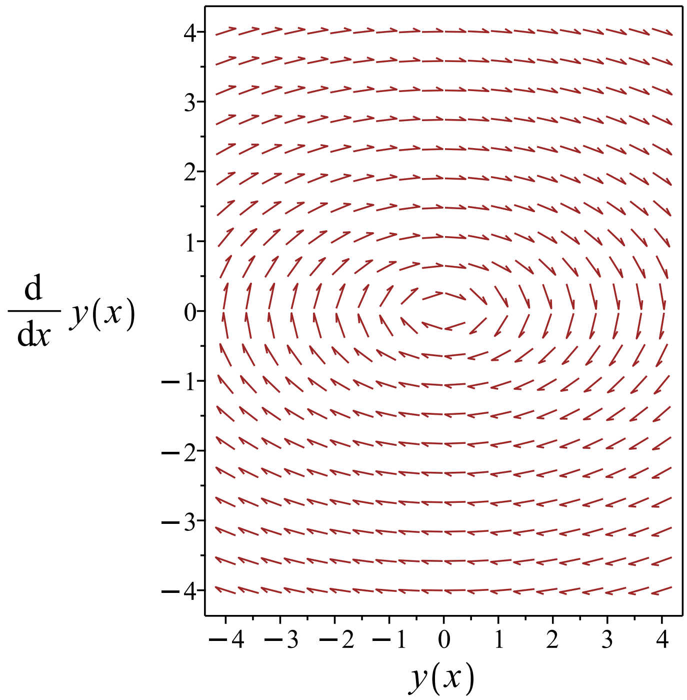


Figure 379: Slope field plot

Verification of solutions

$$2\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right)=x+c_2$$

Verified OK.

$$-2\arctan\left(\frac{y}{\sqrt{-y^2+2c_1}}\right)=c_3+x$$

Verified OK.

10.12.3 Solving using Kovacic algorithm

Writing the ode as

$$4y''+y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 299: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos\left(\frac{x}{2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\frac{x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos\left(\frac{x}{2}\right) \int \frac{1}{\cos\left(\frac{x}{2}\right)^2}\, dx \\ &= \cos\left(\frac{x}{2}\right) \left(2\tan\left(\frac{x}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\cos\left(\frac{x}{2}\right)\right) + c_2\left(\cos\left(\frac{x}{2}\right)\left(2\tan\left(\frac{x}{2}\right)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\frac{x}{2}\right) + 2c_2 \sin\left(\frac{x}{2}\right) \tag{1}$$

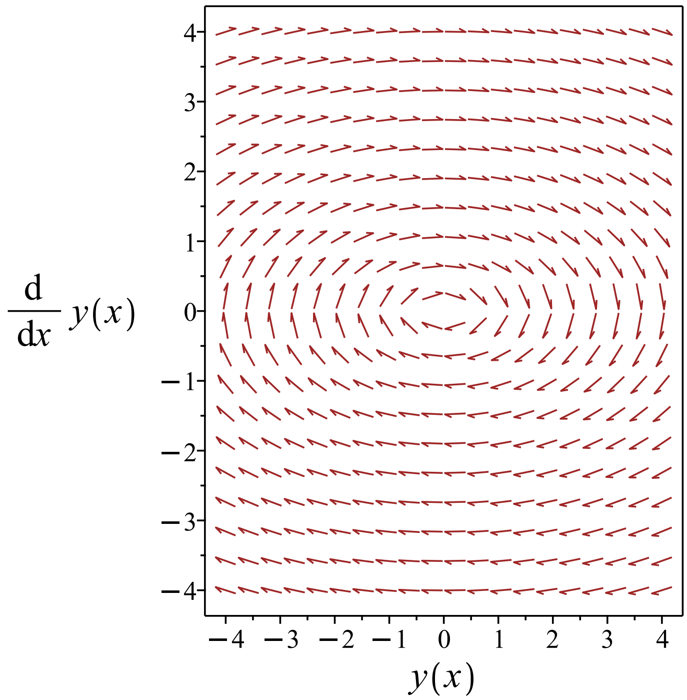


Figure 380: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\frac{x}{2}\right) + 2c_2 \sin\left(\frac{x}{2}\right)$$

Verified OK.

10.12.4 Maple step by step solution

Let's solve

$4y'' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y}{4} = 0$

- Characteristic polynomial of ODE

$r^2 + \frac{1}{4} = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-1})}{2}$
- Roots of the characteristic polynomial

  $r = \left(-\frac{I}{2}, \frac{I}{2}\right)$
- 1st solution of the ODE

  $y_1(x) = \cos\left(\frac{x}{2}\right)$
- 2nd solution of the ODE

  $y_2(x) = \sin\left(\frac{x}{2}\right)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(4*diff(y(x),x$2)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin\left(\frac{x}{2}\right) + c_2 \cos\left(\frac{x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 24

```
DSolve[4*y''[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)$$

10.13 problem 13

Internal problem ID [11743]
Internal file name [`OUTPUT/11752_Thursday_April_11_2024_08_49_22_PM_74938499/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 13.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 5y'' + 7y' - 3y = 0}$$

The characteristic equation is

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

The roots of the above equation are

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + c_2 x\, e^x + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = x\, e^x$$
$$y_3 = e^{3x}$$

Summary
The solution(s) found are the following

$$y = e^{x} c_1 + c_2 x\, e^{x} + c_3 e^{3x} \tag{1}$$

Verification of solutions

$$y = e^{x} c_1 + c_2 x\, e^{x} + c_3 e^{3x}$$

Verified OK.

10.13.1 Maple step by step solution

Let's solve

$y''' - 5y'' + 7y' - 3y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 5y_3(x) - 7y_2(x) + 3y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 5y_3(x) - 7y_2(x) + 3y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -7 & 5 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -7 & 5 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

$$\overrightarrow{y}_1(x) = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

  $$\lambda \mathrm{e}^{\lambda x}\left(x\overrightarrow{v}+\overrightarrow{p}\right)+\mathrm{e}^{\lambda x}\overrightarrow{v}=\mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v}+A\cdot\overrightarrow{p}\right)$$

- Simplify equation

  $$\lambda\overrightarrow{p}+\overrightarrow{v}=A\cdot\overrightarrow{p}$$

- Make use of the identity matrix I

  $$(\lambda\cdot I)\cdot\overrightarrow{p}+\overrightarrow{v}=A\cdot\overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

  $$(A-\lambda\cdot I)\cdot\overrightarrow{p}=\overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

  $$\left(\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 3 & -7 & 5\end{bmatrix}-1\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\cdot\overrightarrow{p}=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p}=\begin{bmatrix}-1\\ 0\\ 0\end{bmatrix}$$

- Second solution from eigenvalue 1

  $$\overrightarrow{y}_2(x)=\mathrm{e}^{x}\cdot\left(x\cdot\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}+\begin{bmatrix}-1\\ 0\\ 0\end{bmatrix}\right)$$

- Consider eigenpair

  $$\left[3,\begin{bmatrix}\frac{1}{9}\\ \frac{1}{3}\\ 1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{y}_3=\mathrm{e}^{3x}\cdot\begin{bmatrix}\frac{1}{9}\\ \frac{1}{3}\\ 1\end{bmatrix}$$

- General solution to the system of ODEs

  $\overrightarrow{y} = c_1 \overrightarrow{y}_1(x) + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3$
- Substitute solutions into the general solution

  $$\overrightarrow{y} = e^x c_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$
- First component of the vector is the solution to the ODE

  $y = \frac{c_3 e^{3x}}{9} + e^x (c_2(x-1) + c_1)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$3)-5*diff(y(x),x$2)+7*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{3x} + e^x (c_3 x + c_2)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 24

```
DSolve[y'''[x]-5*y''[x]+7*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x \left(c_2 x + c_3 e^{2x} + c_1 \right)$$

10.14 problem 14

Internal problem ID [11744]
Internal file name [`OUTPUT/11753_Thursday_April_11_2024_08_49_23_PM_82557195/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 14.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _missing_x]]`

$$\boxed{4y''' + 4y'' - 7y' + 2y = 0}$$

The characteristic equation is

$$4\lambda^3 + 4\lambda^2 - 7\lambda + 2 = 0$$

The roots of the above equation are

$$\lambda_1 = -2$$
$$\lambda_2 = \frac{1}{2}$$
$$\lambda_3 = \frac{1}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + e^{\frac{x}{2}} c_2 + x\, e^{\frac{x}{2}} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = e^{\frac{x}{2}}$$
$$y_3 = e^{\frac{x}{2}} x$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{\frac{x}{2}} c_2 + x \,\mathrm{e}^{\frac{x}{2}} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + \mathrm{e}^{\frac{x}{2}} c_2 + x \,\mathrm{e}^{\frac{x}{2}} c_3$$

Verified OK.

10.14.1 Maple step by step solution

Let's solve

$4y''' + 4y'' - 7y' + 2y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -y'' + \frac{7y'}{4} - \frac{y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + y'' - \frac{7y'}{4} + \frac{y}{2} = 0$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -y_3(x) + \frac{7y_2(x)}{4} - \frac{y_1(x)}{2}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -y_3(x) + \frac{7y_2(x)}{4} - \frac{y_1(x)}{2}\right]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & \frac{7}{4} & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & \frac{7}{4} & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right], \left[\frac{1}{2}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[\frac{1}{2}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue $\frac{1}{2}$

$$\vec{y}_2(x) = \mathrm{e}^{\frac{x}{2}} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = \frac{1}{2}$ is the eigenvalue, an

$$\vec{y}_3(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_3(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \mathrm{e}^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_3(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue $\frac{1}{2}$

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & \frac{7}{4} & -1 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -8 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue $\frac{1}{2}$

$$\vec{y}_3(x) = e^{\frac{x}{2}} \cdot \left(x \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + e^{\frac{x}{2}} c_2 \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{\frac{x}{2}} \cdot \left(x \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ 0 \\ 0 \end{bmatrix}\right)$$

- First component of the vector is the solution to the ODE

$$y = 4\,e^{-2x}\left(\left((x-2)\,c_3 + c_2\right) e^{\frac{5x}{2}} + \frac{c_1}{16}\right)$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(4*diff(y(x),x$3)+4*diff(y(x),x$2)-7*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left((c_3 x + c_2)\,e^{\frac{5x}{2}} + c_1\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 93

```
DSolve[4*y'''[x]+4*y''[x]+7*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) \rightarrow\ & c_1 \exp\left(x\mathrm{Root}\left[4\#1^3+4\#1^2+7\#1+2\&,1\right]\right) \\
& + c_2 \exp\left(x\mathrm{Root}\left[4\#1^3+4\#1^2+7\#1+2\&,2\right]\right) \\
& + c_3 \exp\left(x\mathrm{Root}\left[4\#1^3+4\#1^2+7\#1+2\&,3\right]\right)
\end{aligned}
$$

10.15 problem 15

Internal problem ID [11745]
Internal file name [`OUTPUT/11754_Thursday_April_11_2024_08_49_23_PM_77245304/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 15.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 6y'' + 12y' - 8y = 0}$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= 2\\ \lambda_3 &= 2\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + c_2 e^{2x} x + x^2 e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= e^{2x}\\ y_2 &= e^{2x} x\\ y_3 &= x^2 e^{2x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{2x} x + x^2 e^{2x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{2x} x + x^2 e^{2x} c_3$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+12*diff(y(x),x)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{2x}(c_3 x^2 + c_2 x + c_1)$$

✓ Solution by Mathematica
Time used: 0.003 (sec). Leaf size: 23

```
DSolve[y'''[x]-6*y''[x]+12*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(x(c_3 x + c_2) + c_1)$$

10.16 problem 16

Internal problem ID [11746]
Internal file name [`OUTPUT/11755_Thursday_April_11_2024_08_49_24_PM_2170690/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 16.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 4y'' + 5y' + 6y = 0}$$

The characteristic equation is

$$\lambda^3 + 4\lambda^2 + 5\lambda + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = -3$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{7}}{2}$$
$$\lambda_3 = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-3x} + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{7}}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{7}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-3x}$$
$$y_2 = e^{\left(-\frac{1}{2}-\frac{i\sqrt{7}}{2}\right)x}$$
$$y_3 = e^{\left(-\frac{1}{2}+\frac{i\sqrt{7}}{2}\right)x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-3x} + e^{\left(-\frac{1}{2}-\frac{i\sqrt{7}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{7}}{2}\right)x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-3x} + e^{\left(-\frac{1}{2}-\frac{i\sqrt{7}}{2}\right)x} c_2 + e^{\left(-\frac{1}{2}+\frac{i\sqrt{7}}{2}\right)x} c_3$$

Verified OK.

10.16.1 Maple step by step solution

Let's solve

$y''' + 4y'' + 5y' + 6y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -4y_3(x) - 5y_2(x) - 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -4y_3(x) - 5y_2(x) - 6y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -5 & -4 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -5 & -4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} - \frac{\mathrm{I}\sqrt{7}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{7}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{\mathrm{I}\sqrt{7}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{7}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}+\frac{\mathrm{I}\sqrt{7}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2}-\frac{I\sqrt{7}}{2}, \left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{7}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{7}}{2}} \\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{1}{2}-\frac{I\sqrt{7}}{2}\right)x}\cdot\left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{7}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{7}}{2}} \\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-\frac{x}{2}}\cdot\left(\cos\left(\frac{\sqrt{7}\,x}{2}\right)-I\sin\left(\frac{\sqrt{7}\,x}{2}\right)\right)\cdot\left[\begin{array}{c}\frac{1}{\left(-\frac{1}{2}-\frac{I\sqrt{7}}{2}\right)^{2}} \\ \frac{1}{-\frac{1}{2}-\frac{I\sqrt{7}}{2}} \\ 1\end{array}\right]$$

- Simplify expression

$$e^{-\frac{x}{2}}\cdot\left[\begin{array}{c}\frac{\cos\left(\frac{\sqrt{7}\,x}{2}\right)-I\sin\left(\frac{\sqrt{7}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{I\sqrt{7}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{7}\,x}{2}\right)-I\sin\left(\frac{\sqrt{7}\,x}{2}\right)}{-\frac{1}{2}-\frac{I\sqrt{7}}{2}} \\ \cos\left(\frac{\sqrt{7}\,x}{2}\right)-I\sin\left(\frac{\sqrt{7}\,x}{2}\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{2}(x)=e^{-\frac{x}{2}}\cdot\left[\begin{array}{c}-\frac{3\cos\left(\frac{\sqrt{7}\,x}{2}\right)}{8}-\frac{\sin\left(\frac{\sqrt{7}\,x}{2}\right)\sqrt{7}}{8} \\ -\frac{\cos\left(\frac{\sqrt{7}\,x}{2}\right)}{4}+\frac{\sin\left(\frac{\sqrt{7}\,x}{2}\right)\sqrt{7}}{4} \\ \cos\left(\frac{\sqrt{7}\,x}{2}\right)\end{array}\right], \vec{y}_{3}(x)=e^{-\frac{x}{2}}\cdot\left[\begin{array}{c}-\frac{\cos\left(\frac{\sqrt{7}\,x}{2}\right)\sqrt{7}}{8}+\frac{3\sin\left(\frac{\sqrt{7}\,x}{2}\right)}{8} \\ \frac{\cos\left(\frac{\sqrt{7}\,x}{2}\right)\sqrt{7}}{4}+\frac{\sin\left(\frac{\sqrt{7}\,x}{2}\right)}{4} \\ -\sin\left(\frac{\sqrt{7}\,x}{2}\right)\end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_{1}\vec{y}_{1}+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{-\frac{x}{2}} \cdot \begin{bmatrix} -\frac{3\cos\left(\frac{\sqrt{7}\,x}{2}\right)}{8} - \frac{\sin\left(\frac{\sqrt{7}\,x}{2}\right)\sqrt{7}}{8} \\ -\frac{\cos\left(\frac{\sqrt{7}\,x}{2}\right)}{4} + \frac{\sin\left(\frac{\sqrt{7}\,x}{2}\right)\sqrt{7}}{4} \\ \cos\left(\frac{\sqrt{7}\,x}{2}\right) \end{bmatrix} + c_3 \mathrm{e}^{-\frac{x}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{7}\,x}{2}\right)\sqrt{7}}{8} + \frac{3\sin\left(\frac{\sqrt{7}\,x}{2}\right)}{8} \\ \frac{\cos\left(\frac{\sqrt{7}\,x}{2}\right)\sqrt{7}}{4} + \frac{\sin\left(\frac{\sqrt{7}\,x}{2}\right)}{4} \\ -\sin\left(\frac{\sqrt{7}\,x}{2}\right) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{3\left(\frac{c_3\sqrt{7}}{3}+c_2\right)\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{7}\,x}{2}\right)}{8} - \frac{\mathrm{e}^{-\frac{x}{2}}\left(c_2\sqrt{7}-3c_3\right)\sin\left(\frac{\sqrt{7}\,x}{2}\right)}{8} + \frac{c_1\mathrm{e}^{-3x}}{9}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$3)+4*diff(y(x),x$2)+5*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1\mathrm{e}^{-3x} + c_2\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{7}\,x}{2}\right) + c_3\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{7}\,x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 56

```
DSolve[y'''[x]+4*y''[x]+5*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(c_2e^{5x/2}\cos\left(\frac{\sqrt{7}x}{2}\right) + c_1e^{5x/2}\sin\left(\frac{\sqrt{7}x}{2}\right) + c_3\right)$$

10.17 problem 17

Internal problem ID [11747]
Internal file name [`OUTPUT/11756_Thursday_April_11_2024_08_49_24_PM_84570994/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 17.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - y'' + y' - y = 0}$$

The characteristic equation is

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{x} c_1 + \mathrm{e}^{ix} c_2 + \mathrm{e}^{-ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{x}$$
$$y_2 = \mathrm{e}^{ix}$$
$$y_3 = \mathrm{e}^{-ix}$$

Summary
The solution(s) found are the following

$$y = e^{x} c_1 + e^{ix} c_2 + e^{-ix} c_3 \tag{1}$$

Verification of solutions

$$y = e^{x} c_1 + e^{ix} c_2 + e^{-ix} c_3$$

Verified OK.

10.17.1 Maple step by step solution

Let's solve

$y''' - y'' + y' - y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = y_3(x) - y_2(x) + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = y_3(x) - y_2(x) + y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -1 \\ I \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(x) + I\sin(x) \\ I(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$

- Substitute solutions into the general solution

$$\vec{y} = e^x c_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -c_2\cos(x) + c_3\sin(x) \\ c_2\sin(x) + c_3\cos(x) \\ c_2\cos(x) - c_3\sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$y = e^x c_1 + c_3\sin(x) - c_2\cos(x)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)+diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^x + \sin(x) c_2 + c_3 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 22

```
DSolve[y'''[x]-y''[x]+y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 e^x + c_1 \cos(x) + c_2 \sin(x)$$

10.18 problem 18

Internal problem ID [11748]
Internal file name [`OUTPUT/11757_Thursday_April_11_2024_08_49_24_PM_39924745/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 18.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 8y'' + 16y = 0}$$

The characteristic equation is

$$\lambda^4 + 8\lambda^2 + 16 = 0$$

The roots of the above equation are

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$
$$\lambda_3 = 2i$$
$$\lambda_4 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{2ix} c_1 + x\,\mathrm{e}^{2ix} c_2 + \mathrm{e}^{-2ix} c_3 + x\,\mathrm{e}^{-2ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{2ix}$$
$$y_2 = x\,\mathrm{e}^{2ix}$$
$$y_3 = \mathrm{e}^{-2ix}$$
$$y_4 = x\,\mathrm{e}^{-2ix}$$

Summary
The solution(s) found are the following

$$y = e^{2ix} c_1 + x\, e^{2ix} c_2 + e^{-2ix} c_3 + x\, e^{-2ix} c_4 \tag{1}$$

Verification of solutions

$$y = e^{2ix} c_1 + x\, e^{2ix} c_2 + e^{-2ix} c_3 + x\, e^{-2ix} c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$4)+8*diff(y(x),x$2)+16*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_4 x + c_2) \cos (2x) + \sin (2x) (c_3 x + c_1)$$

✓ Solution by Mathematica
Time used: 0.003 (sec). Leaf size: 30

```
DSolve[y''''[x]+8*y''[x]+16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (c_2 x + c_1) \cos(2x) + (c_4 x + c_3) \sin(2x)$$

10.19 problem 19

Internal problem ID [11749]
Internal file name [OUTPUT/11758_Thursday_April_11_2024_08_49_25_PM_75101065/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 19.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(5)} - 2y'''' + y''' = 0}$$

The characteristic equation is

$$\lambda^5 - 2\lambda^4 + \lambda^3 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 1$$
$$\lambda_5 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_3x^2 + c_2x + c_1 + \mathrm{e}^x c_4 + x\,\mathrm{e}^x c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= x^2 \\
y_4 &= e^x \\
y_5 &= x\,e^x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_3x^2 + c_2x + c_1 + e^x c_4 + x\,e^x c_5 \tag{1}$$

Verification of solutions

$$y = c_3x^2 + c_2x + c_1 + e^x c_4 + x\,e^x c_5$$

Verified OK.

10.19.1 Maple step by step solution

Let's solve

$y^{(5)} - 2y'''' + y''' = 0$

- Highest derivative means the order of the ODE is 5

  $y^{(5)}$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  ◦ Define new variable $y_5(x)$

    $y_5(x) = y''''$

- Isolate for $y_5'(x)$ using original ODE

  $y_5'(x) = 2y_5(x) - y_4(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = 2y_5(x) - y_4(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right], \left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right], \left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right], \left[1, \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right], \left[1, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\begin{bmatrix} 1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- First solution from eigenvalue 1

$$\overrightarrow{y}_4(x) = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\overrightarrow{y}_5(x) = e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_5(x)$ into the homogeneous system

$$\lambda e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = \left(e^{\lambda x}A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda e^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + e^{\lambda x}\overrightarrow{v} = e^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_5(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{y}_5(x) = e^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4(x) + c_5 \vec{y}_5(x)$$

- Substitute solutions into the general solution

$$\vec{y} = e^x c_4 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_5 e^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} c_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = ((x-1) c_5 + c_4) e^x + c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$5)-2*diff(y(x),x$4)+diff(y(x),x$3)=0,y(x), singsol=all)
```

$$y(x) = (c_5x + c_4)\,e^x + c_3x^2 + c_2x + c_1$$

✓ Solution by Mathematica

Time used: 0.096 (sec). Leaf size: 30

```
DSolve[y'''''[x]-2*y''''[x]+y'''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(c_2(x-3) + c_1) + x(c_5x + c_4) + c_3$$

10.20 problem 20

Internal problem ID [11750]
Internal file name [`OUTPUT/11759_Thursday_April_11_2024_08_49_25_PM_94911187/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 20.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - y''' - 3y'' + y' + 2y = 0}$$

The characteristic equation is

$$\lambda^4 - \lambda^3 - 3\lambda^2 + \lambda + 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + c_3 \mathrm{e}^{x} + \mathrm{e}^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = x\,e^{-x}$$
$$y_3 = e^{x}$$
$$y_4 = e^{2x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + x\,e^{-x} c_2 + c_3 e^{x} + e^{2x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + x\,e^{-x} c_2 + c_3 e^{x} + e^{2x} c_4$$

Verified OK.

10.20.1 Maple step by step solution

Let's solve
$y'''' - y''' - 3y'' + y' + 2y = 0$

- Highest derivative means the order of the ODE is 4
  y''''
- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$
    $y_1(x) = y$
  - Define new variable $y_2(x)$
    $y_2(x) = y'$
  - Define new variable $y_3(x)$
    $y_3(x) = y''$
  - Define new variable $y_4(x)$
    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE
    $y_4'(x) = y_4(x) + 3y_3(x) - y_2(x) - 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = y_4(x) + 3y_3(x) - y_2(x) - 2y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -1 & 3 & 1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -1 & 3 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue -1

$$\overrightarrow{y}_1(x) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -1 & 3 & 1 \end{array}\right] - (-1) \cdot \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

$$\vec{y}_2(x) = \mathrm{e}^{-x} \cdot \left(x \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs
  $\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3 + c_4 \vec{y}_4$
- Substitute solutions into the general solution
  $$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-x} \cdot \left(x \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + e^{2x} c_4 \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$
- First component of the vector is the solution to the ODE
  $y = ((-x - 1) c_2 - c_1) e^{-x} + c_3 e^{x} + \frac{e^{2x} c_4}{8}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$4)-diff(y(x),x$3)-3*diff(y(x),x$2)+diff(y(x),x)+2*y(x)=0,y(x), singsol=al
```

$$y(x) = (c_4 x + c_3) e^{-x} + c_1 e^{x} + c_2 e^{2x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 33

```
DSolve[y''''[x]-y'''[x]-3*y''[x]+y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_2 x + e^{2x}(c_4 e^x + c_3) + c_1\right)$$

10.21 problem 21

Internal problem ID [11751]
Internal file name [`OUTPUT/11760_Thursday_April_11_2024_08_49_25_PM_40279363/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 21.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 3y''' - 2y'' + 2y' + 12y = 0}$$

The characteristic equation is

$$\lambda^4 - 3\lambda^3 - 2\lambda^2 + 2\lambda + 12 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= 3 \\
\lambda_3 &= -1 - i \\
\lambda_4 &= -1 + i
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{2x} + \mathrm{e}^{3x} c_2 + \mathrm{e}^{(-1+i)x} c_3 + \mathrm{e}^{(-1-i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{2x} \\
y_2 &= e^{3x} \\
y_3 &= e^{(-1+i)x} \\
y_4 &= e^{(-1-i)x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + e^{3x} c_2 + e^{(-1+i)x} c_3 + e^{(-1-i)x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{2x} + e^{3x} c_2 + e^{(-1+i)x} c_3 + e^{(-1-i)x} c_4$$

Verified OK.

10.21.1 Maple step by step solution

Let's solve

$y'''' - 3y''' - 2y'' + 2y' + 12y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 3y_4(x) + 2y_3(x) - 2y_2(x) - 12y_1(x)$

  Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 3y_4(x) + 2y_3(x) - 2y_2(x) - 12y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -12 & -2 & 2 & 3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -12 & -2 & 2 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-1 - \mathrm{I}, \begin{bmatrix} \frac{1}{4} + \frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[-1 + \mathrm{I}, \begin{bmatrix} \frac{1}{4} - \frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ -\frac{1}{2} - \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c}\frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{2x} \cdot \left[\begin{array}{c}\frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c}\frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{3x} \cdot \left[\begin{array}{c}\frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1 - \mathrm{I}, \left[\begin{array}{c}\frac{1}{4} + \frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2} + \frac{\mathrm{I}}{2} \\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$e^{(-1-I)x} \cdot \begin{bmatrix} \frac{1}{4}+\frac{I}{4} \\ -\frac{I}{2} \\ -\frac{1}{2}+\frac{I}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-x} \cdot (\cos(x) - I\sin(x)) \cdot \begin{bmatrix} \frac{1}{4}+\frac{I}{4} \\ -\frac{I}{2} \\ -\frac{1}{2}+\frac{I}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-x} \cdot \begin{bmatrix} \left(\frac{1}{4}+\frac{I}{4}\right)(\cos(x) - I\sin(x)) \\ -\frac{I}{2}(\cos(x) - I\sin(x)) \\ \left(-\frac{1}{2}+\frac{I}{2}\right)(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{-x} \cdot \begin{bmatrix} \frac{\cos(x)}{4}+\frac{\sin(x)}{4} \\ -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = e^{-x} \cdot \begin{bmatrix} \frac{\cos(x)}{4}-\frac{\sin(x)}{4} \\ -\frac{\cos(x)}{2} \\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + e^{3x} c_2 \cdot \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + c_3 e^{-x} \cdot \begin{bmatrix} \frac{\cos(x)}{4}+\frac{\sin(x)}{4} \\ -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ \cos(x) \end{bmatrix} + c_4 e^{-x} \cdot \begin{bmatrix} \frac{\cos(x)}{4}-\frac{\sin(x)}{4} \\ -\frac{\cos(x)}{2} \\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ -\sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{((c_3+c_4)\cos(x)+\sin(x)(c_3-c_4))\mathrm{e}^{-x}}{4} + \frac{\mathrm{e}^{3x}c_2}{27} + \frac{c_1\mathrm{e}^{2x}}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$4)-3*diff(y(x),x$3)-2*diff(y(x),x$2)+2*diff(y(x),x)+12*y(x)=0,y(x), sings
```

$$y(x) = \mathrm{e}^{2x}c_1 + c_2\mathrm{e}^{3x} + c_3\mathrm{e}^{-x}\sin(x) + c_4\mathrm{e}^{-x}\cos(x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 37

```
DSolve[y''''[x]-3*y'''[x]-2*y''[x]+2*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{-x}\left(e^{3x}(c_4e^x + c_3) + c_2\cos(x) + c_1\sin(x)\right)$$

10.22 problem 22

Internal problem ID [11752]
Internal file name [`OUTPUT/11761_Thursday_April_11_2024_08_49_26_PM_31848990/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 22.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 6y''' + 15y'' + 20y' + 12y = 0}$$

The characteristic equation is

$$\lambda^4 + 6\lambda^3 + 15\lambda^2 + 20\lambda + 12 = 0$$

The roots of the above equation are

$$\begin{aligned} \lambda_1 &= i\sqrt{2} - 1 \\ \lambda_2 &= -1 - i\sqrt{2} \\ \lambda_3 &= -2 \\ \lambda_4 &= -2 \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2 + \mathrm{e}^{\left(i\sqrt{2}-1\right)x} c_3 + \mathrm{e}^{\left(-1-i\sqrt{2}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-2x} \\
y_2 &= x\,\mathrm{e}^{-2x} \\
y_3 &= \mathrm{e}^{\left(i\sqrt{2}-1\right)x} \\
y_4 &= \mathrm{e}^{\left(-1-i\sqrt{2}\right)x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x}c_2 + \mathrm{e}^{\left(i\sqrt{2}-1\right)x}c_3 + \mathrm{e}^{\left(-1-i\sqrt{2}\right)x}c_4 \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x}c_2 + \mathrm{e}^{\left(i\sqrt{2}-1\right)x}c_3 + \mathrm{e}^{\left(-1-i\sqrt{2}\right)x}c_4
$$

Verified OK.

10.22.1 Maple step by step solution

Let's solve

$y'''' + 6y''' + 15y'' + 20y' + 12y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -6y_4(x) - 15y_3(x) - 20y_2(x) - 12y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -6y_4(x) - 15y_3(x) - 20y_2(x) - 12y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -12 & -20 & -15 & -6 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -12 & -20 & -15 & -6 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[-1 - \mathrm{I}\sqrt{2}, \begin{bmatrix} \frac{1}{\left(-1-\mathrm{I}\sqrt{2}\right)^3} \\ \frac{1}{\left(-1-\mathrm{I}\sqrt{2}\right)^2} \\ \frac{1}{-1-\mathrm{I}\sqrt{2}} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{2} - 1, \begin{bmatrix} \frac{1}{\left(\mathrm{I}\sqrt{2}-1\right)^3} \\ \frac{1}{\left(\mathrm{I}\sqrt{2}-1\right)^2} \\ \frac{1}{\mathrm{I}\sqrt{2}-1} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-2, \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue -2

$$\overrightarrow{y}_1(x) = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -2$ is the eigenvalue, a

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -2

$$\left(\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -12 & -20 & -15 & -6 \end{array}\right] - (-2) \cdot \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -2

$$\vec{y}_2(x) = e^{-2x} \cdot \left(x \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1 - I\sqrt{2}, \begin{bmatrix} \frac{1}{\left(-1-I\sqrt{2}\right)^3} \\ \frac{1}{\left(-1-I\sqrt{2}\right)^2} \\ \frac{1}{-1-I\sqrt{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{\left(-1-I\sqrt{2}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(-1-I\sqrt{2}\right)^3} \\ \frac{1}{\left(-1-I\sqrt{2}\right)^2} \\ \frac{1}{-1-I\sqrt{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-x}\cdot\left(\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)\right)\cdot\begin{bmatrix}\frac{1}{\left(-1-\mathrm{I}\sqrt{2}\right)^{3}}\\ \frac{1}{\left(-1-\mathrm{I}\sqrt{2}\right)^{2}}\\ \frac{1}{-1-\mathrm{I}\sqrt{2}}\\ 1\end{bmatrix}$$

- Simplify expression

$$e^{-x}\cdot\begin{bmatrix}\frac{\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{\left(-1-\mathrm{I}\sqrt{2}\right)^{3}}\\ \frac{\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{\left(-1-\mathrm{I}\sqrt{2}\right)^{2}}\\ \frac{\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{-1-\mathrm{I}\sqrt{2}}\\ \cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)\end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{3}(x)=e^{-x}\cdot\begin{bmatrix}\frac{5\cos\left(\sqrt{2}\,x\right)}{27}+\frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{27}\\ -\frac{\cos\left(\sqrt{2}\,x\right)}{9}-\frac{2\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{9}\\ -\frac{\cos\left(\sqrt{2}\,x\right)}{3}+\frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{3}\\ \cos\left(\sqrt{2}\,x\right)\end{bmatrix},\vec{y}_{4}(x)=e^{-x}\cdot\begin{bmatrix}\frac{\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{27}-\frac{5\sin\left(\sqrt{2}\,x\right)}{27}\\ -\frac{2\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{9}+\frac{\sin\left(\sqrt{2}\,x\right)}{9}\\ \frac{\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{3}+\frac{\sin\left(\sqrt{2}\,x\right)}{3}\\ -\sin\left(\sqrt{2}\,x\right)\end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_{1}\vec{y}_{1}(x)+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)+c_{4}\vec{y}_{4}(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_{1}e^{-2x}\cdot\begin{bmatrix}-\frac{1}{8}\\ \frac{1}{4}\\ -\frac{1}{2}\\ 1\end{bmatrix}+c_{2}e^{-2x}\cdot\left(x\cdot\begin{bmatrix}-\frac{1}{8}\\ \frac{1}{4}\\ -\frac{1}{2}\\ 1\end{bmatrix}+\begin{bmatrix}-\frac{1}{16}\\ 0\\ 0\\ 0\end{bmatrix}\right)+c_{3}e^{-x}\cdot\begin{bmatrix}\frac{5\cos\left(\sqrt{2}\,x\right)}{27}+\frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{27}\\ -\frac{\cos\left(\sqrt{2}\,x\right)}{9}-\frac{2\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{9}\\ -\frac{\cos\left(\sqrt{2}\,x\right)}{3}+\frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{3}\\ \cos\left(\sqrt{2}\,x\right)\end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{5\,\mathrm{e}^{-x}\left(\frac{c_4\sqrt{2}}{5}+c_3\right)\cos\left(\sqrt{2}\,x\right)}{27} + \frac{\mathrm{e}^{-x}\left(\sqrt{2}\,c_3-5c_4\right)\sin\left(\sqrt{2}\,x\right)}{27} - \frac{\left(\left(x+\frac{1}{2}\right)c_2+c_1\right)\mathrm{e}^{-2x}}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$4)+6*diff(y(x),x$3)+15*diff(y(x),x$2)+20*diff(y(x),x)+12*y(x)=0,y(x), sin
```

$$y(x) = c_4\mathrm{e}^{-x}\cos\left(x\sqrt{2}\right) + c_3\mathrm{e}^{-x}\sin\left(x\sqrt{2}\right) + \mathrm{e}^{-2x}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 46

```
DSolve[y''''[x]+6*y'''[x]+15*y''[x]+20*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{-2x}\left(c_4x + c_2e^x\cos\left(\sqrt{2}x\right) + c_1e^x\sin\left(\sqrt{2}x\right) + c_3\right)$$

10.23 problem 23

Internal problem ID [11753]
Internal file name [`OUTPUT/11762_Thursday_April_11_2024_08_49_26_PM_21910946/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 23.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + y = 0}$$

The characteristic equation is

$$\lambda^4 + 1 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2} \\
\lambda_2 &= -\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2} \\
\lambda_3 &= -\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2} \\
\lambda_4 &= \frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{\left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)x} c_1 + \mathrm{e}^{\left(-\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right)x} c_2 + \mathrm{e}^{\left(\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}\right)x} c_3 + \mathrm{e}^{\left(-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)x} \\
y_2 &= e^{\left(-\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)x} \\
y_3 &= e^{\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)x} \\
y_4 &= e^{\left(-\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = e^{\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)x} c_1 + e^{\left(-\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)x} c_2 + e^{\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)x} c_3 + e^{\left(-\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)x} c_4 \tag{1}
$$

Verification of solutions

$$
y = e^{\left(\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)x} c_1 + e^{\left(-\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)x} c_2 + e^{\left(\frac{\sqrt{2}}{2}-\frac{i\sqrt{2}}{2}\right)x} c_3 + e^{\left(-\frac{\sqrt{2}}{2}+\frac{i\sqrt{2}}{2}\right)x} c_4
$$

Verified OK.

10.23.1 Maple step by step solution

Let's solve

$y'''' + y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^2} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{\sqrt{2}}{2} + \frac{I\sqrt{2}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{\sqrt{2}}{2}+\frac{I\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}+\frac{I\sqrt{2}}{2}\right)^2} \\ \frac{1}{-\frac{\sqrt{2}}{2}+\frac{I\sqrt{2}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{\sqrt{2}}{2} - \frac{I\sqrt{2}}{2}, \begin{bmatrix} \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{2}} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{2}} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-\frac{\sqrt{2}\,x}{2}} \cdot \left(\cos\left(\frac{\sqrt{2}\,x}{2}\right) - I\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{3}} \\ \frac{1}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{2}} \\ \frac{1}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-I\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{3}} \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-I\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-I\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{-\frac{\sqrt{2}}{2}-\frac{I\sqrt{2}}{2}} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right) - I\sin\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = e^{-\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{\sqrt{2}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right], \vec{y}_2(x) = e^{-\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}-\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ -\cos\left(\frac{\sqrt{2}\,x}{2}\right) \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{\frac{\sqrt{2}\,x}{2}} \cdot \left(\cos\left(\frac{\sqrt{2}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^3} \\ \frac{1}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^2} \\ \frac{1}{\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^{3}} \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\left(\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,x}{2}\right)}{\frac{\sqrt{2}}{2}-\frac{\mathrm{I}\sqrt{2}}{2}} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ \sin\left(\frac{\sqrt{2}\,x}{2}\right) \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right], \vec{y}_4(x) = e^{\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right) \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}-\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{\sqrt{2}\,x}{2}\right) \\ -\frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ \cos\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right] + c_2 e^{-\frac{\sqrt{2}\,x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}-\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ -\cos\left(\frac{\sqrt{2}\,x}{2}\right) \\ \frac{\cos\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2}+\frac{\sin\left(\frac{\sqrt{2}\,x}{2}\right)\sqrt{2}}{2} \\ -\sin\left(\frac{\sqrt{2}\,x}{2}\right) \end{array}\right] + c_3 e^{\frac{\sqrt{2}\,x}{2}}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\sqrt{2}\left(\left((c_1+c_2)e^{-\frac{\sqrt{2}\,x}{2}}-e^{\frac{\sqrt{2}\,x}{2}}(c_3-c_4)\right)\cos\left(\frac{\sqrt{2}\,x}{2}\right)+\left((c_1-c_2)e^{-\frac{\sqrt{2}\,x}{2}}+e^{\frac{\sqrt{2}\,x}{2}}(c_3+c_4)\right)\sin\left(\frac{\sqrt{2}\,x}{2}\right)\right)}{2}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 61

```
dsolve(diff(y(x),x$4)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(-c_1 \mathrm{e}^{-\frac{x\sqrt{2}}{2}} - c_2 \mathrm{e}^{\frac{x\sqrt{2}}{2}}\right) \sin\left(\frac{x\sqrt{2}}{2}\right) + \left(c_3 \mathrm{e}^{-\frac{x\sqrt{2}}{2}} + c_4 \mathrm{e}^{\frac{x\sqrt{2}}{2}}\right) \cos\left(\frac{x\sqrt{2}}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 65

```
DSolve[y''''[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{x}{\sqrt{2}}} \left(\left(c_1 e^{\sqrt{2}x} + c_2\right) \cos\left(\frac{x}{\sqrt{2}}\right) + \left(c_4 e^{\sqrt{2}x} + c_3\right) \sin\left(\frac{x}{\sqrt{2}}\right)\right)$$

10.24 problem 24

Internal problem ID [11754]
Internal file name [`OUTPUT/11763_Thursday_April_11_2024_08_49_26_PM_9690685/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 24.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _quadrature]]
```

$$\boxed{y^{(5)} = 0}$$

The characteristic equation is

$$\lambda^5 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 0$$
$$\lambda_5 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_5x^4 + c_4x^3 + c_3x^2 + c_2x + c_1$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = x^2$$
$$y_4 = x^3$$
$$y_5 = x^4$$

Summary
The solution(s) found are the following

$$y = c_5 x^4 + c_4 x^3 + c_3 x^2 + c_2 x + c_1 \tag{1}$$

Verification of solutions

$$y = c_5 x^4 + c_4 x^3 + c_3 x^2 + c_2 x + c_1$$

Verified OK.

10.24.1 Maple step by step solution

Let's solve
$y^{(5)} = 0$

- Highest derivative means the order of the ODE is 5
  $y^{(5)}$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$
    $y_1(x) = y$
  ◦ Define new variable $y_2(x)$
    $y_2(x) = y'$
  ◦ Define new variable $y_3(x)$
    $y_3(x) = y''$
  ◦ Define new variable $y_4(x)$
    $y_4(x) = y'''$
  ◦ Define new variable $y_5(x)$
    $y_5(x) = y''''$

- Isolate for $y_5'(x)$ using original ODE

  $y_5'(x) = 0$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_5(x) = y_4'(x), y_5'(x) = 0]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right], \left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right], \left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right], \left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right], \left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + c_4 \vec{y}_4 + c_5 \vec{y}_5$$

- Substitute solutions into the general solution

$$\vec{y} = \begin{bmatrix} c_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$5)=0,y(x), singsol=all)
```

$$y(x) = \frac{1}{24}x^4c_1 + \frac{1}{6}c_2x^3 + \frac{1}{2}c_3x^2 + c_4x + c_5$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 27

```
DSolve[y'''''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(x(x(c_5x + c_4) + c_3) + c_2) + c_1$$

10.25 problem 25

Internal problem ID [11755]
Internal file name [`OUTPUT/11764_Thursday_April_11_2024_08_49_27_PM_74442270/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' - 12y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = 5]$$

10.25.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= -12 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - y' - 12y = 0$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -12$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.25.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -12$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 12 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 12 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -12$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-12)} \\ &= \frac{1}{2} \pm \frac{7}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{2} + \frac{7}{2}$$

$$\lambda_2 = \frac{1}{2} - \frac{7}{2}$$

Which simplifies to

$$\lambda_1 = 4$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(4)x} + c_2 e^{(-3)x}$$

Or

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-3x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 4c_1 \mathrm{e}^{4x} - 3c_2 \mathrm{e}^{-3x}$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = 4c_1 - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 1$$

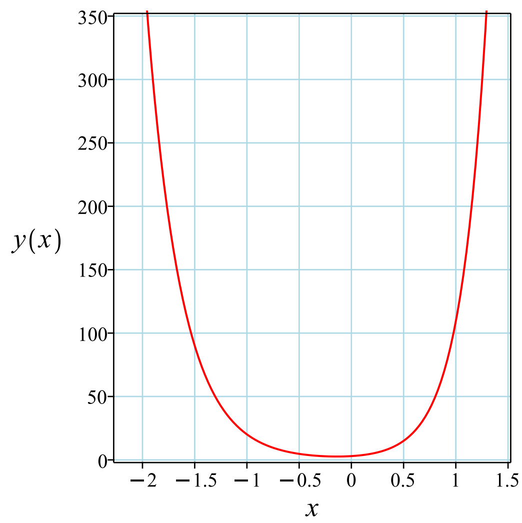
Substituting these values back in above solution results in

$$y = 2\,\mathrm{e}^{4x} + \mathrm{e}^{-3x}$$

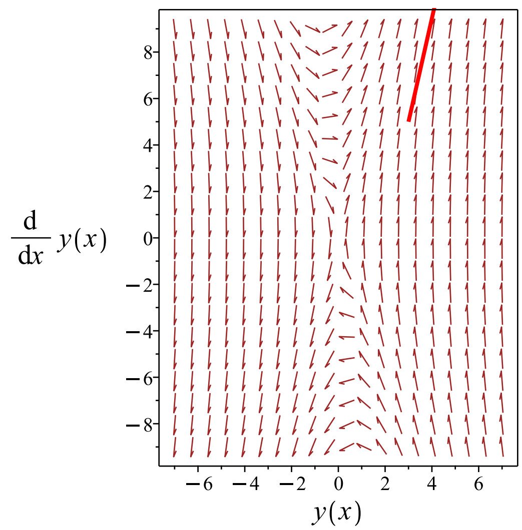
Summary

The solution(s) found are the following

$$y = 2\,\mathrm{e}^{4x} + \mathrm{e}^{-3x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{4x} + \mathrm{e}^{-3x}$$

Verified OK.

10.25.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 12y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{49}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 49$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{49z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 311: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{49}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{7x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{7x}}{7}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{7x}}{7}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{4x}}{7} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$
3 = c_1 + \frac{c_2}{7} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -3c_1 \mathrm{e}^{-3x} + \frac{4c_2 \mathrm{e}^{4x}}{7}
$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$
5 = -3c_1 + \frac{4c_2}{7} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= 14
\end{aligned}
$$

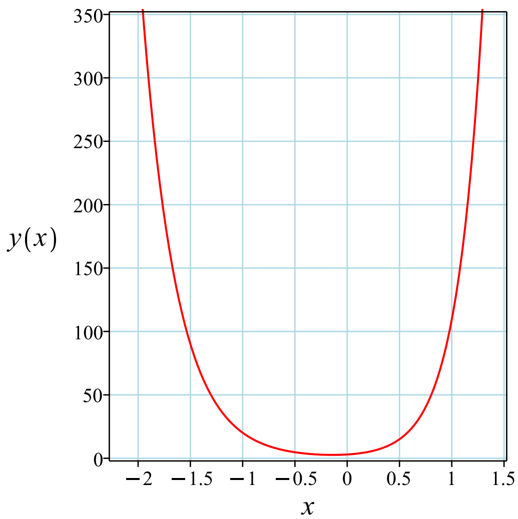
Substituting these values back in above solution results in

$$
y = 2\,\mathrm{e}^{4x} + \mathrm{e}^{-3x}
$$

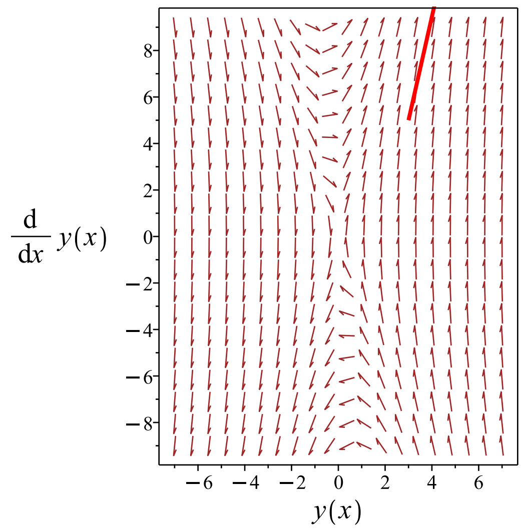
Summary

The solution(s) found are the following

$$
y = 2\,\mathrm{e}^{4x} + \mathrm{e}^{-3x} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,e^{4x} + e^{-3x}$$

Verified OK.

10.25.4 Maple step by step solution

Let's solve

$$\left[y'' - y' - 12y = 0, y(0) = 3, y'\Big|_{\{x=0\}} = 5\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - r - 12 = 0$

- Factor the characteristic polynomial

  $(r + 3)(r - 4) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 4)$

- 1st solution of the ODE

$y_1(x) = e^{-3x}$

- 2nd solution of the ODE

  $y_2(x) = e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-3x} + c_2 e^{4x}$

- □ Check validity of solution $y = c_1 e^{-3x} + c_2 e^{4x}$
  - Use initial condition $y(0) = 3$

    $3 = c_1 + c_2$
  - Compute derivative of the solution

    $y' = -3c_1 e^{-3x} + 4c_2 e^{4x}$
  - Use the initial condition $y'\Big|_{\{x=0\}} = 5$

    $5 = -3c_1 + 4c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 1, c_2 = 2\}$
  - Substitute constant values into general solution and simplify

    $y = (2\,e^{7x} + 1)\,e^{-3x}$

- Solution to the IVP

  $y = (2\,e^{7x} + 1)\,e^{-3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([diff(y(x),x$2)-diff(y(x),x)-12*y(x)=0,y(0) = 3, D(y)(0) = 5],y(x), singsol=all)
```

$$y(x) = \left(2\,e^{7x} + 1\right) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 18

```
DSolve[{y''[x]-y'[x]-12*y[x]==0,{y[0]==3,y'[0]==5}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x} + 2e^{4x}$$

10.26 problem 26

Internal problem ID [11756]
Internal file name [`OUTPUT/11765_Thursday_April_11_2024_08_49_27_PM_74205676/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 7y' + 10y = 0}$$

With initial conditions

$$[y(0) = -4, y'(0) = 2]$$

10.26.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 7 \\ q(x) &= 10 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 7y' + 10y = 0$$

The domain of $p(x) = 7$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 10$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.26.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 7, C = 10$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 7\lambda\, e^{\lambda x} + 10\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 7\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 7, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-7}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{7^2 - (4)(1)(10)}\\ &= -\frac{7}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{7}{2} + \frac{3}{2}$$

$$\lambda_2 = -\frac{7}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = -2$$
$$\lambda_2 = -5$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(-2)x} + c_2 e^{(-5)x}$$

Or

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-5x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-5x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -4$ and $x = 0$ in the above gives

$$-4 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \mathrm{e}^{-2x} - 5c_2 \mathrm{e}^{-5x}$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = -2c_1 - 5c_2 \tag{2A}$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -6$$
$$c_2 = 2$$

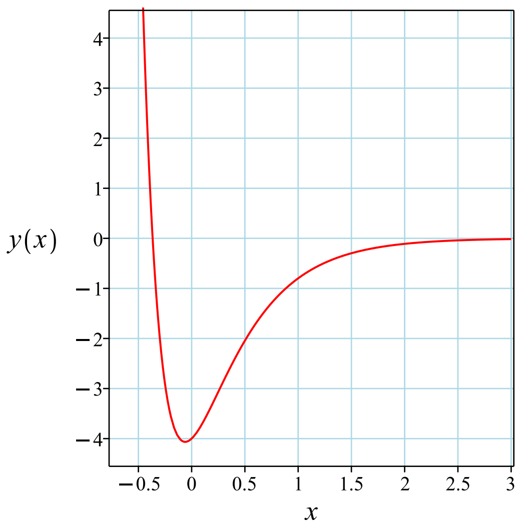
Substituting these values back in above solution results in

$$y = 2\,\mathrm{e}^{-5x} - 6\,\mathrm{e}^{-2x}$$

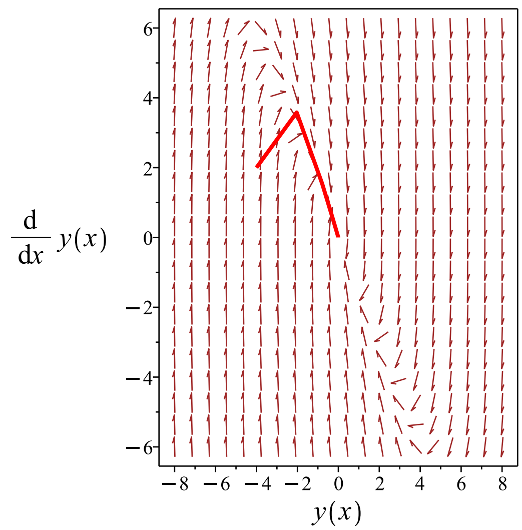
Summary

The solution(s) found are the following

$$y = 2\,e^{-5x} - 6\,e^{-2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,e^{-5x} - 6\,e^{-2x}$$

Verified OK.

10.26.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 7y' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 7 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 313: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{7}{1}\,dx} \\ &= z_1 e^{-\frac{7x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{7x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-5x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{7}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-7x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-5x}\right) + c_2 \left(\mathrm{e}^{-5x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-5x} + \frac{c_2 \mathrm{e}^{-2x}}{3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -4$ and $x = 0$ in the above gives

$$
-4 = c_1 + \frac{c_2}{3} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -5c_1 \mathrm{e}^{-5x} - \frac{2c_2 \mathrm{e}^{-2x}}{3}
$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$
2 = -5c_1 - \frac{2c_2}{3} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= -18
\end{aligned}
$$

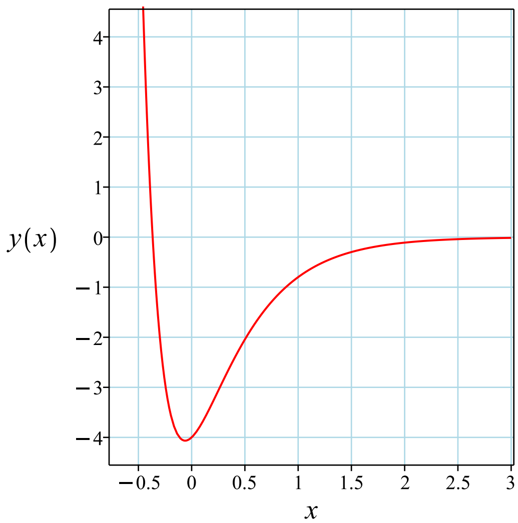
Substituting these values back in above solution results in

$$
y = 2\,\mathrm{e}^{-5x} - 6\,\mathrm{e}^{-2x}
$$

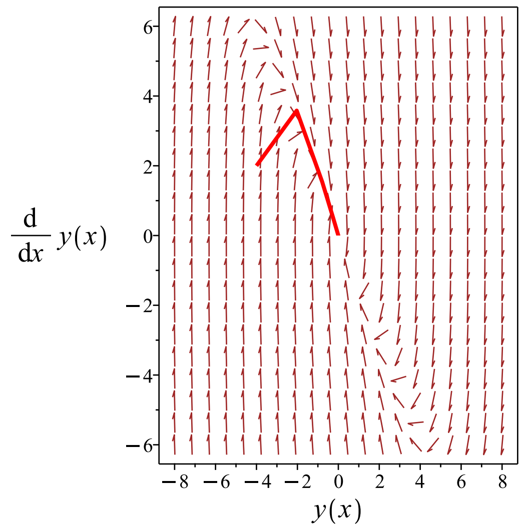
Summary

The solution(s) found are the following

$$
y = 2\,\mathrm{e}^{-5x} - 6\,\mathrm{e}^{-2x} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,e^{-5x} - 6\,e^{-2x}$$

Verified OK.

10.26.4 Maple step by step solution

Let's solve

$$\left[y'' + 7y' + 10y = 0, y(0) = -4, y'\Big|_{\{x=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 7r + 10 = 0$

- Factor the characteristic polynomial

  $(r+5)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-5, -2)$

- 1st solution of the ODE

$y_1(x) = \mathrm{e}^{-5x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-5x} + c_2 \mathrm{e}^{-2x}$

□ Check validity of solution $y = c_1 \mathrm{e}^{-5x} + c_2 \mathrm{e}^{-2x}$

  ◦ Use initial condition $y(0) = -4$

    $-4 = c_1 + c_2$

  ◦ Compute derivative of the solution

    $y' = -5c_1 \mathrm{e}^{-5x} - 2c_2 \mathrm{e}^{-2x}$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 2$

    $2 = -5c_1 - 2c_2$

  ◦ Solve for c_1 and c_2

    $\{c_1 = 2, c_2 = -6\}$

  ◦ Substitute constant values into general solution and simplify

    $y = 2\,\mathrm{e}^{-5x} - 6\,\mathrm{e}^{-2x}$

- Solution to the IVP

  $y = 2\,\mathrm{e}^{-5x} - 6\,\mathrm{e}^{-2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2)+7*diff(y(x),x)+10*y(x)=0,y(0) = -4, D(y)(0) = 2],y(x), singsol=all)
```

$$y(x) = 2\,e^{-5x} - 6\,e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[{y''[x]+7*y'[x]+10*y[x]==0,{y[0]==-4,y'[0]==2}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to e^{-5x}\left(2 - 6e^{3x}\right)$$

10.27 problem 27

Internal problem ID [11757]
Internal file name [`OUTPUT/11766_Thursday_April_11_2024_08_49_28_PM_72234192/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 6y' + 8y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 6]$$

10.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -6 \\ q(x) &= 8 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 6y' + 8y = 0$$

The domain of $p(x) = -6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 8$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.27.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 8$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda\, e^{\lambda x} + 8\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-6^2 - (4)\,(1)\,(8)}\\ &= 3 \pm 1\end{aligned}$$

Hence

$$\lambda_1 = 3 + 1$$
$$\lambda_2 = 3 - 1$$

Which simplifies to

$$\lambda_1 = 4$$
$$\lambda_2 = 2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(4)x} + c_2 e^{(2)x}$$

Or

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 4c_1 \mathrm{e}^{4x} + 2c_2 \mathrm{e}^{2x}$$

substituting $y' = 6$ and $x = 0$ in the above gives

$$6 = 4c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

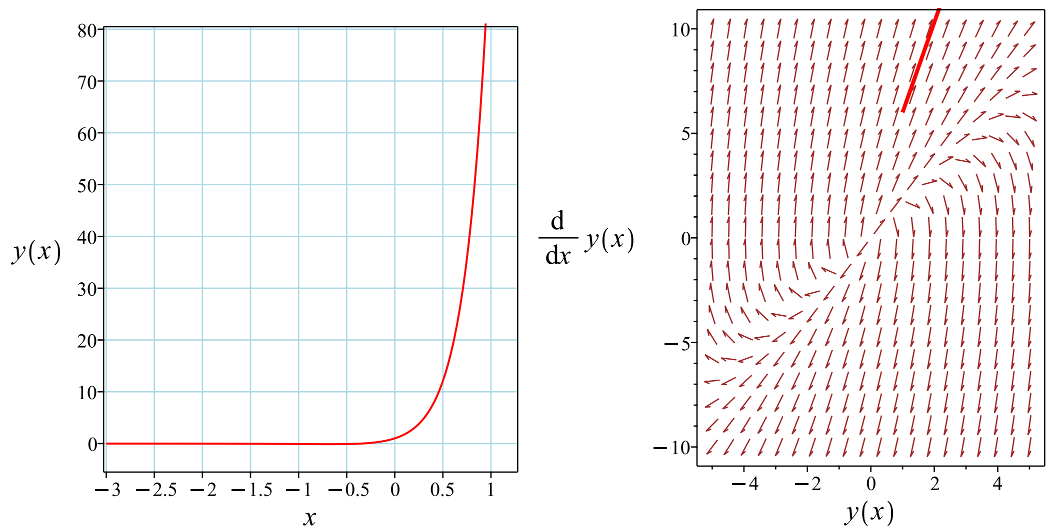
$$c_1 = 2$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = 2\,\mathrm{e}^{4x} - \mathrm{e}^{2x}$$

Summary
The solution(s) found are the following

$$y = 2\,e^{4x} - e^{2x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2\,e^{4x} - e^{2x}$$

Verified OK.

10.27.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 315: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx} \\ &= z_1 e^{3x} \\ &= z_1 \left(\mathrm{e}^{3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{6x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x}\right) + c_2 \left(\mathrm{e}^{2x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{2x} + \frac{c_2 \mathrm{e}^{4x}}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + \frac{c_2}{2} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 2c_1 \mathrm{e}^{2x} + 2c_2 \mathrm{e}^{4x}
$$

substituting $y' = 6$ and $x = 0$ in the above gives

$$
6 = 2c_1 + 2c_2 \tag{2A}
$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -1 \\
c_2 &= 4
\end{aligned}
$$

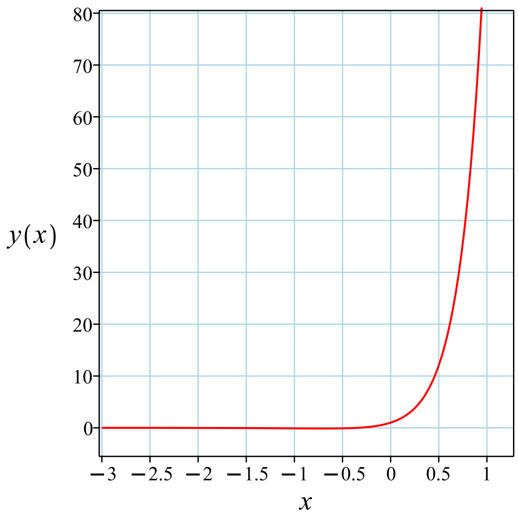
Substituting these values back in above solution results in

$$
y = 2\,\mathrm{e}^{4x} - \mathrm{e}^{2x}
$$

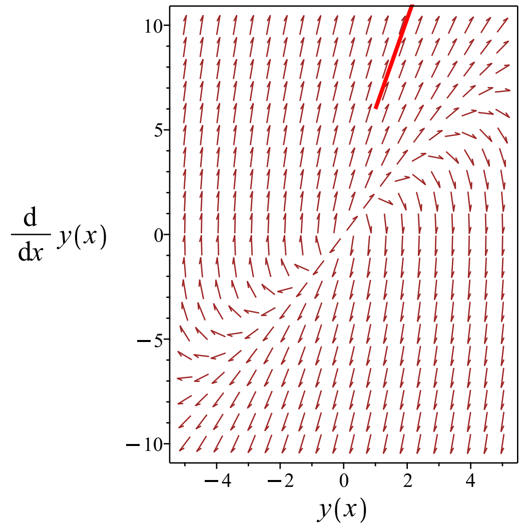
Summary

The solution(s) found are the following

$$
y = 2\,\mathrm{e}^{4x} - \mathrm{e}^{2x} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 2\,e^{4x} - e^{2x}$$

Verified OK.

10.27.4 Maple step by step solution

Let's solve

$$\left[y'' - 6y' + 8y = 0, y(0) = 1, y'\Big|_{\{x=0\}} = 6\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 6r + 8 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-4) = 0$

- Roots of the characteristic polynomial

  $r = (2, 4)$

- 1st solution of the ODE

$$y_1(x) = e^{2x}$$

- 2nd solution of the ODE

$$y_2(x) = e^{4x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{2x} + c_2 e^{4x}$$

□ Check validity of solution $y = c_1 e^{2x} + c_2 e^{4x}$

  - Use initial condition $y(0) = 1$

    $$1 = c_1 + c_2$$

  - Compute derivative of the solution

    $$y' = 2c_1 e^{2x} + 4c_2 e^{4x}$$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 6$

    $$6 = 2c_1 + 4c_2$$

  - Solve for c_1 and c_2

    $$\{c_1 = -1, c_2 = 2\}$$

  - Substitute constant values into general solution and simplify

    $$y = 2 e^{4x} - e^{2x}$$

- Solution to the IVP

$$y = 2 e^{4x} - e^{2x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2)-6*diff(y(x),x)+8*y(x)=0,y(0) = 1, D(y)(0) = 6],y(x), singsol=all)
```

$$y(x) = 2\,\mathrm{e}^{4x} - \mathrm{e}^{2x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 20

```
DSolve[{y''[x]-6*y'[x]+8*y[x]==0,{y[0]==1,y'[0]==6}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{2x}\left(2e^{2x} - 1\right)$$

10.28 problem 28

Internal problem ID [11758]
Internal file name [`OUTPUT/11767_Thursday_April_11_2024_08_49_28_PM_1544632/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{3y'' + 4y' - 4y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = -4]$$

10.28.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = \frac{4}{3}$$
$$q(x) = -\frac{4}{3}$$
$$F = 0$$

Hence the ode is

$$y'' + \frac{4y'}{3} - \frac{4y}{3} = 0$$

The domain of $p(x) = \frac{4}{3}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -\frac{4}{3}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.28.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 3, B = 4, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$3\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} - 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$3\lambda^2 + 4\lambda - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 3, B = 4, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(3)} \pm \frac{1}{(2)\,(3)}\sqrt{4^2 - (4)\,(3)\,(-4)}\\ &= -\frac{2}{3} \pm \frac{4}{3}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{2}{3} + \frac{4}{3}$$

$$\lambda_2 = -\frac{2}{3} - \frac{4}{3}$$

Which simplifies to

$$\lambda_1 = \frac{2}{3}$$
$$\lambda_2 = -2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\frac{2}{3}\right)x} + c_2 e^{(-2)x}$$

Or

$$y = c_1 \mathrm{e}^{\frac{2x}{3}} + c_2 \mathrm{e}^{-2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{\frac{2x}{3}} + c_2 \mathrm{e}^{-2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2c_1 \mathrm{e}^{\frac{2x}{3}}}{3} - 2c_2 \mathrm{e}^{-2x}$$

substituting $y' = -4$ and $x = 0$ in the above gives

$$-4 = \frac{2c_1}{3} - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

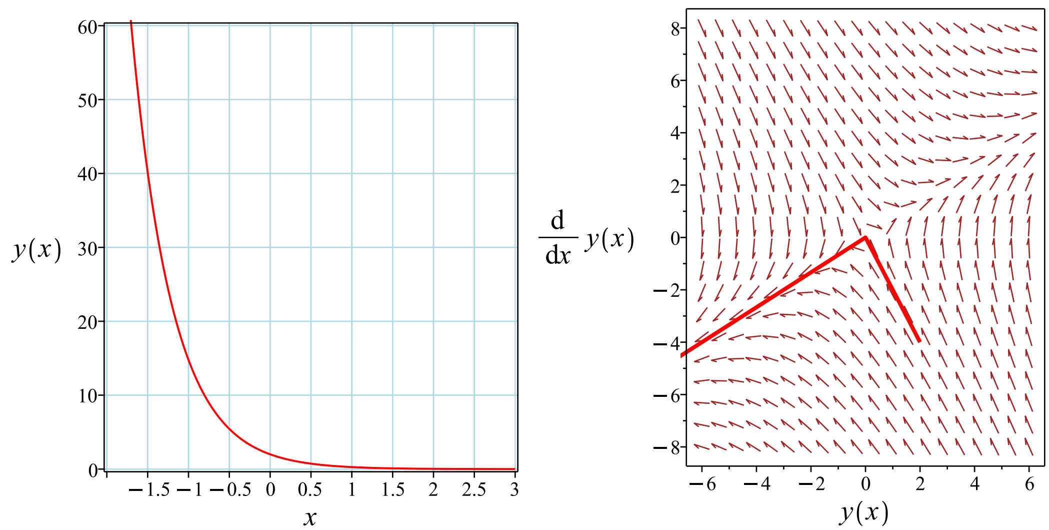
$$c_1 = 0$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = 2\,\mathrm{e}^{-2x}$$

Summary
The solution(s) found are the following

$$y = 2\,\mathrm{e}^{-2x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{-2x}$$

Verified OK.

10.28.3 Solving using Kovacic algorithm

Writing the ode as

$$3y'' + 4y' - 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3 \\ B &= 4 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16}{9} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 16 \\ t &= 9 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{16z(x)}{9} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 317: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{16}{9}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = e^{-\frac{4x}{3}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{3}\,dx} \\
&= z_1 e^{-\frac{2x}{3}} \\
&= z_1 \left(e^{-\frac{2x}{3}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = e^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{3}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{4x}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{3\,e^{\frac{8x}{3}}}{8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{3\,\mathrm{e}^{\frac{8x}{3}}}{8}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-2x} + \frac{3 c_2 \mathrm{e}^{\frac{2x}{3}}}{8} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$
2 = c_1 + \frac{3 c_2}{8} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2 c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{\frac{2x}{3}}}{4}
$$

substituting $y' = -4$ and $x = 0$ in the above gives

$$
-4 = -2 c_1 + \frac{c_2}{4} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= 0
\end{aligned}
$$

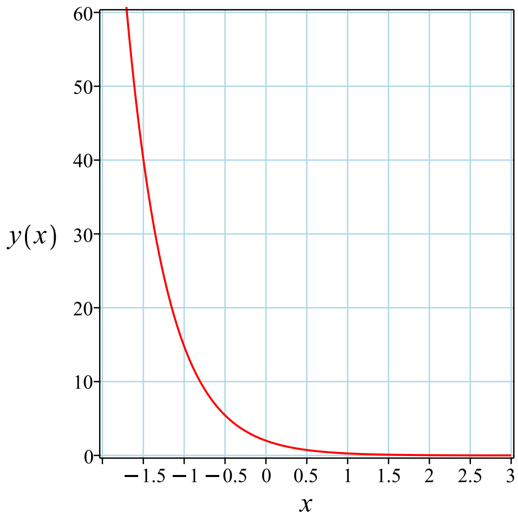
Substituting these values back in above solution results in

$$
y = 2\,\mathrm{e}^{-2x}
$$

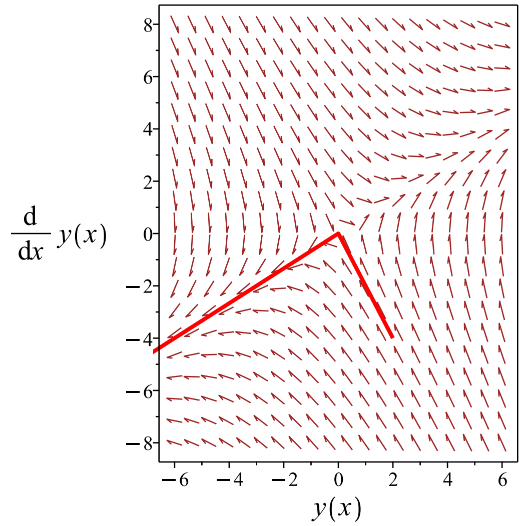
Summary

The solution(s) found are the following

$$
y = 2\,\mathrm{e}^{-2x} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 2\,\mathrm{e}^{-2x}$$

Verified OK.

10.28.4 Maple step by step solution

Let's solve

$$\left[3y'' + 4y' - 4y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = -4\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y'}{3} + \frac{4y}{3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{4y'}{3} - \frac{4y}{3} = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{4}{3}r - \frac{4}{3} = 0$

- Factor the characteristic polynomial

$\frac{(r+2)(3r-2)}{3} = 0$

- Roots of the characteristic polynomial

  $r = \left(-2, \frac{2}{3}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{2x}{3}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} + c_2 e^{\frac{2x}{3}}$

□ Check validity of solution $y = c_1 e^{-2x} + c_2 e^{\frac{2x}{3}}$

  ◦ Use initial condition $y(0) = 2$

    $2 = c_1 + c_2$

  ◦ Compute derivative of the solution

    $y' = -2c_1 e^{-2x} + \frac{2c_2 e^{\frac{2x}{3}}}{3}$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = -4$

    $-4 = -2c_1 + \frac{2c_2}{3}$

  ◦ Solve for c_1 and c_2

    $\{c_1 = 2, c_2 = 0\}$

  ◦ Substitute constant values into general solution and simplify

    $y = 2\,e^{-2x}$

- Solution to the IVP

  $y = 2\,e^{-2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve([3*diff(y(x),x$2)+4*diff(y(x),x)-4*y(x)=0,y(0) = 2, D(y)(0) = -4],y(x), singsol=all)
```

$$y(x) = 2\,\mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 12

```
DSolve[{3*y''[x]+4*y'[x]-4*y[x]==0,{y[0]==2,y'[0]==-4}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to 2e^{-2x}$$

10.29 problem 29

Internal problem ID [11759]
Internal file name [`OUTPUT/11768_Thursday_April_11_2024_08_49_29_PM_39302860/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 6y' + 9y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = -3]$$

10.29.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 6 \\ q(x) &= 9 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 9y = 0$$

The domain of $p(x) = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.29.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 6\lambda\,\mathrm{e}^{\lambda x} + 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(6)^2 - (4)(1)(9)}\\ &= -3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$y = c_1 e^{-3x} + c_2 x\, e^{-3x} \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-3x} + c_2 x\, e^{-3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x\, e^{-3x}$$

substituting $y' = -3$ and $x = 0$ in the above gives

$$-3 = -3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}c_1 &= 2\\ c_2 &= 3\end{aligned}$$

Substituting these values back in above solution results in

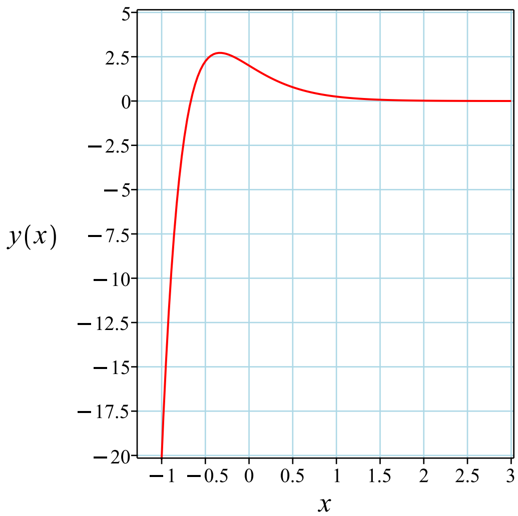
$$y = 3x\, e^{-3x} + 2\, e^{-3x}$$

Which simplifies to

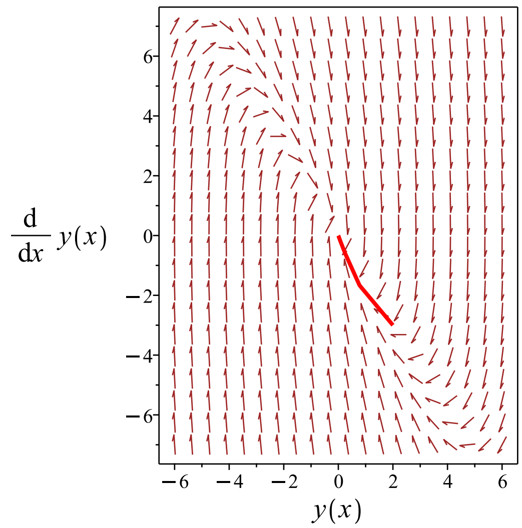
$$y = e^{-3x}(3x + 2)$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{-3x}(3x + 2) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{-3x}(3x+2)$$

Verified OK.

10.29.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 6\, dx} \\ &= e^{3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 0$$
$$\left(e^{3x} y\right)'' = 0$$

Integrating once gives

$$\left(e^{3x}y\right)' = c_1$$

Integrating again gives

$$\left(e^{3x}y\right) = c_1x + c_2$$

Hence the solution is

$$y = \frac{c_1x + c_2}{e^{3x}}$$

Or

$$y = c_1x\,e^{-3x} + c_2e^{-3x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1x\,e^{-3x} + c_2e^{-3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1e^{-3x} - 3c_1x\,e^{-3x} - 3c_2e^{-3x}$$

substituting $y' = -3$ and $x = 0$ in the above gives

$$-3 = c_1 - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 2$$

Substituting these values back in above solution results in

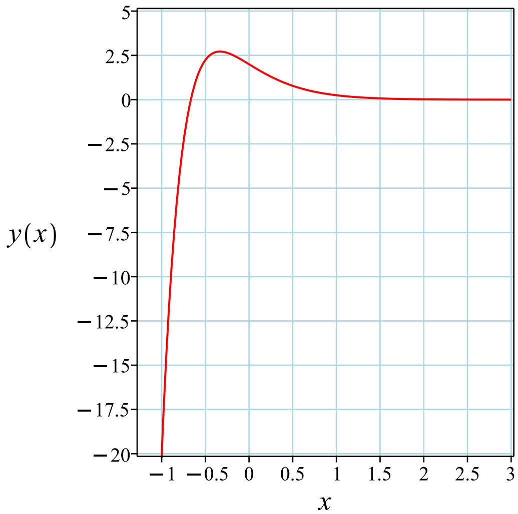
$$y = 3x\,e^{-3x} + 2\,e^{-3x}$$

Which simplifies to

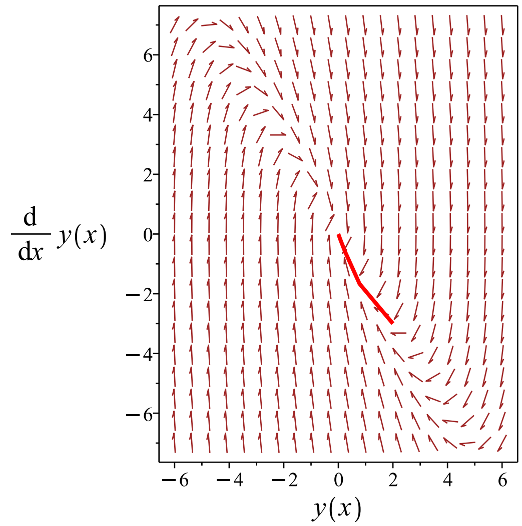
$$y = e^{-3x}(3x + 2)$$

Summary
The solution(s) found are the following

$$y = e^{-3x}(3x + 2) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-3x}(3x + 2)$$

Verified OK.

10.29.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 319: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\
&= z_1 e^{-3x} \\
&= z_1\left(\mathrm{e}^{-3x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}(x)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x} - 3c_2 x\,\mathrm{e}^{-3x}$$

substituting $y' = -3$ and $x = 0$ in the above gives

$$-3 = -3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 2 \\ c_2 &= 3 \end{aligned}$$

Substituting these values back in above solution results in

$$y = 3x\,\mathrm{e}^{-3x} + 2\,\mathrm{e}^{-3x}$$

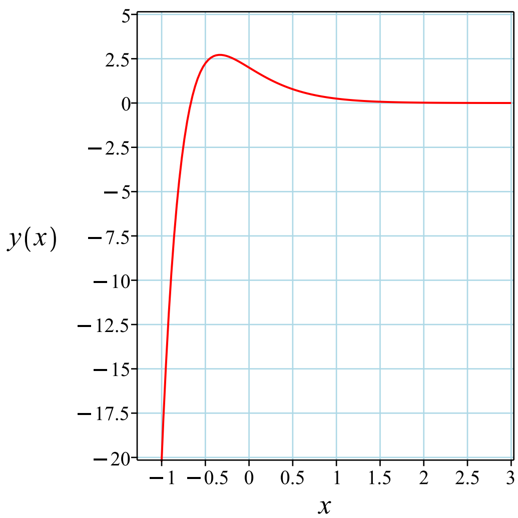
Which simplifies to

$$y = \mathrm{e}^{-3x}(3x + 2)$$

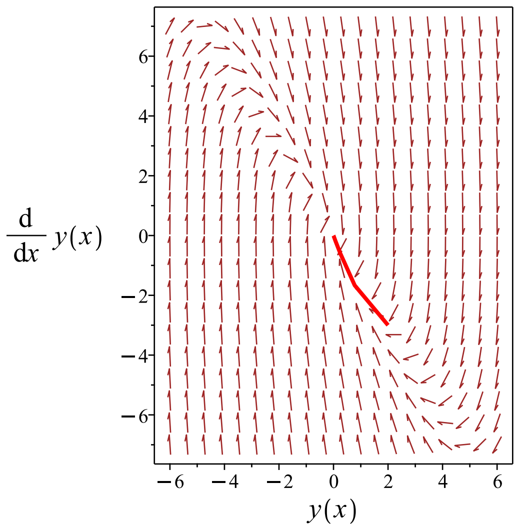
Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-3x}(3x + 2) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{-3x}(3x + 2)$$

Verified OK.

10.29.5 Maple step by step solution

Let's solve

$$\left[y'' + 6y' + 9y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = -3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r + 3)^2 = 0$

- Root of the characteristic polynomial

  $r = -3$

- 1st solution of the ODE

$y_1(x) = \mathrm{e}^{-3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x}$

□ Check validity of solution $y = c_1 \mathrm{e}^{-3x} + c_2 x \mathrm{e}^{-3x}$

  - Use initial condition $y(0) = 2$

    $2 = c_1$

  - Compute derivative of the solution

    $y' = -3c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x} - 3c_2 x\,\mathrm{e}^{-3x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = -3$

    $-3 = -3c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 2, c_2 = 3\}$

  - Substitute constant values into general solution and simplify

    $y = \mathrm{e}^{-3x}(3x + 2)$

- Solution to the IVP

  $y = \mathrm{e}^{-3x}(3x + 2)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([diff(y(x),x$2)+6*diff(y(x),x)+9*y(x)=0,y(0) = 2, D(y)(0) = -3],y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-3x}(3x + 2)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 16

```
DSolve[{y''[x]+6*y'[x]+9*y[x]==0,{y[0]==2,y'[0]==-3}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{-3x}(3x + 2)$$

10.30 problem 30

Internal problem ID [11760]
Internal file name [`OUTPUT/11769_Thursday_April_11_2024_08_49_29_PM_26050774/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' - 12y' + 9y = 0}$$

With initial conditions

$$[y(0) = 4, y'(0) = 9]$$

10.30.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= \frac{9}{4} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 3y' + \frac{9y}{4} = 0$$

The domain of $p(x) = -3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{9}{4}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.30.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = -12, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 \mathrm{e}^{\lambda x} - 12\lambda\, \mathrm{e}^{\lambda x} + 9\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 - 12\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = -12, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{12}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{(-12)^2 - (4)(4)(9)} \\ &= \frac{3}{2}\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -\frac{3}{2}$. Therefore the solution is

$$y = c_1 e^{\frac{3x}{2}} + c_2 x\, e^{\frac{3x}{2}} \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{\frac{3x}{2}} + c_2 x\, e^{\frac{3x}{2}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 0$ in the above gives

$$4 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{3c_1 e^{\frac{3x}{2}}}{2} + e^{\frac{3x}{2}} c_2 + \frac{3c_2 x\, e^{\frac{3x}{2}}}{2}$$

substituting $y' = 9$ and $x = 0$ in the above gives

$$9 = \frac{3c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}c_1 &= 4 \\ c_2 &= 3\end{aligned}$$

Substituting these values back in above solution results in

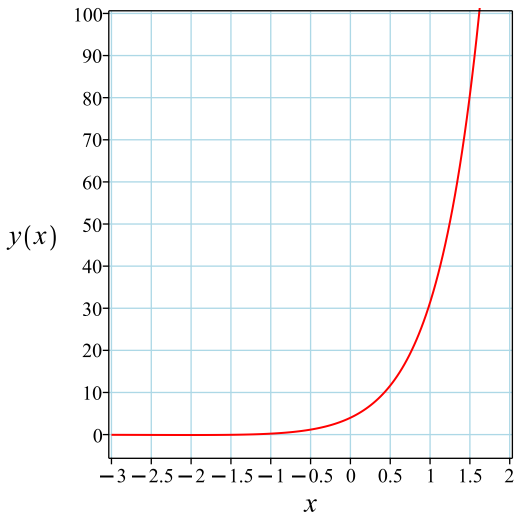
$$y = 3x\, e^{\frac{3x}{2}} + 4\, e^{\frac{3x}{2}}$$

Which simplifies to

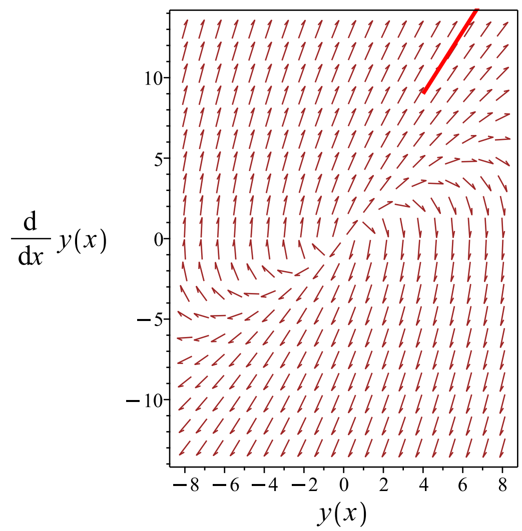
$$y = e^{\frac{3x}{2}}(4 + 3x)$$

Summary
The solution(s) found are the following

$$y = e^{\frac{3x}{2}}(4+3x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{\frac{3x}{2}}(4+3x)$$

Verified OK.

10.30.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -3$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -3\,dx} \\ &= e^{-\frac{3x}{2}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\mathrm{e}^{-\frac{3x}{2}} y\right)'' &= 0
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{-\frac{3x}{2}} y\right)' = c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{-\frac{3x}{2}} y\right) = c_1 x + c_2
$$

Hence the solution is

$$
y = \frac{c_1 x + c_2}{\mathrm{e}^{-\frac{3x}{2}}}
$$

Or

$$
y = c_1 x \,\mathrm{e}^{\frac{3x}{2}} + \mathrm{e}^{\frac{3x}{2}} c_2
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 x \,\mathrm{e}^{\frac{3x}{2}} + \mathrm{e}^{\frac{3x}{2}} c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 0$ in the above gives

$$
4 = c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 \mathrm{e}^{\frac{3x}{2}} + \frac{3 c_1 x \,\mathrm{e}^{\frac{3x}{2}}}{2} + \frac{3 \,\mathrm{e}^{\frac{3x}{2}} c_2}{2}
$$

substituting $y' = 9$ and $x = 0$ in the above gives

$$
9 = c_1 + \frac{3 c_2}{2} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3\\
c_2 &= 4
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 3x\,\mathrm{e}^{\frac{3x}{2}} + 4\,\mathrm{e}^{\frac{3x}{2}}
$$

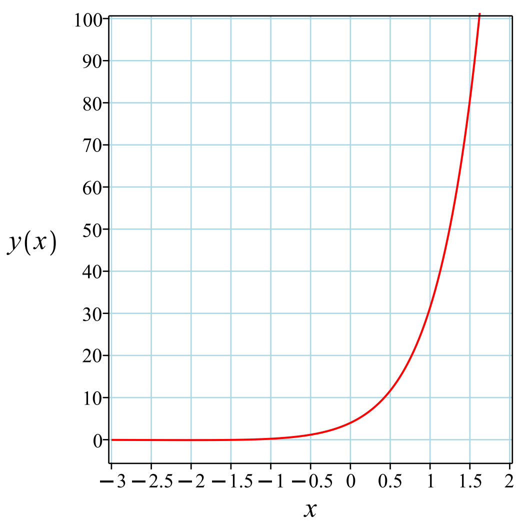
Which simplifies to

$$
y = \mathrm{e}^{\frac{3x}{2}}(4 + 3x)
$$

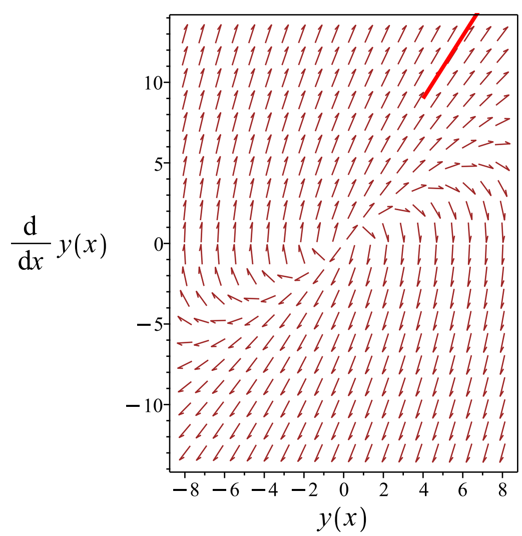
Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{\frac{3x}{2}}(4 + 3x) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{\frac{3x}{2}}(4 + 3x)
$$

Verified OK.

10.30.4 Solving using Kovacic algorithm

Writing the ode as

$$4y'' - 12y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= -12 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 321: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-12}{4}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{3x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-12}{4}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{3x}{2}}\right) + c_2 \left(\mathrm{e}^{\frac{3x}{2}} (x)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{\frac{3x}{2}} + c_2 x\, \mathrm{e}^{\frac{3x}{2}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 0$ in the above gives

$$
4 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{3c_1 \mathrm{e}^{\frac{3x}{2}}}{2} + \mathrm{e}^{\frac{3x}{2}} c_2 + \frac{3c_2 x\, \mathrm{e}^{\frac{3x}{2}}}{2}
$$

substituting $y' = 9$ and $x = 0$ in the above gives

$$9 = \frac{3c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = 3x\,\mathrm{e}^{\frac{3x}{2}} + 4\,\mathrm{e}^{\frac{3x}{2}}$$

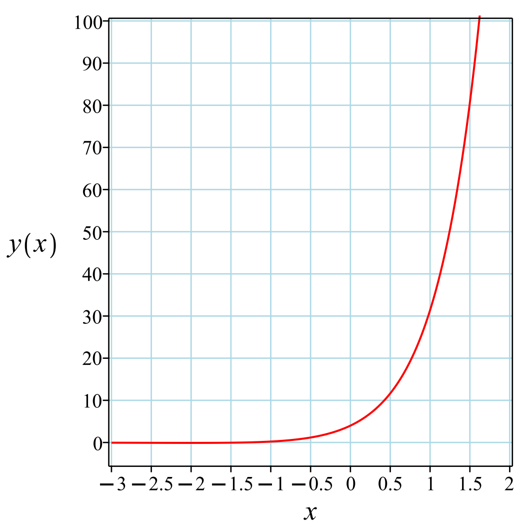
Which simplifies to

$$y = \mathrm{e}^{\frac{3x}{2}}(4 + 3x)$$

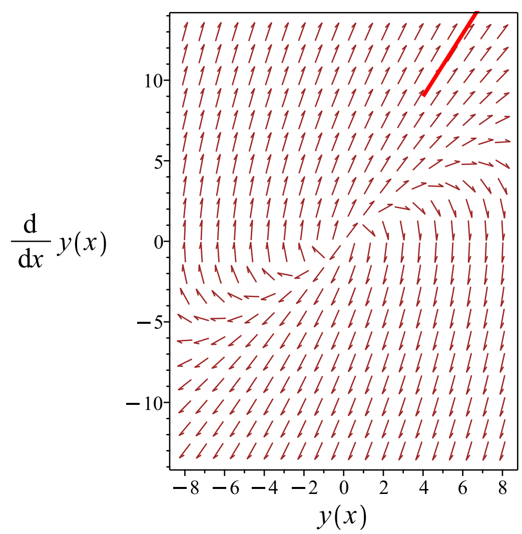
Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{3x}{2}}(4 + 3x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{3x}{2}}(4 + 3x)$$

Verified OK.

10.30.5 Maple step by step solution

Let's solve

$\left[4y''-12y'+9y=0, y(0)=4, y'\Big|_{\{x=0\}}=9\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=3y'-\frac{9y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-3y'+\frac{9y}{4}=0$

- Characteristic polynomial of ODE

  $r^2-3r+\frac{9}{4}=0$

- Factor the characteristic polynomial

  $\frac{(2r-3)^2}{4}=0$

- Root of the characteristic polynomial

  $r=\frac{3}{2}$

- 1st solution of the ODE

  $y_1(x)=e^{\frac{3x}{2}}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x)=x\,e^{\frac{3x}{2}}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)$

- Substitute in solutions

  $y=c_1e^{\frac{3x}{2}}+c_2x\,e^{\frac{3x}{2}}$

□ Check validity of solution $y=c_1e^{\frac{3x}{2}}+c_2xe^{\frac{3x}{2}}$

  - Use initial condition $y(0)=4$

    $4=c_1$

  - Compute derivative of the solution

    $y'=\frac{3c_1e^{\frac{3x}{2}}}{2}+e^{\frac{3x}{2}}c_2+\frac{3c_2x\,e^{\frac{3x}{2}}}{2}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 9$

  $9 = \frac{3c_1}{2} + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 4, c_2 = 3\}$

- Substitute constant values into general solution and simplify

  $y = e^{\frac{3x}{2}}(4 + 3x)$

- Solution to the IVP

  $y = e^{\frac{3x}{2}}(4 + 3x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([4*diff(y(x),x$2)-12*diff(y(x),x)+9*y(x)=0,y(0) = 4, D(y)(0) = 9],y(x), singsol=all)
```

$$y(x) = e^{\frac{3x}{2}}(3x + 4)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 18

```
DSolve[{4*y''[x]-12*y'[x]+9*y[x]==0,{y[0]==4,y'[0]==9}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{3x/2}(3x + 4)$$

10.31 problem 31

Internal problem ID [11761]
Internal file name [`OUTPUT/11770_Thursday_April_11_2024_08_49_30_PM_15626544/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y' + 4y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = 7]$$

10.31.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 4 \\ q(x) &= 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 4y = 0$$

The domain of $p(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.31.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(4)^2 - (4)(1)(4)} \\
&= -2
\end{aligned}
$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$
y = c_1 e^{-2x} + c_2 x\, e^{-2x} \tag{1}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 e^{-2x} + x\, e^{-2x} c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$
3 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 e^{-2x} + c_2 e^{-2x} - 2x\, e^{-2x} c_2
$$

substituting $y' = 7$ and $x = 0$ in the above gives

$$
7 = -2c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= 13
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 13x\, e^{-2x} + 3\, e^{-2x}
$$

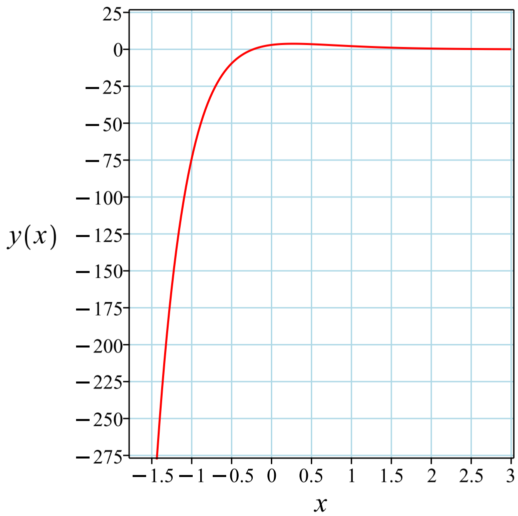
Which simplifies to

$$
y = e^{-2x}(3 + 13x)
$$

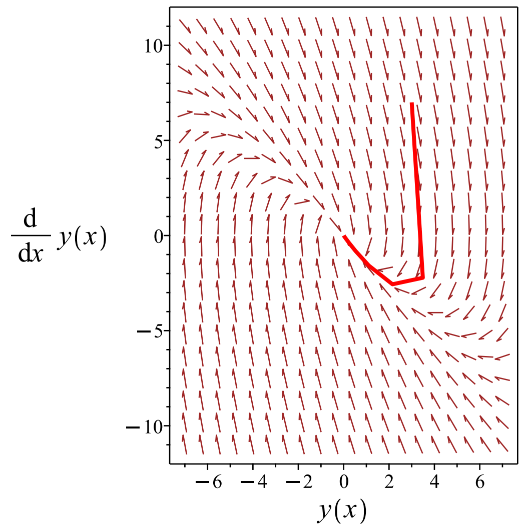
Summary

The solution(s) found are the following

$$
y = e^{-2x}(3 + 13x) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-2x}(3 + 13x)$$

Verified OK.

10.31.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 4\,dx} \\ &= e^{2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(y\,e^{2x}\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(y\,\mathrm{e}^{2x}\right)' = c_1$$

Integrating again gives

$$\left(y\,\mathrm{e}^{2x}\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{2x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{-2x} + c_2 \mathrm{e}^{-2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x\,\mathrm{e}^{-2x} + c_2 \mathrm{e}^{-2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \mathrm{e}^{-2x} - 2c_1 x\,\mathrm{e}^{-2x} - 2c_2 \mathrm{e}^{-2x}$$

substituting $y' = 7$ and $x = 0$ in the above gives

$$7 = c_1 - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 13$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = 13x\,\mathrm{e}^{-2x} + 3\,\mathrm{e}^{-2x}$$

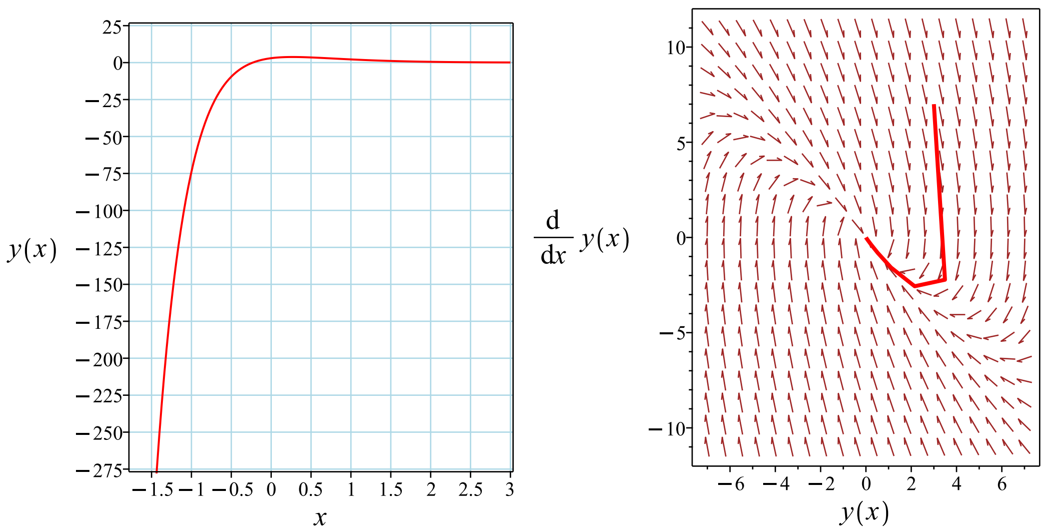
Which simplifies to

$$y = \mathrm{e}^{-2x}(3 + 13x)$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(3 + 13x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{-2x}(3 + 13x)$$

Verified OK.

10.31.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 323: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\ &= z_1 e^{-2x} \\ &= z_1 \left(\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{\mathrm{e}^{-4x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(x)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x} c_2$$

substituting $y' = 7$ and $x = 0$ in the above gives

$$7 = -2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 3 \\ c_2 &= 13 \end{aligned}$$

Substituting these values back in above solution results in

$$y = 13x\,\mathrm{e}^{-2x} + 3\,\mathrm{e}^{-2x}$$

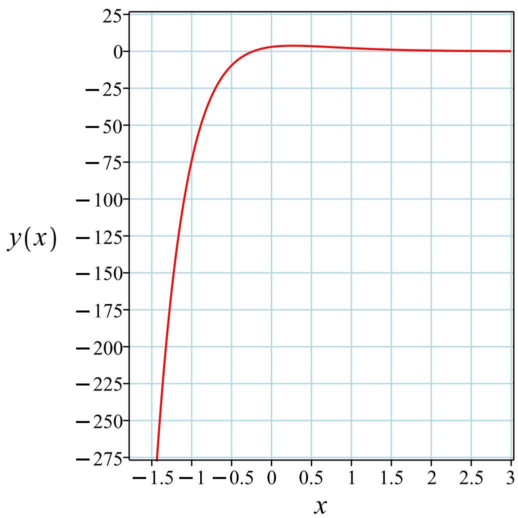
Which simplifies to

$$y = \mathrm{e}^{-2x}(3 + 13x)$$

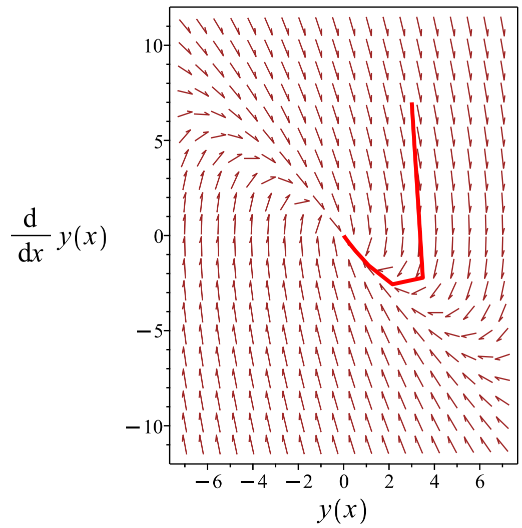
Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x}(3 + 13x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{-2x}(3 + 13x)$$

Verified OK.

10.31.5 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 4y = 0, y(0) = 3, y'\Big|_{\{x=0\}} = 7\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r + 2)^2 = 0$

- Root of the characteristic polynomial

  $r = -2$

- 1st solution of the ODE

$y_1(x) = e^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} + x\,e^{-2x} c_2$

- ☐ Check validity of solution $y = c_1 e^{-2x} + x e^{-2x} c_2$
  - Use initial condition $y(0) = 3$

    $3 = c_1$
  - Compute derivative of the solution

    $y' = -2c_1 e^{-2x} + c_2 e^{-2x} - 2x\,e^{-2x} c_2$
  - Use the initial condition $y'\Big|_{\{x=0\}} = 7$

    $7 = -2c_1 + c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = 13\}$
  - Substitute constant values into general solution and simplify

    $y = e^{-2x}(3 + 13x)$

- Solution to the IVP

  $y = e^{-2x}(3 + 13x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=0,y(0) = 3, D(y)(0) = 7],y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-2x}(3 + 13x)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 16

```
DSolve[{y''[x]+4*y'[x]+4*y[x]==0,{y[0]==3,y'[0]==7}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{-2x}(13x + 3)$$

10.32 problem 32

Internal problem ID [11762]
Internal file name [`OUTPUT/11771_Thursday_April_11_2024_08_49_31_PM_45171481/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{9y'' - 6y' + y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = -1]$$

10.32.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{3} \\ q(x) &= \frac{1}{9} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{2y'}{3} + \frac{y}{9} = 0$$

The domain of $p(x) = -\frac{2}{3}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1}{9}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.32.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 9, B = -6, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$9\lambda^2 e^{\lambda x} - 6\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$9\lambda^2 - 6\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 9, B = -6, C = 1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{6}{(2)(9)} \pm \frac{1}{(2)(9)}\sqrt{(-6)^2 - (4)(9)(1)} \\
&= \frac{1}{3}
\end{aligned}
$$

Hence this is the case of a double root $\lambda_{1,2} = -\frac{1}{3}$. Therefore the solution is

$$
y = c_1 e^{\frac{x}{3}} + c_2 x\, e^{\frac{x}{3}} \tag{1}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 e^{\frac{x}{3}} + c_2 x\, e^{\frac{x}{3}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$
3 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{c_1 e^{\frac{x}{3}}}{3} + c_2 e^{\frac{x}{3}} + \frac{c_2 x\, e^{\frac{x}{3}}}{3}
$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$
-1 = \frac{c_1}{3} + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= -2
\end{aligned}
$$

Substituting these values back in above solution results in

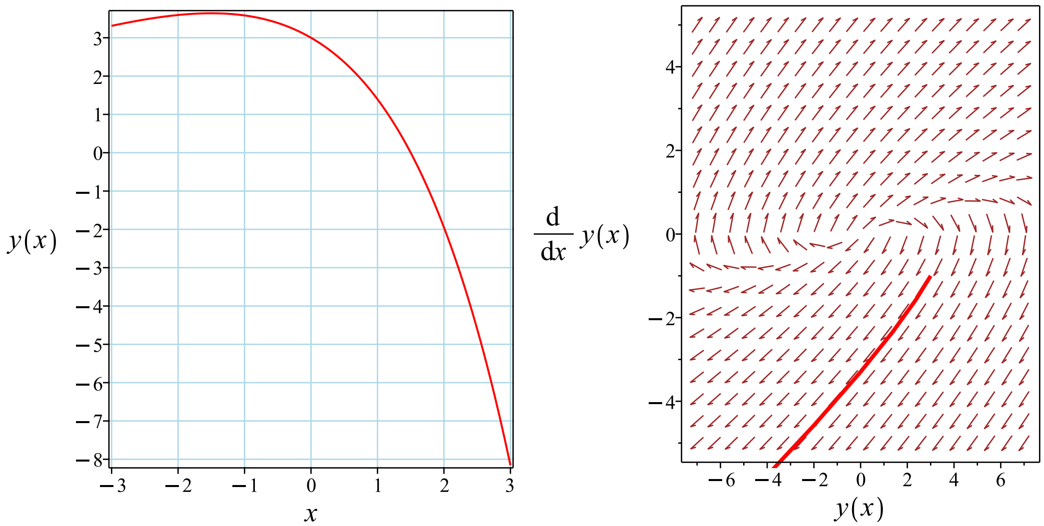
$$
y = -2x\, e^{\frac{x}{3}} + 3\, e^{\frac{x}{3}}
$$

Which simplifies to

$$
y = e^{\frac{x}{3}}(-2x + 3)
$$

Summary
The solution(s) found are the following

$$y = e^{\frac{x}{3}}(-2x+3) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = e^{\frac{x}{3}}(-2x+3)$$

Verified OK.

10.32.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -\frac{2}{3}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -\frac{2}{3}\,dx} \\ &= e^{-\frac{x}{3}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 0 \\
\left(e^{-\frac{x}{3}}y\right)'' &= 0
\end{aligned}
$$

Integrating once gives

$$
\left(e^{-\frac{x}{3}}y\right)' = c_1
$$

Integrating again gives

$$
\left(e^{-\frac{x}{3}}y\right) = c_1x + c_2
$$

Hence the solution is

$$
y = \frac{c_1x + c_2}{e^{-\frac{x}{3}}}
$$

Or

$$
y = c_1x\,e^{\frac{x}{3}} + c_2e^{\frac{x}{3}}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1x\,e^{\frac{x}{3}} + c_2e^{\frac{x}{3}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$
3 = c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1e^{\frac{x}{3}} + \frac{c_1x\,e^{\frac{x}{3}}}{3} + \frac{c_2e^{\frac{x}{3}}}{3}
$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$
-1 = c_1 + \frac{c_2}{3} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -2 \\
c_2 &= 3
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = -2x\,e^{\frac{x}{3}} + 3\,e^{\frac{x}{3}}$$

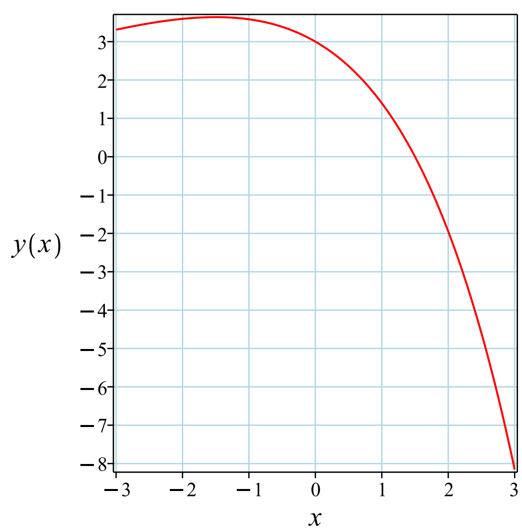
Which simplifies to

$$y = e^{\frac{x}{3}}(-2x+3)$$

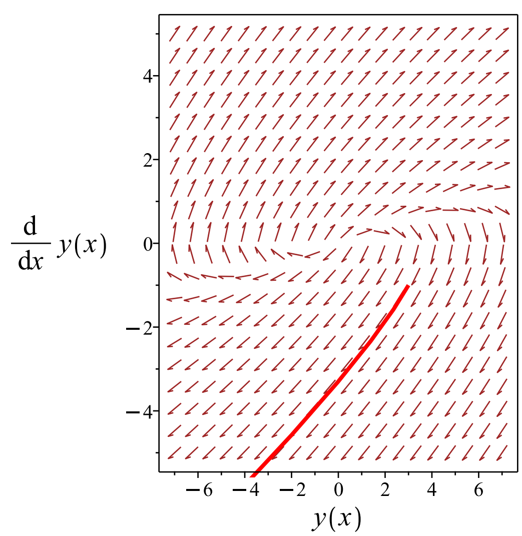
Summary

The solution(s) found are the following

$$y = e^{\frac{x}{3}}(-2x+3) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{\frac{x}{3}}(-2x+3)$$

Verified OK.

10.32.4 Solving using Kovacic algorithm

Writing the ode as

$$9y'' - 6y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9 \\ B &= -6 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 325: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6}{9}\,dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{\frac{x}{3}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{3}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{x}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{9} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\frac{2x}{3}}}{(y_1)^2} \, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x}{3}}\right) + c_2 \left(\mathrm{e}^{\frac{x}{3}}(x)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{\frac{x}{3}} + c_2 x \, \mathrm{e}^{\frac{x}{3}} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$
3 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{c_1 \mathrm{e}^{\frac{x}{3}}}{3} + c_2 \mathrm{e}^{\frac{x}{3}} + \frac{c_2 x \, \mathrm{e}^{\frac{x}{3}}}{3}
$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = \frac{c_1}{3} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = -2$$

Substituting these values back in above solution results in

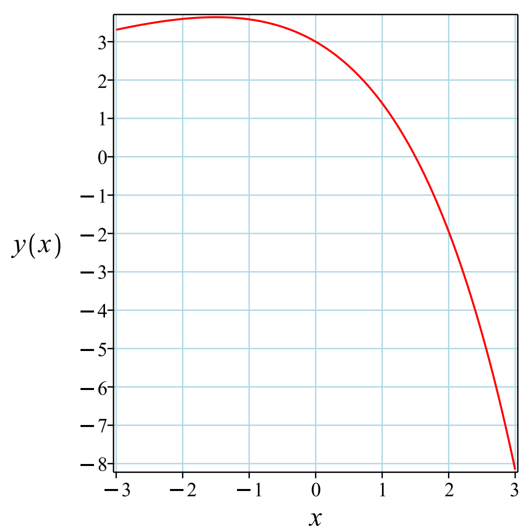
$$y = -2x\,\mathrm{e}^{\frac{x}{3}} + 3\,\mathrm{e}^{\frac{x}{3}}$$

Which simplifies to

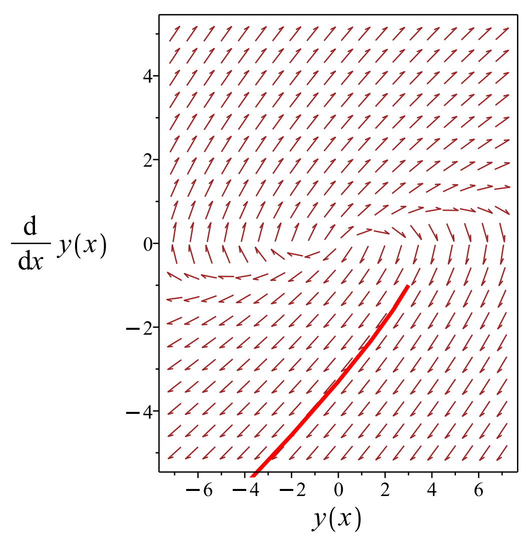
$$y = \mathrm{e}^{\frac{x}{3}}(-2x + 3)$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{\frac{x}{3}}(-2x + 3) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{x}{3}}(-2x + 3)$$

Verified OK.

10.32.5 Maple step by step solution

Let's solve

$\left[9y''-6y'+y=0, y(0)=3, y'\Big|_{\{x=0\}}=-1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{2y'}{3}-\frac{y}{9}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2y'}{3}+\frac{y}{9}=0$

- Characteristic polynomial of ODE

  $r^2-\frac{2}{3}r+\frac{1}{9}=0$

- Factor the characteristic polynomial

  $\frac{(3r-1)^2}{9}=0$

- Root of the characteristic polynomial

  $r=\frac{1}{3}$

- 1st solution of the ODE

  $y_1(x)=e^{\frac{x}{3}}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x)=x\,e^{\frac{x}{3}}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)$

- Substitute in solutions

  $y=c_1e^{\frac{x}{3}}+c_2x\,e^{\frac{x}{3}}$

□ Check validity of solution $y=c_1e^{\frac{x}{3}}+c_2xe^{\frac{x}{3}}$

  ◦ Use initial condition $y(0)=3$

    $3=c_1$

  ◦ Compute derivative of the solution

    $y'=\frac{c_1e^{\frac{x}{3}}}{3}+c_2e^{\frac{x}{3}}+\frac{c_2x\,e^{\frac{x}{3}}}{3}$

- Use the initial condition $y'\Big|_{\{x=0\}} = -1$

  $-1 = \frac{c_1}{3} + c_2$
- Solve for c_1 and c_2

  $\{c_1 = 3, c_2 = -2\}$
- Substitute constant values into general solution and simplify

  $y = e^{\frac{x}{3}}(-2x + 3)$

- Solution to the IVP

  $y = e^{\frac{x}{3}}(-2x + 3)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([9*diff(y(x),x$2)-6*diff(y(x),x)+y(x)=0,y(0) = 3, D(y)(0) = -1],y(x), singsol=all)
```

$$y(x) = e^{\frac{x}{3}}(-2x + 3)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 18

```
DSolve[{9*y''[x]-6*y'[x]+y[x]==0,{y[0]==3,y'[0]==-1}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{x/3}(3 - 2x)$$

10.33 problem 33

Internal problem ID [11763]
Internal file name [`OUTPUT/11772_Thursday_April_11_2024_08_49_31_PM_4033505/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' + 29y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 5]$$

10.33.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= 29 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 29y = 0$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 29$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.33.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 29$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 29\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 29 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 29$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(29)} \\
&= 2 \pm 5i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= 2 + 5i \\
\lambda_2 &= 2 - 5i
\end{aligned}$$

Which simplifies to

$$\lambda_1 = 2 + 5i$$
$$\lambda_2 = 2 - 5i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 2$ and $\beta = 5$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{2x}(c_1 \cos(5x) + c_2 \sin(5x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{2x}(c_1 \cos(5x) + c_2 \sin(5x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2\,e^{2x}(c_1 \cos(5x) + c_2 \sin(5x)) + e^{2x}(-5c_1 \sin(5x) + 5c_2 \cos(5x))$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = 2c_1 + 5c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

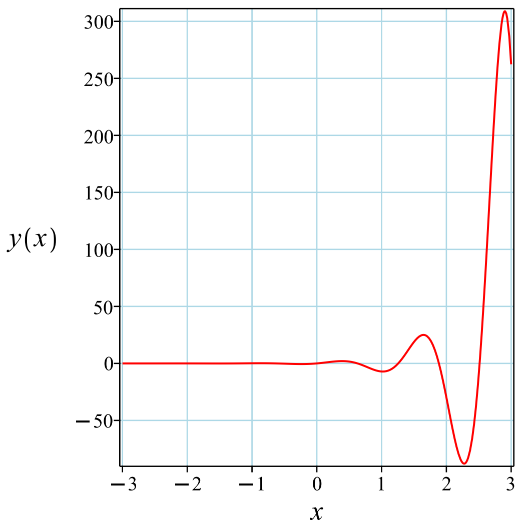
$$c_1 = 0$$
$$c_2 = 1$$

Substituting these values back in above solution results in

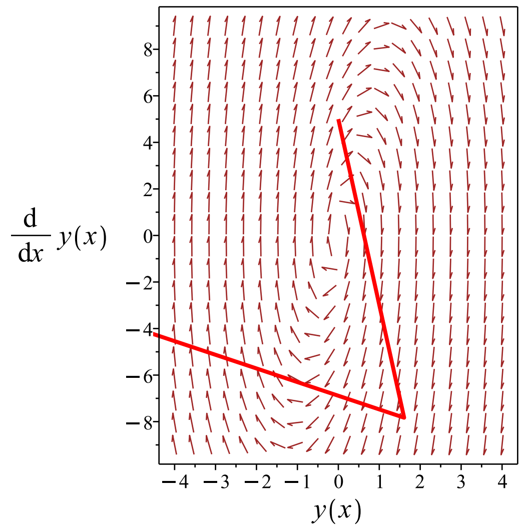
$$y = e^{2x}\sin(5x)$$

Summary
The solution(s) found are the following

$$y = e^{2x} \sin(5x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{2x} \sin(5x)$$

Verified OK.

10.33.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 29y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 29 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-25}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -25 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -25z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 327: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -25$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(5x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}\cos(5x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\tan(5x)}{5}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\cos(5x)\right) + c_2\left(\mathrm{e}^{2x}\cos(5x)\left(\frac{\tan(5x)}{5}\right)\right)
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{2x}\cos(5x) + \frac{\mathrm{e}^{2x} c_2 \sin(5x)}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 \mathrm{e}^{2x}\cos(5x) - 5c_1 \mathrm{e}^{2x}\sin(5x) + \frac{2\,\mathrm{e}^{2x} c_2 \sin(5x)}{5} + \mathrm{e}^{2x} c_2 \cos(5x)$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = 2c_1 + c_2 \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned}
c_1 &= 0 \\
c_2 &= 5
\end{aligned}$$

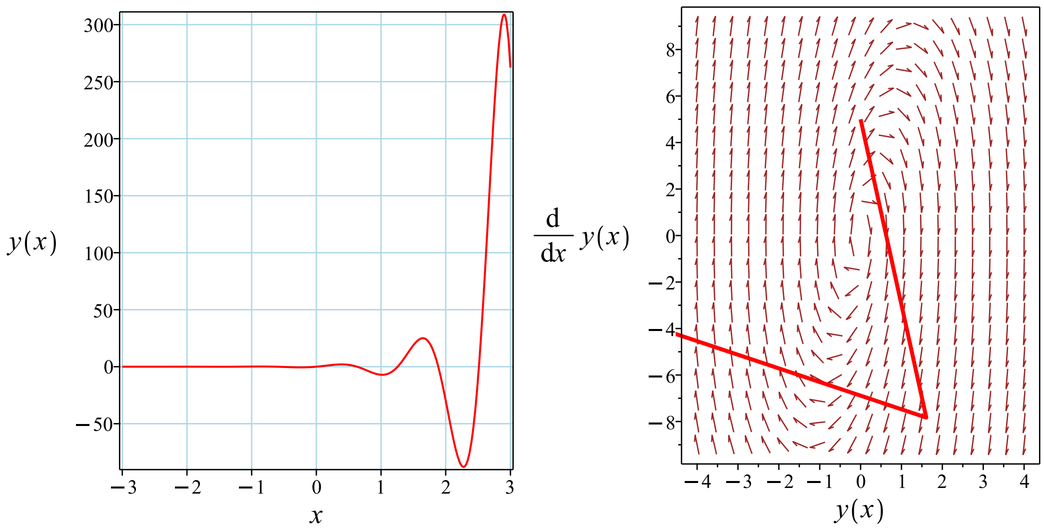
Substituting these values back in above solution results in

$$y = \mathrm{e}^{2x}\sin(5x)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2x}\sin(5x) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = e^{2x} \sin(5x)$$

Verified OK.

10.33.4 Maple step by step solution

Let's solve

$$\left[y'' - 4y' + 29y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 5\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 4r + 29 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm (\sqrt{-100})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - 5\,\mathrm{I}, 2 + 5\,\mathrm{I})$

- 1st solution of the ODE

$y_1(x) = e^{2x} \cos(5x)$

- 2nd solution of the ODE

  $y_2(x) = e^{2x} \sin(5x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{2x} \cos(5x) + e^{2x} c_2 \sin(5x)$

- ☐ Check validity of solution $y = c_1 e^{2x} \cos(5x) + e^{2x} c_2 \sin(5x)$

  - Use initial condition $y(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $y' = 2c_1 e^{2x} \cos(5x) - 5c_1 e^{2x} \sin(5x) + 2 e^{2x} c_2 \sin(5x) + 5 e^{2x} c_2 \cos(5x)$

  - Use the initial condition $y'\big|_{\{x=0\}} = 5$

    $5 = 2c_1 + 5c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 1\}$

  - Substitute constant values into general solution and simplify

    $y = e^{2x} \sin(5x)$

- Solution to the IVP

  $y = e^{2x} \sin(5x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([diff(y(x),x$2)-4*diff(y(x),x)+29*y(x)=0,y(0) = 0, D(y)(0) = 5],y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{2x} \sin(5x)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 15

```
DSolve[{y''[x]-4*y'[x]+29*y[x]==0,{y[0]==0,y'[0]==5}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{2x} \sin(5x)$$

10.34 problem 34

Internal problem ID [11764]
Internal file name [`OUTPUT/11773_Thursday_April_11_2024_08_49_32_PM_86141505/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 6y' + 58y = 0}$$

With initial conditions

$$[y(0) = -1, y'(0) = 5]$$

10.34.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 6 \\ q(x) &= 58 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 58y = 0$$

The domain of $p(x) = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 58$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.34.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 58$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 6\lambda\,\mathrm{e}^{\lambda x} + 58\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 58 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 58$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{6^2 - (4)\,(1)\,(58)} \\ &= -3 \pm 7i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 7i \\ \lambda_2 &= -3 - 7i\end{aligned}$$

Which simplifies to

$$\lambda_1 = -3 + 7i$$
$$\lambda_2 = -3 - 7i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = 7$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-3x}(c_1 \cos(7x) + c_2 \sin(7x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-3x}(c_1 \cos(7x) + c_2 \sin(7x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$-1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3\,\mathrm{e}^{-3x}(c_1 \cos(7x) + c_2 \sin(7x)) + \mathrm{e}^{-3x}(-7c_1 \sin(7x) + 7c_2 \cos(7x))$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = -3c_1 + 7c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = \frac{2}{7}$$

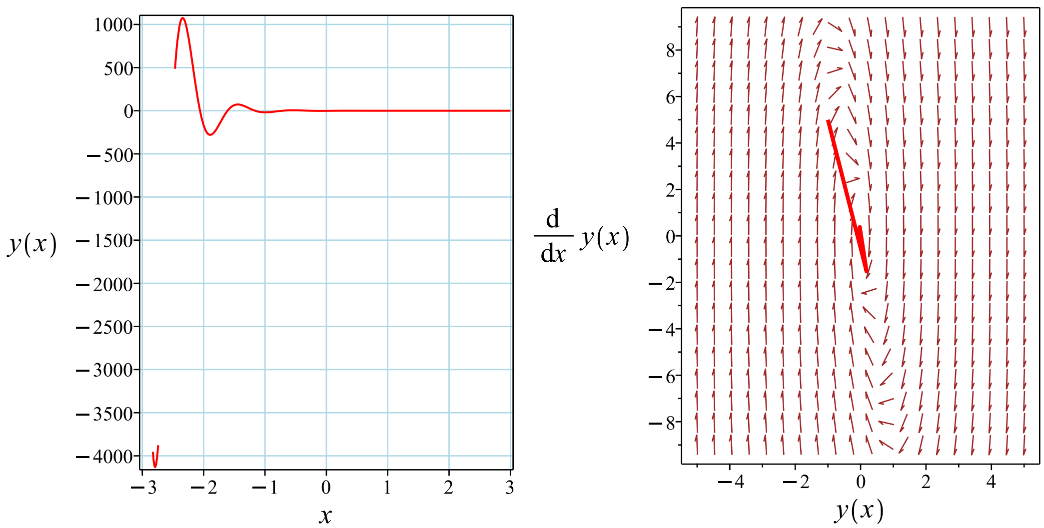
Substituting these values back in above solution results in

$$y = -\frac{e^{-3x}(7\cos(7x) - 2\sin(7x))}{7}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-3x}(7\cos(7x) - 2\sin(7x))}{7} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{e^{-3x}(7\cos(7x) - 2\sin(7x))}{7}$$

Verified OK.

10.34.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 58y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 6 \\
C &= 58
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-49}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -49 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -49 z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 329: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -49$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(7x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{6}{1}\, dx} \\
&= z_1 e^{-3x} \\
&= z_1 \left(\mathrm{e}^{-3x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x} \cos(7x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-6x}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{\tan(7x)}{7} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3x} \cos(7x) \right) + c_2 \left(\mathrm{e}^{-3x} \cos(7x) \left(\frac{\tan(7x)}{7} \right) \right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-3x} \cos(7x) + \frac{c_2 \mathrm{e}^{-3x} \sin(7x)}{7} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$-1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 \mathrm{e}^{-3x} \cos(7x) - 7c_1 \mathrm{e}^{-3x} \sin(7x) - \frac{3c_2 \mathrm{e}^{-3x} \sin(7x)}{7} + c_2 \mathrm{e}^{-3x} \cos(7x)$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = -3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = -\mathrm{e}^{-3x} \cos(7x) + \frac{2\,\mathrm{e}^{-3x} \sin(7x)}{7}$$

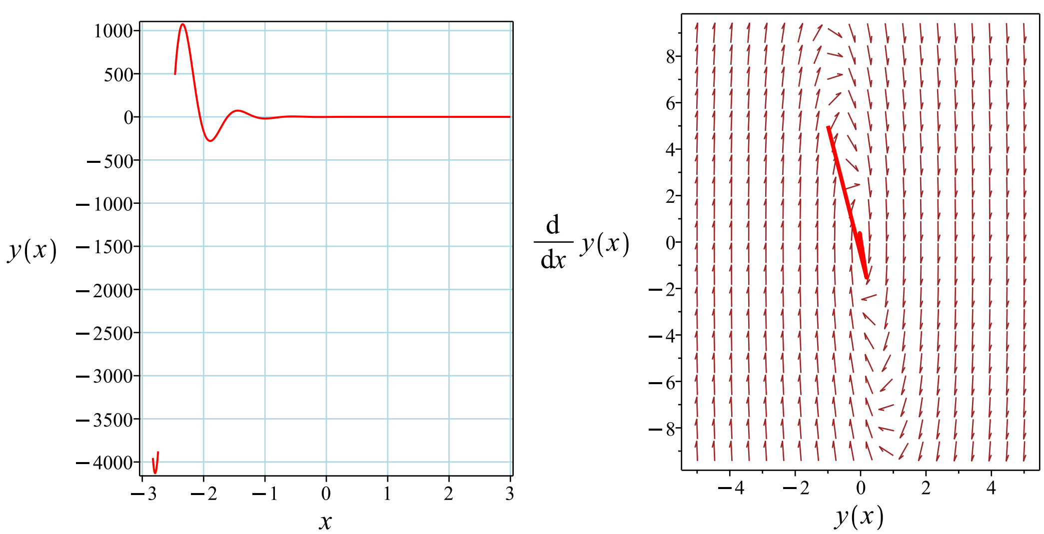
Which simplifies to

$$y = -\frac{\mathrm{e}^{-3x}(7\cos(7x) - 2\sin(7x))}{7}$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-3x}(7\cos(7x) - 2\sin(7x))}{7} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{\mathrm{e}^{-3x}(7\cos(7x) - 2\sin(7x))}{7}$$

Verified OK.

10.34.4 Maple step by step solution

Let's solve

$\left[y'' + 6y' + 58y = 0, y(0) = -1, y'\Big|_{\{x=0\}} = 5\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 6r + 58 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-196})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - 7\,\mathrm{I}, -3 + 7\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-3x} \cos(7x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-3x} \sin(7x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-3x} \cos(7x) + c_2 \mathrm{e}^{-3x} \sin(7x)$

□ Check validity of solution $y = c_1 \mathrm{e}^{-3x} \cos(7x) + c_2 \mathrm{e}^{-3x} \sin(7x)$

  - Use initial condition $y(0) = -1$

    $-1 = c_1$

  - Compute derivative of the solution

    $y' = -3c_1 \mathrm{e}^{-3x} \cos(7x) - 7c_1 \mathrm{e}^{-3x} \sin(7x) - 3c_2 \mathrm{e}^{-3x} \sin(7x) + 7c_2 \mathrm{e}^{-3x} \cos(7x)$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 5$

    $5 = -3c_1 + 7c_2$

  - Solve for c_1 and c_2

    $\left\{c_1 = -1, c_2 = \frac{2}{7}\right\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{e^{-3x}(7\cos(7x)-2\sin(7x))}{7}$

- Solution to the IVP

  $y = -\frac{e^{-3x}(7\cos(7x)-2\sin(7x))}{7}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$2)+6*diff(y(x),x)+58*y(x)=0,y(0) = -1, D(y)(0) = 5],y(x), singsol=all)
```

$$y(x) = \frac{e^{-3x}(2\sin(7x) - 7\cos(7x))}{7}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 27

```
DSolve[{y''[x]+6*y'[x]+58*y[x]==0,{y[0]==-1,y'[0]==5}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{7}e^{-3x}(2\sin(7x) - 7\cos(7x))$$

10.35 problem 35

Internal problem ID [11765]
Internal file name [`OUTPUT/11774_Thursday_April_11_2024_08_49_32_PM_47626021/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' + 6y' + 13y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = -1]$$

10.35.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 6 \\ q(x) &= 13 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 13y = 0$$

The domain of $p(x) = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 13$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.35.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 13$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda\, e^{\lambda x} + 13\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{6^2 - (4)\,(1)\,(13)}\\ &= -3 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 2i\\ \lambda_2 &= -3 - 2i\end{aligned}$$

Which simplifies to

$$
\lambda_1 = -3 + 2i
$$
$$
\lambda_2 = -3 - 2i
$$

Since roots are complex conjugate of each others, then let the roots be

$$
\lambda_{1,2} = \alpha \pm i\beta
$$

Where $\alpha = -3$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$
y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))
$$

Which becomes

$$
y = e^{-3x}(c_1 \cos(2x) + c_2 \sin(2x))
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \mathrm{e}^{-3x}(c_1 \cos(2x) + c_2 \sin(2x)) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$
3 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -3\,\mathrm{e}^{-3x}(c_1 \cos(2x) + c_2 \sin(2x)) + \mathrm{e}^{-3x}(-2c_1 \sin(2x) + 2c_2 \cos(2x))
$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$
-1 = -3c_1 + 2c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
c_1 = 3
$$
$$
c_2 = 4
$$

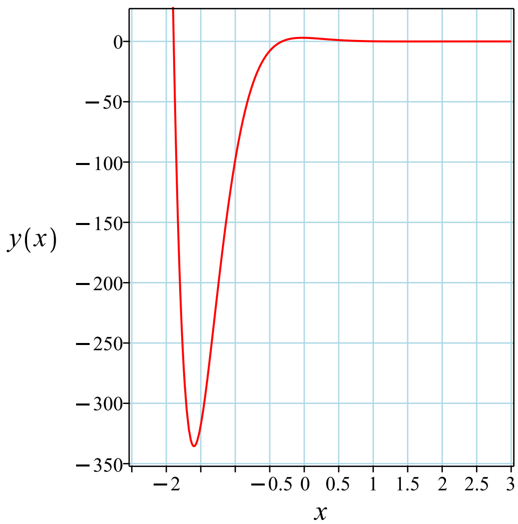
Substituting these values back in above solution results in

$$
y = \mathrm{e}^{-3x}(3\cos(2x) + 4\sin(2x))
$$

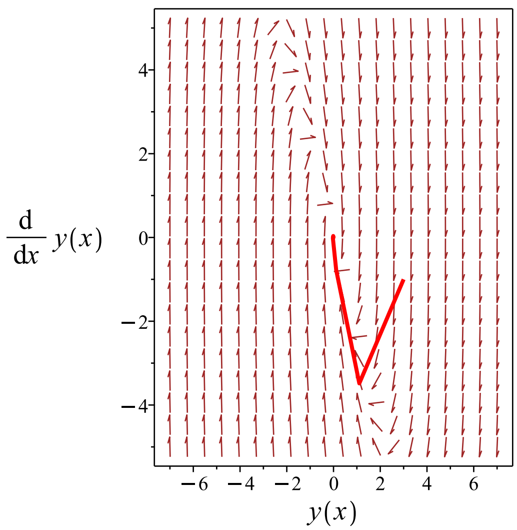
Summary

The solution(s) found are the following

$$y = e^{-3x}(3\cos(2x) + 4\sin(2x)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-3x}(3\cos(2x) + 4\sin(2x))$$

Verified OK.

10.35.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 13y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 331: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\ &= z_1 e^{-3x} \\ &= z_1\left(\mathrm{e}^{-3x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}\cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(2x)}{2}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{-3x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-3x} \cos(2x) + \frac{c_2 \mathrm{e}^{-3x} \sin(2x)}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$
3 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -3c_1 \mathrm{e}^{-3x} \cos(2x) - 2c_1 \mathrm{e}^{-3x} \sin(2x) - \frac{3c_2 \mathrm{e}^{-3x} \sin(2x)}{2} + c_2 \mathrm{e}^{-3x} \cos(2x)
$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$
-1 = -3c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= 8
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 3\,\mathrm{e}^{-3x} \cos(2x) + 4\,\mathrm{e}^{-3x} \sin(2x)
$$

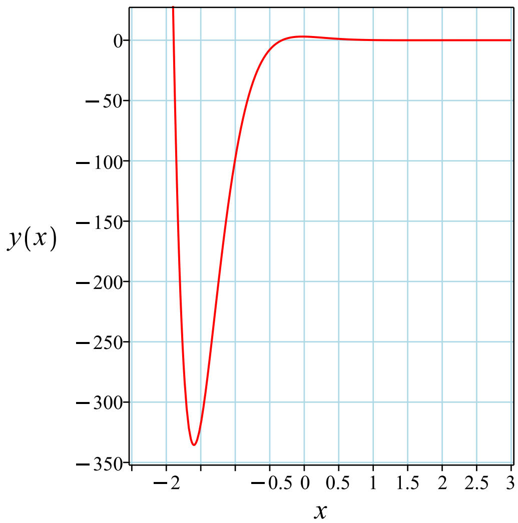
Which simplifies to

$$
y = \mathrm{e}^{-3x} (3 \cos(2x) + 4 \sin(2x))
$$

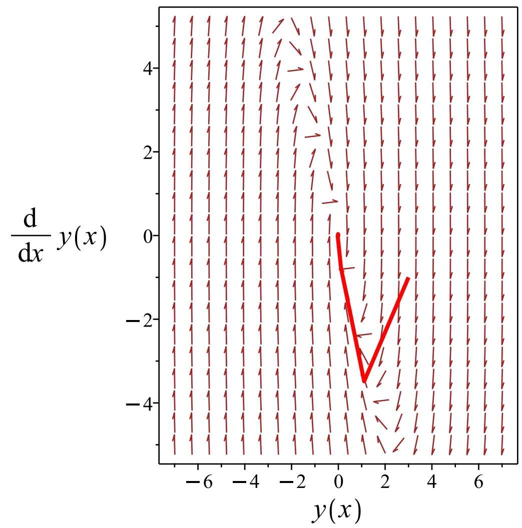
Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-3x} (3 \cos(2x) + 4 \sin(2x)) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-3x}(3\cos(2x) + 4\sin(2x))$$

Verified OK.

10.35.4 Maple step by step solution

Let's solve

$\left[y'' + 6y' + 13y = 0, y(0) = 3, y'\Big|_{\{x=0\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 6r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - 2\,\mathrm{I}, -3 + 2\,\mathrm{I})$

- 1st solution of the ODE

$y_1(x) = e^{-3x} \cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = e^{-3x} \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-3x} \cos(2x) + c_2 e^{-3x} \sin(2x)$

☐ Check validity of solution $y = c_1 e^{-3x} \cos(2x) + c_2 e^{-3x} \sin(2x)$

  ◦ Use initial condition $y(0) = 3$

    $3 = c_1$

  ◦ Compute derivative of the solution

    $y' = -3c_1 e^{-3x} \cos(2x) - 2c_1 e^{-3x} \sin(2x) - 3c_2 e^{-3x} \sin(2x) + 2c_2 e^{-3x} \cos(2x)$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = -1$

    $-1 = -3c_1 + 2c_2$

  ◦ Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = 4\}$

  ◦ Substitute constant values into general solution and simplify

    $y = e^{-3x}(3\cos(2x) + 4\sin(2x))$

- Solution to the IVP

  $y = e^{-3x}(3\cos(2x) + 4\sin(2x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve([diff(y(x),x$2)+6*diff(y(x),x)+13*y(x)=0,y(0) = 3, D(y)(0) = -1],y(x), singsol=all)
```

$$y(x) = e^{-3x}(4\sin(2x) + 3\cos(2x))$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 24

```
DSolve[{y''[x]+6*y'[x]+13*y[x]==0,{y[0]==3,y'[0]==-1}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to e^{-3x}(4\sin(2x) + 3\cos(2x))$$

10.36 problem 36

Internal problem ID [11766]
Internal file name [`OUTPUT/11775_Thursday_April_11_2024_08_49_33_PM_53540331/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + 5y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 6]$$

10.36.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= 5 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2y' + 5y = 0$$

The domain of $p(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.36.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(5)}\\ &= -1 \pm 2i\end{aligned}$$

Hence

$$\lambda_1 = -1 + 2i$$
$$\lambda_2 = -1 - 2i$$

Which simplifies to

$$\lambda_1 = -1 + 2i$$
$$\lambda_2 = -1 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\mathrm{e}^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) + \mathrm{e}^{-x}(-2c_1 \sin(2x) + 2c_2 \cos(2x))$$

substituting $y' = 6$ and $x = 0$ in the above gives

$$6 = -c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 4$$

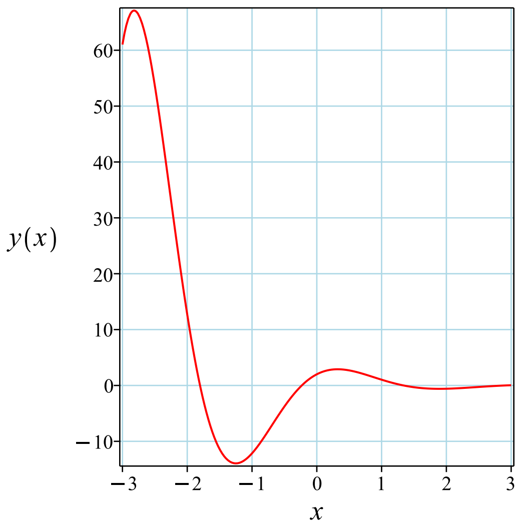
Substituting these values back in above solution results in

$$y = 2\,\mathrm{e}^{-x}(\cos(2x) + 2\sin(2x))$$

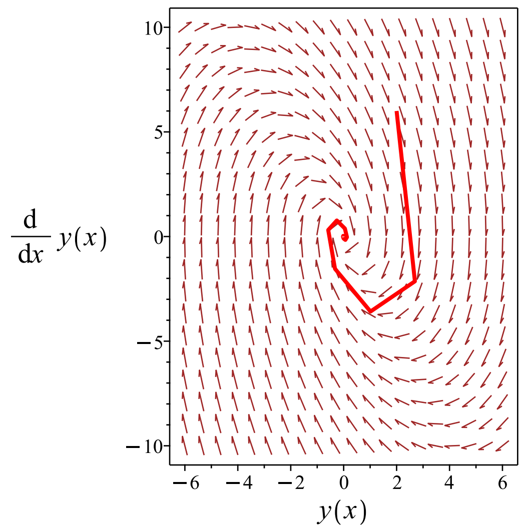
Summary

The solution(s) found are the following

$$y = 2\,e^{-x}(\cos(2x) + 2\sin(2x)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,e^{-x}(\cos(2x) + 2\sin(2x))$$

Verified OK.

10.36.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 333: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}\cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\cos(2x)\right) + c_2\left(\mathrm{e}^{-x}\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x}\cos(2x) + \frac{c_2 \mathrm{e}^{-x}\sin(2x)}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$
2 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-x}\cos(2x) - 2c_1 \mathrm{e}^{-x}\sin(2x) - \frac{c_2 \mathrm{e}^{-x}\sin(2x)}{2} + c_2 \mathrm{e}^{-x}\cos(2x)
$$

substituting $y' = 6$ and $x = 0$ in the above gives

$$
6 = -c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= 8
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 2\,\mathrm{e}^{-x}\cos(2x) + 4\,\mathrm{e}^{-x}\sin(2x)
$$

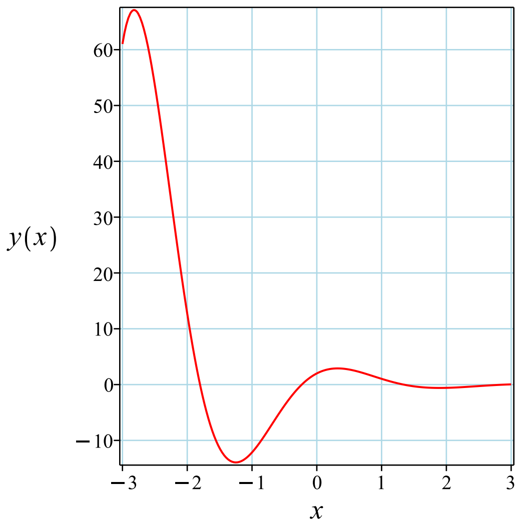
Which simplifies to

$$
y = 2\,\mathrm{e}^{-x}(\cos(2x) + 2\sin(2x))
$$

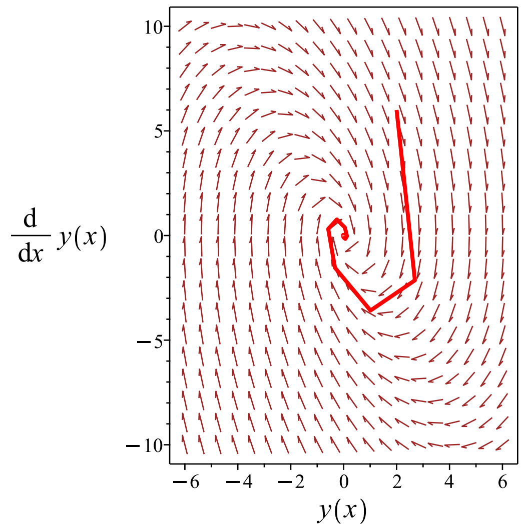
Summary

The solution(s) found are the following

$$
y = 2\,\mathrm{e}^{-x}(\cos(2x) + 2\sin(2x)) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{-x}(\cos(2x) + 2\sin(2x))$$

Verified OK.

10.36.4 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + 5y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = 6\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - 2\,\mathrm{I}, -1 + 2\,\mathrm{I})$

- 1st solution of the ODE

$$y_1(x) = e^{-x} \cos(2x)$$

- 2nd solution of the ODE

$$y_2(x) = e^{-x} \sin(2x)$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{-x} \cos(2x) + c_2 e^{-x} \sin(2x)$$

□ Check validity of solution $y = c_1 e^{-x} \cos(2x) + c_2 e^{-x} \sin(2x)$

  ◦ Use initial condition $y(0) = 2$

$$2 = c_1$$

  ◦ Compute derivative of the solution

$$y' = -c_1 e^{-x} \cos(2x) - 2c_1 e^{-x} \sin(2x) - c_2 e^{-x} \sin(2x) + 2c_2 e^{-x} \cos(2x)$$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 6$

$$6 = -c_1 + 2c_2$$

  ◦ Solve for c_1 and c_2

$$\{c_1 = 2, c_2 = 4\}$$

  ◦ Substitute constant values into general solution and simplify

$$y = 2 e^{-x} (\cos(2x) + 2 \sin(2x))$$

- Solution to the IVP

$$y = 2 e^{-x} (\cos(2x) + 2 \sin(2x))$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$2)+2*diff(y(x),x)+5*y(x)=0,y(0) = 2, D(y)(0) = 6],y(x), singsol=all)
```

$$y(x) = 2\,\mathrm{e}^{-x}(2\sin(2x) + \cos(2x))$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 23

```
DSolve[{y''[x]+2*y'[x]+5*y[x]==0,{y[0]==2,y'[0]==6}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to 2e^{-x}(2\sin(2x) + \cos(2x))$$

10.37 problem 37

Internal problem ID [11767]
Internal file name [`OUTPUT/11776_Thursday_April_11_2024_08_49_34_PM_41951789/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{9y'' + 6y' + 5y = 0}$$

With initial conditions

$$[y(0) = 6, y'(0) = 0]$$

10.37.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{3} \\ q(x) &= \frac{5}{9} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{2y'}{3} + \frac{5y}{9} = 0$$

The domain of $p(x) = \frac{2}{3}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{5}{9}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.37.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 9, B = 6, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$9\lambda^2 e^{\lambda x} + 6\lambda \, e^{\lambda x} + 5 \, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$9\lambda^2 + 6\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 9, B = 6, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(9)} \pm \frac{1}{(2)\,(9)}\sqrt{6^2 - (4)\,(9)\,(5)} \\ &= -\frac{1}{3} \pm \frac{2i}{3}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{3} + \frac{2i}{3} \\ \lambda_2 &= -\frac{1}{3} - \frac{2i}{3}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{3} + \frac{2i}{3}$$
$$\lambda_2 = -\frac{1}{3} - \frac{2i}{3}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{3}$ and $\beta = \frac{2}{3}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{3}}\left(c_1 \cos\left(\frac{2x}{3}\right) + c_2 \sin\left(\frac{2x}{3}\right)\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-\frac{x}{3}}\left(c_1 \cos\left(\frac{2x}{3}\right) + c_2 \sin\left(\frac{2x}{3}\right)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 6$ and $x = 0$ in the above gives

$$6 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\mathrm{e}^{-\frac{x}{3}}\left(c_1 \cos\left(\frac{2x}{3}\right) + c_2 \sin\left(\frac{2x}{3}\right)\right)}{3} + \mathrm{e}^{-\frac{x}{3}}\left(-\frac{2c_1 \sin\left(\frac{2x}{3}\right)}{3} + \frac{2c_2 \cos\left(\frac{2x}{3}\right)}{3}\right)$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -\frac{c_1}{3} + \frac{2c_2}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 6 \\
c_2 &= 3
\end{aligned}
$$

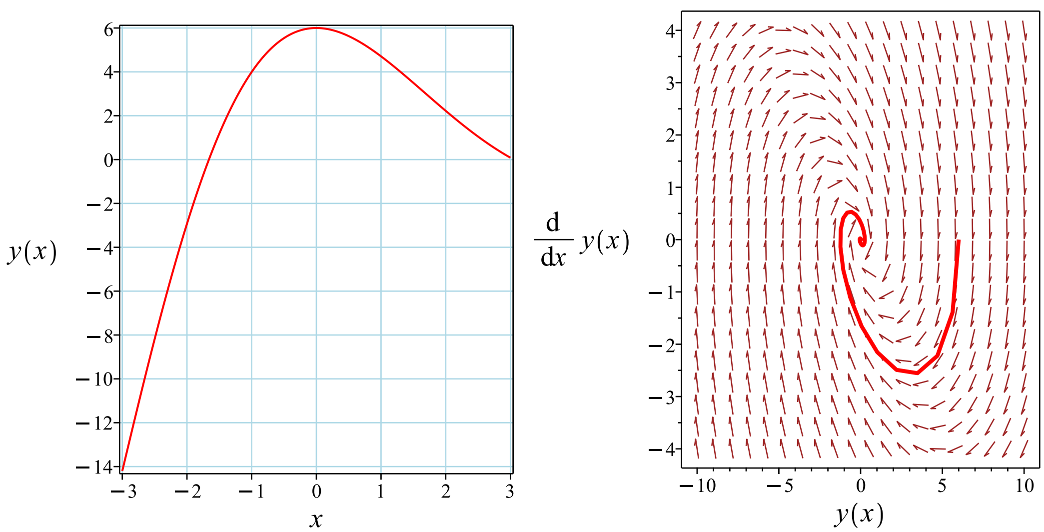
Substituting these values back in above solution results in

$$
y = 3\,\mathrm{e}^{-\frac{x}{3}} \left(2\cos\left(\frac{2x}{3}\right) + \sin\left(\frac{2x}{3}\right)\right)
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = 3\,\mathrm{e}^{-\frac{x}{3}} \left(2\cos\left(\frac{2x}{3}\right) + \sin\left(\frac{2x}{3}\right)\right) \tag{1}
$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$
y = 3\,\mathrm{e}^{-\frac{x}{3}} \left(2\cos\left(\frac{2x}{3}\right) + \sin\left(\frac{2x}{3}\right)\right)
$$

Verified OK.

10.37.3 Solving using Kovacic algorithm

Writing the ode as

$$9y'' + 6y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9 \\ B &= 6 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{9} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 9 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{4z(x)}{9} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 335: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{4}{9}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{2x}{3}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{9}\,dx} \\
&= z_1 e^{-\frac{x}{3}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{3}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{2x}{3}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{9}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{2x}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{3\tan\left(\frac{2x}{3}\right)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{3}}\cos\left(\frac{2x}{3}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{3}}\cos\left(\frac{2x}{3}\right)\left(\frac{3\tan\left(\frac{2x}{3}\right)}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{2x}{3}\right) + \frac{3c_2 \mathrm{e}^{-\frac{x}{3}} \sin\left(\frac{2x}{3}\right)}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 6$ and $x = 0$ in the above gives

$$6 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1 \mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{2x}{3}\right)}{3} - \frac{2c_1 \mathrm{e}^{-\frac{x}{3}} \sin\left(\frac{2x}{3}\right)}{3} - \frac{c_2 \mathrm{e}^{-\frac{x}{3}} \sin\left(\frac{2x}{3}\right)}{2} + c_2 \mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{2x}{3}\right)$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -\frac{c_1}{3} + c_2 \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 6 \\ c_2 &= 2 \end{aligned}$$

Substituting these values back in above solution results in

$$y = 6\,\mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{2x}{3}\right) + 3\,\mathrm{e}^{-\frac{x}{3}} \sin\left(\frac{2x}{3}\right)$$

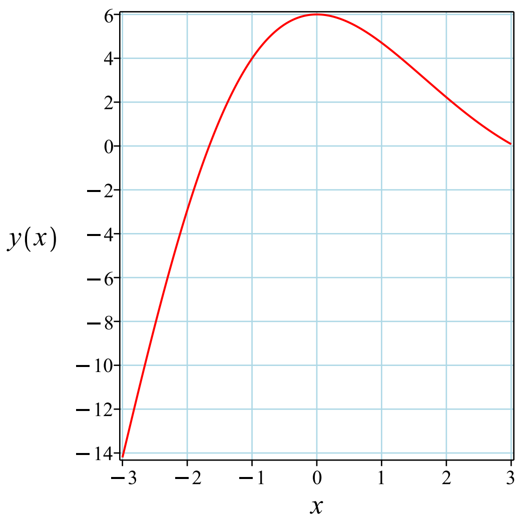
Which simplifies to

$$y = 3\,\mathrm{e}^{-\frac{x}{3}} \left(2 \cos\left(\frac{2x}{3}\right) + \sin\left(\frac{2x}{3}\right)\right)$$

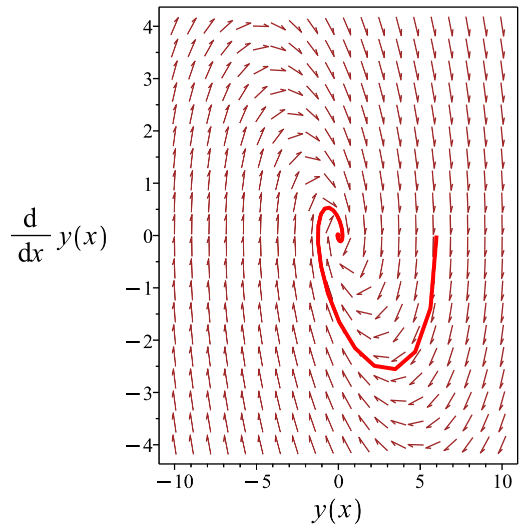
Summary

The solution(s) found are the following

$$y = 3\,\mathrm{e}^{-\frac{x}{3}} \left(2 \cos\left(\frac{2x}{3}\right) + \sin\left(\frac{2x}{3}\right)\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 3\,\mathrm{e}^{-\frac{x}{3}}\left(2\cos\left(\frac{2x}{3}\right)+\sin\left(\frac{2x}{3}\right)\right)$$

Verified OK.

10.37.4 Maple step by step solution

Let's solve

$$\left[9y''+6y'+5y=0, y(0)=6, y'\Big|_{\{x=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y'}{3}-\frac{5y}{9}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2y'}{3}+\frac{5y}{9}=0$

- Characteristic polynomial of ODE

  $r^2+\frac{2}{3}r+\frac{5}{9}=0$

- Use quadratic formula to solve for r

$r = \frac{\left(-\frac{2}{3}\right)\pm\left(\sqrt{-\frac{16}{9}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{3}-\frac{2\,\mathrm{I}}{3}, -\frac{1}{3}+\frac{2\,\mathrm{I}}{3}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{3}}\cos\left(\frac{2x}{3}\right)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{3}}\sin\left(\frac{2x}{3}\right)$

- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x)$

- Substitute in solutions

  $y = c_1\mathrm{e}^{-\frac{x}{3}}\cos\left(\frac{2x}{3}\right) + c_2\mathrm{e}^{-\frac{x}{3}}\sin\left(\frac{2x}{3}\right)$

☐ Check validity of solution $y = c_1\mathrm{e}^{-\frac{x}{3}}\cos\left(\frac{2x}{3}\right) + c_2\mathrm{e}^{-\frac{x}{3}}\sin\left(\frac{2x}{3}\right)$

  - Use initial condition $y(0) = 6$

    $6 = c_1$

  - Compute derivative of the solution

    $y' = -\frac{c_1\mathrm{e}^{-\frac{x}{3}}\cos\left(\frac{2x}{3}\right)}{3} - \frac{2c_1\mathrm{e}^{-\frac{x}{3}}\sin\left(\frac{2x}{3}\right)}{3} - \frac{c_2\mathrm{e}^{-\frac{x}{3}}\sin\left(\frac{2x}{3}\right)}{3} + \frac{2c_2\mathrm{e}^{-\frac{x}{3}}\cos\left(\frac{2x}{3}\right)}{3}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 0$

    $0 = -\frac{c_1}{3} + \frac{2c_2}{3}$

  - Solve for c_1 and c_2

    $\{c_1 = 6, c_2 = 3\}$

  - Substitute constant values into general solution and simplify

    $y = 3\,\mathrm{e}^{-\frac{x}{3}}\left(2\cos\left(\frac{2x}{3}\right) + \sin\left(\frac{2x}{3}\right)\right)$

- Solution to the IVP

  $y = 3\,\mathrm{e}^{-\frac{x}{3}}\left(2\cos\left(\frac{2x}{3}\right) + \sin\left(\frac{2x}{3}\right)\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([9*diff(y(x),x$2)+6*diff(y(x),x)+5*y(x)=0,y(0) = 6, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = 3\,\mathrm{e}^{-\frac{x}{3}}\left(\sin\left(\frac{2x}{3}\right)+2\cos\left(\frac{2x}{3}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 29

```
DSolve[{9*y''[x]+6*y'[x]+5*y[x]==0,{y[0]==6,y'[0]==0}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x)\to 3e^{-x/3}\left(\sin\left(\frac{2x}{3}\right)+2\cos\left(\frac{2x}{3}\right)\right)$$

10.38 problem 38

Internal problem ID [11768]
Internal file name [`OUTPUT/11777_Thursday_April_11_2024_08_49_34_PM_8249898/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{4y'' + 4y' + 37y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = -4]$$

10.38.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = 1$$
$$q(x) = \frac{37}{4}$$
$$F = 0$$

Hence the ode is

$$y'' + y' + \frac{37y}{4} = 0$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{37}{4}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.38.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = 4, C = 37$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 37\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 + 4\lambda + 37 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 4, C = 37$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{4^2 - (4)(4)(37)}\\ &= -\frac{1}{2} \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + 3i\\ \lambda_2 &= -\frac{1}{2} - 3i\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + 3i$$
$$\lambda_2 = -\frac{1}{2} - 3i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}(c_1 \cos(3x) + c_2 \sin(3x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-\frac{x}{2}}(c_1 \cos(3x) + c_2 \sin(3x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\mathrm{e}^{-\frac{x}{2}}(c_1 \cos(3x) + c_2 \sin(3x))}{2} + \mathrm{e}^{-\frac{x}{2}}(-3c_1 \sin(3x) + 3c_2 \cos(3x))$$

substituting $y' = -4$ and $x = 0$ in the above gives

$$-4 = -\frac{c_1}{2} + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -1$$

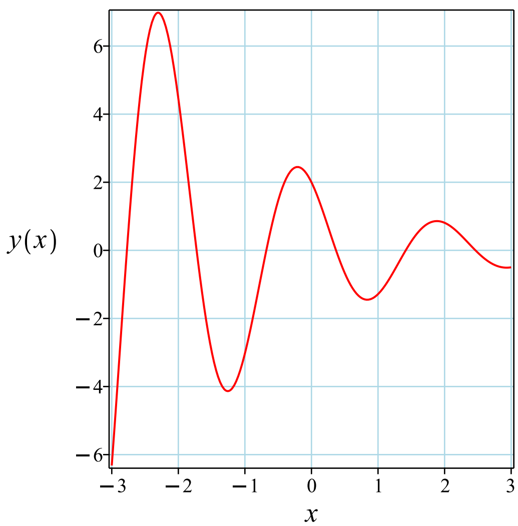
Substituting these values back in above solution results in

$$y = e^{-\frac{x}{2}}(2\cos(3x) - \sin(3x))$$

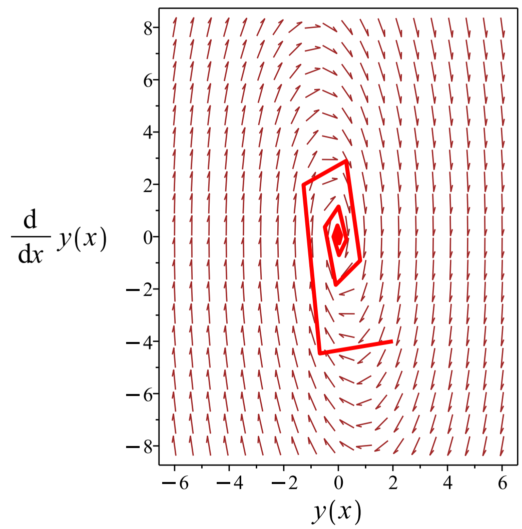
Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{2}}(2\cos(3x) - \sin(3x)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}}(2\cos(3x) - \sin(3x))$$

Verified OK.

10.38.3 Solving using Kovacic algorithm

Writing the ode as

$$4y'' + 4y' + 37y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 4$$
$$B = 4 \tag{3}$$
$$C = 37$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 337: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{4}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{4} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos(3x)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos(3x) \left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos(3x) + \frac{c_2 \mathrm{e}^{-\frac{x}{2}} \sin(3x)}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1 \mathrm{e}^{-\frac{x}{2}} \cos(3x)}{2} - 3c_1 \mathrm{e}^{-\frac{x}{2}} \sin(3x) - \frac{c_2 \mathrm{e}^{-\frac{x}{2}} \sin(3x)}{6} + c_2 \mathrm{e}^{-\frac{x}{2}} \cos(3x)$$

substituting $y' = -4$ and $x = 0$ in the above gives

$$-4 = -\frac{c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -3$$

Substituting these values back in above solution results in

$$y = 2\,\mathrm{e}^{-\frac{x}{2}} \cos(3x) - \mathrm{e}^{-\frac{x}{2}} \sin(3x)$$

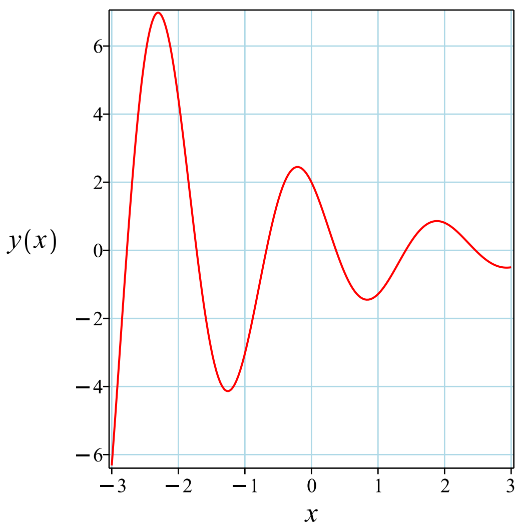
Which simplifies to

$$y = \mathrm{e}^{-\frac{x}{2}} (2\cos(3x) - \sin(3x))$$

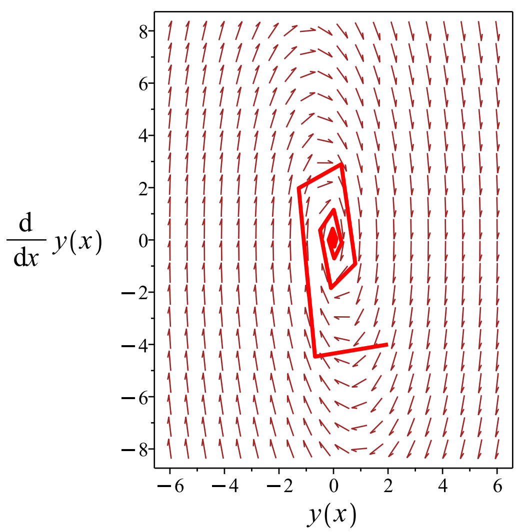
Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{x}{2}} (2\cos(3x) - \sin(3x)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{x}{2}} (2\cos(3x) - \sin(3x))$$

Verified OK.

10.38.4 Maple step by step solution

Let's solve

$$\left[4y'' + 4y' + 37y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = -4\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' - \frac{37y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' + \frac{37y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 + r + \frac{37}{4} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - 3\,\mathrm{I}, -\frac{1}{2} + 3\,\mathrm{I}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos(3x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos(3x) + c_2 \mathrm{e}^{-\frac{x}{2}} \sin(3x)$

□ Check validity of solution $y = c_1 \mathrm{e}^{-\frac{x}{2}} \cos(3x) + c_2 \mathrm{e}^{-\frac{x}{2}} \sin(3x)$

  ◦ Use initial condition $y(0) = 2$

    $2 = c_1$

  ◦ Compute derivative of the solution

    $y' = -\frac{c_1 \mathrm{e}^{-\frac{x}{2}} \cos(3x)}{2} - 3c_1 \mathrm{e}^{-\frac{x}{2}} \sin(3x) - \frac{c_2 \mathrm{e}^{-\frac{x}{2}} \sin(3x)}{2} + 3c_2 \mathrm{e}^{-\frac{x}{2}} \cos(3x)$

- Use the initial condition $y'\Big|_{\{x=0\}} = -4$

  $-4 = -\frac{c_1}{2} + 3c_2$
- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = -1\}$
- Substitute constant values into general solution and simplify

  $y = e^{-\frac{x}{2}}(2\cos(3x) - \sin(3x))$

- Solution to the IVP

  $y = e^{-\frac{x}{2}}(2\cos(3x) - \sin(3x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve([4*diff(y(x),x$2)+4*diff(y(x),x)+37*y(x)=0,y(0) = 2, D(y)(0) = -4],y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}}(-\sin(3x) + 2\cos(3x))$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 26

```
DSolve[{4*y''[x]+4*y'[x]+37*y[x]==0,{y[0]==2,y'[0]==-4}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to e^{-x/2}(2\cos(3x) - \sin(3x))$$

10.39 problem 39

Internal problem ID [11769]
Internal file name [`OUTPUT/11778_Thursday_April_11_2024_08_49_35_PM_59199865/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 39.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 6y'' + 11y' - 6y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0, y''(0) = 2]$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + c_2 e^{2x} + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{x} \\
y_2 &= e^{2x} \\
y_3 &= e^{3x}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = e^{x} c_1 + c_2 e^{2x} + c_3 e^{3x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 + c_2 + c_3 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = e^{x} c_1 + 2 c_2 e^{2x} + 3 c_3 e^{3x}
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = c_1 + 2c_2 + 3c_3 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = e^{x} c_1 + 4 c_2 e^{2x} + 9 c_3 e^{3x}
$$

substituting $y'' = 2$ and $x = 0$ in the above gives

$$
2 = c_1 + 4c_2 + 9c_3 \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= -2 \\
c_3 &= 1
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = e^{x} - 2\, e^{2x} + e^{3x}
$$

Summary
The solution(s) found are the following

$$y = e^{x} - 2\,e^{2x} + e^{3x} \tag{1}$$

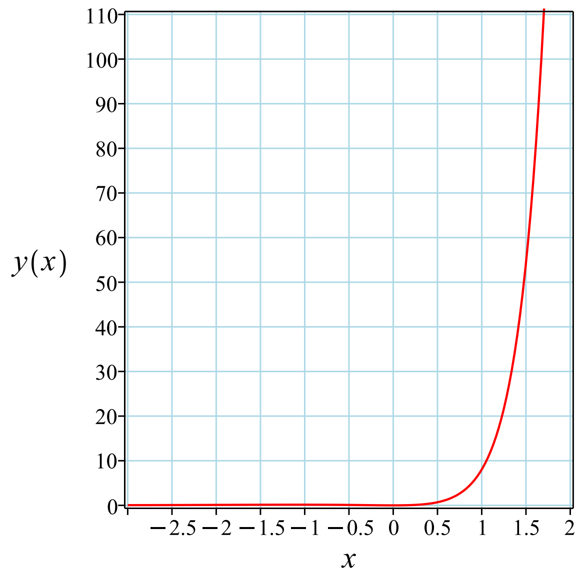


Figure 413: Solution plot

Verification of solutions

$$y = e^{x} - 2\,e^{2x} + e^{3x}$$

Verified OK.

10.39.1 Maple step by step solution

Let's solve

$$\left[y''' - 6y'' + 11y' - 6y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 0, y''\Big|_{\{x=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

$y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 6y_3(x) - 11y_2(x) + 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 6y_3(x) - 11y_2(x) + 6y_1(x)]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{array}\right] \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{array}\right]$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right], \left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{x} \cdot \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c}\frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{2x} \cdot \left[\begin{array}{c}\frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c}\frac{1}{9} \\ \frac{1}{3} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{3x} \cdot \left[\begin{array}{c}\frac{1}{9} \\ \frac{1}{3} \\ 1\end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = e^x c_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = e^x c_1 + \frac{c_2 e^{2x}}{4} + \frac{c_3 e^{3x}}{9}$
- Use the initial condition $y(0) = 0$

  $0 = c_1 + \frac{c_2}{4} + \frac{c_3}{9}$
- Calculate the 1st derivative of the solution

  $y' = e^x c_1 + \frac{c_2 e^{2x}}{2} + \frac{c_3 e^{3x}}{3}$
- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = c_1 + \frac{c_2}{2} + \frac{c_3}{3}$
- Calculate the 2nd derivative of the solution

  $y'' = e^x c_1 + c_2 e^{2x} + c_3 e^{3x}$
- Use the initial condition $y''\Big|_{\{x=0\}} = 2$

  $2 = c_1 + c_2 + c_3$
- Solve for the unknown coefficients

  $\{c_1 = 1, c_2 = -8, c_3 = 9\}$
- Solution to the IVP

  $y = e^x - 2\,e^{2x} + e^{3x}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$3)-6*diff(y(x),x$2)+11*diff(y(x),x)-6*y(x)=0,y(0) = 0, D(y)(0) = 0, (D@@
```

$$y(x) = e^{x} - 2e^{2x} + e^{3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 16

```
DSolve[{y'''[x]-6*y''[x]+11*y'[x]-6*y[x]==0,{y[0]==0,y'[0]==0,y''[0]==2}},y[x],x,IncludeSing
```

$$y(x) \to e^{x}(e^{x} - 1)^{2}$$

10.40 problem 40

Internal problem ID [11770]
Internal file name [`OUTPUT/11779_Thursday_April_11_2024_08_49_35_PM_64211150/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 40.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 2y'' + 4y' - 8y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0, y''(0) = 0]$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 + 4\lambda - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 2i$$
$$\lambda_3 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + e^{2ix} c_2 + e^{-2ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{2x} \\
y_2 &= \mathrm{e}^{2ix} \\
y_3 &= \mathrm{e}^{-2ix}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{2x} + \mathrm{e}^{2ix} c_2 + \mathrm{e}^{-2ix} c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$
2 = c_1 + c_2 + c_3 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 2c_1 \mathrm{e}^{2x} + 2i\mathrm{e}^{2ix} c_2 - 2i\mathrm{e}^{-2ix} c_3
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = 2c_2 i - 2c_3 i + 2c_1 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = 4c_1 \mathrm{e}^{2x} - 4\,\mathrm{e}^{2ix} c_2 - 4\,\mathrm{e}^{-2ix} c_3
$$

substituting $y'' = 0$ and $x = 0$ in the above gives

$$
0 = 4c_1 - 4c_2 - 4c_3 \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= \frac{1}{2} + \frac{i}{2} \\
c_3 &= \frac{1}{2} - \frac{i}{2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \mathrm{e}^{2x} + \cos(2x) - \sin(2x)
$$

Summary

The solution(s) found are the following

$$y = e^{2x} + \cos(2x) - \sin(2x) \tag{1}$$

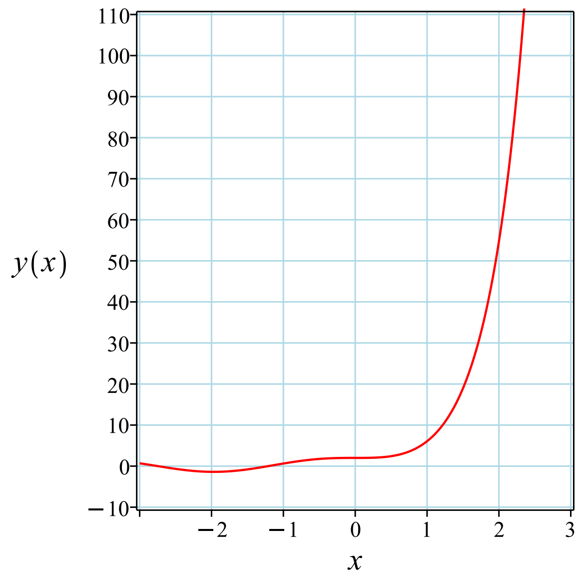


Figure 414: Solution plot

Verification of solutions

$$y = e^{2x} + \cos(2x) - \sin(2x)$$

Verified OK.

10.40.1 Maple step by step solution

Let's solve

$$\left[y''' - 2y'' + 4y' - 8y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = 0, y''\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

$y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 2y_3(x) - 4y_2(x) + 8y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 2y_3(x) - 4y_2(x) + 8y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -4 & 2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -4 & 2 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \left[\begin{array}{c} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}x} \cdot \left[\begin{array}{c} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(2x) - \mathrm{I}\sin(2x)) \cdot \left[\begin{array}{c} -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\frac{\cos(2x)}{4} + \frac{\mathrm{I}\sin(2x)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \left[\begin{array}{c} -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{array}\right], \vec{y}_3(x) = \left[\begin{array}{c} \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{array}\right]\right]$$

- General solution to the system of ODEs

  $\overrightarrow{y}=c_1\overrightarrow{y}_1+c_2\overrightarrow{y}_2(x)+c_3\overrightarrow{y}_3(x)$

- Substitute solutions into the general solution

  $$\overrightarrow{y}=c_1\mathrm{e}^{2x}\cdot\begin{bmatrix}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{bmatrix}+\begin{bmatrix}-\frac{c_2\cos(2x)}{4}+\frac{c_3\sin(2x)}{4}\\ \frac{c_2\sin(2x)}{2}+\frac{c_3\cos(2x)}{2}\\ c_2\cos(2x)-c_3\sin(2x)\end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y=\frac{c_1\mathrm{e}^{2x}}{4}+\frac{c_3\sin(2x)}{4}-\frac{c_2\cos(2x)}{4}$

- Use the initial condition $y(0)=2$

  $2=\frac{c_1}{4}-\frac{c_2}{4}$

- Calculate the 1st derivative of the solution

  $y'=\frac{c_1\mathrm{e}^{2x}}{2}+\frac{c_3\cos(2x)}{2}+\frac{c_2\sin(2x)}{2}$

- Use the initial condition $y'\Big|_{\{x=0\}}=0$

  $0=\frac{c_1}{2}+\frac{c_3}{2}$

- Calculate the 2nd derivative of the solution

  $y''=c_1\mathrm{e}^{2x}-c_3\sin(2x)+c_2\cos(2x)$

- Use the initial condition $y''\Big|_{\{x=0\}}=0$

  $0=c_1+c_2$

- Solve for the unknown coefficients

  $\{c_1=4,c_2=-4,c_3=-4\}$

- Solution to the IVP

  $y=\mathrm{e}^{2x}+\cos(2x)-\sin(2x)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$3)-2*diff(y(x),x$2)+4*diff(y(x),x)-8*y(x)=0,y(0) = 2, D(y)(0) = 0, (D@@2
```

$$y(x) = e^{2x} - \sin(2x) + \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 21

```
DSolve[{y'''[x]-2*y''[x]+4*y'[x]-8*y[x]==0,{y[0]==2,y'[0]==0,y''[0]==0}},y[x],x,IncludeSingu
```

$$y(x) \to e^{2x} - \sin(2x) + \cos(2x)$$

10.41 problem 41

Internal problem ID [11771]
Internal file name [`OUTPUT/11780_Thursday_April_11_2024_08_49_36_PM_34242024/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 41.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 3y'' + 4y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -8, y''(0) = -4]$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x} + x\, e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= \mathrm{e}^{2x} \\
y_3 &= \mathrm{e}^{2x} x
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + x\,\mathrm{e}^{2x} c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-x} + 2c_2 \mathrm{e}^{2x} + \mathrm{e}^{2x} c_3 + 2x\,\mathrm{e}^{2x} c_3
$$

substituting $y' = -8$ and $x = 0$ in the above gives

$$
-8 = -c_1 + 2c_2 + c_3 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = c_1 \mathrm{e}^{-x} + 4c_2 \mathrm{e}^{2x} + 4\,\mathrm{e}^{2x} c_3 + 4x\,\mathrm{e}^{2x} c_3
$$

substituting $y'' = -4$ and $x = 0$ in the above gives

$$
-4 = c_1 + 4c_2 + 4c_3 \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{32}{9} \\
c_2 &= -\frac{23}{9} \\
c_3 &= \frac{2}{3}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{32\,\mathrm{e}^{-x}}{9} - \frac{23\,\mathrm{e}^{2x}}{9} + \frac{2\,\mathrm{e}^{2x} x}{3}
$$

Which simplifies to

$$y = \frac{(6x - 23)\,\mathrm{e}^{2x}}{9} + \frac{32\,\mathrm{e}^{-x}}{9}$$

Summary

The solution(s) found are the following

$$y = \frac{(6x - 23)\,\mathrm{e}^{2x}}{9} + \frac{32\,\mathrm{e}^{-x}}{9} \tag{1}$$

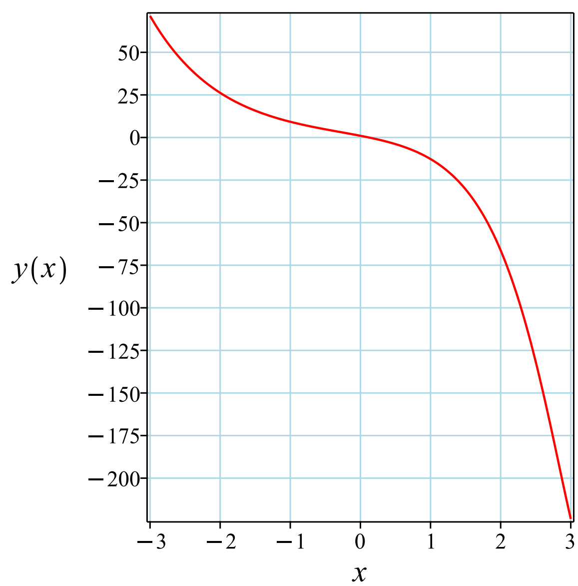


Figure 415: Solution plot

Verification of solutions

$$y = \frac{(6x - 23)\,\mathrm{e}^{2x}}{9} + \frac{32\,\mathrm{e}^{-x}}{9}$$

Verified OK.

10.41.1 Maple step by step solution

Let's solve

$$\left[y''' - 3y'' + 4y = 0, y(0) = 1, y'\Big|_{\{x=0\}} = -8, y''\Big|_{\{x=0\}} = -4\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 3y_3(x) - 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3y_3(x) - 4y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix}$$

- Rewrite the system as

$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right], \left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right], \left[2, \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 2

$$\overrightarrow{y}_2(x) = \mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, an

$\overrightarrow{y}_3(x) = \mathrm{e}^{\lambda x} \left(x \overrightarrow{v} + \overrightarrow{p}\right)$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_3(x)$ into the homogeneous system

$$\lambda \mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v}=\left(\mathrm{e}^{\lambda x}A\right)\cdot\left(x\vec{v}+\vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda \mathrm{e}^{\lambda x}\left(x\vec{v}+\vec{p}\right)+\mathrm{e}^{\lambda x}\vec{v}=\mathrm{e}^{\lambda x}\left(\lambda x\vec{v}+A\cdot\vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_3(x)$ to be a solution to the homogeneous system

$$(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\left[\begin{array}{ccc}0 & 1 & 0\\ 0 & 0 & 1\\ -4 & 0 & 3\end{array}\right]-2\cdot\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]\right)\cdot\vec{p}=\left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p}=\left[\begin{array}{c}-\frac{1}{8}\\ 0\\ 0\end{array}\right]$$

- Second solution from eigenvalue 2

$$\vec{y}_3(x)=\mathrm{e}^{2x}\cdot\left(x\cdot\left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]+\left[\begin{array}{c}-\frac{1}{8}\\ 0\\ 0\end{array}\right]\right)$$

- General solution to the system of ODEs

$$\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_1\mathrm{e}^{-x}\cdot\left[\begin{array}{c}1\\ -1\\ 1\end{array}\right]+c_2\mathrm{e}^{2x}\cdot\left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]+\mathrm{e}^{2x}c_3\cdot\left(x\cdot\left[\begin{array}{c}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{array}\right]+\left[\begin{array}{c}-\frac{1}{8}\\ 0\\ 0\end{array}\right]\right)$$

- First component of the vector is the solution to the ODE

  $y = \frac{((2x-1)c_3+2c_2)e^{2x}}{8} + c_1 e^{-x}$

- Use the initial condition $y(0) = 1$

  $1 = -\frac{c_3}{8} + \frac{c_2}{4} + c_1$

- Calculate the 1st derivative of the solution

  $y' = \frac{e^{2x}c_3}{4} + \frac{((2x-1)c_3+2c_2)e^{2x}}{4} - c_1 e^{-x}$

- Use the initial condition $y'\Big|_{\{x=0\}} = -8$

  $-8 = \frac{c_2}{2} - c_1$

- Calculate the 2nd derivative of the solution

  $y'' = e^{2x}c_3 + \frac{((2x-1)c_3+2c_2)e^{2x}}{2} + c_1 e^{-x}$

- Use the initial condition $y''\Big|_{\{x=0\}} = -4$

  $-4 = \frac{c_3}{2} + c_2 + c_1$

- Solve for the unknown coefficients

  $\left\{c_1 = \frac{32}{9}, c_2 = -\frac{80}{9}, c_3 = \frac{8}{3}\right\}$

- Solution to the IVP

  $y = \frac{(6x-23)e^{2x}}{9} + \frac{32\,e^{-x}}{9}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve([diff(y(x),x$3)-3*diff(y(x),x$2)+4*y(x)=0,y(0) = 1, D(y)(0) = -8, (D@@2)(y)(0) = -4],
```

$$y(x) = \frac{(6x-23)\,e^{2x}}{9} + \frac{32\,e^{-x}}{9}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 27

```
DSolve[{y'''[x]-3*y''[x]+4*y[x]==0,{y[0]==1,y'[0]==-8,y''[0]==-4}},y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{1}{9}e^{-x}\left(e^{3x}(6x-23)+32\right)$$

10.42 problem 42

Internal problem ID [11772]
Internal file name [`OUTPUT/11781_Thursday_April_11_2024_08_49_36_PM_84460581/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 42.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 5y'' + 9y' - 5y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1, y''(0) = 6]$$

The characteristic equation is

$$\lambda^3 - 5\lambda^2 + 9\lambda - 5 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2 - i$$
$$\lambda_3 = 2 + i$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + e^{(2+i)x} c_2 + e^{(2-i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{x} \\
y_2 &= e^{(2+i)x} \\
y_3 &= e^{(2-i)x}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = e^{x} c_1 + e^{(2+i)x} c_2 + e^{(2-i)x} c_3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 + c_2 + c_3 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = e^{x} c_1 + (2+i)\, e^{(2+i)x} c_2 + (2-i)\, e^{(2-i)x} c_3
$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + (2+i)\, c_2 + (2-i)\, c_3 \tag{2A}
$$

Taking two derivatives of the solution gives

$$
y'' = e^{x} c_1 + (3+4i)\, e^{(2+i)x} c_2 + (3-4i)\, e^{(2-i)x} c_3
$$

substituting $y'' = 6$ and $x = 0$ in the above gives

$$
6 = c_1 + (3+4i)\, c_2 + (3-4i)\, c_3 \tag{3A}
$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= -\frac{1}{2} - i \\
c_3 &= -\frac{1}{2} + i
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = e^{x} - \frac{e^{(2+i)x}}{2} - i e^{(2+i)x} - \frac{e^{(2-i)x}}{2} + i e^{(2-i)x}
$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{1}{2}+i\right) \mathrm{e}^{(2-i)x} + \left(-\frac{1}{2}-i\right) \mathrm{e}^{(2+i)x} + \mathrm{e}^{x} \tag{1}$$

Verification of solutions

$$y = \left(-\frac{1}{2}+i\right) \mathrm{e}^{(2-i)x} + \left(-\frac{1}{2}-i\right) \mathrm{e}^{(2+i)x} + \mathrm{e}^{x}$$

Verified OK.

10.42.1 Maple step by step solution

Let's solve

$\left[y''' - 5y'' + 9y' - 5y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 1, y''\Big|_{\{x=0\}} = 6\right]$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 5y_3(x) - 9y_2(x) + 5y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 5y_3(x) - 9y_2(x) + 5y_1(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -9 & 5 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -9 & 5 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2-\mathrm{I}, \begin{bmatrix} \frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ \frac{2}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right], \left[2+\mathrm{I}, \begin{bmatrix} \frac{3}{25}-\frac{4\,\mathrm{I}}{25} \\ \frac{2}{5}-\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2-\mathrm{I}, \begin{bmatrix} \frac{3}{25}+\frac{4\,\mathrm{I}}{25} \\ \frac{2}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(2-I)x} \cdot \begin{bmatrix} \frac{3}{25}+\frac{4\,I}{25} \\ \frac{2}{5}+\frac{I}{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{2x} \cdot (\cos(x) - I\sin(x)) \cdot \begin{bmatrix} \frac{3}{25}+\frac{4\,I}{25} \\ \frac{2}{5}+\frac{I}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{2x} \cdot \begin{bmatrix} \left(\frac{3}{25}+\frac{4\,I}{25}\right)(\cos(x) - I\sin(x)) \\ \left(\frac{2}{5}+\frac{I}{5}\right)(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{2x} \cdot \begin{bmatrix} \frac{3\cos(x)}{25}+\frac{4\sin(x)}{25} \\ \frac{2\cos(x)}{5}+\frac{\sin(x)}{5} \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = e^{2x} \cdot \begin{bmatrix} -\frac{3\sin(x)}{25}+\frac{4\cos(x)}{25} \\ -\frac{2\sin(x)}{5}+\frac{\cos(x)}{5} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = e^{x}c_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2x} \cdot \begin{bmatrix} \frac{3\cos(x)}{25}+\frac{4\sin(x)}{25} \\ \frac{2\cos(x)}{5}+\frac{\sin(x)}{5} \\ \cos(x) \end{bmatrix} + e^{2x}c_3 \cdot \begin{bmatrix} -\frac{3\sin(x)}{25}+\frac{4\cos(x)}{25} \\ -\frac{2\sin(x)}{5}+\frac{\cos(x)}{5} \\ -\sin(x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left((3c_2+4c_3)\cos(x)+4\left(c_2-\frac{3c_3}{4}\right)\sin(x)\right)e^{2x}}{25} + e^{x}c_1$$

- Use the initial condition $y(0) = 0$

$$0 = \frac{3c_2}{25} + \frac{4c_3}{25} + c_1$$

- Calculate the 1st derivative of the solution

$$y' = \frac{\left(-(3c_2+4c_3)\sin(x)+4\left(c_2-\frac{3c_3}{4}\right)\cos(x)\right)e^{2x}}{25} + \frac{2\left((3c_2+4c_3)\cos(x)+4\left(c_2-\frac{3c_3}{4}\right)\sin(x)\right)e^{2x}}{25} + e^{x}c_1$$

- Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = \frac{2c_2}{5} + \frac{c_3}{5} + c_1$

- Calculate the 2nd derivative of the solution

  $y'' = \frac{\left(-(3c_2+4c_3)\cos(x)-4\left(c_2-\frac{3c_3}{4}\right)\sin(x)\right)\mathrm{e}^{2x}}{25} + \frac{4\left(-(3c_2+4c_3)\sin(x)+4\left(c_2-\frac{3c_3}{4}\right)\cos(x)\right)\mathrm{e}^{2x}}{25} + \frac{4\left((3c_2+4c_3)\cos(x)+4\left(c\right.\right.}{25}$

- Use the initial condition $y''\Big|_{\{x=0\}} = 6$

  $6 = c_1 + c_2$

- Solve for the unknown coefficients

  $\{c_1 = 1, c_2 = 5, c_3 = -10\}$

- Solution to the IVP

  $y = (-\cos(x) + 2\sin(x))\,\mathrm{e}^{2x} + \mathrm{e}^{x}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$3)-5*diff(y(x),x$2)+9*diff(y(x),x)-5*y(x)=0,y(0) = 0, D(y)(0) = 1, (D@@2
```

$$y(x) = \mathrm{e}^{x} + (2\sin(x) - \cos(x))\,\mathrm{e}^{2x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 25

```
DSolve[{y'''[x]-5*y''[x]+9*y'[x]-5*y[x]==0,{y[0]==0,y'[0]==1,y''[0]==6}},y[x],x,IncludeSingu
```

$$y(x) \to e^{x}(2e^{x}\sin(x) - e^{x}\cos(x) + 1)$$

10.43 problem 45

Internal problem ID [11773]
Internal file name [`OUTPUT/11782_Thursday_April_11_2024_08_49_37_PM_10735052/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 45.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 2y''' + 6y'' + 2y' + 5y = 0}$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 + 6\lambda^2 + 2\lambda + 5 = 0$$

The roots of the above equation are

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = -1 - 2i$$
$$\lambda_4 = -1 + 2i$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{(-1-2i)x} c_1 + \mathrm{e}^{ix} c_2 + \mathrm{e}^{(-1+2i)x} c_3 + \mathrm{e}^{-ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{(-1-2i)x} \\
y_2 &= \mathrm{e}^{ix} \\
y_3 &= \mathrm{e}^{(-1+2i)x} \\
y_4 &= \mathrm{e}^{-ix}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \mathrm{e}^{(-1-2i)x} c_1 + \mathrm{e}^{ix} c_2 + \mathrm{e}^{(-1+2i)x} c_3 + \mathrm{e}^{-ix} c_4 \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{(-1-2i)x} c_1 + \mathrm{e}^{ix} c_2 + \mathrm{e}^{(-1+2i)x} c_3 + \mathrm{e}^{-ix} c_4
$$

Verified OK.

10.43.1 Maple step by step solution

Let's solve

$y'''' + 2y''' + 6y'' + 2y' + 5y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -2y_4(x) - 6y_3(x) - 2y_2(x) - 5y_1(x)$

  Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -2y_4(x) - 6y_3(x) - 2y_2(x) - 5y_1(x)]$$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -2 & -6 & -2 \end{array}\right] \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -2 & -6 & -2 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}, \left[\begin{array}{c} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{array}\right]\right], \left[-1-2\,\mathrm{I}, \left[\begin{array}{c} \frac{11}{125} - \frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25} - \frac{4\,\mathrm{I}}{25} \\ -\frac{1}{5} + \frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right], \left[-1+2\,\mathrm{I}, \left[\begin{array}{c} \frac{11}{125} + \frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ -\frac{1}{5} - \frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \left[\begin{array}{c} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}x} \cdot \left[\begin{array}{c} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \left[\begin{array}{c} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = \left[\begin{array}{c} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{array}\right], \vec{y}_2(x) = \left[\begin{array}{c} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{array}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-2\mathrm{I}, \left[\begin{array}{c} \frac{11}{125}-\frac{2\mathrm{I}}{125} \\ -\frac{3}{25}-\frac{4\mathrm{I}}{25} \\ -\frac{1}{5}+\frac{2\mathrm{I}}{5} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-2\mathrm{I})x} \cdot \left[\begin{array}{c} \frac{11}{125}-\frac{2\mathrm{I}}{125} \\ -\frac{3}{25}-\frac{4\mathrm{I}}{25} \\ -\frac{1}{5}+\frac{2\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-x} \cdot (\cos(2x) - \mathrm{I}\sin(2x)) \cdot \left[\begin{array}{c} \frac{11}{125}-\frac{2\mathrm{I}}{125} \\ -\frac{3}{25}-\frac{4\mathrm{I}}{25} \\ -\frac{1}{5}+\frac{2\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-x} \cdot \left[\begin{array}{c} \left(\frac{11}{125}-\frac{2\mathrm{I}}{125}\right)(\cos(2x)-\mathrm{I}\sin(2x)) \\ \left(-\frac{3}{25}-\frac{4\mathrm{I}}{25}\right)(\cos(2x)-\mathrm{I}\sin(2x)) \\ \left(-\frac{1}{5}+\frac{2\mathrm{I}}{5}\right)(\cos(2x)-\mathrm{I}\sin(2x)) \\ \cos(2x)-\mathrm{I}\sin(2x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} \frac{11\cos(2x)}{125}-\frac{2\sin(2x)}{125} \\ -\frac{3\cos(2x)}{25}-\frac{4\sin(2x)}{25} \\ -\frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5} \\ \cos(2x) \end{array}\right], \vec{y}_4(x) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -\frac{11\sin(2x)}{125}-\frac{2\cos(2x)}{125} \\ \frac{3\sin(2x)}{25}-\frac{4\cos(2x)}{25} \\ \frac{\sin(2x)}{5}+\frac{2\cos(2x)}{5} \\ -\sin(2x) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_3 \mathrm{e}^{-x} \cdot \begin{bmatrix} \frac{11\cos(2x)}{125} - \frac{2\sin(2x)}{125} \\ -\frac{3\cos(2x)}{25} - \frac{4\sin(2x)}{25} \\ -\frac{\cos(2x)}{5} + \frac{2\sin(2x)}{5} \\ \cos(2x) \end{bmatrix} + c_4 \mathrm{e}^{-x} \cdot \begin{bmatrix} -\frac{11\sin(2x)}{125} - \frac{2\cos(2x)}{125} \\ \frac{3\sin(2x)}{25} - \frac{4\cos(2x)}{25} \\ \frac{\sin(2x)}{5} + \frac{2\cos(2x)}{5} \\ -\sin(2x) \end{bmatrix} + \begin{bmatrix} -\sin(x)\, c_1 - c_2 \cos(\\ -c_1 \cos(x) + c_2 \sin(\\ \sin(x)\, c_1 + c_2 \cos(x \\ c_1 \cos(x) - c_2 \sin(x \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left((22c_3 - 4c_4)\cos(x)^2 - 4\left(c_3 + \frac{11c_4}{2}\right)\sin(x)\cos(x) - 11c_3 + 2c_4\right)\mathrm{e}^{-x}}{125} - \sin(x)\, c_1 - c_2 \cos(x)$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$3)+6*diff(y(x),x$2)+2*diff(y(x),x)+5*y(x)=0,y(x), singso
```

$$y(x) = \left(2c_3 \cos(x)\sin(x) + 2c_4 \cos(x)^2 - c_4\right)\mathrm{e}^{-x} + c_1 \sin(x) + c_2 \cos(x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 40

```
DSolve[y''''[x]+2*y'''[x]+6*y''[x]+2*y'[x]+5*y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to c_3 \cos(x) + e^{-x}(c_2 \cos(2x) + c_4 e^x \sin(x) + c_1 \sin(2x))$$

10.44 problem 46

Internal problem ID [11774]
Internal file name [`OUTPUT/11783_Thursday_April_11_2024_08_49_37_PM_9949308/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.2. The homogeneous linear equation with constant coefficients. Exercises page 135
Problem number: 46.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 3y''' + y'' + 13y' + 30y = 0}$$

The characteristic equation is

$$\lambda^4 + 3\lambda^3 + \lambda^2 + 13\lambda + 30 = 0$$

The roots of the above equation are

$$\lambda_1 = -3$$
$$\lambda_2 = -2$$
$$\lambda_3 = 1 - 2i$$
$$\lambda_4 = 1 + 2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-3x} + \mathrm{e}^{(1+2i)x} c_3 + \mathrm{e}^{(1-2i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{-2x} \\
y_2 &= e^{-3x} \\
y_3 &= e^{(1+2i)x} \\
y_4 &= e^{(1-2i)x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{-3x} + e^{(1+2i)x} c_3 + e^{(1-2i)x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-3x} + e^{(1+2i)x} c_3 + e^{(1-2i)x} c_4$$

Verified OK.

10.44.1 Maple step by step solution

Let's solve

$y'''' + 3y''' + y'' + 13y' + 30y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -3y_4(x) - y_3(x) - 13y_2(x) - 30y_1(x)$

  Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -3y_4(x) - y_3(x) - 13y_2(x) - 30y_1(x)]$$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -30 & -13 & -1 & -3 \end{array}\right] \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -30 & -13 & -1 & -3 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \left[\begin{array}{c} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]\right], \left[-2, \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right], \left[1 - 2\,\mathrm{I}, \left[\begin{array}{c} -\frac{11}{125} - \frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ \frac{1}{5} + \frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right], \left[1 + 2\,\mathrm{I}, \left[\begin{array}{c} -\frac{11}{125} + \frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25} - \frac{4\,\mathrm{I}}{25} \\ \frac{1}{5} - \frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-3, \left[\begin{array}{c} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-3x} \cdot \left[\begin{array}{c} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{-2x} \cdot \left[\begin{array}{c} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1 - 2\,\mathrm{I}, \left[\begin{array}{c} -\frac{11}{125} - \frac{2\,\mathrm{I}}{125} \\ -\frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ \frac{1}{5} + \frac{2\,\mathrm{I}}{5} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{(1-2I)x} \cdot \begin{bmatrix} -\frac{11}{125}-\frac{2I}{125} \\ -\frac{3}{25}+\frac{4I}{25} \\ \frac{1}{5}+\frac{2I}{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{x} \cdot (\cos(2x) - I\sin(2x)) \cdot \begin{bmatrix} -\frac{11}{125}-\frac{2I}{125} \\ -\frac{3}{25}+\frac{4I}{25} \\ \frac{1}{5}+\frac{2I}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{x} \cdot \begin{bmatrix} \left(-\frac{11}{125}-\frac{2I}{125}\right)(\cos(2x) - I\sin(2x)) \\ \left(-\frac{3}{25}+\frac{4I}{25}\right)(\cos(2x) - I\sin(2x)) \\ \left(\frac{1}{5}+\frac{2I}{5}\right)(\cos(2x) - I\sin(2x)) \\ \cos(2x) - I\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{x} \cdot \begin{bmatrix} -\frac{11\cos(2x)}{125}-\frac{2\sin(2x)}{125} \\ -\frac{3\cos(2x)}{25}+\frac{4\sin(2x)}{25} \\ \frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = e^{x} \cdot \begin{bmatrix} \frac{11\sin(2x)}{125}-\frac{2\cos(2x)}{125} \\ \frac{3\sin(2x)}{25}+\frac{4\cos(2x)}{25} \\ -\frac{\sin(2x)}{5}+\frac{2\cos(2x)}{5} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-3x} \cdot \begin{bmatrix} -\frac{1}{27} \\ \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + c_2 e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^{x} \cdot \begin{bmatrix} -\frac{11\cos(2x)}{125}-\frac{2\sin(2x)}{125} \\ -\frac{3\cos(2x)}{25}+\frac{4\sin(2x)}{25} \\ \frac{\cos(2x)}{5}+\frac{2\sin(2x)}{5} \\ \cos(2x) \end{bmatrix} + e^{x} c_4 \cdot \begin{bmatrix} \frac{11\sin(2x)}{125} - \\ \frac{3\sin(2x)}{25} + \\ -\frac{\sin(2x)}{5} + \\ -\sin(\end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{2\,\mathrm{e}^{-3x} \left(\left(\left(\frac{11c_3}{2}+c_4\right)\cos(2x)+\sin(2x)\left(c_3-\frac{11c_4}{2}\right)\right)\mathrm{e}^{4x}+\frac{125c_2\,\mathrm{e}^{x}}{16}+\frac{125c_1}{54}\right)}{125}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$4)+3*diff(y(x),x$3)+diff(y(x),x$2)+13*diff(y(x),x)+30*y(x)=0,y(x), singso
```

$$y(x) = \left(c_3\mathrm{e}^{4x}\sin(2x) + c_4\mathrm{e}^{4x}\cos(2x) + c_2\mathrm{e}^{x} + c_1\right)\mathrm{e}^{-3x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 43

```
DSolve[y''''[x]+3*y'''[x]+y''[x]+13*y'[x]+30*y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to c_2 e^{x}\cos(2x) + e^{-3x}\left(c_4 e^{x} + c_1 e^{4x}\sin(2x) + c_3\right)$$

11 Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151

11.1 problem 1

Internal problem ID [11775]
Internal file name [`OUTPUT/11784_Thursday_April_11_2024_08_49_37_PM_56146256/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 3y' + 8y = 4x^2}$$

11.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 8, f(x) = 4x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 8$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda\, e^{\lambda x} + 8\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(8)} \\ &= \frac{3}{2} \pm \frac{i\sqrt{23}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{i\sqrt{23}}{2} \\ \lambda_2 &= \frac{3}{2} - \frac{i\sqrt{23}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{i\sqrt{23}}{2} \\ \lambda_2 &= \frac{3}{2} - \frac{i\sqrt{23}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{3}{2}$ and $\beta = \frac{\sqrt{23}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{\frac{3x}{2}}\left(c_1 \cos\left(\frac{\sqrt{23}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{23}\, x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{\frac{3x}{2}} \left(c_1 \cos\left(\frac{\sqrt{23}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{23}\, x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right), e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{23}\, x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_3 x^2 + 8A_2 x - 6x A_3 + 8A_1 - 3A_2 + 2A_3 = 4x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{64}, A_2 = \frac{3}{8}, A_3 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2} x^2 + \frac{3}{8} x + \frac{1}{64}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{\frac{3x}{2}}\left(c_1 \cos\left(\frac{\sqrt{23}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{23}\, x}{2}\right)\right)\right) + \left(\frac{1}{2}x^2 + \frac{3}{8}x + \frac{1}{64}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = e^{\frac{3x}{2}}\left(c_1 \cos\left(\frac{\sqrt{23}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{23}\, x}{2}\right)\right) + \frac{x^2}{2} + \frac{3x}{8} + \frac{1}{64} \tag{1}
$$

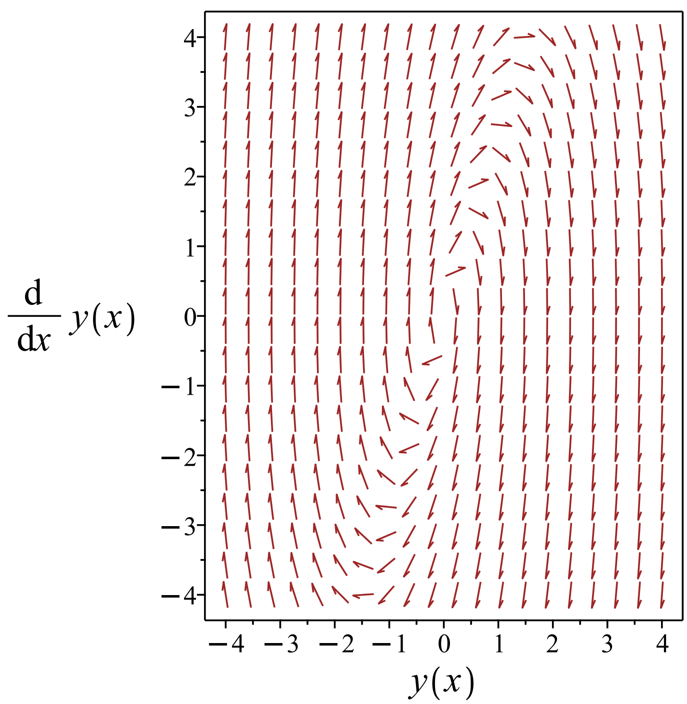


Figure 416: Slope field plot

Verification of solutions

$$
y = e^{\frac{3x}{2}}\left(c_1 \cos\left(\frac{\sqrt{23}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{23}\, x}{2}\right)\right) + \frac{x^2}{2} + \frac{3x}{8} + \frac{1}{64}
$$

Verified OK.

11.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-23}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -23 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{23z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 345: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{23}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{23}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{23}\,\tan\left(\frac{\sqrt{23}\,x}{2}\right)}{23}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\,x}{2}\right) \left(\frac{2\sqrt{23}\,\tan\left(\frac{\sqrt{23}\,x}{2}\right)}{23}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 8y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\,x}{2}\right) c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{23}\,x}{2}\right) \mathrm{e}^{\frac{3x}{2}} \sqrt{23}}{23}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\,x}{2}\right), \frac{2\sin\left(\frac{\sqrt{23}\,x}{2}\right) \mathrm{e}^{\frac{3x}{2}} \sqrt{23}}{23}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_3x^2 + 8A_2x - 6xA_3 + 8A_1 - 3A_2 + 2A_3 = 4x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{64}, A_2 = \frac{3}{8}, A_3 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2}x^2 + \frac{3}{8}x + \frac{1}{64}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\,x}{2}\right) c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{23}\,x}{2}\right) e^{\frac{3x}{2}} \sqrt{23}}{23}\right) + \left(\frac{1}{2}x^2 + \frac{3}{8}x + \frac{1}{64}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\,x}{2}\right) c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{23}\,x}{2}\right) e^{\frac{3x}{2}} \sqrt{23}}{23} + \frac{x^2}{2} + \frac{3x}{8} + \frac{1}{64} \tag{1}$$

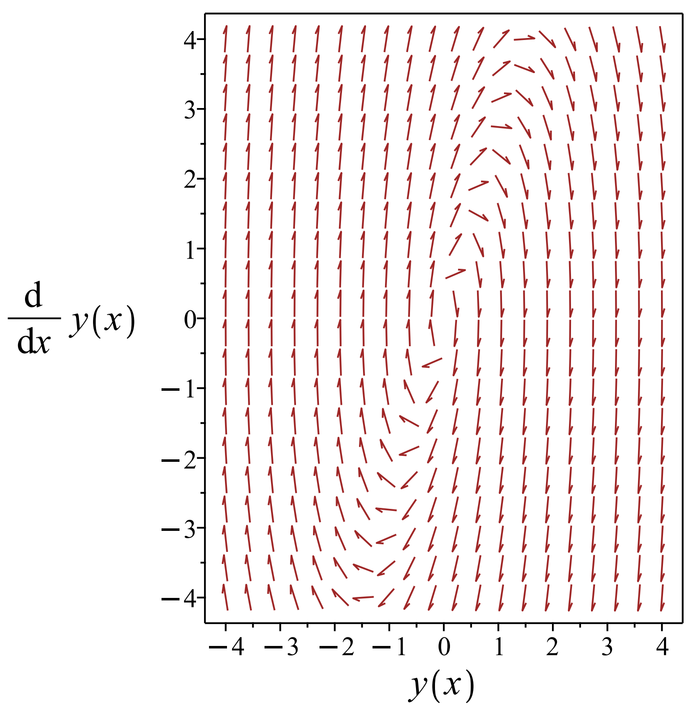


Figure 417: Slope field plot

Verification of solutions

$$y = e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right) c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{23}\, x}{2}\right) e^{\frac{3x}{2}} \sqrt{23}}{23} + \frac{x^2}{2} + \frac{3x}{8} + \frac{1}{64}$$

Verified OK.

11.1.3 Maple step by step solution

Let's solve

$y'' - 3y' + 8y = 4x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 8 = 0$

- Use quadratic formula to solve for r

  $r = \frac{3 \pm (\sqrt{-23})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{3}{2} - \frac{I\sqrt{23}}{2}, \frac{3}{2} + \frac{I\sqrt{23}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{23}\, x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\, x}{2}\right) c_1 + e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{23}\, x}{2}\right) c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4x^2\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\,x}{2}\right) & e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{23}\,x}{2}\right) \\ \frac{3\,e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{23}\,x}{2}\right) e^{\frac{3x}{2}} \sqrt{23}}{2} & \frac{3\,e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{23}\,x}{2}\right)}{2} + \frac{e^{\frac{3x}{2}} \sqrt{23}\cos\left(\frac{\sqrt{23}\,x}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{23}\,e^{3x}}{2}$
- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\frac{8\,e^{\frac{3x}{2}} \sqrt{23}\left(\cos\left(\frac{\sqrt{23}\,x}{2}\right)\left(\int x^2 e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{23}\,x}{2}\right)dx\right) - \sin\left(\frac{\sqrt{23}\,x}{2}\right)\left(\int x^2 e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\,x}{2}\right)dx\right)\right)}{23}$$
- Compute integrals

  $y_p(x) = \frac{1}{2}x^2 + \frac{3}{8}x + \frac{1}{64}$

- Substitute particular solution into general solution to ODE

  $$y = e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\,x}{2}\right) c_1 + e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{23}\,x}{2}\right) c_2 + \frac{x^2}{2} + \frac{3x}{8} + \frac{1}{64}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+8*y(x)=4*x^2,y(x), singsol=all)
```

$$y(x) = e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{23}\,x}{2}\right) c_2 + e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{23}\,x}{2}\right) c_1 + \frac{x^2}{2} + \frac{3x}{8} + \frac{1}{64}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 63

```
DSolve[y''[x]-3*y'[x]+8*y[x]==4*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} + \frac{3x}{8} + c_2 e^{3x/2} \cos\left(\frac{\sqrt{23}x}{2}\right) + c_1 e^{3x/2} \sin\left(\frac{\sqrt{23}x}{2}\right) + \frac{1}{64}$$

11.2 problem 2

Internal problem ID [11776]
Internal file name [`OUTPUT/11785_Thursday_April_11_2024_08_49_38_PM_3825123/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' - 8y = 4\,\mathrm{e}^{2x} - 21\,\mathrm{e}^{-3x}}$$

11.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = -8, f(x) = (4\,\mathrm{e}^{5x} - 21)\,\mathrm{e}^{-3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' - 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -8$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\, \mathrm{e}^{\lambda x} - 8\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(-8)}\\ &= 1 \pm 3\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 3\\ \lambda_2 &= 1 - 3\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(4)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\left(4\,\mathrm{e}^{5x}-21\right)\mathrm{e}^{-3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-3x}\},\{\mathrm{e}^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x},\mathrm{e}^{4x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{-3x} + A_2\mathrm{e}^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$7A_1\mathrm{e}^{-3x} - 8A_2\mathrm{e}^{2x} = \left(4\,\mathrm{e}^{5x}-21\right)\mathrm{e}^{-3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -3, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -3\,\mathrm{e}^{-3x} - \frac{\mathrm{e}^{2x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{4x} + c_2\mathrm{e}^{-2x}\right) + \left(-3\,\mathrm{e}^{-3x} - \frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1\mathrm{e}^{4x} + c_2\mathrm{e}^{-2x} - 3\,\mathrm{e}^{-3x} - \frac{\mathrm{e}^{2x}}{2} \tag{1}$$

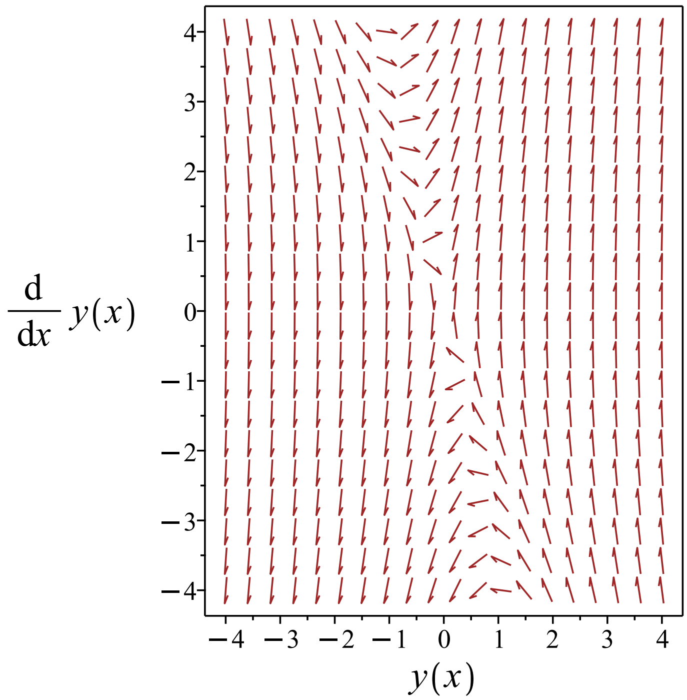


Figure 418: Slope field plot

Verification of solutions

$$y = c_1 e^{4x} + c_2 e^{-2x} - 3\,e^{-3x} - \frac{e^{2x}}{2}$$

Verified OK.

11.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 347: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-3x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2}{1} \, dx} \\ &= z_1 e^{x} \\ &= z_1 (\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{\mathrm{e}^{6x}}{6} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-2x}\right) + c_2 \left(e^{-2x} \left(\frac{e^{6x}}{6}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' - 8y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 e^{-2x} + \frac{c_2 e^{4x}}{6}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\left(4\,e^{5x} - 21\right) e^{-3x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^{-3x}\}, \{e^{2x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{e^{4x}}{6}, e^{-2x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 e^{-3x} + A_2 e^{2x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$7A_1 e^{-3x} - 8A_2 e^{2x} = \left(4\,e^{5x} - 21\right) e^{-3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -3, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -3\,e^{-3x} - \frac{e^{2x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{4x}}{6}\right) + \left(-3\,e^{-3x} - \frac{e^{2x}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{4x}}{6} - 3\,e^{-3x} - \frac{e^{2x}}{2} \tag{1}$$

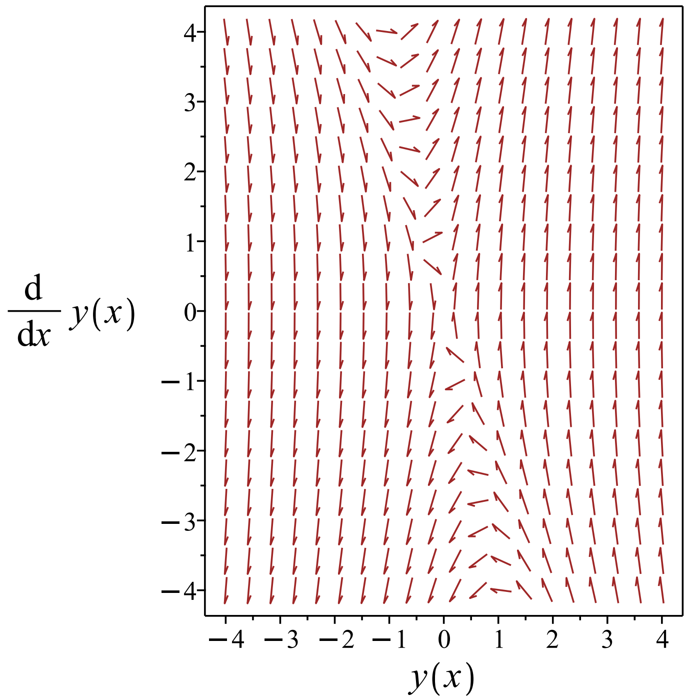


Figure 419: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{4x}}{6} - 3\,e^{-3x} - \frac{e^{2x}}{2}$$

Verified OK.

11.2.3 Maple step by step solution

Let's solve

$y'' - 2y' - 8y = (4\,e^{5x} - 21)\,e^{-3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r - 8 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r - 4) = 0$

- Roots of the characteristic polynomial

$r = (-2, 4)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{4x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = (4\,e^{5x} - 21)\,e^{-3x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{4x} \\ -2\,e^{-2x} & 4\,e^{4x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 6\,e^{2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{e^{-2x}\left(\int (4\,e^{4x} - 21\,e^{-x})dx\right)}{6} + \frac{e^{4x}\left(\int (4\,e^{5x} - 21)e^{-7x}dx\right)}{6}$

  ◦ Compute integrals

  $y_p(x) = -\frac{(e^{5x}+6)e^{-3x}}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{4x} - \frac{(e^{5x}+6)e^{-3x}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-8*y(x)=4*exp(2*x)-21*exp(-3*x),y(x), singsol=all)
```

$$y(x) = \frac{(2c_2 e^{7x} - e^{5x} + 2c_1 e^{x} - 6)\, e^{-3x}}{2}$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 38

```
DSolve[y''[x]-2*y'[x]-8*y[x]==4*Exp[2*x]-21*Exp[-3*x],y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to -\frac{1}{2}e^{-3x}\left(e^{5x}+6\right)+c_1e^{-2x}+c_2e^{4x}$$

11.3 problem 3

Internal problem ID [11777]
Internal file name [`OUTPUT/11786_Thursday_April_11_2024_08_49_38_PM_85345288/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 5y = 6\sin(2x) + 7\cos(2x)}$$

11.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 5, f(x) = 6\sin(2x) + 7\cos(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(5)}\\ &= -1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 2i\\ \lambda_2 &= -1 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 2i\\ \lambda_2 &= -1 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$6 \sin(2x) + 7 \cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x} \cos(2x), e^{-x} \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \cos(2x) + A_2 \sin(2x) - 4A_1 \sin(2x) + 4A_2 \cos(2x) = 6 \sin(2x) + 7 \cos(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(2x) + 2 \sin(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x}(c_1 \cos(2x) + c_2 \sin(2x))\right) + (-\cos(2x) + 2 \sin(2x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) - \cos(2x) + 2 \sin(2x) \tag{1}$$

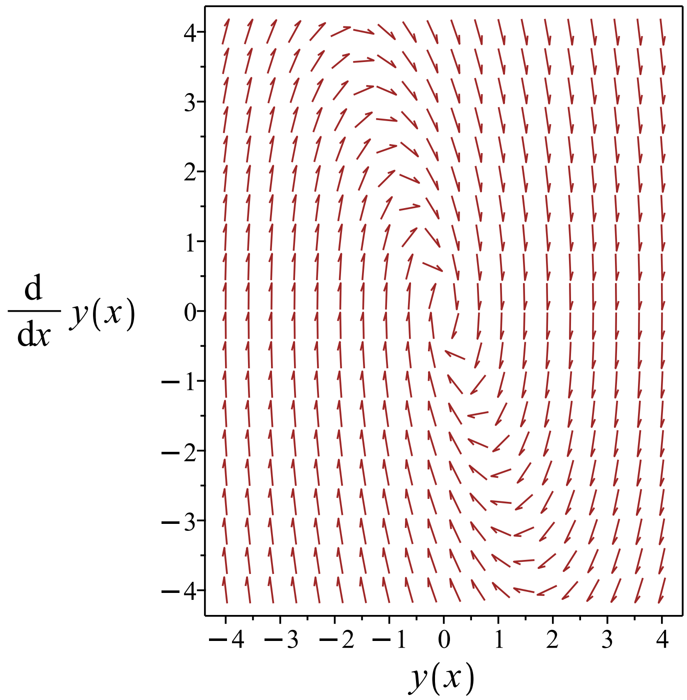


Figure 420: Slope field plot

<u>Verification of solutions</u>

$$y = e^{-x}(c_1 \cos(2x) + c_2 \sin(2x)) - \cos(2x) + 2\sin(2x)$$

Verified OK.

11.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 349: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}\cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\cos\left(2x\right)\right) + c_2\left(\mathrm{e}^{-x}\cos\left(2x\right)\left(\frac{\tan\left(2x\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\mathrm{e}^{-x}\cos\left(2x\right) + \frac{c_2\mathrm{e}^{-x}\sin\left(2x\right)}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
6\sin\left(2x\right) + 7\cos\left(2x\right)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos\left(2x\right), \sin\left(2x\right)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{-x}\cos\left(2x\right), \frac{\mathrm{e}^{-x}\sin\left(2x\right)}{2}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1\cos\left(2x\right) + A_2\sin\left(2x\right)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \cos(2x) + A_2 \sin(2x) - 4A_1 \sin(2x) + 4A_2 \cos(2x) = 6 \sin(2x) + 7 \cos(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(2x) + 2 \sin(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} \cos(2x) + \frac{c_2 e^{-x} \sin(2x)}{2}\right) + (-\cos(2x) + 2 \sin(2x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} \cos(2x) + \frac{c_2 e^{-x} \sin(2x)}{2} - \cos(2x) + 2 \sin(2x) \tag{1}$$

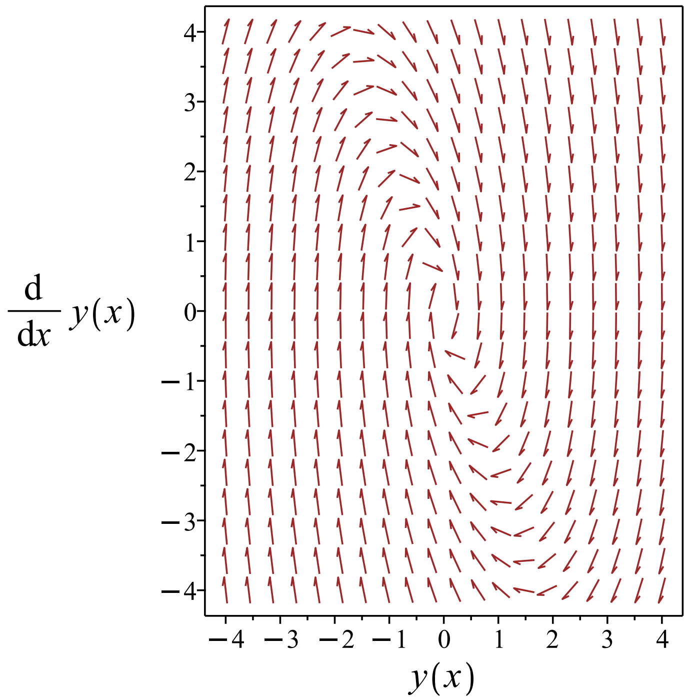


Figure 421: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} \cos(2x) + \frac{c_2 e^{-x} \sin(2x)}{2} - \cos(2x) + 2\sin(2x)$$

Verified OK.

11.3.3 Maple step by step solution

Let's solve

$y'' + 2y' + 5y = 6\sin(2x) + 7\cos(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$r = (-1 - 2\,\mathrm{I}, -1 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x} \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x} \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} \cos(2x) + c_2 \mathrm{e}^{-x} \sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 6 \sin(2x) + 7 \cos(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} \cos(2x) & \mathrm{e}^{-x} \sin(2x) \\ -\mathrm{e}^{-x} \cos(2x) - 2\,\mathrm{e}^{-x} \sin(2x) & -\mathrm{e}^{-x} \sin(2x) + 2\,\mathrm{e}^{-x} \cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{\mathrm{e}^{-x}\left(\cos(2x)\left(\int(-7\sin(4x)-6+6\cos(4x))\mathrm{e}^{x} dx\right)+\sin(2x)\left(\int(7\cos(4x)+7+6\sin(4x))\mathrm{e}^{x} dx\right)\right)}{4}$

  - Compute integrals

    $y_p(x) = -\cos(2x) + 2\sin(2x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} \cos(2x) + c_2 \mathrm{e}^{-x} \sin(2x) - \cos(2x) + 2\sin(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+5*y(x)=6*sin(2*x)+7*cos(2*x),y(x), singsol=all)
```

$$y(x) = (\cos(2x) c_1 + c_2 \sin(2x)) e^{-x} - \cos(2x) + 2\sin(2x)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 38

```
DSolve[y''[x]+2*y'[x]+5*y[x]==6*Sin[2*x]+7*Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}((-e^x + c_2)\cos(2x) + (2e^x + c_1)\sin(2x))$$

11.4 problem 4

Internal problem ID [11778]
Internal file name [`OUTPUT/11787_Thursday_April_11_2024_08_49_39_PM_78115731/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = 10\sin(4x)}$$

11.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 2, f(x) = 10\sin(4x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda \,\mathrm{e}^{\lambda x} + 2 \,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(2)} \\ &= -1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + i \\ \lambda_2 &= -1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + i \\ \lambda_2 &= -1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10 \sin (4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos (4x), \sin (4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x} \cos (x), e^{-x} \sin (x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos (4x) + A_2 \sin (4x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-14A_1 \cos (4x) - 14A_2 \sin (4x) - 8A_1 \sin (4x) + 8A_2 \cos (4x) = 10 \sin (4x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{4}{13}, A_2 = -\frac{7}{13}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{4 \cos (4x)}{13} - \frac{7 \sin (4x)}{13}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x}(c_1 \cos (x) + c_2 \sin (x))\right) + \left(-\frac{4 \cos (4x)}{13} - \frac{7 \sin (4x)}{13}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{4\cos(4x)}{13} - \frac{7\sin(4x)}{13} \tag{1}$$

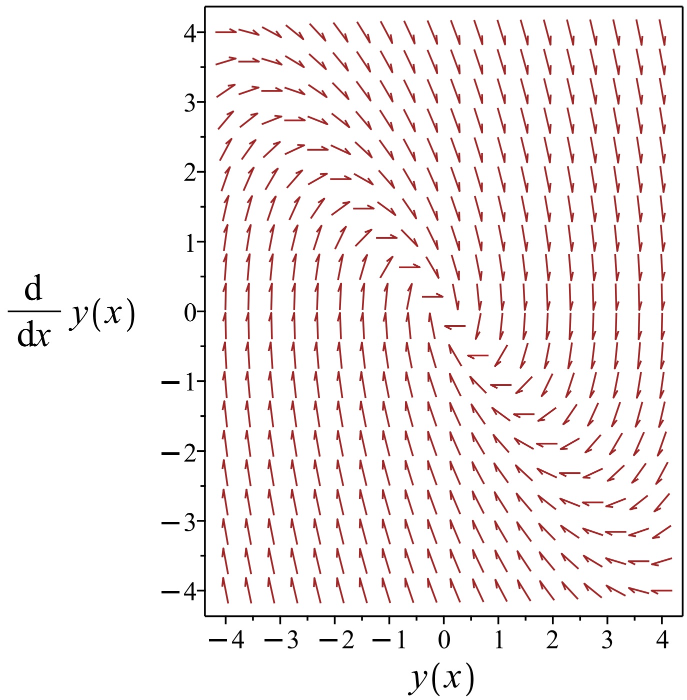


Figure 422: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{4\cos(4x)}{13} - \frac{7\sin(4x)}{13}$$

Verified OK.

11.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 2 \\
C &= 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 351: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x} \cos(x)\right) + c_2\left(\mathrm{e}^{-x} \cos(x) (\tan(x))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\,\mathrm{e}^{-x} c_1 + \sin(x)\,\mathrm{e}^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10 \sin(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}\cos(x), e^{-x}\sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(4x) + A_2 \sin(4x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-14A_1 \cos(4x) - 14A_2 \sin(4x) - 8A_1 \sin(4x) + 8A_2 \cos(4x) = 10\sin(4x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{4}{13}, A_2 = -\frac{7}{13}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{4\cos(4x)}{13} - \frac{7\sin(4x)}{13}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos(x)\, e^{-x} c_1 + \sin(x)\, e^{-x} c_2\right) + \left(-\frac{4\cos(4x)}{13} - \frac{7\sin(4x)}{13}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{4\cos(4x)}{13} - \frac{7\sin(4x)}{13}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{4\cos(4x)}{13} - \frac{7\sin(4x)}{13} \tag{1}$$

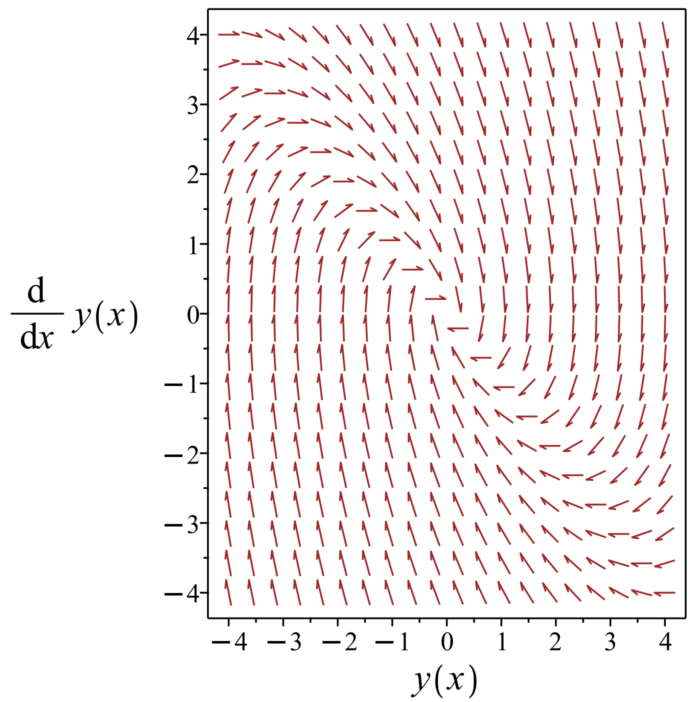


Figure 423: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{4\cos(4x)}{13} - \frac{7\sin(4x)}{13}$$

Verified OK.

11.4.3 Maple step by step solution

Let's solve

$y'' + 2y' + 2y = 10\sin(4x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2+2r+2=0$

- Use quadratic formula to solve for r

  $r=\frac{(-2)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r=(-1-\mathrm{I},-1+\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{-x}\cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\mathrm{e}^{-x}\sin(x)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=\cos(x)\,\mathrm{e}^{-x}c_1+\sin(x)\,\mathrm{e}^{-x}c_2+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=10\sin(4x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x),y_2(x))=\left[\begin{array}{cc}\mathrm{e}^{-x}\cos(x) & \mathrm{e}^{-x}\sin(x)\\ -\mathrm{e}^{-x}\cos(x)-\mathrm{e}^{-x}\sin(x) & -\mathrm{e}^{-x}\sin(x)+\mathrm{e}^{-x}\cos(x)\end{array}\right]$

  - Compute Wronskian

    $W(y_1(x),y_2(x))=\mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x)=-10\,\mathrm{e}^{-x}\left(\cos(x)\left(\int\sin(x)\sin(4x)\,\mathrm{e}^{x}dx\right)-\sin(x)\left(\int\cos(x)\sin(4x)\,\mathrm{e}^{x}dx\right)\right)$

  - Compute integrals

    $y_p(x)=-\frac{4\cos(4x)}{13}-\frac{7\sin(4x)}{13}$

- Substitute particular solution into general solution to ODE

  $y=\cos(x)\,\mathrm{e}^{-x}c_1+\sin(x)\,\mathrm{e}^{-x}c_2-\frac{7\sin(4x)}{13}-\frac{4\cos(4x)}{13}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+2*y(x)=10*sin(4*x),y(x), singsol=all)
```

$$y(x) = e^{-x} \sin(x) c_2 + e^{-x} \cos(x) c_1 - \frac{7 \sin(4x)}{13} - \frac{4 \cos(4x)}{13}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 42

```
DSolve[y''[x]+2*y'[x]+2*y[x]==10*Sin[4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{7}{13} \sin(4x) - \frac{4}{13} \cos(4x) + c_2 e^{-x} \cos(x) + c_1 e^{-x} \sin(x)$$

11.5 problem 5

Internal problem ID [11779]
Internal file name [`OUTPUT/11788_Thursday_April_11_2024_08_49_40_PM_7302965/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 4y = \cos(4x)}$$

11.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 4, f(x) = \cos(4x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(4)} \\ &= -1 \pm i\sqrt{3}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + i\sqrt{3} \\ \lambda_2 &= -1 - i\sqrt{3}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{3} - 1 \\ \lambda_2 &= -i\sqrt{3} - 1\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = \sqrt{3}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}\left(c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x}\left(c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-x} \cos\left(\sqrt{3}\, x\right), e^{-x} \sin\left(\sqrt{3}\, x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(4x) + A_2 \sin(4x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-12A_1 \cos(4x) - 12A_2 \sin(4x) - 8A_1 \sin(4x) + 8A_2 \cos(4x) = \cos(4x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{52}, A_2 = \frac{1}{26}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(4x)}{52} + \frac{\sin(4x)}{26}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x}\left(c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right)\right)\right) + \left(-\frac{3\cos(4x)}{52} + \frac{\sin(4x)}{26}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-x}\left(c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right)\right) - \frac{3\cos(4x)}{52} + \frac{\sin(4x)}{26} \tag{1}$$

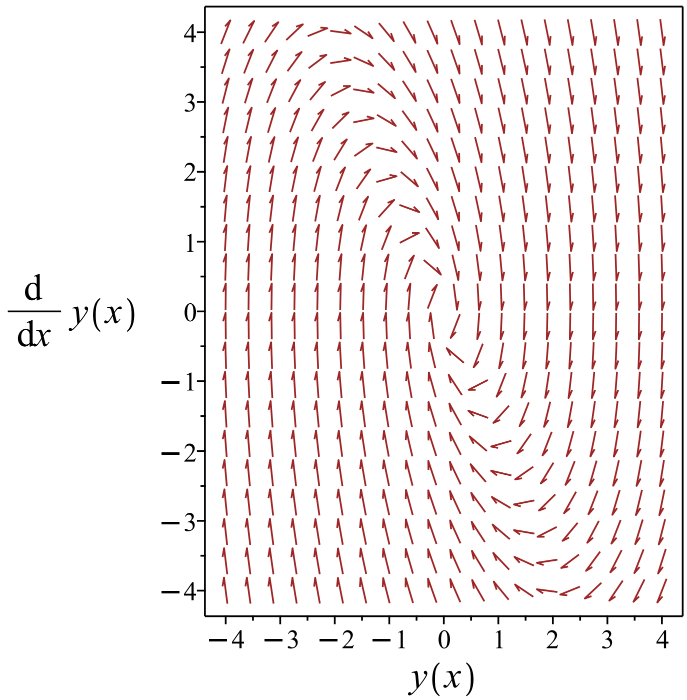


Figure 424: Slope field plot

Verification of solutions

$$y = e^{-x}\left(c_1 \cos\left(\sqrt{3}\, x\right) + c_2 \sin\left(\sqrt{3}\, x\right)\right) - \frac{3\cos(4x)}{52} + \frac{\sin(4x)}{26}$$

Verified OK.

11.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 2 \\
C &= 4
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-3}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -3 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -3z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 353: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -3$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\sqrt{3}\,x\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x} \cos\left(\sqrt{3}\, x\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{3}\, \tan\left(\sqrt{3}\, x\right)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x} \cos\left(\sqrt{3}\, x\right)\right) + c_2 \left(\mathrm{e}^{-x} \cos\left(\sqrt{3}\, x\right) \left(\frac{\sqrt{3}\, \tan\left(\sqrt{3}\, x\right)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-x} \cos\left(\sqrt{3}\, x\right) c_1 + \frac{c_2 \sin\left(\sqrt{3}\, x\right) \mathrm{e}^{-x} \sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-x} \cos\left(\sqrt{3}\, x\right), \frac{\sin\left(\sqrt{3}\, x\right) e^{-x} \sqrt{3}}{3} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(4x) + A_2 \sin(4x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-12A_1 \cos(4x) - 12A_2 \sin(4x) - 8A_1 \sin(4x) + 8A_2 \cos(4x) = \cos(4x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{52}, A_2 = \frac{1}{26}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(4x)}{52} + \frac{\sin(4x)}{26}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x} \cos\left(\sqrt{3}\, x\right) c_1 + \frac{c_2 \sin\left(\sqrt{3}\, x\right) e^{-x} \sqrt{3}}{3} \right) + \left(-\frac{3\cos(4x)}{52} + \frac{\sin(4x)}{26} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-x} \cos\left(\sqrt{3}\, x\right) c_1 + \frac{c_2 \sin\left(\sqrt{3}\, x\right) e^{-x} \sqrt{3}}{3} - \frac{3 \cos(4x)}{52} + \frac{\sin(4x)}{26} \tag{1}$$

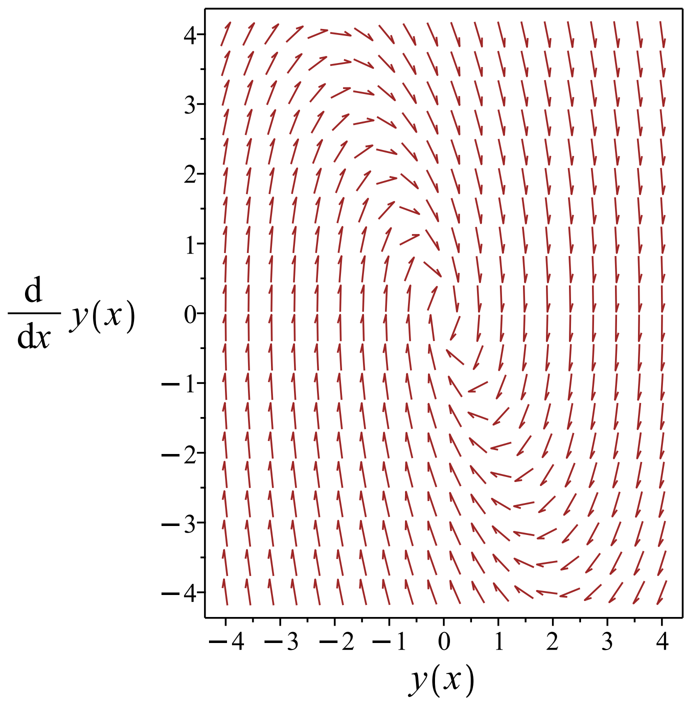


Figure 425: Slope field plot

Verification of solutions

$$y = e^{-x} \cos\left(\sqrt{3}\, x\right) c_1 + \frac{c_2 \sin\left(\sqrt{3}\, x\right) e^{-x} \sqrt{3}}{3} - \frac{3 \cos(4x)}{52} + \frac{\sin(4x)}{26}$$

Verified OK.

11.5.3 Maple step by step solution

Let's solve

$y'' + 2y' + 4y = \cos(4x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 2r + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-12})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\mathrm{I}\sqrt{3}-1, \mathrm{I}\sqrt{3}-1\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}\cos\left(\sqrt{3}\, x\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}\sin\left(\sqrt{3}\, x\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-x}\cos\left(\sqrt{3}\, x\right) c_1 + \mathrm{e}^{-x}\sin\left(\sqrt{3}\, x\right) c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \cos(4x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-x}\cos\left(\sqrt{3}\, x\right) & \mathrm{e}^{-x}\sin\left(\sqrt{3}\, x\right) \\ -\mathrm{e}^{-x}\cos\left(\sqrt{3}\, x\right) - \sin\left(\sqrt{3}\, x\right)\mathrm{e}^{-x}\sqrt{3} & -\mathrm{e}^{-x}\sin\left(\sqrt{3}\, x\right) + \mathrm{e}^{-x}\sqrt{3}\cos\left(\sqrt{3}\, x\right) \end{array}\right.$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \sqrt{3}\,\mathrm{e}^{-2x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\mathrm{e}^{-x}\sqrt{3}\left(\cos\left(\sqrt{3}\, x\right)\left(\int \cos(4x)\mathrm{e}^{x}\sin\left(\sqrt{3}\, x\right)dx\right) - \sin\left(\sqrt{3}\, x\right)\left(\int \cos(4x)\mathrm{e}^{x}\cos\left(\sqrt{3}\, x\right)dx\right)\right)}{3}$

- ○ Compute integrals

  $y_p(x) = -\frac{3\cos(4x)}{52} + \frac{\sin(4x)}{26}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-x}\cos\left(\sqrt{3}\, x\right) c_1 + \mathrm{e}^{-x}\sin\left(\sqrt{3}\, x\right) c_2 - \frac{3\cos(4x)}{52} + \frac{\sin(4x)}{26}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+4*y(x)=cos(4*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x} \sin\left(\sqrt{3}\, x\right) c_2 + \mathrm{e}^{-x} \cos\left(\sqrt{3}\, x\right) c_1 + \frac{\sin(4x)}{26} - \frac{3\cos(4x)}{52}$$

✓ Solution by Mathematica

Time used: 1.15 (sec). Leaf size: 54

```
DSolve[y''[x]+2*y'[x]+4*y[x]==Cos[4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{26} \sin(4x) - \frac{3}{52} \cos(4x) + c_2 e^{-x} \cos\left(\sqrt{3}x\right) + c_1 e^{-x} \sin\left(\sqrt{3}x\right)$$

11.6 problem 6

Internal problem ID [11780]
Internal file name [`OUTPUT/11789_Thursday_April_11_2024_08_49_40_PM_29232317/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 3y' - 4y = 16x - 12\,e^{2x}}$$

11.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = -4, f(x) = 16x - 12\,e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=-3, C=-4$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda\, \mathrm{e}^{\lambda x} - 4\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1, B=-3, C=-4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-3^2 - (4)(1)(-4)}\\ &= \frac{3}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{5}{2}\\ \lambda_2 &= \frac{3}{2} - \frac{5}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(4)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$16x - 12\,\mathrm{e}^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{2x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-x}, \mathrm{e}^{4x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{2x} + A_2 + A_3 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \mathrm{e}^{2x} - 3A_3 - 4A_2 - 4A_3 x = 16x - 12\,\mathrm{e}^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 3, A_3 = -4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{2x} + 3 - 4x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-x}\right) + \left(2\,\mathrm{e}^{2x} + 3 - 4x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-x} + 2\,\mathrm{e}^{2x} + 3 - 4x \tag{1}$$

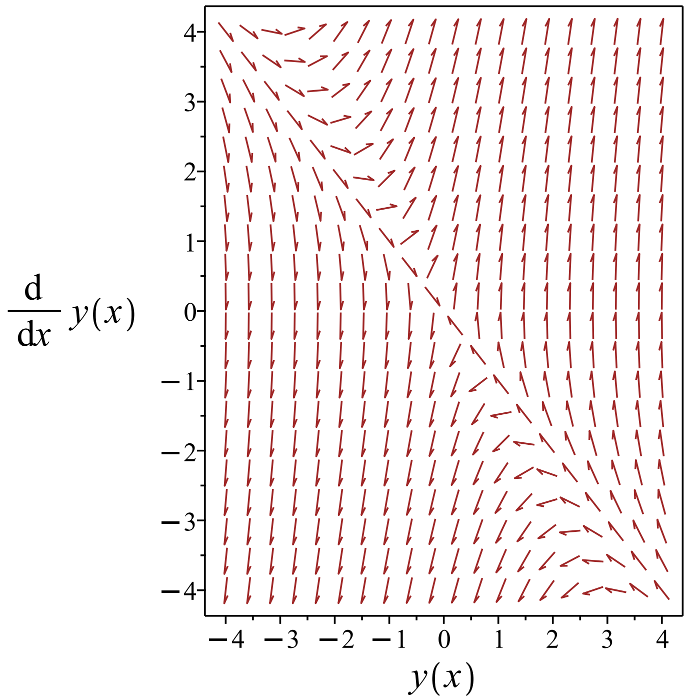


Figure 426: Slope field plot

Verification of solutions

$$y = c_1 e^{4x} + c_2 e^{-x} + 2 e^{2x} + 3 - 4x$$

Verified OK.

11.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 355: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-3}{1} \, dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{\mathrm{e}^{5x}}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{5x}}{5}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 3y' - 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{4x}}{5}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
16x - 12\,\mathrm{e}^{2x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{2x}\}, \{1, x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{4x}}{5}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{2x} + A_2 + A_3 x
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 e^{2x} - 3A_3 - 4A_2 - 4A_3 x = 16x - 12\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 3, A_3 = -4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,e^{2x} + 3 - 4x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{4x}}{5}\right) + \left(2\,e^{2x} + 3 - 4x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{4x}}{5} + 2\,e^{2x} + 3 - 4x \tag{1}$$

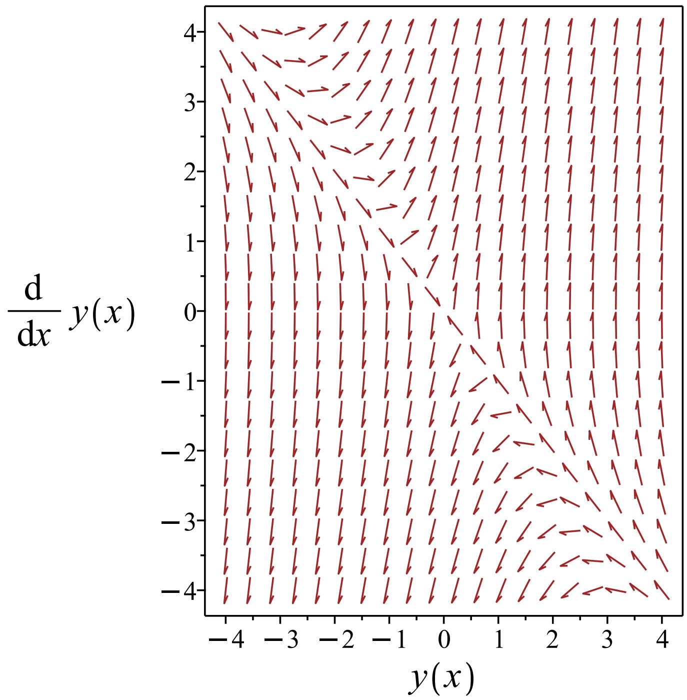


Figure 427: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{4x}}{5} + 2\,e^{2x} + 3 - 4x$$

Verified OK.

11.6.3 Maple step by step solution

Let's solve

$y'' - 3y' - 4y = 16x - 12\,e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r - 4 = 0$

- Factor the characteristic polynomial

  $(r + 1)(r - 4) = 0$

- Roots of the characteristic polynomial

$r = (-1, 4)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{4x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 16x - 12\,e^{2x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{4x} \\ -e^{-x} & 4\,e^{4x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 5\,e^{3x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{4\,e^{-x}\left(\int (3\,e^{2x}-4x)e^{x}dx\right)}{5} - \frac{4\,e^{4x}\left(\int (3\,e^{2x}-4x)e^{-4x}dx\right)}{5}$

  - Compute integrals

    $y_p(x) = 2\,e^{2x} + 3 - 4x$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 e^{4x} + 2\,e^{2x} + 3 - 4x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)-4*y(x)=16*x-12*exp(2*x),y(x), singsol=all)
```

$$y(x) = c_2 e^{4x} + c_1 e^{-x} + 2 e^{2x} - 4x + 3$$

✓ Solution by Mathematica

Time used: 0.136 (sec). Leaf size: 33

```
DSolve[y''[x]-3*y'[x]-4*y[x]==16*x-12*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -4x + 2e^{2x} + c_1 e^{-x} + c_2 e^{4x} + 3$$

11.7 problem 7

Internal problem ID [11781]
Internal file name [`OUTPUT/11790_Thursday_April_11_2024_08_49_41_PM_4744995/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''+6y'+5y=2\,\mathrm{e}^{x}+10\,\mathrm{e}^{5x}}$$

11.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=1,B=6,C=5,f(x)=2\,\mathrm{e}^{x}+10\,\mathrm{e}^{5x}$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+6y'+5y=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A = 1, B = 6, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{6^2 - (4)\,(1)\,(5)} \\ &= -3 \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 2 \\ \lambda_2 &= -3 - 2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -5\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(-1)x} + c_2 e^{(-5)x}\end{aligned}$$

Or

$$y = c_1 e^{-x} + c_2 e^{-5x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 e^{-5x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,e^{x} + 10\,e^{5x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}, \{e^{5x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-5x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x} + A_2 e^{5x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_1 e^{x} + 60A_2 e^{5x} = 2\,e^{x} + 10\,e^{5x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{x}}{6} + \frac{e^{5x}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-5x}\right) + \left(\frac{e^{x}}{6} + \frac{e^{5x}}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-5x} + \frac{e^{x}}{6} + \frac{e^{5x}}{6} \tag{1}$$

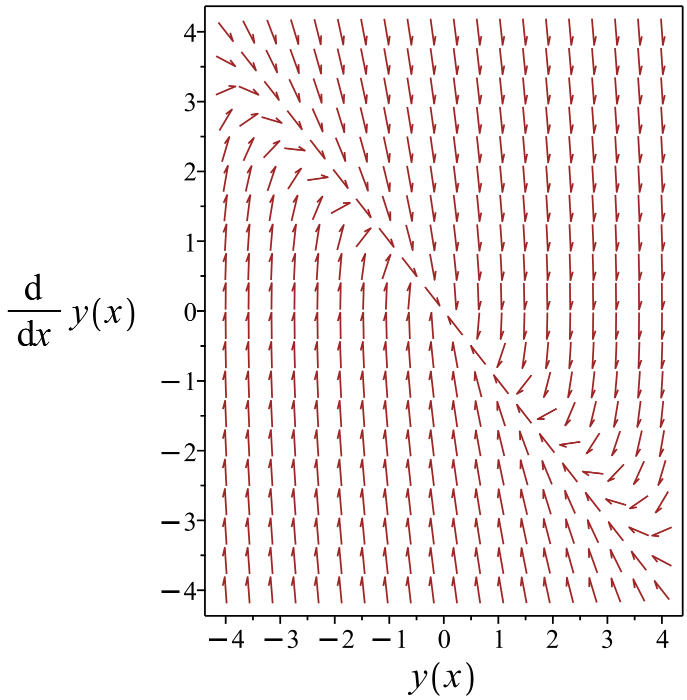


Figure 428: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-5x} + \frac{e^x}{6} + \frac{e^{5x}}{6}$$

Verified OK.

11.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 357: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\ &= z_1 e^{-3x} \\ &= z_1\left(\mathrm{e}^{-3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-5x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-5x}\right) + c_2 \left(e^{-5x} \left(\frac{e^{4x}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 6y' + 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 e^{-5x} + \frac{c_2 e^{-x}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2\,e^{x} + 10\,e^{5x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^{x}\}, \{e^{5x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{e^{-x}}{4}, e^{-5x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 e^{x} + A_2 e^{5x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_1\mathrm{e}^{x} + 60A_2\mathrm{e}^{5x} = 2\,\mathrm{e}^{x} + 10\,\mathrm{e}^{5x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\mathrm{e}^{x}}{6} + \frac{\mathrm{e}^{5x}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-5x} + \frac{c_2\mathrm{e}^{-x}}{4}\right) + \left(\frac{\mathrm{e}^{x}}{6} + \frac{\mathrm{e}^{5x}}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-5x} + \frac{c_2\mathrm{e}^{-x}}{4} + \frac{\mathrm{e}^{x}}{6} + \frac{\mathrm{e}^{5x}}{6} \tag{1}$$

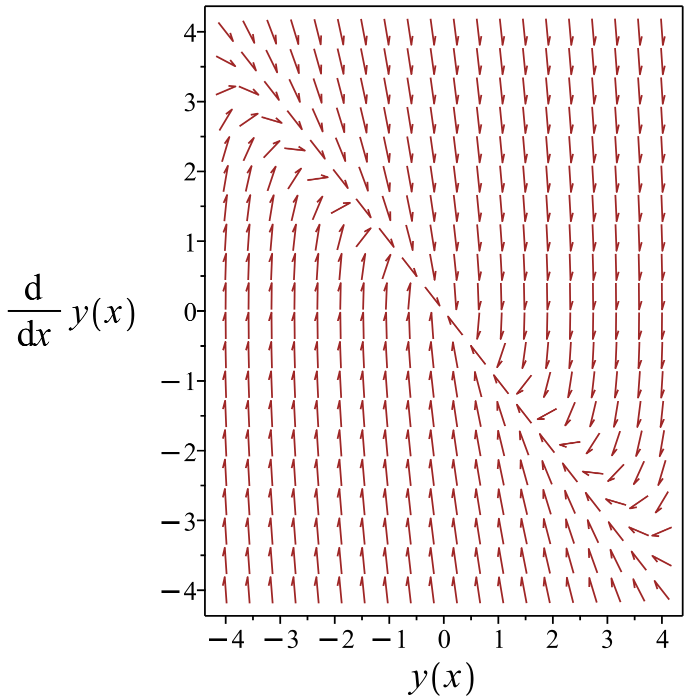


Figure 429: Slope field plot

Verification of solutions

$$y = c_1 e^{-5x} + \frac{c_2 e^{-x}}{4} + \frac{e^x}{6} + \frac{e^{5x}}{6}$$

Verified OK.

11.7.3 Maple step by step solution

Let's solve

$y'' + 6y' + 5y = 2\,e^x + 10\,e^{5x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 5 = 0$

- Factor the characteristic polynomial

  $(r + 5)(r + 1) = 0$

- Roots of the characteristic polynomial

$r = (-5, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-5x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-5x} + c_2 \mathrm{e}^{-x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\,\mathrm{e}^{x} + 10\,\mathrm{e}^{5x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-5x} & \mathrm{e}^{-x} \\ -5\,\mathrm{e}^{-5x} & -\mathrm{e}^{-x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4\,\mathrm{e}^{-6x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\mathrm{e}^{-5x}\left(\int \mathrm{e}^{6x}(5\,\mathrm{e}^{4x}+1) dx\right)}{2} + \frac{\mathrm{e}^{-x}\left(\int \mathrm{e}^{2x}(5\,\mathrm{e}^{4x}+1) dx\right)}{2}$

  ◦ Compute integrals

  $y_p(x) = \frac{\mathrm{e}^{x}}{6} + \frac{\mathrm{e}^{5x}}{6}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-5x} + c_2 \mathrm{e}^{-x} + \frac{\mathrm{e}^{x}}{6} + \frac{\mathrm{e}^{5x}}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+5*y(x)=2*exp(x)+10*exp(5*x),y(x), singsol=all)
```

$$y(x) = \frac{(e^{10x} + e^{6x} + 6c_1 e^{4x} + 6c_2)\, e^{-5x}}{6}$$

✓ Solution by Mathematica

Time used: 0.089 (sec). Leaf size: 36

```
DSolve[y''[x]+6*y'[x]+5*y[x]==2*Exp[x]+10*Exp[5*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}e^x\left(e^{4x} + 1\right) + c_1 e^{-5x} + c_2 e^{-x}$$

11.8 problem 8

Internal problem ID [11782]
Internal file name [`OUTPUT/11791_Thursday_April_11_2024_08_49_42_PM_27254097/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 10y = 5x\,\mathrm{e}^{-2x}}$$

11.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 10, f(x) = 5x\,\mathrm{e}^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 10y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 10$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 10\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(10)}\\ &= -1 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 3i\\ \lambda_2 &= -1 - 3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 3i\\ \lambda_2 &= -1 - 3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x}(c_1 \cos(3x) + c_2 \sin(3x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5x\,\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-x}\cos(3x), \mathrm{e}^{-x}\sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-2x} + A_2\mathrm{e}^{-2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\mathrm{e}^{-2x} + 10A_1 x\,\mathrm{e}^{-2x} + 10A_2\mathrm{e}^{-2x} = 5x\,\mathrm{e}^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,\mathrm{e}^{-2x}}{2} + \frac{\mathrm{e}^{-2x}}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-x}(c_1\cos(3x) + c_2\sin(3x))\right) + \left(\frac{x\,\mathrm{e}^{-2x}}{2} + \frac{\mathrm{e}^{-2x}}{10}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_1\cos(3x) + c_2\sin(3x)) + \frac{x\,\mathrm{e}^{-2x}}{2} + \frac{\mathrm{e}^{-2x}}{10} \tag{1}$$

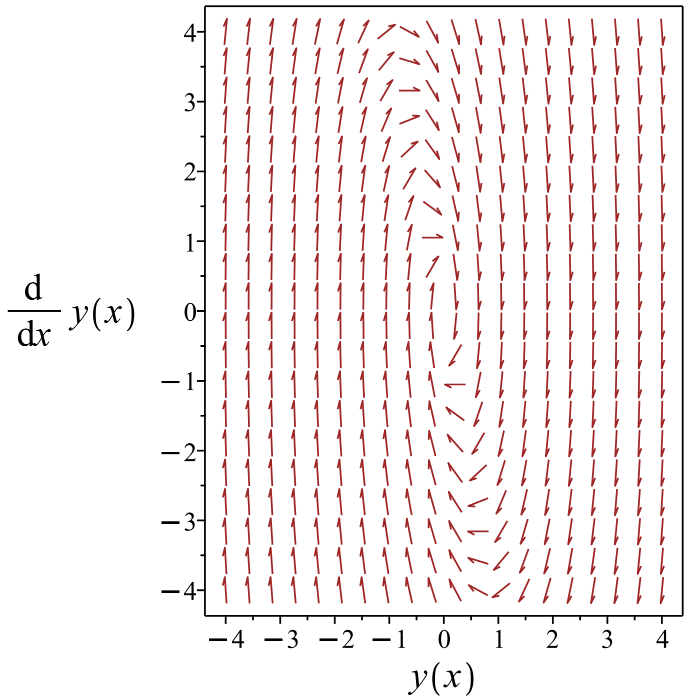


Figure 430: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(3x) + c_2 \sin(3x)) + \frac{x\,e^{-2x}}{2} + \frac{e^{-2x}}{10}$$

Verified OK.

11.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 359: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}\cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\tan(3x)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1y_1 + c_2y_2\\ &= c_1\left(\mathrm{e}^{-x}\cos(3x)\right) + c_2\left(\mathrm{e}^{-x}\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 10y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-x}\cos(3x)\,c_1 + \frac{\mathrm{e}^{-x}\sin(3x)\,c_2}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5x\,\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-x}\cos(3x), \frac{\mathrm{e}^{-x}\sin(3x)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1x\,\mathrm{e}^{-2x} + A_2\mathrm{e}^{-2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^{-2x} + 10A_1 x\, e^{-2x} + 10A_2 e^{-2x} = 5x\, e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\, e^{-2x}}{2} + \frac{e^{-2x}}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x} \cos(3x)\, c_1 + \frac{e^{-x} \sin(3x)\, c_2}{3}\right) + \left(\frac{x\, e^{-2x}}{2} + \frac{e^{-2x}}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-x} \cos(3x)\, c_1 + \frac{e^{-x} \sin(3x)\, c_2}{3} + \frac{x\, e^{-2x}}{2} + \frac{e^{-2x}}{10} \tag{1}$$

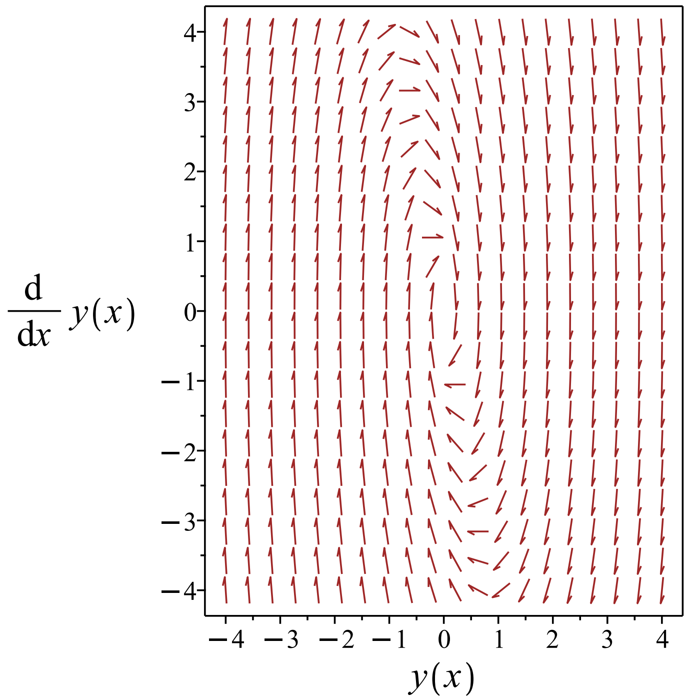


Figure 431: Slope field plot

<u>Verification of solutions</u>

$$y = e^{-x} \cos(3x) c_1 + \frac{e^{-x} \sin(3x) c_2}{3} + \frac{x e^{-2x}}{2} + \frac{e^{-2x}}{10}$$

Verified OK.

11.8.3 Maple step by step solution

Let's solve

$y'' + 2y' + 10y = 5x e^{-2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 10 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$r = (-1 - 3\,\mathrm{I}, -1 + 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x} \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x} \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-x} \cos(3x)\, c_1 + \mathrm{e}^{-x} \sin(3x)\, c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 5x\, \mathrm{e}^{-2x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} \cos(3x) & \mathrm{e}^{-x} \sin(3x) \\ -\mathrm{e}^{-x} \cos(3x) - 3\,\mathrm{e}^{-x} \sin(3x) & -\mathrm{e}^{-x} \sin(3x) + 3\,\mathrm{e}^{-x} \cos(3x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{5\,\mathrm{e}^{-x} \left(\cos(3x) \left(\int \mathrm{e}^{-x} \sin(3x) x dx\right) - \sin(3x) \left(\int \mathrm{e}^{-x} \cos(3x) x dx\right)\right)}{3}$

  - Compute integrals

    $y_p(x) = \frac{(5x+1)\mathrm{e}^{-2x}}{10}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-x} \cos(3x)\, c_1 + \mathrm{e}^{-x} \sin(3x)\, c_2 + \frac{(5x+1)\mathrm{e}^{-2x}}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+10*y(x)=5*x*exp(-2*x),y(x), singsol=all)
```

$$y(x) = \frac{(10c_1 \cos(3x) + 10c_2 \sin(3x)) e^{-x}}{10} + \frac{(5x+1) e^{-2x}}{10}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 41

```
DSolve[y''[x]+2*y'[x]+10*y[x]==5*x*Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{10} e^{-2x} (5x + 10c_2 e^x \cos(3x) + 10c_1 e^x \sin(3x) + 1)$$

11.9 problem 9

Internal problem ID [11783]
Internal file name [`OUTPUT/11792_Thursday_April_11_2024_08_49_42_PM_60476993/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 9.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{y''' + 4y'' + y' - 6y = -18x^2 + 1}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 4y'' + y' - 6y = 0$$

The characteristic equation is

$$\lambda^3 + 4\lambda^2 + \lambda - 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -3$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{x} + e^{-3x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{-3x}$$

Now the particular solution to the given ODE is found

$$y''' + 4y'' + y' - 6y = -18x^2 + 1$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-3x}, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_3 x^2 - 6A_2 x + 2xA_3 - 6A_1 + A_2 + 8A_3 = -18x^2 + 1$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 4, A_2 = 1, A_3 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x^2 + x + 4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{x} + e^{-3x} c_3\right) + \left(3x^2 + x + 4\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{x} + e^{-3x} c_3 + 3x^2 + x + 4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{x} + e^{-3x} c_3 + 3x^2 + x + 4$$

Verified OK.

11.9.1 Maple step by step solution

Let's solve

$y''' + 4y'' + y' - 6y = -18x^2 + 1$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ∘ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ∘ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ∘ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ∘ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -18x^2 - 4y_3(x) - y_2(x) + 6y_1(x) + 1$

  Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -18x^2 - 4y_3(x) - y_2(x) + 6y_1(x) + 1]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ -18x^2 + 1 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ -18x^2 + 1 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} -2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3 + \overrightarrow{y}_p(x)$$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{\mathrm{e}^{-3x}}{9} & \frac{\mathrm{e}^{-2x}}{4} & \mathrm{e}^{x} \\ -\frac{\mathrm{e}^{-3x}}{3} & -\frac{\mathrm{e}^{-2x}}{2} & \mathrm{e}^{x} \\ \mathrm{e}^{-3x} & \mathrm{e}^{-2x} & \mathrm{e}^{x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

  $$\Phi(x) = \begin{bmatrix} \frac{e^{-3x}}{9} & \frac{e^{-2x}}{4} & e^{x} \\ -\frac{e^{-3x}}{3} & -\frac{e^{-2x}}{2} & e^{x} \\ e^{-3x} & e^{-2x} & e^{x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{9} & \frac{1}{4} & 1 \\ -\frac{1}{3} & -\frac{1}{2} & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(x) = \begin{bmatrix} \frac{(e^{4x}+2e^{x}-1)e^{-3x}}{2} & -\frac{(-5e^{4x}+8e^{x}-3)e^{-3x}}{12} & \frac{(e^{4x}-4e^{x}+3)e^{-3x}}{12} \\ \frac{(e^{4x}-4e^{x}+3)e^{-3x}}{2} & \frac{(5e^{4x}+16e^{x}-9)e^{-3x}}{12} & \frac{(e^{4x}+8e^{x}-9)e^{-3x}}{12} \\ \frac{(e^{4x}+8e^{x}-9)e^{-3x}}{2} & -\frac{(-5e^{4x}+32e^{x}-27)e^{-3x}}{12} & \frac{(e^{4x}-16e^{x}+27)e^{-3x}}{12} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} e^{-3x}\left((3x^2+x+4)\,e^{3x} - \frac{4\,e^x}{3} - \frac{35\,e^{4x}}{12} + \frac{1}{4}\right) \\ \frac{(-35\,e^{4x}+72x\,e^{3x}+12\,e^{3x}+32\,e^x-9)e^{-3x}}{12} \\ \frac{(-35\,e^{4x}+72\,e^{3x}-64\,e^x+27)e^{-3x}}{12} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \left[\begin{array}{c} e^{-3x}\left((3x^2+x+4)\,e^{3x} - \frac{4\,e^x}{3} - \frac{35\,e^{4x}}{12} + \frac{1}{4}\right) \\ \frac{(-35\,e^{4x}+72x\,e^{3x}+12\,e^{3x}+32\,e^x-9)e^{-3x}}{12} \\ \frac{(-35\,e^{4x}+72\,e^{3x}-64\,e^x+27)e^{-3x}}{12} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = e^{-3x}\left((3x^2+x+4)\,e^{3x} + \frac{c_2 e^x}{4} + e^{4x}c_3 + \frac{c_1}{9} - \frac{4\,e^x}{3} - \frac{35\,e^{4x}}{12} + \frac{1}{4}\right)$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$3)+4*diff(y(x),x$2)+diff(y(x),x)-6*y(x)=-18*x^2+1,y(x), singsol=all)
```

$$y(x) = e^{-3x}\left((3x^2+x+4)\,e^{3x} + c_1 e^{4x} + c_3 e^x + c_2\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 35

```
DSolve[y'''[x]+4*y''[x]+y'[x]-6*y[x]==-18*x^2+1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3x^2 + x + c_1 e^{-3x} + c_2 e^{-2x} + c_3 e^x + 4$$

11.10 problem 10

Internal problem ID [11784]
Internal file name [`OUTPUT/11793_Thursday_April_11_2024_08_49_43_PM_4198183/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 10.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + 2y'' - 3y' - 10y = 8x\,\mathrm{e}^{-2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 2y'' - 3y' - 10y = 0$$

The characteristic equation is

$$\lambda^3 + 2\lambda^2 - 3\lambda - 10 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -2 - i$$
$$\lambda_3 = -2 + i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + e^{(-2-i)x} c_2 + e^{(-2+i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{2x}$$
$$y_2 = e^{(-2-i)x}$$
$$y_3 = e^{(-2+i)x}$$

Now the particular solution to the given ODE is found

$$y''' + 2y'' - 3y' - 10y = 8x\,e^{-2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8x\,e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{-2x}, e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{(-2-i)x}, e^{(-2+i)x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{-2x} + A_2 e^{-2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{-2x} - 4A_1 x\,e^{-2x} - 4A_2 e^{-2x} = 8x\,e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -2, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x\,\mathrm{e}^{-2x} - \frac{\mathrm{e}^{-2x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{2x} + \mathrm{e}^{(-2-i)x}c_2 + \mathrm{e}^{(-2+i)x}c_3\right) + \left(-2x\,\mathrm{e}^{-2x} - \frac{\mathrm{e}^{-2x}}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\mathrm{e}^{2x} + \mathrm{e}^{(-2-i)x}c_2 + \mathrm{e}^{(-2+i)x}c_3 - 2x\,\mathrm{e}^{-2x} - \frac{\mathrm{e}^{-2x}}{2} \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{2x} + \mathrm{e}^{(-2-i)x}c_2 + \mathrm{e}^{(-2+i)x}c_3 - 2x\,\mathrm{e}^{-2x} - \frac{\mathrm{e}^{-2x}}{2}$$

Verified OK.

11.10.1 Maple step by step solution

Let's solve

$y''' + 2y'' - 3y' - 10y = 8x\,\mathrm{e}^{-2x}$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = 8x\,\mathrm{e}^{-2x} - 2y_3(x) + 3y_2(x) + 10y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = 8x\,\mathrm{e}^{-2x} - 2y_3(x) + 3y_2(x) + 10y_1(x)]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & 3 & -2 \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ 8x\,\mathrm{e}^{-2x} \end{array}\right]$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ 8x\,\mathrm{e}^{-2x} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & 3 & -2 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right], \left[-2-\mathrm{I}, \left[\begin{array}{c} \frac{3}{25} - \frac{4\,\mathrm{I}}{25} \\ -\frac{2}{5} + \frac{\mathrm{I}}{5} \\ 1 \end{array}\right]\right], \left[-2+\mathrm{I}, \left[\begin{array}{c} \frac{3}{25} + \frac{4\,\mathrm{I}}{25} \\ -\frac{2}{5} - \frac{\mathrm{I}}{5} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2 - I, \left[\begin{array}{c} \frac{3}{25} - \frac{4\,I}{25} \\ -\frac{2}{5} + \frac{I}{5} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{(-2-I)x} \cdot \left[\begin{array}{c} \frac{3}{25} - \frac{4\,I}{25} \\ -\frac{2}{5} + \frac{I}{5} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-2x} \cdot (\cos(x) - I\sin(x)) \cdot \left[\begin{array}{c} \frac{3}{25} - \frac{4\,I}{25} \\ -\frac{2}{5} + \frac{I}{5} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-2x} \cdot \left[\begin{array}{c} \left(\frac{3}{25} - \frac{4\,I}{25}\right)(\cos(x) - I\sin(x)) \\ \left(-\frac{2}{5} + \frac{I}{5}\right)(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{-2x} \cdot \left[\begin{array}{c} \frac{3\cos(x)}{25} - \frac{4\sin(x)}{25} \\ -\frac{2\cos(x)}{5} + \frac{\sin(x)}{5} \\ \cos(x) \end{array}\right], \vec{y}_3(x) = e^{-2x} \cdot \left[\begin{array}{c} -\frac{3\sin(x)}{25} - \frac{4\cos(x)}{25} \\ \frac{2\sin(x)}{5} + \frac{\cos(x)}{5} \\ -\sin(x) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

  $\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3(x) + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{2x}}{4} & e^{-2x}\left(\frac{3\cos(x)}{25} - \frac{4\sin(x)}{25}\right) & e^{-2x}\left(-\frac{3\sin(x)}{25} - \frac{4\cos(x)}{25}\right) \\ \frac{e^{2x}}{2} & e^{-2x}\left(-\frac{2\cos(x)}{5} + \frac{\sin(x)}{5}\right) & e^{-2x}\left(\frac{2\sin(x)}{5} + \frac{\cos(x)}{5}\right) \\ e^{2x} & \cos(x)\, e^{-2x} & -e^{-2x}\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{2x}}{4} & e^{-2x}\left(\frac{3\cos(x)}{25} - \frac{4\sin(x)}{25}\right) & e^{-2x}\left(-\frac{3\sin(x)}{25} - \frac{4\cos(x)}{25}\right) \\ \frac{e^{2x}}{2} & e^{-2x}\left(-\frac{2\cos(x)}{5} + \frac{\sin(x)}{5}\right) & e^{-2x}\left(\frac{2\sin(x)}{5} + \frac{\cos(x)}{5}\right) \\ e^{2x} & \cos(x)\, e^{-2x} & -e^{-2x}\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & \frac{3}{25} & -\frac{4}{25} \\ \frac{1}{2} & -\frac{2}{5} & \frac{1}{5} \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{2(6\cos(x)+7\sin(x))e^{-2x}}{17} + \frac{5e^{2x}}{17} & \frac{(-4\cos(x)+\sin(x))e^{-2x}}{17} + \frac{4e^{2x}}{17} & \frac{(-\cos(x)-4\sin(x))e^{-2x}}{17} + \frac{e^{2x}}{17} \\ \frac{10(-\cos(x)-4\sin(x))e^{-2x}}{17} + \frac{10e^{2x}}{17} & \frac{(9\cos(x)+2\sin(x))e^{-2x}}{17} + \frac{8e^{2x}}{17} & \frac{(-2\cos(x)+9\sin(x))e^{-2x}}{17} + \frac{2e^{2x}}{17} \\ \frac{10(-2\cos(x)+9\sin(x))e^{-2x}}{17} + \frac{20e^{2x}}{17} & \frac{(-16\cos(x)-13\sin(x))e^{-2x}}{17} + \frac{16e^{2x}}{17} & \frac{(13\cos(x)-16\sin(x))e^{-2x}}{17} + \frac{4e^{2x}}{17} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$$

- Cancel like terms

  $$\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$$

- Multiply by the inverse of the fundamental matrix

  $$\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$$

- Integrate to solve for $\overrightarrow{v}(x)$

  $$\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $$\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \frac{\mathrm{e}^{-2x}\left(\mathrm{e}^{4x}+16\cos(x)+64\sin(x)-68x-17\right)}{34} \\ \frac{\mathrm{e}^{-2x}\left(\mathrm{e}^{4x}+16\cos(x)-72\sin(x)+68x-17\right)}{17} \\ \frac{2\,\mathrm{e}^{-2x}\left(\mathrm{e}^{4x}-52\cos(x)+64\sin(x)-68x+51\right)}{17} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3(x) + \left[\begin{array}{c} \frac{\mathrm{e}^{-2x}\left(\mathrm{e}^{4x}+16\cos(x)+64\sin(x)-68x-17\right)}{34} \\ \frac{\mathrm{e}^{-2x}\left(\mathrm{e}^{4x}+16\cos(x)-72\sin(x)+68x-17\right)}{17} \\ \frac{2\,\mathrm{e}^{-2x}\left(\mathrm{e}^{4x}-52\cos(x)+64\sin(x)-68x+51\right)}{17} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $$y = \frac{\left((204c_2-272c_3+800)\cos(x)+(-272c_2-204c_3+3200)\sin(x)-3400x-850\right)\mathrm{e}^{-2x}}{1700} + \frac{(425c_1+50)\mathrm{e}^{2x}}{1700}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$3)+2*diff(y(x),x$2)-3*diff(y(x),x)-10*y(x)=8*x*exp(-2*x),y(x), singsol=al
```

$$y(x) = \frac{(2c_2 \cos(x) + 2c_3 \sin(x) - 4x - 1)\,e^{-2x}}{2} + e^{2x} c_1$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 40

```
DSolve[y'''[x]+2*y''[x]-3*y'[x]-10*y[x]==8*x*Exp[-2*x],y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{2} e^{-2x} \left(-4x + 2c_3 e^{4x} + 2c_2 \cos(x) + 2c_1 \sin(x) - 1\right)$$

11.11 problem 11

Internal problem ID [11785]
Internal file name [`OUTPUT/11794_Thursday_April_11_2024_08_49_43_PM_74629381/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 11.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + y'' + 3y' - 5y = 5\sin(2x) + 10x^2 + 3x + 7}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' + 3y' - 5y = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 + 3\lambda - 5 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = -1 - 2i$$
$$\lambda_3 = -1 + 2i$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + e^{(-1-2i)x} c_2 + e^{(-1+2i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{(-1-2i)x}$$
$$y_3 = e^{(-1+2i)x}$$

Now the particular solution to the given ODE is found

$$y''' + y'' + 3y' - 5y = 5\sin(2x) + 10x^2 + 3x + 7$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5\sin(2x) + 10x^2 + 3x + 7$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{(-1-2i)x}, e^{(-1+2i)x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(2x) + A_2\sin(2x) + A_3 + A_4x + A_5x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_1\sin(2x) - 2A_2\cos(2x) - 9A_1\cos(2x) - 9A_2\sin(2x) + 2A_5 \\ &+ 3A_4 + 6A_5x - 5A_3 - 5A_4x - 5A_5x^2 = 5\sin(2x) + 10x^2 + 3x + 7\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2}{17}, A_2 = -\frac{9}{17}, A_3 = -4, A_4 = -3, A_5 = -2\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{2\cos(2x)}{17} - \frac{9\sin(2x)}{17} - 4 - 3x - 2x^2$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(e^x c_1 + e^{(-1-2i)x} c_2 + e^{(-1+2i)x} c_3\right) + \left(\frac{2\cos(2x)}{17} - \frac{9\sin(2x)}{17} - 4 - 3x - 2x^2\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x c_1 + e^{(-1-2i)x} c_2 + e^{(-1+2i)x} c_3 + \frac{2\cos(2x)}{17} - \frac{9\sin(2x)}{17} - 4 - 3x - 2x^2 \tag{1}$$

Verification of solutions

$$y = e^x c_1 + e^{(-1-2i)x} c_2 + e^{(-1+2i)x} c_3 + \frac{2\cos(2x)}{17} - \frac{9\sin(2x)}{17} - 4 - 3x - 2x^2$$

Verified OK.

11.11.1 Maple step by step solution

Let's solve

$y''' + y'' + 3y' - 5y = 5\sin(2x) + 10x^2 + 3x + 7$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 10x^2 - y_3(x) - 3y_2(x) + 5y_1(x) + 5\sin(2x) + 3x + 7$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 10x^2 - y_3(x) - 3y_2(x) + 5y_1(x) + 5\sin(2x) + 3x + 7]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -3 & -1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 5\sin(2x) + 10x^2 + 3x + 7 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 5\sin(2x) + 10x^2 + 3x + 7 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -3 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c}1\\1\\1\end{array}\right]\right], \left[-1-2\,\mathrm{I}, \left[\begin{array}{c}-\frac{3}{25}-\frac{4\,\mathrm{I}}{25}\\-\frac{1}{5}+\frac{2\,\mathrm{I}}{5}\\1\end{array}\right]\right], \left[-1+2\,\mathrm{I}, \left[\begin{array}{c}-\frac{3}{25}+\frac{4\,\mathrm{I}}{25}\\-\frac{1}{5}-\frac{2\,\mathrm{I}}{5}\\1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1\\1\\1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{x} \cdot \left[\begin{array}{c}1\\1\\1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-2\,\mathrm{I}, \left[\begin{array}{c}-\frac{3}{25}-\frac{4\,\mathrm{I}}{25}\\-\frac{1}{5}+\frac{2\,\mathrm{I}}{5}\\1\end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-2\,\mathrm{I})x} \cdot \left[\begin{array}{c}-\frac{3}{25}-\frac{4\,\mathrm{I}}{25}\\-\frac{1}{5}+\frac{2\,\mathrm{I}}{5}\\1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-x} \cdot (\cos(2x) - \mathrm{I}\sin(2x)) \cdot \left[\begin{array}{c}-\frac{3}{25}-\frac{4\,\mathrm{I}}{25}\\-\frac{1}{5}+\frac{2\,\mathrm{I}}{5}\\1\end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-x} \cdot \left[\begin{array}{c}\left(-\frac{3}{25}-\frac{4\,\mathrm{I}}{25}\right)(\cos(2x)-\mathrm{I}\sin(2x))\\\left(-\frac{1}{5}+\frac{2\,\mathrm{I}}{5}\right)(\cos(2x)-\mathrm{I}\sin(2x))\\\cos(2x)-\mathrm{I}\sin(2x)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{-x} \cdot \begin{bmatrix} -\frac{3\cos(2x)}{25} - \frac{4\sin(2x)}{25} \\ -\frac{\cos(2x)}{5} + \frac{2\sin(2x)}{5} \\ \cos(2x) \end{bmatrix}, \vec{y}_3(x) = e^{-x} \cdot \begin{bmatrix} \frac{3\sin(2x)}{25} - \frac{4\cos(2x)}{25} \\ \frac{\sin(2x)}{5} + \frac{2\cos(2x)}{5} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^x & e^{-x}\left(-\frac{3\cos(2x)}{25} - \frac{4\sin(2x)}{25}\right) & e^{-x}\left(\frac{3\sin(2x)}{25} - \frac{4\cos(2x)}{25}\right) \\ e^x & e^{-x}\left(-\frac{\cos(2x)}{5} + \frac{2\sin(2x)}{5}\right) & e^{-x}\left(\frac{\sin(2x)}{5} + \frac{2\cos(2x)}{5}\right) \\ e^x & e^{-x}\cos(2x) & -e^{-x}\sin(2x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^x & e^{-x}\left(-\frac{3\cos(2x)}{25} - \frac{4\sin(2x)}{25}\right) & e^{-x}\left(\frac{3\sin(2x)}{25} - \frac{4\cos(2x)}{25}\right) \\ e^x & e^{-x}\left(-\frac{\cos(2x)}{5} + \frac{2\sin(2x)}{5}\right) & e^{-x}\left(\frac{\sin(2x)}{5} + \frac{2\cos(2x)}{5}\right) \\ e^x & e^{-x}\cos(2x) & -e^{-x}\sin(2x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -\frac{3}{25} & -\frac{4}{25} \\ 1 & -\frac{1}{5} & \frac{2}{5} \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{(-\sin(2x)+3\cos(2x))e^{-x}}{8} + \frac{5e^x}{8} & \frac{(\sin(2x)-\cos(2x))e^{-x}}{4} + \frac{e^x}{4} & \frac{(-\sin(2x)-\cos(2x))e^{-x}}{8} + \frac{e^x}{8} \\ \frac{(-5\sin(2x)-5\cos(2x))e^{-x}}{8} + \frac{5e^x}{8} & \frac{(\sin(2x)+3\cos(2x))e^{-x}}{4} + \frac{e^x}{4} & \frac{(3\sin(2x)-\cos(2x))e^{-x}}{8} + \frac{e^x}{8} \\ \frac{5(3\sin(2x)-\cos(2x))e^{-x}}{8} + \frac{5e^x}{8} & \frac{(-\cos(2x)-7\sin(2x))e^{-x}}{4} + \frac{e^x}{4} & \frac{(7\cos(2x)-\sin(2x))e^{-x}}{8} + \frac{e^x}{8} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

$\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{(-4\,\mathrm{e}^{-x}+4)\cos(2x)}{34} - 2x^2 - \frac{\mathrm{e}^{-x}\sin(2x)}{34} - 3x + 4\,\mathrm{e}^{x} - \frac{9\sin(2x)}{17} - 4 \\ \frac{(2\,\mathrm{e}^{-x}-36)\cos(2x)}{34} + \frac{9\,\mathrm{e}^{-x}\sin(2x)}{34} - 4x + 4\,\mathrm{e}^{x} - \frac{4\sin(2x)}{17} - 3 \\ -4 + \frac{8(-1+\mathrm{e}^{-x})\cos(2x)}{17} - \frac{13\,\mathrm{e}^{-x}\sin(2x)}{34} + 4\,\mathrm{e}^{x} + \frac{36\sin(2x)}{17} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \left[\begin{array}{c} \frac{(-4\,\mathrm{e}^{-x}+4)\cos(2x)}{34} - 2x^2 - \frac{\mathrm{e}^{-x}\sin(2x)}{34} - 3x + 4\,\mathrm{e}^{x} - \frac{9\sin(2x)}{17} - \\ \frac{(2\,\mathrm{e}^{-x}-36)\cos(2x)}{34} + \frac{9\,\mathrm{e}^{-x}\sin(2x)}{34} - 4x + 4\,\mathrm{e}^{x} - \frac{4\sin(2x)}{17} - 3 \\ -4 + \frac{8(-1+\mathrm{e}^{-x})\cos(2x)}{17} - \frac{13\,\mathrm{e}^{-x}\sin(2x)}{34} + 4\,\mathrm{e}^{x} + \frac{36\sin(2x)}{17} \end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y = -4 + \frac{\left((-50-51c_2-68c_3)\cos(2x)-68\sin(2x)\left(c_2-\frac{3c_3}{4}+\frac{25}{136}\right)\right)\mathrm{e}^{-x}}{425} + \frac{2\cos(2x)}{17} - \frac{9\sin(2x)}{17} + (c_1+4)\,\mathrm{e}^{x} - 2x^2 -$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)+3*diff(y(x),x)-5*y(x)=5*sin(2*x)+10*x^2+3*x+7,y(x), sin
```

$$y(x) = \frac{(17c_3\mathrm{e}^{-x} - 9)\sin(2x)}{17} + c_2\mathrm{e}^{-x}\cos(2x) - 2x^2 + c_1\mathrm{e}^{x} - 3x + \frac{2\cos(2x)}{17} - 4$$

✓ Solution by Mathematica

Time used: 0.419 (sec). Leaf size: 55

```
DSolve[y'''[x]+y''[x]+3*y'[x]-5*y[x]==5*Sin[2*x]+10*x^2+3*x+7,y[x],x,IncludeSingularSolution
```

$$y(x) \to -2x^2 - 3x + c_3e^x + \left(\frac{2}{17} + c_2e^{-x}\right)\cos(2x) + \left(-\frac{9}{17} + c_1e^{-x}\right)\sin(2x) - 4$$

11.12 problem 12

Internal problem ID [11786]
Internal file name [`OUTPUT/11795_Thursday_April_11_2024_08_49_44_PM_54708509/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 12.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4y''' - 4y'' - 5y' + 3y = 3x^3 - 8x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$4y''' - 4y'' - 5y' + 3y = 0$$

The characteristic equation is

$$4\lambda^3 - 4\lambda^2 - 5\lambda + 3 = 0$$

The roots of the above equation are

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = \frac{3}{2}$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{\frac{x}{2}} c_2 + c_3 \mathrm{e}^{\frac{3x}{2}}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{\frac{x}{2}}$$
$$y_3 = \mathrm{e}^{\frac{3x}{2}}$$

Now the particular solution to the given ODE is found

$$4y''' - 4y'' - 5y' + 3y = 3x^3 - 8x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-x}, \mathrm{e}^{\frac{x}{2}}, \mathrm{e}^{\frac{3x}{2}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4 x^3 + A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_4x^3 + 3A_3x^2 - 15x^2A_4 + 3A_2x - 10xA_3 - 24xA_4 + 3A_1 - 5A_2 - 8A_3 + 24A_4 \\ = 3x^3 - 8x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 42, A_2 = 22, A_3 = 5, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^3 + 5x^2 + 22x + 42$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + e^{\frac{x}{2}} c_2 + c_3 e^{\frac{3x}{2}}\right) + \left(x^3 + 5x^2 + 22x + 42\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + e^{\frac{x}{2}} c_2 + c_3 e^{\frac{3x}{2}} + x^3 + 5x^2 + 22x + 42 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^{\frac{x}{2}} c_2 + c_3 e^{\frac{3x}{2}} + x^3 + 5x^2 + 22x + 42$$

Verified OK.

11.12.1 Maple step by step solution

Let's solve

$4y''' - 4y'' - 5y' + 3y = 3x^3 - 8x$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = y'' + \frac{5y'}{4} - \frac{3y}{4} + \frac{3x^3}{4} - 2x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - y'' - \frac{5y'}{4} + \frac{3y}{4} = \frac{3}{4}x^3 - 2x$

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = \frac{3x^3}{4} - 2x + y_3(x) + \frac{5y_2(x)}{4} - \frac{3y_1(x)}{4}$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \frac{3x^3}{4} - 2x + y_3(x) + \frac{5y_2(x)}{4} - \frac{3y_1(x)}{4}\right]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

  $$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{3}{4} & \frac{5}{4} & 1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \frac{3}{4}x^3 - 2x \end{bmatrix}$$

- Define the forcing function

  $$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \frac{3}{4}x^3 - 2x \end{bmatrix}$$

- Define the coefficient matrix

  $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{3}{4} & \frac{5}{4} & 1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right], \left[\frac{1}{2}, \left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right]\right], \left[\frac{3}{2}, \left[\begin{array}{c} \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[\frac{1}{2}, \left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{\frac{x}{2}} \cdot \left[\begin{array}{c} 4 \\ 2 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[\frac{3}{2}, \left[\begin{array}{c} \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{\frac{3x}{2}} \cdot \left[\begin{array}{c} \frac{4}{9} \\ \frac{2}{3} \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

  $\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \mathrm{e}^{-x} & 4\,\mathrm{e}^{\frac{x}{2}} & \frac{4\,\mathrm{e}^{\frac{3x}{2}}}{9} \\ -\mathrm{e}^{-x} & 2\,\mathrm{e}^{\frac{x}{2}} & \frac{2\,\mathrm{e}^{\frac{3x}{2}}}{3} \\ \mathrm{e}^{-x} & \mathrm{e}^{\frac{x}{2}} & \mathrm{e}^{\frac{3x}{2}} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \mathrm{e}^{-x} & 4\,\mathrm{e}^{\frac{x}{2}} & \frac{4\,\mathrm{e}^{\frac{3x}{2}}}{9} \\ -\mathrm{e}^{-x} & 2\,\mathrm{e}^{\frac{x}{2}} & \frac{2\,\mathrm{e}^{\frac{3x}{2}}}{3} \\ \mathrm{e}^{-x} & \mathrm{e}^{\frac{x}{2}} & \mathrm{e}^{\frac{3x}{2}} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 4 & \frac{4}{9} \\ -1 & 2 & \frac{2}{3} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -\frac{\left(\mathrm{e}^{\frac{5x}{2}} - 5\,\mathrm{e}^{\frac{3x}{2}} - 1\right)\mathrm{e}^{-x}}{5} & \frac{\left(3\,\mathrm{e}^{\frac{5x}{2}} + 5\,\mathrm{e}^{\frac{3x}{2}} - 8\right)\mathrm{e}^{-x}}{15} & \frac{2\left(3\,\mathrm{e}^{\frac{5x}{2}} - 5\,\mathrm{e}^{\frac{3x}{2}} + 2\right)\mathrm{e}^{-x}}{15} \\ -\frac{\left(3\,\mathrm{e}^{\frac{5x}{2}} - 5\,\mathrm{e}^{\frac{3x}{2}} + 2\right)\mathrm{e}^{-x}}{10} & \frac{\left(9\,\mathrm{e}^{\frac{5x}{2}} + 5\,\mathrm{e}^{\frac{3x}{2}} + 16\right)\mathrm{e}^{-x}}{30} & \frac{\left(9\,\mathrm{e}^{\frac{5x}{2}} - 5\,\mathrm{e}^{\frac{3x}{2}} - 4\right)\mathrm{e}^{-x}}{15} \\ -\frac{\left(9\,\mathrm{e}^{\frac{5x}{2}} - 5\,\mathrm{e}^{\frac{3x}{2}} - 4\right)\mathrm{e}^{-x}}{20} & \frac{\left(27\,\mathrm{e}^{\frac{5x}{2}} + 5\,\mathrm{e}^{\frac{3x}{2}} - 32\right)\mathrm{e}^{-x}}{60} & \frac{\left(27\,\mathrm{e}^{\frac{5x}{2}} - 5\,\mathrm{e}^{\frac{3x}{2}} + 8\right)\mathrm{e}^{-x}}{30} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

$$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \mathrm{e}^{-x}\left(-\frac{128\,\mathrm{e}^{\frac{3x}{2}}}{3} + \frac{2}{3} + (x^3 + 5x^2 + 22x + 42)\,\mathrm{e}^{x}\right) \\ \frac{\left(9x^2\mathrm{e}^{x} - 64\,\mathrm{e}^{\frac{3x}{2}} + 30x\,\mathrm{e}^{x} + 66\,\mathrm{e}^{x} - 2\right)\mathrm{e}^{-x}}{3} \\ \frac{2\left(-16\,\mathrm{e}^{\frac{3x}{2}} + 9x\,\mathrm{e}^{x} + 15\,\mathrm{e}^{x} + 1\right)\mathrm{e}^{-x}}{3} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \begin{bmatrix} \mathrm{e}^{-x}\left(-\frac{128\,\mathrm{e}^{\frac{3x}{2}}}{3} + \frac{2}{3} + (x^3 + 5x^2 + 22x + 42)\,\mathrm{e}^{x}\right) \\ \frac{\left(9x^2\mathrm{e}^{x} - 64\,\mathrm{e}^{\frac{3x}{2}} + 30x\,\mathrm{e}^{x} + 66\,\mathrm{e}^{x} - 2\right)\mathrm{e}^{-x}}{3} \\ \frac{2\left(-16\,\mathrm{e}^{\frac{3x}{2}} + 9x\,\mathrm{e}^{x} + 15\,\mathrm{e}^{x} + 1\right)\mathrm{e}^{-x}}{3} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \mathrm{e}^{-x}\left(\frac{2}{3} + 4\left(-\frac{32}{3} + c_2\right)\mathrm{e}^{\frac{3x}{2}} + \frac{4c_3\mathrm{e}^{\frac{5x}{2}}}{9} + (x^3 + 5x^2 + 22x + 42)\,\mathrm{e}^{x} + c_1\right)$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(4*diff(y(x),x$3)-4*diff(y(x),x$2)-5*diff(y(x),x)+3*y(x)=3*x^3-8*x,y(x), singsol=all)
```

$$y(x) = \left(c_2 \mathrm{e}^{\frac{3x}{2}} + c_3 \mathrm{e}^{\frac{5x}{2}} + \left(x^3 + 5x^2 + 22x + 42\right) \mathrm{e}^{x} + c_1\right) \mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 46

```
DSolve[4*y'''[x]-4*y''[x]-5*y'[x]+3*y[x]==3*x^3-8*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3 + 5x^2 + 22x + c_1 e^{x/2} + c_2 e^{3x/2} + c_3 e^{-x} + 42$$

11.13 problem 13

Internal problem ID [11787]
Internal file name [`OUTPUT/11796_Thursday_April_11_2024_08_49_44_PM_37991172/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' - 6y = 10\,e^{2x} - 18\,e^{3x} - 6x - 11}$$

11.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -6, f(x) = 10\,e^{2x} - 18\,e^{3x} - 6x - 11$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 6 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -6$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(-6)} \\
&= -\frac{1}{2} \pm \frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{5}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{5}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -3
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(2)x} + c_2 e^{(-3)x}
\end{aligned}$$

Or

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10\,\mathrm{e}^{2x} - 18\,\mathrm{e}^{3x} - 6x - 11$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{2x}\}, \{\mathrm{e}^{3x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-3x}, \mathrm{e}^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^{2x}x\}, \{\mathrm{e}^{3x}\}, \{1, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1\mathrm{e}^{2x}x + A_2\mathrm{e}^{3x} + A_3 + A_4x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1\mathrm{e}^{2x} + 6A_2\mathrm{e}^{3x} + A_4 - 6A_3 - 6A_4x = 10\,\mathrm{e}^{2x} - 18\,\mathrm{e}^{3x} - 6x - 11$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = -3, A_3 = 2, A_4 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{2x}x - 3\,\mathrm{e}^{3x} + 2 + x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{2x} + c_2\mathrm{e}^{-3x}\right) + \left(2\,\mathrm{e}^{2x}x - 3\,\mathrm{e}^{3x} + 2 + x\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{-3x} + 2 e^{2x} x - 3 e^{3x} + 2 + x \tag{1}$$

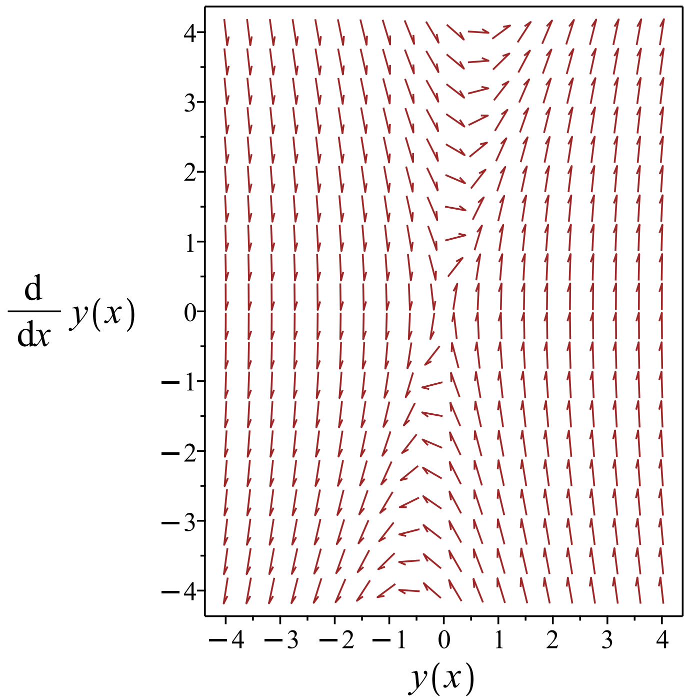


Figure 432: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-3x} + 2 e^{2x} x - 3 e^{3x} + 2 + x$$

Verified OK.

11.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 6y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 365: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{5x}}{5}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{5x}}{5}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{2x}}{5}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-3x}$$
$$y_2 = \frac{e^{2x}}{5}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-3x} & \frac{e^{2x}}{5} \\ \frac{d}{dx}\left(e^{-3x}\right) & \frac{d}{dx}\left(\frac{e^{2x}}{5}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-3x} & \frac{e^{2x}}{5} \\ -3\,e^{-3x} & \frac{2\,e^{2x}}{5} \end{vmatrix}$$

Therefore

$$W = \left(e^{-3x}\right)\left(\frac{2\,e^{2x}}{5}\right) - \left(\frac{e^{2x}}{5}\right)\left(-3\,e^{-3x}\right)$$

Which simplifies to

$$W = e^{-3x} e^{2x}$$

Which simplifies to

$$W = e^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{2x}(10\,e^{2x}-18\,e^{3x}-6x-11)}{5}}{e^{-x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(10\,e^{2x} - 18\,e^{3x} - 6x - 11)\,e^{3x}}{5}dx$$

Hence

$$u_1 = \frac{3\,e^{3x}}{5} - \frac{2\,e^{5x}}{5} + \frac{3\,e^{6x}}{5} + \frac{2x\,e^{3x}}{5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-3x}(10\,e^{2x} - 18\,e^{3x} - 6x - 11)}{e^{-x}}\,dx$$

Which simplifies to

$$u_2 = \int \left(10\,e^{2x} - 18\,e^{3x} - 6x - 11\right) e^{-2x} dx$$

Hence

$$u_2 = 10x + 7\,e^{-2x} + 3x\,e^{-2x} - 18\,e^{x}$$

Which simplifies to

$$u_1 = \frac{(3+2x)\,e^{3x}}{5} - \frac{2\,e^{5x}}{5} + \frac{3\,e^{6x}}{5}$$
$$u_2 = \left(-18\,e^{3x} + 10\,e^{2x}x + 3x + 7\right) e^{-2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{(3+2x)\,e^{3x}}{5} - \frac{2\,e^{5x}}{5} + \frac{3\,e^{6x}}{5}\right) e^{-3x} + \frac{(-18\,e^{3x} + 10\,e^{2x}x + 3x + 7)\,e^{-2x}e^{2x}}{5}$$

Which simplifies to

$$y_p(x) = 2 - \frac{2\,e^{2x}}{5} - 3\,e^{3x} + x + 2\,e^{2x}x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + \frac{c_2 e^{2x}}{5}\right) + \left(2 - \frac{2\,e^{2x}}{5} - 3\,e^{3x} + x + 2\,e^{2x}x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + \frac{c_2 e^{2x}}{5} + 2 - \frac{2\,e^{2x}}{5} - 3\,e^{3x} + x + 2\,e^{2x}x \tag{1}$$

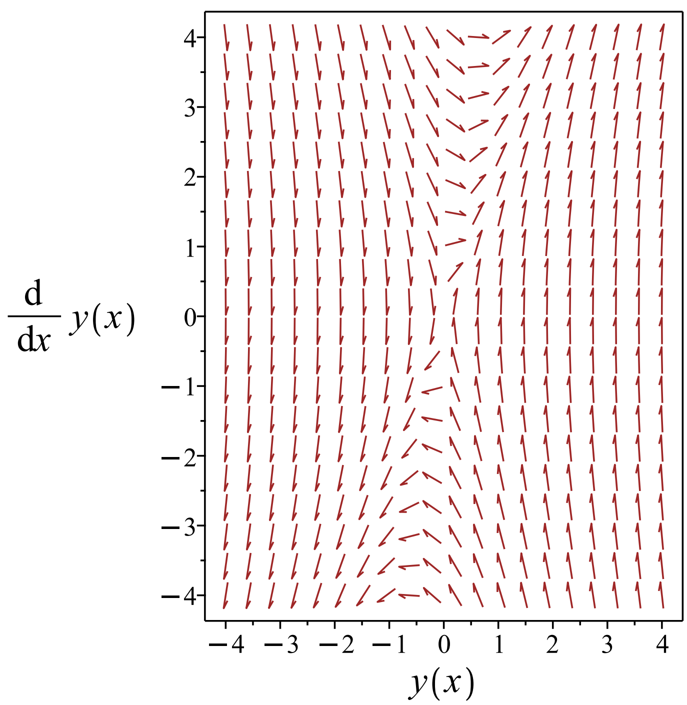


Figure 433: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + \frac{c_2 e^{2x}}{5} + 2 - \frac{2\,e^{2x}}{5} - 3\,e^{3x} + x + 2\,e^{2x}x$$

Verified OK.

11.13.3 Maple step by step solution

Let's solve

$y'' + y' - 6y = 10\,e^{2x} - 18\,e^{3x} - 6x - 11$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 6 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3x} + c_2 e^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 10\,e^{2x} - 18\,e^{3x} - 6x -\right.$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-3x} & e^{2x} \\ -3\,e^{-3x} & 2\,e^{2x} \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = 5\,e^{-x}$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{\left(e^{5x}\left(\int \left(10\,e^{2x}-18\,e^{3x}-6x-11\right)e^{-2x}dx\right)-\left(\int \left(10\,e^{2x}-18\,e^{3x}-6x-11\right)e^{3x}dx\right)\right)e^{-3x}}{5}$$

- Compute integrals

$$y_p(x) = 2 - \frac{2\,e^{2x}}{5} - 3\,e^{3x} + x + 2\,e^{2x}x$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-3x} + c_2 e^{2x} + 2 - \frac{2\,e^{2x}}{5} - 3\,e^{3x} + x + 2\,e^{2x}x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-6*y(x)=10*exp(2*x)-18*exp(3*x)-6*x-11,y(x), singsol=all)
```

$$y(x) = e^{-3x}\left(\left(2x + c_1 - \frac{2}{5}\right)e^{5x} + (x+2)\,e^{3x} + c_2 - 3\,e^{6x}\right)$$

✓ Solution by Mathematica

Time used: 0.299 (sec). Leaf size: 38

```
DSolve[y''[x]+y'[x]-6*y[x]==10*Exp[2*x]-18*Exp[3*x]-6*x-11,y[x],x,IncludeSingularSolutions -
```

$$y(x) \to x - 3e^{3x} + c_1 e^{-3x} + e^{2x}\left(2x - \frac{2}{5} + c_2\right) + 2$$

11.14 problem 14

Internal problem ID [11788]
Internal file name [`OUTPUT/11797_Thursday_April_11_2024_08_49_45_PM_81940993/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' - 2y = 6\,\mathrm{e}^{-2x} + 3\,\mathrm{e}^{x} - 4x^2}$$

11.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -2, f(x) = (-4x^2\mathrm{e}^{2x} + 3\,\mathrm{e}^{3x} + 6)\,\mathrm{e}^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(-2)} \\ &= -\frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{3}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = e^{x} c_1 + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{x} c_1 + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\left(-4x^2 e^{2x} + 3\,e^{3x} + 6\right) e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{e^{-2x}\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-2x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^x\}, \{e^{-2x}\}, \{1, x, x^2\}]$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^x\}, \{x\,e^{-2x}\}, \{1, x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^x + A_2 x\,e^{-2x} + A_3 + A_4 x + A_5 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^x - 3A_2 e^{-2x} + 2A_5 + A_4 + 2A_5 x - 2A_3 - 2A_4 x - 2A_5 x^2 = \left(-4x^2 e^{2x} + 3\,e^{3x} + 6\right) e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -2, A_3 = 3, A_4 = 2, A_5 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\,e^x - 2x\,e^{-2x} + 3 + 2x + 2x^2$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(e^x c_1 + c_2 e^{-2x}\right) + \left(x\,e^x - 2x\,e^{-2x} + 3 + 2x + 2x^2\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^x c_1 + c_2 e^{-2x} + x\,e^x - 2x\,e^{-2x} + 3 + 2x + 2x^2 \tag{1}$$

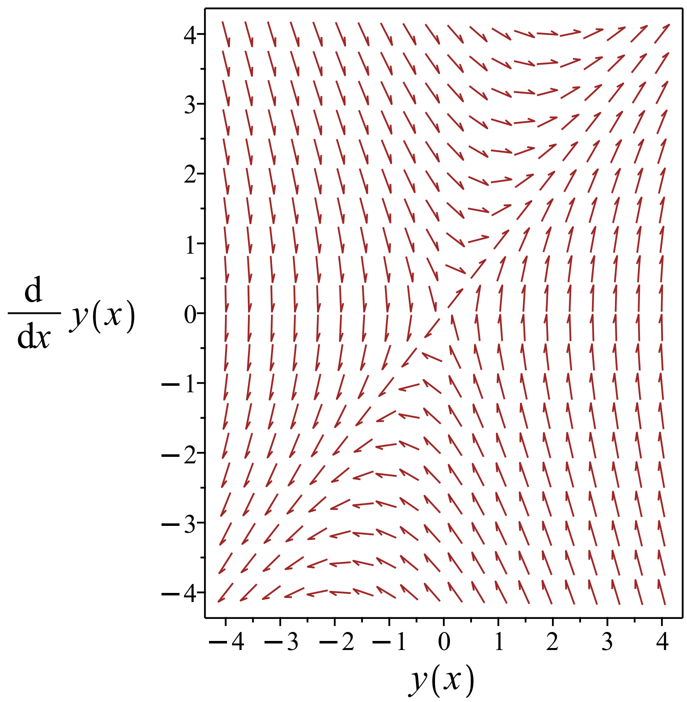


Figure 434: Slope field plot

Verification of solutions

$$y = e^x c_1 + c_2 e^{-2x} + x\,e^x - 2x\,e^{-2x} + 3 + 2x + 2x^2$$

Verified OK.

11.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 367: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx}\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{x}}{3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \frac{\mathrm{e}^{x}}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \frac{\mathrm{e}^{x}}{3} \\ \frac{d}{dx}(\mathrm{e}^{-2x}) & \frac{d}{dx}\left(\frac{\mathrm{e}^{x}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \frac{\mathrm{e}^{x}}{3} \\ -2\,\mathrm{e}^{-2x} & \frac{\mathrm{e}^{x}}{3} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-2x}\right)\left(\frac{\mathrm{e}^{x}}{3}\right) - \left(\frac{\mathrm{e}^{x}}{3}\right)\left(-2\,\mathrm{e}^{-2x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-2x}\mathrm{e}^{x}$$

Which simplifies to

$$W = \mathrm{e}^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{x}\left(-4x^2\mathrm{e}^{2x}+3\,\mathrm{e}^{3x}+6\right)\mathrm{e}^{-2x}}{3}}{\mathrm{e}^{-x}}\,dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{4x^2\mathrm{e}^{2x}}{3}+\mathrm{e}^{3x}+2\right)dx$$

Hence

$$u_1 = -2x+\frac{2x^2\mathrm{e}^{2x}}{3}-\frac{2\,\mathrm{e}^{2x}x}{3}+\frac{\mathrm{e}^{2x}}{3}-\frac{\mathrm{e}^{3x}}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-4x}\left(-4x^2\mathrm{e}^{2x}+3\,\mathrm{e}^{3x}+6\right)}{\mathrm{e}^{-x}}\,dx$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{-3x}\left(-4x^2\mathrm{e}^{2x}+3\,\mathrm{e}^{3x}+6\right)dx$$

Hence

$$u_2 = 3x-2\,\mathrm{e}^{-3x}+4x^2\mathrm{e}^{-x}+8x\,\mathrm{e}^{-x}+8\,\mathrm{e}^{-x}$$

Which simplifies to

$$u_1 = \frac{\left(2x^2-2x+1\right)\mathrm{e}^{2x}}{3}-2x-\frac{\mathrm{e}^{3x}}{3}$$

$$u_2 = \left(4x^2+8x+8\right)\mathrm{e}^{-x}+3x-2\,\mathrm{e}^{-3x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\left(2x^2-2x+1\right)\mathrm{e}^{2x}}{3}-2x-\frac{\mathrm{e}^{3x}}{3}\right)\mathrm{e}^{-2x}+\frac{\left(\left(4x^2+8x+8\right)\mathrm{e}^{-x}+3x-2\,\mathrm{e}^{-3x}\right)\mathrm{e}^{x}}{3}$$

Which simplifies to

$$y_p(x) = e^{-2x}\left(\left(2x^2+2x+3\right)e^{2x}+x\,e^{3x}-2x-\frac{e^{3x}}{3}-\frac{2}{3}\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x}+\frac{c_2 e^{x}}{3}\right)+\left(e^{-2x}\left(\left(2x^2+2x+3\right)e^{2x}+x\,e^{3x}-2x-\frac{e^{3x}}{3}-\frac{2}{3}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x}+\frac{c_2 e^{x}}{3}+e^{-2x}\left(\left(2x^2+2x+3\right)e^{2x}+x\,e^{3x}-2x-\frac{e^{3x}}{3}-\frac{2}{3}\right) \tag{1}$$

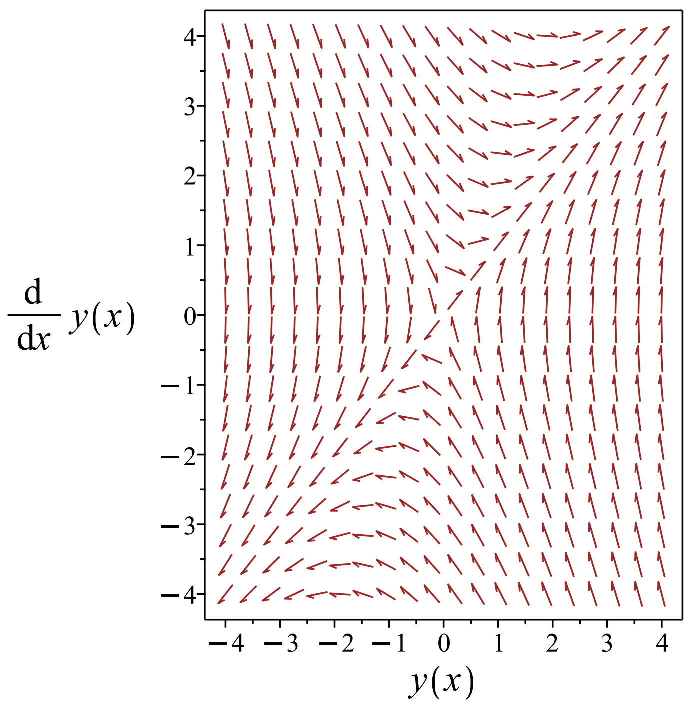


Figure 435: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x}+\frac{c_2 e^{x}}{3}+e^{-2x}\left(\left(2x^2+2x+3\right)e^{2x}+x\,e^{3x}-2x-\frac{e^{3x}}{3}-\frac{2}{3}\right)$$

Verified OK.

11.14.3 Maple step by step solution

Let's solve

$y'' + y' - 2y = (-4x^2 e^{2x} + 3\,e^{3x} + 6)\,e^{-2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -4\,e^{2x} e^{-2x} x^2 + 3\,e^{3x} e^{-2x} - y' + 2y + 6\,e^{-2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' - 2y = -(4x^2 e^{2x} - 3\,e^{3x} - 6)\,e^{-2x}$

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -(4x^2 e^{2x} - 3\,e^{3x} - 6)\,e\right.$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{x} \\ -2\,e^{-2x} & e^{x} \end{bmatrix}$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = 3\,e^{-x}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\frac{\left(e^{3x}\left(\int e^{-3x}\left(4x^2 e^{2x} - 3\,e^{3x} - 6\right) dx\right) + \int \left(-4x^2 e^{2x} + 3\,e^{3x} + 6\right) dx\right) e^{-2x}}{3}$$

- Compute integrals

  $$y_p(x) = e^{-2x}\left(\left(2x^2 + 2x + 3\right) e^{2x} + x\,e^{3x} - 2x - \frac{e^{3x}}{3} - \frac{2}{3}\right)$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 e^{-2x} + c_2 e^{x} + e^{-2x}\left(\left(2x^2 + 2x + 3\right) e^{2x} + x\,e^{3x} - 2x - \frac{e^{3x}}{3} - \frac{2}{3}\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-2*y(x)=6*exp(-2*x)+3*exp(x)-4*x^2,y(x), singsol=all)
```

$$y(x) = e^{-2x}\left(\left(2x^2 + 2x + 3\right) e^{2x} + \left(c_2 + x - \frac{1}{3}\right) e^{3x} - 2x + c_1 - \frac{2}{3}\right)$$

✓ Solution by Mathematica

Time used: 0.163 (sec). Leaf size: 54

```
DSolve[y''[x]+y'[x]-2*y[x]==6*Exp[-2*x]+3*Exp[x]-4*x^2,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{3}e^{-2x}\left(e^{2x}\left(6x^2+6x+9\right)-6x+e^{3x}(3x-1+3c_2)-2+3c_1\right)$$

11.15 problem 15

Internal problem ID [11789]
Internal file name [`OUTPUT/11798_Thursday_April_11_2024_08_49_45_PM_5507649/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 15.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 3y'' + 4y = 4\,e^{x} - 18\,e^{-x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 3y'' + 4y = 0$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x} + x\, e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{2x}$$
$$y_3 = e^{2x} x$$

Now the particular solution to the given ODE is found

$$y''' - 3y'' + 4y = 4\, e^{x} - 18\, e^{-x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\, e^{x} - 18\, e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}, \{e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x} x, e^{-x}, e^{2x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{x}\}, \{x\, e^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{x} + A_2 x\, e^{-x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{x} + 9A_2 e^{-x} = 4\, e^{x} - 18\, e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{x} - 2x\,\mathrm{e}^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + x\,\mathrm{e}^{2x} c_3\right) + \left(2\,\mathrm{e}^{x} - 2x\,\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y = (c_3 x + c_2)\,\mathrm{e}^{2x} + c_1 \mathrm{e}^{-x} + 2\,\mathrm{e}^{x} - 2x\,\mathrm{e}^{-x}$$

Summary
The solution(s) found are the following

$$y = (c_3 x + c_2)\,\mathrm{e}^{2x} + c_1 \mathrm{e}^{-x} + 2\,\mathrm{e}^{x} - 2x\,\mathrm{e}^{-x} \tag{1}$$

Verification of solutions

$$y = (c_3 x + c_2)\,\mathrm{e}^{2x} + c_1 \mathrm{e}^{-x} + 2\,\mathrm{e}^{x} - 2x\,\mathrm{e}^{-x}$$

Verified OK.

11.15.1 Maple step by step solution

Let's solve

$y''' - 3y'' + 4y = 4\,\mathrm{e}^{x} - 18\,\mathrm{e}^{-x}$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

$y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 4\,\mathrm{e}^{x} - 18\,\mathrm{e}^{-x} + 3y_3(x) - 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = 4\,\mathrm{e}^{x} - 18\,\mathrm{e}^{-x} + 3y_3(x) - 4y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 4\,\mathrm{e}^{x} - 18\,\mathrm{e}^{-x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 4\,\mathrm{e}^{x} - 18\,\mathrm{e}^{-x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]\right], \left[2, \left[\begin{array}{c}\frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]\right], \left[2, \left[\begin{array}{c}0 \\ 0 \\ 0\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \left[\begin{array}{c}\frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]\right]$$

- First solution from eigenvalue 2

$$\overrightarrow{y}_2(x) = \mathrm{e}^{2x} \cdot \left[\begin{array}{c}\frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, an

$$\overrightarrow{y}_3(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_3(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\vec{y}_3(x)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

  $$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

  $$\vec{p} = \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

  $$\vec{y}_3(x) = \mathrm{e}^{2x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 0 \end{bmatrix}\right)$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(x) = \begin{bmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{4} & \mathrm{e}^{2x}\left(\frac{x}{4} - \frac{1}{8}\right) \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{2} & \frac{\mathrm{e}^{2x}x}{2} \\ \mathrm{e}^{-x} & \mathrm{e}^{2x} & \mathrm{e}^{2x}x \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & \frac{e^{2x}}{4} & e^{2x}\left(\frac{x}{4}-\frac{1}{8}\right) \\ -e^{-x} & \frac{e^{2x}}{2} & \frac{e^{2x}x}{2} \\ e^{-x} & e^{2x} & e^{2x}x \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{8} \\ -1 & \frac{1}{2} & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} (-2x+1)e^{2x} & -\frac{2e^{-x}}{3}+\frac{2e^{2x}}{3}-e^{2x}x & \frac{e^{-x}}{3}-\frac{e^{2x}}{3}+e^{2x}x \\ -4e^{2x}x & \frac{2e^{-x}}{3}+\frac{e^{2x}}{3}-2e^{2x}x & -\frac{e^{-x}}{3}+\frac{e^{2x}}{3}+2e^{2x}x \\ -8e^{2x}x & -\frac{2e^{-x}}{3}+\frac{2e^{2x}}{3}-4e^{2x}x & \frac{e^{-x}}{3}+\frac{2e^{2x}}{3}+4e^{2x}x \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{(-18x-14)e^{-x}}{3} + \frac{(-6x-4)e^{2x}}{3} + 6e^x \\ \frac{(18x-4)e^{-x}}{3} + \frac{(-12x-14)e^{2x}}{3} + 6e^x \\ \frac{(-18x-14)e^{-x}}{3} + \frac{(-24x-28)e^{2x}}{3} + 14e^x \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \begin{bmatrix} \frac{(-18x-14)e^{-x}}{3} + \frac{(-6x-4)e^{2x}}{3} + 6e^x \\ \frac{(18x-4)e^{-x}}{3} + \frac{(-12x-14)e^{2x}}{3} + 6e^x \\ \frac{(-18x-14)e^{-x}}{3} + \frac{(-24x-28)e^{2x}}{3} + 14e^x \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-32+6(-8+c_3)x+6c_2-3c_3)e^{2x}}{24} + \frac{(-14-18x+3c_1)e^{-x}}{3} + 6e^x$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)+4*y(x)=4*exp(x)-18*exp(-x),y(x), singsol=all)
```

$$y(x) = \frac{(-6x+3c_1-4)e^{-x}}{3} + (c_3x+c_2)e^{2x} + 2e^x$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 44

```
DSolve[y'''[x]-3*y''[x]+4*y[x]==4*Exp[x]-18*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}e^{-x}\left(-6x + 6e^{2x} + 3e^{3x}(c_3x + c_2) - 4 + 3c_1\right)$$

11.16 problem 16

Internal problem ID [11790]
Internal file name [`OUTPUT/11799_Thursday_April_11_2024_08_49_46_PM_29813267/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 16.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 2y'' - y' + 2y = 9\,\mathrm{e}^{2x} - 8\,\mathrm{e}^{3x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 2y'' - y' + 2y = 0$$

The characteristic equation is

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y''' - 2y'' - y' + 2y = 9\,e^{2x} - 8\,e^{3x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$9\,e^{2x} - 8\,e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x} x\}, \{e^{3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{2x} x + A_2 e^{3x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{2x} + 8A_2 e^{3x} = 9\,e^{2x} - 8\,e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3\,e^{2x}x - e^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{x} + e^{2x} c_3\right) + \left(3\,e^{2x}x - e^{3x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{x} + e^{2x} c_3 + 3\,e^{2x}x - e^{3x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{x} + e^{2x} c_3 + 3\,e^{2x}x - e^{3x}$$

Verified OK.

11.16.1 Maple step by step solution

Let's solve

$y''' - 2y'' - y' + 2y = 9\,e^{2x} - 8\,e^{3x}$

- Highest derivative means the order of the ODE is 3

  y'''

☐ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 9\,e^{2x} - 8\,e^{3x} + 2y_3(x) + y_2(x) - 2y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 9\,e^{2x} - 8\,e^{3x} + 2y_3(x) + y_2(x) - 2y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 9\,e^{2x} - 8\,e^{3x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 9\,e^{2x} - 8\,e^{3x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$$

☐ Fundamental matrix

- ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix} e^{-x} & e^{x} & \frac{e^{2x}}{4} \\ -e^{-x} & e^{x} & \frac{e^{2x}}{2} \\ e^{-x} & e^{x} & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t
  $\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$
  $$\Phi(x)=\begin{bmatrix} e^{-x} & e^{x} & \frac{e^{2x}}{4} \\ -e^{-x} & e^{x} & \frac{e^{2x}}{2} \\ e^{-x} & e^{x} & e^{2x} \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} 1 & 1 & \frac{1}{4} \\ -1 & 1 & \frac{1}{2} \\ 1 & 1 & 1 \end{bmatrix}}$$
- Evaluate and simplify to get the fundamental matrix
  $$\Phi(x)=\begin{bmatrix} \frac{e^{-x}}{3}+e^{x}-\frac{e^{2x}}{3} & -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{e^{-x}}{6}-\frac{e^{x}}{2}+\frac{e^{2x}}{3} \\ -\frac{e^{-x}}{3}+e^{x}-\frac{2e^{2x}}{3} & \frac{e^{-x}}{2}+\frac{e^{x}}{2} & -\frac{e^{-x}}{6}-\frac{e^{x}}{2}+\frac{2e^{2x}}{3} \\ \frac{e^{-x}}{3}+e^{x}-\frac{4e^{2x}}{3} & -\frac{e^{-x}}{2}+\frac{e^{x}}{2} & \frac{e^{-x}}{6}-\frac{e^{x}}{2}+\frac{4e^{2x}}{3} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$
  $\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$
- Take the derivative of the particular solution
  $\vec{y}'_p(x)=\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs
  $\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der
  $A\cdot\Phi(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$
- Cancel like terms
  $\Phi(x)\cdot\vec{v}'(x)=\vec{f}(x)$
- Multiply by the inverse of the fundamental matrix
  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot\vec{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} -\frac{4\,e^{2x}}{3}+3\,e^{2x}x-\frac{e^{-x}}{6}+\frac{5\,e^{x}}{2}-e^{3x} \\ \frac{e^{2x}}{3}+6\,e^{2x}x+\frac{e^{-x}}{6}+\frac{5\,e^{x}}{2}-3\,e^{3x} \\ \frac{20\,e^{2x}}{3}+12\,e^{2x}x-\frac{e^{-x}}{6}+\frac{5\,e^{x}}{2}-9\,e^{3x} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \left[\begin{array}{c} -\frac{4\,e^{2x}}{3}+3\,e^{2x}x-\frac{e^{-x}}{6}+\frac{5\,e^{x}}{2}-e^{3x} \\ \frac{e^{2x}}{3}+6\,e^{2x}x+\frac{e^{-x}}{6}+\frac{5\,e^{x}}{2}-3\,e^{3x} \\ \frac{20\,e^{2x}}{3}+12\,e^{2x}x-\frac{e^{-x}}{6}+\frac{5\,e^{x}}{2}-9\,e^{3x} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{(36x+3c_3-16)\,e^{2x}}{12} + \frac{(-1+6c_1)\,e^{-x}}{6} - e^{3x} + \frac{(5+2c_2)\,e^{x}}{2}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$3)-2*diff(y(x),x$2)-diff(y(x),x)+2*y(x)=9*exp(2*x)-8*exp(3*x),y(x), sings
```

$$y(x) = (3x + c_3 - 4)\,e^{2x} + c_1 e^{x} + c_2 e^{-x} - e^{3x}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 44

```
DSolve[y'''[x]-2*y''[x]-y'[x]+2*y[x]==9*Exp[2*x]-8*Exp[3*x],y[x],x,IncludeSingularSolutions
```

$$y(x) \to -e^{3x} + c_1 e^{-x} + \left(\frac{81}{32} + c_2\right) e^{x} + e^{2x}(3x - 4 + c_3)$$

11.17 problem 17

Internal problem ID [11791]
Internal file name [`OUTPUT/11800_Thursday_April_11_2024_08_49_46_PM_28207374/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 17.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + y' = 2x^2 + 4\sin(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y' = 0$$

The characteristic equation is

$$\lambda^3 + \lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + e^{ix} c_2 + e^{-ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= 1 \\ y_2 &= e^{ix} \\ y_3 &= e^{-ix} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' + y' = 2x^2 + 4\sin(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} 1 & e^{ix} & e^{-ix} \\ 0 & i e^{ix} & -i e^{-ix} \\ 0 & -e^{ix} & -e^{-ix} \end{bmatrix}$$

$$|W| = -2i e^{ix} e^{-ix}$$

The determinant simplifies to

$$|W| = -2i$$

Now we determine W_i for each U_i.

$$\begin{aligned}
W_1(x) &= \det \begin{bmatrix} e^{ix} & e^{-ix} \\ ie^{ix} & -ie^{-ix} \end{bmatrix} \\
&= -2i
\end{aligned}$$

$$\begin{aligned}
W_2(x) &= \det \begin{bmatrix} 1 & e^{-ix} \\ 0 & -ie^{-ix} \end{bmatrix} \\
&= -ie^{-ix}
\end{aligned}$$

$$\begin{aligned}
W_3(x) &= \det \begin{bmatrix} 1 & e^{ix} \\ 0 & ie^{ix} \end{bmatrix} \\
&= ie^{ix}
\end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\
&= (-1)^2 \int \frac{(2x^2 + 4\sin(x))(-2i)}{(1)(-2i)}\, dx \\
&= \int \frac{-2i(2x^2 + 4\sin(x))}{-2i}\, dx \\
&= \int \left(2x^2 + 4\sin(x)\right)\, dx \\
&= \frac{2x^3}{3} - 4\cos(x)
\end{aligned}$$

$$\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\
&= (-1)^1 \int \frac{(2x^2 + 4\sin(x))(-ie^{-ix})}{(1)(-2i)}\, dx \\
&= -\int \frac{-i(2x^2 + 4\sin(x))\, e^{-ix}}{-2i}\, dx \\
&= -\int \left(\left(x^2 + 2\sin(x)\right) e^{-ix}\right)\, dx \\
&= -\left(\int \left(x^2 + 2\sin(x)\right) e^{-ix} dx\right)
\end{aligned}$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^0 \int \frac{(2x^2+4\sin(x))\,(i\mathrm{e}^{ix})}{(1)\,(-2i)}\, dx \\
&= \int \frac{i(2x^2+4\sin(x))\,\mathrm{e}^{ix}}{-2i}\, dx \\
&= \int \left(-(x^2+2\sin(x))\,\mathrm{e}^{ix}\right)\, dx \\
&= \frac{-4i\mathrm{e}^{ix}+(-2-i)\,x\,\mathrm{e}^{ix}+(-4+2i)\,\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right)+ix^2\mathrm{e}^{ix}+(-2+i)\,x\,\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right)^2+ix^2\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right)^2 -}{1+\tan\left(\frac{x}{2}\right)^2} \\
&= \frac{-4i\mathrm{e}^{ix}+(-2-i)\,x\,\mathrm{e}^{ix}+(-4+2i)\,\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right)+ix^2\mathrm{e}^{ix}+(-2+i)\,x\,\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right)^2+ix^2\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right)^2 -}{1+\tan\left(\frac{x}{2}\right)^2}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3
$$

Hence

$$
\begin{aligned}
y_p =& \left(\frac{2x^3}{3} - 4\cos(x)\right) \\
&+ \left(-\left(\int (x^2+2\sin(x))\,\mathrm{e}^{-ix}dx\right)\right)(\mathrm{e}^{ix}) \\
&+ \left(\frac{-4i\mathrm{e}^{ix}+(-2-i)\,x\,\mathrm{e}^{ix}+(-4+2i)\,\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right)+ix^2\mathrm{e}^{ix}+(-2+i)\,x\,\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right)^2+ix^2\mathrm{e}^{ix}\tan\left(\frac{x}{2}\right)^2}{1+\tan\left(\frac{x}{2}\right)^2}\right.
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = -2i-\left(\int (x^2+2\sin(x))\,\mathrm{e}^{-ix}dx\right)\mathrm{e}^{ix}+(-ix-2i-4)\cos(x)+(-2+i-x)\sin(x)+ix^2+\frac{2x^3}{3}-2x
$$

Which simplifies to

$$
y_p = -2i-\left(\int (x^2+2\sin(x))\cos(x)\,dx\right)(i\sin(x)+\cos(x))+(-\sin(x)+i\cos(x))\left(\int (x^2+2\sin(x))\right.
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1+\mathrm{e}^{ix}c_2+\mathrm{e}^{-ix}c_3\right) + \left(-2i - \left(\int (x^2+2\sin(x))\cos(x)\,dx\right)(i\sin(x)+\cos(x))\right. \\
&\quad + (-\sin(x)+i\cos(x))\left(\int (x^2+2\sin(x))\sin(x)\,dx\right) + (-ix-2i-4)\cos(x) \\
&\quad \left. + (-2+i-x)\sin(x) + ix^2 + \frac{2x^3}{3} - 2x\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 + \mathrm{e}^{ix} c_2 + \mathrm{e}^{-ix} c_3 - 2i - \left(\int \left(x^2 + 2\sin(x)\right)\cos(x)\, dx\right)\left(i\sin(x) + \cos(x)\right) \\
& + \left(-\sin(x) + i\cos(x)\right)\left(\int \left(x^2 + 2\sin(x)\right)\sin(x)\, dx\right) \\
& + \left(-ix - 2i - 4\right)\cos(x) + \left(-2 + i - x\right)\sin(x) + i x^2 + \frac{2x^3}{3} - 2x
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 + \mathrm{e}^{ix} c_2 + \mathrm{e}^{-ix} c_3 - 2i - \left(\int \left(x^2 + 2\sin(x)\right)\cos(x)\, dx\right)\left(i\sin(x) + \cos(x)\right) \\
& + \left(-\sin(x) + i\cos(x)\right)\left(\int \left(x^2 + 2\sin(x)\right)\sin(x)\, dx\right) \\
& + \left(-ix - 2i - 4\right)\cos(x) + \left(-2 + i - x\right)\sin(x) + i x^2 + \frac{2x^3}{3} - 2x
\end{aligned}
$$

Verified OK.

11.17.1 Maple step by step solution

Let's solve

$y''' + y' = 2x^2 + 4\sin(x)$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 2x^2 + 4\sin(x) - y_2(x)$

  Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 2x^2 + 4\sin(x) - y_2(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 2x^2 + 4\sin(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 2x^2 + 4\sin(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}x} \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

  ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 1 & -\cos(x) & \sin(x) \\ 0 & \sin(x) & \cos(x) \\ 0 & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} 1 & -\cos(x) & \sin(x) \\ 0 & \sin(x) & \cos(x) \\ 0 & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & \sin(x) & 1-\cos(x) \\ 0 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} -2x \sin(x) + 4 \sin(x) + \frac{2x^3}{3} - 4 \cos(x) - 4x + 4 \\ (-2x + 4) \cos(x) + 2x^2 + 2 \sin(x) - 4 \\ (2x - 4) \sin(x) + 4x \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + \begin{bmatrix} -2x \sin(x) + 4 \sin(x) + \frac{2x^3}{3} - 4 \cos(x) - 4x + 4 \\ (-2x + 4) \cos(x) + 2x^2 + 2 \sin(x) - 4 \\ (2x - 4) \sin(x) + 4x \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = 4 + (-2x + 4 + c_3) \sin(x) + (-4 - c_2) \cos(x) + \frac{2x^3}{3} - 4x + c_1$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 2*_a^2+4*sin(_a)-_b(_a), _b(_
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$3)+diff(y(x),x)=2*x^2+4*sin(x),y(x), singsol=all)
```

$$y(x) = (-2 - c_2)\cos(x) + (-2x + c_1)\sin(x) + \frac{2x^3}{3} - 4x + c_3$$

✓ Solution by Mathematica

Time used: 0.233 (sec). Leaf size: 35

```
DSolve[y'''[x]+y'[x]==2*x^2+4*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2x^3}{3} - 4x - (2 + c_2)\cos(x) + (-2x + c_1)\sin(x) + c_3$$

11.18 problem 18

Internal problem ID [11792]
Internal file name [`OUTPUT/11801_Thursday_April_11_2024_08_49_47_PM_42252830/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 18.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_y]]
```

$$\boxed{y'''' - 3y''' + 2y'' = 3\,\mathrm{e}^{-x} + 6\,\mathrm{e}^{2x} - 6x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 3y''' + 2y'' = 0$$

The characteristic equation is

$$\lambda^4 - 3\lambda^3 + 2\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 2$$
$$\lambda_4 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + c_3 e^x + e^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = e^x$$
$$y_4 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y'''' - 3y''' + 2y'' = 3\,e^{-x} + 6\,e^{2x} - 6x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\,e^{-x} + 6\,e^{2x} - 6x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}, \{e^{2x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, e^x, e^{2x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-x}\}, \{e^{2x}\}, \{x, x^2\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-x}\}, \{e^{2x}\}, \{x^2, x^3\}]$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-x}\}, \{e^{2x} x\}, \{x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{-x} + A_2 e^{2x} x + A_3 x^2 + A_4 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^{-x} + 4A_2 e^{2x} - 18A_4 + 4A_3 + 12A_4 x = 3\, e^{-x} + 6\, e^{2x} - 6x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = \frac{3}{2}, A_3 = -\frac{9}{4}, A_4 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-x}}{2} + \frac{3\, e^{2x} x}{2} - \frac{9x^2}{4} - \frac{x^3}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_2 x + c_1 + c_3 e^{x} + e^{2x} c_4\right) + \left(\frac{e^{-x}}{2} + \frac{3\, e^{2x} x}{2} - \frac{9x^2}{4} - \frac{x^3}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x + c_1 + c_3 e^{x} + e^{2x} c_4 + \frac{e^{-x}}{2} + \frac{3\, e^{2x} x}{2} - \frac{9x^2}{4} - \frac{x^3}{2} \tag{1}$$

Verification of solutions

$$y = c_2 x + c_1 + c_3 e^{x} + e^{2x} c_4 + \frac{e^{-x}}{2} + \frac{3\, e^{2x} x}{2} - \frac{9x^2}{4} - \frac{x^3}{2}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 3*(diff(_b(_a), _a))-2*_b(_a)
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   <- double symmetry of the form [xi=0, eta=F(x)] successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$4)-3*diff(y(x),x$3)+2*diff(y(x),x$2)=3*exp(-x)+6*exp(2*x)-6*x,y(x), sings
```

$$y(x) = \frac{(6x + c_1 - 12)\,\mathrm{e}^{2x}}{4} - \frac{x^3}{2} - \frac{9x^2}{4} + c_3x + c_2\mathrm{e}^{x} + c_4 + \frac{\mathrm{e}^{-x}}{2}$$

✓ Solution by Mathematica

Time used: 0.372 (sec). Leaf size: 54

```
DSolve[y''''[x]-3*y'''[x]+2*y''[x]==3*Exp[-x]+6*Exp[2*x]-6*x,y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{1}{4}\left(-\left((2x+9)x^2\right) + 2e^{-x} + 4c_1e^{x} + e^{2x}(6x - 12 + c_2)\right) + c_4x + c_3$$

11.19 problem 19

Internal problem ID [11793]
Internal file name [`OUTPUT/11802_Thursday_April_11_2024_08_49_47_PM_75614766/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 19.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 6y'' + 11y' - 6y = x\,\mathrm{e}^{x} - 4\,\mathrm{e}^{2x} + 6\,\mathrm{e}^{4x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 6y'' + 11y' - 6y = 0$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + c_2 e^{2x} + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{2x}$$
$$y_3 = e^{3x}$$

Now the particular solution to the given ODE is found

$$y''' - 6y'' + 11y' - 6y = x\,e^x - 4\,e^{2x} + 6\,e^{4x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,e^x - 4\,e^{2x} + 6\,e^{4x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{e^{4x}\}, \{x\,e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}, e^{3x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x}\}, \{e^{4x}\}, \{x\,e^x, x^2 e^x\}]$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x} x\}, \{e^{4x}\}, \{x\,e^x, x^2 e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{2x} x + A_2 e^{4x} + A_3 x\,e^x + A_4 x^2 e^x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_4x\,\mathrm{e}^{x} - A_1\mathrm{e}^{2x} + 2A_3\mathrm{e}^{x} - 6A_4\mathrm{e}^{x} + 6A_2\mathrm{e}^{4x} = x\,\mathrm{e}^{x} - 4\,\mathrm{e}^{2x} + 6\,\mathrm{e}^{4x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 4, A_2 = 1, A_3 = \frac{3}{4}, A_4 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4\,\mathrm{e}^{2x}x + \mathrm{e}^{4x} + \frac{3x\,\mathrm{e}^{x}}{4} + \frac{x^2\mathrm{e}^{x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{x}c_1 + c_2\mathrm{e}^{2x} + c_3\mathrm{e}^{3x}\right) + \left(4\,\mathrm{e}^{2x}x + \mathrm{e}^{4x} + \frac{3x\,\mathrm{e}^{x}}{4} + \frac{x^2\mathrm{e}^{x}}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x}c_1 + c_2\mathrm{e}^{2x} + c_3\mathrm{e}^{3x} + 4\,\mathrm{e}^{2x}x + \mathrm{e}^{4x} + \frac{3x\,\mathrm{e}^{x}}{4} + \frac{x^2\mathrm{e}^{x}}{4} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{x}c_1 + c_2\mathrm{e}^{2x} + c_3\mathrm{e}^{3x} + 4\,\mathrm{e}^{2x}x + \mathrm{e}^{4x} + \frac{3x\,\mathrm{e}^{x}}{4} + \frac{x^2\mathrm{e}^{x}}{4}$$

Verified OK.

11.19.1 Maple step by step solution

Let's solve

$y''' - 6y'' + 11y' - 6y = x\,\mathrm{e}^{x} - 4\,\mathrm{e}^{2x} + 6\,\mathrm{e}^{4x}$

- Highest derivative means the order of the ODE is 3

  y'''

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

  $y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = x\,\mathrm{e}^{x} - 4\,\mathrm{e}^{2x} + 6\,\mathrm{e}^{4x} + 6y_3(x) - 11y_2(x) + 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x\,\mathrm{e}^{x} - 4\,\mathrm{e}^{2x} + 6\,\mathrm{e}^{4x} + 6y_3(x) - 11y_2(x) + 6y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x\,\mathrm{e}^{x} - 4\,\mathrm{e}^{2x} + 6\,\mathrm{e}^{4x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x\,\mathrm{e}^{x} - 4\,\mathrm{e}^{2x} + 6\,\mathrm{e}^{4x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right], \left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^x & \frac{e^{2x}}{4} & \frac{e^{3x}}{9} \\ e^x & \frac{e^{2x}}{2} & \frac{e^{3x}}{3} \\ e^x & e^{2x} & e^{3x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^x & \frac{e^{2x}}{4} & \frac{e^{3x}}{9} \\ e^x & \frac{e^{2x}}{2} & \frac{e^{3x}}{3} \\ e^x & e^{2x} & e^{3x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{9} \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 3e^x - 3e^{2x} + e^{3x} & -\frac{5e^x}{2} + 4e^{2x} - \frac{3e^{3x}}{2} & \frac{e^x}{2} - e^{2x} + \frac{e^{3x}}{2} \\ 3e^x - 6e^{2x} + 3e^{3x} & -\frac{5e^x}{2} + 8e^{2x} - \frac{9e^{3x}}{2} & \frac{e^x}{2} - 2e^{2x} + \frac{3e^{3x}}{2} \\ 3e^x - 12e^{2x} + 9e^{3x} & -\frac{5e^x}{2} + 16e^{2x} - \frac{27e^{3x}}{2} & \frac{e^x}{2} - 4e^{2x} + \frac{9e^{3x}}{2} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x)\cdot \vec{v}(x)+\Phi(x)\cdot \vec{v}'(x)=A\cdot \Phi(x)\cdot \vec{v}(x)+\vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot \Phi(x)\cdot \vec{v}(x)+\Phi(x)\cdot \vec{v}'(x)=A\cdot \Phi(x)\cdot \vec{v}(x)+\vec{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot \vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x)=\Phi(x)\cdot \left(\int_0^x \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x)=\left[\begin{array}{c} 2(2x+1)\,\mathrm{e}^{2x}-\frac{39\,\mathrm{e}^{3x}}{8}+\mathrm{e}^{4x}+\frac{(2x^2+6x+15)\mathrm{e}^{x}}{8} \\ 8(1+x)\,\mathrm{e}^{2x}-\frac{117\,\mathrm{e}^{3x}}{8}+4\,\mathrm{e}^{4x}+\frac{(2x^2+10x+21)\mathrm{e}^{x}}{8} \\ 8(3+2x)\,\mathrm{e}^{2x}-\frac{351\,\mathrm{e}^{3x}}{8}+16\,\mathrm{e}^{4x}+\frac{(2x^2+14x+31)\mathrm{e}^{x}}{8} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3+\left[\begin{array}{c} 2(2x+1)\,\mathrm{e}^{2x}-\frac{39\,\mathrm{e}^{3x}}{8}+\mathrm{e}^{4x}+\frac{(2x^2+6x+15)\mathrm{e}^{x}}{8} \\ 8(1+x)\,\mathrm{e}^{2x}-\frac{117\,\mathrm{e}^{3x}}{8}+4\,\mathrm{e}^{4x}+\frac{(2x^2+10x+21)\mathrm{e}^{x}}{8} \\ 8(3+2x)\,\mathrm{e}^{2x}-\frac{351\,\mathrm{e}^{3x}}{8}+16\,\mathrm{e}^{4x}+\frac{(2x^2+14x+31)\mathrm{e}^{x}}{8} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y=\frac{(8+16x+c_2)\mathrm{e}^{2x}}{4}+\frac{(8c_3-351)\mathrm{e}^{3x}}{72}+\mathrm{e}^{4x}+\frac{(2x^2+8c_1+6x+15)\mathrm{e}^{x}}{8}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 42

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+11*diff(y(x),x)-6*y(x)=x*exp(x)-4*exp(2*x)+6*exp(4*x)
```

$$y(x) = (4x + c_2)\,e^{2x} + c_3 e^{3x} + e^{4x} + \frac{(2x^2 + 8c_1 + 6x + 7)\,e^x}{8}$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 53

```
DSolve[y'''[x]-6*y''[x]+11*y'[x]-6*y[x]==x*Exp[x]-4*Exp[2*x]+6*Exp[4*x],y[x],x,IncludeSingul
```

$$y(x) \to \frac{1}{8}e^x\left(2x^2 + 6x + 8e^{3x} + 8e^x(4x + c_2) + 8c_3 e^{2x} + 7 + 8c_1\right)$$

11.20 problem 20

Internal problem ID [11794]
Internal file name [`OUTPUT/11803_Thursday_April_11_2024_08_49_48_PM_81427022/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 20.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 4y'' + 5y' - 2y = 3x^2 \mathrm{e}^x - 7\,\mathrm{e}^x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y'' + 5y' - 2y = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + c_2 x\, e^x + e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = x\, e^x$$
$$y_3 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y''' - 4y'' + 5y' - 2y = 3x^2 e^x - 7\, e^x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3x^2 e^x - 7\, e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^x, x^2 e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^x, e^x, e^{2x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^x, x^2 e^x, x^3 e^x\}]$$

Since $x\, e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^x, x^3 e^x, e^x x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^x + A_2 x^3 e^x + A_3 e^x x^4$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_2 x\,\mathrm{e}^x - 12A_3\mathrm{e}^x x^2 + 24A_3\mathrm{e}^x x + 6A_2\mathrm{e}^x - 2A_1\mathrm{e}^x = 3x^2\mathrm{e}^x - 7\,\mathrm{e}^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -1, A_3 = -\frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2\mathrm{e}^x}{2} - x^3\mathrm{e}^x - \frac{\mathrm{e}^x x^4}{4}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^x c_1 + c_2 x\,\mathrm{e}^x + \mathrm{e}^{2x} c_3\right) + \left(\frac{x^2\mathrm{e}^x}{2} - x^3\mathrm{e}^x - \frac{\mathrm{e}^x x^4}{4}\right)
\end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{2x} c_3 + \mathrm{e}^x (c_2 x + c_1) + \frac{x^2\mathrm{e}^x}{2} - x^3\mathrm{e}^x - \frac{\mathrm{e}^x x^4}{4}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2x} c_3 + \mathrm{e}^x (c_2 x + c_1) + \frac{x^2\mathrm{e}^x}{2} - x^3\mathrm{e}^x - \frac{\mathrm{e}^x x^4}{4} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{2x} c_3 + \mathrm{e}^x (c_2 x + c_1) + \frac{x^2\mathrm{e}^x}{2} - x^3\mathrm{e}^x - \frac{\mathrm{e}^x x^4}{4}$$

Verified OK.

11.20.1 Maple step by step solution

Let's solve

$y''' - 4y'' + 5y' - 2y = 3x^2 e^x - 7 e^x$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs

  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$

  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$

  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$

  - ○ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 3x^2 e^x + 4y_3(x) - 5y_2(x) + 2y_1(x) - 7 e^x$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3x^2 e^x + 4y_3(x) - 5y_2(x) + 2y_1(x) - 7 e^x]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 3x^2 e^x - 7 e^x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 3x^2 e^x - 7 e^x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

$$\vec{y}_1(x) = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\vec{y}_2(x) = e^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = \left(e^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = e^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$\lambda\vec{p}+\vec{v}=A\cdot\vec{p}$

- Make use of the identity matrix I

  $(\lambda\cdot I)\cdot\vec{p}+\vec{v}=A\cdot\vec{p}$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

  $(A-\lambda\cdot I)\cdot\vec{p}=\vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & -5 & 4\end{bmatrix}-1\cdot\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\cdot\vec{p}=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p}=\begin{bmatrix}-1\\ 0\\ 0\end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{y}_2(x)=\mathrm{e}^{x}\cdot\left(x\cdot\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}+\begin{bmatrix}-1\\ 0\\ 0\end{bmatrix}\right)$$

- Consider eigenpair

$$\left[2,\begin{bmatrix}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3=\mathrm{e}^{2x}\cdot\begin{bmatrix}\frac{1}{4}\\ \frac{1}{2}\\ 1\end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x)=c_1\vec{y}_1(x)+c_2\vec{y}_2(x)+c_3\vec{y}_3+\vec{y}_p(x)$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix} e^{x} & e^{x}(x-1) & \frac{e^{2x}}{4} \\ e^{x} & x\,e^{x} & \frac{e^{2x}}{2} \\ e^{x} & x\,e^{x} & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

$\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix} e^{x} & e^{x}(x-1) & \frac{e^{2x}}{4} \\ e^{x} & x\,e^{x} & \frac{e^{2x}}{2} \\ e^{x} & x\,e^{x} & e^{2x} \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} 1 & -1 & \frac{1}{4} \\ 1 & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} -e^{x}(x-1) & -\frac{e^{2x}}{2}+\frac{e^{x}(3x+1)}{2} & \frac{e^{2x}}{2}+\frac{e^{x}(-x-1)}{2} \\ -x\,e^{x} & 2\,e^{x}+\frac{3x\,e^{x}}{2}-e^{2x} & -e^{x}-\frac{x\,e^{x}}{2}+e^{2x} \\ -x\,e^{x} & 2\,e^{x}+\frac{3x\,e^{x}}{2}-2\,e^{2x} & -e^{x}-\frac{x\,e^{x}}{2}+2\,e^{2x} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$

- Take the derivative of the particular solution

$\vec{y}_p'(x)=\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A\cdot\Phi(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- Cancel like terms

$\Phi(x)\cdot\vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} -\frac{e^{2x}}{2} + \frac{(-x^4-4x^3+2x^2+4x+4)e^x}{8} \\ -e^{2x} + \frac{(-x^4-8x^3-10x^2+8x+8)e^x}{8} \\ -2\,e^{2x} + \frac{(-x^4-8x^3-34x^2-40x+16)e^x}{8} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3 + \begin{bmatrix} -\frac{e^{2x}}{2} + \frac{(-x^4-4x^3+2x^2+4x+4)e^x}{8} \\ -e^{2x} + \frac{(-x^4-8x^3-10x^2+8x+8)e^x}{8} \\ -2\,e^{2x} + \frac{(-x^4-8x^3-34x^2-40x+16)e^x}{8} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(2c_3-4)e^{2x}}{8} - \frac{(x^4+4x^3-2x^2+(-8c_2-4)x-8c_1+8c_2-4)e^x}{8}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x$2)+5*diff(y(x),x)-2*y(x)=3*x^2*exp(x)-7*exp(x),y(x), sin
```

$$y(x) = -\frac{e^{x}\left(x^{4}+4x^{3}-4c_{2}e^{x}-4c_{3}x-2x^{2}-4c_{1}\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 47

```
DSolve[y'''[x]-4*y''[x]+5*y'[x]-2*y[x]==3*x^2*Exp[x]-7*Exp[x],y[x],x,IncludeSingularSolution
```

$$y(x) \to \frac{1}{4}e^{x}\left(-x^{4}-4x^{3}+2x^{2}+4(1+c_{2})x+4(c_{3}e^{x}+1+c_{1})\right)$$

11.21 problem 21

Internal problem ID [11795]
Internal file name [`OUTPUT/11804_Thursday_April_11_2024_08_49_48_PM_227895/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = x \sin (x)}$$

11.21.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = x \sin (x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \sin(x), \cos(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \sin(x), \cos(x) x, \cos(x) x^2, \sin(x) x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \sin(x) + A_2 \cos(x) x + A_3 \cos(x) x^2 + A_4 \sin(x) x^2$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \cos(x) - 2A_2 \sin(x) - 4A_3 \sin(x) x + 2A_3 \cos(x) + 4A_4 \cos(x) x + 2A_4 \sin(x) = x \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}, A_2 = 0, A_3 = -\frac{1}{4}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x \sin(x)}{4} - \frac{\cos(x) x^2}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{x \sin(x)}{4} - \frac{\cos(x) x^2}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) + \frac{x \sin(x)}{4} - \frac{\cos(x) x^2}{4} \tag{1}
$$

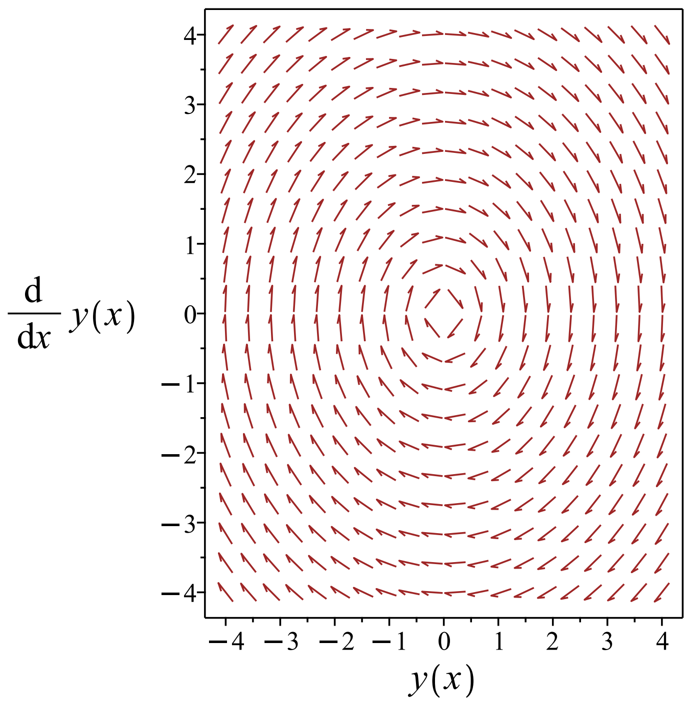


Figure 436: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) + \frac{x \sin(x)}{4} - \frac{\cos(x) x^2}{4}
$$

Verified OK.

11.21.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
y'' + y &= 0 \qquad (1)\\
Ay'' + By' + Cy &= 0 \qquad (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1\\
B &= 0 \qquad (3)\\
C &= 1
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \qquad (5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1\\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 374: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\sin(x), \cos(x)x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\sin(x), \cos(x)x, \cos(x)x^2, \sin(x)x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x\sin(x) + A_2\cos(x)x + A_3\cos(x)x^2 + A_4\sin(x)x^2$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_1\cos(x) - 2A_2\sin(x) - 4A_3\sin(x)x + 2A_3\cos(x) + 4A_4\cos(x)x + 2A_4\sin(x)\\&= x\sin(x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}, A_2 = 0, A_3 = -\frac{1}{4}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\sin(x)}{4} - \frac{\cos(x)x^2}{4}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= (c_1\cos(x) + c_2\sin(x)) + \left(\frac{x\sin(x)}{4} - \frac{\cos(x)x^2}{4}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{x \sin(x)}{4} - \frac{\cos(x) x^2}{4} \tag{1}$$

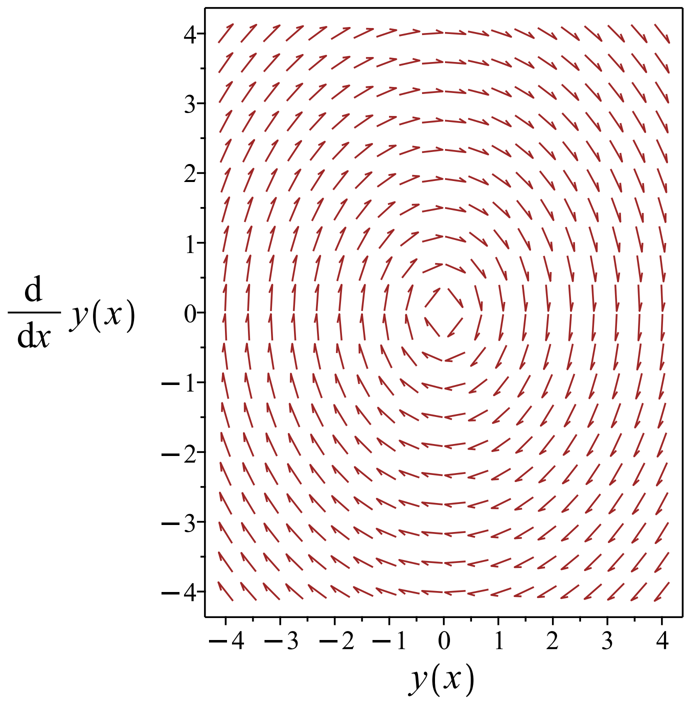


Figure 437: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{x \sin(x)}{4} - \frac{\cos(x) x^2}{4}$$

Verified OK.

11.21.3 Maple step by step solution

Let's solve

$y'' + y = x \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x \sin(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \sin(x)^2 x dx\right) + \frac{\sin(x)\left(\int \sin(2x) x dx\right)}{2}$

  ◦ Compute integrals

    $y_p(x) = \frac{x(-\cos(x) x + \sin(x))}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \frac{x(-\cos(x) x + \sin(x))}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+y(x)=x*sin(x),y(x), singsol=all)
```

$$y(x) = \frac{(-x^2 + 4c_1)\cos(x)}{4} + \frac{\sin(x)(4c_2 + x)}{4}$$

✓ Solution by Mathematica
Time used: 0.046 (sec). Leaf size: 34

```
DSolve[y''[x]+y[x]==x*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}\left((-2x^2 + 1 + 8c_1)\cos(x) + 2(x + 4c_2)\sin(x)\right)$$

11.22 problem 22

Internal problem ID [11796]
Internal file name [`OUTPUT/11805_Thursday_April_11_2024_08_49_49_PM_63250674/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = 12x^2 - 16x\cos(2x)}$$

11.22.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = 12x^2 - 16x\cos(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12x^2 - 16x \cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}, \{x \cos(2x), \sin(2x) x, \cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1, x, x^2\}, \{x \cos(2x), x^2 \cos(2x), x^2 \sin(2x), \sin(2x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3x^2 + A_2x + A_1 + A_4x \cos(2x) + A_5x^2 \cos(2x) + A_6x^2 \sin(2x) + A_7 \sin(2x) x$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_3 - 4A_4 \sin(2x) + 2A_5 \cos(2x) - 8A_5x \sin(2x) + 2A_6 \sin(2x)\\ &+ 8A_6x \cos(2x) + 4A_7 \cos(2x) + 4A_3x^2 + 4A_2x + 4A_1 = 12x^2 - 16x \cos(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{2}, A_2 = 0, A_3 = 3, A_4 = -1, A_5 = 0, A_6 = -2, A_7 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x^2 - \frac{3}{2} - x\cos(2x) - 2x^2\sin(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(2x) + c_2\sin(2x)) + \left(3x^2 - \frac{3}{2} - x\cos(2x) - 2x^2\sin(2x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(2x) + c_2\sin(2x) + 3x^2 - \frac{3}{2} - x\cos(2x) - 2x^2\sin(2x) \tag{1}$$

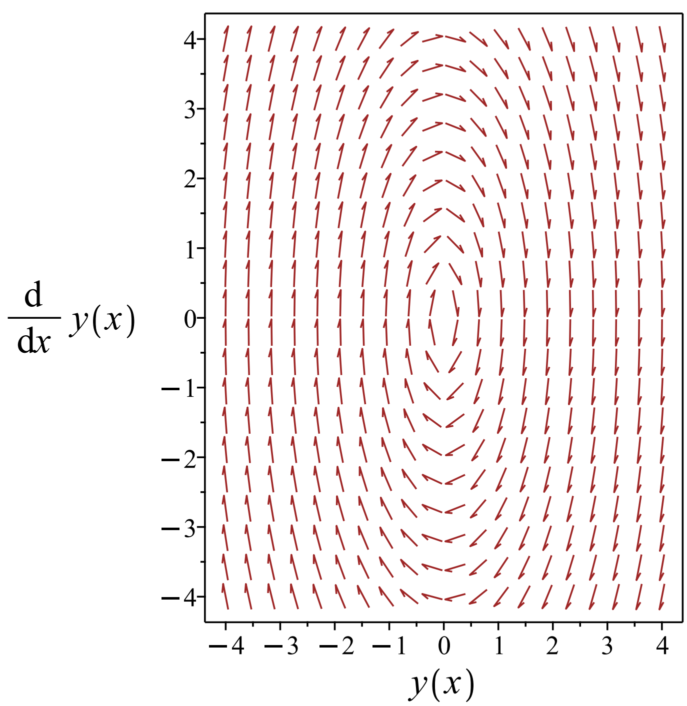


Figure 438: Slope field plot

Verification of solutions

$$y = c_1\cos(2x) + c_2\sin(2x) + 3x^2 - \frac{3}{2} - x\cos(2x) - 2x^2\sin(2x)$$

Verified OK.

11.22.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 376: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(2x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(2x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\
&= \cos(2x) \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12x^2 - 16x \cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}, \{x \cos(2x), \sin(2x)\, x, \cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1, x, x^2\}, \{x \cos(2x), x^2 \cos(2x), x^2 \sin(2x), \sin(2x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3x^2 + A_2x + A_1 + A_4x \cos(2x) + A_5x^2 \cos(2x) + A_6x^2 \sin(2x) + A_7 \sin(2x)\, x$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_3 - 4A_4 \sin(2x) + 2A_5 \cos(2x) - 8A_5x \sin(2x) + 2A_6 \sin(2x)\\&+ 8A_6x \cos(2x) + 4A_7 \cos(2x) + 4A_3x^2 + 4A_2x + 4A_1 = 12x^2 - 16x \cos(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{2}, A_2 = 0, A_3 = 3, A_4 = -1, A_5 = 0, A_6 = -2, A_7 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x^2 - \frac{3}{2} - x\cos(2x) - 2x^2\sin(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(2x) + \frac{c_2\sin(2x)}{2}\right) + \left(3x^2 - \frac{3}{2} - x\cos(2x) - 2x^2\sin(2x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + 3x^2 - \frac{3}{2} - x\cos(2x) - 2x^2\sin(2x) \tag{1}$$

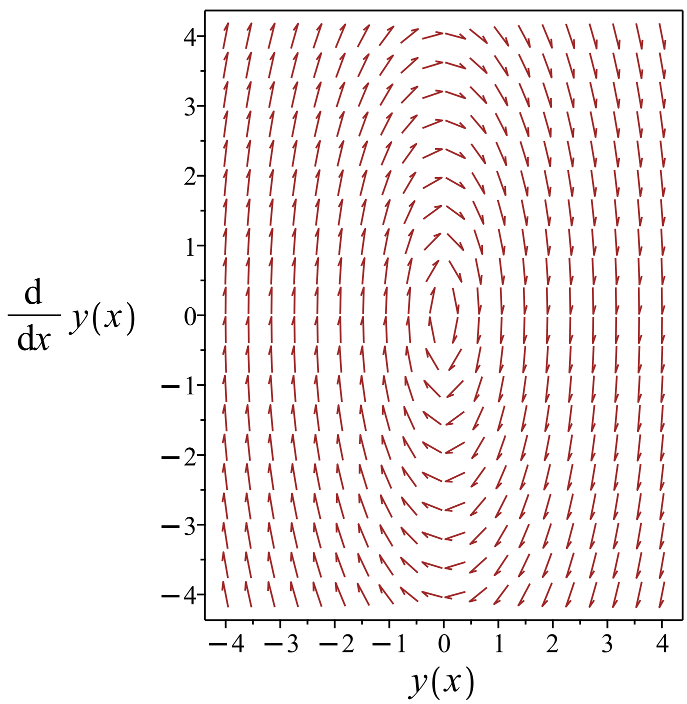


Figure 439: Slope field plot

Verification of solutions

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + 3x^2 - \frac{3}{2} - x\cos(2x) - 2x^2\sin(2x)$$

Verified OK.

11.22.3 Maple step by step solution

Let's solve

$y'' + 4y = 12x^2 - 16x\cos(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 12x^2 - 16x\cos(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = 2\cos(2x)\left(\int \sin(2x)\, x(4\cos(2x) - 3x)\, dx\right) - 2\sin(2x)\left(\int \cos(2x)\, x(4\cos(2x) - 3x)\, dx\right)$$

- Compute integrals

$$y_p(x) = -2x^2 \sin(2x) + \frac{\sin(2x)}{2} - x\cos(2x) + 3x^2 - \frac{3}{2}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2x) + c_2 \sin(2x) - 2x^2 \sin(2x) + \frac{\sin(2x)}{2} - x\cos(2x) + 3x^2 - \frac{3}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)+4*y(x)=12*x^2-16*x*cos(2*x),y(x), singsol=all)
```

$$y(x) = -\frac{3}{2} + \frac{(-8x^2 + 4c_2 + 1)\sin(2x)}{4} + (c_1 - x)\cos(2x) + 3x^2$$

✓ Solution by Mathematica

Time used: 0.251 (sec). Leaf size: 44

```
DSolve[y''[x]+4*y[x]==12*x^2-16*x*Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3x^2 + \frac{1}{4}(-8x^2 + 1 + 4c_2)\sin(2x) + (-x + c_1)\cos(2x) - \frac{3}{2}$$

11.23 problem 23

Internal problem ID [11797]
Internal file name [`OUTPUT/11806_Thursday_April_11_2024_08_49_50_PM_86836542/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 23.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_y]]`

$$\boxed{y'''' + 2y''' - 3y'' = 18x^2 + 16x\,e^x + 4\,e^{3x} - 9}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y''' - 3y'' = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^3 - 3\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 1$$
$$\lambda_4 = -3$$

Therefore the homogeneous solution is

$$y_h(x) = c_2x + c_1 + c_3\mathrm{e}^x + \mathrm{e}^{-3x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= \mathrm{e}^x \\
y_4 &= \mathrm{e}^{-3x}
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 2y''' - 3y'' = 18x^2 + 16x\,\mathrm{e}^x + 4\,\mathrm{e}^{3x} - 9$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$18x^2 + 16x\,\mathrm{e}^x + 4\,\mathrm{e}^{3x} - 9$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{3x}\}, \{x\,\mathrm{e}^x, \mathrm{e}^x\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, \mathrm{e}^x, \mathrm{e}^{-3x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^{3x}\}, \{x\,\mathrm{e}^x, \mathrm{e}^x\}, \{x, x^2, x^3\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^{3x}\}, \{x\,\mathrm{e}^x, \mathrm{e}^x\}, \{x^2, x^3, x^4\}]$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^{3x}\}, \{x\,\mathrm{e}^x, x^2\mathrm{e}^x\}, \{x^2, x^3, x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{3x} + A_2 x\, e^{x} + A_3 x^2 e^{x} + A_4 x^2 + A_5 x^3 + A_6 x^4$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&108A_1 e^{3x} + 4A_2 e^{x} + 18A_3 e^{x} + 8A_3 x\, e^{x} + 24A_6 + 12A_5 + 48A_6 x \\ &- 6A_4 - 18A_5 x - 36A_6 x^2 = 18x^2 + 16x\, e^{x} + 4\, e^{3x} - 9\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{27}, A_2 = -9, A_3 = 2, A_4 = -\frac{19}{6}, A_5 = -\frac{4}{3}, A_6 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{3x}}{27} - 9x\, e^{x} + 2x^2 e^{x} - \frac{19x^2}{6} - \frac{4x^3}{3} - \frac{x^4}{2}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(c_2 x + c_1 + c_3 e^{x} + e^{-3x} c_4\right) + \left(\frac{e^{3x}}{27} - 9x\, e^{x} + 2x^2 e^{x} - \frac{19x^2}{6} - \frac{4x^3}{3} - \frac{x^4}{2}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x + c_1 + c_3 e^{x} + e^{-3x} c_4 + \frac{e^{3x}}{27} - 9x\, e^{x} + 2x^2 e^{x} - \frac{19x^2}{6} - \frac{4x^3}{3} - \frac{x^4}{2} \tag{1}$$

Verification of solutions

$$y = c_2 x + c_1 + c_3 e^{x} + e^{-3x} c_4 + \frac{e^{3x}}{27} - 9x\, e^{x} + 2x^2 e^{x} - \frac{19x^2}{6} - \frac{4x^3}{3} - \frac{x^4}{2}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 16*_a*exp(_a)+18*_a^2-2*(diff
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 4; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 64

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$3)-3*diff(y(x),x$2)=18*x^2+16*x*exp(x)+4*exp(3*x)-9,y(x)
```

$$y(x) = -\frac{\left(\left(x^4+\frac{8}{3}x^3+\frac{19}{3}x^2-2c_3x-2c_4\right)e^{3x}+\left(-4x^2+18x-2c_2-\frac{57}{2}\right)e^{4x}-\frac{2c_1}{9}-\frac{2\,e^{6x}}{27}\right)e^{-3x}}{2}$$

✓ Solution by Mathematica

Time used: 1.232 (sec). Leaf size: 70

```
DSolve[y''''[x]+2*y'''[x]-3*y''[x]==18*x^2+16*x*Exp[x]+4*Exp[3*x]-9,y[x],x,IncludeSingularSo
```

$$y(x) \to -\frac{1}{6}\left(3x^2+8x+19\right)x^2+\frac{1}{4}e^x\left(8x^2-36x+57+4c_2\right)+\frac{e^{3x}}{27}+c_4x+\frac{1}{9}c_1e^{-3x}+c_3$$

11.24 problem 24

Internal problem ID [11798]
Internal file name [`OUTPUT/11807_Thursday_April_11_2024_08_49_50_PM_12791522/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 24.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 5y''' + 7y'' - 5y' + 6y = 5\sin(x) - 12\sin(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 5y''' + 7y'' - 5y' + 6y = 0$$

The characteristic equation is

$$\lambda^4 - 5\lambda^3 + 7\lambda^2 - 5\lambda + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + e^{3x} c_2 + e^{ix} c_3 + e^{-ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{2x}$$
$$y_2 = e^{3x}$$
$$y_3 = e^{ix}$$
$$y_4 = e^{-ix}$$

Now the particular solution to the given ODE is found

$$y'''' - 5y''' + 7y'' - 5y' + 6y = 5\sin(x) - 12\sin(2x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{2x} & e^{3x} & e^{ix} & e^{-ix} \\ 2\,e^{2x} & 3\,e^{3x} & ie^{ix} & -ie^{-ix} \\ 4\,e^{2x} & 9\,e^{3x} & -e^{ix} & -e^{-ix} \\ 8\,e^{2x} & 27\,e^{3x} & -ie^{ix} & ie^{-ix} \end{bmatrix}$$

$$|W| = -100i e^{2x} e^{3x} e^{ix} e^{-ix}$$

The determinant simplifies to

$$|W| = -100i\mathrm{e}^{5x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} \mathrm{e}^{3x} & \mathrm{e}^{ix} & \mathrm{e}^{-ix} \\ 3\,\mathrm{e}^{3x} & i\mathrm{e}^{ix} & -i\mathrm{e}^{-ix} \\ 9\,\mathrm{e}^{3x} & -\mathrm{e}^{ix} & -\mathrm{e}^{-ix} \end{bmatrix} \\ &= -20i\mathrm{e}^{3x} \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} \mathrm{e}^{2x} & \mathrm{e}^{ix} & \mathrm{e}^{-ix} \\ 2\,\mathrm{e}^{2x} & i\mathrm{e}^{ix} & -i\mathrm{e}^{-ix} \\ 4\,\mathrm{e}^{2x} & -\mathrm{e}^{ix} & -\mathrm{e}^{-ix} \end{bmatrix} \\ &= -10i\mathrm{e}^{2x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} \mathrm{e}^{2x} & \mathrm{e}^{3x} & \mathrm{e}^{-ix} \\ 2\,\mathrm{e}^{2x} & 3\,\mathrm{e}^{3x} & -i\mathrm{e}^{-ix} \\ 4\,\mathrm{e}^{2x} & 9\,\mathrm{e}^{3x} & -\mathrm{e}^{-ix} \end{bmatrix} \\ &= (5+5i)\,\mathrm{e}^{(5-i)x} \end{aligned}$$

$$\begin{aligned} W_4(x) &= \det \begin{bmatrix} \mathrm{e}^{2x} & \mathrm{e}^{3x} & \mathrm{e}^{ix} \\ 2\,\mathrm{e}^{2x} & 3\,\mathrm{e}^{3x} & i\mathrm{e}^{ix} \\ 4\,\mathrm{e}^{2x} & 9\,\mathrm{e}^{3x} & -\mathrm{e}^{ix} \end{bmatrix} \\ &= (5-5i)\,\mathrm{e}^{(5+i)x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^3 \int \frac{(5\sin(x) - 12\sin(2x))\left(-20i\mathrm{e}^{3x}\right)}{(1)\left(-100i\mathrm{e}^{5x}\right)}\,dx \\
&= -\int \frac{-20i(5\sin(x) - 12\sin(2x))\,\mathrm{e}^{3x}}{-100i\mathrm{e}^{5x}}\,dx \\
&= -\int \left(-\frac{(-5\sin(x) + 12\sin(2x))\,\mathrm{e}^{-2x}}{5}\right)dx \\
&= \frac{3\,\mathrm{e}^{-2x}(-2\sin(2x) - 2\cos(2x))}{10} + \frac{\cos(x)\,\mathrm{e}^{-2x}}{5} + \frac{2\,\mathrm{e}^{-2x}\sin(x)}{5} \\
&= \frac{3\,\mathrm{e}^{-2x}(-2\sin(2x) - 2\cos(2x))}{10} + \frac{\cos(x)\,\mathrm{e}^{-2x}}{5} + \frac{2\,\mathrm{e}^{-2x}\sin(x)}{5}
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{(5\sin(x) - 12\sin(2x))\left(-10i\mathrm{e}^{2x}\right)}{(1)\left(-100i\mathrm{e}^{5x}\right)}\,dx \\
&= \int \frac{-10i(5\sin(x) - 12\sin(2x))\,\mathrm{e}^{2x}}{-100i\mathrm{e}^{5x}}\,dx \\
&= \int \left(-\frac{(-5\sin(x) + 12\sin(2x))\,\mathrm{e}^{-3x}}{10}\right)dx \\
&= -\frac{6\,\mathrm{e}^{-3x}(-3\sin(2x) - 2\cos(2x))}{65} - \frac{\mathrm{e}^{-3x}\cos(x)}{20} - \frac{3\,\mathrm{e}^{-3x}\sin(x)}{20} \\
&= -\frac{6\,\mathrm{e}^{-3x}(-3\sin(2x) - 2\cos(2x))}{65} - \frac{\mathrm{e}^{-3x}\cos(x)}{20} - \frac{3\,\mathrm{e}^{-3x}\sin(x)}{20}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(5\sin(x) - 12\sin(2x))\left((5+5i)\,\mathrm{e}^{(5-i)x}\right)}{(1)\left(-100i\mathrm{e}^{5x}\right)}\,dx \\
&= -\int \frac{(5+5i)\,(5\sin(x) - 12\sin(2x))\,\mathrm{e}^{(5-i)x}}{-100i\mathrm{e}^{5x}}\,dx \\
&= -\int \left(\left(\frac{1}{20} - \frac{i}{20}\right)(-5\sin(x) + 12\sin(2x))\,\mathrm{e}^{-ix}\right)dx \\
&= -\left(\int \left(\frac{1}{20} - \frac{i}{20}\right)(-5\sin(x) + 12\sin(2x))\,\mathrm{e}^{-ix}dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(5\sin(x) - 12\sin(2x))\left((5-5i)\,\mathrm{e}^{(5+i)x}\right)}{(1)\left(-100i\mathrm{e}^{5x}\right)}\,dx \\
&= \int \frac{(5-5i)\,(5\sin(x) - 12\sin(2x))\,\mathrm{e}^{(5+i)x}}{-100i\mathrm{e}^{5x}}\,dx \\
&= \int \left(\left(-\frac{1}{20} - \frac{i}{20}\right)(-5\sin(x) + 12\sin(2x))\,\mathrm{e}^{ix}\right)dx \\
&= \int \left(-\frac{1}{20} - \frac{i}{20}\right)(-5\sin(x) + 12\sin(2x))\,\mathrm{e}^{ix}dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{3\,\mathrm{e}^{-2x}(-2\sin(2x) - 2\cos(2x))}{10} + \frac{\cos(x)\,\mathrm{e}^{-2x}}{5} + \frac{2\,\mathrm{e}^{-2x}\sin(x)}{5}\right)\left(\mathrm{e}^{2x}\right) \\
&+ \left(-\frac{6\,\mathrm{e}^{-3x}(-3\sin(2x) - 2\cos(2x))}{65} - \frac{\mathrm{e}^{-3x}\cos(x)}{20} - \frac{3\,\mathrm{e}^{-3x}\sin(x)}{20}\right)\left(\mathrm{e}^{3x}\right) \\
&+ \left(-\left(\int \left(\frac{1}{20} - \frac{i}{20}\right)(-5\sin(x) + 12\sin(2x))\,\mathrm{e}^{-ix}dx\right)\right)\left(\mathrm{e}^{ix}\right) \\
&+ \left(\int \left(-\frac{1}{20} - \frac{i}{20}\right)(-5\sin(x) + 12\sin(2x))\,\mathrm{e}^{ix}dx\right)\left(\mathrm{e}^{-ix}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \left(-\frac{1}{20} + \frac{i}{20}\right)\left(\int (-5\sin(x) + 12\sin(2x))\,\mathrm{e}^{-ix}dx\right)\mathrm{e}^{ix} + \left(-\frac{1}{20} - \frac{i}{20}\right)\left(\int (-5\sin(x) + 12\sin(2x)
$$

Which simplifies to

$$
y_p = \frac{27}{65} + \frac{(\cos(x) - \sin(x))\left(\int (24\cos(x) - 5)\sin(x)^2\,dx\right)}{10} + \frac{(-\cos(x) - \sin(x))\left(\int (24\cos(x) - 5)\sin(x
}{10}
$$

Therefore the general solution is

$$
y = y_h + y_p
$$

$$
\begin{aligned}
&= \left(c_1 e^{2x} + e^{3x} c_2 + e^{ix} c_3 + e^{-ix} c_4\right) \\
&+ \left(\frac{27}{65} + \frac{(\cos(x) - \sin(x)) \left(\int (24\cos(x) - 5)\sin(x)^2 dx\right)}{10}\right. \\
&+ \frac{(-\cos(x) - \sin(x)) \left(\int (24\cos(x) - 5)\sin(x)\cos(x) dx\right)}{10} - \frac{54\cos(x)^2}{65} \\
&\left. + \frac{3(13 - 56\sin(x))\cos(x)}{260} + \frac{\sin(x)}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 e^{2x} + e^{3x} c_2 + e^{ix} c_3 + e^{-ix} c_4 + \frac{27}{65} \\
&+ \frac{(\cos(x) - \sin(x)) \left(\int (24\cos(x) - 5)\sin(x)^2 dx\right)}{10} \\
&+ \frac{(-\cos(x) - \sin(x)) \left(\int (24\cos(x) - 5)\sin(x)\cos(x) dx\right)}{10} \\
&- \frac{54\cos(x)^2}{65} + \frac{3(13 - 56\sin(x))\cos(x)}{260} + \frac{\sin(x)}{4}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 e^{2x} + e^{3x} c_2 + e^{ix} c_3 + e^{-ix} c_4 + \frac{27}{65} + \frac{(\cos(x) - \sin(x)) \left(\int (24\cos(x) - 5)\sin(x)^2 dx\right)}{10} \\
&+ \frac{(-\cos(x) - \sin(x)) \left(\int (24\cos(x) - 5)\sin(x)\cos(x) dx\right)}{10} \\
&- \frac{54\cos(x)^2}{65} + \frac{3(13 - 56\sin(x))\cos(x)}{260} + \frac{\sin(x)}{4}
\end{aligned}
$$

Verified OK.

11.24.1 Maple step by step solution

Let's solve

$y'''' - 5y''' + 7y'' - 5y' + 6y = 5\sin(x) - 12\sin(2x)$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

$y_1(x) = y$

- Define new variable $y_2(x)$

  $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Define new variable $y_4(x)$

  $y_4(x) = y'''$

- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = 5\sin(x) - 12\sin(2x) + 5y_4(x) - 7y_3(x) + 5y_2(x) - 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 5\sin(x) - 12\sin(2x) + 5y_4(x) - 7y_3(x) +$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 5 & -7 & 5 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5\sin(x) - 12\sin(2x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5\sin(x) - 12\sin(2x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 5 & -7 & 5 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{3x} \cdot \begin{bmatrix} \frac{1}{27} \\ \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}x} \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{2x}}{8} & \frac{e^{3x}}{27} & -\sin(x) & -\cos(x) \\ \frac{e^{2x}}{4} & \frac{e^{3x}}{9} & -\cos(x) & \sin(x) \\ \frac{e^{2x}}{2} & \frac{e^{3x}}{3} & \sin(x) & \cos(x) \\ e^{2x} & e^{3x} & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{2x}}{8} & \frac{e^{3x}}{27} & -\sin(x) & -\cos(x) \\ \frac{e^{2x}}{4} & \frac{e^{3x}}{9} & -\cos(x) & \sin(x) \\ \frac{e^{2x}}{2} & \frac{e^{3x}}{3} & \sin(x) & \cos(x) \\ e^{2x} & e^{3x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{8} & \frac{1}{27} & 0 & -1 \\ \frac{1}{4} & \frac{1}{9} & -1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{3e^{2x}}{5} - \frac{e^{3x}}{5} - \frac{3\sin(x)}{5} + \frac{3\cos(x)}{5} & -\frac{e^{2x}}{5} + \frac{e^{3x}}{10} + \frac{11\sin(x)}{10} + \frac{\cos(x)}{10} & \frac{3e^{2x}}{5} - \frac{e^{3x}}{5} - \frac{3\sin(x)}{5} \\ \frac{6e^{2x}}{5} - \frac{3e^{3x}}{5} - \frac{3\cos(x)}{5} - \frac{3\sin(x)}{5} & -\frac{2e^{2x}}{5} + \frac{3e^{3x}}{10} + \frac{11\cos(x)}{10} - \frac{\sin(x)}{10} & \frac{6e^{2x}}{5} - \frac{3e^{3x}}{5} - \frac{3\cos(x}{5} \\ \frac{12e^{2x}}{5} - \frac{9e^{3x}}{5} + \frac{3\sin(x)}{5} - \frac{3\cos(x)}{5} & -\frac{4e^{2x}}{5} + \frac{9e^{3x}}{10} - \frac{11\sin(x)}{10} - \frac{\cos(x)}{10} & \frac{12e^{2x}}{5} - \frac{9e^{3x}}{5} + \frac{3\sin(x}{5} \\ \frac{24e^{2x}}{5} - \frac{27e^{3x}}{5} + \frac{3\cos(x)}{5} + \frac{3\sin(x)}{5} & -\frac{8e^{2x}}{5} + \frac{27e^{3x}}{10} - \frac{11\cos(x)}{10} + \frac{\sin(x)}{10} & \frac{24e^{2x}}{5} - \frac{27e^{3x}}{5} + \frac{3\cos(}{5} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} -\frac{5}{13}+\frac{2\,\mathrm{e}^{2x}}{5}-\frac{7\,\mathrm{e}^{3x}}{52}+\frac{10\cos(x)^2}{13}+\frac{(-65x+40\sin(x)-169)\cos(x)}{260}+\frac{(-6+5x)\sin(x)}{20} \\ -\frac{2}{13}+\frac{4\,\mathrm{e}^{2x}}{5}-\frac{21\,\mathrm{e}^{3x}}{52}+\frac{4\cos(x)^2}{13}+\frac{(65x-400\sin(x)-143)\cos(x)}{260}+\frac{(18+5x)\sin(x)}{20} \\ \frac{20}{13}+\frac{8\,\mathrm{e}^{2x}}{5}-\frac{63\,\mathrm{e}^{3x}}{52}-\frac{40\cos(x)^2}{13}+\frac{(65x-160\sin(x)+299)\cos(x)}{260}+\frac{(16-5x)\sin(x)}{20} \\ \frac{8}{13}+\frac{16\,\mathrm{e}^{2x}}{5}-\frac{189\,\mathrm{e}^{3x}}{52}-\frac{16\cos(x)^2}{13}+\frac{(-65x+1600\sin(x)+273)\cos(x)}{260}+\frac{(-28-5x)\sin(x)}{20} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \left[\begin{array}{c} -\frac{5}{13}+\frac{2\,\mathrm{e}^{2x}}{5}-\frac{7\,\mathrm{e}^{3x}}{52}+\frac{10\cos(x)^2}{13}+\frac{(-65x+40\sin(x)-169}{260} \\ -\frac{2}{13}+\frac{4\,\mathrm{e}^{2x}}{5}-\frac{21\,\mathrm{e}^{3x}}{52}+\frac{4\cos(x)^2}{13}+\frac{(65x-400\sin(x)-143}{260} \\ \frac{20}{13}+\frac{8\,\mathrm{e}^{2x}}{5}-\frac{63\,\mathrm{e}^{3x}}{52}-\frac{40\cos(x)^2}{13}+\frac{(65x-160\sin(x)+299)}{260} \\ \frac{8}{13}+\frac{16\,\mathrm{e}^{2x}}{5}-\frac{189\,\mathrm{e}^{3x}}{52}-\frac{16\cos(x)^2}{13}+\frac{(-65x+1600\sin(x)+27}{260} \end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y = -\frac{5}{13} + \frac{(16+5c_1)\,\mathrm{e}^{2x}}{40} + \frac{(-189+52c_2)\,\mathrm{e}^{3x}}{1404} + \frac{10\cos(x)^2}{13} + \frac{(-169-65x-260c_4+40\sin(x))\cos(x)}{260} + \frac{(-6+5x-20c_3)\sin(x)}{20}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 51

```
dsolve(diff(y(x),x$4)-5*diff(y(x),x$3)+7*diff(y(x),x$2)-5*diff(y(x),x)+6*y(x)=5*sin(x)-12*si
```

$$y(x) = \frac{5\cos(2x)}{13} + c_3\mathrm{e}^{2x} + c_4\mathrm{e}^{3x} + \frac{\sin(2x)}{13} + \frac{(-2-5x+20c_1)\cos(x)}{20} + \frac{(1+x+4c_2)\sin(x)}{4}$$

✓ Solution by Mathematica

Time used: 0.232 (sec). Leaf size: 71

```
DSolve[y''''[x]-5*y'''[x]+7*y''[x]-5*y'[x]+6*y[x]==5*Sin[x]-12*Sin[2*x],y[x],x,IncludeSingul
```

$$y(x) \to -\frac{5\sin^2(x)}{13} + \frac{5\cos^2(x)}{13} + e^{2x}(c_4e^x + c_3) + \left(\frac{x}{4} + \frac{3}{8} + c_2\right)\sin(x) + \cos(x)\left(-\frac{x}{4} + \frac{2\sin(x)}{13} - \frac{1}{10} + c_1\right)$$

11.25 problem 25

Internal problem ID [11799]
Internal file name [`OUTPUT/11808_Thursday_April_11_2024_08_49_52_PM_42387168/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 4y' + 3y = 9x^2 + 4}$$

With initial conditions

$$[y(0) = 6, y'(0) = 8]$$

11.25.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= 3 \\ F &= 9x^2 + 4 \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 3y = 9x^2 + 4$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 9x^2 + 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.25.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 3, f(x) = 9x^2 + 4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 3 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-4^2 - (4)(1)(3)} \\ &= 2 \pm 1\end{aligned}$$

Hence

$$\lambda_1 = 2 + 1$$
$$\lambda_2 = 2 - 1$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_3x^2 + 3A_2x - 8xA_3 + 3A_1 - 4A_2 + 2A_3 = 9x^2 + 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 10, A_2 = 8, A_3 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x^2 + 8x + 10$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1e^{3x} + c_2e^x) + (3x^2 + 8x + 10) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1e^{3x} + c_2e^x + 3x^2 + 8x + 10 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 6$ and $x = 0$ in the above gives

$$6 = c_1 + c_2 + 10 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1 e^{3x} + c_2 e^x + 6x + 8$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = 3c_1 + c_2 + 8 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -6$$

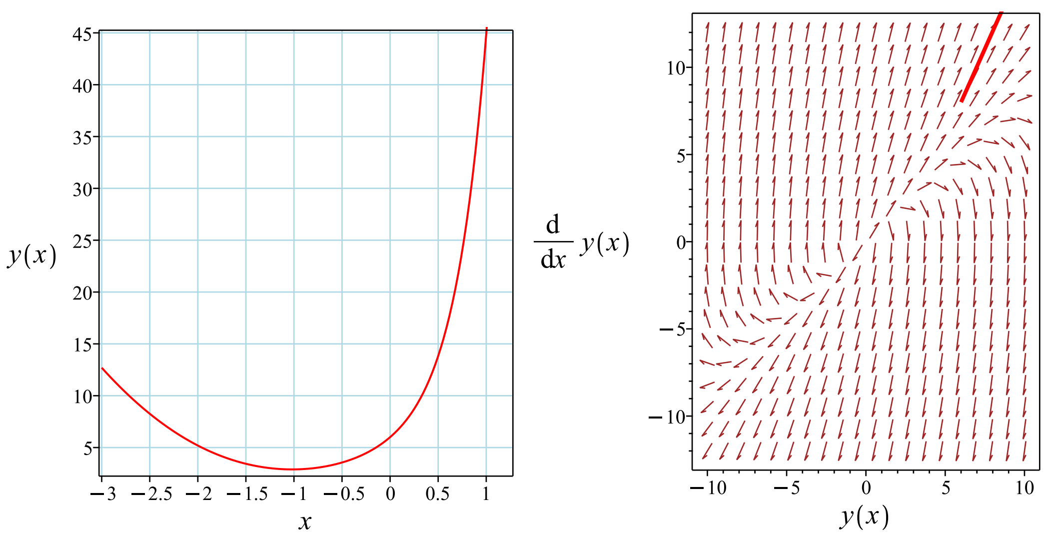
Substituting these values back in above solution results in

$$y = 10 + 3x^2 + 2\,e^{3x} - 6\,e^x + 8x$$

Summary

The solution(s) found are the following

$$y = 10 + 3x^2 + 2\,e^{3x} - 6\,e^x + 8x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 10 + 3x^2 + 2\,e^{3x} - 6\,e^x + 8x$$

Verified OK.

11.25.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 379: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{4x}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (\mathrm{e}^{x}) + c_2 \left(\mathrm{e}^{x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{x} c_1 + \frac{\mathrm{e}^{3x} c_2}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{3x}}{2}, e^x\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_3x^2 + 3A_2x - 8xA_3 + 3A_1 - 4A_2 + 2A_3 = 9x^2 + 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 10, A_2 = 8, A_3 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x^2 + 8x + 10$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x c_1 + \frac{e^{3x} c_2}{2}\right) + (3x^2 + 8x + 10) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{x} c_1 + \frac{e^{3x} c_2}{2} + 3x^2 + 8x + 10 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 6$ and $x = 0$ in the above gives

$$6 = c_1 + \frac{c_2}{2} + 10 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x} c_1 + \frac{3\,e^{3x} c_2}{2} + 6x + 8$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = c_1 + \frac{3c_2}{2} + 8 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -6$$
$$c_2 = 4$$

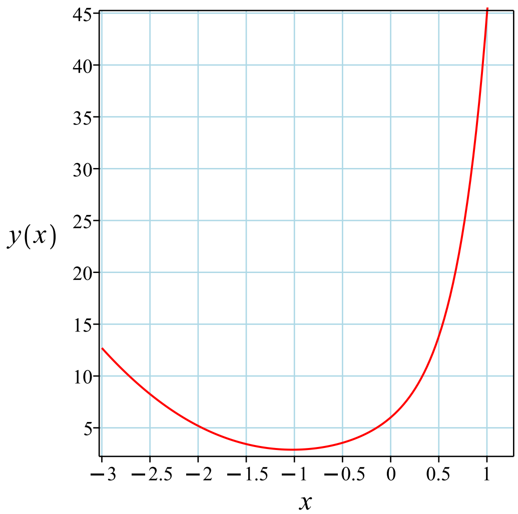
Substituting these values back in above solution results in

$$y = 10 + 3x^2 + 2\,e^{3x} - 6\,e^{x} + 8x$$

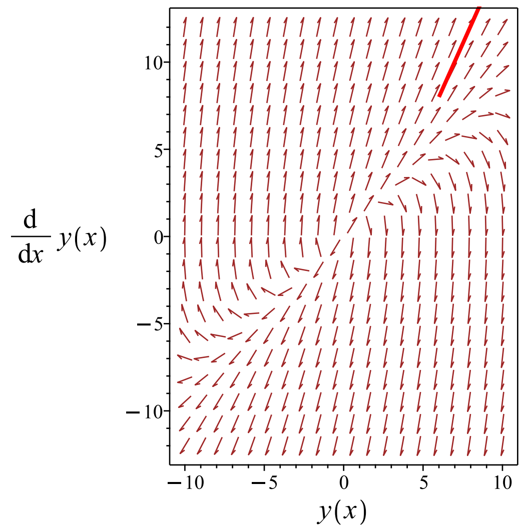
Summary

The solution(s) found are the following

$$y = 10 + 3x^2 + 2\,e^{3x} - 6\,e^{x} + 8x \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 10 + 3x^2 + 2\,e^{3x} - 6\,e^{x} + 8x$$

Verified OK.

11.25.4 Maple step by step solution

Let's solve

$$\left[y'' - 4y' + 3y = 9x^2 + 4, y(0) = 6, y'\Big|_{\{x=0\}} = 8\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (1, 3)$

- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^{x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{x} c_1 + \mathrm{e}^{3x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 9x^2 + 4\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{x} & \mathrm{e}^{3x} \\ \mathrm{e}^{x} & 3\,\mathrm{e}^{3x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{4x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\mathrm{e}^{x}\left(\int (9x^2+4)\mathrm{e}^{-x} dx\right)}{2} + \frac{\mathrm{e}^{3x}\left(\int \mathrm{e}^{-3x}(9x^2+4) dx\right)}{2}$

  ◦ Compute integrals

  $y_p(x) = 3x^2 + 8x + 10$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{x} c_1 + \mathrm{e}^{3x} c_2 + 3x^2 + 8x + 10$

☐ Check validity of solution $y = \mathrm{e}^{x} c_1 + \mathrm{e}^{3x} c_2 + 3x^2 + 8x + 10$

  ◦ Use initial condition $y(0) = 6$

  $6 = c_1 + c_2 + 10$

  ◦ Compute derivative of the solution

  $y' = \mathrm{e}^{x} c_1 + 3\,\mathrm{e}^{3x} c_2 + 6x + 8$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 8$

  $8 = c_1 + 3c_2 + 8$

- Solve for c_1 and c_2

  $\{c_1 = -6, c_2 = 2\}$

- Substitute constant values into general solution and simplify

  $y = 10 + 3x^2 + 2\,e^{3x} - 6\,e^{x} + 8x$

- Solution to the IVP

  $y = 10 + 3x^2 + 2\,e^{3x} - 6\,e^{x} + 8x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve([diff(y(x),x$2)-4*diff(y(x),x)+3*y(x)=9*x^2+4,y(0) = 6, D(y)(0) = 8],y(x), singsol=al
```

$$y(x) = -6\,e^{x} + 2\,e^{3x} + 3x^2 + 8x + 10$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 27

```
DSolve[{y''[x]-4*y'[x]+3*y[x]==9*x^2+4,{y[0]==6,y'[0]==8}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to 3x^2 + 8x - 6e^x + 2e^{3x} + 10$$

11.26 problem 26

Internal problem ID [11800]
Internal file name [`OUTPUT/11809_Thursday_April_11_2024_08_49_52_PM_16421415/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 5y' + 4y = 16x + 20\,\mathrm{e}^{x}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 3]$$

11.26.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 5 \\ q(x) &= 4 \\ F &= 16x + 20\,\mathrm{e}^{x} \end{aligned}$$

Hence the ode is

$$y'' + 5y' + 4y = 16x + 20\,\mathrm{e}^{x}$$

The domain of $p(x) = 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 16x + 20\,\mathrm{e}^{x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.26.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 5, C = 4, f(x) = 16x + 20\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 5y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 5, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 5\lambda\,\mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{5^2 - (4)(1)(4)} \\ &= -\frac{5}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{5}{2} + \frac{3}{2} \\ \lambda_2 &= -\frac{5}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(-1)x} + c_2 e^{(-4)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-4x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-4x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$16x + 20\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-4x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x + A_2 + A_3 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1 e^x + 5A_3 + 4A_2 + 4A_3 x = 16x + 20\, e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = -5, A_3 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\, e^x - 5 + 4x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-4x}\right) + \left(2\, e^x - 5 + 4x\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + c_2 e^{-4x} + 2\, e^x - 5 + 4x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 - 3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} - 4c_2 e^{-4x} + 2\,e^{x} + 4$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = -c_1 - 4c_2 + 6 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 0$$

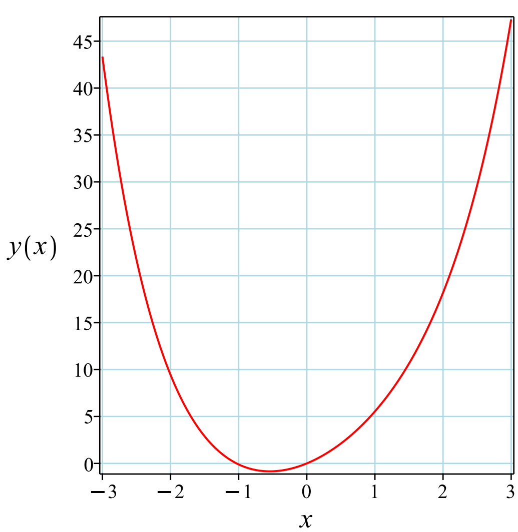
Substituting these values back in above solution results in

$$y = -5 + 2\,e^{x} + 3\,e^{-x} + 4x$$

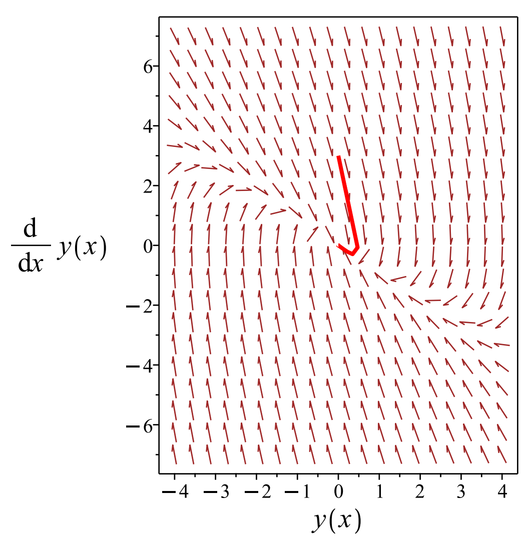
Summary

The solution(s) found are the following

$$y = -5 + 2\,e^{x} + 3\,e^{-x} + 4x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -5 + 2\,e^{x} + 3\,e^{-x} + 4x$$

Verified OK.

11.26.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 5 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 381: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-\frac{5x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{5x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-4x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-5x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-4x}\right) + c_2 \left(\mathrm{e}^{-4x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 5y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-4x} + \frac{c_2 \mathrm{e}^{-x}}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$16x + 20\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{-x}}{3}, \mathrm{e}^{-4x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{x} + A_2 + A_3x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_1\mathrm{e}^{x} + 5A_3 + 4A_2 + 4A_3x = 16x + 20\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = -5, A_3 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{x} - 5 + 4x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-4x} + \frac{c_2\mathrm{e}^{-x}}{3}\right) + (2\,\mathrm{e}^{x} - 5 + 4x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-4x} + \frac{c_2 \mathrm{e}^{-x}}{3} + 2\,\mathrm{e}^{x} - 5 + 4x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{3} - 3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1 \mathrm{e}^{-4x} - \frac{c_2 \mathrm{e}^{-x}}{3} + 2\,\mathrm{e}^{x} + 4$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = -4c_1 - \frac{c_2}{3} + 6 \tag{2A}$$

Equations $\{1\mathrm{A}, 2\mathrm{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 9$$

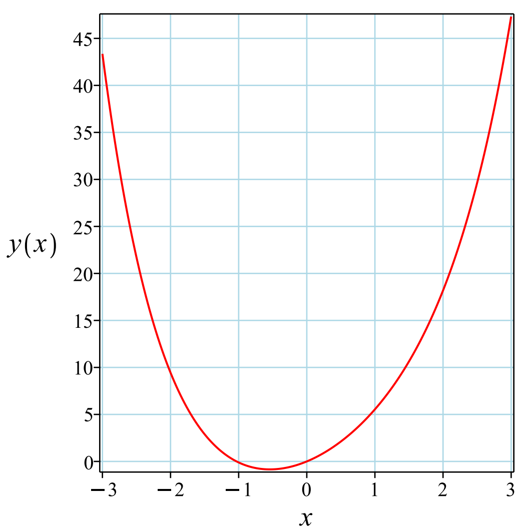
Substituting these values back in above solution results in

$$y = -5 + 2\,\mathrm{e}^{x} + 3\,\mathrm{e}^{-x} + 4x$$

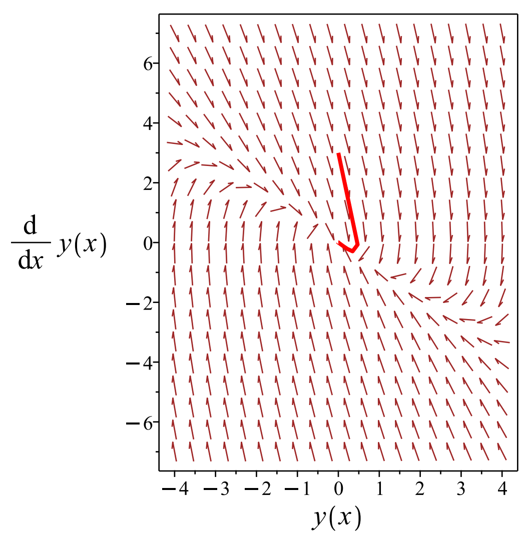
Summary

The solution(s) found are the following

$$y = -5 + 2\,\mathrm{e}^{x} + 3\,\mathrm{e}^{-x} + 4x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -5 + 2\,e^{x} + 3\,e^{-x} + 4x$$

Verified OK.

11.26.4 Maple step by step solution

Let's solve

$\left[y'' + 5y' + 4y = 16x + 20\,e^{x}, y(0) = 0, y'\Big|_{\{x=0\}} = 3\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 5r + 4 = 0$

- Factor the characteristic polynomial

  $(r+4)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-4, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-4x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-4x} + c_2 e^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 16x + 20\,e^{x}\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-4x} & e^{-x} \\ -4\,e^{-4x} & -e^{-x} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,e^{-5x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{4\,e^{-4x}\left(\int (4x+5\,e^{x})e^{4x}dx\right)}{3} + \frac{4\,e^{-x}\left(\int (4x+5\,e^{x})e^{x}dx\right)}{3}$

- Compute integrals

  $y_p(x) = 2\,e^{x} - 5 + 4x$

- Substitute particular solution into general solution to ODE

  $y = c_1e^{-4x} + c_2e^{-x} + 2\,e^{x} - 5 + 4x$

☐ Check validity of solution $y = c_1e^{-4x} + c_2e^{-x} + 2e^{x} - 5 + 4x$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 - 3$

- Compute derivative of the solution

  $y' = -4c_1e^{-4x} - c_2e^{-x} + 2\,e^{x} + 4$

- Use the initial condition $y'\Big|_{\{x=0\}} = 3$

  $3 = -4c_1 - c_2 + 6$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 3\}$

- Substitute constant values into general solution and simplify

  $y = -5 + 2\,e^{x} + 3\,e^{-x} + 4x$

- Solution to the IVP

  $y = -5 + 2\,e^{x} + 3\,e^{-x} + 4x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$2)+5*diff(y(x),x)+4*y(x)=16*x+20*exp(x),y(0) = 0, D(y)(0) = 3],y(x), sin
```

$$y(x) = 3\,\mathrm{e}^{-x} - 5 + 2\,\mathrm{e}^{x} + 4x$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 22

```
DSolve[{y''[x]+5*y'[x]+4*y[x]==16*x+20*Exp[x],{y[0]==0,y'[0]==3}},y[x],x,IncludeSingularSolu
```

$$y(x) \to 4x + 3e^{-x} + 2e^{x} - 5$$

11.27 problem 27

Internal problem ID [11801]
Internal file name [`OUTPUT/11810_Thursday_April_11_2024_08_49_53_PM_59993326/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 8y' + 15y = 9\,\mathrm{e}^{2x} x}$$

With initial conditions

$$[y(0) = 5, y'(0) = 10]$$

11.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -8 \\ q(x) &= 15 \\ F &= 9\,\mathrm{e}^{2x} x \end{aligned}$$

Hence the ode is

$$y'' - 8y' + 15y = 9\,e^{2x}x$$

The domain of $p(x) = -8$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 15$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 9\,e^{2x}x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.27.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -8, C = 15, f(x) = 9\,e^{2x}x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 8y' + 15y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -8, C = 15$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 8\lambda\,e^{\lambda x} + 15\,e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 8\lambda + 15 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -8, C = 15$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{8}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-8^2 - (4)(1)(15)} \\
&= 4 \pm 1
\end{aligned}$$

Hence

$$\lambda_1 = 4 + 1$$
$$\lambda_2 = 4 - 1$$

Which simplifies to

$$\lambda_1 = 5$$
$$\lambda_2 = 3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(5)x} + c_2 e^{(3)x}$$

Or

$$y = c_1 e^{5x} + e^{3x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{5x} + e^{3x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$9 e^{2x} x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x} x, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{3x}, e^{5x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x} x + A_2 e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{2x} x - 4A_1 e^{2x} + 3A_2 e^{2x} = 9\, e^{2x} x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3, A_2 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3\, e^{2x} x + 4\, e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{5x} + e^{3x} c_2\right) + \left(3\, e^{2x} x + 4\, e^{2x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{5x} + e^{3x} c_2 + 3\, e^{2x} x + 4\, e^{2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 0$ in the above gives

$$5 = c_1 + c_2 + 4 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 5c_1 e^{5x} + 3\, e^{3x} c_2 + 6\, e^{2x} x + 11\, e^{2x}$$

substituting $y' = 10$ and $x = 0$ in the above gives

$$10 = 5c_1 + 3c_2 + 11 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 3$$

Substituting these values back in above solution results in

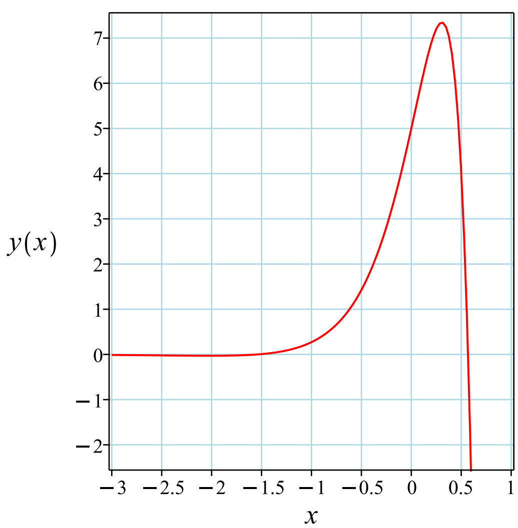
$$y = 3\,\mathrm{e}^{2x}x + 4\,\mathrm{e}^{2x} - 2\,\mathrm{e}^{5x} + 3\,\mathrm{e}^{3x}$$

Which simplifies to

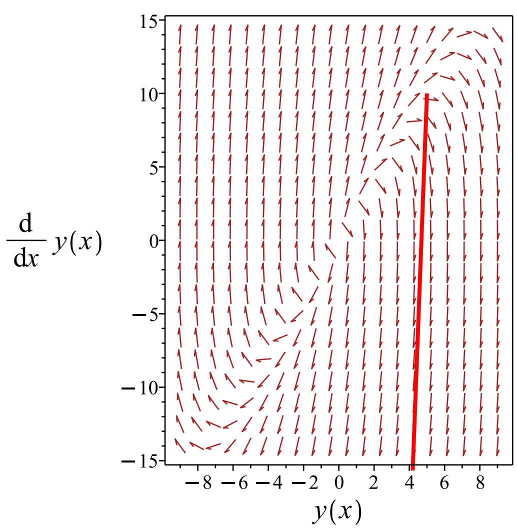
$$y = (4 + 3x)\,\mathrm{e}^{2x} + 3\,\mathrm{e}^{3x} - 2\,\mathrm{e}^{5x}$$

Summary
The solution(s) found are the following

$$y = (4 + 3x)\,\mathrm{e}^{2x} + 3\,\mathrm{e}^{3x} - 2\,\mathrm{e}^{5x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (4 + 3x)\,\mathrm{e}^{2x} + 3\,\mathrm{e}^{3x} - 2\,\mathrm{e}^{5x}$$

Verified OK.

11.27.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 8y' + 15y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -8 \\ C &= 15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 383: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-8}{1}\,dx}\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{4x} \\
&= z_1\left(\mathrm{e}^{4x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-8}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{8x}}{\left(y_1\right)^2}\, dx \\
&= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{3x}\right) + c_2\left(\mathrm{e}^{3x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 8y' + 15y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{3x} + \frac{c_2 \mathrm{e}^{5x}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$9\,\mathrm{e}^{2x}x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{2x}x, \mathrm{e}^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{5x}}{2}, \mathrm{e}^{3x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{2x}x + A_2\mathrm{e}^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1\mathrm{e}^{2x}x - 4A_1\mathrm{e}^{2x} + 3A_2\mathrm{e}^{2x} = 9\,\mathrm{e}^{2x}x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3, A_2 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3\,\mathrm{e}^{2x}x + 4\,\mathrm{e}^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{3x} + \frac{c_2\mathrm{e}^{5x}}{2}\right) + \left(3\,\mathrm{e}^{2x}x + 4\,\mathrm{e}^{2x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{3x} + \frac{c_2 e^{5x}}{2} + 3\,e^{2x} x + 4\,e^{2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 0$ in the above gives

$$5 = c_1 + \frac{c_2}{2} + 4 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1 e^{3x} + \frac{5c_2 e^{5x}}{2} + 6\,e^{2x} x + 11\,e^{2x}$$

substituting $y' = 10$ and $x = 0$ in the above gives

$$10 = 3c_1 + \frac{5c_2}{2} + 11 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 3 \\ c_2 &= -4 \end{aligned}$$

Substituting these values back in above solution results in

$$y = 3\,e^{2x} x + 4\,e^{2x} - 2\,e^{5x} + 3\,e^{3x}$$

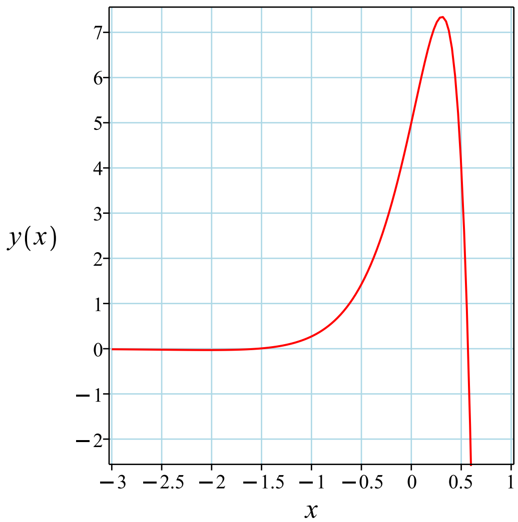
Which simplifies to

$$y = (4 + 3x)\,e^{2x} + 3\,e^{3x} - 2\,e^{5x}$$

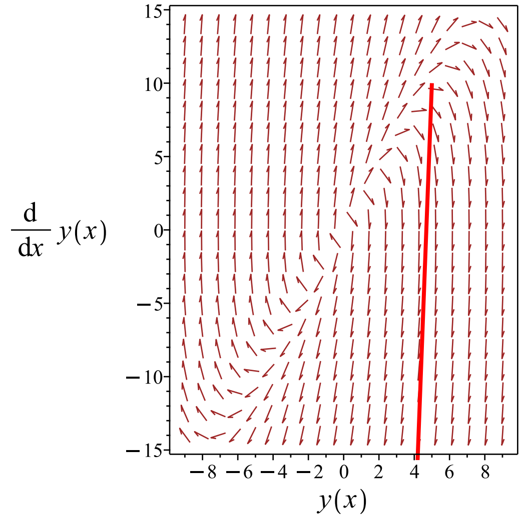
Summary

The solution(s) found are the following

$$y = (4 + 3x)\,e^{2x} + 3\,e^{3x} - 2\,e^{5x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = (4 + 3x)\, e^{2x} + 3\, e^{3x} - 2\, e^{5x}$$

Verified OK.

11.27.4 Maple step by step solution

Let's solve

$$\left[y'' - 8y' + 15y = 9\, e^{2x} x, y(0) = 5, y'\Big|_{\{x=0\}} = 10\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 8r + 15 = 0$

- Factor the characteristic polynomial

  $(r - 3)(r - 5) = 0$

- Roots of the characteristic polynomial

  $r = (3, 5)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{5x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{5x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 9\,\mathrm{e}^{2x} x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{3x} & \mathrm{e}^{5x} \\ 3\,\mathrm{e}^{3x} & 5\,\mathrm{e}^{5x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{8x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{9\,\mathrm{e}^{3x}\left(\int x\,\mathrm{e}^{-x}dx\right)}{2} + \frac{9\,\mathrm{e}^{5x}\left(\int x\,\mathrm{e}^{-3x}dx\right)}{2}$

  - Compute integrals

    $y_p(x) = (4 + 3x)\,\mathrm{e}^{2x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{5x} + (4 + 3x)\,\mathrm{e}^{2x}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{5x} + (4 + 3x)\,\mathrm{e}^{2x}$

  - Use initial condition $y(0) = 5$

    $5 = c_1 + c_2 + 4$

  - Compute derivative of the solution

    $y' = 3c_1 \mathrm{e}^{3x} + 5c_2 \mathrm{e}^{5x} + 3\,\mathrm{e}^{2x} + 2(4 + 3x)\,\mathrm{e}^{2x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 10$

    $10 = 3c_1 + 5c_2 + 11$

  - Solve for c_1 and c_2

$\{c_1 = 3, c_2 = -2\}$

- Substitute constant values into general solution and simplify

  $y = (4 + 3x) e^{2x} + 3 e^{3x} - 2 e^{5x}$

- Solution to the IVP

  $y = (4 + 3x) e^{2x} + 3 e^{3x} - 2 e^{5x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve([diff(y(x),x$2)-8*diff(y(x),x)+15*y(x)=9*x*exp(2*x),y(0) = 5, D(y)(0) = 10],y(x), sin
```

$$y(x) = -2 e^{5x} + 3 e^{3x} + (3x + 4) e^{2x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 28

```
DSolve[{y''[x]-8*y'[x]+15*y[x]==9*x*Exp[2*x],{y[0]==5,y'[0]==10}},y[x],x,IncludeSingularSolu
```

$$y(x) \rightarrow e^{2x}(3x + 3e^{x} - 2e^{3x} + 4)$$

11.28 problem 28

Internal problem ID [11802]
Internal file name [`OUTPUT/11811_Thursday_April_11_2024_08_49_54_PM_95445400/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 7y' + 10y = 4x\,\mathrm{e}^{-3x}}$$

With initial conditions

$$[y(0) = 0, y'(0) = -1]$$

11.28.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 7 \\ q(x) &= 10 \\ F &= 4x\,\mathrm{e}^{-3x} \end{aligned}$$

Hence the ode is

$$y'' + 7y' + 10y = 4x\,e^{-3x}$$

The domain of $p(x) = 7$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 10$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 4x\,e^{-3x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.28.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 7, C = 10, f(x) = 4x\,e^{-3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 7y' + 10y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 7, C = 10$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 7\lambda\,e^{\lambda x} + 10\,e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 7\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 7, C = 10$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-7}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{7^2 - (4)(1)(10)} \\
&= -\frac{7}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{7}{2} + \frac{3}{2} \\
\lambda_2 &= -\frac{7}{2} - \frac{3}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -2 \\
\lambda_2 &= -5
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(-2)x} + c_2 e^{(-5)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-5x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-5x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4x\,\mathrm{e}^{-3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-3x},\mathrm{e}^{-3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-5x},\mathrm{e}^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-3x} + A_2 \mathrm{e}^{-3x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1\mathrm{e}^{-3x} - 2A_1 x\,\mathrm{e}^{-3x} - 2A_2\mathrm{e}^{-3x} = 4x\,\mathrm{e}^{-3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x\,\mathrm{e}^{-3x} - \mathrm{e}^{-3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-5x}\right) + \left(-2x\,\mathrm{e}^{-3x} - \mathrm{e}^{-3x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-5x} - 2x\,\mathrm{e}^{-3x} - \mathrm{e}^{-3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2x} - 5c_2 e^{-5x} + e^{-3x} + 6x\, e^{-3x}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = -2c_1 - 5c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = -2x\, e^{-3x} - e^{-3x} + e^{-2x}$$

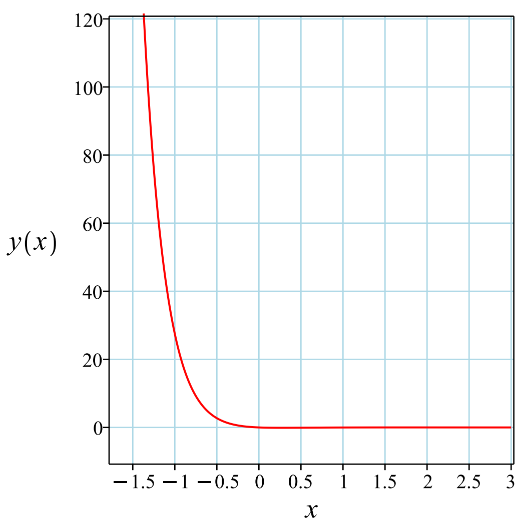
Which simplifies to

$$y = (-2x - 1)\, e^{-3x} + e^{-2x}$$

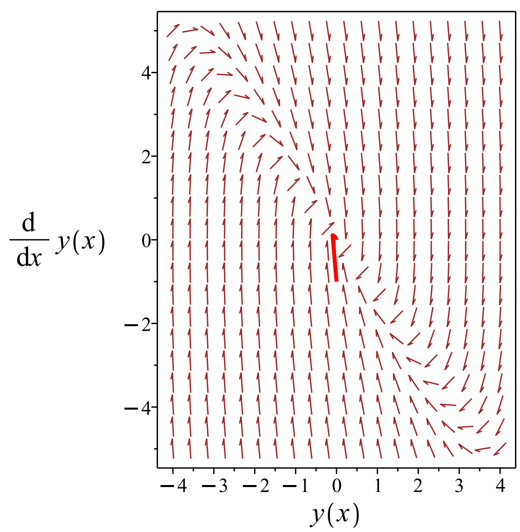
Summary

The solution(s) found are the following

$$y = (-2x - 1)\, e^{-3x} + e^{-2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (-2x - 1)\, e^{-3x} + e^{-2x}$$

Verified OK.

11.28.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 7y' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 7 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 385: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-\frac{7x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{7x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-5x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-7x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-5x}\right) + c_2 \left(\mathrm{e}^{-5x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 7y' + 10y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-5x} + \frac{c_2 \mathrm{e}^{-2x}}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4x\,e^{-3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{-3x}, e^{-3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{-2x}}{3}, e^{-5x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{-3x} + A_2 e^{-3x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{-3x} - 2A_1 x\,e^{-3x} - 2A_2 e^{-3x} = 4x\,e^{-3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x\,e^{-3x} - e^{-3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-5x} + \frac{c_2 e^{-2x}}{3}\right) + \left(-2x\,e^{-3x} - e^{-3x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-5x} + \frac{c_2 \mathrm{e}^{-2x}}{3} - 2x\,\mathrm{e}^{-3x} - \mathrm{e}^{-3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{3} - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -5c_1 \mathrm{e}^{-5x} - \frac{2c_2 \mathrm{e}^{-2x}}{3} + \mathrm{e}^{-3x} + 6x\,\mathrm{e}^{-3x}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = -5c_1 - \frac{2c_2}{3} + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = -2x\,\mathrm{e}^{-3x} - \mathrm{e}^{-3x} + \mathrm{e}^{-2x}$$

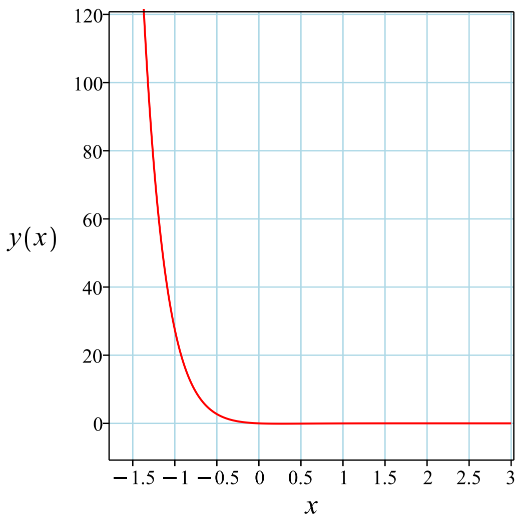
Which simplifies to

$$y = (-2x - 1)\,\mathrm{e}^{-3x} + \mathrm{e}^{-2x}$$

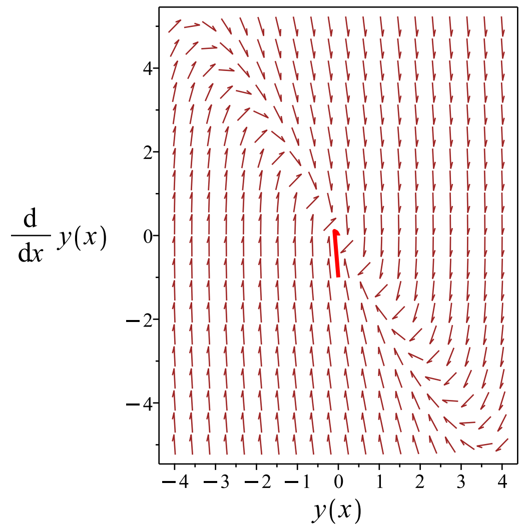
Summary

The solution(s) found are the following

$$y = (-2x - 1)\,\mathrm{e}^{-3x} + \mathrm{e}^{-2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = (-2x - 1)\,\mathrm{e}^{-3x} + \mathrm{e}^{-2x}$$

Verified OK.

11.28.4 Maple step by step solution

Let's solve

$$\left[y'' + 7y' + 10y = 4x\,\mathrm{e}^{-3x}, y(0) = 0, y'\Big|_{\{x=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 7r + 10 = 0$

- Factor the characteristic polynomial

  $(r + 5)(r + 2) = 0$

- Roots of the characteristic polynomial

  $r = (-5, -2)$

- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^{-5x}$

- 2nd solution of the homogeneous ODE

$y_2(x) = \mathrm{e}^{-2x}$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y = c_1 \mathrm{e}^{-5x} + c_2 \mathrm{e}^{-2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4x\,\mathrm{e}^{-3x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-5x} & \mathrm{e}^{-2x} \\ -5\,\mathrm{e}^{-5x} & -2\,\mathrm{e}^{-2x} \end{bmatrix}$

- ○ Compute Wronskian

$W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{-7x}$

- ○ Substitute functions into equation for $y_p(x)$

$y_p(x) = -\frac{4\,\mathrm{e}^{-5x}\left(\int \mathrm{e}^{2x} x dx\right)}{3} + \frac{4\,\mathrm{e}^{-2x}\left(\int x\,\mathrm{e}^{-x} dx\right)}{3}$

- ○ Compute integrals

$y_p(x) = (-2x-1)\,\mathrm{e}^{-3x}$

- Substitute particular solution into general solution to ODE

$y = c_1 \mathrm{e}^{-5x} + c_2 \mathrm{e}^{-2x} + (-2x-1)\,\mathrm{e}^{-3x}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-5x} + c_2 \mathrm{e}^{-2x} + (-2x-1)\,\mathrm{e}^{-3x}$

- ○ Use initial condition $y(0) = 0$

$0 = c_1 + c_2 - 1$

- ○ Compute derivative of the solution

$y' = -5c_1 \mathrm{e}^{-5x} - 2c_2 \mathrm{e}^{-2x} - 2\,\mathrm{e}^{-3x} - 3(-2x-1)\,\mathrm{e}^{-3x}$

- ○ Use the initial condition $y'\Big|_{\{x=0\}} = -1$

$-1 = -5c_1 - 2c_2 + 1$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 1\}$
- Substitute constant values into general solution and simplify

  $y = (-2x - 1)\,\mathrm{e}^{-3x} + \mathrm{e}^{-2x}$

- Solution to the IVP

  $y = (-2x - 1)\,\mathrm{e}^{-3x} + \mathrm{e}^{-2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$2)+7*diff(y(x),x)+10*y(x)=4*x*exp(-3*x),y(0) = 0, D(y)(0) = -1],y(x), si
```

$$y(x) = \mathrm{e}^{-2x} + (-2x - 1)\,\mathrm{e}^{-3x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 19

```
DSolve[{y''[x]+7*y'[x]+10*y[x]==4*x*Exp[-3*x],{y[0]==0,y'[0]==-1}},y[x],x,IncludeSingularSol
```

$$y(x) \to e^{-3x}(-2x + e^x - 1)$$

11.29 problem 29

Internal problem ID [11803]
Internal file name [`OUTPUT/11812_Thursday_April_11_2024_08_49_54_PM_52142107/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 8y' + 16y = 8\,\mathrm{e}^{-2x}}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0]$$

11.29.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 8 \\ q(x) &= 16 \\ F &= 8\,\mathrm{e}^{-2x} \end{aligned}$$

Hence the ode is

$$y'' + 8y' + 16y = 8\,\mathrm{e}^{-2x}$$

The domain of $p(x) = 8$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 16$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 8\,\mathrm{e}^{-2x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.29.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 8, C = 16, f(x) = 8\,\mathrm{e}^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 8y' + 16y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 8, C = 16$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 8\lambda e^{\lambda x} + 16 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 8\lambda + 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 8, C = 16$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-8}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(8)^2 - (4)(1)(16)} \\ &= -4\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 4$. Therefore the solution is

$$y = c_1 e^{-4x} + c_2 e^{-4x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-4x} + c_2 x\, e^{-4x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\, e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-4x} x, e^{-4x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^{-2x} = 8\,e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,e^{-2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-4x} + c_2 x\,e^{-4x}\right) + \left(2\,e^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-4x}(c_2 x + c_1) + 2\,e^{-2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-4x}(c_2 x + c_1) + 2\,e^{-2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = 2 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4\,e^{-4x}(c_2 x + c_1) + c_2 e^{-4x} - 4\,e^{-2x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -4 - 4c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 4$$

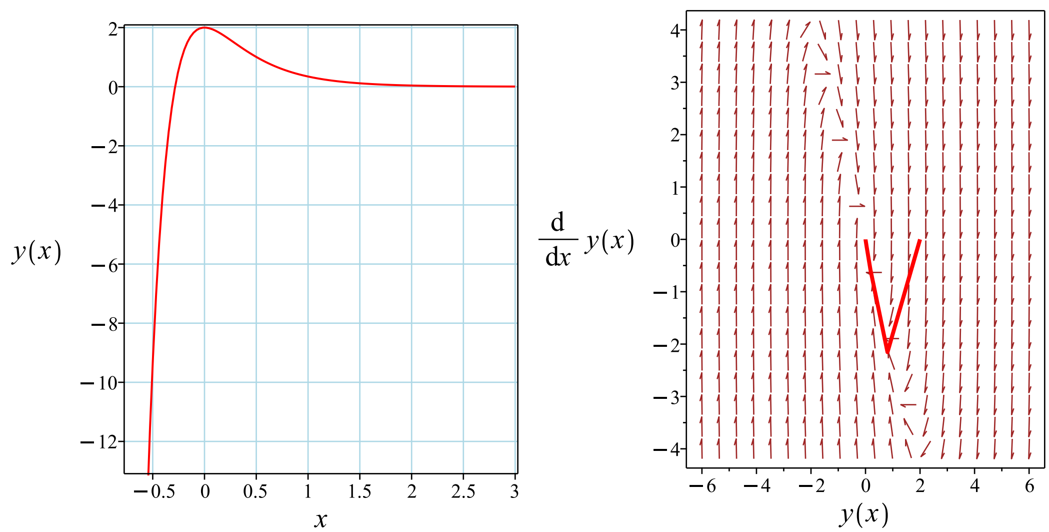
Substituting these values back in above solution results in

$$y = 4\,\mathrm{e}^{-4x}x + 2\,\mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = 4\,\mathrm{e}^{-4x}x + 2\,\mathrm{e}^{-2x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 4\,\mathrm{e}^{-4x}x + 2\,\mathrm{e}^{-2x}$$

Verified OK.

11.29.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y''+p(x)\,y'+\frac{\left(p(x)^{2}+p'(x)\right)y}{2}=f(x)$$

Where $p(x)=8$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x)&=e^{\frac{1}{2}\int p\,dx}\\&=e^{\int 8\,dx}\\&=\mathrm{e}^{4x}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}\left(M(x)y\right)''&=8\,\mathrm{e}^{4x}\mathrm{e}^{-2x}\\\left(\mathrm{e}^{4x}y\right)''&=8\,\mathrm{e}^{4x}\mathrm{e}^{-2x}\end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{4x}y\right)'=4\,\mathrm{e}^{2x}+c_1$$

Integrating again gives

$$\left(\mathrm{e}^{4x}y\right)=c_1x+2\,\mathrm{e}^{2x}+c_2$$

Hence the solution is

$$y=\frac{c_1x+2\,\mathrm{e}^{2x}+c_2}{\mathrm{e}^{4x}}$$

Or

$$y=c_1x\,\mathrm{e}^{-4x}+2\,\mathrm{e}^{-2x}+c_2\mathrm{e}^{-4x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y=c_1x\,\mathrm{e}^{-4x}+2\,\mathrm{e}^{-2x}+c_2\mathrm{e}^{-4x}\tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=2$ and $x=0$ in the above gives

$$2=c_2+2\tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^{-4x} - 4c_1 x\, e^{-4x} - 4\, e^{-2x} - 4c_2 e^{-4x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -4 + c_1 - 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = 0$$

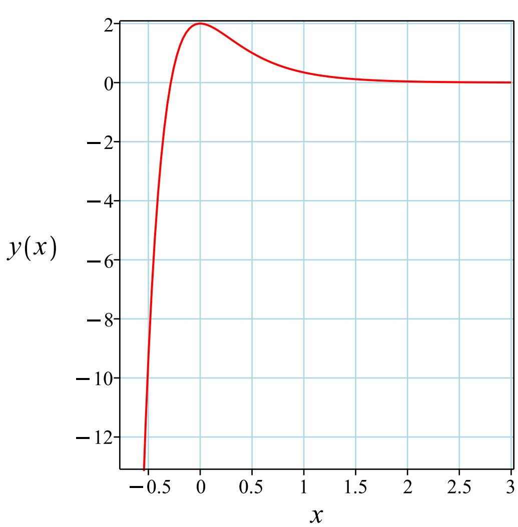
Substituting these values back in above solution results in

$$y = 4\, e^{-4x} x + 2\, e^{-2x}$$

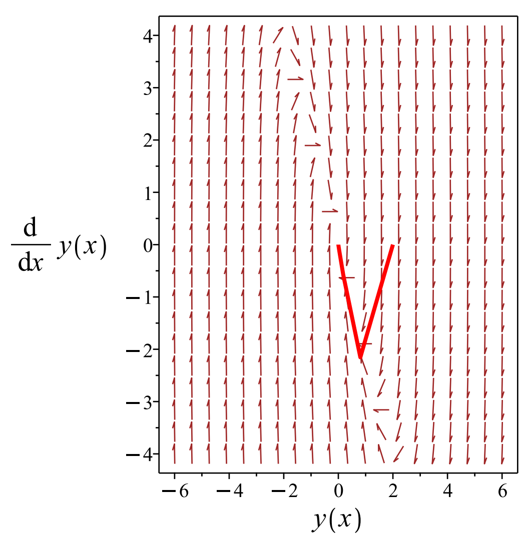
Summary

The solution(s) found are the following

$$y = 4\, e^{-4x} x + 2\, e^{-2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 4\, e^{-4x} x + 2\, e^{-2x}$$

Verified OK.

11.29.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 8y' + 16y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 8 \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 387: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{8}{1} \, dx}
\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-4x} \\
&= z_1\left(\mathrm{e}^{-4x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-4x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8}{1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-8x}}{\left(y_1\right)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-4x}\right) + c_2\left(\mathrm{e}^{-4x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 8y' + 16y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-4x} + c_2 x\,\mathrm{e}^{-4x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\,\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-4x}x, \mathrm{e}^{-4x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 \mathrm{e}^{-2x} = 8\,\mathrm{e}^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{-2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-4x} + c_2 x\,\mathrm{e}^{-4x}\right) + \left(2\,\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-4x}(c_2 x + c_1) + 2\,\mathrm{e}^{-2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-4x}(c_2 x + c_1) + 2\,e^{-2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = 2 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4\,e^{-4x}(c_2 x + c_1) + c_2 e^{-4x} - 4\,e^{-2x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -4 - 4c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 4$$

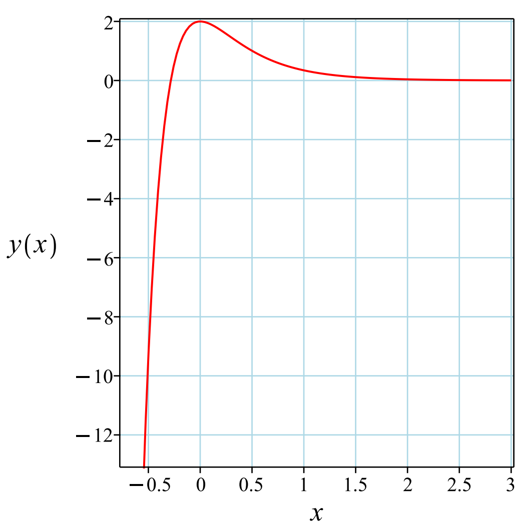
Substituting these values back in above solution results in

$$y = 4\,e^{-4x}x + 2\,e^{-2x}$$

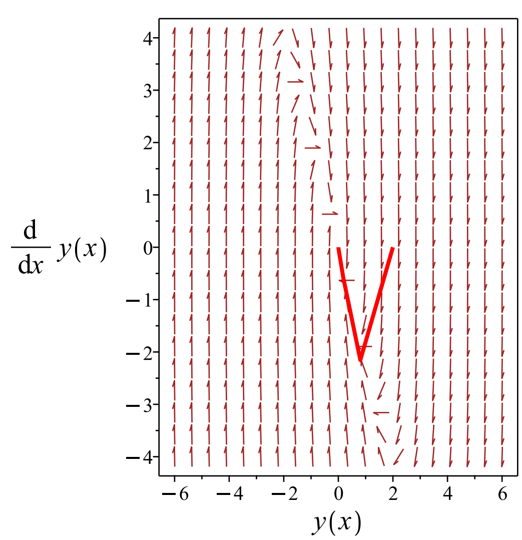
Summary

The solution(s) found are the following

$$y = 4\,e^{-4x}x + 2\,e^{-2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 4\,\mathrm{e}^{-4x} x + 2\,\mathrm{e}^{-2x}$$

Verified OK.

11.29.5 Maple step by step solution

Let's solve

$\left[y'' + 8y' + 16y = 8\,\mathrm{e}^{-2x}, y(0) = 2, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 + 8r + 16 = 0$
- Factor the characteristic polynomial

  $(r+4)^2 = 0$
- Root of the characteristic polynomial

  $r = -4$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-4x}$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = \mathrm{e}^{-4x} x$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-4x} + c_2 x\,\mathrm{e}^{-4x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 8\,\mathrm{e}^{-2x}\right]$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-4x} & e^{-4x}x \\ -4\,e^{-4x} & -4\,e^{-4x}x + e^{-4x} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-8x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -8\,e^{-4x}\left(\int e^{2x}x dx - \left(\int e^{2x}dx\right)x\right)$
- Compute integrals

  $y_p(x) = 2\,e^{-2x}$

- Substitute particular solution into general solution to ODE

  $y = c_1e^{-4x} + c_2x\,e^{-4x} + 2\,e^{-2x}$

☐ Check validity of solution $y = c_1e^{-4x} + c_2xe^{-4x} + 2e^{-2x}$

- Use initial condition $y(0) = 2$

  $2 = 2 + c_1$
- Compute derivative of the solution

  $y' = -4c_1e^{-4x} + c_2e^{-4x} - 4c_2x\,e^{-4x} - 4\,e^{-2x}$
- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = -4 - 4c_1 + c_2$
- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 4\}$
- Substitute constant values into general solution and simplify

  $y = 4\,e^{-4x}x + 2\,e^{-2x}$

- Solution to the IVP

  $y = 4\,e^{-4x}x + 2\,e^{-2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve([diff(y(x),x$2)+8*diff(y(x),x)+16*y(x)=8*exp(-2*x),y(0) = 2, D(y)(0) = 0],y(x), sings
```

$$y(x) = 4\,\mathrm{e}^{-4x}x + 2\,\mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 21

```
DSolve[{y''[x]+8*y'[x]+16*y[x]==8*Exp[-2*x],{y[0]==2,y'[0]==0}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to 2e^{-4x}\left(2x + e^{2x}\right)$$

11.30 problem 30

Internal problem ID [11804]
Internal file name [`OUTPUT/11813_Thursday_April_11_2024_08_49_55_PM_97443435/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 6y' + 9y = 27\,\mathrm{e}^{-6x}}$$

With initial conditions

$$[y(0) = -2, y'(0) = 0]$$

11.30.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 6 \\ q(x) &= 9 \\ F &= 27\,\mathrm{e}^{-6x} \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 9y = 27\,\mathrm{e}^{-6x}$$

The domain of $p(x) = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 27\,\mathrm{e}^{-6x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.30.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 6, C = 9, f(x) = 27\,\mathrm{e}^{-6x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(6)^2 - (4)\,(1)\,(9)}\\ &= -3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$y = c_1 e^{-3x} + c_2 x\, e^{-3x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-3x} + c_2 x\, e^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$27\, e^{-6x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-6x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-3x}, e^{-3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-6x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 e^{-6x} = 27\,e^{-6x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3\,e^{-6x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + c_2 x\,e^{-3x}\right) + \left(3\,e^{-6x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-3x}(c_2 x + c_1) + 3\,e^{-6x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-3x}(c_2 x + c_1) + 3\,e^{-6x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 0$ in the above gives

$$-2 = 3 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3\,e^{-3x}(c_2 x + c_1) + c_2 e^{-3x} - 18\,e^{-6x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -18 - 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -5$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = 3x\,e^{-3x} + 3\,e^{-6x} - 5\,e^{-3x}$$

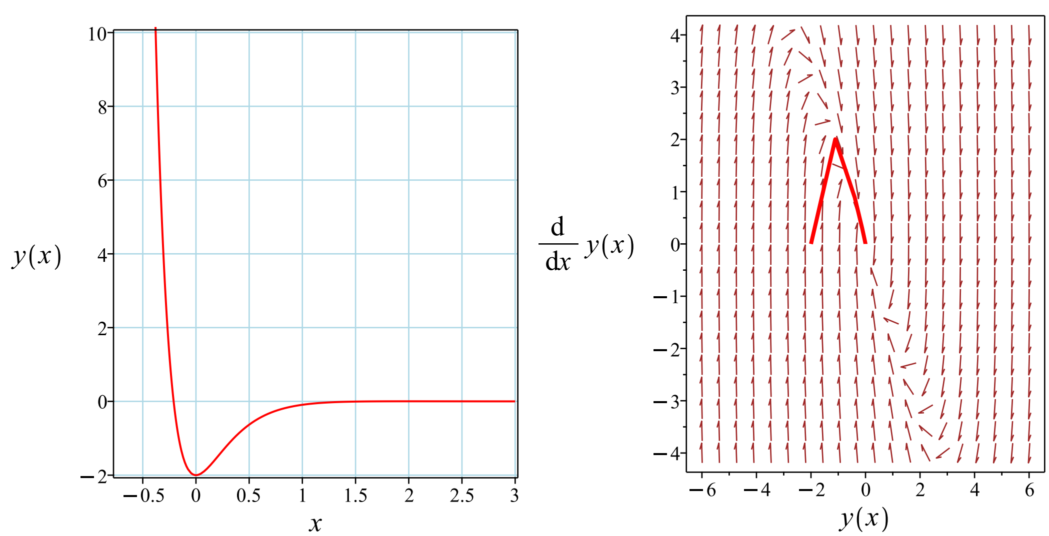
Which simplifies to

$$y = (3x - 5)\,e^{-3x} + 3\,e^{-6x}$$

Summary

The solution(s) found are the following

$$y = (3x - 5)\,e^{-3x} + 3\,e^{-6x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = (3x - 5)\,e^{-3x} + 3\,e^{-6x}$$

Verified OK.

11.30.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 6\,dx} \\ &= \mathrm{e}^{3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 27\,\mathrm{e}^{3x}\mathrm{e}^{-6x} \\ \left(\mathrm{e}^{3x}y\right)'' &= 27\,\mathrm{e}^{3x}\mathrm{e}^{-6x} \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{3x}y\right)' = -9\,\mathrm{e}^{-3x} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{3x}y\right) = c_1x + 3\,\mathrm{e}^{-3x} + c_2$$

Hence the solution is

$$y = \frac{c_1x + 3\,\mathrm{e}^{-3x} + c_2}{\mathrm{e}^{3x}}$$

Or

$$y = c_1x\,\mathrm{e}^{-3x} + c_2\mathrm{e}^{-3x} + 3\,\mathrm{e}^{-6x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1x\,\mathrm{e}^{-3x} + c_2\mathrm{e}^{-3x} + 3\,\mathrm{e}^{-6x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 0$ in the above gives

$$-2 = c_2 + 3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \mathrm{e}^{-3x} - 3c_1 x\, \mathrm{e}^{-3x} - 3c_2 \mathrm{e}^{-3x} - 18\, \mathrm{e}^{-6x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_1 - 3c_2 - 18 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = -5$$

Substituting these values back in above solution results in

$$y = 3x\, \mathrm{e}^{-3x} + 3\, \mathrm{e}^{-6x} - 5\, \mathrm{e}^{-3x}$$

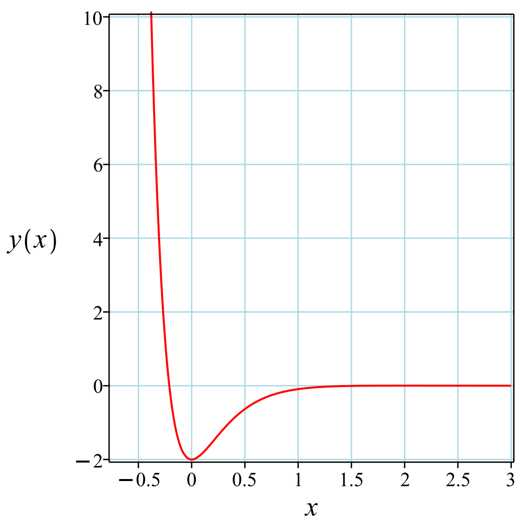
Which simplifies to

$$y = (3x - 5)\, \mathrm{e}^{-3x} + 3\, \mathrm{e}^{-6x}$$

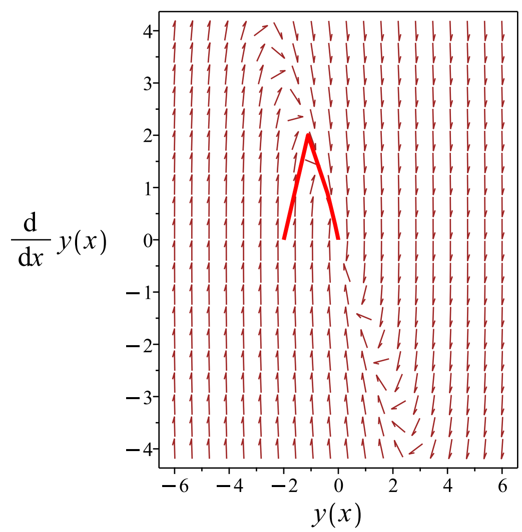
Summary

The solution(s) found are the following

$$y = (3x - 5)\, \mathrm{e}^{-3x} + 3\, \mathrm{e}^{-6x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (3x - 5)\, \mathrm{e}^{-3x} + 3\, \mathrm{e}^{-6x}$$

Verified OK.

11.30.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 389: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-3x} \\
&= z_1\left(\mathrm{e}^{-3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-6x}}{\left(y_1\right)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-3x} + c_2 x\, \mathrm{e}^{-3x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$27\,\mathrm{e}^{-6x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-6x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-3x}, \mathrm{e}^{-3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{-6x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1\mathrm{e}^{-6x} = 27\,\mathrm{e}^{-6x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3\,\mathrm{e}^{-6x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-3x} + c_2x\,\mathrm{e}^{-3x}\right) + \left(3\,\mathrm{e}^{-6x}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-3x}(c_2x + c_1) + 3\,\mathrm{e}^{-6x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-3x}(c_2 x + c_1) + 3\,e^{-6x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 0$ in the above gives

$$-2 = 3 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3\,e^{-3x}(c_2 x + c_1) + c_2 e^{-3x} - 18\,e^{-6x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -18 - 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -5$$
$$c_2 = 3$$

Substituting these values back in above solution results in

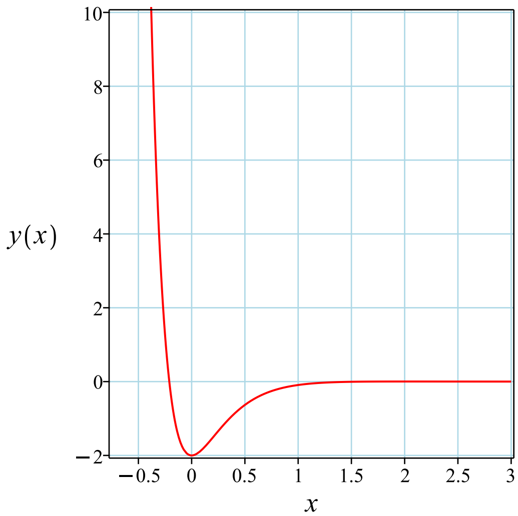
$$y = 3x\,e^{-3x} + 3\,e^{-6x} - 5\,e^{-3x}$$

Which simplifies to

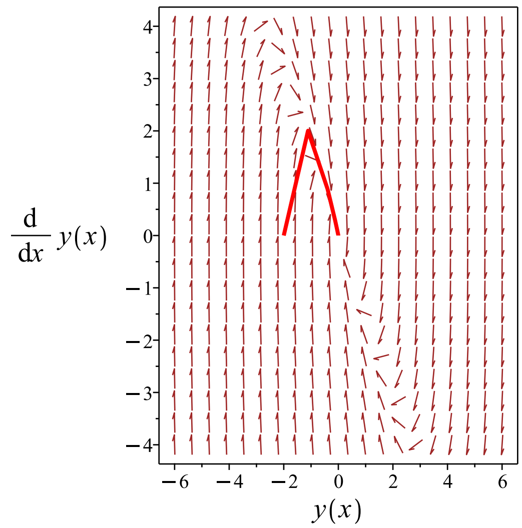
$$y = (3x - 5)\,e^{-3x} + 3\,e^{-6x}$$

Summary
The solution(s) found are the following

$$y = (3x - 5)\,e^{-3x} + 3\,e^{-6x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = (3x - 5)\,\mathrm{e}^{-3x} + 3\,\mathrm{e}^{-6x}$$

Verified OK.

11.30.5 Maple step by step solution

Let's solve

$$\left[y'' + 6y' + 9y = 27\,\mathrm{e}^{-6x}, y(0) = -2, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r + 3)^2 = 0$

- Root of the characteristic polynomial

  $r = -3$

- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^{-3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 27\,\mathrm{e}^{-6x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & x\,\mathrm{e}^{-3x} \\ -3\,\mathrm{e}^{-3x} & \mathrm{e}^{-3x} - 3x\,\mathrm{e}^{-3x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-6x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -27\,\mathrm{e}^{-3x}\left(\int x\,\mathrm{e}^{-3x}dx - \left(\int \mathrm{e}^{-3x}dx\right)x\right)$

  ◦ Compute integrals

  $y_p(x) = 3\,\mathrm{e}^{-6x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x} + 3\,\mathrm{e}^{-6x}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-3x} + c_2 x \mathrm{e}^{-3x} + 3\mathrm{e}^{-6x}$

  ◦ Use initial condition $y(0) = -2$

  $-2 = 3 + c_1$

  ◦ Compute derivative of the solution

  $y' = -3c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x} - 3c_2 x\,\mathrm{e}^{-3x} - 18\,\mathrm{e}^{-6x}$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = -18 - 3c_1 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = -5, c_2 = 3\}$

- Substitute constant values into general solution and simplify

  $y = (3x - 5)\,\mathrm{e}^{-3x} + 3\,\mathrm{e}^{-6x}$

- Solution to the IVP

  $y = (3x - 5)\,\mathrm{e}^{-3x} + 3\,\mathrm{e}^{-6x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$2)+6*diff(y(x),x)+9*y(x)=27*exp(-6*x),y(0) = -2, D(y)(0) = 0],y(x), sing
```

$$y(x) = (3x - 5)\,\mathrm{e}^{-3x} + 3\,\mathrm{e}^{-6x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 24

```
DSolve[{y''[x]+6*y'[x]+9*y[x]==27*Exp[-6*x],{y[0]==-2,y'[0]==0}},y[x],x,IncludeSingularSolut
```

$$y(x) \to e^{-6x}\left(e^{3x}(3x - 5) + 3\right)$$

11.31 problem 31

Internal problem ID [11805]
Internal file name [`OUTPUT/11814_Thursday_April_11_2024_08_49_56_PM_89938605/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + 4y' + 13y = 18\,\mathrm{e}^{-2x}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 4]$$

11.31.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 4 \\ q(x) &= 13 \\ F &= 18\,\mathrm{e}^{-2x} \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 13y = 18\,\mathrm{e}^{-2x}$$

The domain of $p(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 13$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 18\,\mathrm{e}^{-2x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.31.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 13, f(x) = 18\,\mathrm{e}^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 13y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 13$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\lambda\,\mathrm{e}^{\lambda x} + 13\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(13)} \\ &= -2 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 3i \\ \lambda_2 &= -2 - 3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + 3i \\ \lambda_2 &= -2 - 3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-2x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-2x}(c_1 \cos(3x) + c_2 \sin(3x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$18\,\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x} \cos(3x), e^{-2x} \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 e^{-2x} = 18 e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2 e^{-2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2x}(c_1 \cos(3x) + c_2 \sin(3x))\right) + \left(2 e^{-2x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-2x}(c_1 \cos(3x) + c_2 \sin(3x)) + 2 e^{-2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2\,\mathrm{e}^{-2x}(c_1 \cos(3x) + c_2 \sin(3x)) + \mathrm{e}^{-2x}(-3c_1 \sin(3x) + 3c_2 \cos(3x)) - 4\,\mathrm{e}^{-2x}$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = -2c_1 + 3c_2 - 4 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = \frac{4}{3}$$

Substituting these values back in above solution results in

$$y = \frac{4\,\mathrm{e}^{-2x} \sin(3x)}{3} - 2\,\mathrm{e}^{-2x} \cos(3x) + 2\,\mathrm{e}^{-2x}$$

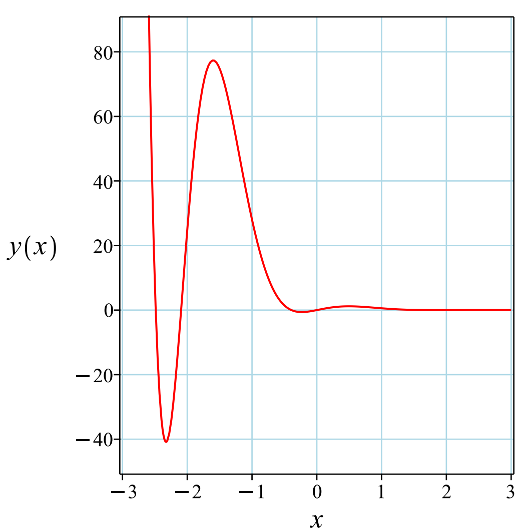
Which simplifies to

$$y = -\frac{2\,\mathrm{e}^{-2x}(-3 - 2\sin(3x) + 3\cos(3x))}{3}$$

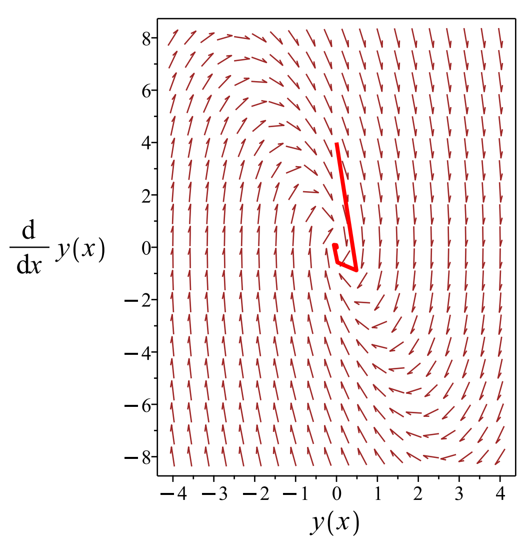
Summary

The solution(s) found are the following

$$y = -\frac{2\,\mathrm{e}^{-2x}(-3 - 2\sin(3x) + 3\cos(3x))}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2\,e^{-2x}(-3 - 2\sin(3x) + 3\cos(3x))}{3}$$

Verified OK.

11.31.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 13y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 391: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x} \cos(3x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\tan(3x)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x} \cos(3x)\right) + c_2 \left(\mathrm{e}^{-2x} \cos(3x) \left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 13y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} \cos(3x) + \frac{\sin(3x) e^{-2x} c_2}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$18 e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-2x} \cos(3x), \frac{e^{-2x} \sin(3x)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 e^{-2x} = 18 e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2 e^{-2x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-2x} \cos(3x) + \frac{\sin(3x)\,\mathrm{e}^{-2x} c_2}{3}\right) + \left(2\,\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-2x} \cos(3x) + \frac{\sin(3x)\,\mathrm{e}^{-2x} c_2}{3} + 2\,\mathrm{e}^{-2x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 + 2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \mathrm{e}^{-2x} \cos(3x) - 3c_1 \mathrm{e}^{-2x} \sin(3x) + \cos(3x)\,\mathrm{e}^{-2x} c_2 - \frac{2\sin(3x)\,\mathrm{e}^{-2x} c_2}{3} - 4\,\mathrm{e}^{-2x}
$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$
4 = -2c_1 - 4 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -2 \\
c_2 &= 4
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{4\,\mathrm{e}^{-2x} \sin(3x)}{3} - 2\,\mathrm{e}^{-2x} \cos(3x) + 2\,\mathrm{e}^{-2x}
$$

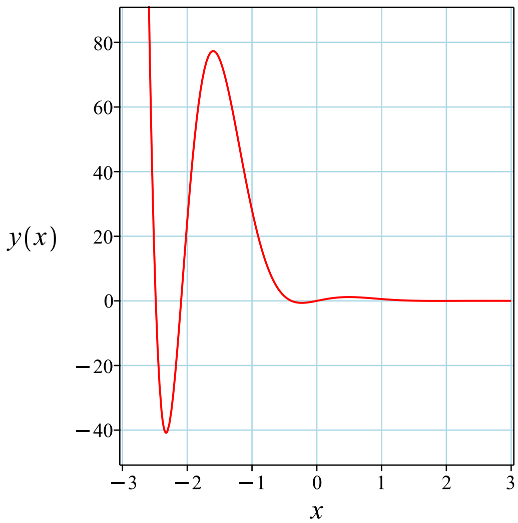
Which simplifies to

$$
y = -\frac{2\,\mathrm{e}^{-2x}(-3 - 2\sin(3x) + 3\cos(3x))}{3}
$$

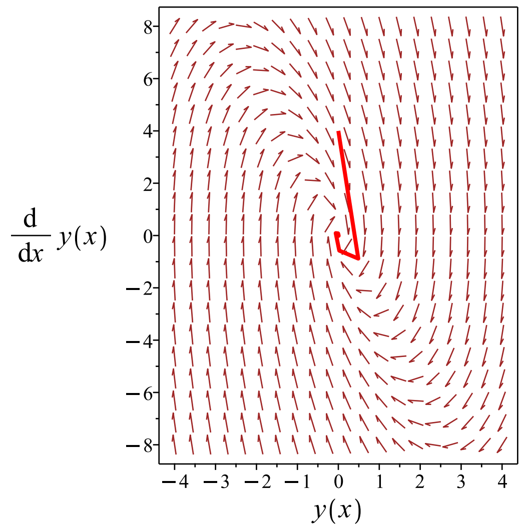
Summary

The solution(s) found are the following

$$
y = -\frac{2\,\mathrm{e}^{-2x}(-3 - 2\sin(3x) + 3\cos(3x))}{3} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2\,\mathrm{e}^{-2x}(-3-2\sin(3x)+3\cos(3x))}{3}$$

Verified OK.

11.31.4 Maple step by step solution

Let's solve

$$\left[y''+4y'+13y=18\,\mathrm{e}^{-2x}, y(0)=0, y'\Big|_{\{x=0\}}=4\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+4r+13=0$

- Use quadratic formula to solve for r

  $r=\frac{(-4)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r=(-2-3\,\mathrm{I}, -2+3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^{-2x} \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-2x} \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} \cos(3x) + \sin(3x) \mathrm{e}^{-2x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 18\,\mathrm{e}^{-2x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} \cos(3x) & \mathrm{e}^{-2x} \sin(3x) \\ -2\,\mathrm{e}^{-2x} \cos(3x) - 3\,\mathrm{e}^{-2x} \sin(3x) & -2\,\mathrm{e}^{-2x} \sin(3x) + 3\,\mathrm{e}^{-2x} \cos(3x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{-4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -6\,\mathrm{e}^{-2x}\left(\cos(3x)\left(\int \sin(3x)\, dx\right) - \sin(3x)\left(\int \cos(3x)\, dx\right)\right)$

  - Compute integrals

    $y_p(x) = 2\,\mathrm{e}^{-2x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} \cos(3x) + \sin(3x) \mathrm{e}^{-2x} c_2 + 2\,\mathrm{e}^{-2x}$

□ Check validity of solution $y = c_1 \mathrm{e}^{-2x} \cos(3x) + \sin(3x) \mathrm{e}^{-2x} c_2 + 2\mathrm{e}^{-2x}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + 2$

  - Compute derivative of the solution

    $y' = -2c_1 \mathrm{e}^{-2x} \cos(3x) - 3c_1 \mathrm{e}^{-2x} \sin(3x) + 3\cos(3x) \mathrm{e}^{-2x} c_2 - 2\sin(3x) \mathrm{e}^{-2x} c_2 - 4\,\mathrm{e}^{-2x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 4$

    $4 = -2c_1 + 3c_2 - 4$

- Solve for c_1 and c_2

  $\{c_1 = -2, c_2 = \frac{4}{3}\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{2\,e^{-2x}(-3-2\sin(3x)+3\cos(3x))}{3}$

- Solution to the IVP

  $y = -\frac{2\,e^{-2x}(-3-2\sin(3x)+3\cos(3x))}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve([diff(y(x),x$2)+4*diff(y(x),x)+13*y(x)=18*exp(-2*x),y(0) = 0, D(y)(0) = 4],y(x), sing
```

$$y(x) = \frac{2\,e^{-2x}(2\sin(3x) - 3\cos(3x) + 3)}{3}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 28

```
DSolve[{y''[x]+4*y'[x]+13*y[x]==18*Exp[-2*x],{y[0]==0,y'[0]==4}},y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{1}{3}e^{-2x}(4\sin(3x) - 6\cos(3x) + 6)$$

11.32 problem 32

Internal problem ID [11806]
Internal file name [`OUTPUT/11815_Thursday_April_11_2024_08_49_57_PM_19006565/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 10y' + 29y = 8\,e^{5x}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 8]$$

11.32.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -10 \\ q(x) &= 29 \\ F &= 8\,e^{5x} \end{aligned}$$

Hence the ode is

$$y'' - 10y' + 29y = 8\,\mathrm{e}^{5x}$$

The domain of $p(x) = -10$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 29$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 8\,\mathrm{e}^{5x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.32.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -10, C = 29, f(x) = 8\,\mathrm{e}^{5x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 10y' + 29y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -10, C = 29$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 10\lambda\,\mathrm{e}^{\lambda x} + 29\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 10\lambda + 29 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -10, C = 29$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{10}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-10^2 - (4)(1)(29)} \\ &= 5 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 5 + 2i \\ \lambda_2 &= 5 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 5 + 2i \\ \lambda_2 &= 5 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 5$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{5x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{5x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\,\mathrm{e}^{5x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{5x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{5x} \cos(2x), e^{5x} \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{5x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^{5x} = 8\,e^{5x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,e^{5x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{5x}(c_1 \cos(2x) + c_2 \sin(2x))\right) + \left(2\,e^{5x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{5x}(c_1 \cos(2x) + c_2 \sin(2x)) + 2\,e^{5x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 5\,e^{5x}(c_1 \cos(2x) + c_2 \sin(2x)) + e^{5x}(-2c_1 \sin(2x) + 2c_2 \cos(2x)) + 10\,e^{5x}$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = 5c_1 + 2c_2 + 10 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 4$$

Substituting these values back in above solution results in

$$y = -2\,e^{5x} \cos(2x) + 4\,e^{5x} \sin(2x) + 2\,e^{5x}$$

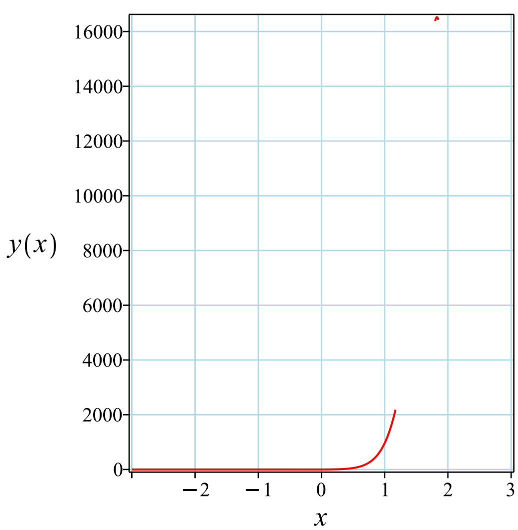
Which simplifies to

$$y = -2\,e^{5x}(-1 + \cos(2x) - 2\sin(2x))$$

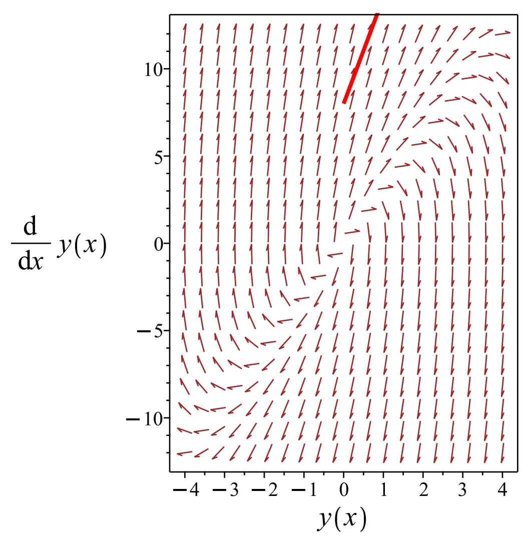
Summary

The solution(s) found are the following

$$y = -2\,e^{5x}(-1 + \cos(2x) - 2\sin(2x)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2\,e^{5x}(-1 + \cos(2x) - 2\sin(2x))$$

Verified OK.

11.32.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 10y' + 29y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -10 \\ C &= 29 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 393: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{5x} \\
&= z_1 \left(\mathrm{e}^{5x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{5x} \cos(2x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{10x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{5x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{5x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 10y' + 29y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{5x} \cos(2x) + \frac{\sin(2x)\, \mathrm{e}^{5x} c_2}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\,\mathrm{e}^{5x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{5x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{5x}\cos(2x), \frac{\mathrm{e}^{5x}\sin(2x)}{2}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{5x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1\mathrm{e}^{5x} = 8\,\mathrm{e}^{5x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{5x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{5x}\cos(2x) + \frac{\sin(2x)\,\mathrm{e}^{5x}c_2}{2}\right) + \left(2\,\mathrm{e}^{5x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{5x} \cos(2x) + \frac{\sin(2x)\,\mathrm{e}^{5x} c_2}{2} + 2\,\mathrm{e}^{5x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 5c_1 \mathrm{e}^{5x} \cos(2x) - 2c_1 \mathrm{e}^{5x} \sin(2x) + \cos(2x)\,\mathrm{e}^{5x} c_2 + \frac{5\sin(2x)\,\mathrm{e}^{5x} c_2}{2} + 10\,\mathrm{e}^{5x}$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = 5c_1 + 10 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 8$$

Substituting these values back in above solution results in

$$y = -2\,\mathrm{e}^{5x} \cos(2x) + 4\,\mathrm{e}^{5x} \sin(2x) + 2\,\mathrm{e}^{5x}$$

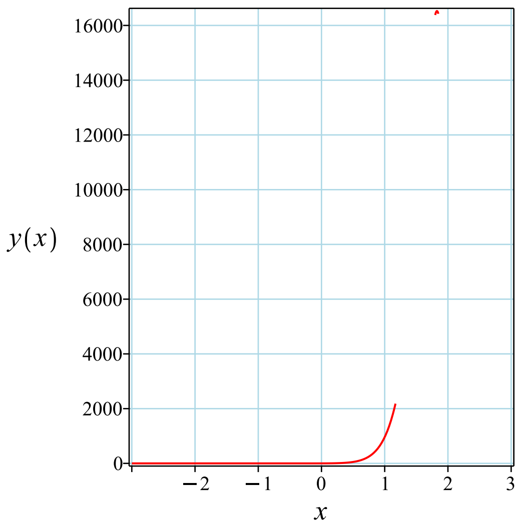
Which simplifies to

$$y = -2\,\mathrm{e}^{5x}(-1 + \cos(2x) - 2\sin(2x))$$

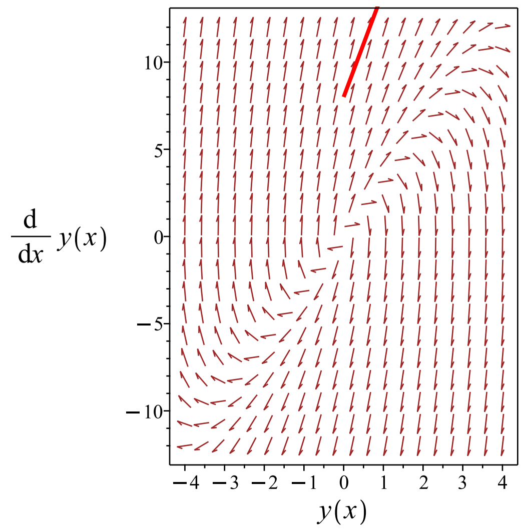
Summary

The solution(s) found are the following

$$y = -2\,\mathrm{e}^{5x}(-1 + \cos(2x) - 2\sin(2x)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -2\,\mathrm{e}^{5x}(-1 + \cos(2x) - 2\sin(2x))$$

Verified OK.

11.32.4 Maple step by step solution

Let's solve

$$\left[y'' - 10y' + 29y = 8\,\mathrm{e}^{5x}, y(0) = 0, y'\Big|_{\{x=0\}} = 8\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 10r + 29 = 0$

- Use quadratic formula to solve for r

  $r = \frac{10 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (5 - 2\,\mathrm{I}, 5 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = e^{5x} \cos(2x)$

- 2nd solution of the homogeneous ODE

$y_2(x) = e^{5x} \sin(2x)$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y = c_1 e^{5x} \cos(2x) + \sin(2x) e^{5x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 8\,e^{5x}\right]$

- ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{5x}\cos(2x) & e^{5x}\sin(2x) \\ 5\,e^{5x}\cos(2x) - 2\,e^{5x}\sin(2x) & 5\,e^{5x}\sin(2x) + 2\,e^{5x}\cos(2x) \end{bmatrix}$$

- ◦ Compute Wronskian

$W(y_1(x), y_2(x)) = 2\,e^{10x}$

- ◦ Substitute functions into equation for $y_p(x)$

$y_p(x) = -4\,e^{5x}\left(\cos(2x)\left(\int \sin(2x)\,dx\right) - \sin(2x)\left(\int \cos(2x)\,dx\right)\right)$

- ◦ Compute integrals

$y_p(x) = 2\,e^{5x}$

- Substitute particular solution into general solution to ODE

$y = c_1 e^{5x} \cos(2x) + \sin(2x) e^{5x} c_2 + 2\,e^{5x}$

□ Check validity of solution $y = c_1 e^{5x} \cos(2x) + \sin(2x) e^{5x} c_2 + 2e^{5x}$

- ◦ Use initial condition $y(0) = 0$

$0 = c_1 + 2$

- ◦ Compute derivative of the solution

$y' = 5c_1 e^{5x} \cos(2x) - 2c_1 e^{5x} \sin(2x) + 2\cos(2x) e^{5x} c_2 + 5\sin(2x) e^{5x} c_2 + 10\,e^{5x}$

- ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 8$

$8 = 5c_1 + 2c_2 + 10$

- Solve for c_1 and c_2

  $\{c_1 = -2, c_2 = 4\}$

- Substitute constant values into general solution and simplify

  $y = -2\,\mathrm{e}^{5x}(-1 + \cos(2x) - 2\sin(2x))$

- Solution to the IVP

  $y = -2\,\mathrm{e}^{5x}(-1 + \cos(2x) - 2\sin(2x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve([diff(y(x),x$2)-10*diff(y(x),x)+29*y(x)=8*exp(5*x),y(0) = 0, D(y)(0) = 8],y(x), sings
```

$$y(x) = -2\,\mathrm{e}^{5x}(-1 - 2\sin(2x) + \cos(2x))$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 24

```
DSolve[{y''[x]-10*y'[x]+29*y[x]==8*Exp[5*x],{y[0]==0,y'[0]==8}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to -2e^{5x}(-2\sin(2x) + \cos(2x) - 1)$$

11.33 problem 33

Internal problem ID [11807]
Internal file name [`OUTPUT/11816_Thursday_April_11_2024_08_49_58_PM_34915414/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y' + 13y = 8 \sin(3x)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 2]$$

11.33.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= 13 \\ F &= 8 \sin(3x) \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 13y = 8 \sin (3x)$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 13$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 8 \sin (3x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.33.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 13, f(x) = 8 \sin (3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 13y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 13$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 13 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(13)} \\ &= 2 \pm 3i\end{aligned}$$

Hence

$$\lambda_1 = 2 + 3i$$
$$\lambda_2 = 2 - 3i$$

Which simplifies to

$$\lambda_1 = 2 + 3i$$
$$\lambda_2 = 2 - 3i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 2$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{2x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{2x}(c_1 \cos(3x) + c_2 \sin(3x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 \sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}\cos(3x), e^{2x}\sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(3x) + A_2\sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1\cos(3x) + 4A_2\sin(3x) + 12A_1\sin(3x) - 12A_2\cos(3x) = 8\sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{5}, A_2 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3\cos(3x)}{5} + \frac{\sin(3x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{2x}(c_1\cos(3x) + c_2\sin(3x))\right) + \left(\frac{3\cos(3x)}{5} + \frac{\sin(3x)}{5}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{2x}(c_1\cos(3x) + c_2\sin(3x)) + \frac{3\cos(3x)}{5} + \frac{\sin(3x)}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + \frac{3}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2\,\mathrm{e}^{2x}(c_1 \cos(3x) + c_2 \sin(3x)) + \mathrm{e}^{2x}(-3c_1 \sin(3x) + 3c_2 \cos(3x)) - \frac{9\sin(3x)}{5} + \frac{3\cos(3x)}{5}$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = 2c_1 + \frac{3}{5} + 3c_2 \tag{2A}$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{2}{5}$$
$$c_2 = \frac{1}{5}$$

Substituting these values back in above solution results in

$$y = \frac{2\,\mathrm{e}^{2x}\cos(3x)}{5} + \frac{\mathrm{e}^{2x}\sin(3x)}{5} + \frac{3\cos(3x)}{5} + \frac{\sin(3x)}{5}$$

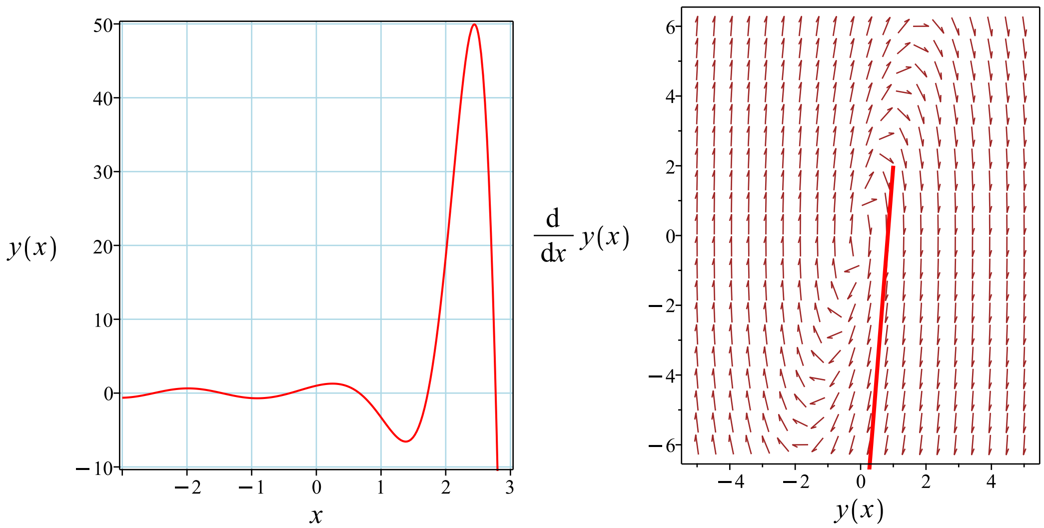
Which simplifies to

$$y = \frac{(2\,\mathrm{e}^{2x} + 3)\cos(3x)}{5} + \frac{\sin(3x)(\mathrm{e}^{2x} + 1)}{5}$$

Summary

The solution(s) found are the following

$$y = \frac{(2\,\mathrm{e}^{2x} + 3)\cos(3x)}{5} + \frac{\sin(3x)(\mathrm{e}^{2x} + 1)}{5} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{(2\,e^{2x}+3)\cos(3x)}{5} + \frac{\sin(3x)\,(e^{2x}+1)}{5}$$

Verified OK.

11.33.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 13y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -9$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 395: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\ &= z_1 e^{2x} \\ &= z_1\left(\mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}\cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{2x}\cos(3x)\right) + c_2\left(e^{2x}\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - 4y' + 13y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 e^{2x}\cos(3x) + \frac{e^{2x} c_2 \sin(3x)}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
8\sin(3x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(3x), \sin(3x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{e^{2x}\cos(3x), \frac{e^{2x}\sin(3x)}{3}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1\cos(3x) + A_2\sin(3x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 \cos(3x) + 4A_2 \sin(3x) + 12A_1 \sin(3x) - 12A_2 \cos(3x) = 8 \sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{5}, A_2 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3\cos(3x)}{5} + \frac{\sin(3x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{2x} \cos(3x) + \frac{\mathrm{e}^{2x} c_2 \sin(3x)}{3}\right) + \left(\frac{3\cos(3x)}{5} + \frac{\sin(3x)}{5}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{2x} \cos(3x) + \frac{\mathrm{e}^{2x} c_2 \sin(3x)}{3} + \frac{3\cos(3x)}{5} + \frac{\sin(3x)}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + \frac{3}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 \mathrm{e}^{2x} \cos(3x) - 3c_1 \mathrm{e}^{2x} \sin(3x) + \frac{2\,\mathrm{e}^{2x} c_2 \sin(3x)}{3} + \mathrm{e}^{2x} c_2 \cos(3x) - \frac{9\sin(3x)}{5} + \frac{3\cos(3x)}{5}$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = 2c_1 + \frac{3}{5} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{2}{5}$$
$$c_2 = \frac{3}{5}$$

Substituting these values back in above solution results in

$$y = \frac{2\,e^{2x}\cos(3x)}{5} + \frac{e^{2x}\sin(3x)}{5} + \frac{3\cos(3x)}{5} + \frac{\sin(3x)}{5}$$

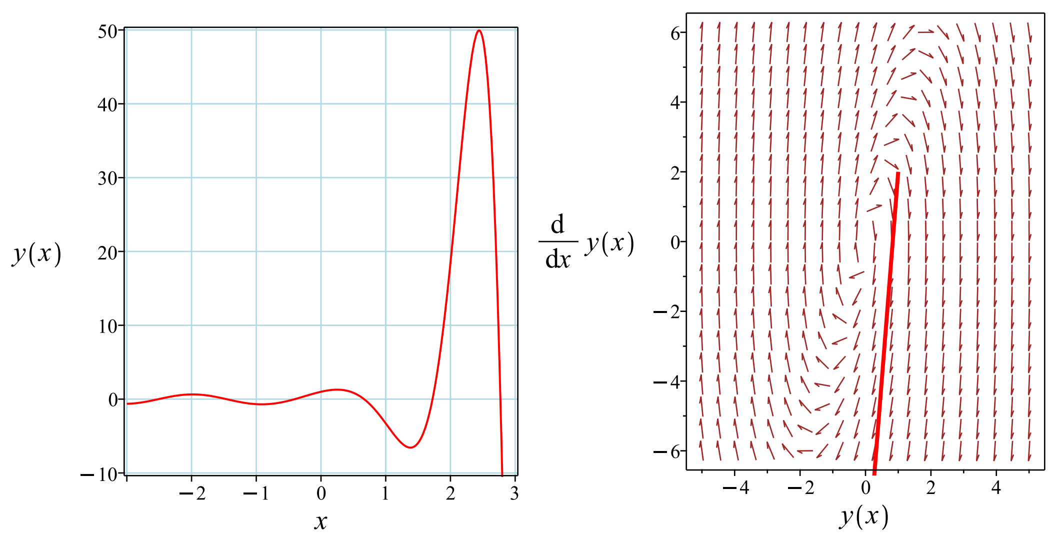
Which simplifies to

$$y = \frac{(2\,e^{2x}+3)\cos(3x)}{5} + \frac{\sin(3x)\,(e^{2x}+1)}{5}$$

Summary

The solution(s) found are the following

$$y = \frac{(2\,e^{2x}+3)\cos(3x)}{5} + \frac{\sin(3x)\,(e^{2x}+1)}{5} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{(2\,e^{2x}+3)\cos(3x)}{5} + \frac{\sin(3x)\,(e^{2x}+1)}{5}$$

Verified OK.

11.33.4 Maple step by step solution

Let's solve

$$\left[y''-4y'+13y=8\sin\left(3x\right), y(0)=1, y'\Big|_{\{x=0\}}=2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2-4r+13=0$

- Use quadratic formula to solve for r

  $r=\frac{4\pm\left(\sqrt{-36}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(2-3\,\mathrm{I}, 2+3\,\mathrm{I}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{2x}\cos\left(3x\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\mathrm{e}^{2x}\sin\left(3x\right)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{2x}\cos\left(3x\right)+\mathrm{e}^{2x}c_2\sin\left(3x\right)+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=8\sin\left(3x\right)\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W\left(y_1(x), y_2(x)\right)=\left[\begin{array}{cc}\mathrm{e}^{2x}\cos\left(3x\right) & \mathrm{e}^{2x}\sin\left(3x\right)\\ 2\,\mathrm{e}^{2x}\cos\left(3x\right)-3\,\mathrm{e}^{2x}\sin\left(3x\right) & 2\,\mathrm{e}^{2x}\sin\left(3x\right)+3\,\mathrm{e}^{2x}\cos\left(3x\right)\end{array}\right]$

- ◦ Compute Wronskian

  $W\left(y_1(x), y_2(x)\right)=3\,\mathrm{e}^{4x}$

- ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{4\,\mathrm{e}^{2x}\left(-2\cos(3x)\left(\int \mathrm{e}^{-2x}\sin(3x)^2 dx\right)+\sin(3x)\left(\int \sin(6x)\mathrm{e}^{-2x}dx\right)\right)}{3}$$

- ◦ Compute integrals

$$y_p(x) = \frac{3\cos(3x)}{5} + \frac{\sin(3x)}{5}$$

- • Substitute particular solution into general solution to ODE

$$y = c_1\mathrm{e}^{2x}\cos(3x) + \mathrm{e}^{2x}c_2\sin(3x) + \frac{3\cos(3x)}{5} + \frac{\sin(3x)}{5}$$

- □ Check validity of solution $y = c_1\mathrm{e}^{2x}\cos(3x) + \mathrm{e}^{2x}c_2\sin(3x) + \frac{3\cos(3x)}{5} + \frac{\sin(3x)}{5}$
  - ◦ Use initial condition $y(0) = 1$

$$1 = c_1 + \frac{3}{5}$$

  - ◦ Compute derivative of the solution

$$y' = 2c_1\mathrm{e}^{2x}\cos(3x) - 3c_1\mathrm{e}^{2x}\sin(3x) + 2\,\mathrm{e}^{2x}c_2\sin(3x) + 3\,\mathrm{e}^{2x}c_2\cos(3x) - \frac{9\sin(3x)}{5} + \frac{3\cos(3x)}{5}$$

  - ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 2$

$$2 = 2c_1 + \frac{3}{5} + 3c_2$$

  - ◦ Solve for c_1 and c_2

$$\left\{c_1 = \frac{2}{5}, c_2 = \frac{1}{5}\right\}$$

  - ◦ Substitute constant values into general solution and simplify

$$y = \frac{(2\,\mathrm{e}^{2x}+3)\cos(3x)}{5} + \frac{\sin(3x)(\mathrm{e}^{2x}+1)}{5}$$

- • Solution to the IVP

$$y = \frac{(2\,\mathrm{e}^{2x}+3)\cos(3x)}{5} + \frac{\sin(3x)(\mathrm{e}^{2x}+1)}{5}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 31

```
dsolve([diff(y(x),x$2)-4*diff(y(x),x)+13*y(x)=8*sin(3*x),y(0) = 1, D(y)(0) = 2],y(x), singso
```

$$y(x) = \frac{(2\,e^{2x}+3)\cos(3x)}{5} + \frac{\sin(3x)\,(e^{2x}+1)}{5}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 36

```
DSolve[{y''[x]-4*y'[x]+13*y[x]==8*Sin[3*x],{y[0]==1,y'[0]==2}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{1}{5}\left(\left(e^{2x}+1\right)\sin(3x) + \left(2e^{2x}+3\right)\cos(3x)\right)$$

11.34 problem 34

Internal problem ID [11808]
Internal file name [`OUTPUT/11817_Thursday_April_11_2024_08_49_58_PM_20923427/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y' - 6y = 8\,\mathrm{e}^{2x} - 5\,\mathrm{e}^{3x}}$$

With initial conditions

$$[y(0) = 3, y'(0) = 5]$$

11.34.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned}p(x) &= -1\\ q(x) &= -6\\ F &= 8\,\mathrm{e}^{2x} - 5\,\mathrm{e}^{3x}\end{aligned}$$

Hence the ode is

$$y'' - y' - 6y = 8\,e^{2x} - 5\,e^{3x}$$

The domain of $p(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 8\,e^{2x} - 5\,e^{3x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.34.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -6, f(x) = 8\,e^{2x} - 5\,e^{3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda\,e^{\lambda x} - 6\,e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-6)} \\ &= \frac{1}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{2} + \frac{5}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{5}{2}$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = -2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(-2)x}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\,\mathrm{e}^{2x} - 5\,\mathrm{e}^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{3x}\}$$

Since e^{3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x}\}, \{x\,e^{3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{2x} + A_2 x\,e^{3x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{2x} + 5A_2 e^{3x} = 8\,e^{2x} - 5\,e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2\,e^{2x} - x\,e^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + c_2 e^{-2x}\right) + \left(-2\,e^{2x} - x\,e^{3x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{3x} + c_2 e^{-2x} - 2\,e^{2x} - x\,e^{3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 + c_2 - 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1 e^{3x} - 2c_2 e^{-2x} - 4\,e^{2x} - e^{3x} - 3x\,e^{3x}$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = 3c_1 - 2c_2 - 5 \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = -x\,e^{3x} - 2\,e^{2x} + 4\,e^{3x} + e^{-2x}$$

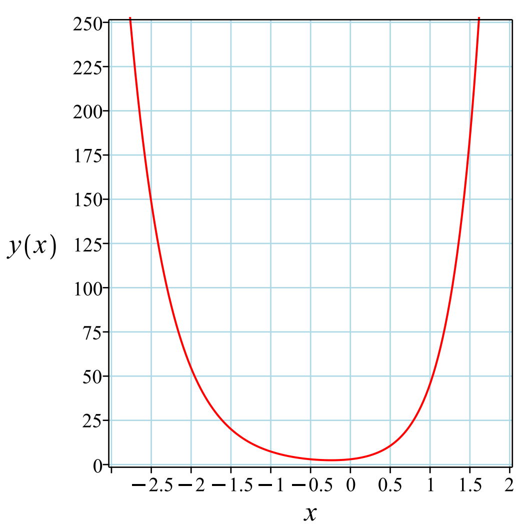
Which simplifies to

$$y = -\left((-4 + x)\,e^{5x} + 2\,e^{4x} - 1\right) e^{-2x}$$

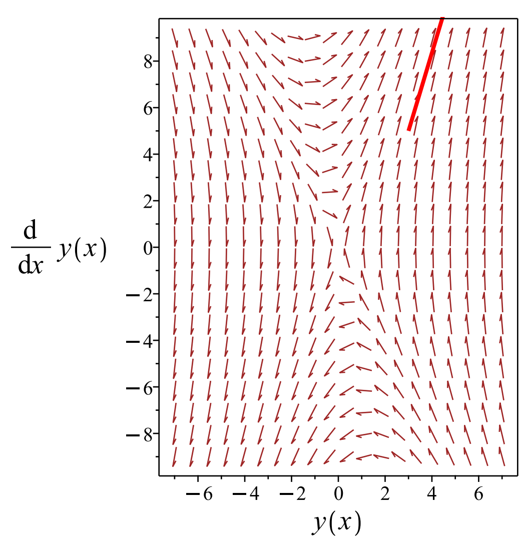
Summary

The solution(s) found are the following

$$y = -\left((-4 + x)\,e^{5x} + 2\,e^{4x} - 1\right) e^{-2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\left((-4+x)\,\mathrm{e}^{5x} + 2\,\mathrm{e}^{4x} - 1\right)\mathrm{e}^{-2x}$$

Verified OK.

11.34.3 Solving using Kovacic algorithm

Writing the ode as

$$\begin{align*} y'' - y' - 6y &= 0 \tag{1} \\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= 1 \\ B &= -1 \tag{3} \\ C &= -6 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{align*} s &= 25 \\ t &= 4 \end{align*}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 397: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(e^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = e^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{e^{5x}}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-2x}\right) + c_2\left(e^{-2x}\left(\frac{e^{5x}}{5}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} + \frac{e^{3x} c_2}{5}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-2x}$$
$$y_2 = \frac{e^{3x}}{5}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-2x} & \frac{e^{3x}}{5} \\ \frac{d}{dx}\left(e^{-2x}\right) & \frac{d}{dx}\left(\frac{e^{3x}}{5}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2x} & \frac{e^{3x}}{5} \\ -2\,e^{-2x} & \frac{3\,e^{3x}}{5} \end{vmatrix}$$

Therefore

$$W = \left(e^{-2x}\right)\left(\frac{3\,e^{3x}}{5}\right) - \left(\frac{e^{3x}}{5}\right)\left(-2\,e^{-2x}\right)$$

Which simplifies to

$$W = e^{3x}e^{-2x}$$

Which simplifies to

$$W = e^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{3x}\left(8\,e^{2x}-5\,e^{3x}\right)}{5}}{e^{x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{4x}(5\,e^{x}-8)}{5}dx$$

Hence

$$u_1 = -\frac{2\,e^{4x}}{5} + \frac{e^{5x}}{5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2x}(8\,e^{2x}-5\,e^{3x})}{e^{x}}\,dx$$

Which simplifies to

$$u_2 = \int \left(-5 + 8\,e^{-x}\right)dx$$

Hence

$$u_2 = -5x - 8\,e^{-x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{2\,\mathrm{e}^{4x}}{5} + \frac{\mathrm{e}^{5x}}{5}\right)\mathrm{e}^{-2x} + \frac{(-5x - 8\,\mathrm{e}^{-x})\,\mathrm{e}^{3x}}{5}$$

Which simplifies to

$$y_p(x) = -2\,\mathrm{e}^{2x} + \frac{\mathrm{e}^{3x}}{5} - x\,\mathrm{e}^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-2x} + \frac{\mathrm{e}^{3x}c_2}{5}\right) + \left(-2\,\mathrm{e}^{2x} + \frac{\mathrm{e}^{3x}}{5} - x\,\mathrm{e}^{3x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\mathrm{e}^{-2x} + \frac{\mathrm{e}^{3x}c_2}{5} - 2\,\mathrm{e}^{2x} + \frac{\mathrm{e}^{3x}}{5} - x\,\mathrm{e}^{3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 + \frac{c_2}{5} - \frac{9}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1\mathrm{e}^{-2x} + \frac{3\,\mathrm{e}^{3x}c_2}{5} - 4\,\mathrm{e}^{2x} - \frac{2\,\mathrm{e}^{3x}}{5} - 3x\,\mathrm{e}^{3x}$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = -2c_1 + \frac{3c_2}{5} - \frac{22}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 19 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -x\,e^{3x} - 2\,e^{2x} + 4\,e^{3x} + e^{-2x}$$

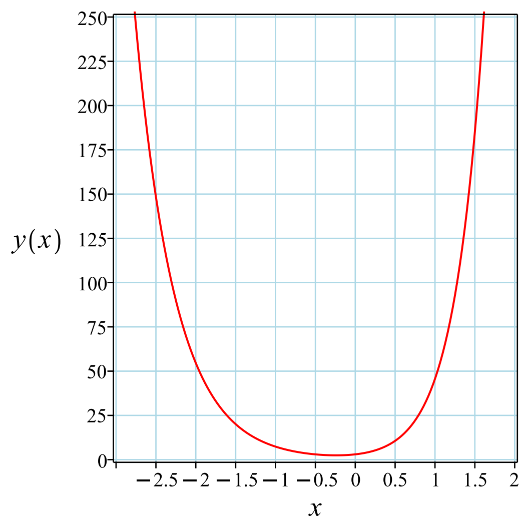
Which simplifies to

$$y = -\left((-4 + x)\,e^{5x} + 2\,e^{4x} - 1\right) e^{-2x}$$

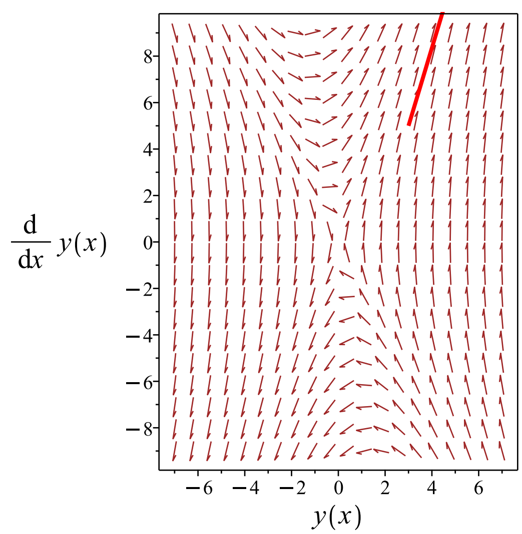
Summary

The solution(s) found are the following

$$y = -\left((-4 + x)\,e^{5x} + 2\,e^{4x} - 1\right) e^{-2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\left((-4 + x)\,e^{5x} + 2\,e^{4x} - 1\right) e^{-2x}$$

Verified OK.

11.34.4 Maple step by step solution

Let's solve

$$\left[y''-y'-6y=8\,\mathrm{e}^{2x}-5\,\mathrm{e}^{3x}, y(0)=3, y'\Big|_{\{x=0\}}=5\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2-r-6=0$

- Factor the characteristic polynomial

  $(r+2)(r-3)=0$

- Roots of the characteristic polynomial

  $r=(-2,3)$

- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\mathrm{e}^{3x}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-2x}+\mathrm{e}^{3x}c_2+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=8\,\mathrm{e}^{2x}-5\,\mathrm{e}^{3x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x))=\begin{bmatrix}\mathrm{e}^{-2x} & \mathrm{e}^{3x}\\ -2\,\mathrm{e}^{-2x} & 3\,\mathrm{e}^{3x}\end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x))=5\,\mathrm{e}^{x}$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{\left(\mathrm{e}^{5x}\left(\int \left(-5+8\,\mathrm{e}^{-x}\right)dx\right)+\int \mathrm{e}^{4x}\left(5\,\mathrm{e}^{x}-8\right)dx\right)\mathrm{e}^{-2x}}{5}$$

- ∘ Compute integrals

$$y_p(x) = \frac{\left(-10+\left(-5x+1\right)\mathrm{e}^{x}\right)\mathrm{e}^{2x}}{5}$$

- • Substitute particular solution into general solution to ODE

$$y = c_1\mathrm{e}^{-2x} + \mathrm{e}^{3x}c_2 + \frac{\left(-10+\left(-5x+1\right)\mathrm{e}^{x}\right)\mathrm{e}^{2x}}{5}$$

- □ Check validity of solution $y = c_1\mathrm{e}^{-2x} + \mathrm{e}^{3x}c_2 + \frac{\left(-10+\left(-5x+1\right)\mathrm{e}^{x}\right)\mathrm{e}^{2x}}{5}$
  - ∘ Use initial condition $y(0) = 3$

$$3 = c_1 + c_2 - \frac{9}{5}$$

  - ∘ Compute derivative of the solution

$$y' = -2c_1\mathrm{e}^{-2x} + 3\,\mathrm{e}^{3x}c_2 + \frac{\left(-5\,\mathrm{e}^{x}+\left(-5x+1\right)\mathrm{e}^{x}\right)\mathrm{e}^{2x}}{5} + \frac{2\left(-10+\left(-5x+1\right)\mathrm{e}^{x}\right)\mathrm{e}^{2x}}{5}$$

  - ∘ Use the initial condition $y'\Big|_{\{x=0\}} = 5$

$$5 = -2c_1 + 3c_2 - \frac{22}{5}$$

  - ∘ Solve for c_1 and c_2

$$\left\{c_1 = 1, c_2 = \frac{19}{5}\right\}$$

  - ∘ Substitute constant values into general solution and simplify

$$y = -\left(\left(-4+x\right)\mathrm{e}^{5x} + 2\,\mathrm{e}^{4x} - 1\right)\mathrm{e}^{-2x}$$

- • Solution to the IVP

$$y = -\left(\left(-4+x\right)\mathrm{e}^{5x} + 2\,\mathrm{e}^{4x} - 1\right)\mathrm{e}^{-2x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve([diff(y(x),x$2)-diff(y(x),x)-6*y(x)=8*exp(2*x)-5*exp(3*x),y(0) = 3, D(y)(0) = 5],y(x)
```

$$y(x) = -\left((-4 + x)\,\mathrm{e}^{5x} + 2\,\mathrm{e}^{4x} - 1\right)\mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 28

```
DSolve[{y''[x]-y'[x]-6*y[x]==8*Exp[2*x]-5*Exp[3*x],{y[0]==3,y'[0]==5}},y[x],x,IncludeSingula
```

$$y(x) \to -e^{3x}(x-4) + e^{-2x} - 2e^{2x}$$

11.35 problem 35

Internal problem ID [11809]
Internal file name [`OUTPUT/11818_Thursday_April_11_2024_08_49_59_PM_2985471/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = 2\,\mathrm{e}^{2x} x + 6\,\mathrm{e}^{x}}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

11.35.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 1 \\ F &= 2\,\mathrm{e}^{2x}x + 6\,\mathrm{e}^{x} \end{aligned}$$

Hence the ode is

$$y'' - 2y' + y = 2\,\mathrm{e}^{2x}x + 6\,\mathrm{e}^{x}$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 2\,\mathrm{e}^{2x}x + 6\,\mathrm{e}^{x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.35.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = 2\,\mathrm{e}^{2x}x + 6\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)} \\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^{x} + c_2 x\, e^{x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{x} c_1 + c_2 x\, e^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\, e^{2x} x + 6\, e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}, \{e^{2x} x, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{x}, e^{x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^x\}, \{e^{2x}x, e^{2x}\}]$$

Since $x\,e^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2e^x\}, \{e^{2x}x, e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2e^x + A_2e^{2x}x + A_3e^{2x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1e^x + A_2e^{2x}x + 2A_2e^{2x} + A_3e^{2x} = 2\,e^{2x}x + 6\,e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3, A_2 = 2, A_3 = -4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x^2e^x + 2\,e^{2x}x - 4\,e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + c_2 x\,e^x) + \left(3x^2e^x + 2\,e^{2x}x - 4\,e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^x(c_2x + c_1) + 3x^2e^x + 2\,e^{2x}x - 4\,e^{2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{x}(c_2 x + c_1) + 3x^2 e^{x} + 2\,e^{2x} x - 4\,e^{2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = -4 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x}(c_2 x + c_1) + c_2 e^{x} + 6x\,e^{x} + 3x^2 e^{x} + 4\,e^{2x} x - 6\,e^{2x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -6 + c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 5$$
$$c_2 = 1$$

Substituting these values back in above solution results in

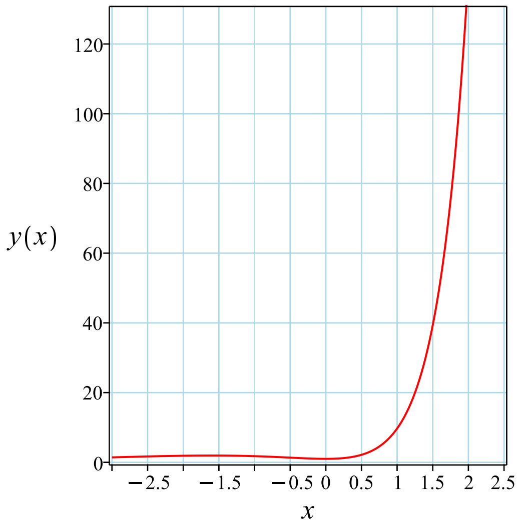
$$y = 3x^2 e^{x} + 2\,e^{2x} x + x\,e^{x} - 4\,e^{2x} + 5\,e^{x}$$

Which simplifies to

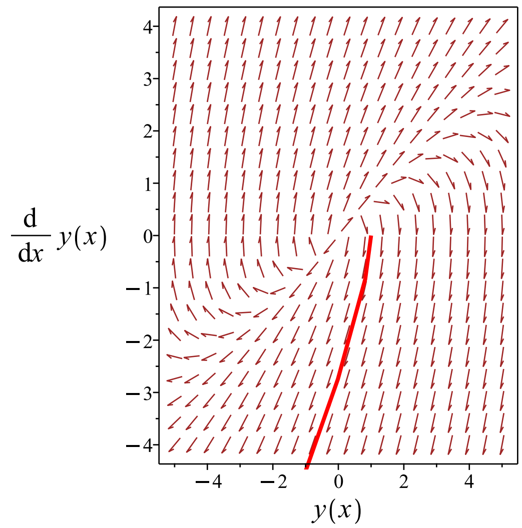
$$y = (2x - 4)\,e^{2x} + e^{x}\left(3x^2 + x + 5\right)$$

Summary
The solution(s) found are the following

$$y = (2x - 4)\,e^{2x} + e^{x}\left(3x^2 + x + 5\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (2x-4)\,\mathrm{e}^{2x} + \mathrm{e}^{x}\left(3x^{2}+x+5\right)$$

Verified OK.

11.35.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^{2}+p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= \mathrm{e}^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \mathrm{e}^{-x}\left(2\,\mathrm{e}^{2x}x + 6\,\mathrm{e}^{x}\right) \\ \left(\mathrm{e}^{-x}y\right)'' &= \mathrm{e}^{-x}\left(2\,\mathrm{e}^{2x}x + 6\,\mathrm{e}^{x}\right) \end{aligned}$$

Integrating once gives

$$\left(e^{-x}y\right)' = (2x-2)\,e^{x} + 6x + c_1$$

Integrating again gives

$$\left(e^{-x}y\right) = (2x-4)\,e^{x} + 3x^2 + c_1x + c_2$$

Hence the solution is

$$y = \frac{(2x-4)\,e^{x} + 3x^2 + c_1x + c_2}{e^{-x}}$$

Or

$$y = c_1x\,e^{x} + 3x^2e^{x} + 2\,e^{2x}x + c_2e^{x} - 4\,e^{2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1x\,e^{x} + 3x^2e^{x} + 2\,e^{2x}x + c_2e^{x} - 4\,e^{2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = -4 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x}c_1 + c_1x\,e^{x} + 6x\,e^{x} + 3x^2e^{x} - 6\,e^{2x} + 4\,e^{2x}x + c_2e^{x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -6 + c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 5$$

Substituting these values back in above solution results in

$$y = 3x^2e^{x} + 2\,e^{2x}x + x\,e^{x} - 4\,e^{2x} + 5\,e^{x}$$

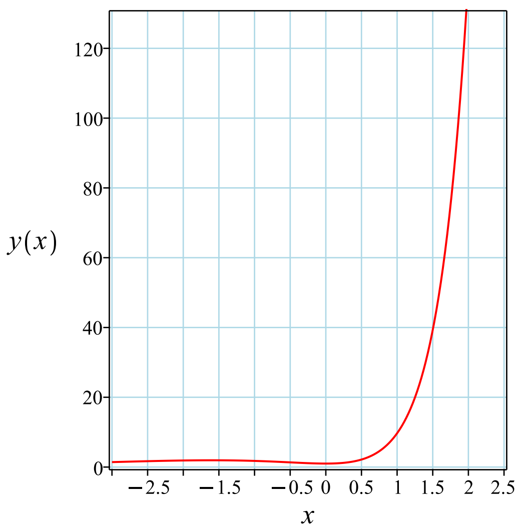
Which simplifies to

$$y = (2x-4)\,e^{2x} + e^{x}\left(3x^2 + x + 5\right)$$

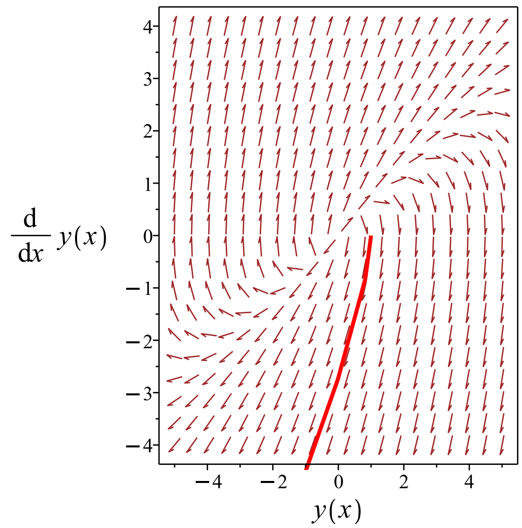
Summary

The solution(s) found are the following

$$y = (2x - 4)\,e^{2x} + e^{x}\left(3x^{2} + x + 5\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (2x - 4)\,e^{2x} + e^{x}\left(3x^{2} + x + 5\right)$$

Verified OK.

11.35.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 399: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}\\ &= z_1 e^{x}\\ &= z_1(\mathrm{e}^{x})\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x c_1 + c_2 x\,\mathrm{e}^x
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2\,\mathrm{e}^{2x} x + 6\,\mathrm{e}^x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^x\}, \{\mathrm{e}^{2x} x, \mathrm{e}^{2x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^x, \mathrm{e}^x\}
$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\,\mathrm{e}^x\}, \{\mathrm{e}^{2x} x, \mathrm{e}^{2x}\}]
$$

Since $x\,\mathrm{e}^x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^2 \mathrm{e}^x\}, \{\mathrm{e}^{2x} x, \mathrm{e}^{2x}\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^x + A_2 e^{2x} x + A_3 e^{2x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x + A_2 e^{2x} x + 2A_2 e^{2x} + A_3 e^{2x} = 2\,e^{2x} x + 6\,e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3, A_2 = 2, A_3 = -4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x^2 e^x + 2\,e^{2x} x - 4\,e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + c_2 x\,e^x) + \left(3x^2 e^x + 2\,e^{2x} x - 4\,e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^x (c_2 x + c_1) + 3x^2 e^x + 2\,e^{2x} x - 4\,e^{2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^x (c_2 x + c_1) + 3x^2 e^x + 2\,e^{2x} x - 4\,e^{2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = -4 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x}(c_2 x + c_1) + c_2 e^{x} + 6x\,e^{x} + 3x^2 e^{x} + 4\,e^{2x} x - 6\,e^{2x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -6 + c_1 + c_2 \tag{2A}$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 5$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = 3x^2 e^{x} + 2\,e^{2x} x + x\,e^{x} - 4\,e^{2x} + 5\,e^{x}$$

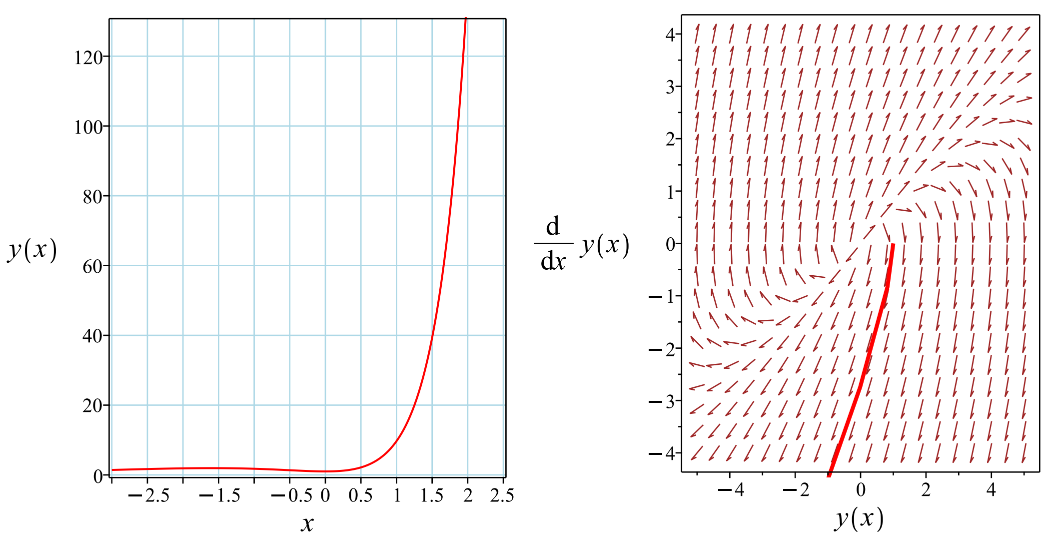
Which simplifies to

$$y = (2x - 4)\,e^{2x} + e^{x}\left(3x^2 + x + 5\right)$$

Summary

The solution(s) found are the following

$$y = (2x - 4)\,e^{2x} + e^{x}\left(3x^2 + x + 5\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = (2x - 4)\,e^{2x} + e^{x}\left(3x^2 + x + 5\right)$$

Verified OK.

11.35.5 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + y = 2\,\mathrm{e}^{2x} x + 6\,\mathrm{e}^{x}, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 - 2r + 1 = 0$
- Factor the characteristic polynomial
  $(r-1)^2 = 0$
- Root of the characteristic polynomial
  $r = 1$
- 1st solution of the homogeneous ODE
  $y_1(x) = \mathrm{e}^{x}$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence
  $y_2(x) = x\,\mathrm{e}^{x}$
- General solution of the ODE
  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y = \mathrm{e}^{x} c_1 + c_2 x\,\mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function
  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\,\mathrm{e}^{2x} x + 6\,\mathrm{e}^{x}\right]$
- ○ Wronskian of solutions of the homogeneous equation
  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{x} & x\,\mathrm{e}^{x} \\ \mathrm{e}^{x} & x\,\mathrm{e}^{x} + \mathrm{e}^{x} \end{bmatrix}$
- ○ Compute Wronskian
  $W(y_1(x), y_2(x)) = \mathrm{e}^{2x}$
- ○ Substitute functions into equation for $y_p(x)$

$y_p(x) = -2\,\mathrm{e}^x\left(\int x(x\,\mathrm{e}^x+3)\,dx - \left(\int (x\,\mathrm{e}^x+3)\,dx\right)x\right)$

- Compute integrals

  $y_p(x) = (2x-4)\left(\mathrm{e}^x\right)^2 + 3x^2\mathrm{e}^x$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^x c_1 + c_2 x\,\mathrm{e}^x + (2x-4)\left(\mathrm{e}^x\right)^2 + 3x^2\mathrm{e}^x$

- ☐ Check validity of solution $y = \mathrm{e}^x c_1 + c_2 x\mathrm{e}^x + (2x-4)\left(\mathrm{e}^x\right)^2 + 3x^2\mathrm{e}^x$
  - Use initial condition $y(0) = 1$

    $1 = -4 + c_1$
  - Compute derivative of the solution

    $y' = \mathrm{e}^x c_1 + c_2\mathrm{e}^x + c_2 x\,\mathrm{e}^x + 2(\mathrm{e}^x)^2 + 2(2x-4)\left(\mathrm{e}^x\right)^2 + 6x\,\mathrm{e}^x + 3x^2\mathrm{e}^x$
  - Use the initial condition $y'\Big|_{\{x=0\}} = 0$

    $0 = -6 + c_1 + c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 5, c_2 = 1\}$
  - Substitute constant values into general solution and simplify

    $y = (2x-4)\,\mathrm{e}^{2x} + \mathrm{e}^x(3x^2+x+5)$

- Solution to the IVP

  $y = (2x-4)\,\mathrm{e}^{2x} + \mathrm{e}^x(3x^2+x+5)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+y(x)=2*x*exp(2*x)+6*exp(x),y(0) = 1, D(y)(0) = 0],y(x)
```

$$y(x) = (2x - 4)\,e^{2x} + e^{x}\left(3x^2 + x + 5\right)$$

✓ Solution by Mathematica

Time used: 0.104 (sec). Leaf size: 25

```
DSolve[{y''[x]-2*y'[x]+y[x]==2*x*Exp[2*x]+6*Exp[x],{y[0]==1,y'[0]==0}},y[x],x,IncludeSingula
```

$$y(x) \to e^{x}\left(3x^2 + x + 2e^{x}(x - 2) + 5\right)$$

11.36 problem 36

Internal problem ID [11810]
Internal file name [`OUTPUT/11819_Thursday_April_11_2024_08_50_00_PM_59531729/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - y = 3x^2 e^x}$$

With initial conditions

$$[y(0) = 1, y'(0) = 2]$$

11.36.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -1 \\ F &= 3x^2 e^x \end{aligned}$$

Hence the ode is

$$y'' - y = 3x^2 e^x$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 3x^2 e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.36.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = 3x^2 e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = e^x c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = e^x c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3x^2 e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\, e^x, x^2 e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{x}, x^{2}e^{x}, x^{3}e^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{x} + A_2 x^{2} e^{x} + A_3 x^{3} e^{x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{x} + 2A_2 e^{x} + 4A_2 x\,e^{x} + 6A_3 x\,e^{x} + 6A_3 x^{2} e^{x} = 3x^{2} e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{4}, A_2 = -\frac{3}{4}, A_3 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3x\,e^{x}}{4} - \frac{3x^{2}e^{x}}{4} + \frac{x^{3}e^{x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{x} c_1 + c_2 e^{-x}\right) + \left(\frac{3x\,e^{x}}{4} - \frac{3x^{2}e^{x}}{4} + \frac{x^{3}e^{x}}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{x} c_1 + c_2 e^{-x} + \frac{3x\,e^{x}}{4} - \frac{3x^{2}e^{x}}{4} + \frac{x^{3}e^{x}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \mathrm{e}^{x} c_1 - c_2 \mathrm{e}^{-x} - \frac{3x\,\mathrm{e}^{x}}{4} + \frac{3\,\mathrm{e}^{x}}{4} + \frac{3x^2 \mathrm{e}^{x}}{4} + \frac{x^3 \mathrm{e}^{x}}{2}$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = c_1 - c_2 + \frac{3}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{9}{8}$$
$$c_2 = -\frac{1}{8}$$

Substituting these values back in above solution results in

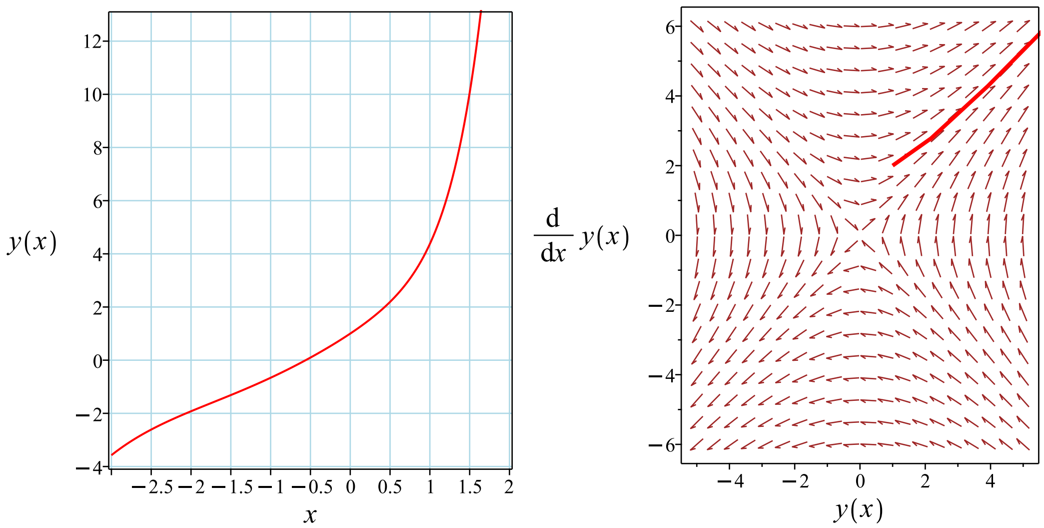
$$y = \frac{9\,\mathrm{e}^{x}}{8} - \frac{\mathrm{e}^{-x}}{8} + \frac{3x\,\mathrm{e}^{x}}{4} - \frac{3x^2 \mathrm{e}^{x}}{4} + \frac{x^3 \mathrm{e}^{x}}{2}$$

Which simplifies to

$$y = -\frac{\mathrm{e}^{-x}}{8} + \frac{(4x^3 - 6x^2 + 6x + 9)\,\mathrm{e}^{x}}{8}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-x}}{8} + \frac{(4x^3 - 6x^2 + 6x + 9)\,\mathrm{e}^{x}}{8} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{e^{-x}}{8} + \frac{(4x^3 - 6x^2 + 6x + 9)\, e^{x}}{8}$$

Verified OK.

11.36.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 401: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\ &= \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_2 e^{x}}{2} + c_1 e^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= e^{-x} \\
y_2 &= \frac{e^{x}}{2}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ \frac{d}{dx}\left(e^{-x}\right) & \frac{d}{dx}\left(\frac{e^x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ -e^{-x} & \frac{e^x}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\right)\left(\frac{e^x}{2}\right) - \left(\frac{e^x}{2}\right)\left(-e^{-x}\right)$$

Which simplifies to

$$W = e^{-x} e^x$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{3x^2 e^{2x}}{2}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{3x^2 e^{2x}}{2} dx$$

Hence

$$u_1 = -\frac{3(2x^2 - 2x + 1) e^{2x}}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{3 e^{-x} x^2 e^x}{1} dx$$

Which simplifies to

$$u_2 = \int 3x^2 dx$$

Hence

$$u_2 = x^3$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{3(2x^2 - 2x + 1)\,\mathrm{e}^{2x}\mathrm{e}^{-x}}{8} + \frac{x^3 \mathrm{e}^x}{2}$$

Which simplifies to

$$y_p(x) = \frac{\mathrm{e}^x (4x^3 - 6x^2 + 6x - 3)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_2 \mathrm{e}^x}{2} + c_1 \mathrm{e}^{-x}\right) + \left(\frac{\mathrm{e}^x (4x^3 - 6x^2 + 6x - 3)}{8}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_2 \mathrm{e}^x}{2} + c_1 \mathrm{e}^{-x} + \frac{\mathrm{e}^x (4x^3 - 6x^2 + 6x - 3)}{8} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = \frac{c_2}{2} + c_1 - \frac{3}{8} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_2 \mathrm{e}^x}{2} - c_1 \mathrm{e}^{-x} + \frac{\mathrm{e}^x (4x^3 - 6x^2 + 6x - 3)}{8} + \frac{(12x^2 - 12x + 6)\,\mathrm{e}^x}{8}$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = \frac{c_2}{2} - c_1 + \frac{3}{8} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{8}$$
$$c_2 = 3$$

Substituting these values back in above solution results in

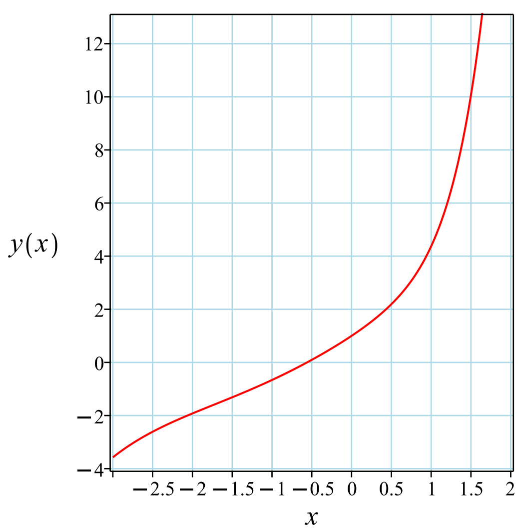
$$y = \frac{9\,\mathrm{e}^{x}}{8} - \frac{\mathrm{e}^{-x}}{8} + \frac{3x\,\mathrm{e}^{x}}{4} - \frac{3x^2\mathrm{e}^{x}}{4} + \frac{x^3\mathrm{e}^{x}}{2}$$

Which simplifies to

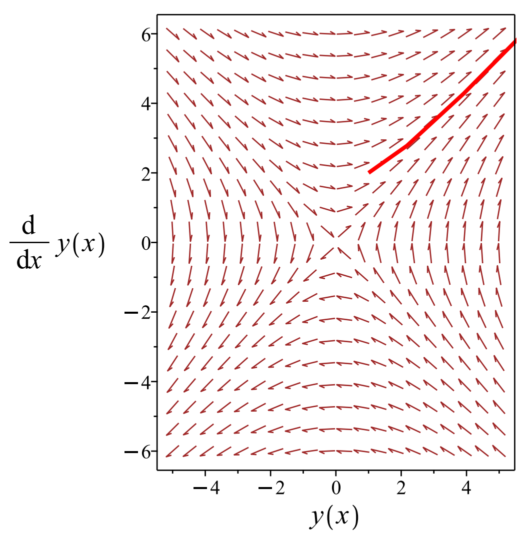
$$y = -\frac{\mathrm{e}^{-x}}{8} + \frac{(4x^3 - 6x^2 + 6x + 9)\,\mathrm{e}^{x}}{8}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-x}}{8} + \frac{(4x^3 - 6x^2 + 6x + 9)\,\mathrm{e}^{x}}{8} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{\mathrm{e}^{-x}}{8} + \frac{(4x^3 - 6x^2 + 6x + 9)\,\mathrm{e}^{x}}{8}$$

Verified OK.

11.36.4 Maple step by step solution

Let's solve

$$\left[y'' - y = 3x^2 e^x, y(0) = 1, y'\Big|_{\{x=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 3x^2 e^x\right]$$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

  ◦ Substitute functions into equation for $y_p(x)$

$y_p(x) = -\frac{3\,\mathrm{e}^{-x}\left(\int x^2 \mathrm{e}^{2x}dx\right)}{2} + \frac{3\,\mathrm{e}^{x}\left(\int x^2 dx\right)}{2}$

- ◦ Compute integrals

  $y_p(x) = \frac{\mathrm{e}^{x}\left(4x^3-6x^2+6x-3\right)}{8}$

- • Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \frac{\mathrm{e}^{x}\left(4x^3-6x^2+6x-3\right)}{8}$

- □ Check validity of solution $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \frac{\mathrm{e}^{x}\left(4x^3-6x^2+6x-3\right)}{8}$

  - ◦ Use initial condition $y(0) = 1$

    $1 = c_1 + c_2 - \frac{3}{8}$

  - ◦ Compute derivative of the solution

    $y' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \frac{\mathrm{e}^{x}\left(4x^3-6x^2+6x-3\right)}{8} + \frac{\left(12x^2-12x+6\right)\mathrm{e}^{x}}{8}$

  - ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 2$

    $2 = -c_1 + c_2 + \frac{3}{8}$

  - ◦ Solve for c_1 and c_2

    $\left\{c_1 = -\frac{1}{8}, c_2 = \frac{3}{2}\right\}$

  - ◦ Substitute constant values into general solution and simplify

    $y = -\frac{\mathrm{e}^{-x}}{8} + \frac{\left(4x^3-6x^2+6x+9\right)\mathrm{e}^{x}}{8}$

- • Solution to the IVP

  $y = -\frac{\mathrm{e}^{-x}}{8} + \frac{\left(4x^3-6x^2+6x+9\right)\mathrm{e}^{x}}{8}$

<u>Maple trace</u>

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve([diff(y(x),x$2)-y(x)=3*x^2*exp(x),y(0) = 1, D(y)(0) = 2],y(x), singsol=all)
```

$$y(x) = -\frac{e^{-x}}{8} + \frac{(4x^3 - 6x^2 + 6x + 9)\,e^{x}}{8}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 37

```
DSolve[{y''[x]-y[x]==3*x^2*Exp[x],{y[0]==1,y'[0]==2}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{8}e^{-x}\left(e^{2x}\left(4x^3 - 6x^2 + 6x + 9\right) - 1\right)$$

11.37 problem 37

Internal problem ID [11811]
Internal file name [`OUTPUT/11820_Thursday_April_11_2024_08_50_01_PM_87746551/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = 3x^2 - 4\sin(x)}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

11.37.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ F &= 3x^2 - 4\sin(x) \end{aligned}$$

Hence the ode is

$$y'' + y = 3x^2 - 4\sin(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 3x^2 - 4\sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.37.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 3x^2 - 4\sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3x^2 - 4\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\sin(x), \cos(x)\, x\}, \{1, x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \sin(x) + A_2 \cos(x)\, x + A_3 + A_4 x + A_5 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \cos(x) - 2A_2 \sin(x) + 2A_5 + A_3 + A_4 x + A_5 x^2 = 3x^2 - 4\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 2, A_3 = -6, A_4 = 0, A_5 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -6 + 2\cos(x)\, x + 3x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-6 + 2\cos(x)\, x + 3x^2\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) - 6 + 2\cos(x)\, x + 3x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 - 6 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\sin(x) c_1 + c_2 \cos(x) - 2x \sin(x) + 2\cos(x) + 6x$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = c_2 + 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 6$$
$$c_2 = -1$$

Substituting these values back in above solution results in

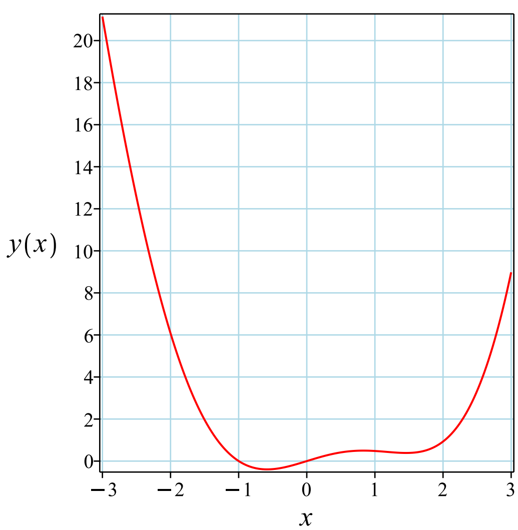
$$y = -6 + 2\cos(x) x + 3x^2 - \sin(x) + 6\cos(x)$$

Which simplifies to

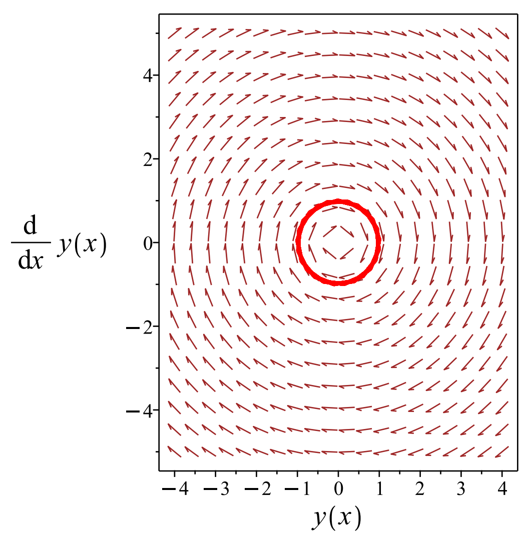
$$y = (2x + 6)\cos(x) + 3x^2 - \sin(x) - 6$$

<u>Summary</u>
The solution(s) found are the following

$$y = (2x + 6)\cos(x) + 3x^2 - \sin(x) - 6 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (2x+6)\cos(x) + 3x^2 - \sin(x) - 6$$

Verified OK.

11.37.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 403: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3x^2 - 4\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\sin(x), \cos(x)\, x\}, \{1, x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \sin(x) + A_2 \cos(x)\, x + A_3 + A_4 x + A_5 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \cos(x) - 2A_2 \sin(x) + 2A_5 + A_3 + A_4 x + A_5 x^2 = 3x^2 - 4\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 2, A_3 = -6, A_4 = 0, A_5 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -6 + 2\cos(x)\, x + 3x^2$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(-6 + 2\cos(x)\, x + 3x^2\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(x) + c_2 \sin(x) - 6 + 2\cos(x)\, x + 3x^2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 - 6 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\sin(x)\, c_1 + c_2 \cos(x) - 2x \sin(x) + 2\cos(x) + 6x
$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$
1 = c_2 + 2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 6 \\
c_2 &= -1
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -6 + 2\cos(x)\, x + 3x^2 - \sin(x) + 6\cos(x)
$$

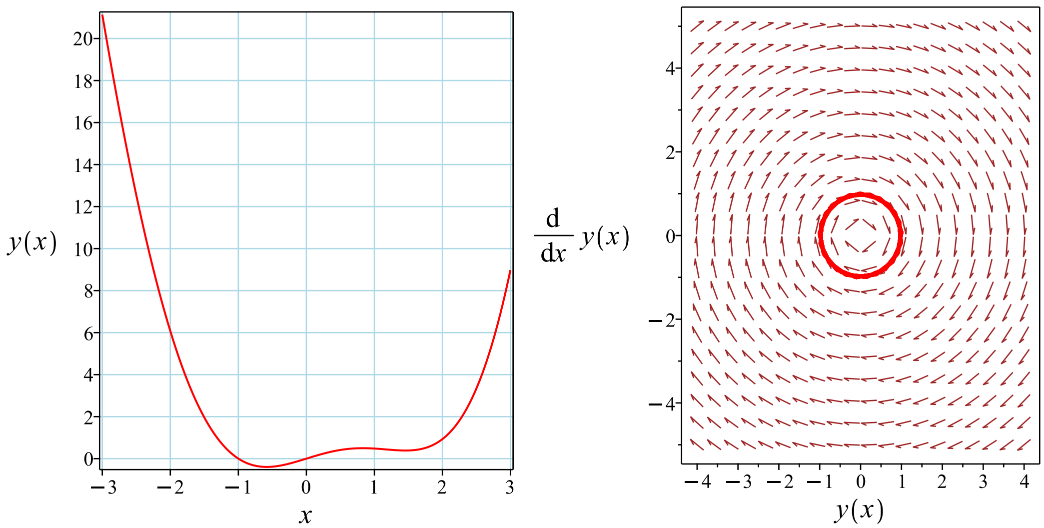
Which simplifies to

$$
y = (2x + 6)\cos(x) + 3x^2 - \sin(x) - 6
$$

Summary

The solution(s) found are the following

$$
y = (2x + 6)\cos(x) + 3x^2 - \sin(x) - 6 \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = (2x+6)\cos(x) + 3x^2 - \sin(x) - 6$$

Verified OK.

11.37.4 Maple step by step solution

Let's solve

$$\left[y'' + y = 3x^2 - 4\sin(x), y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

$y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 3x^2 - 4\sin(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \cos(x)\left(\int \left(4\sin(x)^2 - 3\sin(x)x^2\right)dx\right) - \sin(x)\left(\int \cos(x)\left(-3x^2 + 4\sin(x)\right)dx\right)$

  - Compute integrals

    $y_p(x) = -6 + 2\cos(x)x + 3x^2$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - 6 + 2\cos(x)x + 3x^2$

☐ Check validity of solution $y = c_1 \cos(x) + c_2 \sin(x) - 6 + 2\cos(x)x + 3x^2$

  - Use initial condition $y(0) = 0$

    $0 = c_1 - 6$

  - Compute derivative of the solution

    $y' = -\sin(x)c_1 + c_2 \cos(x) - 2x\sin(x) + 2\cos(x) + 6x$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 1$

    $1 = c_2 + 2$

- Solve for c_1 and c_2

  $\{c_1 = 6, c_2 = -1\}$

- Substitute constant values into general solution and simplify

  $y = (2x + 6)\cos(x) + 3x^2 - \sin(x) - 6$

- Solution to the IVP

  $y = (2x + 6)\cos(x) + 3x^2 - \sin(x) - 6$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve([diff(y(x),x$2)+y(x)=3*x^2-4*sin(x),y(0) = 0, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = (2x + 6)\cos(x) + 3x^2 - \sin(x) - 6$$

✓ Solution by Mathematica

Time used: 0.158 (sec). Leaf size: 23

```
DSolve[{y''[x]+y[x]==3*x^2-4*Sin[x],{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to 3x^2 - \sin(x) + 2(x + 3)\cos(x) - 6$$

11.38 problem 38

Internal problem ID [11812]
Internal file name [`OUTPUT/11821_Thursday_April_11_2024_08_50_01_PM_54548519/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y = 8\sin(2x)}$$

With initial conditions

$$[y(0) = 6, y'(0) = 8]$$

11.38.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 4 \\ F &= 8\sin(2x) \end{aligned}$$

Hence the ode is

$$y'' + 4y = 8 \sin(2x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 8 \sin(2x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.38.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = 8 \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(2x), \sin(2x)\,x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\cos(2x) + A_2 \sin(2x)\,x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 \sin(2x) + 4A_2 \cos(2x) = 8\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x\cos(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + (-2x\cos(2x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2x) + c_2 \sin(2x) - 2x\cos(2x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 6$ and $x = 0$ in the above gives

$$6 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2x) + 2c_2 \cos(2x) - 2\cos(2x) + 4\sin(2x)\, x$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = -2 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 6$$
$$c_2 = 5$$

Substituting these values back in above solution results in

$$y = -2x\cos(2x) + 5\sin(2x) + 6\cos(2x)$$

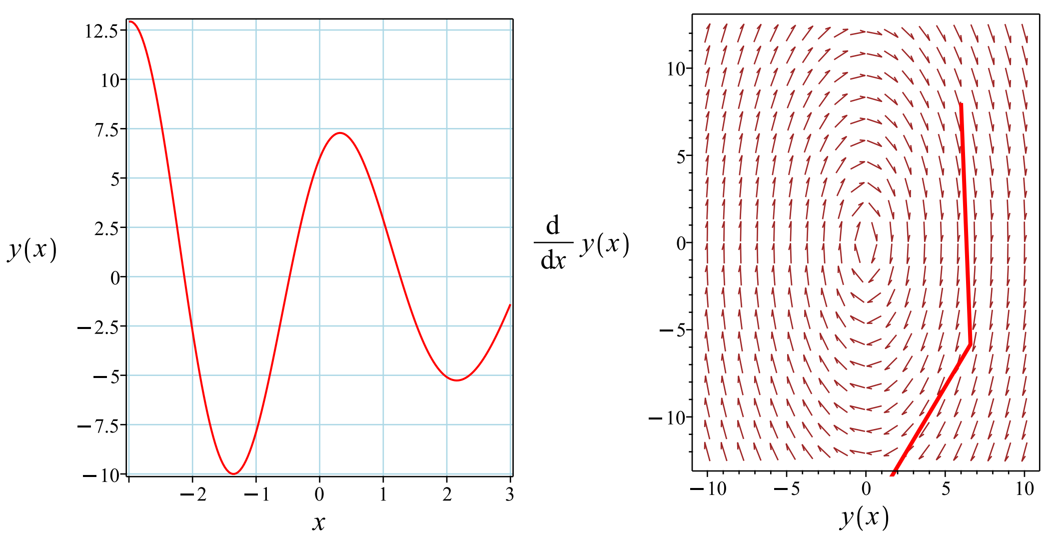
Which simplifies to

$$y = (-2x + 6)\cos(2x) + 5\sin(2x)$$

Summary

The solution(s) found are the following

$$y = (-2x + 6)\cos(2x) + 5\sin(2x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = (-2x + 6)\cos(2x) + 5\sin(2x)$$

Verified OK.

11.38.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 405: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos(2x), \sin(2x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(2x) + A_2 \sin(2x) x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 \sin(2x) + 4A_2 \cos(2x) = 8 \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x\cos(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(2x) + \frac{c_2\sin(2x)}{2}\right) + (-2x\cos(2x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} - 2x\cos(2x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 6$ and $x = 0$ in the above gives

$$6 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1\sin(2x) + c_2\cos(2x) - 2\cos(2x) + 4\sin(2x)\,x$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = c_2 - 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 6 \\ c_2 &= 10 \end{aligned}$$

Substituting these values back in above solution results in

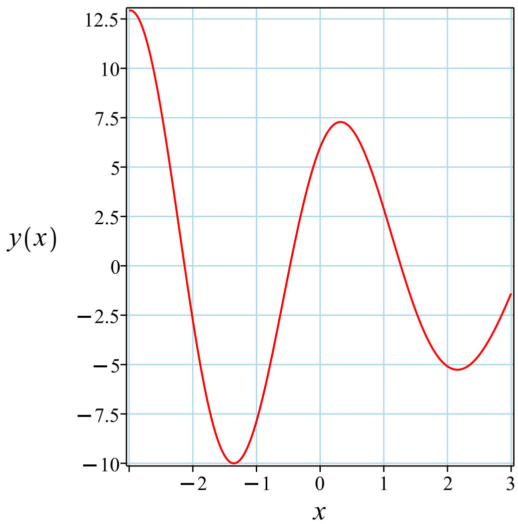
$$y = -2x\cos(2x) + 5\sin(2x) + 6\cos(2x)$$

Which simplifies to

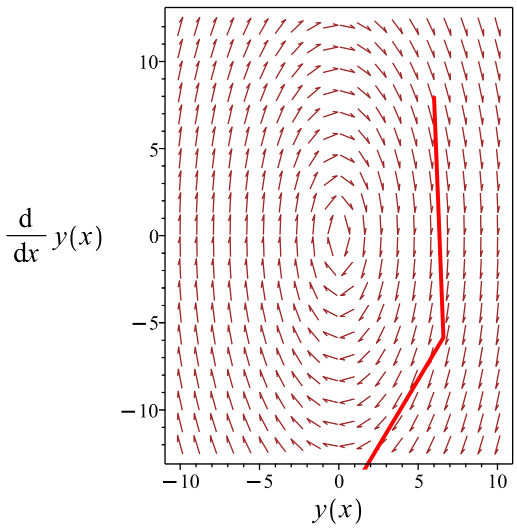
$$y = (-2x + 6)\cos(2x) + 5\sin(2x)$$

Summary
The solution(s) found are the following

$$y = (-2x + 6)\cos(2x) + 5\sin(2x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = (-2x + 6)\cos(2x) + 5\sin(2x)$$

Verified OK.

11.38.4 Maple step by step solution

Let's solve

$$\left[y'' + 4y = 8\sin(2x), y(0) = 6, y'\Big|_{\{x=0\}} = 8\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 8\sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\cos(2x)\left(\int (1 - \cos(4x))\, dx\right) + 2\sin(2x)\left(\int \sin(4x)\, dx\right)$

  - Compute integrals

    $y_p(x) = \frac{\sin(2x)}{2} - 2x\cos(2x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(2x)}{2} - 2x\cos(2x)$

☐ Check validity of solution $y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(2x)}{2} - 2x\cos(2x)$

  - Use initial condition $y(0) = 6$

    $6 = c_1$

  - Compute derivative of the solution

    $y' = -2c_1 \sin(2x) + 2c_2 \cos(2x) - \cos(2x) + 4\sin(2x)\, x$

- Use the initial condition $y'\Big|_{\{x=0\}} = 8$

  $8 = -1 + 2c_2$
- Solve for c_1 and c_2

  $\left\{c_1 = 6, c_2 = \frac{9}{2}\right\}$
- Substitute constant values into general solution and simplify

  $y = (-2x + 6)\cos(2x) + 5\sin(2x)$

- Solution to the IVP

  $y = (-2x + 6)\cos(2x) + 5\sin(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$2)+4*y(x)=8*sin(2*x),y(0) = 6, D(y)(0) = 8],y(x), singsol=all)
```

$$y(x) = (-2x + 6)\cos(2x) + 5\sin(2x)$$

✓ Solution by Mathematica

Time used: 0.108 (sec). Leaf size: 19

```
DSolve[{y''[x]+4*y[x]==8*Sin[2*x],{y[0]==0,y'[0]==1}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to 3\sin(x)\cos(x) - 2x\cos(2x)$$

11.39 problem 39

Internal problem ID [11813]
Internal file name [`OUTPUT/11822_Thursday_April_11_2024_08_50_02_PM_7433673/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 39.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 4y'' + y' + 6y = 3x\,e^{x} + 2\,e^{x} - \sin(x)}$$

With initial conditions

$$\left[y(0) = \frac{33}{40}, y'(0) = 0, y''(0) = 0\right]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y'' + y' + 6y = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{2x}$$
$$y_3 = e^{3x}$$

Now the particular solution to the given ODE is found

$$y''' - 4y'' + y' + 6y = 3x e^x + 2 e^x - \sin(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3x e^x + 2 e^x - \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^x, e^x\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{2x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x e^x + A_2 e^x + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^x + 4A_1 x e^x + 4A_2 e^x + 10A_3 \cos(x) + 10A_4 \sin(x) = 3x e^x + 2 e^x - \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{4}, A_2 = \frac{5}{4}, A_3 = 0, A_4 = -\frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3x\,\mathrm{e}^x}{4} + \frac{5\,\mathrm{e}^x}{4} - \frac{\sin(x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{2x} + c_3\mathrm{e}^{3x}\right) + \left(\frac{3x\,\mathrm{e}^x}{4} + \frac{5\,\mathrm{e}^x}{4} - \frac{\sin(x)}{10}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{2x} + c_3\mathrm{e}^{3x} + \frac{3x\,\mathrm{e}^x}{4} + \frac{5\,\mathrm{e}^x}{4} - \frac{\sin(x)}{10} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{33}{40}$ and $x = 0$ in the above gives

$$\frac{33}{40} = c_1 + c_2 + c_3 + \frac{5}{4} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1\mathrm{e}^{-x} + 2c_2\mathrm{e}^{2x} + 3c_3\mathrm{e}^{3x} + \frac{3x\,\mathrm{e}^x}{4} + 2\,\mathrm{e}^x - \frac{\cos(x)}{10}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -c_1 + 2c_2 + 3c_3 + \frac{19}{10} \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = c_1\mathrm{e}^{-x} + 4c_2\mathrm{e}^{2x} + 9c_3\mathrm{e}^{3x} + \frac{3x\,\mathrm{e}^x}{4} + \frac{11\,\mathrm{e}^x}{4} + \frac{\sin(x)}{10}$$

substituting $y'' = 0$ and $x = 0$ in the above gives

$$0 = c_1 + 4c_2 + 9c_3 + \frac{11}{4} \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{7}{20} \\
c_2 &= -\frac{31}{40} \\
c_3 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{7\,\mathrm{e}^{-x}}{20} - \frac{31\,\mathrm{e}^{2x}}{40} + \frac{3x\,\mathrm{e}^{x}}{4} + \frac{5\,\mathrm{e}^{x}}{4} - \frac{\sin(x)}{10}
$$

Summary

The solution(s) found are the following

$$
y = \frac{7\,\mathrm{e}^{-x}}{20} - \frac{31\,\mathrm{e}^{2x}}{40} + \frac{3x\,\mathrm{e}^{x}}{4} + \frac{5\,\mathrm{e}^{x}}{4} - \frac{\sin(x)}{10} \tag{1}
$$

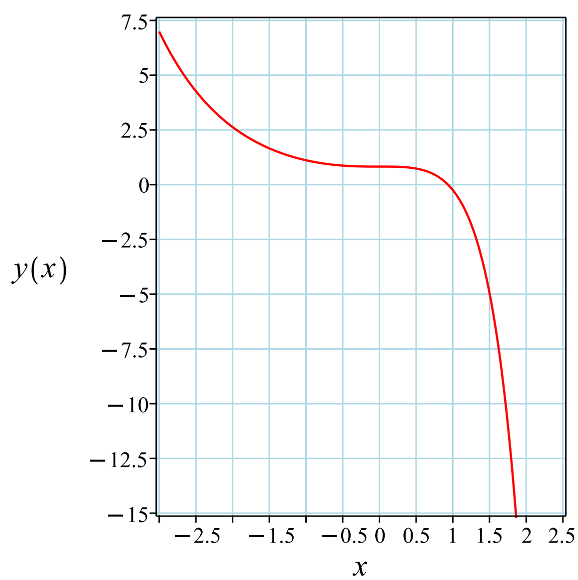


Figure 471: Solution plot

Verification of solutions

$$
y = \frac{7\,\mathrm{e}^{-x}}{20} - \frac{31\,\mathrm{e}^{2x}}{40} + \frac{3x\,\mathrm{e}^{x}}{4} + \frac{5\,\mathrm{e}^{x}}{4} - \frac{\sin(x)}{10}
$$

Verified OK.

11.39.1 Maple step by step solution

Let's solve

$$\left[y''' - 4y'' + y' + 6y = 3x\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} - \sin(x), y(0) = \frac{33}{40}, y'\Big|_{\{x=0\}} = 0, y''\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 3x\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} - \sin(x) + 4y_3(x) - y_2(x) - 6y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 3x\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} - \sin(x) + 4y_3(x) - y_2(x) - 6y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 3x\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} - \sin(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 3x\,\mathrm{e}^{x} + 2\,\mathrm{e}^{x} - \sin(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{array}\right]$$

- Rewrite the system as

$${\overrightarrow{y}}^{\prime}(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right], \left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{2} = \mathrm{e}^{2x} \cdot \left[\begin{array}{c} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{3x} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \vec{y}_p(x)$$

☐ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{4} & \frac{\mathrm{e}^{3x}}{9} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{2} & \frac{\mathrm{e}^{3x}}{3} \\ \mathrm{e}^{-x} & \mathrm{e}^{2x} & \mathrm{e}^{3x} \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{4} & \frac{\mathrm{e}^{3x}}{9} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{2} & \frac{\mathrm{e}^{3x}}{3} \\ \mathrm{e}^{-x} & \mathrm{e}^{2x} & \mathrm{e}^{3x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{9} \\ -1 & \frac{1}{2} & \frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{\mathrm{e}^{-x}}{2} + \mathrm{e}^{2x} - \frac{\mathrm{e}^{3x}}{2} & -\frac{5\,\mathrm{e}^{-x}}{12} + \frac{2\,\mathrm{e}^{2x}}{3} - \frac{\mathrm{e}^{3x}}{4} & \frac{\mathrm{e}^{-x}}{12} - \frac{\mathrm{e}^{2x}}{3} + \frac{\mathrm{e}^{3x}}{4} \\ -\frac{\mathrm{e}^{-x}}{2} + 2\,\mathrm{e}^{2x} - \frac{3\,\mathrm{e}^{3x}}{2} & \frac{5\,\mathrm{e}^{-x}}{12} + \frac{4\,\mathrm{e}^{2x}}{3} - \frac{3\,\mathrm{e}^{3x}}{4} & -\frac{\mathrm{e}^{-x}}{12} - \frac{2\,\mathrm{e}^{2x}}{3} + \frac{3\,\mathrm{e}^{3x}}{4} \\ \frac{\mathrm{e}^{-x}}{2} + 4\,\mathrm{e}^{2x} - \frac{9\,\mathrm{e}^{3x}}{2} & -\frac{5\,\mathrm{e}^{-x}}{12} + \frac{8\,\mathrm{e}^{2x}}{3} - \frac{9\,\mathrm{e}^{3x}}{4} & \frac{\mathrm{e}^{-x}}{12} - \frac{4\,\mathrm{e}^{2x}}{3} + \frac{9\,\mathrm{e}^{3x}}{4} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{y}_p(x) = \begin{bmatrix} -\frac{e^{-x}}{16} - \frac{8\,e^{2x}}{5} + \frac{33\,e^{3x}}{80} + \frac{(3x+5)e^{x}}{4} - \frac{\sin(x)}{10} \\ \frac{99\,e^{3x}}{80} - \frac{16\,e^{2x}}{5} + \frac{3x\,e^{x}}{4} + 2\,e^{x} - \frac{\cos(x)}{10} + \frac{e^{-x}}{16} \\ -\frac{e^{-x}}{16} - \frac{32\,e^{2x}}{5} + \frac{297\,e^{3x}}{80} + \frac{(3x+11)e^{x}}{4} + \frac{\sin(x)}{10} \end{bmatrix}$$

- Plug particular solution back into general solution

  $$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \begin{bmatrix} -\frac{e^{-x}}{16} - \frac{8\,e^{2x}}{5} + \frac{33\,e^{3x}}{80} + \frac{(3x+5)e^{x}}{4} - \frac{\sin(x)}{10} \\ \frac{99\,e^{3x}}{80} - \frac{16\,e^{2x}}{5} + \frac{3x\,e^{x}}{4} + 2\,e^{x} - \frac{\cos(x)}{10} + \frac{e^{-x}}{16} \\ -\frac{e^{-x}}{16} - \frac{32\,e^{2x}}{5} + \frac{297\,e^{3x}}{80} + \frac{(3x+11)e^{x}}{4} + \frac{\sin(x)}{10} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = \frac{(720c_1-45)e^{-x}}{720} + \frac{(180c_2-1152)e^{2x}}{720} + \frac{(80c_3+297)e^{3x}}{720} + \frac{(540x+900)e^{x}}{720} - \frac{\sin(x)}{10}$

- Use the initial condition $y(0) = \frac{33}{40}$

$\frac{33}{40} = c_1 + \frac{c_2}{4} + \frac{c_3}{9}$

- Calculate the 1st derivative of the solution

  $y' = -\frac{(720c_1-45)\mathrm{e}^{-x}}{720} + \frac{(180c_2-1152)\mathrm{e}^{2x}}{360} + \frac{(80c_3+297)\mathrm{e}^{3x}}{240} + \frac{3\,\mathrm{e}^{x}}{4} + \frac{(540x+900)\mathrm{e}^{x}}{720} - \frac{\cos(x)}{10}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = -c_1 + \frac{c_2}{2} + \frac{c_3}{3}$

- Calculate the 2nd derivative of the solution

  $y'' = \frac{(720c_1-45)\mathrm{e}^{-x}}{720} + \frac{(180c_2-1152)\mathrm{e}^{2x}}{180} + \frac{(80c_3+297)\mathrm{e}^{3x}}{80} + \frac{3\,\mathrm{e}^{x}}{2} + \frac{(540x+900)\mathrm{e}^{x}}{720} + \frac{\sin(x)}{10}$

- Use the initial condition $y''\Big|_{\{x=0\}} = 0$

  $0 = c_1 + c_2 + c_3$

- Solve for the unknown coefficients

  $\left\{c_1 = \frac{33}{80}, c_2 = \frac{33}{10}, c_3 = -\frac{297}{80}\right\}$

- Solution to the IVP

  $y = \frac{7\,\mathrm{e}^{-x}}{20} - \frac{31\,\mathrm{e}^{2x}}{40} + \frac{(3x+5)\mathrm{e}^{x}}{4} - \frac{\sin(x)}{10}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 30

```
dsolve([diff(y(x),x$3)-4*diff(y(x),x$2)+diff(y(x),x)+6*y(x)=3*x*exp(x)+2*exp(x)-sin(x),y(0)
```

$$y(x) = \frac{7\,\mathrm{e}^{-x}}{20} - \frac{31\,\mathrm{e}^{2x}}{40} + \frac{(3x+5)\,\mathrm{e}^{x}}{4} - \frac{\sin(x)}{10}$$

✓ Solution by Mathematica

Time used: 0.285 (sec). Leaf size: 38

```
DSolve[{y'''[x]-4*y''[x]+y'[x]+6*y[x]==3*x*Exp[x]+2*Exp[x]-Sin[x],{y[0]==33/40,y'[0]==0,y''[
```

$$y(x) \to \frac{1}{40}\left(10e^x(3x+5) + 14e^{-x} - 31e^{2x} - 4\sin(x)\right)$$

11.40 problem 40

Internal problem ID [11814]
Internal file name [`OUTPUT/11823_Thursday_April_11_2024_08_50_03_PM_30210734/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 40.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 6y'' + 9y' - 4y = 8x^2 + 3 - 6\,\mathrm{e}^{2x}}$$

With initial conditions

$$[y(0) = 1, y'(0) = 7, y''(0) = 0]$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 6y'' + 9y' - 4y = 0$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 4$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = e^x c_1 + c_2 x\, e^x + e^{4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = x\, e^x$$
$$y_3 = e^{4x}$$

Now the particular solution to the given ODE is found

$$y''' - 6y'' + 9y' - 4y = 8x^2 + 3 - 6\, e^{2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8x^2 + 3 - 6\, e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^x, e^x, e^{4x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x} + A_2 + A_3 x + A_4 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^{2x} - 12A_4 + 9A_3 + 18A_4 x - 4A_2 - 4A_3 x - 4A_4 x^2 = 8x^2 + 3 - 6\, e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3, A_2 = -15, A_3 = -9, A_4 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3\,\mathrm{e}^{2x} - 15 - 9x - 2x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{x} c_1 + c_2 x\,\mathrm{e}^{x} + \mathrm{e}^{4x} c_3\right) + \left(3\,\mathrm{e}^{2x} - 15 - 9x - 2x^2\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{4x} c_3 + \mathrm{e}^{x}(c_2 x + c_1) + 3\,\mathrm{e}^{2x} - 15 - 9x - 2x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{4x} c_3 + \mathrm{e}^{x}(c_2 x + c_1) + 3\,\mathrm{e}^{2x} - 15 - 9x - 2x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = -12 + c_3 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 4\,\mathrm{e}^{4x} c_3 + \mathrm{e}^{x}(c_2 x + c_1) + c_2\mathrm{e}^{x} + 6\,\mathrm{e}^{2x} - 9 - 4x$$

substituting $y' = 7$ and $x = 0$ in the above gives

$$7 = -3 + 4c_3 + c_1 + c_2 \tag{2A}$$

Taking two derivatives of the solution gives

$$y'' = 16\,\mathrm{e}^{4x} c_3 + \mathrm{e}^{x}(c_2 x + c_1) + 2c_2\mathrm{e}^{x} + 12\,\mathrm{e}^{2x} - 4$$

substituting $y'' = 0$ and $x = 0$ in the above gives

$$0 = 8 + 16c_3 + c_1 + 2c_2 \tag{3A}$$

Equations {1A,2A,3A} are now solved for $\{c_1, c_2, c_3\}$. Solving for the constants gives

$$c_1 = \frac{44}{3}$$
$$c_2 = 2$$
$$c_3 = -\frac{5}{3}$$

Substituting these values back in above solution results in

$$y = -15 - \frac{5\,\mathrm{e}^{4x}}{3} + 2x\,\mathrm{e}^{x} + \frac{44\,\mathrm{e}^{x}}{3} + 3\,\mathrm{e}^{2x} - 9x - 2x^2$$

Summary
The solution(s) found are the following

$$y = -15 - \frac{5\,\mathrm{e}^{4x}}{3} + 2x\,\mathrm{e}^{x} + \frac{44\,\mathrm{e}^{x}}{3} + 3\,\mathrm{e}^{2x} - 9x - 2x^2 \tag{1}$$

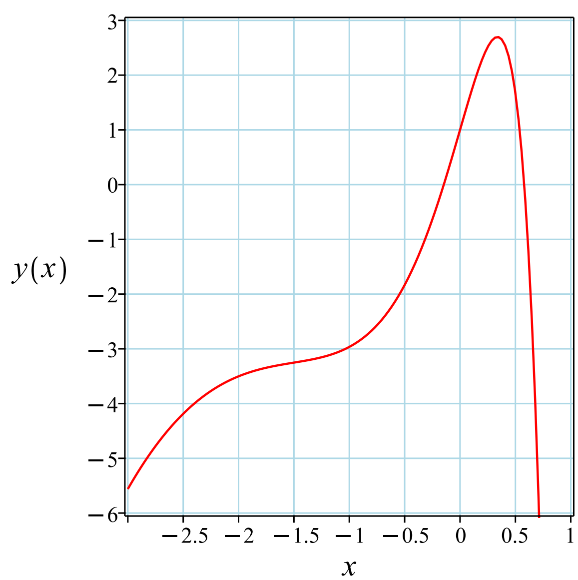


Figure 472: Solution plot

Verification of solutions

$$y = -15 - \frac{5\,\mathrm{e}^{4x}}{3} + 2x\,\mathrm{e}^{x} + \frac{44\,\mathrm{e}^{x}}{3} + 3\,\mathrm{e}^{2x} - 9x - 2x^2$$

Verified OK.

11.40.1 Maple step by step solution

Let's solve

$$\left[y''' - 6y'' + 9y' - 4y = 8x^2 + 3 - 6\,e^{2x}, y(0) = 1, y'\Big|_{\{x=0\}} = 7, y''\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 3

y'''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = 8x^2 - 6\,e^{2x} + 6y_3(x) - 9y_2(x) + 4y_1(x) + 3$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 8x^2 - 6\,e^{2x} + 6y_3(x) - 9y_2(x) + 4y_1(x) + 3]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -9 & 6 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 8x^2 + 3 - 6\,e^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 8x^2 + 3 - 6\,e^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -9 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

$$\vec{y}_1(x) = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\vec{y}_2(x) = e^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = \left(e^{\lambda x}A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = e^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$\lambda \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Make use of the identity matrix I

$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -9 & 6 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \overrightarrow{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\overrightarrow{y}_2(x) = \mathrm{e}^{x} \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{4x} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1(x) + c_2 \overrightarrow{y}_2(x) + c_3 \overrightarrow{y}_3 + \overrightarrow{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{x} & e^{x}(x-1) & \frac{e^{4x}}{16} \\ e^{x} & x\,e^{x} & \frac{e^{4x}}{4} \\ e^{x} & x\,e^{x} & e^{4x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{x} & e^{x}(x-1) & \frac{e^{4x}}{16} \\ e^{x} & x\,e^{x} & \frac{e^{4x}}{4} \\ e^{x} & x\,e^{x} & e^{4x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -1 & \frac{1}{16} \\ 1 & 0 & \frac{1}{4} \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -e^{x}(x-1) & -\frac{e^{4x}}{12} + \frac{(15x+1)e^{x}}{12} & \frac{e^{4x}}{12} + \frac{(-3x-1)e^{x}}{12} \\ -x\,e^{x} & \frac{4\,e^{x}}{3} + \frac{5x\,e^{x}}{4} - \frac{e^{4x}}{3} & \frac{e^{4x}}{3} + \frac{(-4-3x)e^{x}}{12} \\ -x\,e^{x} & \frac{4\,e^{x}}{3} + \frac{5x\,e^{x}}{4} - \frac{4\,e^{4x}}{3} & \frac{4\,e^{4x}}{3} + \frac{(-4-3x)e^{x}}{12} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

$\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} -\frac{45}{4} + \frac{9\,e^{2x}}{4} - \frac{e^{4x}}{6} + \frac{5(22-15x)e^{x}}{12} - \frac{3x^2}{2} - \frac{27x}{4} \\ -\frac{27}{4} + \frac{9\,e^{2x}}{2} - \frac{2\,e^{4x}}{3} + \frac{5(7-15x)e^{x}}{12} - 3x \\ -\frac{31}{4} + \frac{15\,e^{2x}}{2} - \frac{8\,e^{4x}}{3} + \frac{5(7-15x)e^{x}}{12} - 2x^2 - 4x \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3 + \begin{bmatrix} -\frac{45}{4} + \frac{9\,e^{2x}}{4} - \frac{e^{4x}}{6} + \frac{5(22-15x)e^{x}}{12} - \frac{3x^2}{2} - \frac{27x}{4} \\ -\frac{27}{4} + \frac{9\,e^{2x}}{2} - \frac{2\,e^{4x}}{3} + \frac{5(7-15x)e^{x}}{12} - 3x \\ -\frac{31}{4} + \frac{15\,e^{2x}}{2} - \frac{8\,e^{4x}}{3} + \frac{5(7-15x)e^{x}}{12} - 2x^2 - 4x \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = \frac{(3c_3-8)e^{4x}}{48} + \frac{9\,e^{2x}}{4} + \frac{((48c_2-300)x+48c_1-48c_2+440)e^{x}}{48} - \frac{3x^2}{2} - \frac{27x}{4} - \frac{45}{4}$

- Use the initial condition $y(0) = 1$

  $1 = \frac{c_3}{16} + c_1 - c_2$

- Calculate the 1st derivative of the solution

  $y' = \frac{(3c_3-8)e^{4x}}{12} + \frac{9\,e^{2x}}{2} + \frac{(48c_2-300)e^{x}}{48} + \frac{((48c_2-300)x+48c_1-48c_2+440)e^{x}}{48} - 3x - \frac{27}{4}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 7$

  $7 = \frac{c_3}{4} + c_1$

- Calculate the 2nd derivative of the solution

  $y'' = \frac{(3c_3-8)e^{4x}}{3} + 9\,e^{2x} + \frac{(48c_2-300)e^{x}}{24} + \frac{((48c_2-300)x+48c_1-48c_2+440)e^{x}}{48} - 3$

- Use the initial condition $y''\Big|_{\{x=0\}} = 0$

  $0 = c_1 + c_2 + c_3$

- Solve for the unknown coefficients

$$\left\{c_1 = \frac{115}{9}, c_2 = \frac{31}{3}, c_3 = -\frac{208}{9}\right\}$$

- Solution to the IVP

$$y = \frac{9\,e^{2x}}{4} - \frac{29\,e^{4x}}{18} + \frac{(147x+418)e^{x}}{36} - \frac{3x^2}{2} - \frac{27x}{4} - \frac{45}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
dsolve([diff(y(x),x$3)-6*diff(y(x),x$2)+9*diff(y(x),x)-4*y(x)=8*x^2+3-6*exp(2*x),y(0) = 1, D
```

$$y(x) = -2x^2 - 9x + 3\,e^{2x} - 15 + \frac{44\,e^{x}}{3} - \frac{5\,e^{4x}}{3} + 2\,e^{x}x$$

✓ Solution by Mathematica

Time used: 0.21 (sec). Leaf size: 42

```
DSolve[{y'''[x]-6*y''[x]+9*y'[x]-4*y[x]==8*x^2+3-6*Exp[2*x],{y[0]==1,y'[0]==7,y''[0]==0}},y[
```

$$y(x) \to -2x^2 - 9x + 3e^{2x} - \frac{5e^{4x}}{3} + e^{x}\left(2x + \frac{44}{3}\right) - 15$$

11.41 problem 41

Internal problem ID [11815]
Internal file name [`OUTPUT/11824_Thursday_April_11_2024_08_50_03_PM_64035439/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 6y' + 8y = x^3 + x + e^{-2x}}$$

11.41.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 8, f(x) = x^3 + x + e^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 8$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 6\lambda\, \mathrm{e}^{\lambda x} + 8\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-6^2 - (4)(1)(8)}\\ &= 3 \pm 1\end{aligned}$$

Hence

$$\lambda_1 = 3 + 1$$
$$\lambda_2 = 3 - 1$$

Which simplifies to

$$\lambda_1 = 4$$
$$\lambda_2 = 2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(4)x} + c_2 e^{(2)x}$$

Or

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x + e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}, \{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}, e^{4x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2x} + A_2 + A_3 x + A_4 x^2 + A_5 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&24A_1 e^{-2x} + 2A_4 + 6A_5 x - 6A_3 - 12A_4 x - 18A_5 x^2 + 8A_2 + 8A_3 x + 8A_4 x^2 + 8A_5 x^3 \\ &= x^3 + x + e^{-2x}\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{24}, A_2 = \frac{69}{256}, A_3 = \frac{29}{64}, A_4 = \frac{9}{32}, A_5 = \frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-2x}}{24} + \frac{69}{256} + \frac{29x}{64} + \frac{9x^2}{32} + \frac{x^3}{8}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(c_1 e^{4x} + c_2 e^{2x}\right) + \left(\frac{e^{-2x}}{24} + \frac{69}{256} + \frac{29x}{64} + \frac{9x^2}{32} + \frac{x^3}{8}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{4x} + c_2 e^{2x} + \frac{e^{-2x}}{24} + \frac{69}{256} + \frac{29x}{64} + \frac{9x^2}{32} + \frac{x^3}{8} \quad (1)$$

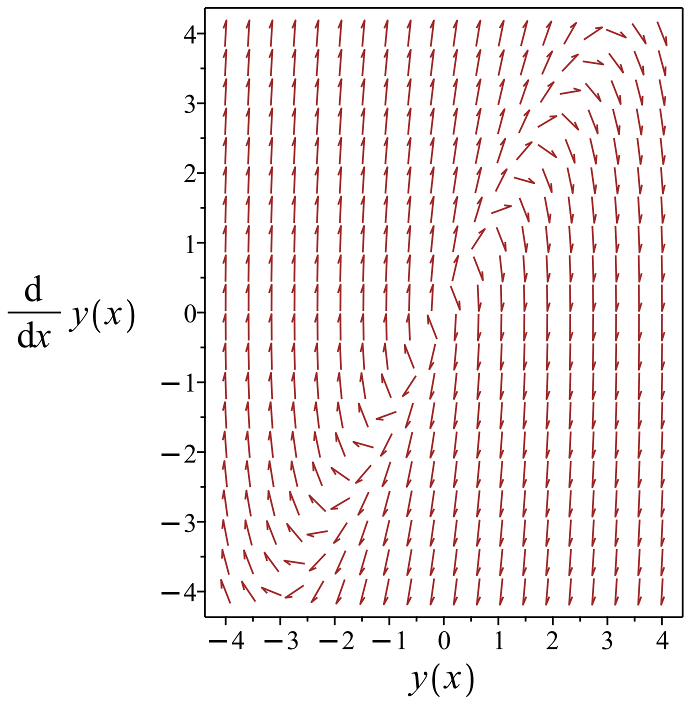


Figure 473: Slope field plot

Verification of solutions

$$y = c_1 e^{4x} + c_2 e^{2x} + \frac{e^{-2x}}{24} + \frac{69}{256} + \frac{29x}{64} + \frac{9x^2}{32} + \frac{x^3}{8}$$

Verified OK.

11.41.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 8y = 0 \quad (1)$$
$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 409: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\, dx} \\
&= z_1 e^{3x} \\
&= z_1 \left(\mathrm{e}^{3x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{6x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{e^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{2x}\right) + c_2 \left(e^{2x} \left(\frac{e^{2x}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 8y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{2x} + \frac{c_2 e^{4x}}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x + e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}, \{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{4x}}{2}, e^{2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2x} + A_2 + A_3 x + A_4 x^2 + A_5 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&24A_1 e^{-2x} + 2A_4 + 6A_5 x - 6A_3 - 12A_4 x - 18A_5 x^2 + 8A_2 + 8A_3 x + 8A_4 x^2 + 8A_5 x^3 \\ &= x^3 + x + e^{-2x}\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{24}, A_2 = \frac{69}{256}, A_3 = \frac{29}{64}, A_4 = \frac{9}{32}, A_5 = \frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-2x}}{24} + \frac{69}{256} + \frac{29x}{64} + \frac{9x^2}{32} + \frac{x^3}{8}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(c_1 e^{2x} + \frac{c_2 e^{4x}}{2}\right) + \left(\frac{e^{-2x}}{24} + \frac{69}{256} + \frac{29x}{64} + \frac{9x^2}{32} + \frac{x^3}{8}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + \frac{c_2 e^{4x}}{2} + \frac{e^{-2x}}{24} + \frac{69}{256} + \frac{29x}{64} + \frac{9x^2}{32} + \frac{x^3}{8} \tag{1}$$

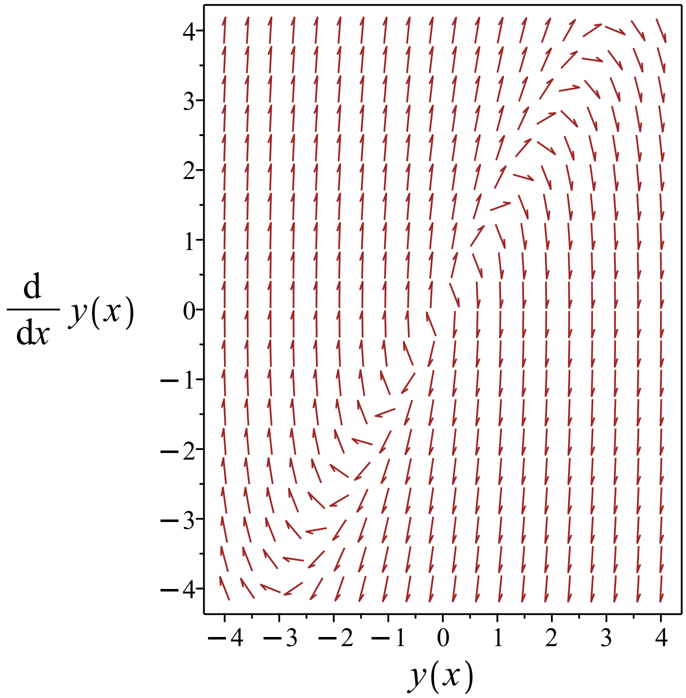


Figure 474: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + \frac{c_2 e^{4x}}{2} + \frac{e^{-2x}}{24} + \frac{69}{256} + \frac{29x}{64} + \frac{9x^2}{32} + \frac{x^3}{8}$$

Verified OK.

11.41.3 Maple step by step solution

Let's solve

$y'' - 6y' + 8y = x^3 + x + e^{-2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 6r + 8 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-4) = 0$

- Roots of the characteristic polynomial

  $r = (2, 4)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{2x} + c_2 e^{4x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^3 + x + e^{-2x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & e^{4x} \\ 2\,e^{2x} & 4\,e^{4x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,e^{6x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{e^{2x}\left(\int e^{-2x}(x^3+x+e^{-2x})dx\right)}{2} + \frac{e^{4x}\left(\int e^{-4x}(x^3+x+e^{-2x})dx\right)}{2}$

  - Compute integrals

    $y_p(x) = \frac{e^{-2x}}{24} + \frac{69}{256} + \frac{29x}{64} + \frac{9x^2}{32} + \frac{x^3}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{2x} + c_2 e^{4x} + \frac{e^{-2x}}{24} + \frac{69}{256} + \frac{29x}{64} + \frac{9x^2}{32} + \frac{x^3}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+8*y(x)=x^3+x+exp(-2*x),y(x), singsol=all)
```

$$y(x) = \frac{c_1 e^{4x}}{2} + \frac{e^{-2x}}{24} + \frac{69}{256} + \frac{29x}{64} + \frac{9x^2}{32} + \frac{x^3}{8} + c_2 e^{2x}$$

✓ Solution by Mathematica

Time used: 0.699 (sec). Leaf size: 50

```
DSolve[y''[x]-6*y'[x]+8*y[x]==x^3+x+Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{256}(32x^3 + 72x^2 + 116x + 69) + \frac{e^{-2x}}{24} + c_1 e^{2x} + c_2 e^{4x}$$

11.42 problem 42

Internal problem ID [11816]
Internal file name [`OUTPUT/11825_Thursday_April_11_2024_08_50_04_PM_71030995/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = e^{3x} + e^{-3x} + e^{3x} \sin(3x)}$$

11.42.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = e^{3x} + e^{-3x} + e^{3x} \sin(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)} \\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i \\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i \\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{3x} + e^{-3x} + e^{3x} \sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-3x}\}, \{e^{3x}\}, \{e^{3x} \cos(3x), e^{3x} \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-3x} + A_2 e^{3x} + A_3 e^{3x} \cos(3x) + A_4 e^{3x} \sin(3x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&18A_1 e^{-3x} + 18A_2 e^{3x} - 18A_3 e^{3x} \sin(3x) + 18A_4 e^{3x} \cos(3x)\\ &+ 9A_3 e^{3x} \cos(3x) + 9A_4 e^{3x} \sin(3x) = e^{3x} + e^{-3x} + e^{3x} \sin(3x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{18}, A_2 = \frac{1}{18}, A_3 = -\frac{2}{45}, A_4 = \frac{1}{45}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-3x}}{18} + \frac{e^{3x}}{18} - \frac{2 e^{3x} \cos(3x)}{45} + \frac{e^{3x} \sin(3x)}{45}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(3x) + c_2 \sin(3x)) + \left(\frac{e^{-3x}}{18} + \frac{e^{3x}}{18} - \frac{2 e^{3x} \cos(3x)}{45} + \frac{e^{3x} \sin(3x)}{45}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{e^{-3x}}{18} + \frac{e^{3x}}{18} - \frac{2 e^{3x} \cos(3x)}{45} + \frac{e^{3x} \sin(3x)}{45} \tag{1}
$$

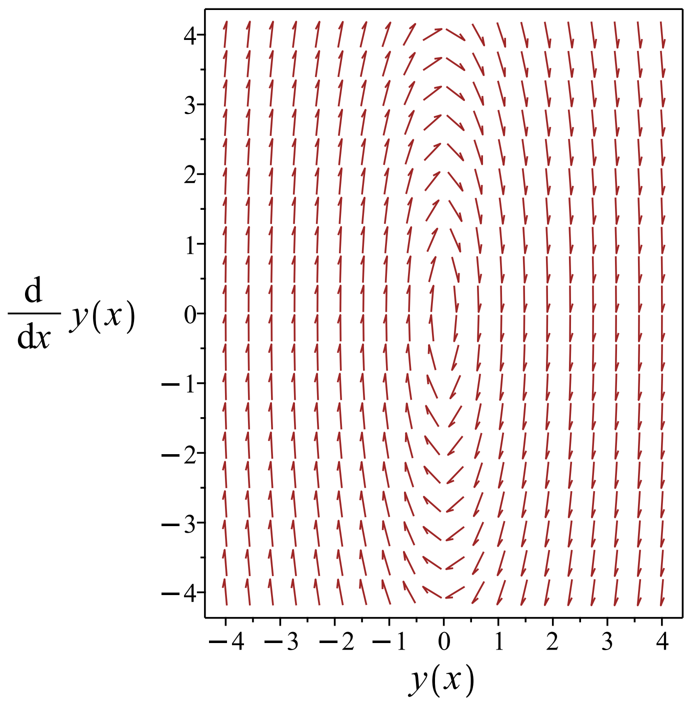


Figure 475: Slope field plot

Verification of solutions

$$
y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{e^{-3x}}{18} + \frac{e^{3x}}{18} - \frac{2 e^{3x} \cos(3x)}{45} + \frac{e^{3x} \sin(3x)}{45}
$$

Verified OK.

11.42.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 411: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2} \, dx \\ &= \cos(3x) \left(\frac{\tan(3x)}{3} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2 \left(\cos(3x) \left(\frac{\tan(3x)}{3} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{3x} + e^{-3x} + e^{3x} \sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-3x}\}, \{e^{3x}\}, \{e^{3x} \cos(3x), e^{3x} \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-3x} + A_2 e^{3x} + A_3 e^{3x} \cos(3x) + A_4 e^{3x} \sin(3x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}
&18A_1 e^{-3x} + 18A_2 e^{3x} - 18A_3 e^{3x} \sin(3x) + 18A_4 e^{3x} \cos(3x) \\
&\quad + 9A_3 e^{3x} \cos(3x) + 9A_4 e^{3x} \sin(3x) = e^{3x} + e^{-3x} + e^{3x} \sin(3x)
\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{18}, A_2 = \frac{1}{18}, A_3 = -\frac{2}{45}, A_4 = \frac{1}{45}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{-3x}}{18} + \frac{e^{3x}}{18} - \frac{2 e^{3x} \cos(3x)}{45} + \frac{e^{3x} \sin(3x)}{45}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + \left(\frac{e^{-3x}}{18} + \frac{e^{3x}}{18} - \frac{2\,e^{3x} \cos(3x)}{45} + \frac{e^{3x} \sin(3x)}{45}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{e^{-3x}}{18} + \frac{e^{3x}}{18} - \frac{2\,e^{3x} \cos(3x)}{45} + \frac{e^{3x} \sin(3x)}{45} \tag{1}
$$

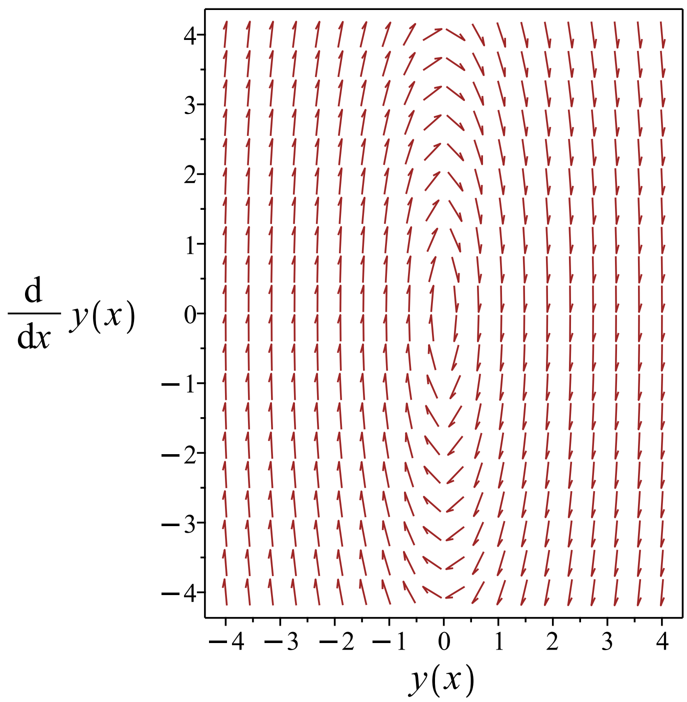


Figure 476: Slope field plot

Verification of solutions

$$
y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{e^{-3x}}{18} + \frac{e^{3x}}{18} - \frac{2\,e^{3x} \cos(3x)}{45} + \frac{e^{3x} \sin(3x)}{45}
$$

Verified OK.

11.42.3 Maple step by step solution

Let's solve

$y'' + 9y = e^{3x} + e^{-3x} + e^{3x}\sin(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = e^{3x} + e^{-3x} + e^{3x}\sin(3x\right.$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(3x)\left(\int \sin(3x)\left(e^{3x}+e^{-3x}+e^{3x}\sin(3x)\right)dx\right)}{3} + \frac{\sin(3x)\left(\int \cos(3x)\left(e^{3x}+e^{-3x}+e^{3x}\sin(3x)\right)dx\right)}{3}$$

- ◦ Compute integrals

$$y_p(x) = \frac{(5-4\cos(3x)+2\sin(3x))e^{3x}}{90} + \frac{e^{-3x}}{18}$$

- Substitute particular solution into general solution to ODE

$$y = \frac{(5-4\cos(3x)+2\sin(3x))e^{3x}}{90} + c_1\cos(3x) + c_2\sin(3x) + \frac{e^{-3x}}{18}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$2)+9*y(x)=exp(3*x)+exp(-3*x)+exp(3*x)*sin(3*x),y(x), singsol=all)
```

$$y(x) = c_2\sin(3x) + c_1\cos(3x) + \frac{(2\sin(3x) - 4\cos(3x) + 5)e^{3x}}{90} + \frac{e^{-3x}}{18}$$

✓ Solution by Mathematica

Time used: 0.997 (sec). Leaf size: 57

```
DSolve[y''[x]+9*y[x]==Exp[3*x]+Exp[-3*x]+Exp[3*x]*Sin[3*x],y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{1}{90}\left(5e^{-3x}\left(e^{6x}+1\right) + \left(-4e^{3x}+90c_1\right)\cos(3x) + 2\left(e^{3x}+45c_2\right)\sin(3x)\right)$$

11.43 problem 43

Internal problem ID [11817]
Internal file name [`OUTPUT/11826_Thursday_April_11_2024_08_50_05_PM_73129953/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 43.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 5y = e^{-2x}(\cos(x) + 1)}$$

11.43.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 5, f(x) = e^{-2x}(\cos(x) + 1)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(5)}\\ &= -2 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + i\\ \lambda_2 &= -2 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + i\\ \lambda_2 &= -2 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-2x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-2x}(\cos(x) + 1)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}, \{\cos(x) e^{-2x}, e^{-2x} \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x) e^{-2x}, e^{-2x} \sin(x)\}$$

Since $\cos(x) e^{-2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-2x}\}, \{x \cos(x) e^{-2x}, x e^{-2x} \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{-2x} + A_2 x \cos(x) e^{-2x} + A_3 x e^{-2x} \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{-2x} - 2A_2 \sin(x) e^{-2x} + 2A_3 e^{-2x} \cos(x) = e^{-2x}(\cos(x) + 1)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 0, A_3 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{-2x} + \frac{x e^{-2x} \sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2x}(c_1 \cos(x) + c_2 \sin(x))\right) + \left(e^{-2x} + \frac{x e^{-2x} \sin(x)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-2x} + \frac{x e^{-2x} \sin(x)}{2} \tag{1}$$

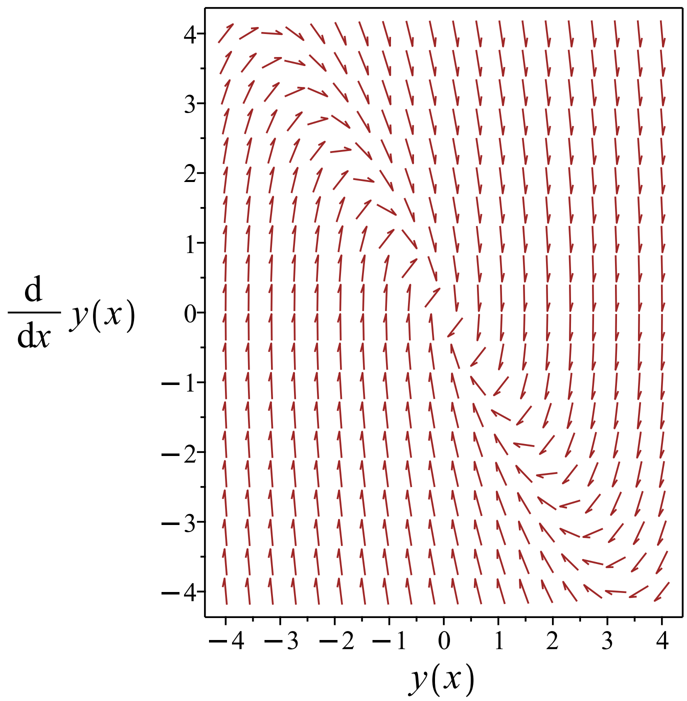


Figure 477: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-2x} + \frac{x e^{-2x} \sin(x)}{2}$$

Verified OK.

11.43.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 413: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\ &= z_1 e^{-2x} \\ &= z_1\left(\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\, e^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-4x}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\cos(x)\, e^{-2x}\right) + c_2\left(\cos(x)\, e^{-2x}(\tan(x))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\, e^{-2x} c_1 + c_2 e^{-2x} \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-2x}(\cos(x) + 1)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}, \{\cos(x)\, e^{-2x}, e^{-2x} \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x)\, e^{-2x}, e^{-2x} \sin(x)\}$$

Since $\cos(x)\, e^{-2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-2x}\}, \{x \cos(x)\, e^{-2x}, x\, e^{-2x} \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{-2x} + A_2 x \cos(x)\, e^{-2x} + A_3 x\, e^{-2x} \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{-2x} - 2A_2 \sin(x)\, e^{-2x} + 2A_3 e^{-2x} \cos(x) = e^{-2x}(\cos(x) + 1)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 0, A_3 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{-2x} + \frac{x\, e^{-2x} \sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos(x)\, e^{-2x} c_1 + c_2 e^{-2x} \sin(x)\right) + \left(e^{-2x} + \frac{x\, e^{-2x} \sin(x)}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-2x} + \frac{x\,e^{-2x} \sin(x)}{2}$$

Summary
The solution(s) found are the following

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-2x} + \frac{x\,e^{-2x} \sin(x)}{2} \tag{1}$$

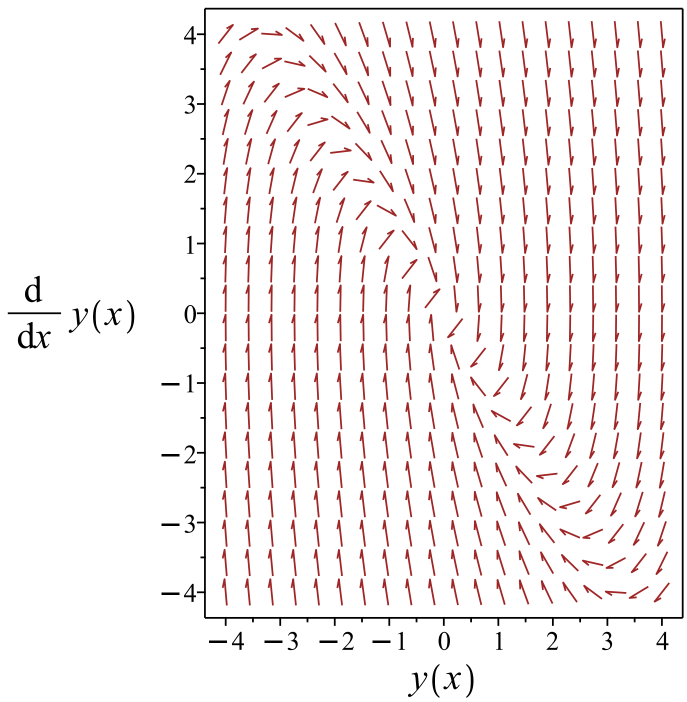


Figure 478: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-2x} + \frac{x\,e^{-2x} \sin(x)}{2}$$

Verified OK.

11.43.3 Maple step by step solution

Let's solve

$y'' + 4y' + 5y = e^{-2x}(\cos(x) + 1)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - I, -2 + I)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)\, e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-2x} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x)\, e^{-2x} c_1 + c_2 e^{-2x} \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{-2x}(\cos(x) + 1)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x)\, e^{-2x} & e^{-2x} \sin(x) \\ -e^{-2x} \sin(x) - 2\cos(x)\, e^{-2x} & -2\, e^{-2x} \sin(x) + \cos(x)\, e^{-2x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-4x}$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\mathrm{e}^{-2x}\left(\cos(x)\left(\int \sin(x)(\cos(x)+1)\,dx\right) - \sin(x)\left(\int \left(\cos(x)^2+\cos(x)\right)dx\right)\right)$$

- Compute integrals

$$y_p(x) = \frac{(x\sin(x)+\cos(x)+2)\,\mathrm{e}^{-2x}}{2}$$

- Substitute particular solution into general solution to ODE

$$y = \cos(x)\,\mathrm{e}^{-2x}c_1 + c_2\mathrm{e}^{-2x}\sin(x) + \frac{(x\sin(x)+\cos(x)+2)\,\mathrm{e}^{-2x}}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+5*y(x)=exp(-2*x)*(1+cos(x)),y(x), singsol=all)
```

$$y(x) = \frac{((2c_1+1)\cos(x) + 2 + (2c_2+x)\sin(x))\,\mathrm{e}^{-2x}}{2}$$

✓ Solution by Mathematica

Time used: 0.122 (sec). Leaf size: 35

```
DSolve[y''[x]+4*y'[x]+5*y[x]==Exp[-2*x]*(1+Cos[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-2x}((1+4c_2)\cos(x) + 2(x+2c_1)\sin(x) + 4)$$

11.44 problem 44

Internal problem ID [11818]
Internal file name [`OUTPUT/11827_Thursday_April_11_2024_08_50_05_PM_44040111/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 44.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 6y' + 9y = e^{x}x^{4} + x^{3}e^{2x} + e^{3x}x^{2}}$$

11.44.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 9, f(x) = e^{x}x^{2}(e^{2x} + x\,e^{x} + x^{2})$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 6\lambda\,\mathrm{e}^{\lambda x} + 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-6)^2 - (4)(1)(9)}\\ &= 3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$y = c_1 \mathrm{e}^{3x} + c_2 x\,\mathrm{e}^{3x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{3x} + \mathrm{e}^{3x} c_2 x$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^x x^2 \left(\mathrm{e}^{2x} + x\,\mathrm{e}^x + x^2\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{3x}, \mathrm{e}^{3x} x^2, \mathrm{e}^{3x}\}, \{x^2 \mathrm{e}^{2x}, x^3 \mathrm{e}^{2x}, \mathrm{e}^{2x} x, \mathrm{e}^{2x}\}, \{x\,\mathrm{e}^x, x^2 \mathrm{e}^x, x^3 \mathrm{e}^x, \mathrm{e}^x x^4, \mathrm{e}^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{3x}, \mathrm{e}^{3x}\}$$

Since e^{3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{3x}, x^3\mathrm{e}^{3x}, \mathrm{e}^{3x}x^2\}, \{x^2\mathrm{e}^{2x}, x^3\mathrm{e}^{2x}, \mathrm{e}^{2x}x, \mathrm{e}^{2x}\}, \{x\,\mathrm{e}^{x}, x^2\mathrm{e}^{x}, x^3\mathrm{e}^{x}, \mathrm{e}^{x}x^4, \mathrm{e}^{x}\}]$$

Since $x\,\mathrm{e}^{3x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3\mathrm{e}^{3x}, x^4\mathrm{e}^{3x}, \mathrm{e}^{3x}x^2\}, \{x^2\mathrm{e}^{2x}, x^3\mathrm{e}^{2x}, \mathrm{e}^{2x}x, \mathrm{e}^{2x}\}, \{x\,\mathrm{e}^{x}, x^2\mathrm{e}^{x}, x^3\mathrm{e}^{x}, \mathrm{e}^{x}x^4, \mathrm{e}^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$\begin{aligned} y_p = {} & A_1x^3\mathrm{e}^{3x} + A_2x^4\mathrm{e}^{3x} + A_3\mathrm{e}^{3x}x^2 + A_4x^2\mathrm{e}^{2x} + A_5x^3\mathrm{e}^{2x} + A_6\mathrm{e}^{2x}x \\ & + A_7\mathrm{e}^{2x} + A_8x\,\mathrm{e}^{x} + A_9x^2\mathrm{e}^{x} + A_{10}x^3\mathrm{e}^{x} + A_{11}\mathrm{e}^{x}x^4 + A_{12}\mathrm{e}^{x} \end{aligned}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} & -6A_5x^2\mathrm{e}^{2x} - 8A_9x\,\mathrm{e}^{x} - 12A_{10}x^2\mathrm{e}^{x} - 16A_{11}\mathrm{e}^{x}x^3 + 6A_1x\,\mathrm{e}^{3x} + 12A_2x^2\mathrm{e}^{3x} \\ & + 6A_5x\,\mathrm{e}^{2x} + 6A_{10}x\,\mathrm{e}^{x} + 12A_{11}\mathrm{e}^{x}x^2 - 4A_4x\,\mathrm{e}^{2x} - 2A_6\mathrm{e}^{2x} - 4A_8\mathrm{e}^{x} \\ & + 4A_{11}\mathrm{e}^{x}x^4 + A_4x^2\mathrm{e}^{2x} + A_5x^3\mathrm{e}^{2x} + A_6\mathrm{e}^{2x}x + 4A_8x\,\mathrm{e}^{x} + 4A_9x^2\mathrm{e}^{x} + 4A_{10}x^3\mathrm{e}^{x} \\ & + 4A_{12}\mathrm{e}^{x} + A_7\mathrm{e}^{2x} + 2A_3\mathrm{e}^{3x} + 2A_4\mathrm{e}^{2x} + 2A_9\mathrm{e}^{x} = \mathrm{e}^{x}x^2\left(\mathrm{e}^{2x} + x\,\mathrm{e}^{x} + x^2\right) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{12}, A_3 = 0, A_4 = 6, A_5 = 1, A_6 = 18, A_7 = 24, A_8 = 3, A_9 = \frac{9}{4}, A_{10} = 1, A_{11} = \frac{1}{4}, A_{12} = \frac{15}{8}\right.$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^4\mathrm{e}^{3x}}{12} + 6x^2\mathrm{e}^{2x} + x^3\mathrm{e}^{2x} + 18\,\mathrm{e}^{2x}x + 24\,\mathrm{e}^{2x} + 3x\,\mathrm{e}^{x} + \frac{9x^2\mathrm{e}^{x}}{4} + x^3\mathrm{e}^{x} + \frac{\mathrm{e}^{x}x^4}{4} + \frac{15\,\mathrm{e}^{x}}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{3x} + \mathrm{e}^{3x}c_2x\right) \\ &\quad + \left(\frac{x^4\mathrm{e}^{3x}}{12} + 6x^2\mathrm{e}^{2x} + x^3\mathrm{e}^{2x} + 18\,\mathrm{e}^{2x}x + 24\,\mathrm{e}^{2x} + 3x\,\mathrm{e}^{x} + \frac{9x^2\mathrm{e}^{x}}{4} + x^3\mathrm{e}^{x} + \frac{\mathrm{e}^{x}x^4}{4} + \frac{15\,\mathrm{e}^{x}}{8}\right) \end{aligned}$$

Which simplifies to

$$y = e^{3x}(c_2x + c_1) + \frac{x^4 e^{3x}}{12} + 6x^2 e^{2x} + x^3 e^{2x} + 18\, e^{2x} x + 24\, e^{2x} + 3x\, e^x + \frac{9x^2 e^x}{4} + x^3 e^x + \frac{e^x x^4}{4} + \frac{15\, e^x}{8}$$

Summary

The solution(s) found are the following

$$y = e^{3x}(c_2x + c_1) + \frac{x^4 e^{3x}}{12} + 6x^2 e^{2x} + x^3 e^{2x} + 18\, e^{2x} x + 24\, e^{2x} + 3x\, e^x + \frac{9x^2 e^x}{4} + x^3 e^x + \frac{e^x x^4}{4} + \frac{15\, e^x}{8} \tag{1}$$

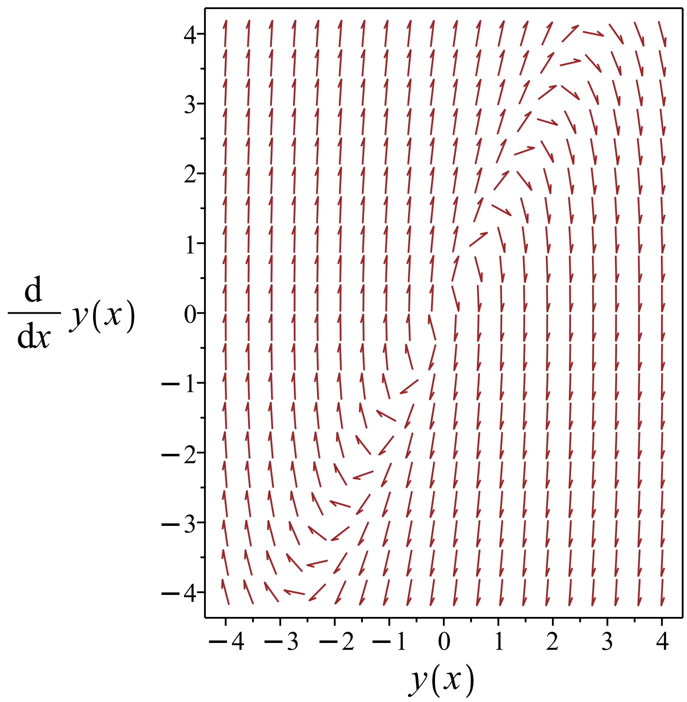


Figure 479: Slope field plot

Verification of solutions

$$y = e^{3x}(c_2x + c_1) + \frac{x^4 e^{3x}}{12} + 6x^2 e^{2x} + x^3 e^{2x} + 18\, e^{2x} x + 24\, e^{2x} + 3x\, e^x + \frac{9x^2 e^x}{4} + x^3 e^x + \frac{e^x x^4}{4} + \frac{15\, e^x}{8}$$

Verified OK.

11.44.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y''+p(x)\,y'+\frac{\left(p(x)^{2}+p'(x)\right)y}{2}=f(x)$$

Where $p(x)=-6$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x)&=e^{\frac{1}{2}\int p\,dx}\\&=e^{\int -6\,dx}\\&=\mathrm{e}^{-3x}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}(M(x)y)''&=\mathrm{e}^{-3x}\mathrm{e}^{x}x^{2}\left(\mathrm{e}^{2x}+x\,\mathrm{e}^{x}+x^{2}\right)\\\left(\mathrm{e}^{-3x}y\right)''&=\mathrm{e}^{-3x}\mathrm{e}^{x}x^{2}\left(\mathrm{e}^{2x}+x\,\mathrm{e}^{x}+x^{2}\right)\end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-3x}y\right)'=\frac{\left(-2x^{4}-4x^{3}-6x^{2}-6x-3\right)\mathrm{e}^{-2x}}{4}+\left(-x^{3}-3x^{2}-6x-6\right)\mathrm{e}^{-x}+\frac{x^{3}}{3}+c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-3x}y\right)=\frac{\left(2x^{4}+8x^{3}+18x^{2}+24x+15\right)\mathrm{e}^{-2x}}{8}+\left(x^{3}+6x^{2}+18x+24\right)\mathrm{e}^{-x}+\frac{x^{4}}{12}+c_1x+c_2$$

Hence the solution is

$$y=\frac{\frac{\left(2x^{4}+8x^{3}+18x^{2}+24x+15\right)\mathrm{e}^{-2x}}{8}+\left(x^{3}+6x^{2}+18x+24\right)\mathrm{e}^{-x}+\frac{x^{4}}{12}+c_1x+c_2}{\mathrm{e}^{-3x}}$$

Or

$$\begin{aligned}y=&\frac{x^{4}\mathrm{e}^{3x}}{12}+c_1x\,\mathrm{e}^{3x}+\frac{x^{4}\mathrm{e}^{x}}{4}+x^{3}\mathrm{e}^{2x}+\mathrm{e}^{3x}c_2+x^{3}\mathrm{e}^{x}\\&+6x^{2}\mathrm{e}^{2x}+\frac{9x^{2}\mathrm{e}^{x}}{4}+18\,\mathrm{e}^{2x}x+3x\,\mathrm{e}^{x}+24\,\mathrm{e}^{2x}+\frac{15\,\mathrm{e}^{x}}{8}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{x^4 e^{3x}}{12} + c_1 x\, e^{3x} + \frac{x^4 e^x}{4} + x^3 e^{2x} + e^{3x} c_2 + x^3 e^x + 6x^2 e^{2x} + \frac{9x^2 e^x}{4} + 18\, e^{2x} x + 3x\, e^x + 24\, e^{2x} + \frac{15\, e^x}{8} \tag{1}$$

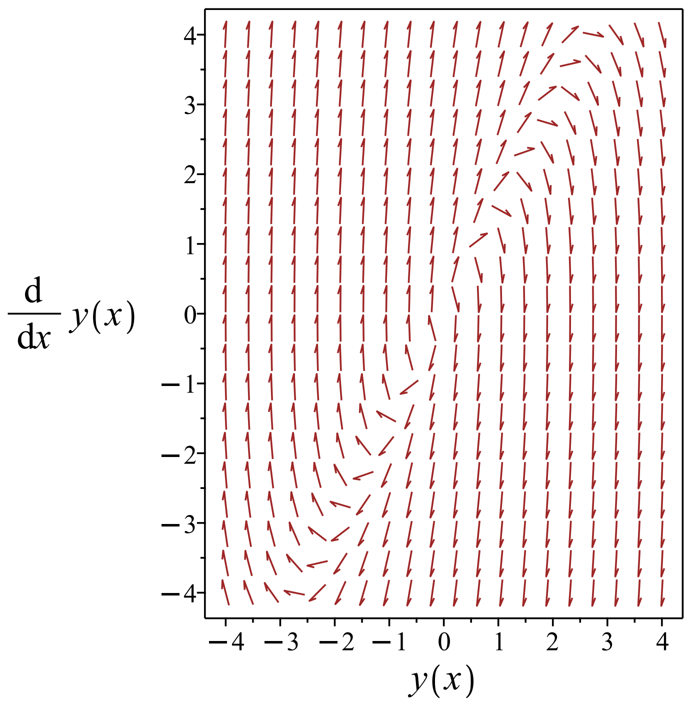


Figure 480: Slope field plot

Verification of solutions

$$y = \frac{x^4 e^{3x}}{12} + c_1 x\, e^{3x} + \frac{x^4 e^x}{4} + x^3 e^{2x} + e^{3x} c_2 + x^3 e^x + 6x^2 e^{2x} + \frac{9x^2 e^x}{4} + 18\, e^{2x} x + 3x\, e^x + 24\, e^{2x} + \frac{15\, e^x}{8}$$

Verified OK.

11.44.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 415: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{3x} \\
&= z_1 \left(\mathrm{e}^{3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{6x}}{\left(y_1\right)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{3x}\right) + c_2 \left(\mathrm{e}^{3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{3x} + \mathrm{e}^{3x} c_2 x
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{x} x^{2} \left(e^{2x} + x\,e^{x} + x^{2}\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{3x}, e^{3x} x^{2}, e^{3x}\}, \{x^{2} e^{2x}, x^{3} e^{2x}, e^{2x} x, e^{2x}\}, \{x\,e^{x}, x^{2} e^{x}, x^{3} e^{x}, x^{4} e^{x}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{3x}, e^{3x}\}$$

Since e^{3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{3x}, x^{3} e^{3x}, e^{3x} x^{2}\}, \{x^{2} e^{2x}, x^{3} e^{2x}, e^{2x} x, e^{2x}\}, \{x\,e^{x}, x^{2} e^{x}, x^{3} e^{x}, x^{4} e^{x}, e^{x}\}]$$

Since $x\,e^{3x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^{3} e^{3x}, x^{4} e^{3x}, e^{3x} x^{2}\}, \{x^{2} e^{2x}, x^{3} e^{2x}, e^{2x} x, e^{2x}\}, \{x\,e^{x}, x^{2} e^{x}, x^{3} e^{x}, x^{4} e^{x}, e^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$\begin{aligned} y_p = {} & A_1 x^{3} e^{3x} + A_2 x^{4} e^{3x} + A_3 e^{3x} x^{2} + A_4 x^{2} e^{2x} + A_5 x^{3} e^{2x} + A_6 e^{2x} x \\ & + A_7 e^{2x} + A_8 x\,e^{x} + A_9 x^{2} e^{x} + A_{10} x^{3} e^{x} + A_{11} x^{4} e^{x} + A_{12} e^{x} \end{aligned}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} & 2A_3 e^{3x} + 2A_4 e^{2x} + 2A_9 e^{x} - 2A_6 e^{2x} - 4A_8 e^{x} + A_7 e^{2x} + 4A_{12} e^{x} + A_6 e^{2x} x \\ & + 4A_8 x\,e^{x} + 4A_9 x^{2} e^{x} + 4A_{10} x^{3} e^{x} + 4A_{11} x^{4} e^{x} + A_4 x^{2} e^{2x} + A_5 x^{3} e^{2x} \\ & - 6A_5 x^{2} e^{2x} - 8A_9 x\,e^{x} - 12A_{10} x^{2} e^{x} - 16A_{11} x^{3} e^{x} + 6A_1 x\,e^{3x} + 12A_2 x^{2} e^{3x} \\ & + 6A_5 x\,e^{2x} + 6A_{10} x\,e^{x} + 12A_{11} x^{2} e^{x} - 4A_4 x\,e^{2x} = e^{x} x^{2} \left(e^{2x} + x\,e^{x} + x^{2}\right) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{12}, A_3 = 0, A_4 = 6, A_5 = 1, A_6 = 18, A_7 = 24, A_8 = 3, A_9 = \frac{9}{4}, A_{10} = 1, A_{11} = \frac{1}{4}, A_{12} = \frac{15}{8}\right.$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^4 e^{3x}}{12} + 6x^2 e^{2x} + x^3 e^{2x} + 18\,e^{2x} x + 24\,e^{2x} + 3x\,e^{x} + \frac{9x^2 e^{x}}{4} + x^3 e^{x} + \frac{x^4 e^{x}}{4} + \frac{15\,e^{x}}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + e^{3x} c_2 x\right) \\ &\quad + \left(\frac{x^4 e^{3x}}{12} + 6x^2 e^{2x} + x^3 e^{2x} + 18\,e^{2x} x + 24\,e^{2x} + 3x\,e^{x} + \frac{9x^2 e^{x}}{4} + x^3 e^{x} + \frac{x^4 e^{x}}{4} + \frac{15\,e^{x}}{8}\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} y = {} & e^{3x}(c_2 x + c_1) + \frac{x^4 e^{3x}}{12} + 6x^2 e^{2x} + x^3 e^{2x} + 18\,e^{2x} x \\ & + 24\,e^{2x} + 3x\,e^{x} + \frac{9x^2 e^{x}}{4} + x^3 e^{x} + \frac{x^4 e^{x}}{4} + \frac{15\,e^{x}}{8} \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & e^{3x}(c_2 x + c_1) + \frac{x^4 e^{3x}}{12} + 6x^2 e^{2x} + x^3 e^{2x} + 18\,e^{2x} x \\ & + 24\,e^{2x} + 3x\,e^{x} + \frac{9x^2 e^{x}}{4} + x^3 e^{x} + \frac{x^4 e^{x}}{4} + \frac{15\,e^{x}}{8} \end{aligned} \tag{1}$$

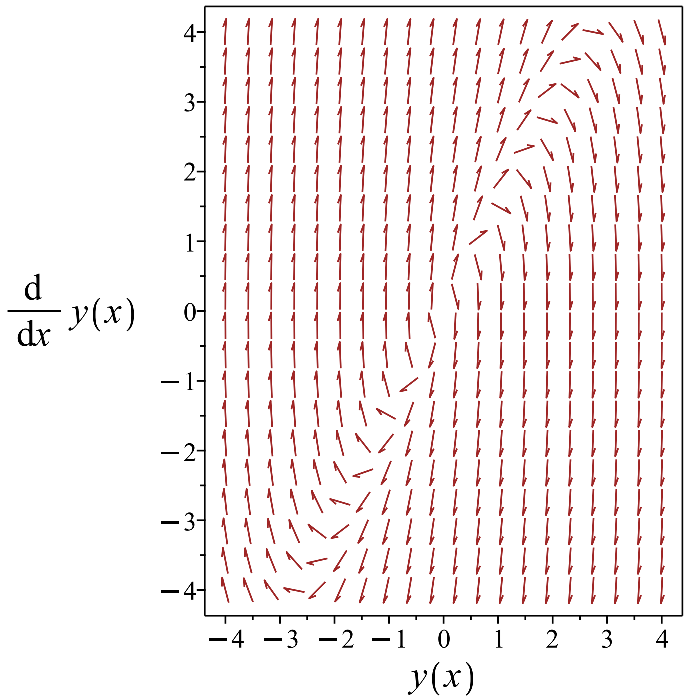


Figure 481: Slope field plot

Verification of solutions

$$y = e^{3x}(c_2x + c_1) + \frac{x^4 e^{3x}}{12} + 6x^2 e^{2x} + x^3 e^{2x} + 18\,e^{2x}x + 24\,e^{2x} + 3x\,e^{x} + \frac{9x^2 e^{x}}{4} + x^3 e^{x} + \frac{x^4 e^{x}}{4} + \frac{15\,e^{x}}{8}$$

Verified OK.

11.44.4 Maple step by step solution

Let's solve

$y'' - 6y' + 9y = e^{x}x^2(e^{2x} + x\,e^{x} + x^2)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r-3)^2 = 0$

- Root of the characteristic polynomial

  $r = 3$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{3x} + e^{3x} c_2 x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = e^x x^2 (e^{2x} + x\,e^x + x^2)\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{3x} & x\,e^{3x} \\ 3\,e^{3x} & e^{3x} + 3x\,e^{3x} \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{6x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = e^{3x}\left(-\left(\int x^3 e^{-2x}(e^{2x} + x\,e^x + x^2)\,dx\right) + \left(\int x^2 e^{-2x}(e^{2x} + x\,e^x + x^2)\,dx\right) x\right)$

  - Compute integrals

    $y_p(x) = \frac{e^x\left(\frac{45}{2} + e^{2x}x^4 + 12(x^3+6x^2+18x+24)e^x + 3x^4 + 12x^3 + 27x^2 + 36x\right)}{12}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{3x} + e^{3x} c_2 x + \frac{e^x\left(\frac{45}{2} + e^{2x}x^4 + 12(x^3+6x^2+18x+24)e^x + 3x^4 + 12x^3 + 27x^2 + 36x\right)}{12}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 62

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+9*y(x)=x^4*exp(x)+x^3*exp(2*x)+x^2*exp(3*x),y(x), sings
```

$$y(x) = (x^3 + 6x^2 + 18x + 24) e^{2x} + \frac{(x^4 + 12c_1x + 12c_2) e^{3x}}{12} + \frac{\left(x^4 + 4x^3 + 9x^2 + 12x + \frac{15}{2}\right) e^{x}}{4}$$

✓ Solution by Mathematica

Time used: 1.391 (sec). Leaf size: 70

```
DSolve[y''[x]-6*y'[x]+9*y[x]==x^4*Exp[x]+x^3*Exp[2*x]+x^2*Exp[3*x],y[x],x,IncludeSingularSol
```

$$y(x) \to e^x \left(\frac{x^4}{4} + e^{2x}\left(\frac{x^4}{12} + c_2x + c_1\right) + x^3 + \frac{9x^2}{4} + e^x(x^3 + 6x^2 + 18x + 24) + 3x + \frac{15}{8}\right)$$

11.45 problem 45

Internal problem ID [11819]
Internal file name [`OUTPUT/11828_Thursday_April_11_2024_08_50_07_PM_24839565/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 45.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 6y' + 13y = x\,\mathrm{e}^{-3x} \sin(2x) + x^2 \mathrm{e}^{-2x} \sin(3x)}$$

11.45.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 6, C = 13, f(x) = x\,\mathrm{e}^{-2x}(x \sin(3x) + \mathrm{e}^{-x} \sin(2x))$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 6y' + 13y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 13$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda e^{\lambda x} + 13 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(13)} \\ &= -3 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 2i \\ \lambda_2 &= -3 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -3 + 2i \\ \lambda_2 &= -3 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-3x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-3x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-3x} \cos(2x)$$
$$y_2 = e^{-3x} \sin(2x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-3x} \cos(2x) & e^{-3x} \sin(2x) \\ \frac{d}{dx}(e^{-3x} \cos(2x)) & \frac{d}{dx}(e^{-3x} \sin(2x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-3x} \cos(2x) & e^{-3x} \sin(2x) \\ -3\,e^{-3x} \cos(2x) - 2\,e^{-3x} \sin(2x) & -3\,e^{-3x} \sin(2x) + 2\,e^{-3x} \cos(2x) \end{vmatrix}$$

Therefore

$$W = \left(e^{-3x}\cos(2x)\right)\left(-3\,e^{-3x}\sin(2x) + 2\,e^{-3x}\cos(2x)\right) - \left(e^{-3x}\sin(2x)\right)\left(-3\,e^{-3x}\cos(2x) - 2\,e^{-3x}\sin(2x)\right)$$

Which simplifies to

$$W = 2\,e^{-6x}\cos(2x)^2 + 2\,e^{-6x}\sin(2x)^2$$

Which simplifies to

$$W = 2\,e^{-6x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-3x}\sin(2x)\,x\,e^{-2x}(x\sin(3x) + e^{-x}\sin(2x))}{2\,e^{-6x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x\sin(2x)\left(\sin(3x)\,x\,e^{x} + \sin(2x)\right)}{2}dx$$

Hence

$$\begin{aligned} u_1 = &-\frac{\left(\frac{x^2}{2} - \frac{1}{2}\right)e^{x}\cos(x)}{4} + \frac{\left(-\frac{1}{2}x^2 + x - \frac{1}{2}\right)e^{x}\sin(x)}{4} \\ &+ \frac{\left(\frac{1}{26}x^2 + \frac{12}{169}x - \frac{37}{4394}\right)e^{x}\cos(5x)}{4} - \frac{\left(-\frac{5}{26}x^2 + \frac{5}{169}x + \frac{55}{4394}\right)e^{x}\sin(5x)}{4} \\ &- 2x\left(-\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}\right) + \frac{x^2}{8} - \frac{\sin(x)^2}{2} \\ &+ 2x\left(-\frac{\left(\sin(x)^3 + \frac{3\sin(x)}{2}\right)\cos(x)}{4} + \frac{3x}{8}\right) + \frac{\left(2\cos(x)^2 - 5\right)^2}{32} \end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-3x}\cos(2x)\,x\,e^{-2x}(x\sin(3x) + e^{-x}\sin(2x))}{2\,e^{-6x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x \cos(2x) \left(\sin(3x)\, x\, \mathrm{e}^{x} + \sin(2x)\right)}{2} dx$$

Hence

$$\begin{aligned} u_2 = {} & \frac{\left(-\frac{5}{26}x^2 + \frac{5}{169}x + \frac{55}{4394}\right) \mathrm{e}^{x} \cos(5x)}{4} + \frac{\left(\frac{1}{26}x^2 + \frac{12}{169}x - \frac{37}{4394}\right) \mathrm{e}^{x} \sin(5x)}{4} \\ & + \frac{\left(-\frac{1}{2}x^2 + x - \frac{1}{2}\right) \mathrm{e}^{x} \cos(x)}{4} + \frac{\left(\frac{x^2}{2} - \frac{1}{2}\right) \mathrm{e}^{x} \sin(x)}{4} - \frac{x \cos(x)^4}{2} \\ & + \frac{\left(\cos(x)^3 + \frac{3\cos(x)}{2}\right) \sin(x)}{8} - \frac{x}{16} + \frac{\cos(x)^2 x}{2} - \frac{\cos(x)\sin(x)}{4} \end{aligned}$$

Which simplifies to

$$\begin{aligned} u_1 = {} & \frac{\mathrm{e}^{x} \left(x^2 + \frac{24}{13}x - \frac{37}{169}\right) \cos(5x)}{104} + \frac{5\,\mathrm{e}^{x} \left(x^2 - \frac{2}{13}x - \frac{11}{169}\right) \sin(5x)}{104} \\ & - \frac{\left((1+x)\cos(x) + \sin(x)(x-1)\right)(x-1)\,\mathrm{e}^{x}}{8} + \frac{\cos(x)^4}{8} \\ & - \frac{5\cos(x)^2}{8} - \frac{x\left(\sin(x)^2 - \frac{1}{2}\right)\sin(x)\cos(x)}{2} - \frac{x^2}{8} - \frac{\sin(x)^2}{2} + \frac{25}{32} \\ u_2 = {} & -\frac{5\,\mathrm{e}^{x} \left(x^2 - \frac{2}{13}x - \frac{11}{169}\right) \cos(5x)}{104} + \frac{\left(338x^2 + 624x - 74\right) \mathrm{e}^{x} \sin(5x)}{35152} \\ & - \frac{\left((x-1)\cos(x) - \sin(x)(1+x)\right)(x-1)\,\mathrm{e}^{x}}{8} - \frac{x\cos(x)^4}{2} \\ & + \frac{\cos(x)^3 \sin(x)}{8} + \frac{\cos(x)^2 x}{2} - \frac{\cos(x)\sin(x)}{16} - \frac{x}{16} \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{e^x\left(x^2+\frac{24}{13}x-\frac{37}{169}\right)\cos(5x)}{104} + \frac{5\,e^x\left(x^2-\frac{2}{13}x-\frac{11}{169}\right)\sin(5x)}{104} - \frac{((1+x)\cos(x)+\sin(x)(x-1))(x-1)\,e^x}{8} + \frac{\cos(x)^4}{8} - \frac{5\cos(x)^2}{8} - \frac{x\left(\sin(x)^2-\frac{1}{2}\right)\sin(x)\cos(x)}{2} - \frac{x^2}{8} - \frac{\sin(x)^2}{2} + \frac{25}{32}\right)e^{-3x}\cos(2x) + \left(-\frac{5\,e^x\left(x^2-\frac{2}{13}x-\frac{11}{169}\right)\cos(5x)}{104} + \frac{(338x^2+624x-74)\,e^x\sin(5x)}{35152} - \frac{((x-1)\cos(x)-\sin(x)(1+x))(x-1)\,e^x}{8} - \frac{x\cos(x)^4}{2} + \frac{\cos(x)^3\sin(x)}{8} + \frac{\cos(x)^2 x}{2} - \frac{\cos(x)\sin(x)}{16} - \frac{x}{16}\right)e^{-3x}\sin(2x)$$

Which simplifies to

$$y_p(x) = -\frac{6\left(e^x\left(x^2-\frac{2}{13}x-\frac{180}{169}\right)\cos(x)^3 + \left(\frac{2\left(x^2-\frac{41}{13}x+\frac{563}{338}\right)\sin(x)e^x}{3} + \frac{13x^2}{24} - \frac{39}{32}\right)\cos(x)^2 + \left(\left(-\frac{3}{4}x^2+\frac{3}{26}x+\frac{135}{169}\right)e\right.}{13}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(e^{-3x}(c_1\cos(2x)+c_2\sin(2x))\right) + \left(-\frac{6\left(e^x\left(x^2-\frac{2}{13}x-\frac{180}{169}\right)\cos(x)^3 + \left(\frac{2\left(x^2-\frac{41}{13}x+\frac{563}{338}\right)\sin(x)e^x}{3} + \frac{13x^2}{24} - \frac{39}{32}\right)\cos(x)^2 + \left(\left(-\frac{3}{4}x^2+\frac{3}{26}x+\right.\right.}{13}\right.$$

Summary

The solution(s) found are the following

$$y = e^{-3x}(c_1\cos(2x)+c_2\sin(2x)) - \frac{6\left(e^x\left(x^2-\frac{2}{13}x-\frac{180}{169}\right)\cos(x)^3 + \left(\frac{2\left(x^2-\frac{41}{13}x+\frac{563}{338}\right)\sin(x)e^x}{3} + \frac{13x^2}{24} - \frac{39}{32}\right)\cos(x)^2 + \left(\left(-\frac{3}{4}x^2+\frac{3}{26}x+\frac{135}{169}\right)e\right.}{13} \quad (1)$$

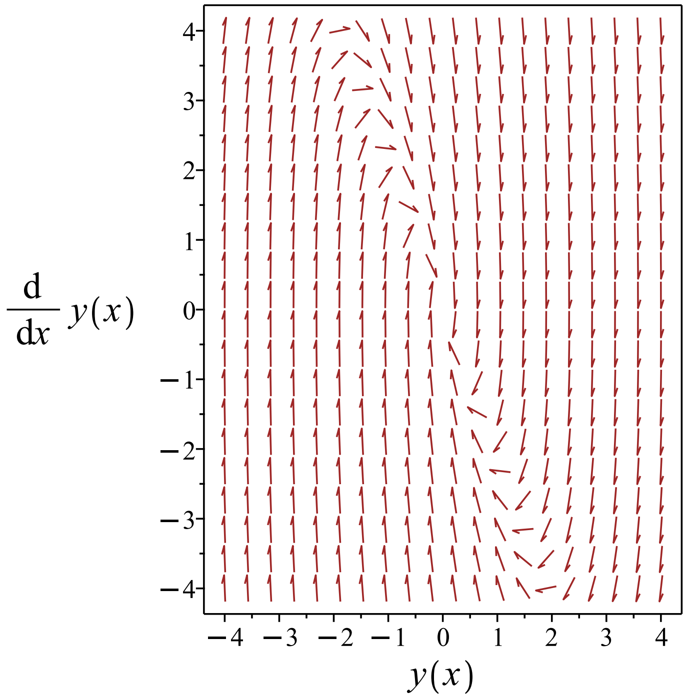


Figure 482: Slope field plot

Verification of solutions

$$y = e^{-3x}(c_1 \cos(2x) + c_2 \sin(2x)) - \frac{6\left(e^x\left(x^2 - \frac{2}{13}x - \frac{180}{169}\right)\cos(x)^3 + \left(\frac{2\left(x^2 - \frac{41}{13}x + \frac{563}{338}\right)\sin(x)e^x}{3} + \frac{13x^2}{24} - \frac{39}{32}\right)\cos(x)^2 + \left(\left(-\frac{3}{4}x^2 + \frac{3}{26}x + \frac{135}{169}\right)e^{x}\right.}{13}$$

Verified OK.

11.45.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 13y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 417: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\ &= z_1 e^{-3x} \\ &= z_1\left(\mathrm{e}^{-3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x} \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{-3x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 6y' + 13y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-3x} \cos(2x) + \frac{\mathrm{e}^{-3x} \sin(2x)\, c_2}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-3x} \cos(2x)$$
$$y_2 = \frac{e^{-3x} \sin(2x)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-3x} \cos(2x) & \frac{e^{-3x} \sin(2x)}{2} \\ \frac{d}{dx}\left(e^{-3x} \cos(2x)\right) & \frac{d}{dx}\left(\frac{e^{-3x} \sin(2x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-3x} \cos(2x) & \frac{e^{-3x} \sin(2x)}{2} \\ -3\,e^{-3x} \cos(2x) - 2\,e^{-3x} \sin(2x) & -\frac{3\,e^{-3x} \sin(2x)}{2} + e^{-3x} \cos(2x) \end{vmatrix}$$

Therefore

$$W = \left(e^{-3x} \cos(2x)\right)\left(-\frac{3\,e^{-3x} \sin(2x)}{2} + e^{-3x} \cos(2x)\right) - \left(\frac{e^{-3x} \sin(2x)}{2}\right)\left(-3\,e^{-3x} \cos(2x) - 2\,e^{-3x} \sin(2x)\right)$$

Which simplifies to

$$W = \mathrm{e}^{-6x} \cos(2x)^2 + \mathrm{e}^{-6x} \sin(2x)^2$$

Which simplifies to

$$W = \mathrm{e}^{-6x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{-3x} \sin(2x) x\, \mathrm{e}^{-2x} (x \sin(3x) + \mathrm{e}^{-x} \sin(2x))}{2}}{\mathrm{e}^{-6x}} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{x \sin(2x) (\sin(3x)\, x\, \mathrm{e}^{x} + \sin(2x))}{2} dx$$

Hence

$$\begin{aligned} u_1 = &-\frac{\left(\frac{x^2}{2} - \frac{1}{2}\right) \mathrm{e}^{x} \cos(x)}{4} + \frac{\left(-\frac{1}{2}x^2 + x - \frac{1}{2}\right) \mathrm{e}^{x} \sin(x)}{4} \\ &+ \frac{\left(\frac{1}{26}x^2 + \frac{12}{169}x - \frac{37}{4394}\right) \mathrm{e}^{x} \cos(5x)}{4} - \frac{\left(-\frac{5}{26}x^2 + \frac{5}{169}x + \frac{55}{4394}\right) \mathrm{e}^{x} \sin(5x)}{4} \\ &- 2x\left(-\frac{\cos(x) \sin(x)}{2} + \frac{x}{2}\right) + \frac{x^2}{8} - \frac{\sin(x)^2}{2} \\ &+ 2x\left(-\frac{\left(\sin(x)^3 + \frac{3\sin(x)}{2}\right) \cos(x)}{4} + \frac{3x}{8}\right) + \frac{\left(2\cos(x)^2 - 5\right)^2}{32} \end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-3x} \cos(2x)\, x\, \mathrm{e}^{-2x} (x \sin(3x) + \mathrm{e}^{-x} \sin(2x))}{\mathrm{e}^{-6x}} \, dx$$

Which simplifies to

$$u_2 = \int x \cos(2x) (\sin(3x)\, x\, \mathrm{e}^{x} + \sin(2x)) \, dx$$

Hence

$$u_2 = \frac{\left(-\frac{5}{26}x^2 + \frac{5}{169}x + \frac{55}{4394}\right) e^x \cos(5x)}{2} + \frac{\left(\frac{1}{26}x^2 + \frac{12}{169}x - \frac{37}{4394}\right) e^x \sin(5x)}{2} + \frac{\left(-\frac{1}{2}x^2 + x - \frac{1}{2}\right) e^x \cos(x)}{2} + \frac{\left(\frac{x^2}{2} - \frac{1}{2}\right) e^x \sin(x)}{2} - x\cos(x)^4 + \frac{\left(\cos(x)^3 + \frac{3\cos(x)}{2}\right)\sin(x)}{4} - \frac{x}{8} + \cos(x)^2 x - \frac{\cos(x)\sin(x)}{2}$$

Which simplifies to

$$u_1 = \frac{e^x\left(x^2 + \frac{24}{13}x - \frac{37}{169}\right)\cos(5x)}{104} + \frac{5\,e^x\left(x^2 - \frac{2}{13}x - \frac{11}{169}\right)\sin(5x)}{104} - \frac{((1+x)\cos(x) + \sin(x)(x-1))(x-1)e^x}{8} + \frac{\cos(x)^4}{8} - \frac{5\cos(x)^2}{8} - \frac{x\left(\sin(x)^2 - \frac{1}{2}\right)\sin(x)\cos(x)}{2} - \frac{x^2}{8} - \frac{\sin(x)^2}{2} + \frac{25}{32}$$

$$u_2 = -\frac{5\,e^x\left(x^2 - \frac{2}{13}x - \frac{11}{169}\right)\cos(5x)}{52} + \frac{(169x^2 + 312x - 37)e^x \sin(5x)}{8788} - \frac{((x-1)\cos(x) - \sin(x)(1+x))(x-1)e^x}{4} - x\cos(x)^4 + \frac{\cos(x)^3\sin(x)}{4} + \cos(x)^2 x - \frac{\cos(x)\sin(x)}{8} - \frac{x}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{e^x\left(x^2 + \frac{24}{13}x - \frac{37}{169}\right)\cos(5x)}{104} + \frac{5\,e^x\left(x^2 - \frac{2}{13}x - \frac{11}{169}\right)\sin(5x)}{104} - \frac{((1+x)\cos(x) + \sin(x)(x-1))(x-1)e^x}{8} + \frac{\cos(x)^4}{8} - \frac{5\cos(x)^2}{8} - \frac{x\left(\sin(x)^2 - \frac{1}{2}\right)\sin(x)\cos(x)}{2} - \frac{x^2}{8} - \frac{\sin(x)^2}{2} + \frac{25}{32}\right) e^{-3x}\cos(2x) + \frac{\left(-\frac{5\,e^x\left(x^2 - \frac{2}{13}x - \frac{11}{169}\right)\cos(5x)}{52} + \frac{(169x^2+312x-37)e^x\sin(5x)}{8788} - \frac{((x-1)\cos(x) - \sin(x)(1+x))(x-1)e^x}{4} - x\cos(x)^4 + \frac{\cos(x)^3 \sin}{4}\right.}{2}$$

Which simplifies to

$$y_p(x) = -\frac{6\left(e^x\left(x^2 - \frac{2}{13}x - \frac{180}{169}\right)\cos(x)^3 + \left(\frac{2\left(x^2 - \frac{41}{13}x + \frac{563}{338}\right)\sin(x)e^x}{3} + \frac{13x^2}{24} - \frac{39}{32}\right)\cos(x)^2 + \left(\left(-\frac{3}{4}x^2 + \frac{3}{26}x + \frac{135}{169}\right)e\right.\right.}{13}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(c_1 \mathrm{e}^{-3x} \cos(2x) + \frac{\mathrm{e}^{-3x} \sin(2x) c_2}{2}\right)$$

$$+ \left(-\frac{6\left(\mathrm{e}^x \left(x^2 - \frac{2}{13}x - \frac{180}{169}\right) \cos(x)^3 + \left(\frac{2\left(x^2 - \frac{41}{13}x + \frac{563}{338}\right) \sin(x) \mathrm{e}^x}{3} + \frac{13x^2}{24} - \frac{39}{32}\right) \cos(x)^2 + \left(\left(-\frac{3}{4}x^2 + \frac{3}{26}x + \right.\right.\right.}{13}\right.$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-3x} \cos(2x) + \frac{\mathrm{e}^{-3x} \sin(2x) c_2}{2} \tag{1}$$
$$- \frac{6\left(\mathrm{e}^x \left(x^2 - \frac{2}{13}x - \frac{180}{169}\right) \cos(x)^3 + \left(\frac{2\left(x^2 - \frac{41}{13}x + \frac{563}{338}\right) \sin(x) \mathrm{e}^x}{3} + \frac{13x^2}{24} - \frac{39}{32}\right) \cos(x)^2 + \left(\left(-\frac{3}{4}x^2 + \frac{3}{26}x + \frac{135}{169}\right) \mathrm{e}\right.\right.}{13}$$

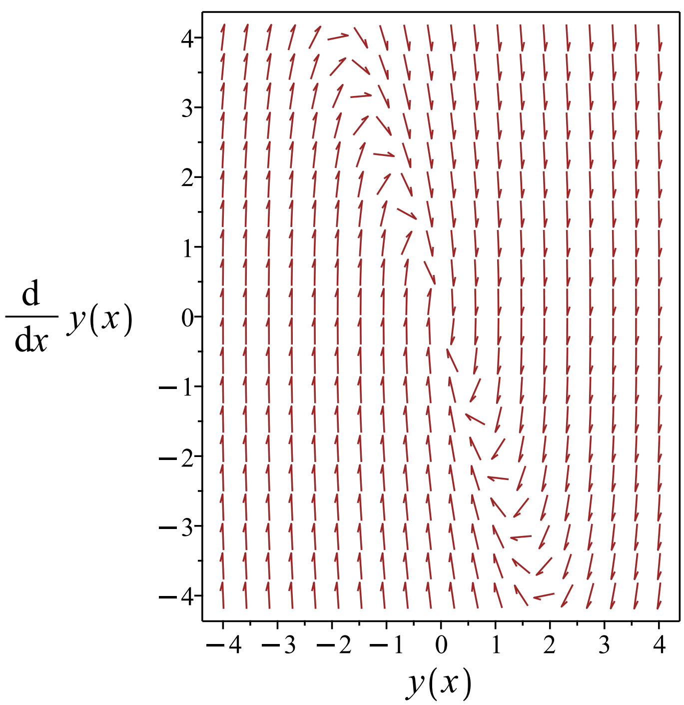


Figure 483: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-3x} \cos(2x) + \frac{\mathrm{e}^{-3x} \sin(2x) c_2}{2} - \frac{6\left(\mathrm{e}^{x}\left(x^2 - \frac{2}{13}x - \frac{180}{169}\right)\cos(x)^3 + \left(\frac{2\left(x^2 - \frac{41}{13}x + \frac{563}{338}\right)\sin(x)\mathrm{e}^{x}}{3} + \frac{13x^2}{24} - \frac{39}{32}\right)\cos(x)^2 + \left(\left(-\frac{3}{4}x^2 + \frac{3}{26}x + \frac{135}{169}\right)\mathrm{e}\right.\right.}{13}$$

Verified OK.

11.45.3 Maple step by step solution

Let's solve

$y'' + 6y' + 13y = x\,\mathrm{e}^{-2x}(x \sin(3x) + \mathrm{e}^{-x} \sin(2x))$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - 2\,\mathrm{I}, -3 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x} \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-3x} \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} \cos(2x) + \mathrm{e}^{-3x} \sin(2x) c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\,\mathrm{e}^{-2x}(x \sin(3x) + \mathrm{e}^{-x}\right.$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-3x}\cos(2x) & e^{-3x}\sin(2x) \\ -3e^{-3x}\cos(2x) - 2e^{-3x}\sin(2x) & -3e^{-3x}\sin(2x) + 2e^{-3x}\cos(2x) \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = 2e^{-6x}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\frac{e^{-3x}\left(\cos(2x)\left(\int x\sin(2x)(\sin(3x)x\,e^{x}+\sin(2x))dx\right)-\sin(2x)\left(\int x\cos(2x)(\sin(3x)x\,e^{x}+\sin(2x))dx\right)\right)}{2}$$

- Compute integrals

  $$y_p(x) = -\frac{6\left(e^{x}\left(x^2-\frac{2}{13}x-\frac{180}{169}\right)\cos(x)^3+\left(\frac{2\left(x^2-\frac{41}{13}x+\frac{563}{338}\right)\sin(x)e^{x}}{3}+\frac{13x^2}{24}-\frac{39}{32}\right)\cos(x)^2+\left(\left(-\frac{3}{4}x^2+\frac{3}{26}x+\frac{135}{169}\right)e^{x}-\frac{13x\sin(x)}{48}\right)\text{co}}{13}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1e^{-3x}\cos(2x) + e^{-3x}\sin(2x)c_2 - \frac{6\left(e^{x}\left(x^2-\frac{2}{13}x-\frac{180}{169}\right)\cos(x)^3+\left(\frac{2\left(x^2-\frac{41}{13}x+\frac{563}{338}\right)\sin(x)e^{x}}{3}+\frac{13x^2}{24}-\frac{39}{32}\right)\cos(x)^2}{}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 68

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+13*y(x)=x*exp(-3*x)*sin(2*x)+x^2*exp(-2*x)*sin(3*x),y(x
```

$$y(x) = -\frac{3\left(\left(\frac{13x^2}{12}-\frac{26c_1}{3}-\frac{39}{16}\right)\cos(2x)+e^{x}\left(x^2-\frac{2}{13}x-\frac{180}{169}\right)\cos(3x)+\frac{2e^{x}\left(x^2-\frac{41}{13}x+\frac{563}{338}\right)\sin(3x)}{3}-\frac{13\sin(2x)(x+16c_2)}{24}\right.}{26}$$

✓ Solution by Mathematica

Time used: 1.921 (sec). Leaf size: 82

```
DSolve[y''[x]+6*y'[x]+13*y[x]==x*Exp[-3*x]*Sin[2*x]+x^2*Exp[-2*x]*Sin[3*x],y[x],x,IncludeSin
```

$$y(x) \to \frac{e^{-3x}(-32e^{x}(338x^2 - 1066x + 563)\sin(3x) - 96e^{x}(169x^2 - 26x - 180)\cos(3x) - 2197(8x^2 - 1 - 64c_2)}{140608}$$

11.46 problem 46

Internal problem ID [11820]
Internal file name [`OUTPUT/11829_Thursday_April_11_2024_08_51_25_PM_72330630/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 46.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 3y'' + 2y' = x^2 e^x + 3\,e^{2x} x + 5x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 3y'' + 2y' = 0$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = 1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^x + e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^x$$
$$y_3 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y''' - 3y'' + 2y' = x^2 e^x + 3\,e^{2x} x + 5x^2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 e^x + 3\,e^{2x} x + 5x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x} x, e^{2x}\}, \{1, x, x^2\}, \{x\,e^x, x^2 e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^x, e^{2x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x} x, e^{2x}\}, \{x, x^2, x^3\}, \{x\,e^x, x^2 e^x, e^x\}]$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x} x, e^{2x}\}, \{x, x^2, x^3\}, \{x\,e^x, x^2 e^x, x^3 e^x\}]$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{2x}, e^{2x} x\}, \{x, x^2, x^3\}, \{x\,e^x, x^2 e^x, x^3 e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{2x} + A_2 e^{2x} x + A_3 x + A_4 x^2 + A_5 x^3 + A_6 x\,e^x + A_7 x^2 e^x + A_8 x^3 e^x$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1x\,e^{2x} - 2A_7x\,e^{x} - 3A_8x^2e^{x} + 2A_3 - 6A_4 + 6A_5 - 18A_5x + 4A_4x \\ + 6A_5x^2 + 6A_8e^{x} + 6A_1e^{2x} + 2A_2e^{2x} - A_6e^{x} = x^2e^{x} + 3\,e^{2x}x + 5x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{4}, A_2 = -\frac{9}{4}, A_3 = \frac{35}{4}, A_4 = \frac{15}{4}, A_5 = \frac{5}{6}, A_6 = -2, A_7 = 0, A_8 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3x^2e^{2x}}{4} - \frac{9\,e^{2x}x}{4} + \frac{35x}{4} + \frac{15x^2}{4} + \frac{5x^3}{6} - 2x\,e^{x} - \frac{x^3e^{x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2e^{x} + e^{2x}c_3\right) + \left(\frac{3x^2e^{2x}}{4} - \frac{9\,e^{2x}x}{4} + \frac{35x}{4} + \frac{15x^2}{4} + \frac{5x^3}{6} - 2x\,e^{x} - \frac{x^3e^{x}}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2e^{x} + e^{2x}c_3 + \frac{3x^2e^{2x}}{4} - \frac{9\,e^{2x}x}{4} + \frac{35x}{4} + \frac{15x^2}{4} + \frac{5x^3}{6} - 2x\,e^{x} - \frac{x^3e^{x}}{3} \quad (1)$$

Verification of solutions

$$y = c_1 + c_2e^{x} + e^{2x}c_3 + \frac{3x^2e^{2x}}{4} - \frac{9\,e^{2x}x}{4} + \frac{35x}{4} + \frac{15x^2}{4} + \frac{5x^3}{6} - 2x\,e^{x} - \frac{x^3e^{x}}{3}$$

Verified OK.

11.46.1 Maple step by step solution

Let's solve

$y''' - 3y'' + 2y' = x^2 e^x + 3\,e^{2x} x + 5x^2$

- Highest derivative means the order of the ODE is 3

  y'''

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = x^2 e^x + 3\,e^{2x} x + 5x^2 + 3y_3(x) - 2y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = x^2 e^x + 3\,e^{2x} x + 5x^2 + 3y_3(x) - 2y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x^2 e^x + 3\,e^{2x} x + 5x^2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x^2 e^x + 3\,e^{2x} x + 5x^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \vec{y}_p(x)$$

☐ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} 1 & e^{x} & \frac{e^{2x}}{4} \\ 0 & e^{x} & \frac{e^{2x}}{2} \\ 0 & e^{x} & e^{2x} \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} 1 & e^{x} & \frac{e^{2x}}{4} \\ 0 & e^{x} & \frac{e^{2x}}{2} \\ 0 & e^{x} & e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & \frac{1}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{3}{2} + 2\,e^{x} - \frac{e^{2x}}{2} & \frac{1}{2} - e^{x} + \frac{e^{2x}}{2} \\ 0 & 2\,e^{x} - e^{2x} & -e^{x} + e^{2x} \\ 0 & 2\,e^{x} - 2\,e^{2x} & -e^{x} + 2\,e^{2x} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}_p^{\prime}(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}^{\prime}(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}^{\prime}(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}^{\prime}(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}^{\prime}(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}^{\prime}(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \frac{35}{4}+\frac{(3x^2-9x+17)\mathrm{e}^{2x}}{4}+\frac{(-x^3-6x-39)\mathrm{e}^{x}}{3}+\frac{5x^3}{6}+\frac{15x^2}{4}+\frac{35x}{4} \\ \frac{35}{4}+\frac{(6x^2-12x+25)\mathrm{e}^{2x}}{4}+\frac{(-x^3-3x^2-6x-45)\mathrm{e}^{x}}{3}+\frac{5x^2}{2}+\frac{15x}{2} \\ \frac{15}{2}+\frac{(6x^2-6x+19)\mathrm{e}^{2x}}{2}+\frac{(-x^3-6x^2-12x-51)\mathrm{e}^{x}}{3}+5x \end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \left[\begin{array}{c} \frac{35}{4}+\frac{(3x^2-9x+17)\mathrm{e}^{2x}}{4}+\frac{(-x^3-6x-39)\mathrm{e}^{x}}{3}+\frac{5x^3}{6}+\frac{15x^2}{4}+\frac{35x}{4} \\ \frac{35}{4}+\frac{(6x^2-12x+25)\mathrm{e}^{2x}}{4}+\frac{(-x^3-3x^2-6x-45)\mathrm{e}^{x}}{3}+\frac{5x^2}{2}+\frac{15x}{2} \\ \frac{15}{2}+\frac{(6x^2-6x+19)\mathrm{e}^{2x}}{2}+\frac{(-x^3-6x^2-12x-51)\mathrm{e}^{x}}{3}+5x \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y = \frac{35}{4} + \frac{(3x^2+c_3-9x+17)\mathrm{e}^{2x}}{4} + \frac{(-x^3+3c_2-6x-39)\mathrm{e}^{x}}{3} + \frac{5x^3}{6} + \frac{15x^2}{4} + \frac{35x}{4} + c_1$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _a^2*exp(_a)+3*exp(2*_a)*_a+5
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   <- double symmetry of the form [xi=0, eta=F(x)] successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)+2*diff(y(x),x)=x^2*exp(x)+3*x*exp(2*x)+5*x^2,y(x), si
```

$$y(x) = \frac{(6x^2 + 4c_1 - 18x + 21)\,e^{2x}}{8} + \frac{(-x^3 + 3c_2 - 6x + 6)\,e^{x}}{3} + \frac{5x^3}{6} + \frac{15x^2}{4} + \frac{35x}{4} + c_3$$

✓ Solution by Mathematica

Time used: 0.885 (sec). Leaf size: 67

```
DSolve[y'''[x]-3*y''[x]+2*y'[x]==x^2*Exp[x]+3*x*Exp[2*x]+5*x^2,y[x],x,IncludeSingularSolutio
```

$$y(x) \to \frac{5x^3}{6} + e^x\left(-\frac{x^3}{3} - 2x + c_1\right) + \frac{15x^2}{4} + \frac{1}{8}e^{2x}(6x^2 - 18x + 21 + 4c_2) + \frac{35x}{4} + c_3$$

11.47 problem 47

Internal problem ID [11821]
Internal file name [`OUTPUT/11830_Thursday_April_11_2024_08_51_26_PM_89373344/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 47.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_3rd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''' - 6y'' + 12y' - 8y = e^{2x}x + e^{3x}x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 6y'' + 12y' - 8y = 0$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + c_2 e^{2x} x + x^2 e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= e^{2x} \\
y_2 &= e^{2x} x \\
y_3 &= x^2 e^{2x}
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - 6y'' + 12y' - 8y = e^{2x} x + e^{3x} x^2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x} x + e^{3x} x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x} x, e^{2x}\}, \{x\, e^{3x}, e^{3x} x^2, e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x^2 e^{2x}, e^{2x} x, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{2x}, e^{2x} x\}, \{x\, e^{3x}, e^{3x} x^2, e^{3x}\}]$$

Since $e^{2x} x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{2x}, x^3 e^{2x}\}, \{x\, e^{3x}, e^{3x} x^2, e^{3x}\}]$$

Since $x^2 e^{2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3 e^{2x}, e^{2x} x^4\}, \{x\, e^{3x}, e^{3x} x^2, e^{3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^3 e^{2x} + A_2 e^{2x} x^4 + A_3 x\, e^{3x} + A_4 e^{3x} x^2 + A_5 e^{3x}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&6A_4 e^{3x} x + 24A_2 e^{2x} x + A_3 x\,e^{3x} + A_4 e^{3x} x^2 + A_5 e^{3x} + 3A_3 e^{3x} + 6A_4 e^{3x} + 6A_1 e^{2x}\\ &= e^{2x} x + e^{3x} x^2\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{24}, A_3 = -6, A_4 = 1, A_5 = 12\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{2x} x^4}{24} - 6x\,e^{3x} + e^{3x} x^2 + 12\,e^{3x}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= \left(c_1 e^{2x} + c_2 e^{2x} x + x^2 e^{2x} c_3\right) + \left(\frac{e^{2x} x^4}{24} - 6x\,e^{3x} + e^{3x} x^2 + 12\,e^{3x}\right)\end{aligned}$$

Which simplifies to

$$y = e^{2x}\left(c_3 x^2 + c_2 x + c_1\right) + \frac{e^{2x} x^4}{24} - 6x\,e^{3x} + e^{3x} x^2 + 12\,e^{3x}$$

Summary

The solution(s) found are the following

$$y = e^{2x}\left(c_3 x^2 + c_2 x + c_1\right) + \frac{e^{2x} x^4}{24} - 6x\,e^{3x} + e^{3x} x^2 + 12\,e^{3x} \tag{1}$$

Verification of solutions

$$y = e^{2x}\left(c_3 x^2 + c_2 x + c_1\right) + \frac{e^{2x} x^4}{24} - 6x\,e^{3x} + e^{3x} x^2 + 12\,e^{3x}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+12*diff(y(x),x)-8*y(x)=x*exp(2*x)+x^2*exp(3*x),y(x),
```

$$y(x) = \frac{(x^4 + 24c_3x^2 + 24c_2x + 24c_1)\,\mathrm{e}^{2x}}{24} + \mathrm{e}^{3x}(x^2 - 6x + 12)$$

✓ Solution by Mathematica

Time used: 0.084 (sec). Leaf size: 47

```
DSolve[y'''[x]-6*y''[x]+12*y'[x]-8*y[x]==x*Exp[2*x]+x^2*Exp[3*x],y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{1}{24}e^{2x}\left(x^4 + 24e^x(x^2 - 6x + 12) + 24c_3x^2 + 24c_2x + 24c_1\right)$$

11.48 problem 48

Internal problem ID [11822]
Internal file name [`OUTPUT/11831_Thursday_April_11_2024_08_51_27_PM_70264925/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 48.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$y'''' + 3y''' + 4y'' + 3y' + y = x^2 e^{-x} + 3\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 3y''' + 4y'' + 3y' + y = 0$$

The characteristic equation is

$$\lambda^4 + 3\lambda^3 + 4\lambda^2 + 3\lambda + 1 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2}-\frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2}+\frac{i\sqrt{3}}{2} \\
\lambda_3 &= -1 \\
\lambda_4 &= -1
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= x\,\mathrm{e}^{-x} \\
y_3 &= \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} \\
y_4 &= \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 3y''' + 4y'' + 3y' + y = x^2 \mathrm{e}^{-x} + 3\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{-x} & x\,e^{-x} & e^{-\frac{(1+i\sqrt{3})x}{2}} & e^{\frac{(i\sqrt{3}-1)x}{2}} \\ -e^{-x} & e^{-x}(1-x) & -\frac{(1+i\sqrt{3})e^{-\frac{(1+i\sqrt{3})x}{2}}}{2} & \frac{(i\sqrt{3}-1)e^{\frac{(i\sqrt{3}-1)x}{2}}}{2} \\ e^{-x} & e^{-x}(x-2) & \frac{(1+i\sqrt{3})^2 e^{-\frac{(1+i\sqrt{3})x}{2}}}{4} & \frac{(i\sqrt{3}-1)^2 e^{\frac{(i\sqrt{3}-1)x}{2}}}{4} \\ -e^{-x} & e^{-x}(-x+3) & e^{-\frac{(1+i\sqrt{3})x}{2}} & e^{\frac{(i\sqrt{3}-1)x}{2}} \end{bmatrix}$$

$$|W| = ie^{-2x}\sqrt{3}\,e^{\frac{(i\sqrt{3}-1)x}{2}}\,e^{-\frac{(1+i\sqrt{3})x}{2}}$$

The determinant simplifies to

$$|W| = i\sqrt{3}\,e^{-3x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det\begin{bmatrix} x\,e^{-x} & e^{-\frac{(1+i\sqrt{3})x}{2}} & e^{\frac{(i\sqrt{3}-1)x}{2}} \\ e^{-x}(1-x) & -\frac{(1+i\sqrt{3})e^{-\frac{(1+i\sqrt{3})x}{2}}}{2} & \frac{(i\sqrt{3}-1)e^{\frac{(i\sqrt{3}-1)x}{2}}}{2} \\ e^{-x}(x-2) & \frac{(1+i\sqrt{3})^2 e^{-\frac{(1+i\sqrt{3})x}{2}}}{4} & \frac{(i\sqrt{3}-1)^2 e^{\frac{(i\sqrt{3}-1)x}{2}}}{4} \end{bmatrix} \\ &= i\sqrt{3}\,e^{-2x}(x-1) \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det\begin{bmatrix} e^{-x} & e^{-\frac{(1+i\sqrt{3})x}{2}} & e^{\frac{(i\sqrt{3}-1)x}{2}} \\ -e^{-x} & -\frac{(1+i\sqrt{3})e^{-\frac{(1+i\sqrt{3})x}{2}}}{2} & \frac{(i\sqrt{3}-1)e^{\frac{(i\sqrt{3}-1)x}{2}}}{2} \\ e^{-x} & \frac{(1+i\sqrt{3})^2 e^{-\frac{(1+i\sqrt{3})x}{2}}}{4} & \frac{(i\sqrt{3}-1)^2 e^{\frac{(i\sqrt{3}-1)x}{2}}}{4} \end{bmatrix} \\ &= i\sqrt{3}\,e^{-2x} \end{aligned}$$

$$W_3(x) = \det \begin{bmatrix} e^{-x} & x\,e^{-x} & e^{\frac{\left(i\sqrt{3}-1\right)x}{2}} \\ -e^{-x} & e^{-x}(1-x) & \frac{\left(i\sqrt{3}-1\right)e^{\frac{\left(i\sqrt{3}-1\right)x}{2}}}{2} \\ e^{-x} & e^{-x}(x-2) & \frac{\left(i\sqrt{3}-1\right)^2 e^{\frac{\left(i\sqrt{3}-1\right)x}{2}}}{4} \end{bmatrix}$$

$$= \frac{e^{\frac{x\left(i\sqrt{3}-5\right)}{2}}\left(i\sqrt{3}-1\right)}{2}$$

$$W_4(x) = \det \begin{bmatrix} e^{-x} & x\,e^{-x} & e^{-\frac{\left(1+i\sqrt{3}\right)x}{2}} \\ -e^{-x} & e^{-x}(1-x) & -\frac{\left(1+i\sqrt{3}\right)e^{-\frac{\left(1+i\sqrt{3}\right)x}{2}}}{2} \\ e^{-x} & e^{-x}(x-2) & \frac{\left(1+i\sqrt{3}\right)^2 e^{-\frac{\left(1+i\sqrt{3}\right)x}{2}}}{4} \end{bmatrix}$$

$$= -\frac{e^{-\frac{x\left(i\sqrt{3}+5\right)}{2}}\left(1+i\sqrt{3}\right)}{2}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^3 \int \frac{\left(x^2 e^{-x} + 3\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(i\sqrt{3}\,e^{-2x}(x-1)\right)}{(1)\left(i\sqrt{3}\,e^{-3x}\right)}\,dx \\
&= -\int \frac{i\left(x^2 e^{-x} + 3\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right)\sqrt{3}\,e^{-2x}(x-1)}{i\sqrt{3}\,e^{-3x}}\,dx \\
&= -\int \left((x-1)\left(3\,e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + x^2\right)\right)dx \\
&= -3\left(\frac{x}{2}+\frac{1}{2}\right)e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + 3\left(-\frac{\sqrt{3}\,x}{2}+\frac{\sqrt{3}}{2}\right)e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) - \frac{x^4}{4} + \frac{3\,e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{3\sqrt{3}\,e^{\ldots}}{\ldots} \\
&= -3\left(\frac{x}{2}+\frac{1}{2}\right)e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + 3\left(-\frac{\sqrt{3}\,x}{2}+\frac{\sqrt{3}}{2}\right)e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) - \frac{x^4}{4} + \frac{3\,e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{3\sqrt{3}\,e^{\ldots}}{\ldots}
\end{aligned}$$

$$U_2 = (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx$$

$$= (-1)^2 \int \frac{\left(x^2 e^{-x} + 3\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(i\sqrt{3}\,e^{-2x}\right)}{(1)\left(i\sqrt{3}\,e^{-3x}\right)}\,dx$$

$$= \int \frac{i\left(x^2 e^{-x} + 3\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right)\sqrt{3}\,e^{-2x}}{i\sqrt{3}\,e^{-3x}}\,dx$$

$$= \int \left(3\,e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + x^2\right)dx$$

$$= \frac{x^3}{3} + \frac{3\,e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{3\sqrt{3}\,e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}$$

$$U_3 = (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx$$

$$= (-1)^1 \int \frac{\left(x^2 e^{-x} + 3\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right)\left(\frac{e^{\frac{x\left(i\sqrt{3}-5\right)}{2}}\left(i\sqrt{3}-1\right)}{2}\right)}{(1)\left(i\sqrt{3}\,e^{-3x}\right)}\,dx$$

$$= -\int \frac{\frac{\left(x^2 e^{-x}+3\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right)e^{\frac{x\left(i\sqrt{3}-5\right)}{2}}\left(i\sqrt{3}-1\right)}{2}}{i\sqrt{3}\,e^{-3x}}\,dx$$

$$= -\int \left(\frac{\sqrt{3}\,e^{\frac{i\sqrt{3}\,x}{2}}\left(\sqrt{3}+i\right)\left(e^{-\frac{x}{2}}x^2 + 3\cos\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{6}\right)dx$$

$$= -\frac{3x}{4} - \frac{i\sqrt{3}\,x}{4} + \frac{ie^{i\sqrt{3}\,x}\sqrt{3}}{4} - \frac{e^{i\sqrt{3}\,x}}{4} + \frac{\left(i\sqrt{3}\,x + i\sqrt{3} + x^2 + x - 1\right)\left(1 + i\sqrt{3}\right)\sqrt{3}\left(\sqrt{3}+i\right)e^{\frac{\left(i\sqrt{3}-1\right)x}{2}}}{12}$$

$$= -\frac{3x}{4} - \frac{i\sqrt{3}\,x}{4} + \frac{ie^{i\sqrt{3}\,x}\sqrt{3}}{4} - \frac{e^{i\sqrt{3}\,x}}{4} + \frac{\left(i\sqrt{3}\,x + i\sqrt{3} + x^2 + x - 1\right)\left(1 + i\sqrt{3}\right)\sqrt{3}\left(\sqrt{3}+i\right)e^{\frac{\left(i\sqrt{3}-1\right)x}{2}}}{12}$$

$$U_4 = (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)}\, dx$$

$$= (-1)^0 \int \frac{\left(x^2 \mathrm{e}^{-x} + 3\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) \left(-\frac{\mathrm{e}^{-\frac{x\left(i\sqrt{3}+5\right)}{2}} \left(1+i\sqrt{3}\right)}{2}\right)}{(1)\left(i\sqrt{3}\,\mathrm{e}^{-3x}\right)}\, dx$$

$$= \int \frac{-\frac{\left(x^2 \mathrm{e}^{-x}+3\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) \mathrm{e}^{-\frac{x\left(i\sqrt{3}+5\right)}{2}} \left(1+i\sqrt{3}\right)}{2}}{i\sqrt{3}\,\mathrm{e}^{-3x}}\, dx$$

$$= \int \left(-\frac{\sqrt{3}\,\mathrm{e}^{-\frac{i\sqrt{3}\,x}{2}} \left(-i+\sqrt{3}\right) \left(\mathrm{e}^{-\frac{x}{2}} x^2 + 3\cos\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{6}\right) dx$$

$$= -\frac{\left(-i\sqrt{3}\,x - i\sqrt{3} + x^2 + x - 1\right)\left(i\sqrt{3}-1\right)\sqrt{3}\left(-i+\sqrt{3}\right)\mathrm{e}^{-\frac{\left(1+i\sqrt{3}\right)x}{2}}}{12} - \frac{3x}{4} + \frac{i\sqrt{3}\,x}{4} - \frac{i\mathrm{e}^{-i\sqrt{3}\,x}\sqrt{3}}{4} -$$

$$= -\frac{\left(-i\sqrt{3}\,x - i\sqrt{3} + x^2 + x - 1\right)\left(i\sqrt{3}-1\right)\sqrt{3}\left(-i+\sqrt{3}\right)\mathrm{e}^{-\frac{\left(1+i\sqrt{3}\right)x}{2}}}{12} - \frac{3x}{4} + \frac{i\sqrt{3}\,x}{4} - \frac{i\mathrm{e}^{-i\sqrt{3}\,x}\sqrt{3}}{4} -$$

Now that all the U_i functions have been determined, the particular solution is found from

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Hence

$$y_p = \left(-3\left(\frac{x}{2}+\frac{1}{2}\right)\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + 3\left(-\frac{\sqrt{3}\,x}{2}+\frac{\sqrt{3}}{2}\right)\mathrm{e}^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) - \frac{x^4}{4} + \frac{3\,\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{3\sqrt{3}}{}\right.$$

$$+ \left(\frac{x^3}{3} + \frac{3\,\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{3\sqrt{3}\,\mathrm{e}^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right)\left(x\,\mathrm{e}^{-x}\right)$$

$$+ \left(-\frac{3x}{4} - \frac{i\sqrt{3}\,x}{4} + \frac{i\mathrm{e}^{i\sqrt{3}\,x}\sqrt{3}}{4} - \frac{\mathrm{e}^{i\sqrt{3}\,x}}{4} + \frac{\left(i\sqrt{3}\,x + i\sqrt{3} + x^2 + x - 1\right)\left(1+i\sqrt{3}\right)\sqrt{3}\left(\sqrt{3}+i\right)\mathrm{e}^{\frac{\left(i\sqrt{3}-1\right)x}{2}}}{12}\right.$$

$$+ \left(-\frac{\left(-i\sqrt{3}\,x - i\sqrt{3} + x^2 + x - 1\right)\left(i\sqrt{3}-1\right)\sqrt{3}\left(-i+\sqrt{3}\right)\mathrm{e}^{-\frac{\left(1+i\sqrt{3}\right)x}{2}}}{12} - \frac{3x}{4} + \frac{i\sqrt{3}\,x}{4} - \frac{i\mathrm{e}^{-i\sqrt{3}\,x}\sqrt{3}}{4}\right.$$

Therefore the particular solution is

$$y_p = \frac{(-3x-1)\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sqrt{3}\,\mathrm{e}^{-\frac{x}{2}}(-5+x) \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\mathrm{e}^{-x}\left(x^4+4x^3-24x-24\right)}{12}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \mathrm{e}^{\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 + \mathrm{e}^{\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_4\right) + \left(\frac{(-3x-1)\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right. \\
&\quad \left. - \frac{\sqrt{3}\,\mathrm{e}^{-\frac{x}{2}}(-5+x)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\mathrm{e}^{-x}(x^4+4x^3-24x-24)}{12}\right)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y &= \mathrm{e}^{-\frac{\left(1+i\sqrt{3}\right)x}{2}} c_3 + \mathrm{e}^{\frac{\left(i\sqrt{3}-1\right)x}{2}} c_4 + \mathrm{e}^{-x}(c_2 x + c_1) + \frac{(-3x-1)\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\
&\quad - \frac{\sqrt{3}\,\mathrm{e}^{-\frac{x}{2}}(-5+x)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\mathrm{e}^{-x}(x^4+4x^3-24x-24)}{12}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= \mathrm{e}^{-\frac{\left(1+i\sqrt{3}\right)x}{2}} c_3 + \mathrm{e}^{\frac{\left(i\sqrt{3}-1\right)x}{2}} c_4 + \mathrm{e}^{-x}(c_2 x + c_1) + \frac{(-3x-1)\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\
&\quad - \frac{\sqrt{3}\,\mathrm{e}^{-\frac{x}{2}}(-5+x)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\mathrm{e}^{-x}(x^4+4x^3-24x-24)}{12}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= \mathrm{e}^{-\frac{\left(1+i\sqrt{3}\right)x}{2}} c_3 + \mathrm{e}^{\frac{\left(i\sqrt{3}-1\right)x}{2}} c_4 + \mathrm{e}^{-x}(c_2 x + c_1) + \frac{(-3x-1)\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\
&\quad - \frac{\sqrt{3}\,\mathrm{e}^{-\frac{x}{2}}(-5+x)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\mathrm{e}^{-x}(x^4+4x^3-24x-24)}{12}
\end{aligned}
$$

Verified OK.

11.48.1 Maple step by step solution

Let's solve

$y'''' + 3y''' + 4y'' + 3y' + y = x^2 e^{-x} + 3\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ○ Define new variable $y_1(x)$

  $y_1(x) = y$

  ○ Define new variable $y_2(x)$

  $y_2(x) = y'$

  ○ Define new variable $y_3(x)$

  $y_3(x) = y''$

  ○ Define new variable $y_4(x)$

  $y_4(x) = y'''$

  ○ Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = x^2 e^{-x} + 3\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) - 4y_3(x) - 3y_2(x) - y_1(x) - 3y_4(x)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = x^2 e^{-x} + 3\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) - 4y_3(x) - 3y_2(\right.$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -3 & -4 & -3 \end{bmatrix} \cdot \overrightarrow{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x^2 \mathrm{e}^{-x} + 3\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}$$

- Define the forcing function

$$\overrightarrow{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x^2 \mathrm{e}^{-x} + 3\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -3 & -4 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right], \left[-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right], \left[-\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue -1

$$\overrightarrow{y}_1(x) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\overrightarrow{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, a

$$\overrightarrow{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the x multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\overrightarrow{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda x}\overrightarrow{v} = \mathrm{e}^{\lambda x}\left(\lambda x\overrightarrow{v} + A \cdot \overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \overrightarrow{p} + \overrightarrow{v} = A \cdot \overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -3 & -4 & -3 \end{array}\right] - (-1) \cdot \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]\right) \cdot \overrightarrow{p} = \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

$$\vec{y}_2(x) = \mathrm{e}^{-x} \cdot \left(x \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \cdot \begin{bmatrix} \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^3} \\ \frac{1}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-\frac{x}{2}}\cdot\left[\begin{array}{c}\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{3}}\\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}}\\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}\\ \cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{3}(x)=e^{-\frac{x}{2}}\cdot\left[\begin{array}{c}\cos\left(\frac{\sqrt{3}\,x}{2}\right)\\ -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\\ \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\\ \cos\left(\frac{\sqrt{3}\,x}{2}\right)\end{array}\right],\vec{y}_{4}(x)=e^{-\frac{x}{2}}\cdot\left[\begin{array}{c}-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\\ -\frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\\ \frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right)\end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x)=c_1\vec{y}_1(x)+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)+\vec{y}_p(x)$

☐ Fundamental matrix

  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\left[\begin{array}{cccc}-e^{-x} & e^{-x}(-x-1) & e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\\ e^{-x} & x\,e^{-x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(-\frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right)\\ -e^{-x} & -x\,e^{-x} & e^{-\frac{x}{2}}\left(\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) & e^{-\frac{x}{2}}\left(\frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right)\\ e^{-x} & x\,e^{-x} & e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\end{array}\right.$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

$\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -e^{-x} & e^{-x}(-x-1) & e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ e^{-x} & x\,e^{-x} & e^{-\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & e^{-\frac{x}{2}}\left(-\frac{\sqrt{3}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ -e^{-x} & -x\,e^{-x} & e^{-\frac{x}{2}}\left(\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} - \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) & e^{-\frac{x}{2}}\left(\frac{\sqrt{3}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ e^{-x} & x\,e^{-x} & e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} (1+x)\,e^{-x} & -\frac{e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} + 2x\,e^{-x} - e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) + e^{-x} & -e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3} + \\ -x\,e^{-x} & \frac{2\,e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} - 2x\,e^{-x} + e^{-x} & -e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) + e^{-} \\ x\,e^{-x} & e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \frac{e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} + e^{-x}(2x-1) & 2\,e^{-\frac{x}{2}}\cos\Big(\\ -x\,e^{-x} & e^{-x} - 2x\,e^{-x} - e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \frac{e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{3} & e^{-x} - 2x\,e^{-x} - e^{-\frac{x}{2}}\,\mathrm{c} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} -\sqrt{3}\,e^{-\frac{x}{2}}\left(x-\frac{17}{6}\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)-\frac{3\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{e^{-x}(x^4-12x^2-42x+18)}{12} \\ \frac{(-3x+10)e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sqrt{3}\,e^{-\frac{x}{2}}\left(x-\frac{10}{3}\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}-\frac{e^{-x}(x^4-4x^3-12x^2-18x+60)}{12} \\ \frac{(3x-10)e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\left(x+\frac{2}{3}\right)\sqrt{3}\,e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{e^{-x}(x^4-4x^3+6x+60)}{12} \\ -\sqrt{3}\,e^{-\frac{x}{2}}\left(x-\frac{4}{3}\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)+3\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\frac{e^{-x}(x^4-4x^3+12x^2+6x+36)}{12} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1(x)+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+c_4\vec{y}_4(x)+\left[\begin{array}{c} -\sqrt{3}\,e^{-\frac{x}{2}}\left(x-\frac{17}{6}\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)-\frac{3\,e^{-\frac{x}{2}}\cos\left(\right.}{2} \\ \frac{(-3x+10)e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sqrt{3}\,e^{-\frac{x}{2}}\left(x-\frac{10}{3}\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right.}{2} \\ \frac{(3x-10)e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\left(x+\frac{2}{3}\right)\sqrt{3}\,e^{-\frac{x}{2}}\sin\left(\right.}{2} \\ -\sqrt{3}\,e^{-\frac{x}{2}}\left(x-\frac{4}{3}\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)+3\,e^{-\frac{x}{2}}\cos\left(\right. \end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y = -\left(\left(x-\frac{17}{6}\right)\sqrt{3}+c_4\right)e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)+\frac{(12c_3-18)e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{12}+\frac{(x^4-12x^2+(-12c_2-42)x-12c_1-12c_2+18}{12}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 73

```
dsolve(diff(y(x),x$4)+3*diff(y(x),x$3)+4*diff(y(x),x$2)+3*diff(y(x),x)+y(x)=x^2*exp(-x)+3*ex
```

$$y(x) = -\frac{3\,e^{-\frac{x}{2}}\left(x-\frac{2c_3}{3}+\frac{1}{3}\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}}\left((x-5)\sqrt{3}-2c_4\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\left(-24+x^4+4x^3+12(-2+c_2)\,x+12c_1\right)e^{-x}}{12}$$

✓ Solution by Mathematica

Time used: 2.054 (sec). Leaf size: 104

```
DSolve[y''''[x]+3*y'''[x]+4*y''[x]+3*y'[x]+y[x]==x^2*Exp[-x]+3*Exp[-x/2]*Cos[Sqrt[3]/2*x],y[
```

$$y(x) \to \frac{1}{12}e^{-x}\left(x^4+4x^3-24x+12c_4x-6e^{x/2}(3x+1-2c_2)\cos\left(\frac{\sqrt{3}x}{2}\right) - 6e^{x/2}\left(\sqrt{3}x-5\sqrt{3}-2c_1\right)\sin\left(\frac{\sqrt{3}x}{2}\right)-24+12c_3\right)$$

11.49 problem 49

Internal problem ID [11823]
Internal file name [`OUTPUT/11832_Thursday_April_11_2024_08_51_29_PM_74065651/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 49.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - 16y = x^2 \sin(2x) + e^{2x}x^4}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - 16y = 0$$

The characteristic equation is

$$\lambda^4 - 16 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$
$$\lambda_3 = 2i$$
$$\lambda_4 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^{2x} + e^{2ix} c_3 + e^{-2ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-2x} \\ y_2 &= e^{2x} \\ y_3 &= e^{2ix} \\ y_4 &= e^{-2ix} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - 16y = x^2 \sin(2x) + e^{2x} x^4$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{-2x} & e^{2x} & e^{2ix} & e^{-2ix} \\ -2 e^{-2x} & 2 e^{2x} & 2i e^{2ix} & -2i e^{-2ix} \\ 4 e^{-2x} & 4 e^{2x} & -4 e^{2ix} & -4 e^{-2ix} \\ -8 e^{-2x} & 8 e^{2x} & -8i e^{2ix} & 8i e^{-2ix} \end{bmatrix}$$

$$|W| = -1024i e^{-2x} e^{2x} e^{2ix} e^{-2ix}$$

The determinant simplifies to

$$|W| = -1024i$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} e^{2x} & e^{2ix} & e^{-2ix} \\ 2\,e^{2x} & 2ie^{2ix} & -2ie^{-2ix} \\ 4\,e^{2x} & -4\,e^{2ix} & -4\,e^{-2ix} \end{bmatrix} \\ &= -32ie^{2x} \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} e^{-2x} & e^{2ix} & e^{-2ix} \\ -2\,e^{-2x} & 2ie^{2ix} & -2ie^{-2ix} \\ 4\,e^{-2x} & -4\,e^{2ix} & -4\,e^{-2ix} \end{bmatrix} \\ &= -32ie^{-2x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} e^{-2x} & e^{2x} & e^{-2ix} \\ -2\,e^{-2x} & 2\,e^{2x} & -2ie^{-2ix} \\ 4\,e^{-2x} & 4\,e^{2x} & -4\,e^{-2ix} \end{bmatrix} \\ &= -32\,e^{-2ix} \end{aligned}$$

$$\begin{aligned} W_4(x) &= \det \begin{bmatrix} e^{-2x} & e^{2x} & e^{2ix} \\ -2\,e^{-2x} & 2\,e^{2x} & 2ie^{2ix} \\ 4\,e^{-2x} & 4\,e^{2x} & -4\,e^{2ix} \end{bmatrix} \\ &= -32\,e^{2ix} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\
&= (-1)^3 \int \frac{\left(x^2 \sin(2x) + \mathrm{e}^{2x} x^4\right)\left(-32i\mathrm{e}^{2x}\right)}{(1)\,(-1024i)}\, dx \\
&= -\int \frac{-32i(x^2 \sin(2x) + \mathrm{e}^{2x} x^4)\, \mathrm{e}^{2x}}{-1024i}\, dx \\
&= -\int \left(\frac{x^2(x^2\mathrm{e}^{2x} + \sin(2x))\, \mathrm{e}^{2x}}{32}\right) dx \\
&= -\frac{x^4\mathrm{e}^{4x}}{128} + \frac{x^3\mathrm{e}^{4x}}{128} - \frac{3x^2\mathrm{e}^{4x}}{512} + \frac{3x\,\mathrm{e}^{4x}}{1024} - \frac{3\,\mathrm{e}^{4x}}{4096} - \frac{\left(-2x^2 + 2x - \frac{1}{2}\right)\mathrm{e}^{2x}\cos(2x)}{256} - \frac{\left(2x^2 - \frac{1}{2}\right)\mathrm{e}^{2x}\sin(2x)}{256} \\
&= -\frac{x^4\mathrm{e}^{4x}}{128} + \frac{x^3\mathrm{e}^{4x}}{128} - \frac{3x^2\mathrm{e}^{4x}}{512} + \frac{3x\,\mathrm{e}^{4x}}{1024} - \frac{3\,\mathrm{e}^{4x}}{4096} - \frac{\left(-2x^2 + 2x - \frac{1}{2}\right)\mathrm{e}^{2x}\cos(2x)}{256} - \frac{\left(2x^2 - \frac{1}{2}\right)\mathrm{e}^{2x}\sin(2x)}{256}
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\, dx \\
&= (-1)^2 \int \frac{\left(x^2 \sin(2x) + \mathrm{e}^{2x} x^4\right)\left(-32i\mathrm{e}^{-2x}\right)}{(1)\,(-1024i)}\, dx \\
&= \int \frac{-32i(x^2 \sin(2x) + \mathrm{e}^{2x} x^4)\, \mathrm{e}^{-2x}}{-1024i}\, dx \\
&= \int \left(\frac{x^2(x^2 + \mathrm{e}^{-2x}\sin(2x))}{32}\right) dx \\
&= \frac{x^5}{160} + \frac{\left(-\frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{16}\right)\mathrm{e}^{-2x}\cos(2x)}{32} + \frac{\left(-\frac{x^2}{4} + \frac{1}{16}\right)\mathrm{e}^{-2x}\sin(2x)}{32} \\
&= \frac{x^5}{160} + \frac{\left(-\frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{16}\right)\mathrm{e}^{-2x}\cos(2x)}{32} + \frac{\left(-\frac{x^2}{4} + \frac{1}{16}\right)\mathrm{e}^{-2x}\sin(2x)}{32}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^1 \int \frac{\left(x^2 \sin\left(2x\right) + \mathrm{e}^{2x} x^4\right)\left(-32\,\mathrm{e}^{-2ix}\right)}{(1)\left(-1024i\right)}\, dx \\
&= -\int \frac{-32(x^2 \sin\left(2x\right) + \mathrm{e}^{2x} x^4)\,\mathrm{e}^{-2ix}}{-1024i}\, dx \\
&= -\int \left(-\frac{ix^2(x^2 \mathrm{e}^{2x} + \sin\left(2x\right))\,\mathrm{e}^{-2ix}}{32}\right) dx \\
&= \left(-\frac{1}{1024} + \frac{i}{1024}\right)\left(8x^4 - 8ix^3 - 8x^3 + 12ix^2 - 6ix + 6x - 3\right)\mathrm{e}^{(2-2i)x} + \frac{x^3}{192} - \frac{i(8x^2 - 4ix - 1)\,\mathrm{e}^{-4}}{2048} \\
&= \left(-\frac{1}{1024} + \frac{i}{1024}\right)\left(8x^4 - 8ix^3 - 8x^3 + 12ix^2 - 6ix + 6x - 3\right)\mathrm{e}^{(2-2i)x} + \frac{x^3}{192} - \frac{i(8x^2 - 4ix - 1)\,\mathrm{e}^{-4}}{2048}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\, dx \\
&= (-1)^0 \int \frac{\left(x^2 \sin\left(2x\right) + \mathrm{e}^{2x} x^4\right)\left(-32\,\mathrm{e}^{2ix}\right)}{(1)\left(-1024i\right)}\, dx \\
&= \int \frac{-32(x^2 \sin\left(2x\right) + \mathrm{e}^{2x} x^4)\,\mathrm{e}^{2ix}}{-1024i}\, dx \\
&= \int \left(-\frac{ix^2(x^2 \mathrm{e}^{2x} + \sin\left(2x\right))\,\mathrm{e}^{2ix}}{32}\right) dx \\
&= \frac{x^3}{192} - \frac{i\left(\frac{1}{64} - \frac{1}{16}ix - \frac{1}{8}x^2\right)\mathrm{e}^{4ix}}{32} + \left(-\frac{1}{1024} - \frac{i}{1024}\right)\left(8x^4 + 8ix^3 - 8x^3 - 12ix^2 + 6ix + 6x - 3\right)\mathrm{e}^{(2+} \\
&= \frac{x^3}{192} - \frac{i\left(\frac{1}{64} - \frac{1}{16}ix - \frac{1}{8}x^2\right)\mathrm{e}^{4ix}}{32} + \left(-\frac{1}{1024} - \frac{i}{1024}\right)\left(8x^4 + 8ix^3 - 8x^3 - 12ix^2 + 6ix + 6x - 3\right)\mathrm{e}^{(2+}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$y_p = \left(-\frac{x^4 e^{4x}}{128} + \frac{x^3 e^{4x}}{128} - \frac{3x^2 e^{4x}}{512} + \frac{3x\, e^{4x}}{1024} - \frac{3\, e^{4x}}{4096} - \frac{\left(-2x^2 + 2x - \frac{1}{2}\right) e^{2x} \cos(2x)}{256} - \frac{\left(2x^2 - \frac{1}{2}\right) e^{2x} \sin(2x)}{256}\right.$$
$$+ \left(\frac{x^5}{160} + \frac{\left(-\frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{16}\right) e^{-2x} \cos(2x)}{32} + \frac{\left(-\frac{x^2}{4} + \frac{1}{16}\right) e^{-2x} \sin(2x)}{32}\right) \left(e^{2x}\right)$$
$$+ \left(\left(-\frac{1}{1024} + \frac{i}{1024}\right) \left(8x^4 - 8ix^3 - 8x^3 + 12ix^2 - 6ix + 6x - 3\right) e^{(2-2i)x} + \frac{x^3}{192} - \frac{i(8x^2 - 4ix - 1)\, e^{-}}{2048}\right.$$
$$+ \left(\frac{x^3}{192} - \frac{i\left(\frac{1}{64} - \frac{1}{16}ix - \frac{1}{8}x^2\right) e^{4ix}}{32} + \left(-\frac{1}{1024} - \frac{i}{1024}\right) \left(8x^4 + 8ix^3 - 8x^3 - 12ix^2 + 6ix + 6x - 3\right) e^{(2}\right.$$

Therefore the particular solution is

$$y_p = \frac{(128x^5 - 480x^4 + 800x^3 - 600x^2 + 60x + 105)\, e^{2x}}{20480} + \frac{(8x^3 - 15x)\cos(2x)}{768} + \frac{(-24x^2 + 5)\sin(2x)}{1024}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{2x} + e^{2ix} c_3 + e^{-2ix} c_4\right) \\ &\quad + \left(\frac{(128x^5 - 480x^4 + 800x^3 - 600x^2 + 60x + 105)\, e^{2x}}{20480} + \frac{(8x^3 - 15x)\cos(2x)}{768} \right. \\ &\quad \left. + \frac{(-24x^2 + 5)\sin(2x)}{1024}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$\begin{aligned} y = {} & c_1 e^{-2x} + c_2 e^{2x} + e^{2ix} c_3 + e^{-2ix} c_4 \\ & + \frac{(128x^5 - 480x^4 + 800x^3 - 600x^2 + 60x + 105)\, e^{2x}}{20480} \\ & + \frac{(8x^3 - 15x)\cos(2x)}{768} + \frac{(-24x^2 + 5)\sin(2x)}{1024} \end{aligned} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 e^{-2x} + c_2 e^{2x} + e^{2ix} c_3 + e^{-2ix} c_4 + \frac{(128x^5 - 480x^4 + 800x^3 - 600x^2 + 60x + 105)\, e^{2x}}{20480} + \frac{(8x^3 - 15x)\cos(2x)}{768} + \frac{(-24x^2 + 5)\sin(2x)}{1024}$$

Verified OK.

11.49.1 Maple step by step solution

Let's solve

$y'''' - 16y = x^2 \sin(2x) + e^{2x}x^4$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = e^{2x}x^4 + x^2 \sin(2x) + 16y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = e^{2x}x^4 + x^2 \sin(2x) + 16y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x^2 \sin(2x) + e^{2x}x^4 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x^2 \sin(2x) + e^{2x} x^4 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[2\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} -\frac{1}{8} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = e^{2x} \cdot \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \left[\begin{array}{c} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-2\,\mathrm{I}x} \cdot \left[\begin{array}{c} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(2x) - \mathrm{I}\sin(2x)) \cdot \left[\begin{array}{c} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\begin{bmatrix} -\frac{I}{8}(\cos(2x) - I\sin(2x)) \\ -\frac{\cos(2x)}{4} + \frac{I\sin(2x)}{4} \\ \frac{I}{2}(\cos(2x) - I\sin(2x)) \\ \cos(2x) - I\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -\frac{e^{-2x}}{8} & \frac{e^{2x}}{8} & -\frac{\sin(2x)}{8} & -\frac{\cos(2x)}{8} \\ \frac{e^{-2x}}{4} & \frac{e^{2x}}{4} & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} \\ -\frac{e^{-2x}}{2} & \frac{e^{2x}}{2} & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} \\ e^{-2x} & e^{2x} & \cos(2x) & -\sin(2x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -\frac{e^{-2x}}{8} & \frac{e^{2x}}{8} & -\frac{\sin(2x)}{8} & -\frac{\cos(2x)}{8} \\ \frac{e^{-2x}}{4} & \frac{e^{2x}}{4} & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} \\ -\frac{e^{-2x}}{2} & \frac{e^{2x}}{2} & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} \\ e^{-2x} & e^{2x} & \cos(2x) & -\sin(2x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{8} & \frac{1}{8} & 0 & -\frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} + \frac{e^{2x}}{4} + \frac{\cos(2x)}{2} & -\frac{e^{-2x}}{8} + \frac{e^{2x}}{8} + \frac{\sin(2x)}{4} & \frac{e^{-2x}}{16} + \frac{e^{2x}}{16} - \frac{\cos(2x)}{8} & -\frac{e^{-2x}}{32} + \\ -\frac{e^{-2x}}{2} + \frac{e^{2x}}{2} - \sin(2x) & \frac{e^{-2x}}{4} + \frac{e^{2x}}{4} + \frac{\cos(2x)}{2} & -\frac{e^{-2x}}{8} + \frac{e^{2x}}{8} + \frac{\sin(2x)}{4} & \frac{e^{-2x}}{16} + \\ e^{-2x} + e^{2x} - 2\cos(2x) & -\frac{e^{-2x}}{2} + \frac{e^{2x}}{2} - \sin(2x) & \frac{e^{-2x}}{4} + \frac{e^{2x}}{4} + \frac{\cos(2x)}{2} & -\frac{e^{-2x}}{8} + \\ -2e^{-2x} + 2e^{2x} + 4\sin(2x) & e^{-2x} + e^{2x} - 2\cos(2x) & -\frac{e^{-2x}}{2} + \frac{e^{2x}}{2} - \sin(2x) & \frac{e^{-2x}}{4} + \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{3}{512} + \frac{(128x^5 - 480x^4 + 800x^3 - 600x^2 + 60x + 145) e^{2x}}{20480} - \frac{5 e^{-2x}}{4096} + \frac{(16x^3 - 30x - 9)\cos(x)^2}{768} - \frac{3x^2 \cos(x)\sin(\ldots}{64} \\ \frac{5}{256} + \frac{(128x^5 - 160x^4 - 160x^3 + 600x^2 - 540x + 175) e^{2x}}{10240} + \frac{5 e^{-2x}}{2048} + \frac{(-4x^2 - 5)\cos(x)^2}{128} + \frac{(-16x^3 - 6x + 9)\sin\ldots}{384} \\ -\frac{3}{128} + \frac{(128x^5 + 160x^4 - 480x^3 + 360x^2 + 60x - 95) e^{2x}}{5120} - \frac{5 e^{-2x}}{1024} + \frac{(-16x^3 - 18x + 9)\cos(x)^2}{192} + \frac{(-x^2 + 1)\sin(x)\ldots}{16} \\ -\frac{1}{64} + \frac{(128x^5 + 480x^4 - 160x^3 - 360x^2 + 420x - 65) e^{2x}}{2560} + \frac{5 e^{-2x}}{512} + \frac{(-12x^2 + 1)\cos(x)^2}{32} + \frac{(16x^3 + 6x - 9)\sin\ldots}{96} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3(x) + c_4 \vec{y}_4(x) + \left[\begin{array}{c} \frac{3}{512} + \frac{(128x^5-480x^4+800x^3-600x^2+60x+145)\,\mathrm{e}^{2x}}{20480} - \frac{5\,\mathrm{e}^{-2}}{4096} \\ \frac{5}{256} + \frac{(128x^5-160x^4-160x^3+600x^2-540x+175)\,\mathrm{e}^{2x}}{10240} + \frac{5\,\mathrm{e}^{-}}{204} \\ -\frac{3}{128} + \frac{(128x^5+160x^4-480x^3+360x^2+60x-95)\,\mathrm{e}^{2x}}{5120} - \frac{5\,\mathrm{e}^{-2x}}{1024} \\ -\frac{1}{64} + \frac{(128x^5+480x^4-160x^3-360x^2+420x-65)\,\mathrm{e}^{2x}}{2560} + \frac{5\,\mathrm{e}^{-}}{512} \end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y = \frac{3}{512} + \frac{(128x^5-480x^4+800x^3-600x^2+2560c_2+60x+145)\,\mathrm{e}^{2x}}{20480} + \frac{(-5-512c_1)\,\mathrm{e}^{-2x}}{4096} + \frac{(16x^3-192c_4-30x-9)\cos(x)^2}{768} - \frac{3\,\mathrm{s}}{}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 79

```
dsolve(diff(y(x),x$4)-16*y(x)=x^2*sin(2*x)+x^4*exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(128x^5-480x^4+800x^3-600x^2+20480c_3+60x+105)\,\mathrm{e}^{2x}}{20480} + \frac{(8x^3+768c_1-15x)\cos(2x)}{768} + \frac{(-6x^2+256c_4-11)\sin(2x)}{256} + \mathrm{e}^{-2x}c_2$$

✓ Solution by Mathematica

Time used: 0.562 (sec). Leaf size: 92

```
DSolve[y''''[x]-16*y[x]==x^2*Sin[2*x]+x^4*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{768}\left(8x^3 - 15x + 768c_2\right)\cos(2x) - \frac{1}{512}\left(24x^2 - 5 - 1024c_4\right)\sin(x)\cos(x) + \frac{e^{2x}(128x^5 - 480x^4 + 800x^3 - 600x^2 + 60x + 105 + 20480c_1)}{20480} + c_3e^{-2x}$$

11.50 problem 50

Internal problem ID [11824]
Internal file name [`OUTPUT/11833_Thursday_April_11_2024_08_52_05_PM_42327415/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 50.
ODE order: 6.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_y]]`

$$\boxed{y^{(6)} + 2y^{(5)} + 5y'''' = x^3 + x^2 \mathrm{e}^{-x} + \mathrm{e}^{-x} \sin(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y^{(6)} + 2y^{(5)} + 5y'''' = 0$$

The characteristic equation is

$$\lambda^6 + 2\lambda^5 + 5\lambda^4 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$
$$\lambda_4 = 0$$
$$\lambda_5 = -1 + 2i$$
$$\lambda_6 = -1 - 2i$$

Therefore the homogeneous solution is

$$y_h(x) = x^3 c_4 + c_3 x^2 + c_2 x + c_1 + e^{(-1-2i)x} c_5 + e^{(-1+2i)x} c_6$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= 1 \\ y_2 &= x \\ y_3 &= x^2 \\ y_4 &= x^3 \\ y_5 &= e^{(-1-2i)x} \\ y_6 &= e^{(-1+2i)x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y^{(6)} + 2y^{(5)} + 5y'''' = x^3 + x^2 e^{-x} + e^{-x} \sin(2x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4 + U_5 y_5 + U_6 y_6$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ y_1' & y_2' & y_3' & y_4' & y_5' & y_6' \\ y_1'' & y_2'' & y_3'' & y_4'' & y_5'' & y_6'' \\ y_1''' & y_2''' & y_3''' & y_4''' & y_5''' & y_6''' \\ y_1'''' & y_2'''' & y_3'''' & y_4'''' & y_5'''' & y_6'''' \\ y_1''''' & y_2''''' & y_3''''' & y_4''''' & y_5''''' & y_6''''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} 1 & x & x^2 & x^3 & e^{(-1-2i)x} & e^{(-1+2i)x} \\ 0 & 1 & 2x & 3x^2 & (-1-2i)\,e^{(-1-2i)x} & (-1+2i)\,e^{(-1+2i)x} \\ 0 & 0 & 2 & 6x & (-3+4i)\,e^{(-1-2i)x} & (-3-4i)\,e^{(-1+2i)x} \\ 0 & 0 & 0 & 6 & (11+2i)\,e^{(-1-2i)x} & (11-2i)\,e^{(-1+2i)x} \\ 0 & 0 & 0 & 0 & (-7-24i)\,e^{(-1-2i)x} & (-7+24i)\,e^{(-1+2i)x} \\ 0 & 0 & 0 & 0 & (-41+38i)\,e^{(-1-2i)x} & (-41-38i)\,e^{(-1+2i)x} \end{bmatrix}$$

$$|W| = 30000i e^{(-1-2i)x} e^{(-1+2i)x}$$

The determinant simplifies to

$$|W| = 30000i e^{-2x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} x & x^2 & x^3 & e^{(-1-2i)x} & e^{(-1+2i)x} \\ 1 & 2x & 3x^2 & (-1-2i)\,e^{(-1-2i)x} & (-1+2i)\,e^{(-1+2i)x} \\ 0 & 2 & 6x & (-3+4i)\,e^{(-1-2i)x} & (-3-4i)\,e^{(-1+2i)x} \\ 0 & 0 & 6 & (11+2i)\,e^{(-1-2i)x} & (11-2i)\,e^{(-1+2i)x} \\ 0 & 0 & 0 & (-7-24i)\,e^{(-1-2i)x} & (-7+24i)\,e^{(-1+2i)x} \end{bmatrix}$$

$$= 8i e^{-2x}\left(125x^3 + 150x^2 - 30x - 72\right)$$

$$W_2(x) = \det \begin{bmatrix} 1 & x^2 & x^3 & e^{(-1-2i)x} & e^{(-1+2i)x} \\ 0 & 2x & 3x^2 & (-1-2i)\,e^{(-1-2i)x} & (-1+2i)\,e^{(-1+2i)x} \\ 0 & 2 & 6x & (-3+4i)\,e^{(-1-2i)x} & (-3-4i)\,e^{(-1+2i)x} \\ 0 & 0 & 6 & (11+2i)\,e^{(-1-2i)x} & (11-2i)\,e^{(-1+2i)x} \\ 0 & 0 & 0 & (-7-24i)\,e^{(-1-2i)x} & (-7+24i)\,e^{(-1+2i)x} \end{bmatrix}$$

$$= 120i e^{-2x}\left(25x^2 + 20x - 2\right)$$

$$W_3(x) = \det \begin{bmatrix} 1 & x & x^3 & e^{(-1-2i)x} & e^{(-1+2i)x} \\ 0 & 1 & 3x^2 & (-1-2i)\,e^{(-1-2i)x} & (-1+2i)\,e^{(-1+2i)x} \\ 0 & 0 & 6x & (-3+4i)\,e^{(-1-2i)x} & (-3-4i)\,e^{(-1+2i)x} \\ 0 & 0 & 6 & (11+2i)\,e^{(-1-2i)x} & (11-2i)\,e^{(-1+2i)x} \\ 0 & 0 & 0 & (-7-24i)\,e^{(-1-2i)x} & (-7+24i)\,e^{(-1+2i)x} \end{bmatrix}$$
$$= 600ie^{-2x}(2+5x)$$

$$W_4(x) = \det \begin{bmatrix} 1 & x & x^2 & e^{(-1-2i)x} & e^{(-1+2i)x} \\ 0 & 1 & 2x & (-1-2i)\,e^{(-1-2i)x} & (-1+2i)\,e^{(-1+2i)x} \\ 0 & 0 & 2 & (-3+4i)\,e^{(-1-2i)x} & (-3-4i)\,e^{(-1+2i)x} \\ 0 & 0 & 0 & (11+2i)\,e^{(-1-2i)x} & (11-2i)\,e^{(-1+2i)x} \\ 0 & 0 & 0 & (-7-24i)\,e^{(-1-2i)x} & (-7+24i)\,e^{(-1+2i)x} \end{bmatrix}$$
$$= 1000ie^{-2x}$$

$$W_5(x) = \det \begin{bmatrix} 1 & x & x^2 & x^3 & e^{(-1+2i)x} \\ 0 & 1 & 2x & 3x^2 & (-1+2i)\,e^{(-1+2i)x} \\ 0 & 0 & 2 & 6x & (-3-4i)\,e^{(-1+2i)x} \\ 0 & 0 & 0 & 6 & (11-2i)\,e^{(-1+2i)x} \\ 0 & 0 & 0 & 0 & (-7+24i)\,e^{(-1+2i)x} \end{bmatrix}$$
$$= (-84+288i)\,e^{(-1+2i)x}$$

$$W_6(x) = \det \begin{bmatrix} 1 & x & x^2 & x^3 & e^{(-1-2i)x} \\ 0 & 1 & 2x & 3x^2 & (-1-2i)\,e^{(-1-2i)x} \\ 0 & 0 & 2 & 6x & (-3+4i)\,e^{(-1-2i)x} \\ 0 & 0 & 0 & 6 & (11+2i)\,e^{(-1-2i)x} \\ 0 & 0 & 0 & 0 & (-7-24i)\,e^{(-1-2i)x} \end{bmatrix}$$
$$= (-84-288i)\,e^{(-1-2i)x}$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{6-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^5 \int \frac{\left(x^3 + x^2\mathrm{e}^{-x} + \mathrm{e}^{-x}\sin(2x)\right)\left(8i\mathrm{e}^{-2x}(125x^3 + 150x^2 - 30x - 72)\right)}{(1)\left(30000i\mathrm{e}^{-2x}\right)}\,dx \\
&= -\int \frac{8i(x^3 + x^2\mathrm{e}^{-x} + \mathrm{e}^{-x}\sin(2x))\,\mathrm{e}^{-2x}(125x^3 + 150x^2 - 30x - 72)}{30000i\mathrm{e}^{-2x}}\,dx \\
&= -\int \left(\frac{(x^3 + x^2\mathrm{e}^{-x} + \mathrm{e}^{-x}\sin(2x))\,(125x^3 + 150x^2 - 30x - 72)}{3750}\right)dx \\
&= \frac{3x^4}{625} + \frac{12\,\mathrm{e}^{-x}(-\sin(2x) - 2\cos(2x))}{3125} + \frac{3046\,\mathrm{e}^{-x}}{625} + \frac{31x^4\mathrm{e}^{-x}}{150} + \frac{307x^3\mathrm{e}^{-x}}{375} + \frac{x^5}{625} + \frac{3046x\,\mathrm{e}^{-x}}{625} - \frac{x^6}{150}
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{6-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^4 \int \frac{\left(x^3 + x^2\mathrm{e}^{-x} + \mathrm{e}^{-x}\sin(2x)\right)\left(120i\mathrm{e}^{-2x}(25x^2 + 20x - 2)\right)}{(1)\left(30000i\mathrm{e}^{-2x}\right)}\,dx \\
&= \int \frac{120i(x^3 + x^2\mathrm{e}^{-x} + \mathrm{e}^{-x}\sin(2x))\,\mathrm{e}^{-2x}(25x^2 + 20x - 2)}{30000i\mathrm{e}^{-2x}}\,dx \\
&= \int \left(\frac{(x^3 + x^2\mathrm{e}^{-x} + \mathrm{e}^{-x}\sin(2x))\,(25x^2 + 20x - 2)}{250}\right)dx \\
&= \frac{x^6}{60} + \frac{2x^5}{125} - \frac{x^4}{500} - \frac{x^4\mathrm{e}^{-x}}{10} - \frac{12x^3\mathrm{e}^{-x}}{25} - \frac{179x^2\mathrm{e}^{-x}}{125} - \frac{358x\,\mathrm{e}^{-x}}{125} - \frac{358\,\mathrm{e}^{-x}}{125} + \frac{\left(-\frac{2}{5}x^2 - \frac{8}{25}x + \frac{4}{125}\right)\mathrm{e}^{-x}}{10}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{6-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^3 \int \frac{\left(x^3 + x^2\mathrm{e}^{-x} + \mathrm{e}^{-x}\sin(2x)\right)\left(600i\mathrm{e}^{-2x}(2 + 5x)\right)}{(1)\left(30000i\mathrm{e}^{-2x}\right)}\,dx \\
&= -\int \frac{600i(x^3 + x^2\mathrm{e}^{-x} + \mathrm{e}^{-x}\sin(2x))\,\mathrm{e}^{-2x}(2 + 5x)}{30000i\mathrm{e}^{-2x}}\,dx \\
&= -\int \left(\frac{\left((x^2 + \sin(2x))\,\mathrm{e}^{-x} + x^3\right)\left(\frac{2}{5} + x\right)}{10}\right)dx \\
&= -\frac{x^4}{100} - \frac{x^5}{50} + \frac{17x^2\mathrm{e}^{-x}}{50} + \frac{17x\,\mathrm{e}^{-x}}{25} + \frac{17\,\mathrm{e}^{-x}}{25} + \frac{x^3\mathrm{e}^{-x}}{10} - \frac{\mathrm{e}^{-x}(-\sin(2x) - 2\cos(2x))}{125} - \frac{\left(-\frac{2x}{5} - \frac{4}{25}\right)\mathrm{e}}{10} \\
&= -\frac{x^4}{100} - \frac{x^5}{50} + \frac{17x^2\mathrm{e}^{-x}}{50} + \frac{17x\,\mathrm{e}^{-x}}{25} + \frac{17\,\mathrm{e}^{-x}}{25} + \frac{x^3\mathrm{e}^{-x}}{10} - \frac{\mathrm{e}^{-x}(-\sin(2x) - 2\cos(2x))}{125} - \frac{\left(-\frac{2x}{5} - \frac{4}{25}\right)\mathrm{e}}{10}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{6-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{\left(x^3 + x^2\mathrm{e}^{-x} + \mathrm{e}^{-x}\sin(2x)\right)\left(1000i\mathrm{e}^{-2x}\right)}{(1)\left(30000i\mathrm{e}^{-2x}\right)}\,dx \\
&= \int \frac{1000i(x^3 + x^2\mathrm{e}^{-x} + \mathrm{e}^{-x}\sin(2x))\,\mathrm{e}^{-2x}}{30000i\mathrm{e}^{-2x}}\,dx \\
&= \int \left(\frac{(x^2 + \sin(2x))\,\mathrm{e}^{-x}}{30} + \frac{x^3}{30}\right)\,dx \\
&= \frac{x^4}{120} - \frac{x^2\mathrm{e}^{-x}}{30} - \frac{x\,\mathrm{e}^{-x}}{15} - \frac{\mathrm{e}^{-x}}{15} + \frac{\mathrm{e}^{-x}(-\sin(2x) - 2\cos(2x))}{150} \\
&= \frac{x^4}{120} - \frac{x^2\mathrm{e}^{-x}}{30} - \frac{x\,\mathrm{e}^{-x}}{15} - \frac{\mathrm{e}^{-x}}{15} + \frac{\mathrm{e}^{-x}(-\sin(2x) - 2\cos(2x))}{150}
\end{aligned}
$$

$$
\begin{aligned}
U_5 &= (-1)^{6-5} \int \frac{F(x)W_5(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{\left(x^3 + x^2\mathrm{e}^{-x} + \mathrm{e}^{-x}\sin(2x)\right)\left((-84+288i)\,\mathrm{e}^{(-1+2i)x}\right)}{(1)\left(30000i\mathrm{e}^{-2x}\right)}\,dx \\
&= -\int \frac{(-84+288i)\left(x^3 + x^2\mathrm{e}^{-x} + \mathrm{e}^{-x}\sin(2x)\right)\mathrm{e}^{(-1+2i)x}}{30000i\mathrm{e}^{-2x}}\,dx \\
&= -\int \left(\left(\frac{6}{625} + \frac{7i}{2500}\right)\left(\left(x^2 + \sin(2x)\right)\mathrm{e}^{-x} + x^3\right)\mathrm{e}^{(1+2i)x}\right)\,dx \\
&= \left(-\frac{6}{625} - \frac{7i}{2500}\right)\left(\left(-\frac{7}{625} + \frac{24i}{625}\right)\left((-11-2i)\,\mathrm{e}^{(1+2i)x}x^3 + (9-12i)\,\mathrm{e}^{(1+2i)x}x^2 + (6+12i)\,x\,\mathrm{e}^{(1+2i)}\right.\right. \\
&= \left(-\frac{6}{625} - \frac{7i}{2500}\right)\left(\left(-\frac{7}{625} + \frac{24i}{625}\right)\left((-11-2i)\,\mathrm{e}^{(1+2i)x}x^3 + (9-12i)\,\mathrm{e}^{(1+2i)x}x^2 + (6+12i)\,x\,\mathrm{e}^{(1+2i)}\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
U_6 &= (-1)^{6-6} \int \frac{F(x) W_6(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{\left(x^3 + x^2 \mathrm{e}^{-x} + \mathrm{e}^{-x} \sin(2x)\right)\left((-84 - 288i)\, \mathrm{e}^{(-1-2i)x}\right)}{(1)\left(30000 i \mathrm{e}^{-2x}\right)}\, dx \\
&= \int \frac{(-84 - 288i)\left(x^3 + x^2 \mathrm{e}^{-x} + \mathrm{e}^{-x} \sin(2x)\right) \mathrm{e}^{(-1-2i)x}}{30000 i \mathrm{e}^{-2x}}\, dx \\
&= \int \left(\left(-\frac{6}{625} + \frac{7i}{2500}\right)\left(\left(x^2 + \sin(2x)\right) \mathrm{e}^{-x} + x^3\right) \mathrm{e}^{(1-2i)x}\right)\, dx \\
&= \left(-\frac{6}{625} + \frac{7i}{2500}\right)\left(\left(-\frac{7}{625} - \frac{24i}{625}\right)\left((-11 + 2i)\, \mathrm{e}^{(1-2i)x} x^3 + (9 + 12i)\, \mathrm{e}^{(1-2i)x} x^2 + (6 - 12i)\, x\, \mathrm{e}^{(1-2i)} \right.\right. \\
&= \left(-\frac{6}{625} + \frac{7i}{2500}\right)\left(\left(-\frac{7}{625} - \frac{24i}{625}\right)\left((-11 + 2i)\, \mathrm{e}^{(1-2i)x} x^3 + (9 + 12i)\, \mathrm{e}^{(1-2i)x} x^2 + (6 - 12i)\, x\, \mathrm{e}^{(1-2i)} \right.\right.
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4 + U_5 y_5 + U_6 y_6
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{3x^4}{625} + \frac{12\, \mathrm{e}^{-x}(-\sin(2x) - 2\cos(2x))}{3125} + \frac{3046\, \mathrm{e}^{-x}}{625} + \frac{31 x^4 \mathrm{e}^{-x}}{150} + \frac{307 x^3 \mathrm{e}^{-x}}{375} + \frac{x^5}{625} + \frac{3046 x\, \mathrm{e}^{-x}}{625} - \frac{x}{1} \right. \\
&+ \left(\frac{x^6}{60} + \frac{2x^5}{125} - \frac{x^4}{500} - \frac{x^4 \mathrm{e}^{-x}}{10} - \frac{12 x^3 \mathrm{e}^{-x}}{25} - \frac{179 x^2 \mathrm{e}^{-x}}{125} - \frac{358 x\, \mathrm{e}^{-x}}{125} - \frac{358\, \mathrm{e}^{-x}}{125} + \frac{\left(-\frac{2}{5}x^2 - \frac{8}{25}x + \frac{4}{125}\right) \mathrm{e}^{-}}{10} \right. \\
&+ \left(-\frac{x^4}{100} - \frac{x^5}{50} + \frac{17 x^2 \mathrm{e}^{-x}}{50} + \frac{17 x\, \mathrm{e}^{-x}}{25} + \frac{17\, \mathrm{e}^{-x}}{25} + \frac{x^3 \mathrm{e}^{-x}}{10} - \frac{\mathrm{e}^{-x}(-\sin(2x) - 2\cos(2x))}{125} - \frac{\left(-\frac{2x}{5} - \frac{4}{25}\right)}{1} \right. \\
&+ \left(\frac{x^4}{120} - \frac{x^2 \mathrm{e}^{-x}}{30} - \frac{x\, \mathrm{e}^{-x}}{15} - \frac{\mathrm{e}^{-x}}{15} + \frac{\mathrm{e}^{-x}(-\sin(2x) - 2\cos(2x))}{150}\right)\left(x^3\right) \\
&+ \left(\left(-\frac{6}{625} - \frac{7i}{2500}\right)\left(\left(-\frac{7}{625} + \frac{24i}{625}\right)\left((-11 - 2i)\, \mathrm{e}^{(1+2i)x} x^3 + (9 - 12i)\, \mathrm{e}^{(1+2i)x} x^2 + (6 + 12i)\, x\, \mathrm{e}^{(1+2} \right.\right.\right. \\
&+ \left(\left(-\frac{6}{625} + \frac{7i}{2500}\right)\left(\left(-\frac{7}{625} - \frac{24i}{625}\right)\left((-11 + 2i)\, \mathrm{e}^{(1-2i)x} x^3 + (9 + 12i)\, \mathrm{e}^{(1-2i)x} x^2 + (6 - 12i)\, x\, \mathrm{e}^{(1-2} \right.\right.\right.
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = -\frac{1008}{390625} + \frac{(-448 - 339i + (70 - 240i)\, x)\, \mathrm{e}^{(-1-2i)x}}{50000} + \frac{(-448 + 339i + (70 + 240i)\, x)\, \mathrm{e}^{(-1+2i)x}}{50000} + \frac{(2x^2 -}{}
$$

Which simplifies to

$$
y_p = \frac{((183750x - 1176000)\cos(2x) + (-630000x - 889875)\sin(2x) + 16406250x^2 + 131250000x + 3199}{65625000}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(x^3 c_4 + c_3 x^2 + c_2 x + c_1 + \mathrm{e}^{(-1-2i)x} c_5 + \mathrm{e}^{(-1+2i)x} c_6\right) \\
&\quad + \left(\frac{((183750x - 1176000)\cos(2x) + (-630000x - 889875)\sin(2x) + 16406250x^2 + 131250000x + 31}{65625000}\right. \\
&\quad \left. + \frac{x^7}{4200} - \frac{x^6}{1500} - \frac{x^5}{2500} + \frac{3x^4}{625} - \frac{19x^3}{3125} - \frac{66x^2}{15625} + \frac{834x}{78125} - \frac{1008}{390625}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= x^3 c_4 + c_3 x^2 + c_2 x + c_1 + \mathrm{e}^{(-1-2i)x} c_5 + \mathrm{e}^{(-1+2i)x} c_6 \\
&\quad + \frac{((183750x - 1176000)\cos(2x) + (-630000x - 889875)\sin(2x) + 16406250x^2 + 131250000x + 3199}{65625000} \\
&\quad + \frac{x^7}{4200} - \frac{x^6}{1500} - \frac{x^5}{2500} + \frac{3x^4}{625} - \frac{19x^3}{3125} - \frac{66x^2}{15625} + \frac{834x}{78125} - \frac{1008}{390625}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= x^3 c_4 + c_3 x^2 + c_2 x + c_1 + \mathrm{e}^{(-1-2i)x} c_5 + \mathrm{e}^{(-1+2i)x} c_6 \\
&\quad + \frac{((183750x - 1176000)\cos(2x) + (-630000x - 889875)\sin(2x) + 16406250x^2 + 131250000x + 3199}{65625000} \\
&\quad + \frac{x^7}{4200} - \frac{x^6}{1500} - \frac{x^5}{2500} + \frac{3x^4}{625} - \frac{19x^3}{3125} - \frac{66x^2}{15625} + \frac{834x}{78125} - \frac{1008}{390625}
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 6; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _a^2*exp(-_a)+_a^3+exp(-_a)*s
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 6; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 91

```
dsolve(diff(y(x),x$6)+2*diff(y(x),x$5)+5*diff(y(x),x$4)=x^3+x^2*exp(-x)+exp(-x)*sin(2*x),y(x
```

$$y(x) = c_5x + c_6 + \frac{\left(\int \left(\left(\left(-330x + 1320c_1 + 240c_2 + 69\right)\cos\left(2x\right) + \left(60x - 240c_1 + 1320c_2 + 567\right)\sin\left(2x\right) - 3750x^2 - 22\right.\right.\right.}{15000}$$

✓ Solution by Mathematica

Time used: 11.809 (sec). Leaf size: 119

```
DSolve[y''''''[x]+2*y'''''[x]+5*y''''[x]==x^3+x^2*Exp[-x]+Exp[-x]*Sin[2*x],y[x],x,IncludeSin
```

$$y(x) \to c_6x^3 + c_5x^2 + \frac{e^{-x}(10(25e^xx^7 - 70e^xx^6 - 42e^xx^5 + 504e^xx^4 + 26250x^2 + 210000x + 511875) + 84(35x - 2(97 + 24\ldots}{1050000} + c_4x + c_3$$

11.51 problem 51

Internal problem ID [11825]
Internal file name [`OUTPUT/11834_Thursday_April_11_2024_08_52_41_PM_58692852/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 51.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 2y'' + y = \cos(x)\, x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 2y'' + y = 0$$

The characteristic equation is

$$\lambda^4 + 2\lambda^2 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = i$$
$$\lambda_2 = -i$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = e^{ix} c_1 + x\,e^{ix} c_2 + e^{-ix} c_3 + x\,e^{-ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{ix} \\ y_2 &= x\,e^{ix} \\ y_3 &= e^{-ix} \\ y_4 &= x\,e^{-ix} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 2y'' + y = \cos(x)\,x^2$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\,dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{ix} & x\,e^{ix} & e^{-ix} & x\,e^{-ix} \\ i e^{ix} & e^{ix}(ix+1) & -i e^{-ix} & e^{-ix}(-ix+1) \\ -e^{ix} & e^{ix}(2i-x) & -e^{-ix} & e^{-ix}(-2i-x) \\ -i e^{ix} & -e^{ix}(ix+3) & i e^{-ix} & e^{-ix}(ix-3) \end{bmatrix}$$

$$|W| = 16\,e^{2ix} e^{-2ix}$$

The determinant simplifies to

$$|W| = 16$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} x\,e^{ix} & e^{-ix} & x\,e^{-ix} \\ e^{ix}(ix+1) & -ie^{-ix} & e^{-ix}(-ix+1) \\ e^{ix}(2i-x) & -e^{-ix} & e^{-ix}(-2i-x) \end{bmatrix} \\ &= -4\,e^{-ix}(x-i) \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} e^{ix} & e^{-ix} & x\,e^{-ix} \\ ie^{ix} & -ie^{-ix} & e^{-ix}(-ix+1) \\ -e^{ix} & -e^{-ix} & e^{-ix}(-2i-x) \end{bmatrix} \\ &= -4\,e^{-ix} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} e^{ix} & x\,e^{ix} & x\,e^{-ix} \\ ie^{ix} & e^{ix}(ix+1) & e^{-ix}(-ix+1) \\ -e^{ix} & e^{ix}(2i-x) & e^{-ix}(-2i-x) \end{bmatrix} \\ &= -4\,e^{ix}(x+i) \end{aligned}$$

$$\begin{aligned} W_4(x) &= \det \begin{bmatrix} e^{ix} & x\,e^{ix} & e^{-ix} \\ ie^{ix} & e^{ix}(ix+1) & -ie^{-ix} \\ -e^{ix} & e^{ix}(2i-x) & -e^{-ix} \end{bmatrix} \\ &= -4\,e^{ix} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^3 \int \frac{(\cos(x)\,x^2)\,(-4\,e^{-ix}(x-i))}{(1)\,(16)}\,dx \\ &= -\int \frac{-4\cos(x)\,x^2 e^{-ix}(x-i)}{16}\,dx \\ &= -\int \left(-\frac{\cos(x)\,x^2 e^{-ix}(x-i)}{4}\right)dx \\ &= -\left(\int -\frac{\cos(x)\,x^2 e^{-ix}(x-i)}{4}dx\right) \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x) W_2(x)}{a W(x)} \, dx \\
&= (-1)^2 \int \frac{(\cos(x) x^2)(-4\,\mathrm{e}^{-ix})}{(1)(16)} \, dx \\
&= \int \frac{-4\cos(x) x^2 \mathrm{e}^{-ix}}{16} \, dx \\
&= \int \left(-\frac{\cos(x) x^2 \mathrm{e}^{-ix}}{4}\right) dx \\
&= \int -\frac{\cos(x) x^2 \mathrm{e}^{-ix}}{4} dx
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x) W_3(x)}{a W(x)} \, dx \\
&= (-1)^1 \int \frac{(\cos(x) x^2)(-4\,\mathrm{e}^{ix}(x+i))}{(1)(16)} \, dx \\
&= -\int \frac{-4\cos(x) x^2 \mathrm{e}^{ix}(x+i)}{16} \, dx \\
&= -\int \left(-\frac{\cos(x) x^2 \mathrm{e}^{ix}(x+i)}{4}\right) dx \\
&= \frac{x^4}{32} + \frac{i x^3}{24} - \frac{i(4x^3 + 10 i x^2 - 10x - 5i)\,\mathrm{e}^{2ix}}{64}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)} \, dx \\
&= (-1)^0 \int \frac{(\cos(x) x^2)(-4\,\mathrm{e}^{ix})}{(1)(16)} \, dx \\
&= \int \frac{-4\cos(x) x^2 \mathrm{e}^{ix}}{16} \, dx \\
&= \int \left(-\frac{\cos(x) x^2 \mathrm{e}^{ix}}{4}\right) dx \\
&= -\frac{x^3}{24} + \frac{i(2x^2 + 2ix - 1)\,\mathrm{e}^{2ix}}{32}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = {} & \left(-\left(\int -\frac{\cos(x)\, x^2 \mathrm{e}^{-ix} (x-i)}{4} dx\right)\right) \left(\mathrm{e}^{ix}\right) \\
& + \left(\int -\frac{\cos(x)\, x^2 \mathrm{e}^{-ix}}{4} dx\right) \left(x\, \mathrm{e}^{ix}\right) \\
& + \left(\frac{x^4}{32} + \frac{i x^3}{24} - \frac{i(4x^3 + 10 i x^2 - 10x - 5i)\, \mathrm{e}^{2ix}}{64}\right) \left(\mathrm{e}^{-ix}\right) \\
& + \left(-\frac{x^3}{24} + \frac{i(2x^2 + 2ix - 1)\, \mathrm{e}^{2ix}}{32}\right) \left(x\, \mathrm{e}^{-ix}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\cos(x)\left(-4x^4 + 36x^2 + 6ix - 45\right)}{192} - \frac{5\sin(x)\left(-\frac{16}{15}x^3 + i + \frac{18}{5}x\right)}{64}
$$

Which simplifies to

$$
y_p = \frac{\cos(x)\left(-4x^4 + 36x^2 + 6ix - 45\right)}{192} - \frac{5\sin(x)\left(-\frac{16}{15}x^3 + i + \frac{18}{5}x\right)}{64}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{ix} c_1 + x\, \mathrm{e}^{ix} c_2 + \mathrm{e}^{-ix} c_3 + x\, \mathrm{e}^{-ix} c_4\right) \\
&\quad + \left(\frac{\cos(x)\left(-4x^4 + 36x^2 + 6ix - 45\right)}{192} - \frac{5\sin(x)\left(-\frac{16}{15}x^3 + i + \frac{18}{5}x\right)}{64}\right)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y = {} & (c_4 x + c_3)\, \mathrm{e}^{-ix} + (c_2 x + c_1)\, \mathrm{e}^{ix} \\
& + \frac{\cos(x)\left(-4x^4 + 36x^2 + 6ix - 45\right)}{192} - \frac{5\sin(x)\left(-\frac{16}{15}x^3 + i + \frac{18}{5}x\right)}{64}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & (c_4 x + c_3)\, \mathrm{e}^{-ix} + (c_2 x + c_1)\, \mathrm{e}^{ix} \\
& + \frac{\cos(x)\left(-4x^4 + 36x^2 + 6ix - 45\right)}{192} - \frac{5\sin(x)\left(-\frac{16}{15}x^3 + i + \frac{18}{5}x\right)}{64}
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = (c_4x + c_3)\,\mathrm{e}^{-ix} + (c_2x + c_1)\,\mathrm{e}^{ix} + \frac{\cos(x)\left(-4x^4 + 36x^2 + 6ix - 45\right)}{192} - \frac{5\sin(x)\left(-\frac{16}{15}x^3 + i + \frac{18}{5}x\right)}{64}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 46

```
dsolve(diff(y(x),x$4)+2*diff(y(x),x$2)+y(x)=x^2*cos(x),y(x), singsol=all)
```

$$y(x) = \frac{(-4x^4 + 192c_4x + 36x^2 + 192c_1 - 21)\cos(x)}{192} + \frac{(x^3 + (12c_3 - 3)\,x + 12c_2)\sin(x)}{12}$$

✓ Solution by Mathematica

Time used: 0.138 (sec). Leaf size: 56

```
DSolve[y''''[x]+2*y''[x]+y[x]==x^2*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{12}(x^3 + 3(-1 + 4c_4)x + 12c_3)\sin(x) + \left(-\frac{x^4}{48} + \frac{3x^2}{16} + c_2x - \frac{5}{32} + c_1\right)\cos(x)$$

11.52 problem 52

Internal problem ID [11826]
Internal file name [`OUTPUT/11835_Thursday_April_11_2024_08_52_43_PM_27972420/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 52.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$y'''' + 16y = x\,\mathrm{e}^{\sqrt{2}\,x} \sin\left(\sqrt{2}\,x\right) + \mathrm{e}^{-\sqrt{2}\,x} \cos\left(\sqrt{2}\,x\right)$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 16y = 0$$

The characteristic equation is

$$\lambda^4 + 16 = 0$$

The roots of the above equation are

$$\lambda_1 = \sqrt{2} + i\sqrt{2}$$
$$\lambda_2 = -\sqrt{2} + i\sqrt{2}$$
$$\lambda_3 = -\sqrt{2} - i\sqrt{2}$$
$$\lambda_4 = \sqrt{2} - i\sqrt{2}$$

Therefore the homogeneous solution is

$$y_h(x) = e^{\left(\sqrt{2}-i\sqrt{2}\right)x} c_1 + e^{\left(-\sqrt{2}+i\sqrt{2}\right)x} c_2 + e^{\left(-\sqrt{2}-i\sqrt{2}\right)x} c_3 + e^{\left(\sqrt{2}+i\sqrt{2}\right)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{\left(\sqrt{2}-i\sqrt{2}\right)x}$$

$$y_2 = e^{\left(-\sqrt{2}+i\sqrt{2}\right)x}$$

$$y_3 = e^{\left(-\sqrt{2}-i\sqrt{2}\right)x}$$

$$y_4 = e^{\left(\sqrt{2}+i\sqrt{2}\right)x}$$

Now the particular solution to the given ODE is found

$$y'''' + 16y = x\,e^{\sqrt{2}\,x} \sin\left(\sqrt{2}\,x\right) + e^{-\sqrt{2}\,x} \cos\left(\sqrt{2}\,x\right)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \left[\begin{array}{cccc} e^{(1-i)\sqrt{2}\,x} & e^{(-1+i)\sqrt{2}\,x} & e^{(-1-i)\sqrt{2}\,x} & e^{(1+i)\sqrt{2}\,x} \\ (1-i)\sqrt{2}\,e^{(1-i)\sqrt{2}\,x} & (-1+i)\sqrt{2}\,e^{(-1+i)\sqrt{2}\,x} & (-1-i)\sqrt{2}\,e^{(-1-i)\sqrt{2}\,x} & (1+i)\sqrt{2}\,e^{(1+i)\sqrt{2}\,x} \\ -4i e^{(1-i)\sqrt{2}\,x} & -4i e^{(-1+i)\sqrt{2}\,x} & 4i e^{(-1-i)\sqrt{2}\,x} & 4i e^{(1+i)\sqrt{2}\,x} \\ (-4-4i)\sqrt{2}\,e^{(1-i)\sqrt{2}\,x} & (4+4i)\sqrt{2}\,e^{(-1+i)\sqrt{2}\,x} & (4-4i)\sqrt{2}\,e^{(-1-i)\sqrt{2}\,x} & (-4+4i)\sqrt{2}\,e^{(1+i)\sqrt{2}\,x} \end{array}\right]$$

$$|W| = 1024\,e^{(1-i)\sqrt{2}\,x} e^{(-1+i)\sqrt{2}\,x} e^{(-1-i)\sqrt{2}\,x} e^{(1+i)\sqrt{2}\,x}$$

The determinant simplifies to

$$|W| = 1024$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det\left[\begin{array}{ccc} e^{(-1+i)\sqrt{2}\,x} & e^{(-1-i)\sqrt{2}\,x} & e^{(1+i)\sqrt{2}\,x} \\ (-1+i)\sqrt{2}\,e^{(-1+i)\sqrt{2}\,x} & (-1-i)\sqrt{2}\,e^{(-1-i)\sqrt{2}\,x} & (1+i)\sqrt{2}\,e^{(1+i)\sqrt{2}\,x} \\ -4i e^{(-1+i)\sqrt{2}\,x} & 4i e^{(-1-i)\sqrt{2}\,x} & 4i e^{(1+i)\sqrt{2}\,x} \end{array}\right] \\ &= (16-16i)\sqrt{2}\,e^{(-1+i)\sqrt{2}\,x} \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det\left[\begin{array}{ccc} e^{(1-i)\sqrt{2}\,x} & e^{(-1-i)\sqrt{2}\,x} & e^{(1+i)\sqrt{2}\,x} \\ (1-i)\sqrt{2}\,e^{(1-i)\sqrt{2}\,x} & (-1-i)\sqrt{2}\,e^{(-1-i)\sqrt{2}\,x} & (1+i)\sqrt{2}\,e^{(1+i)\sqrt{2}\,x} \\ -4i e^{(1-i)\sqrt{2}\,x} & 4i e^{(-1-i)\sqrt{2}\,x} & 4i e^{(1+i)\sqrt{2}\,x} \end{array}\right] \\ &= (16-16i)\sqrt{2}\,e^{(1-i)\sqrt{2}\,x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det\left[\begin{array}{ccc} e^{(1-i)\sqrt{2}\,x} & e^{(-1+i)\sqrt{2}\,x} & e^{(1+i)\sqrt{2}\,x} \\ (1-i)\sqrt{2}\,e^{(1-i)\sqrt{2}\,x} & (-1+i)\sqrt{2}\,e^{(-1+i)\sqrt{2}\,x} & (1+i)\sqrt{2}\,e^{(1+i)\sqrt{2}\,x} \\ -4i e^{(1-i)\sqrt{2}\,x} & -4i e^{(-1+i)\sqrt{2}\,x} & 4i e^{(1+i)\sqrt{2}\,x} \end{array}\right] \\ &= (-16-16i)\sqrt{2}\,e^{(1+i)\sqrt{2}\,x} \end{aligned}$$

$$\begin{aligned} W_4(x) &= \det\left[\begin{array}{ccc} e^{(1-i)\sqrt{2}\,x} & e^{(-1+i)\sqrt{2}\,x} & e^{(-1-i)\sqrt{2}\,x} \\ (1-i)\sqrt{2}\,e^{(1-i)\sqrt{2}\,x} & (-1+i)\sqrt{2}\,e^{(-1+i)\sqrt{2}\,x} & (-1-i)\sqrt{2}\,e^{(-1-i)\sqrt{2}\,x} \\ -4i e^{(1-i)\sqrt{2}\,x} & -4i e^{(-1+i)\sqrt{2}\,x} & 4i e^{(-1-i)\sqrt{2}\,x} \end{array}\right] \\ &= (-16-16i)\sqrt{2}\,e^{(-1-i)\sqrt{2}\,x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^3 \int \frac{\left(x\,\mathrm{e}^{\sqrt{2}\,x}\sin\left(\sqrt{2}\,x\right) + \mathrm{e}^{-\sqrt{2}\,x}\cos\left(\sqrt{2}\,x\right)\right)\left((16-16i)\,\sqrt{2}\,\mathrm{e}^{(-1+i)\sqrt{2}\,x}\right)}{(1)\,(1024)}\,dx \\
&= -\int \frac{(16-16i)\left(x\,\mathrm{e}^{\sqrt{2}\,x}\sin\left(\sqrt{2}\,x\right) + \mathrm{e}^{-\sqrt{2}\,x}\cos\left(\sqrt{2}\,x\right)\right)\sqrt{2}\,\mathrm{e}^{(-1+i)\sqrt{2}\,x}}{1024}\,dx \\
&= -\int \left(\left(\frac{1}{64}-\frac{i}{64}\right)\sqrt{2}\,\mathrm{e}^{(-2+i)\sqrt{2}\,x}\left(x\,\mathrm{e}^{2\sqrt{2}\,x}\sin\left(\sqrt{2}\,x\right) + \cos\left(\sqrt{2}\,x\right)\right)\right)dx \\
&= -\left(\int \left(\frac{1}{64}-\frac{i}{64}\right)\sqrt{2}\,\mathrm{e}^{(-2+i)\sqrt{2}\,x}\left(x\,\mathrm{e}^{2\sqrt{2}\,x}\sin\left(\sqrt{2}\,x\right) + \cos\left(\sqrt{2}\,x\right)\right)dx\right)
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{\left(x\,\mathrm{e}^{\sqrt{2}\,x}\sin\left(\sqrt{2}\,x\right) + \mathrm{e}^{-\sqrt{2}\,x}\cos\left(\sqrt{2}\,x\right)\right)\left((16-16i)\,\sqrt{2}\,\mathrm{e}^{(1-i)\sqrt{2}\,x}\right)}{(1)\,(1024)}\,dx \\
&= \int \frac{(16-16i)\left(x\,\mathrm{e}^{\sqrt{2}\,x}\sin\left(\sqrt{2}\,x\right) + \mathrm{e}^{-\sqrt{2}\,x}\cos\left(\sqrt{2}\,x\right)\right)\sqrt{2}\,\mathrm{e}^{(1-i)\sqrt{2}\,x}}{1024}\,dx \\
&= \int \left(\left(\frac{1}{64}-\frac{i}{64}\right)\sqrt{2}\,\mathrm{e}^{-i\sqrt{2}\,x}\left(x\,\mathrm{e}^{2\sqrt{2}\,x}\sin\left(\sqrt{2}\,x\right) + \cos\left(\sqrt{2}\,x\right)\right)\right)dx \\
&= \int \left(\frac{1}{64}-\frac{i}{64}\right)\sqrt{2}\,\mathrm{e}^{-i\sqrt{2}\,x}\left(x\,\mathrm{e}^{2\sqrt{2}\,x}\sin\left(\sqrt{2}\,x\right) + \cos\left(\sqrt{2}\,x\right)\right)dx
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{\left(x\,\mathrm{e}^{\sqrt{2}\,x}\sin\left(\sqrt{2}\,x\right) + \mathrm{e}^{-\sqrt{2}\,x}\cos\left(\sqrt{2}\,x\right)\right)\left((-16-16i)\,\sqrt{2}\,\mathrm{e}^{(1+i)\sqrt{2}\,x}\right)}{(1)\,(1024)}\,dx \\
&= -\int \frac{(-16-16i)\left(x\,\mathrm{e}^{\sqrt{2}\,x}\sin\left(\sqrt{2}\,x\right) + \mathrm{e}^{-\sqrt{2}\,x}\cos\left(\sqrt{2}\,x\right)\right)\sqrt{2}\,\mathrm{e}^{(1+i)\sqrt{2}\,x}}{1024}\,dx \\
&= -\int \left(\left(-\frac{1}{64}-\frac{i}{64}\right)\sqrt{2}\,\mathrm{e}^{i\sqrt{2}\,x}\left(x\,\mathrm{e}^{2\sqrt{2}\,x}\sin\left(\sqrt{2}\,x\right) + \cos\left(\sqrt{2}\,x\right)\right)\right)dx \\
&= \frac{\sqrt{2}\,x}{128} + \frac{i\sqrt{2}\,x}{128} + \frac{\mathrm{e}^{2i\sqrt{2}\,x}}{256} - \frac{i\mathrm{e}^{2i\sqrt{2}\,x}}{256} + \left(-\frac{1}{8192}-\frac{i}{8192}\right)\sqrt{2}\left(\sqrt{2}+i\sqrt{2}\right)\left(i\sqrt{2}\,\mathrm{e}^{2i\sqrt{2}\,x} - \sqrt{2}\,\mathrm{e}^{2i\sqrt{2}\,x} + \right. \\
&= \frac{\sqrt{2}\,x}{128} + \frac{i\sqrt{2}\,x}{128} + \frac{\mathrm{e}^{2i\sqrt{2}\,x}}{256} - \frac{i\mathrm{e}^{2i\sqrt{2}\,x}}{256} + \left(-\frac{1}{8192}-\frac{i}{8192}\right)\sqrt{2}\left(\sqrt{2}+i\sqrt{2}\right)\left(i\sqrt{2}\,\mathrm{e}^{2i\sqrt{2}\,x} - \sqrt{2}\,\mathrm{e}^{2i\sqrt{2}\,x} + \right.
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x)W_4(x)}{aW(x)}\,dx \\
&= (-1)^{0} \int \frac{\left(x\,\mathrm{e}^{\sqrt{2}\,x} \sin\left(\sqrt{2}\,x\right) + \mathrm{e}^{-\sqrt{2}\,x} \cos\left(\sqrt{2}\,x\right)\right)\left(\left(-16-16i\right)\sqrt{2}\,\mathrm{e}^{(-1-i)\sqrt{2}\,x}\right)}{(1)\,(1024)}\,dx \\
&= \int \frac{\left(-16-16i\right)\left(x\,\mathrm{e}^{\sqrt{2}\,x} \sin\left(\sqrt{2}\,x\right) + \mathrm{e}^{-\sqrt{2}\,x} \cos\left(\sqrt{2}\,x\right)\right)\sqrt{2}\,\mathrm{e}^{(-1-i)\sqrt{2}\,x}}{1024}\,dx \\
&= \int \left(\left(-\frac{1}{64}-\frac{i}{64}\right)\sqrt{2}\,\mathrm{e}^{(-2-i)\sqrt{2}\,x}\left(x\,\mathrm{e}^{2\sqrt{2}\,x} \sin\left(\sqrt{2}\,x\right) + \cos\left(\sqrt{2}\,x\right)\right)\right) dx \\
&= \int \left(-\frac{1}{64}-\frac{i}{64}\right)\sqrt{2}\,\mathrm{e}^{(-2-i)\sqrt{2}\,x}\left(x\,\mathrm{e}^{2\sqrt{2}\,x} \sin\left(\sqrt{2}\,x\right) + \cos\left(\sqrt{2}\,x\right)\right) dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = {} & \left(-\left(\int \left(\frac{1}{64}-\frac{i}{64}\right)\sqrt{2}\,\mathrm{e}^{(-2+i)\sqrt{2}\,x}\left(x\,\mathrm{e}^{2\sqrt{2}\,x} \sin\left(\sqrt{2}\,x\right) + \cos\left(\sqrt{2}\,x\right)\right) dx\right)\right)\left(\mathrm{e}^{\left(\sqrt{2}-i\sqrt{2}\right)x}\right) \\
& + \left(\int \left(\frac{1}{64}-\frac{i}{64}\right)\sqrt{2}\,\mathrm{e}^{-i\sqrt{2}\,x}\left(x\,\mathrm{e}^{2\sqrt{2}\,x} \sin\left(\sqrt{2}\,x\right) + \cos\left(\sqrt{2}\,x\right)\right) dx\right)\left(\mathrm{e}^{\left(-\sqrt{2}+i\sqrt{2}\right)x}\right) \\
& + \left(\frac{\sqrt{2}\,x}{128} + \frac{i\sqrt{2}\,x}{128} + \frac{\mathrm{e}^{2i\sqrt{2}\,x}}{256} - \frac{i\mathrm{e}^{2i\sqrt{2}\,x}}{256} + \left(-\frac{1}{8192}-\frac{i}{8192}\right)\sqrt{2}\left(\sqrt{2}+i\sqrt{2}\right)\left(i\sqrt{2}\,\mathrm{e}^{2i\sqrt{2}\,x} - \sqrt{2}\,\mathrm{e}^{2i\sqrt{2}\,x} \right.\right. \\
& + \left(\int \left(-\frac{1}{64}-\frac{i}{64}\right)\sqrt{2}\,\mathrm{e}^{(-2-i)\sqrt{2}\,x}\left(x\,\mathrm{e}^{2\sqrt{2}\,x} \sin\left(\sqrt{2}\,x\right) + \cos\left(\sqrt{2}\,x\right)\right) dx\right)\left(\mathrm{e}^{\left(\sqrt{2}+i\sqrt{2}\right)x}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\mathrm{e}^{-\sqrt{2}\,x}\left(\left(4\sqrt{2}\,x-1+i\right)\sin\left(\sqrt{2}\,x\right) + \cos\left(\sqrt{2}\,x\right)\left(4\sqrt{2}\,x+7+i\right)\right)}{256} - \frac{\left(\left(\sqrt{2}\,x^2-3x+\frac{7\sqrt{2}}{8}\right)\sin\left(\sqrt{2}\,x\right)\right.}{12}
$$

Which simplifies to

$$
y_p = \frac{\mathrm{e}^{-\sqrt{2}\,x}\left(\left(4\sqrt{2}\,x-1+i\right)\sin\left(\sqrt{2}\,x\right) + \cos\left(\sqrt{2}\,x\right)\left(4\sqrt{2}\,x+7+i\right)\right)}{256} - \frac{\left(\left(\sqrt{2}\,x^2-3x+\frac{7\sqrt{2}}{8}\right)\sin\left(\sqrt{2}\,x\right)\right.}{12}
$$

Therefore the general solution is

$$
y = y_h + y_p
$$

$$= \left(e^{\left(\sqrt{2}-i\sqrt{2}\right)x} c_1 + e^{\left(-\sqrt{2}+i\sqrt{2}\right)x} c_2 + e^{\left(-\sqrt{2}-i\sqrt{2}\right)x} c_3 + e^{\left(\sqrt{2}+i\sqrt{2}\right)x} c_4\right) + \left(\frac{e^{-\sqrt{2}\,x}\left(\left(4\sqrt{2}\,x-1+i\right)\sin\left(\sqrt{2}\,x\right)+\cos\left(\sqrt{2}\,x\right)\left(4\sqrt{2}\,x+7+i\right)\right)}{256} - \frac{\left(\left(\sqrt{2}\,x^2-3x+\frac{7\sqrt{2}}{8}\right)\sin\left(\sqrt{2}\,x\right)+\cos\left(\sqrt{2}\,x\right)\sqrt{2}\left(x^2-\frac{3}{8}\right)\right)e^{\sqrt{2}\,x}}{128}\right)$$

Which simplifies to

$$y = e^{(1-i)\sqrt{2}\,x} c_1 + e^{(-1+i)\sqrt{2}\,x} c_2 + e^{(-1-i)\sqrt{2}\,x} c_3 + e^{(1+i)\sqrt{2}\,x} c_4 + \frac{e^{-\sqrt{2}\,x}\left(\left(4\sqrt{2}\,x-1+i\right)\sin\left(\sqrt{2}\,x\right)+\cos\left(\sqrt{2}\,x\right)\left(4\sqrt{2}\,x+7+i\right)\right)}{256} - \frac{\left(\left(\sqrt{2}\,x^2-3x+\frac{7\sqrt{2}}{8}\right)\sin\left(\sqrt{2}\,x\right)+\cos\left(\sqrt{2}\,x\right)\sqrt{2}\left(x^2-\frac{3}{8}\right)\right)e^{\sqrt{2}\,x}}{128}$$

Summary

The solution(s) found are the following

$$y = e^{(1-i)\sqrt{2}\,x} c_1 + e^{(-1+i)\sqrt{2}\,x} c_2 + e^{(-1-i)\sqrt{2}\,x} c_3 + e^{(1+i)\sqrt{2}\,x} c_4 + \frac{e^{-\sqrt{2}\,x}\left(\left(4\sqrt{2}\,x-1+i\right)\sin\left(\sqrt{2}\,x\right)+\cos\left(\sqrt{2}\,x\right)\left(4\sqrt{2}\,x+7+i\right)\right)}{256} - \frac{\left(\left(\sqrt{2}\,x^2-3x+\frac{7\sqrt{2}}{8}\right)\sin\left(\sqrt{2}\,x\right)+\cos\left(\sqrt{2}\,x\right)\sqrt{2}\left(x^2-\frac{3}{8}\right)\right)e^{\sqrt{2}\,x}}{128} \tag{1}$$

Verification of solutions

$$y = e^{(1-i)\sqrt{2}\,x} c_1 + e^{(-1+i)\sqrt{2}\,x} c_2 + e^{(-1-i)\sqrt{2}\,x} c_3 + e^{(1+i)\sqrt{2}\,x} c_4 + \frac{e^{-\sqrt{2}\,x}\left(\left(4\sqrt{2}\,x-1+i\right)\sin\left(\sqrt{2}\,x\right)+\cos\left(\sqrt{2}\,x\right)\left(4\sqrt{2}\,x+7+i\right)\right)}{256} - \frac{\left(\left(\sqrt{2}\,x^2-3x+\frac{7\sqrt{2}}{8}\right)\sin\left(\sqrt{2}\,x\right)+\cos\left(\sqrt{2}\,x\right)\sqrt{2}\left(x^2-\frac{3}{8}\right)\right)e^{\sqrt{2}\,x}}{128}$$

Verified OK.

11.52.1 Maple step by step solution

Let's solve

$y'''' + 16y = x\,e^{\sqrt{2}\,x} \sin\left(\sqrt{2}\,x\right) + e^{-\sqrt{2}\,x} \cos\left(\sqrt{2}\,x\right)$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

  $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

  $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

  $y_3(x) = y''$

  ◦ Define new variable $y_4(x)$

  $y_4(x) = y'''$

  ◦ Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = x\,e^{\sqrt{2}\,x} \sin\left(\sqrt{2}\,x\right) + e^{-\sqrt{2}\,x} \cos\left(\sqrt{2}\,x\right) - 16y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = x\,e^{\sqrt{2}\,x} \sin\left(\sqrt{2}\,x\right) + e^{-\sqrt{2}\,x} \cos\left(\sqrt{2}\,x\right) - 1\right.$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ x\,e^{\sqrt{2}\,x} \sin\left(\sqrt{2}\,x\right) + e^{-\sqrt{2}\,x} \cos\left(\sqrt{2}\,x\right) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ x\,\mathrm{e}^{\sqrt{2}\,x} \sin\left(\sqrt{2}\,x\right)+\mathrm{e}^{-\sqrt{2}\,x} \cos\left(\sqrt{2}\,x\right) \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & 0 & 0 & 0 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\sqrt{2}-\mathrm{I}\sqrt{2}, \left[\begin{array}{c} \frac{1}{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{3}} \\ \frac{1}{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{2}} \\ \frac{1}{-\sqrt{2}-\mathrm{I}\sqrt{2}} \\ 1 \end{array}\right]\right], \left[-\sqrt{2}+\mathrm{I}\sqrt{2}, \left[\begin{array}{c} \frac{1}{\left(-\sqrt{2}+\mathrm{I}\sqrt{2}\right)^{3}} \\ \frac{1}{\left(-\sqrt{2}+\mathrm{I}\sqrt{2}\right)^{2}} \\ \frac{1}{-\sqrt{2}+\mathrm{I}\sqrt{2}} \\ 1 \end{array}\right]\right], \left[\sqrt{2}-\mathrm{I}\sqrt{2}, \left[\begin{array}{c} \frac{1}{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{3}} \\ \frac{1}{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{2}} \\ \frac{1}{\sqrt{2}-\mathrm{I}\sqrt{2}} \\ 1 \end{array}\right.\right.$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\sqrt{2}-\mathrm{I}\sqrt{2}, \left[\begin{array}{c} \frac{1}{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{3}} \\ \frac{1}{\left(-\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{2}} \\ \frac{1}{-\sqrt{2}-\mathrm{I}\sqrt{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(-\sqrt{2}-I\sqrt{2}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(-\sqrt{2}-I\sqrt{2}\right)^3} \\ \frac{1}{\left(-\sqrt{2}-I\sqrt{2}\right)^2} \\ \frac{1}{-\sqrt{2}-I\sqrt{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-\sqrt{2}\,x} \cdot \left(\cos\left(\sqrt{2}\,x\right) - I\sin\left(\sqrt{2}\,x\right)\right) \cdot \begin{bmatrix} \frac{1}{\left(-\sqrt{2}-I\sqrt{2}\right)^3} \\ \frac{1}{\left(-\sqrt{2}-I\sqrt{2}\right)^2} \\ \frac{1}{-\sqrt{2}-I\sqrt{2}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-\sqrt{2}\,x} \cdot \begin{bmatrix} \frac{\cos\left(\sqrt{2}\,x\right)-I\sin\left(\sqrt{2}\,x\right)}{\left(-\sqrt{2}-I\sqrt{2}\right)^3} \\ \frac{\cos\left(\sqrt{2}\,x\right)-I\sin\left(\sqrt{2}\,x\right)}{\left(-\sqrt{2}-I\sqrt{2}\right)^2} \\ \frac{\cos\left(\sqrt{2}\,x\right)-I\sin\left(\sqrt{2}\,x\right)}{-\sqrt{2}-I\sqrt{2}} \\ \cos\left(\sqrt{2}\,x\right) - I\sin\left(\sqrt{2}\,x\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = e^{-\sqrt{2}\,x} \cdot \begin{bmatrix} \frac{\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{16} + \frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{16} \\ -\frac{\sin\left(\sqrt{2}\,x\right)}{4} \\ -\frac{\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{4} + \frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{4} \\ \cos\left(\sqrt{2}\,x\right) \end{bmatrix}, \vec{y}_2(x) = e^{-\sqrt{2}\,x} \cdot \begin{bmatrix} \frac{\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{16} - \frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{16} \\ -\frac{\cos\left(\sqrt{2}\,x\right)}{4} \\ \frac{\cos\left(\sqrt{2}\,x\right)\sqrt{2}}{4} + \frac{\sin\left(\sqrt{2}\,x\right)\sqrt{2}}{4} \\ -\sin\left(\sqrt{2}\,x\right) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\sqrt{2}-\mathrm{I}\sqrt{2}, \left[\begin{array}{c} \frac{1}{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{3}} \\ \frac{1}{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{2}} \\ \frac{1}{\sqrt{2}-\mathrm{I}\sqrt{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{3}} \\ \frac{1}{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{2}} \\ \frac{1}{\sqrt{2}-\mathrm{I}\sqrt{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{\sqrt{2}\,x} \cdot \left(\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{3}} \\ \frac{1}{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{2}} \\ \frac{1}{\sqrt{2}-\mathrm{I}\sqrt{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{\sqrt{2}\,x} \cdot \left[\begin{array}{c} \frac{\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{3}} \\ \frac{\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{\left(\sqrt{2}-\mathrm{I}\sqrt{2}\right)^{2}} \\ \frac{\cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right)}{\sqrt{2}-\mathrm{I}\sqrt{2}} \\ \cos\left(\sqrt{2}\,x\right)-\mathrm{I}\sin\left(\sqrt{2}\,x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^{\sqrt{2}\,x}\cdot\left[\begin{array}{c} -\frac{\cos(\sqrt{2}\,x)\sqrt{2}}{16}+\frac{\sin(\sqrt{2}\,x)\sqrt{2}}{16} \\ \frac{\sin(\sqrt{2}\,x)}{4} \\ \frac{\cos(\sqrt{2}\,x)\sqrt{2}}{4}+\frac{\sin(\sqrt{2}\,x)\sqrt{2}}{4} \\ \cos(\sqrt{2}\,x) \end{array}\right], \vec{y}_4(x) = e^{\sqrt{2}\,x}\cdot\left[\begin{array}{c} \frac{\cos(\sqrt{2}\,x)\sqrt{2}}{16}+\frac{\sin(\sqrt{2}\,x)\sqrt{2}}{16} \\ \frac{\cos(\sqrt{2}\,x)}{4} \\ \frac{\cos(\sqrt{2}\,x)\sqrt{2}}{4}-\frac{\sin(\sqrt{2}\,x)\sqrt{2}}{4} \\ -\sin(\sqrt{2}\,x) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

  $\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \left[\begin{array}{ccc} e^{-\sqrt{2}\,x}\left(\frac{\cos(\sqrt{2}\,x)\sqrt{2}}{16}+\frac{\sin(\sqrt{2}\,x)\sqrt{2}}{16}\right) & e^{-\sqrt{2}\,x}\left(\frac{\cos(\sqrt{2}\,x)\sqrt{2}}{16}-\frac{\sin(\sqrt{2}\,x)\sqrt{2}}{16}\right) & e^{\sqrt{2}\,x}\left(-\frac{\cos(\sqrt{}}{1}\right. \\ -\frac{e^{-\sqrt{2}\,x}\sin(\sqrt{2}\,x)}{4} & -\frac{e^{-\sqrt{2}\,x}\cos(\sqrt{2}\,x)}{4} & \frac{e^{\sqrt{2}}}{} \\ e^{-\sqrt{2}\,x}\left(-\frac{\cos(\sqrt{2}\,x)\sqrt{2}}{4}+\frac{\sin(\sqrt{2}\,x)\sqrt{2}}{4}\right) & e^{-\sqrt{2}\,x}\left(\frac{\cos(\sqrt{2}\,x)\sqrt{2}}{4}+\frac{\sin(\sqrt{2}\,x)\sqrt{2}}{4}\right) & e^{\sqrt{2}\,x}\left(\frac{\cos(\sqrt{2}}{4}\right. \\ e^{-\sqrt{2}\,x}\cos(\sqrt{2}\,x) & -e^{-\sqrt{2}\,x}\sin(\sqrt{2}\,x) & e^{\sqrt{2}\,x} \end{array}\right.$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \left[\begin{array}{ccc} e^{-\sqrt{2}\,x}\left(\frac{\cos(\sqrt{2}\,x)\sqrt{2}}{16}+\frac{\sin(\sqrt{2}\,x)\sqrt{2}}{16}\right) & e^{-\sqrt{2}\,x}\left(\frac{\cos(\sqrt{2}\,x)\sqrt{2}}{16}-\frac{\sin(\sqrt{2}\,x)\sqrt{2}}{16}\right) & e^{\sqrt{2}\,x}\left(-\frac{\cos(\sqrt{}}{1}\right. \\ -\frac{e^{-\sqrt{2}\,x}\sin(\sqrt{2}\,x)}{4} & -\frac{e^{-\sqrt{2}\,x}\cos(\sqrt{2}\,x)}{4} & \frac{e^{\sqrt{}}}{} \\ e^{-\sqrt{2}\,x}\left(-\frac{\cos(\sqrt{2}\,x)\sqrt{2}}{4}+\frac{\sin(\sqrt{2}\,x)\sqrt{2}}{4}\right) & e^{-\sqrt{2}\,x}\left(\frac{\cos(\sqrt{2}\,x)\sqrt{2}}{4}+\frac{\sin(\sqrt{2}\,x)\sqrt{2}}{4}\right) & e^{\sqrt{2}\,x}\left(\frac{\cos(\sqrt{2}}{4}\right. \\ e^{-\sqrt{2}\,x}\cos(\sqrt{2}\,x) & -e^{-\sqrt{2}\,x}\sin(\sqrt{2}\,x) & e^{\sqrt{2}\,x} \end{array}\right.$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{\cos(\sqrt{2}\,x)\left(e^{-\sqrt{2}\,x}+e^{\sqrt{2}\,x}\right)}{2} & \frac{\sqrt{2}\left(e^{-}\right.}{} \\ \frac{\left(\left(-\cos(\sqrt{2}\,x)-\sin(\sqrt{2}\,x)\right)e^{-\sqrt{2}\,x}+e^{\sqrt{2}\,x}\left(-\sin(\sqrt{2}\,x)+\cos(\sqrt{2}\,x)\right)\right)\sqrt{2}}{2} & \\ -2\sin\left(\sqrt{2}\,x\right)\left(-e^{-\sqrt{2}\,x}+e^{\sqrt{2}\,x}\right) & \frac{\left(\left(-\cos\right.\right.}{} \\ -2\sqrt{2}\left(e^{-\sqrt{2}\,x}\left(-\cos\left(\sqrt{2}\,x\right)+\sin\left(\sqrt{2}\,x\right)\right)+e^{\sqrt{2}\,x}\left(\cos\left(\sqrt{2}\,x\right)+\sin\left(\sqrt{2}\,x\right)\right)\right) & \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$
- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{\left(\left(8+(16x+1)\sqrt{2}\right)\sin\left(\sqrt{2}\,x\right)+16\cos\left(\sqrt{2}\,x\right)\left(1+\left(x-\frac{3}{16}\right)\sqrt{2}\right)\right)\mathrm{e}^{-\sqrt{2}\,x}}{1024} - \frac{\left(\left(\sqrt{2}\,x^2-3x+\frac{7\sqrt{2}}{8}-1\right)\sin\left(\sqrt{2}\,x\right)+\cos\cdots}{128} \\ -\frac{\mathrm{e}^{-\sqrt{2}\,x}\left(\left(\left(\frac{1}{4}+\frac{(1-x)\sqrt{2}}{4}+x^2\right)\cos\left(\sqrt{2}\,x\right)-\frac{\left(\frac{1}{2}+(x+3)\sqrt{2}\right)\sin\left(\sqrt{2}\,x\right)}{4}\right)\mathrm{e}^{2\sqrt{2}\,x}+\frac{\left(-\sqrt{2}-1\right)\cos\left(\sqrt{2}\,x\right)}{4}+2\left(x+\cdots\right.\right.}{32} \\ \frac{\left(\left(-8+(-16x-1)\sqrt{2}\right)\cos\left(\sqrt{2}\,x\right)+16\sin\left(\sqrt{2}\,x\right)\left(-1+\left(x-\frac{3}{16}\right)\sqrt{2}\right)\right)\mathrm{e}^{-\sqrt{2}\,x}}{256} - \frac{\left(\left(\sqrt{2}\,x^2+x-\frac{\sqrt{2}}{8}-1\right)\cos\left(\sqrt{2}\,x\right)-\left(\cdots\right.\right.}{32} \\ \frac{\left(\left(16x-6\sqrt{2}-1\right)\cos\left(\sqrt{2}\,x\right)+\left(10\sqrt{2}+2\right)\sin\left(\sqrt{2}\,x\right)\right)\mathrm{e}^{-\sqrt{2}\,x}}{64} + \frac{\mathrm{e}^{\sqrt{2}\,x}\left(\left(\frac{1}{8}+\left(-\frac{3x}{4}+\frac{3}{4}\right)\sqrt{2}\right)\cos\left(\sqrt{2}\,x\right)+\left(\left(\frac{3x}{4}+\cdots\right.\right.\right.}{8} \end{array}\right.$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \left[\begin{array}{c} \frac{\left(\left(8+(16x+1)\sqrt{2}\right)\sin\left(\sqrt{2}\,x\right)+16\cos\left(\sqrt{2}\,x\right)\left(1+\left(x-\frac{3}{16}\right)\cdots\right.\right.}{1024} \\ -\frac{\mathrm{e}^{-\sqrt{2}\,x}\left(\left(\left(\frac{1}{4}+\frac{(1-x)\sqrt{2}}{4}+x^2\right)\cos\left(\sqrt{2}\,x\right)-\cdots\right.\right.}{\cdots} \\ \frac{\left(\left(-8+(-16x-1)\sqrt{2}\right)\cos\left(\sqrt{2}\,x\right)+16\sin\left(\sqrt{2}\,x\right)\left(-1+(x-\cdots\right.\right.}{256} \\ \frac{\left(\left(16x-6\sqrt{2}-1\right)\cos\left(\sqrt{2}\,x\right)+\left(10\sqrt{2}+2\right)\sin\left(\sqrt{2}\,x\right)\right)\cdots}{64} \end{array}\right.$$

- First component of the vector is the solution to the ODE

$$y = \frac{\left(\left(8+(16x+64c_1-64c_2+1)\sqrt{2}\right)\sin\left(\sqrt{2}\,x\right)+16\left(1+\left(x+4c_1+4c_2-\frac{3}{16}\right)\sqrt{2}\right)\cos\left(\sqrt{2}\,x\right)\right)\mathrm{e}^{-\sqrt{2}\,x}}{1024} - \frac{\mathrm{e}^{\sqrt{2}\,x}\left(\left(\left(x^2-8c_3-8c_4+\frac{7}{8}\right)\sqrt{\cdots}\right.\right.}{\cdots}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 108

```
dsolve(diff(y(x),x$4)+16*y(x)=x*exp(sqrt(2)*x)*sin(sqrt(2)*x)+exp(-sqrt(2)*x)*cos(sqrt(2)*x)
```

$$y(x) = \frac{\left(\left(2x\sqrt{2}+128c_3+3\right)\cos\left(x\sqrt{2}\right)+2\sin\left(x\sqrt{2}\right)\left(x\sqrt{2}+64c_4\right)\right)\mathrm{e}^{-x\sqrt{2}}}{128} - \frac{\left(\left(x^2\sqrt{2}-128c_1-\frac{5\sqrt{2}}{8}\right)\cos\left(x\sqrt{2}\right)+\sin\left(x\sqrt{2}\right)\left(x^2\sqrt{2}-3x-128c_2+\frac{5\sqrt{2}}{8}\right)\right)\mathrm{e}^{x\sqrt{2}}}{128}$$

✓ Solution by Mathematica

Time used: 2.857 (sec). Leaf size: 140

```
DSolve[y''''[x]+16*y[x]==x*Exp[Sqrt[2]*x]*Sin[Sqrt[2]*x]+Exp[-Sqrt[2]*x]*Cos[Sqrt[2]*x],y[x]
```

$$y(x) \rightarrow \frac{e^{-\sqrt{2}x}\left(\left(e^{2\sqrt{2}x}\left(-8\sqrt{2}x^2+5\sqrt{2}+1024c_1\right)+8\left(2\sqrt{2}x+3+128c_2\right)\right)\cos\left(\sqrt{2}x\right)-\left(e^{2\sqrt{2}x}\left(8\sqrt{2}x^2-24x\right.\right.\right.}{1024}$$

11.53 problem 53

Internal problem ID [11827]
Internal file name [`OUTPUT/11836_Thursday_April_11_2024_08_52_46_PM_21129270/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 53.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _linear, _nonhomogeneous]]`

$$\boxed{y'''' + 3y'' - 4y = \cos(x)^2 - \cosh(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 3y'' - 4y = 0$$

The characteristic equation is

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$
$$\lambda_3 = 1$$
$$\lambda_4 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{2ix} c_3 + \mathrm{e}^{-2ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= \mathrm{e}^{x} \\ y_3 &= \mathrm{e}^{2ix} \\ y_4 &= \mathrm{e}^{-2ix} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' + 3y'' - 4y = \cos(x)^2 - \cosh(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} & \mathrm{e}^{2ix} & \mathrm{e}^{-2ix} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} & 2i\mathrm{e}^{2ix} & -2i\mathrm{e}^{-2ix} \\ \mathrm{e}^{-x} & \mathrm{e}^{x} & -4\,\mathrm{e}^{2ix} & -4\,\mathrm{e}^{-2ix} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} & -8i\mathrm{e}^{2ix} & 8i\mathrm{e}^{-2ix} \end{bmatrix}$$

$$|W| = -200i\mathrm{e}^{-x}\mathrm{e}^{x}\mathrm{e}^{2ix}\mathrm{e}^{-2ix}$$

The determinant simplifies to

$$|W| = -200i$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} e^x & e^{2ix} & e^{-2ix} \\ e^x & 2ie^{2ix} & -2ie^{-2ix} \\ e^x & -4\,e^{2ix} & -4\,e^{-2ix} \end{bmatrix}$$
$$= -20ie^x$$

$$W_2(x) = \det \begin{bmatrix} e^{-x} & e^{2ix} & e^{-2ix} \\ -e^{-x} & 2ie^{2ix} & -2ie^{-2ix} \\ e^{-x} & -4\,e^{2ix} & -4\,e^{-2ix} \end{bmatrix}$$
$$= -20ie^{-x}$$

$$W_3(x) = \det \begin{bmatrix} e^{-x} & e^x & e^{-2ix} \\ -e^{-x} & e^x & -2ie^{-2ix} \\ e^{-x} & e^x & -4\,e^{-2ix} \end{bmatrix}$$
$$= -10\,e^{-2ix}$$

$$W_4(x) = \det \begin{bmatrix} e^{-x} & e^x & e^{2ix} \\ -e^{-x} & e^x & 2ie^{2ix} \\ e^{-x} & e^x & -4\,e^{2ix} \end{bmatrix}$$
$$= -10\,e^{2ix}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1} \int \frac{F(x) W_1(x)}{a W(x)}\, dx \\ &= (-1)^3 \int \frac{\left(\cos(x)^2 - \cosh(x)\right)\left(-20ie^x\right)}{(1)\,(-200i)}\, dx \\ &= -\int \frac{-20i\left(\cos(x)^2 - \cosh(x)\right) e^x}{-200i}\, dx \\ &= -\int \left(\frac{\left(\cos(x)^2 - \cosh(x)\right) e^x}{10}\right) dx \\ &= \frac{x}{20} - \frac{\left(\cos(x) + 2\sin(x)\right) e^x \cos(x)}{50} - \frac{e^x}{25} + \frac{e^{2x}}{40} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x) W_2(x)}{a W(x)}\, dx \\
&= (-1)^2 \int \frac{\left(\cos(x)^2 - \cosh(x)\right)\left(-20 i\,\mathrm{e}^{-x}\right)}{(1)\,(-200 i)}\, dx \\
&= \int \frac{-20 i \left(\cos(x)^2 - \cosh(x)\right) \mathrm{e}^{-x}}{-200 i}\, dx \\
&= \int \left(\frac{\left(\cos(x)^2 - \cosh(x)\right) \mathrm{e}^{-x}}{10}\right) dx \\
&= \frac{\left(-\cos(x) + 2\sin(x)\right) \mathrm{e}^{-x} \cos(x)}{50} - \frac{\mathrm{e}^{-x}}{25} - \frac{x}{20} + \frac{\mathrm{e}^{-2x}}{40}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^1 \int \frac{\left(\cos(x)^2 - \cosh(x)\right)\left(-10\,\mathrm{e}^{-2ix}\right)}{(1)\,(-200 i)}\, dx \\
&= -\int \frac{-10 \left(\cos(x)^2 - \cosh(x)\right) \mathrm{e}^{-2ix}}{-200 i}\, dx \\
&= -\int \left(-\frac{i \left(\cos(x)^2 - \cosh(x)\right) \mathrm{e}^{-2ix}}{20}\right) dx \\
&= \frac{i\mathrm{e}^{(-1-2i)x}}{200} - \frac{\mathrm{e}^{-2ix}}{80} + \frac{\mathrm{e}^{(1-2i)x}}{100} + \frac{\mathrm{e}^{(-1-2i)x}}{100} - \frac{i\mathrm{e}^{(1-2i)x}}{200} + \frac{ix}{80} - \frac{\mathrm{e}^{-4ix}}{320}
\end{aligned}
$$

$$
\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{\left(\cos(x)^2 - \cosh(x)\right)\left(-10\,\mathrm{e}^{2ix}\right)}{(1)\,(-200 i)}\, dx \\
&= \int \frac{-10 \left(\cos(x)^2 - \cosh(x)\right) \mathrm{e}^{2ix}}{-200 i}\, dx \\
&= \int \left(-\frac{i \left(\cos(x)^2 - \cosh(x)\right) \mathrm{e}^{2ix}}{20}\right) dx \\
&= \int -\frac{i \left(\cos(x)^2 - \cosh(x)\right) \mathrm{e}^{2ix}}{20} dx
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{x}{20}-\frac{(\cos(x)+2\sin(x))\,\mathrm{e}^{x}\cos(x)}{50}-\frac{\mathrm{e}^{x}}{25}+\frac{\mathrm{e}^{2x}}{40}\right)\left(\mathrm{e}^{-x}\right)\\
&+\left(\frac{(-\cos(x)+2\sin(x))\,\mathrm{e}^{-x}\cos(x)}{50}-\frac{\mathrm{e}^{-x}}{25}-\frac{x}{20}+\frac{\mathrm{e}^{-2x}}{40}\right)\left(\mathrm{e}^{x}\right)\\
&+\left(\frac{i\,\mathrm{e}^{(-1-2i)x}}{200}-\frac{\mathrm{e}^{-2ix}}{80}+\frac{\mathrm{e}^{(1-2i)x}}{100}+\frac{\mathrm{e}^{(-1-2i)x}}{100}-\frac{i\,\mathrm{e}^{(1-2i)x}}{200}+\frac{ix}{80}-\frac{\mathrm{e}^{-4ix}}{320}\right)\left(\mathrm{e}^{2ix}\right)\\
&+\left(\int -\frac{i\left(\cos(x)^{2}-\cosh(x)\right)\mathrm{e}^{2ix}}{20}dx\right)\left(\mathrm{e}^{-2ix}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = -\frac{1}{8}+\frac{(9+10x)\,\mathrm{e}^{-x}}{200}-\frac{\sin(2x)\,(8x+i)}{320}-\frac{37\cos(2x)}{1600}+\frac{(9-10x)\,\mathrm{e}^{x}}{200}
$$

Which simplifies to

$$
y_p = -\frac{1}{8}+\frac{(9+10x)\,\mathrm{e}^{-x}}{200}-\frac{\sin(2x)\,(8x+i)}{320}-\frac{37\cos(2x)}{1600}+\frac{(9-10x)\,\mathrm{e}^{x}}{200}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p\\
&= \left(c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{x}+\mathrm{e}^{2ix}c_3+\mathrm{e}^{-2ix}c_4\right)\\
&\quad+\left(-\frac{1}{8}+\frac{(9+10x)\,\mathrm{e}^{-x}}{200}-\frac{\sin(2x)\,(8x+i)}{320}-\frac{37\cos(2x)}{1600}+\frac{(9-10x)\,\mathrm{e}^{x}}{200}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{x}+\mathrm{e}^{2ix}c_3+\mathrm{e}^{-2ix}c_4-\frac{1}{8}+\frac{(9+10x)\,\mathrm{e}^{-x}}{200}\\
&-\frac{\sin(2x)\,(8x+i)}{320}-\frac{37\cos(2x)}{1600}+\frac{(9-10x)\,\mathrm{e}^{x}}{200}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1\mathrm{e}^{-x}+c_2\mathrm{e}^{x}+\mathrm{e}^{2ix}c_3+\mathrm{e}^{-2ix}c_4-\frac{1}{8}+\frac{(9+10x)\,\mathrm{e}^{-x}}{200}\\
&-\frac{\sin(2x)\,(8x+i)}{320}-\frac{37\cos(2x)}{1600}+\frac{(9-10x)\,\mathrm{e}^{x}}{200}
\end{aligned}
$$

Verified OK.

11.53.1 Maple step by step solution

Let's solve

$y'''' + 3y'' - 4y = \cos(x)^2 - \cosh(x)$

- Highest derivative means the order of the ODE is 4

  y''''

☐ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = \cos(x)^2 - \cosh(x) - 3y_3(x) + 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = \cos(x)^2 - \cosh(x) - 3y_3(x) + 4y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & -3 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos(x)^2 - \cosh(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \cos(x)^2 - \cosh(x) \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & -3 & 0 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right], \left[-2\,\mathrm{I}, \left[\begin{array}{c} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[2\,\mathrm{I}, \left[\begin{array}{c} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1 \\ 1 \\ 1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{x} \cdot \left[\begin{array}{c}1 \\ 1 \\ 1 \\ 1\end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \left[\begin{array}{c}-\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1\end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-2\,\mathrm{I}x} \cdot \left[\begin{array}{c}-\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\left(\cos\left(2x\right) - \mathrm{I}\sin\left(2x\right)\right) \cdot \left[\begin{array}{c}-\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1\end{array}\right]$$

- Simplify expression

$$\begin{bmatrix} -\frac{I}{8}(\cos(2x) - I\sin(2x)) \\ -\frac{\cos(2x)}{4} + \frac{I\sin(2x)}{4} \\ \frac{I}{2}(\cos(2x) - I\sin(2x)) \\ \cos(2x) - I\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -e^{-x} & e^{x} & -\frac{\sin(2x)}{8} & -\frac{\cos(2x)}{8} \\ e^{-x} & e^{x} & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} \\ -e^{-x} & e^{x} & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} \\ e^{-x} & e^{x} & \cos(2x) & -\sin(2x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -e^{-x} & e^{x} & -\frac{\sin(2x)}{8} & -\frac{\cos(2x)}{8} \\ e^{-x} & e^{x} & -\frac{\cos(2x)}{4} & \frac{\sin(2x)}{4} \\ -e^{-x} & e^{x} & \frac{\sin(2x)}{2} & \frac{\cos(2x)}{2} \\ e^{-x} & e^{x} & \cos(2x) & -\sin(2x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 & 0 & -\frac{1}{8} \\ 1 & 1 & -\frac{1}{4} & 0 \\ -1 & 1 & 0 & \frac{1}{2} \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \left[\begin{array}{cccc} \frac{2\,\mathrm{e}^{-x}}{5}+\frac{2\,\mathrm{e}^{x}}{5}+\frac{\cos(2x)}{5} & -\frac{2\,\mathrm{e}^{-x}}{5}+\frac{2\,\mathrm{e}^{x}}{5}+\frac{\sin(2x)}{10} & \frac{\mathrm{e}^{-x}}{10}+\frac{\mathrm{e}^{x}}{10}-\frac{\cos(2x)}{5} & -\frac{\mathrm{e}^{-x}}{10}+\frac{\mathrm{e}^{x}}{10}-\mathrm{si} \\ -\frac{2\,\mathrm{e}^{-x}}{5}+\frac{2\,\mathrm{e}^{x}}{5}-\frac{2\sin(2x)}{5} & \frac{2\,\mathrm{e}^{-x}}{5}+\frac{2\,\mathrm{e}^{x}}{5}+\frac{\cos(2x)}{5} & -\frac{\mathrm{e}^{-x}}{10}+\frac{\mathrm{e}^{x}}{10}+\frac{2\sin(2x)}{5} & \frac{\mathrm{e}^{-x}}{10}+\frac{\mathrm{e}^{x}}{10}-\mathrm{cos} \\ \frac{2\,\mathrm{e}^{-x}}{5}+\frac{2\,\mathrm{e}^{x}}{5}-\frac{4\cos(2x)}{5} & -\frac{2\,\mathrm{e}^{-x}}{5}+\frac{2\,\mathrm{e}^{x}}{5}-\frac{2\sin(2x)}{5} & \frac{\mathrm{e}^{-x}}{10}+\frac{\mathrm{e}^{x}}{10}+\frac{4\cos(2x)}{5} & -\frac{\mathrm{e}^{-x}}{10}+\frac{\mathrm{e}^{x}}{10}+2\,\mathrm{s} \\ -\frac{2\,\mathrm{e}^{-x}}{5}+\frac{2\,\mathrm{e}^{x}}{5}+\frac{8\sin(2x)}{5} & \frac{2\,\mathrm{e}^{-x}}{5}+\frac{2\,\mathrm{e}^{x}}{5}-\frac{4\cos(2x)}{5} & -\frac{\mathrm{e}^{-x}}{10}+\frac{\mathrm{e}^{x}}{10}-\frac{8\sin(2x)}{5} & \frac{\mathrm{e}^{-x}}{10}+\frac{\mathrm{e}^{x}}{10}+4\,\mathrm{co} \end{array}\right.$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$
  - Take the derivative of the particular solution

    $\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$
  - Cancel like terms

    $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$
  - Multiply by the inverse of the fundamental matrix

    $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$
  - Integrate to solve for $\overrightarrow{v}(x)$

    $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\,ds$
  - Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\,ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \frac{(5x+8)\mathrm{e}^{-x}}{100}-\frac{7\cos(x)^2}{100}-\frac{\sin(x)\cos(x)x}{20}-\frac{9}{100}+\frac{(-5x+8)\mathrm{e}^{x}}{100} \\ \frac{(-5x-3)\mathrm{e}^{-x}}{100}-\frac{\cos(x)^2 x}{10}+\frac{9\cos(x)\sin(x)}{100}+\frac{(-5x+3)\mathrm{e}^{x}}{100}+\frac{x}{20} \\ \frac{(5x-2)\mathrm{e}^{-x}}{100}+\frac{2\cos(x)^2}{25}+\frac{\sin(x)\cos(x)x}{5}-\frac{1}{25}+\frac{(-5x-2)\mathrm{e}^{x}}{100} \\ \frac{(-5x+7)\mathrm{e}^{-x}}{100}+\frac{x\cos(2x)}{5}+\frac{\sin(2x)}{50}+\frac{(-5x-7)\mathrm{e}^{x}}{100} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3(x) + c_4 \vec{y}_4(x) + \begin{bmatrix} \frac{(5x+8)e^{-x}}{100} - \frac{7\cos(x)^2}{100} - \frac{\sin(x)\cos(x)x}{20} - \frac{9}{100} + \frac{(-5x+8)e^{x}}{100} \\ \frac{(-5x-3)e^{-x}}{100} - \frac{\cos(x)^2 x}{10} + \frac{9\cos(x)\sin(x)}{100} + \frac{(-5x+3)e^{x}}{100} + \frac{x}{20} \\ \frac{(5x-2)e^{-x}}{100} + \frac{2\cos(x)^2}{25} + \frac{\sin(x)\cos(x)x}{5} - \frac{1}{25} + \frac{(-5x-2)e^{x}}{100} \\ \frac{(-5x+7)e^{-x}}{100} + \frac{x\cos(2x)}{5} + \frac{\sin(2x)}{50} + \frac{(-5x-7)e^{x}}{100} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{9}{100} + \frac{(8+5x-100c_1)e^{-x}}{100} + \frac{(-7-25c_4)\cos(x)^2}{100} - \frac{\sin(x)(x+5c_3)\cos(x)}{20} + \frac{(8-5x+100c_2)e^{x}}{100} + \frac{c_4}{8}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 56

```
dsolve(diff(y(x),x$4)+3*diff(y(x),x$2)-4*y(x)=cos(x)^2-cosh(x),y(x), singsol=all)
```

$$y(x) = -\frac{1}{8} + \frac{(10x + 200c_3 + 9)\,e^{-x}}{200} + \frac{(200c_2 - 9)\cos(2x)}{200} + \frac{(-x + 40c_4)\sin(2x)}{40} + \frac{(-10x + 200c_1 + 9)\,e^{x}}{200}$$

✓ Solution by Mathematica

Time used: 0.21 (sec). Leaf size: 75

```
DSolve[y''''[x]+3*y''[x]-4*y[x]==Cos[x]^2-Cosh[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{400}e^{-x}\big((-13+400c_1)e^x\cos(2x) + 2\big(10x - 25e^x + e^{2x}(-10x+9+200c_4) - 5e^x(x-40c_2)\sin(2x) + 9 + 200c_3\big)\big)$$

11.54 problem 54

Internal problem ID [11828]
Internal file name [`OUTPUT/11837_Thursday_April_11_2024_08_52_48_PM_71487248/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.3. The method of undetermined coefficients. Exercises page 151
Problem number: 54.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' + 10y'' + 9y = \sin(x)\sin(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' + 10y'' + 9y = 0$$

The characteristic equation is

$$\lambda^4 + 10\lambda^2 + 9 = 0$$

The roots of the above equation are

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$
$$\lambda_3 = i$$
$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = e^{ix}c_1 + e^{3ix}c_2 + e^{-ix}c_3 + e^{-3ix}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{ix}$$

$$y_2 = e^{3ix}$$

$$y_3 = e^{-ix}$$

$$y_4 = e^{-3ix}$$

Now the particular solution to the given ODE is found

$$y'''' + 10y'' + 9y = \sin(x)\sin(2x)$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3 + U_4y_4$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)}\,dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{ix} & e^{3ix} & e^{-ix} & e^{-3ix} \\ ie^{ix} & 3ie^{3ix} & -ie^{-ix} & -3ie^{-3ix} \\ -e^{ix} & -9\,e^{3ix} & -e^{-ix} & -9\,e^{-3ix} \\ -ie^{ix} & -27ie^{3ix} & ie^{-ix} & 27ie^{-3ix} \end{bmatrix}$$

$$|W| = 768\,e^{ix}e^{3ix}e^{-ix}e^{-3ix}$$

The determinant simplifies to

$$|W| = 768$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} e^{3ix} & e^{-ix} & e^{-3ix} \\ 3ie^{3ix} & -ie^{-ix} & -3ie^{-3ix} \\ -9\,e^{3ix} & -e^{-ix} & -9\,e^{-3ix} \end{bmatrix}$$
$$= 48ie^{-ix}$$

$$W_2(x) = \det \begin{bmatrix} e^{ix} & e^{-ix} & e^{-3ix} \\ ie^{ix} & -ie^{-ix} & -3ie^{-3ix} \\ -e^{ix} & -e^{-ix} & -9\,e^{-3ix} \end{bmatrix}$$
$$= 16ie^{-3ix}$$

$$W_3(x) = \det \begin{bmatrix} e^{ix} & e^{3ix} & e^{-3ix} \\ ie^{ix} & 3ie^{3ix} & -3ie^{-3ix} \\ -e^{ix} & -9\,e^{3ix} & -9\,e^{-3ix} \end{bmatrix}$$
$$= -48ie^{ix}$$

$$W_4(x) = \det \begin{bmatrix} e^{ix} & e^{3ix} & e^{-ix} \\ ie^{ix} & 3ie^{3ix} & -ie^{-ix} \\ -e^{ix} & -9\,e^{3ix} & -e^{-ix} \end{bmatrix}$$
$$= -16ie^{3ix}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{4-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^3 \int \frac{(\sin(x)\sin(2x))\,(48ie^{-ix})}{(1)\,(768)}\,dx \\ &= -\int \frac{48i\sin(x)\sin(2x)\,e^{-ix}}{768}\,dx \\ &= -\int \left(\frac{i\sin(x)\sin(2x)\,e^{-ix}}{16}\right)\,dx \\ &= -\left(\int \frac{i\sin(x)\sin(2x)\,e^{-ix}}{16}dx\right) \end{aligned}$$

$$\begin{aligned}
U_2 &= (-1)^{4-2} \int \frac{F(x) W_2(x)}{a W(x)} \, dx \\
&= (-1)^2 \int \frac{(\sin(x)\sin(2x))\left(16i \mathrm{e}^{-3ix}\right)}{(1)(768)} \, dx \\
&= \int \frac{16i \sin(x) \sin(2x) \mathrm{e}^{-3ix}}{768} \, dx \\
&= \int \left(\frac{i \sin(x) \sin(2x) \mathrm{e}^{-3ix}}{48}\right) dx \\
&= \int \frac{i \sin(x) \sin(2x) \mathrm{e}^{-3ix}}{48} dx
\end{aligned}$$

$$\begin{aligned}
U_3 &= (-1)^{4-3} \int \frac{F(x) W_3(x)}{a W(x)} \, dx \\
&= (-1)^1 \int \frac{(\sin(x)\sin(2x))\left(-48i \mathrm{e}^{ix}\right)}{(1)(768)} \, dx \\
&= -\int \frac{-48i \sin(x) \sin(2x) \mathrm{e}^{ix}}{768} \, dx \\
&= -\int \left(-\frac{i \sin(x) \sin(2x) \mathrm{e}^{ix}}{16}\right) dx \\
&= -\left(\int -\frac{i \sin(x) \sin(2x) \mathrm{e}^{ix}}{16} dx\right)
\end{aligned}$$

$$\begin{aligned}
U_4 &= (-1)^{4-4} \int \frac{F(x) W_4(x)}{a W(x)} \, dx \\
&= (-1)^0 \int \frac{(\sin(x)\sin(2x))\left(-16i \mathrm{e}^{3ix}\right)}{(1)(768)} \, dx \\
&= \int \frac{-16i \sin(x) \sin(2x) \mathrm{e}^{3ix}}{768} \, dx \\
&= \int \left(-\frac{i \sin(x) \sin(2x) \mathrm{e}^{3ix}}{48}\right) dx \\
&= \int -\frac{i \sin(x) \sin(2x) \mathrm{e}^{3ix}}{48} dx
\end{aligned}$$

Now that all the U_i functions have been determined, the particular solution is found from

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4$$

Hence

$$
\begin{aligned}
y_p = {} & \left(-\left(\int \frac{i \sin(x) \sin(2x)\, e^{-ix}}{16} dx\right)\right) \left(e^{ix}\right) \\
& + \left(\int \frac{i \sin(x) \sin(2x)\, e^{-3ix}}{48} dx\right) \left(e^{3ix}\right) \\
& + \left(-\left(\int -\frac{i \sin(x) \sin(2x)\, e^{ix}}{16} dx\right)\right) \left(e^{-ix}\right) \\
& + \left(\int -\frac{i \sin(x) \sin(2x)\, e^{3ix}}{48} dx\right) \left(e^{-3ix}\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{i\left(\left(\int \sin(x) \sin(2x)\, e^{-3ix} dx\right) e^{3ix} - 3\left(\int \sin(x) \sin(2x)\, e^{-ix} dx\right) e^{ix} + 3\left(\int \sin(x) \sin(2x)\, e^{ix} dx\right) e^{-ix} -}{48}
$$

Which simplifies to

$$
y_p = \frac{\left(\int \sin(x) \sin(2x) \sin(3x)\, dx\right) \cos(3x)}{24} - \frac{\left(\int \sin(x) \sin(2x) \cos(3x)\, dx\right) \sin(3x)}{24} - \frac{\left(\int \sin(x)^2 \sin(2x)\, d\right.}{8}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{ix} c_1 + e^{3ix} c_2 + e^{-ix} c_3 + e^{-3ix} c_4\right) \\
&\quad + \left(\frac{\left(\int \sin(x) \sin(2x) \sin(3x)\, dx\right) \cos(3x)}{24} - \frac{\left(\int \sin(x) \sin(2x) \cos(3x)\, dx\right) \sin(3x)}{24}\right. \\
&\quad \left. - \frac{\left(\int \sin(x)^2 \sin(2x)\, dx\right) \cos(x)}{8} + \frac{\left(\int (1 - \cos(4x))\, dx\right) \sin(x)}{32}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y = {} & e^{ix} c_1 + e^{3ix} c_2 + e^{-ix} c_3 + e^{-3ix} c_4 + \frac{\left(\int \sin(x) \sin(2x) \sin(3x)\, dx\right) \cos(3x)}{24} \\
& - \frac{\left(\int \sin(x) \sin(2x) \cos(3x)\, dx\right) \sin(3x)}{24} \\
& - \frac{\left(\int \sin(x)^2 \sin(2x)\, dx\right) \cos(x)}{8} + \frac{\left(\int (1 - \cos(4x))\, dx\right) \sin(x)}{32}
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = e^{ix}c_1 + e^{3ix}c_2 + e^{-ix}c_3 + e^{-3ix}c_4 + \frac{\left(\int \sin(x)\sin(2x)\sin(3x)\,dx\right)\cos(3x)}{24} - \frac{\left(\int \sin(x)\sin(2x)\cos(3x)\,dx\right)\sin(3x)}{24} - \frac{\left(\int \sin(x)^2\sin(2x)\,dx\right)\cos(x)}{8} + \frac{\left(\int (1-\cos(4x))\,dx\right)\sin(x)}{32}$$

Verified OK.

11.54.1 Maple step by step solution

Let's solve

$y'''' + 10y'' + 9y = \sin(x)\sin(2x)$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = \sin(x)\sin(2x) - 10y_3(x) - 9y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = \sin(x)\sin(2x) - 10y_3(x) - 9y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & 0 & -10 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(x)\sin(2x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(x)\sin(2x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & 0 & -10 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[3\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ -\frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3\,\mathrm{I}, \left[\begin{array}{c} -\frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{-3\,\mathrm{I}x} \cdot \left[\begin{array}{c} -\frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(3x) - \mathrm{I}\sin(3x)) \cdot \left[\begin{array}{c} -\frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} -\frac{\mathrm{I}}{27}(\cos(3x) - \mathrm{I}\sin(3x)) \\ -\frac{\cos(3x)}{9} + \frac{\mathrm{I}\sin(3x)}{9} \\ \frac{\mathrm{I}}{3}(\cos(3x) - \mathrm{I}\sin(3x)) \\ \cos(3x) - \mathrm{I}\sin(3x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = \left[\begin{array}{c} -\frac{\sin(3x)}{27} \\ -\frac{\cos(3x)}{9} \\ \frac{\sin(3x)}{3} \\ \cos(3x) \end{array}\right], \vec{y}_2(x) = \left[\begin{array}{c} -\frac{\cos(3x)}{27} \\ \frac{\sin(3x)}{9} \\ \frac{\cos(3x)}{3} \\ -\sin(3x) \end{array}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}x} \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix} -\frac{\sin(3x)}{27} & -\frac{\cos(3x)}{27} & -\sin(x) & -\cos(x) \\ -\frac{\cos(3x)}{9} & \frac{\sin(3x)}{9} & -\cos(x) & \sin(x) \\ \frac{\sin(3x)}{3} & \frac{\cos(3x)}{3} & \sin(x) & \cos(x) \\ \cos(3x) & -\sin(3x) & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t

  $\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix} -\frac{\sin(3x)}{27} & -\frac{\cos(3x)}{27} & -\sin(x) & -\cos(x) \\ -\frac{\cos(3x)}{9} & \frac{\sin(3x)}{9} & -\cos(x) & \sin(x) \\ \frac{\sin(3x)}{3} & \frac{\cos(3x)}{3} & \sin(x) & \cos(x) \\ \cos(3x) & -\sin(3x) & \cos(x) & -\sin(x) \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} 0 & -\frac{1}{27} & 0 & -1 \\ -\frac{1}{9} & 0 & -1 & 0 \\ 0 & \frac{1}{3} & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} -\frac{\cos(x)^3}{2}+\frac{3\cos(x)}{2} & -\frac{\sin(3x)}{24}+\frac{9\sin(x)}{8} & \frac{\sin(x)^2\cos(x)}{2} & \frac{\sin(x)^3}{6} \\ -\frac{3\sin(x)^3}{2} & -\frac{\cos(x)^3}{2}+\frac{3\cos(x)}{2} & \frac{3\sin(3x)}{8}-\frac{\sin(x)}{8} & \frac{\sin(x)^2\cos(x)}{2} \\ -\frac{9\sin(x)^2\cos(x)}{2} & -\frac{3\sin(x)^3}{2} & \frac{9\cos(x)^3}{2}-\frac{7\cos(x)}{2} & \frac{3\sin(3x)}{8}-\frac{\sin(x)}{8} \\ -\frac{27\sin(3x)}{8}+\frac{9\sin(x)}{8} & -\frac{9\sin(x)^2\cos(x)}{2} & -\frac{27\sin(3x)}{8}+\frac{\sin(x)}{8} & \frac{9\cos(x)^3}{2}-\frac{7\cos(x)}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x)=\Phi(x)\cdot\vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}'_p(x)=\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(x)\cdot\vec{v}(x)+\Phi(x)\cdot\vec{v}'(x)=A\cdot\Phi(x)\cdot\vec{v}(x)+\vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$
- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \frac{\sin(x)\left(2\cos(x)^2 x-3\cos(x)\sin(x)+x\right)}{48} \\ -\frac{7\cos(x)^2\sin(x)}{48}+\frac{\sin(x)}{12}+\frac{\cos(x)^3 x}{8}-\frac{\cos(x) x}{16} \\ -\frac{3\cos(x)^2\sin(x) x}{8}+\frac{5\sin(x)^2\cos(x)}{16}+\frac{x\sin(x)}{16} \\ \frac{9\cos(x)^2\sin(x)}{16}-\frac{\sin(x)}{4}-\frac{9\cos(x)^3 x}{8}+\frac{13\cos(x) x}{16} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1(x)+c_2\overrightarrow{y}_2(x)+c_3\overrightarrow{y}_3(x)+c_4\overrightarrow{y}_4(x)+\left[\begin{array}{c} \frac{\sin(x)\left(2\cos(x)^2 x-3\cos(x)\sin(x)+x\right)}{48} \\ -\frac{7\cos(x)^2\sin(x)}{48}+\frac{\sin(x)}{12}+\frac{\cos(x)^3 x}{8}-\frac{\cos(x) x}{16} \\ -\frac{3\cos(x)^2\sin(x) x}{8}+\frac{5\sin(x)^2\cos(x)}{16}+\frac{x\sin(x)}{16} \\ \frac{9\cos(x)^2\sin(x)}{16}-\frac{\sin(x)}{4}-\frac{9\cos(x)^3 x}{8}+\frac{13\cos(x) x}{16} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = -\frac{c_2\cos(3x)}{27}-\frac{c_1\sin(3x)}{27}-\frac{\sin(x)^2\cos(x)}{16}+\frac{\left(2\cos(x)^2 x+x-48c_3\right)\sin(x)}{48}-c_4\cos(x)$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
dsolve(diff(y(x),x$4)+10*diff(y(x),x$2)+9*y(x)=sin(x)*sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(11 + 1152c_3)\cos(3x)}{1152} + \frac{(x + 96c_4)\sin(3x)}{96} + \frac{(-1 + 64c_1)\cos(x)}{64} + \frac{\sin(x)(x + 32c_2)}{32}$$

✓ Solution by Mathematica

Time used: 0.079 (sec). Leaf size: 54

```
DSolve[y''''[x]+10*y''[x]+9*y[x]==Sin[x]*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{32}x\sin(x) + \frac{1}{96}x\sin(3x) + \left(-\frac{1}{64} + c_3\right)\cos(x) + \left(\frac{13}{576} + c_1\right)\cos(3x) + c_4\sin(x) + c_2\sin(3x)$$

12 Chapter 4, Section 4.4. Variation of parameters. Exercises page 162

12.1 problem 1

Internal problem ID [11829]
Internal file name [`OUTPUT/11838_Thursday_April_11_2024_08_52_50_PM_83565124/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \cot(x)}$$

12.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \cot(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\cot(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \cos(x)\,dx$$

Hence

$$u_1 = -\sin(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\cot(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int \cos(x)\cot(x)\,dx$$

Hence

$$u_2 = \cos(x) + \ln(\csc(x) - \cot(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)\sin(x) + (\cos(x) + \ln(\csc(x) - \cot(x)))\sin(x)$$

Which simplifies to

$$y_p(x) = \sin(x)\ln(\csc(x) - \cot(x))$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + (\sin(x) \ln(\csc(x) - \cot(x)))
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) + \sin(x) \ln(\csc(x) - \cot(x)) \tag{1}
$$

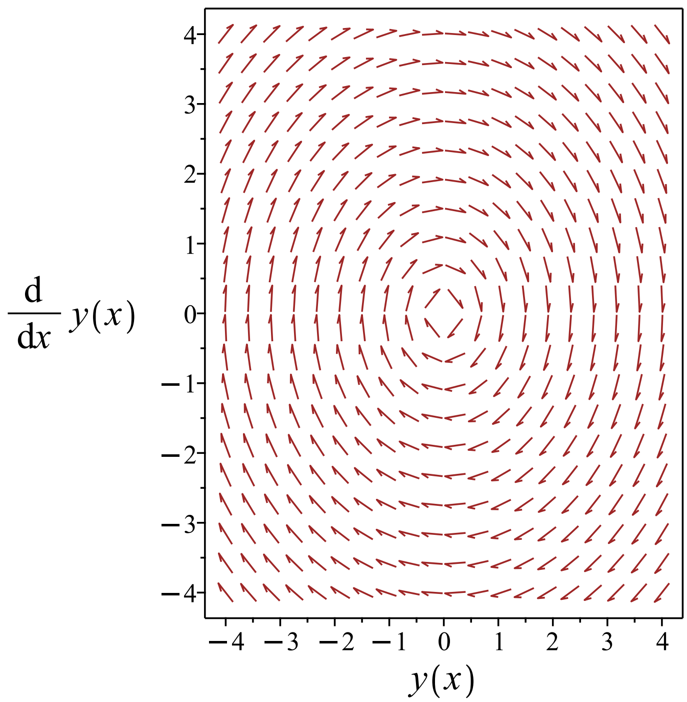


Figure 484: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) + \sin(x) \ln(\csc(x) - \cot(x))
$$

Verified OK.

12.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 425: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x)))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$

$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\cot(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \cos(x)\,dx$$

Hence

$$u_1 = -\sin(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\cot(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int \cos(x)\cot(x)\,dx$$

Hence

$$u_2 = \cos(x) + \ln(\csc(x) - \cot(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)\sin(x) + (\cos(x) + \ln(\csc(x) - \cot(x)))\sin(x)$$

Which simplifies to

$$y_p(x) = \sin(x)\ln(\csc(x) - \cot(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (\sin(x)\ln(\csc(x) - \cot(x))) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \sin(x) \ln(\csc(x) - \cot(x)) \tag{1}$$

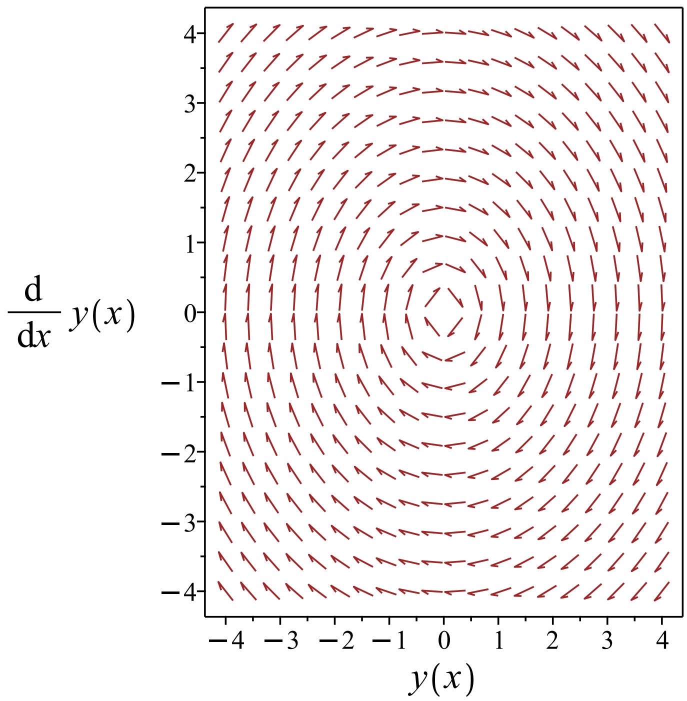


Figure 485: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \sin(x) \ln(\csc(x) - \cot(x))$$

Verified OK.

12.1.3 Maple step by step solution

Let's solve

$y'' + y = \cot(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cot(x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \cos(x)\, dx\right) + \sin(x) \left(\int \cos(x) \cot(x)\, dx\right)$
  - Compute integrals

    $y_p(x) = \sin(x) \ln(\csc(x) - \cot(x))$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \sin(x) \ln(\csc(x) - \cot(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+y(x)=cot(x),y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + c_1 \cos(x) + \sin(x) \ln(\csc(x) - \cot(x))$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 33

```
DSolve[y''[x]+y[x]==Cot[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(x) + \sin(x) \left(\log\left(\sin\left(\frac{x}{2}\right)\right) - \log\left(\cos\left(\frac{x}{2}\right)\right) + c_2\right)$$

12.2 problem 2

Internal problem ID [11830]
Internal file name [`OUTPUT/11839_Thursday_April_11_2024_08_52_51_PM_66482266/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \tan(x)^2}$$

12.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \tan(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\tan(x)^2}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\tan(x)^2\,dx$$

Hence

$$u_1 = -\frac{\sin(x)^4}{\cos(x)} - \left(2 + \sin(x)^2\right)\cos(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\tan(x)^2}{1}\,dx$$

Which simplifies to

$$u_2 = \int \sin(x)\tan(x)\,dx$$

Hence

$$u_2 = -\sin(x) + \ln(\sec(x) + \tan(x))$$

Which simplifies to

$$u_1 = -\cos(x) - \sec(x)$$
$$u_2 = -\sin(x) + \ln(\sec(x) + \tan(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\cos(x) - \sec(x))\cos(x) + (-\sin(x) + \ln(\sec(x) + \tan(x)))\sin(x)$$

Which simplifies to

$$y_p(x) = -2 + \sin(x)\ln(\sec(x) + \tan(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (-2 + \sin(x)\ln(\sec(x) + \tan(x))) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - 2 + \sin(x)\ln(\sec(x) + \tan(x)) \tag{1}$$

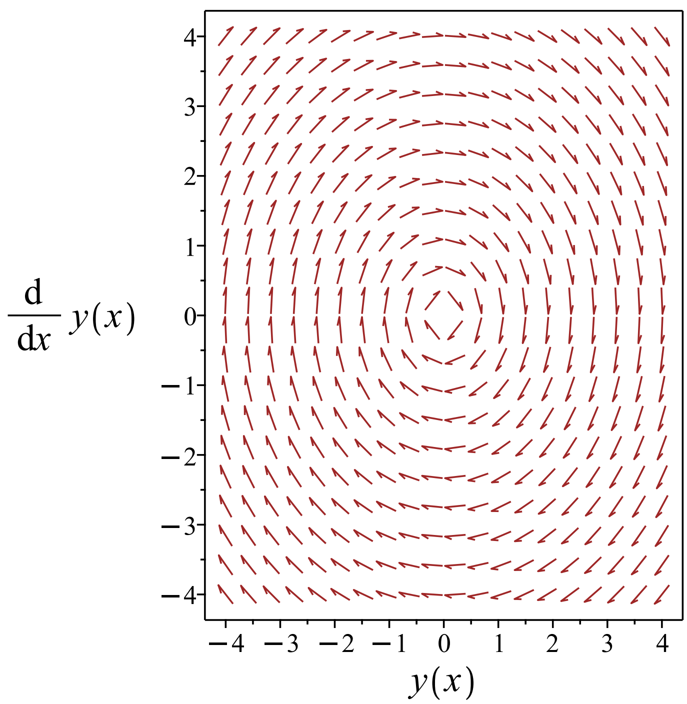


Figure 486: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - 2 + \sin(x)\ln(\sec(x) + \tan(x))$$

Verified OK.

12.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 427: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\tan(x)^2}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\tan(x)^2\,dx$$

Hence

$$u_1 = -\frac{\sin(x)^4}{\cos(x)} - \left(2+\sin(x)^2\right)\cos(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\tan(x)^2}{1}\,dx$$

Which simplifies to

$$u_2 = \int \sin(x)\tan(x)\,dx$$

Hence

$$u_2 = -\sin(x) + \ln(\sec(x)+\tan(x))$$

Which simplifies to

$$u_1 = -\cos(x) - \sec(x)$$
$$u_2 = -\sin(x) + \ln(\sec(x)+\tan(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\cos(x)-\sec(x))\cos(x) + (-\sin(x)+\ln(\sec(x)+\tan(x)))\sin(x)$$

Which simplifies to

$$y_p(x) = -2 + \sin(x)\ln(\sec(x)+\tan(x))$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + (-2 + \sin(x) \ln(\sec(x) + \tan(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) - 2 + \sin(x) \ln(\sec(x) + \tan(x)) \tag{1}
$$

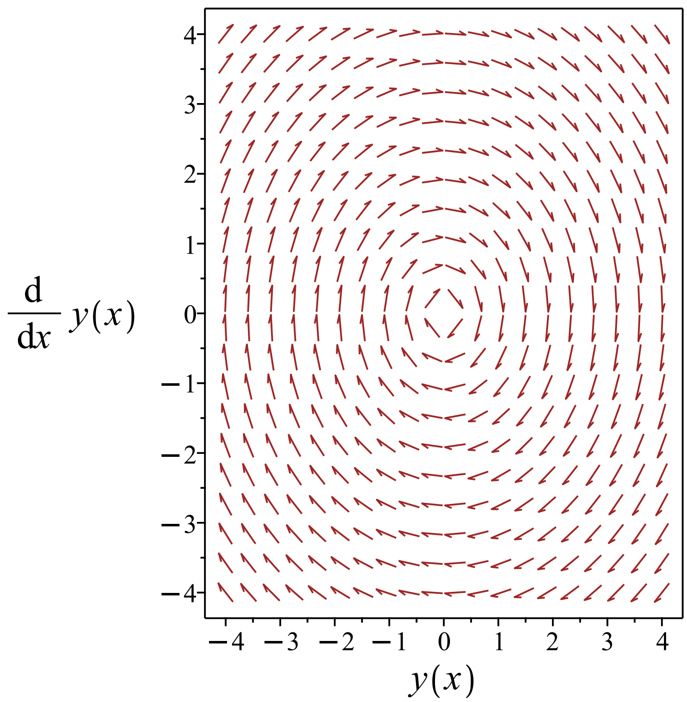


Figure 487: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) - 2 + \sin(x) \ln(\sec(x) + \tan(x))
$$

Verified OK.

12.2.3 Maple step by step solution

Let's solve

$y'' + y = \tan(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \tan(x)^2\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(x)\left(\int \sin(x)\tan(x)^2\,dx\right) + \sin(x)\left(\int \sin(x)\tan(x)\,dx\right)$$

- Compute integrals

  $$y_p(x) = -2 + \sin(x)\ln(\sec(x) + \tan(x))$$

- Substitute particular solution into general solution to ODE

  $$y = c_1\cos(x) + c_2\sin(x) - 2 + \sin(x)\ln(\sec(x) + \tan(x))$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)=tan(x)^2,y(x), singsol=all)
```

$$y(x) = \sin(x)\,c_2 + c_1\cos(x) - 2 + \sin(x)\ln(\sec(x) + \tan(x))$$

✓ Solution by Mathematica

Time used: 0.124 (sec). Leaf size: 23

```
DSolve[y''[x]+y[x]==Tan[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x)\text{arctanh}(\sin(x)) + c_1\cos(x) + c_2\sin(x) - 2$$

12.3 problem 3

Internal problem ID [11831]
Internal file name [`OUTPUT/11840_Thursday_April_11_2024_08_52_52_PM_64402138/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \sec(x)}$$

12.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sec(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)}{1} dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\, dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)}{1} dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(x) + x\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (\ln(\cos(x))\cos(x) + x\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + x \sin(x) \tag{1}$$

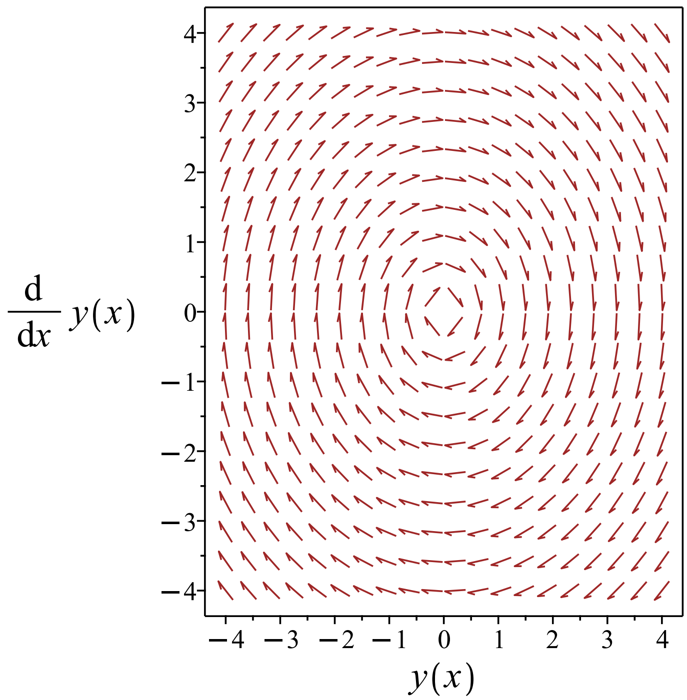


Figure 488: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + x \sin(x)$$

Verified OK.

12.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 429: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x)))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(x) + x\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (\ln(\cos(x))\cos(x) + x\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(x) + c_2\sin(x) + \ln(\cos(x))\cos(x) + x\sin(x) \tag{1}$$

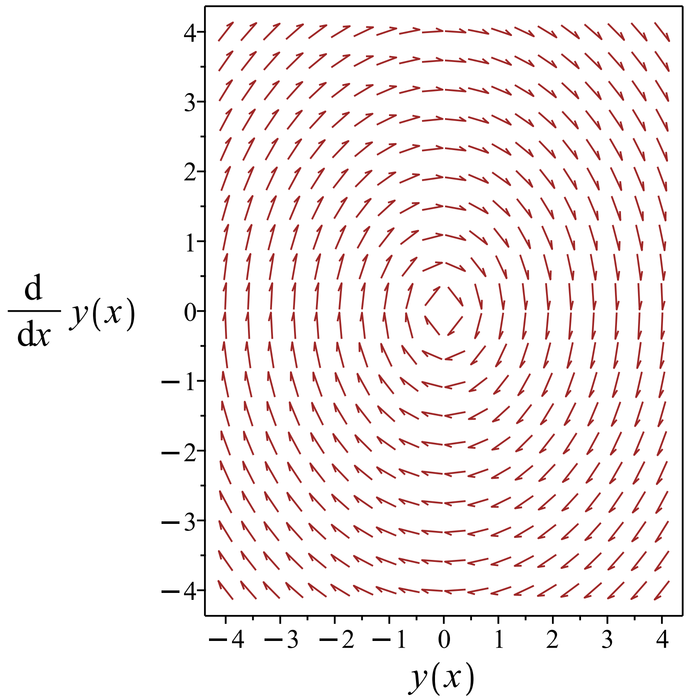


Figure 489: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + x \sin(x)$$

Verified OK.

12.3.3 Maple step by step solution

Let's solve

$y'' + y = \sec(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sec(x)\right]$

  ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ∘ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x)\left(\int \tan(x)\, dx\right) + \sin(x)\left(\int 1 dx\right)$

  ∘ Compute integrals

  $y_p(x) = \ln(\cos(x)) \cos(x) + x \sin(x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \ln(\cos(x)) \cos(x) + x \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+y(x)=sec(x),y(x), singsol=all)
```

$$y(x) = -\ln(\sec(x))\cos(x) + c_1\cos(x) + \sin(x)(c_2 + x)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 22

```
DSolve[y''[x]+y[x]==Sec[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x + c_2)\sin(x) + \cos(x)(\log(\cos(x)) + c_1)$$

12.4 problem 4

Internal problem ID [11832]
Internal file name [`OUTPUT/11841_Thursday_April_11_2024_08_52_53_PM_17409633/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \sec(x)^3}$$

12.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sec(x)^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)^3}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\sec(x)^2\,dx$$

Hence

$$u_1 = -\frac{\sec(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)^3}{1}\,dx$$

Which simplifies to

$$u_2 = \int \sec(x)^2\,dx$$

Hence

$$u_2 = \tan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\sec(x)^2\cos(x)}{2} + \sin(x)\tan(x)$$

Which simplifies to

$$y_p(x) = -\cos(x) + \frac{\sec(x)}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\cos(x) + \frac{\sec(x)}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) + \frac{\sec(x)}{2} \tag{1}
$$

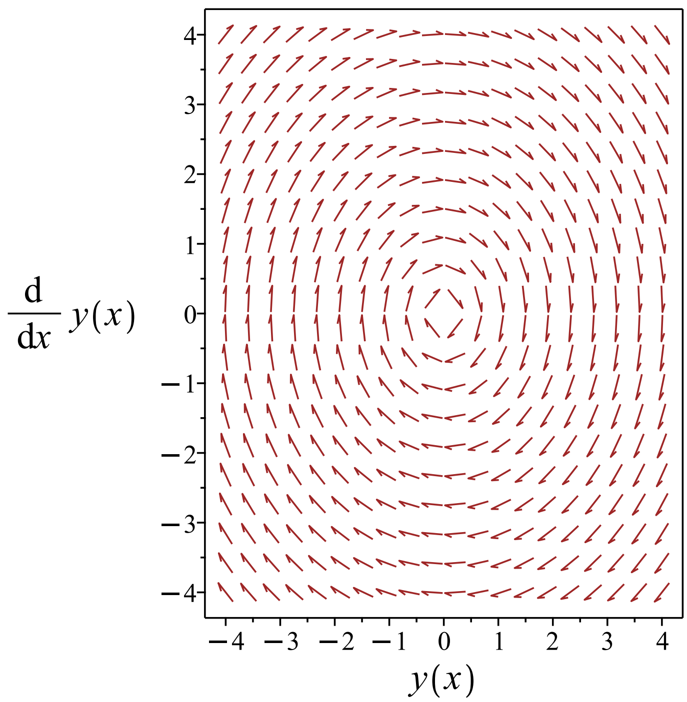


Figure 490: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) + \frac{\sec(x)}{2}
$$

Verified OK.

12.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 431: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2} \, dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x) (\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)^3}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\sec(x)^2\,dx$$

Hence

$$u_1 = -\frac{\sec(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)^3}{1}\,dx$$

Which simplifies to

$$u_2 = \int \sec(x)^2\,dx$$

Hence

$$u_2 = \tan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\sec(x)^2\cos(x)}{2} + \sin(x)\tan(x)$$

Which simplifies to

$$y_p(x) = -\cos(x) + \frac{\sec(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + \left(-\cos(x) + \frac{\sec(x)}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) + \frac{\sec(x)}{2} \tag{1}$$

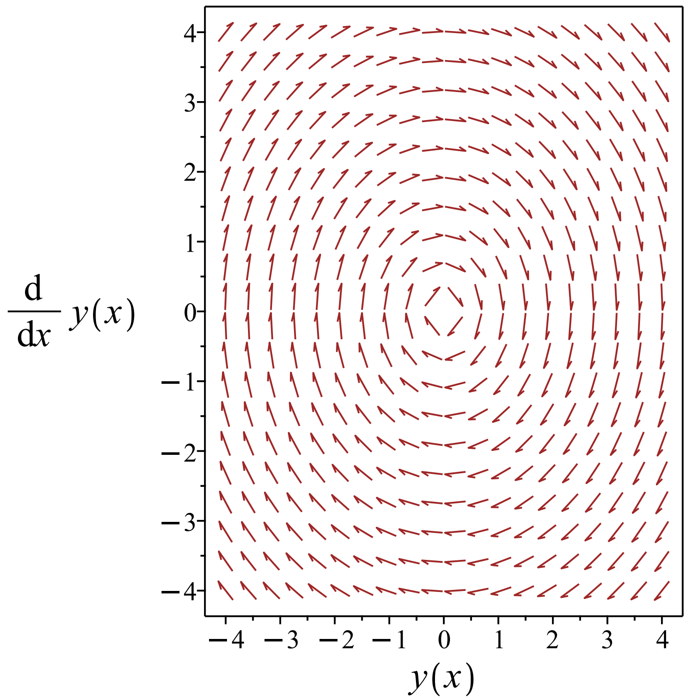


Figure 491: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) + \frac{\sec(x)}{2}$$

Verified OK.

12.4.3 Maple step by step solution

Let's solve

$y'' + y = \sec(x)^3$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sec(x)^3\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \tan(x) \sec(x)^2 dx\right) + \sin(x) \left(\int \sec(x)^2 dx\right)$

  - Compute integrals

    $y_p(x) = -\cos(x) + \frac{\sec(x)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - \cos(x) + \frac{\sec(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+y(x)=sec(x)^3,y(x), singsol=all)
```

$$y(x) = (-1 + c_1)\cos(x) + \sin(x)\, c_2 + \frac{\sec(x)}{2}$$

✓ Solution by Mathematica
Time used: 0.065 (sec). Leaf size: 25

```
DSolve[y''[x]+y[x]==Sec[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sec(x)}{2} + c_1 \cos(x) + \sin(x)(\tan(x) + c_2)$$

12.5 problem 5

Internal problem ID [11833]
Internal file name [`OUTPUT/11842_Thursday_April_11_2024_08_52_55_PM_82310885/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 4y = \sec(x)^2}$$

12.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \sec(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)$$
$$y_2 = \sin(2x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}(\sin(2x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(2\cos(2x)) - (\sin(2x))(-2\sin(2x))$$

Which simplifies to

$$W = 2\cos(2x)^2 + 2\sin(2x)^2$$

Which simplifies to

$$W = 2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(2x)\sec(x)^2}{2} dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\, dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\sec(x)^2}{2} dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(2x)\sec(x)^2}{2} dx$$

Hence

$$u_2 = x - \frac{\tan(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(2x) + \left(x - \frac{\tan(x)}{2}\right)\sin(2x)$$

Which simplifies to

$$y_p(x) = \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)x - \sin(x)^2$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2 \tag{1}$$

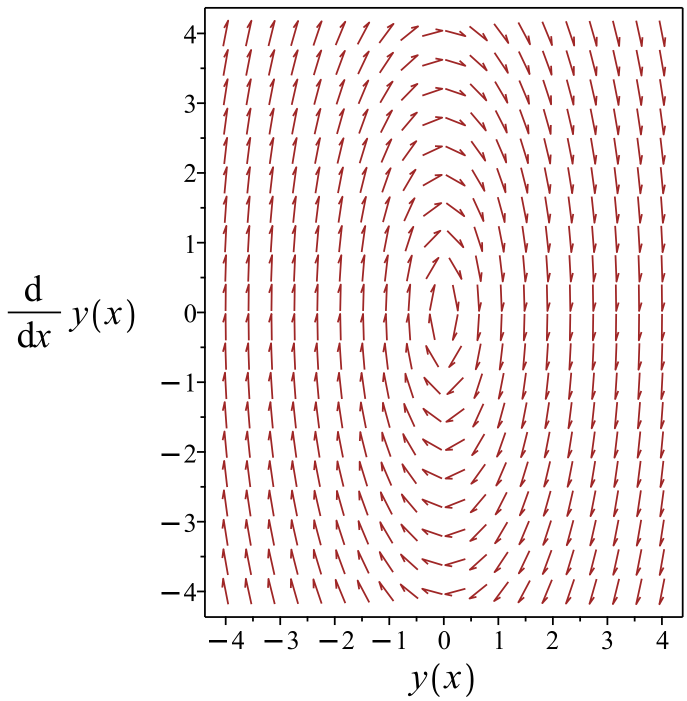


Figure 492: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2$$

Verified OK.

12.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 433: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2} \, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2 \left(\cos(2x) \left(\frac{\tan(2x)}{2} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)$$
$$y_2 = \frac{\sin(2x)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}\left(\frac{\sin(2x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ -2\sin(2x) & \cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(\cos(2x)) - \left(\frac{\sin(2x)}{2}\right)(-2\sin(2x))$$

Which simplifies to

$$W = \sin(2x)^2 + \cos(2x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(2x)\sec(x)^2}{2}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\,dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(2x)\sec(x)^2}{1}\,dx$$

Which simplifies to

$$u_2 = \int \cos(2x)\sec(x)^2\,dx$$

Hence

$$u_2 = 2x - \tan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x))\cos(2x) + \frac{(2x - \tan(x))\sin(2x)}{2}$$

Which simplifies to

$$y_p(x) = \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2$$

Therefore the general solution is

$y = y_h + y_p$

$$= \left(c_1\cos(2x) + \frac{c_2\sin(2x)}{2}\right) + \left(\ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2 \tag{1}$$

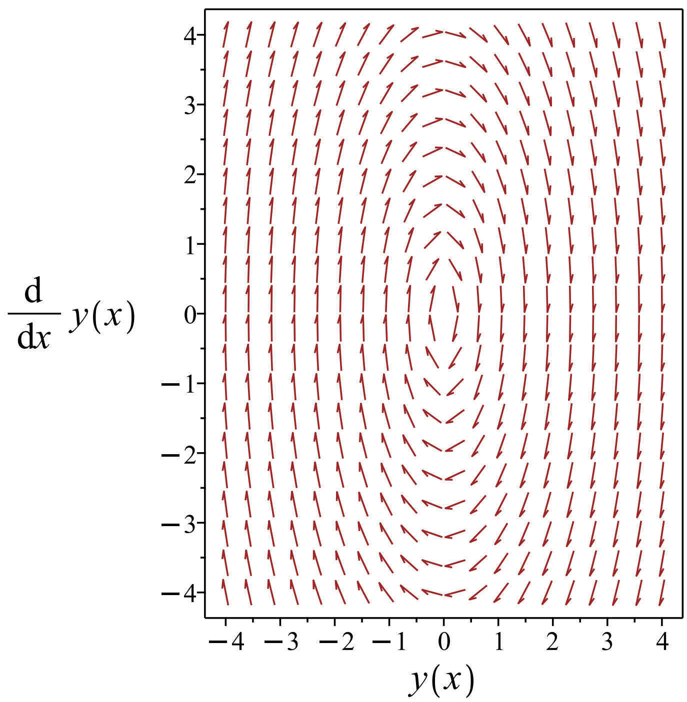


Figure 493: Slope field plot

<u>Verification of solutions</u>

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2$$

Verified OK.

12.5.3 Maple step by step solution

Let's solve

$y'' + 4y = \sec(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sec(x)^2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(2x)\left(\int \tan(x)\,dx\right) + \frac{\sin(2x)\left(\int \cos(2x)\sec(x)^2 dx\right)}{2}$$

- Compute integrals

$$y_p(x) = \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(2x) + c_2\sin(2x) + \ln(\cos(x))\left(2\cos(x)^2 - 1\right) + 2\sin(x)\cos(x)\,x - \sin(x)^2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
dsolve(diff(y(x),x$2)+4*y(x)=sec(x)^2,y(x), singsol=all)
```

$$y(x) = \left(-2\cos(x)^2 + 1\right)\ln(\sec(x)) + 2\cos(x)^2 c_1 + 2\sin(x)(c_2 + x)\cos(x) - \sin(x)^2 - c_1$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 33

```
DSolve[y''[x]+4*y[x]==Sec[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(2x)(\log(\cos(x)) + c_1) + \sin(x)(-\sin(x) + 2(x + c_2)\cos(x))$$

12.6 problem 6

Internal problem ID [11834]
Internal file name [`OUTPUT/11843_Thursday_April_11_2024_08_52_56_PM_90341025/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \sec(x)\tan(x)}$$

12.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sec(x)\tan(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\tan(x)\sec(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \tan(x)^2\,dx$$

Hence

$$u_1 = -\tan(x) + x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)\tan(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int \tan(x)\,dx$$

Hence

$$u_2 = -\ln(\cos(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\tan(x) + x)\cos(x) - \ln(\cos(x))\sin(x)$$

Which simplifies to

$$y_p(x) = -\ln(\cos(x))\sin(x) - \sin(x) + \cos(x)\,x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + (-\ln(\cos(x)) \sin(x) - \sin(x) + \cos(x)\, x)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \ln(\cos(x)) \sin(x) - \sin(x) + \cos(x)\, x \tag{1}$$

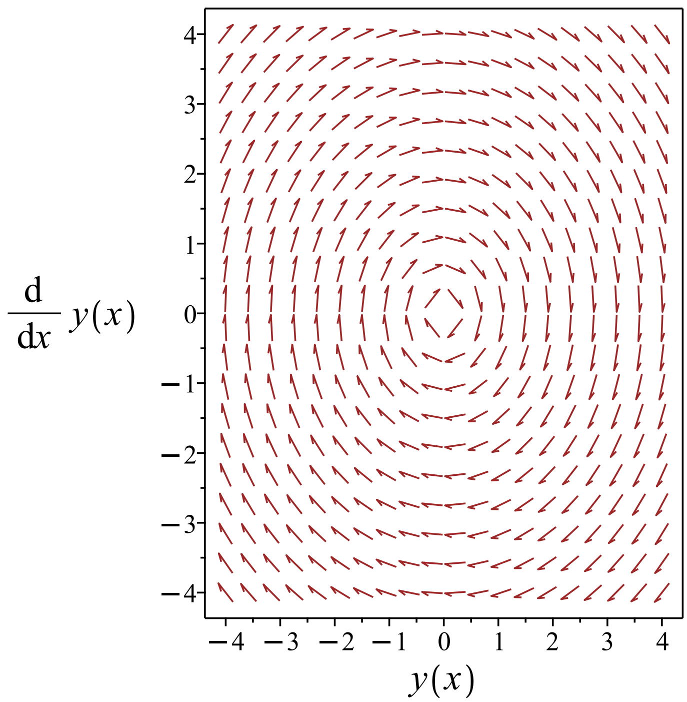


Figure 494: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \ln(\cos(x)) \sin(x) - \sin(x) + \cos(x)\, x$$

Verified OK.

12.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 435: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\tan(x)\sec(x)}{1} dx$$

Which simplifies to

$$u_1 = -\int \tan(x)^2 dx$$

Hence

$$u_1 = -\tan(x) + x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)\tan(x)}{1} dx$$

Which simplifies to

$$u_2 = \int \tan(x) dx$$

Hence

$$u_2 = -\ln(\cos(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-\tan(x) + x)\cos(x) - \ln(\cos(x))\sin(x)$$

Which simplifies to

$$y_p(x) = -\ln(\cos(x))\sin(x) - \sin(x) + \cos(x)x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (-\ln(\cos(x))\sin(x) - \sin(x) + \cos(x)x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \ln(\cos(x)) \sin(x) - \sin(x) + \cos(x) x \tag{1}$$

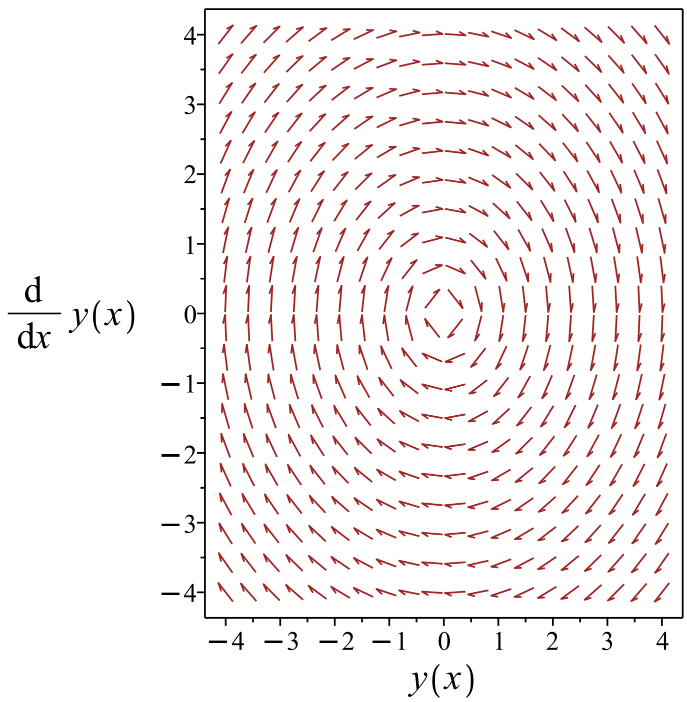


Figure 495: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \ln(\cos(x)) \sin(x) - \sin(x) + \cos(x) x$$

Verified OK.

12.6.3 Maple step by step solution

Let's solve

$y'' + y = \sec(x) \tan(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sec(x)\tan(x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x)\left(\int \tan(x)^2 dx\right) + \sin(x)\left(\int \tan(x) dx\right)$
  - Compute integrals

    $y_p(x) = -\ln(\cos(x))\sin(x) - \sin(x) + \cos(x)x$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - \ln(\cos(x))\sin(x) - \sin(x) + \cos(x)x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)=tan(x)*sec(x),y(x), singsol=all)
```

$$y(x) = \ln(\sec(x))\sin(x) + (c_2 - 1)\sin(x) + \cos(x)(c_1 + x)$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 29

```
DSolve[y''[x]+y[x]==Tan[x]*Sec[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x)\arctan(\tan(x)) + c_1\cos(x) + \sin(x)(-\log(\cos(x)) - 1 + c_2)$$

12.7 problem 7

Internal problem ID [11835]
Internal file name [`OUTPUT/11844_Thursday_April_11_2024_08_52_58_PM_35052809/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y''+4y'+5y = \mathrm{e}^{-2x} \sec(x)}$$

12.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 5, f(x) = \mathrm{e}^{-2x} \sec(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y''+4y'+5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 5$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(5)} \\
&= -2 \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -2 + i \\
\lambda_2 &= -2 - i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -2 + i \\
\lambda_2 &= -2 - i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-2x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)\,\mathrm{e}^{-2x}$$
$$y_2 = \mathrm{e}^{-2x}\sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x)\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x}\sin(x) \\ \frac{d}{dx}\left(\cos(x)\,\mathrm{e}^{-2x}\right) & \frac{d}{dx}\left(\mathrm{e}^{-2x}\sin(x)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x)\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x}\sin(x) \\ -\mathrm{e}^{-2x}\sin(x) - 2\cos(x)\,\mathrm{e}^{-2x} & -2\,\mathrm{e}^{-2x}\sin(x) + \cos(x)\,\mathrm{e}^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(\cos(x)\,\mathrm{e}^{-2x}\right)\left(-2\,\mathrm{e}^{-2x}\sin(x) + \cos(x)\,\mathrm{e}^{-2x}\right) - \left(\mathrm{e}^{-2x}\sin(x)\right)\left(-\mathrm{e}^{-2x}\sin(x) - 2\cos(x)\,\mathrm{e}^{-2x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-4x}\cos(x)^2 + \mathrm{e}^{-4x}\sin(x)^2$$

Which simplifies to

$$W = e^{-4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-4x} \sin(x) \sec(x)}{e^{-4x}} dx$$

Which simplifies to

$$u_1 = -\int \tan(x)\, dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x) e^{-4x} \sec(x)}{e^{-4x}} dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x)) \cos(x) e^{-2x} + x e^{-2x} \sin(x)$$

Which simplifies to

$$y_p(x) = e^{-2x}(\ln(\cos(x)) \cos(x) + x \sin(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2x}(c_1 \cos(x) + c_2 \sin(x))\right) + \left(e^{-2x}(\ln(\cos(x)) \cos(x) + x \sin(x))\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-2x}(\ln(\cos(x)) \cos(x) + x \sin(x)) \tag{1}$$

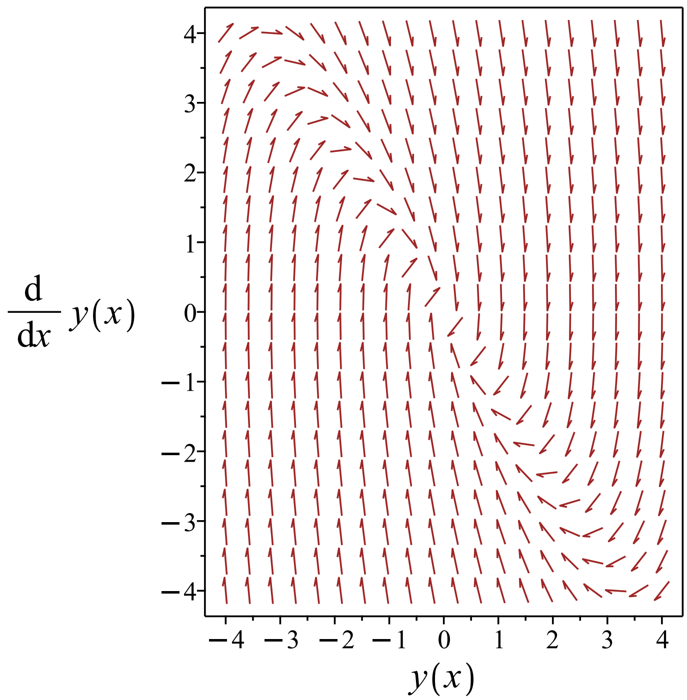


Figure 496: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-2x}(\ln(\cos(x)) \cos(x) + x \sin(x))$$

Verified OK.

12.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 437: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\ &= z_1 e^{-2x} \\ &= z_1\left(\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\,e^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\cos(x)\,e^{-2x}\right) + c_2\left(\cos(x)\,e^{-2x}(\tan(x))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{-2x}\cos(x)\,c_1 + c_2 e^{-2x}\sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)\,\mathrm{e}^{-2x}$$
$$y_2 = \mathrm{e}^{-2x}\sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x)\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x}\sin(x) \\ \frac{d}{dx}\left(\cos(x)\,\mathrm{e}^{-2x}\right) & \frac{d}{dx}\left(\mathrm{e}^{-2x}\sin(x)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x)\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x}\sin(x) \\ -\mathrm{e}^{-2x}\sin(x) - 2\cos(x)\,\mathrm{e}^{-2x} & -2\,\mathrm{e}^{-2x}\sin(x) + \cos(x)\,\mathrm{e}^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(\cos(x)\,\mathrm{e}^{-2x}\right)\left(-2\,\mathrm{e}^{-2x}\sin(x) + \cos(x)\,\mathrm{e}^{-2x}\right) - \left(\mathrm{e}^{-2x}\sin(x)\right)\left(-\mathrm{e}^{-2x}\sin(x) - 2\cos(x)\,\mathrm{e}^{-2x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-4x}\cos(x)^2 + \mathrm{e}^{-4x}\sin(x)^2$$

Which simplifies to

$$W = \mathrm{e}^{-4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{-4x} \sin(x) \sec(x)}{\mathrm{e}^{-4x}} \, dx$$

Which simplifies to

$$u_1 = -\int \tan(x) \, dx$$

Hence

$$u_1 = \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x) \mathrm{e}^{-4x} \sec(x)}{\mathrm{e}^{-4x}} \, dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\cos(x)) \cos(x) \mathrm{e}^{-2x} + x \, \mathrm{e}^{-2x} \sin(x)$$

Which simplifies to

$$y_p(x) = \mathrm{e}^{-2x} (\ln(\cos(x)) \cos(x) + x \sin(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-2x} \cos(x) c_1 + c_2 \mathrm{e}^{-2x} \sin(x)\right) + \left(\mathrm{e}^{-2x} (\ln(\cos(x)) \cos(x) + x \sin(x))\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-2x}(\ln(\cos(x))\cos(x) + x\sin(x))$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-2x}(\ln(\cos(x))\cos(x) + x\sin(x)) \tag{1}$$

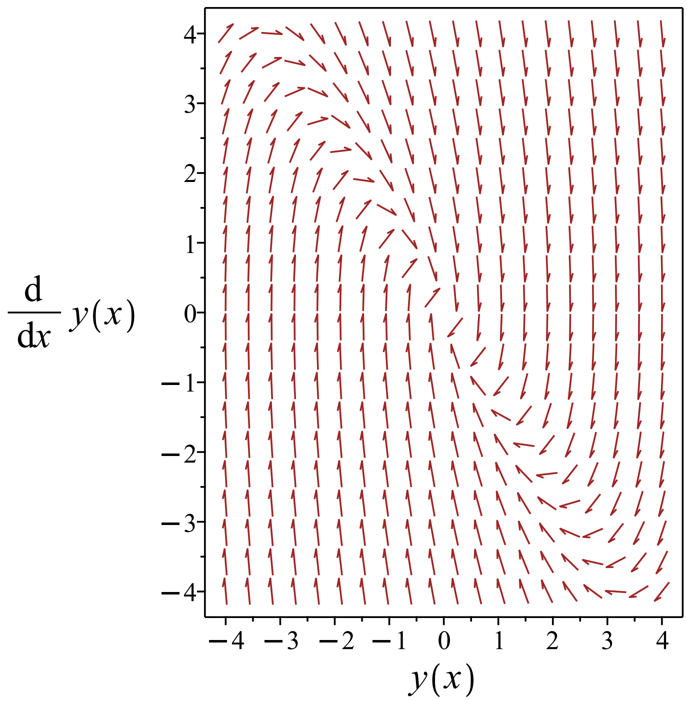


Figure 497: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_1 \cos(x) + c_2 \sin(x)) + e^{-2x}(\ln(\cos(x))\cos(x) + x\sin(x))$$

Verified OK.

12.7.3 Maple step by step solution

Let's solve

$y'' + 4y' + 5y = e^{-2x} \sec(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4) \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - I, -2 + I)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x) e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-2x} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{-2x} \cos(x) c_1 + c_2 e^{-2x} \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{-2x} \sec(x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) e^{-2x} & e^{-2x} \sin(x) \\ -e^{-2x} \sin(x) - 2\cos(x) e^{-2x} & -2 e^{-2x} \sin(x) + \cos(x) e^{-2x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-4x}$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -e^{-2x}\left(\cos(x)\left(\int \tan(x)\,dx\right) - \sin(x)\left(\int 1dx\right)\right)$$

- Compute integrals

$$y_p(x) = e^{-2x}(\ln(\cos(x))\cos(x) + x\sin(x))$$

- Substitute particular solution into general solution to ODE

$$y = e^{-2x}\cos(x)\,c_1 + c_2e^{-2x}\sin(x) + e^{-2x}(\ln(\cos(x))\cos(x) + x\sin(x))$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+5*y(x)=exp(-2*x)*sec(x),y(x), singsol=all)
```

$$y(x) = e^{-2x}(-\ln(\sec(x))\cos(x) + c_1\cos(x) + \sin(x)(c_2 + x))$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 28

```
DSolve[y''[x]+4*y'[x]+5*y[x]==Exp[-2*x]*Sec[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}((x + c_1)\sin(x) + \cos(x)(\log(\cos(x)) + c_2))$$

12.8 problem 8

Internal problem ID [11836]
Internal file name [`OUTPUT/11845_Thursday_April_11_2024_08_52_59_PM_69790256/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 2y' + 5y = e^{x} \tan(2x)}$$

12.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 5, f(x) = e^{x} \tan(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\,\mathrm{e}^{\lambda x} + 5\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(5)}\\ &= 1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 2i\\ \lambda_2 &= 1 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + 2i\\ \lambda_2 &= 1 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x \cos(2x)$$
$$y_2 = e^x \sin(2x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x \cos(2x) & e^x \sin(2x) \\ \frac{d}{dx}(e^x \cos(2x)) & \frac{d}{dx}(e^x \sin(2x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x \cos(2x) & e^x \sin(2x) \\ e^x \cos(2x) - 2 e^x \sin(2x) & e^x \sin(2x) + 2 e^x \cos(2x) \end{vmatrix}$$

Therefore

$$W = (e^x \cos(2x))(e^x \sin(2x) + 2 e^x \cos(2x)) - (e^x \sin(2x))(e^x \cos(2x) - 2 e^x \sin(2x))$$

Which simplifies to

$$W = 2 e^{2x} \sin(2x)^2 + 2 e^{2x} \cos(2x)^2$$

Which simplifies to

$$W = 2\,\mathrm{e}^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{2x} \sin(2x) \tan(2x)}{2\,\mathrm{e}^{2x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2x) \tan(2x)}{2} dx$$

Hence

$$u_1 = \frac{\sin(2x)}{4} - \frac{\ln(\sec(2x) + \tan(2x))}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{2x} \cos(2x) \tan(2x)}{2\,\mathrm{e}^{2x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\sin(2x)}{2} dx$$

Hence

$$u_2 = -\frac{\cos(2x)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\sin(2x)}{4} - \frac{\ln(\sec(2x) + \tan(2x))}{4}\right) \mathrm{e}^{x} \cos(2x) - \frac{\cos(2x)\,\mathrm{e}^{x} \sin(2x)}{4}$$

Which simplifies to

$$y_p(x) = -\frac{\mathrm{e}^{x} \cos(2x) \ln(\sec(2x) + \tan(2x))}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (e^x(c_1 \cos(2x) + c_2 \sin(2x))) + \left(-\frac{e^x \cos(2x) \ln(\sec(2x) + \tan(2x))}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = e^x(c_1 \cos(2x) + c_2 \sin(2x)) - \frac{e^x \cos(2x) \ln(\sec(2x) + \tan(2x))}{4} \tag{1}
$$

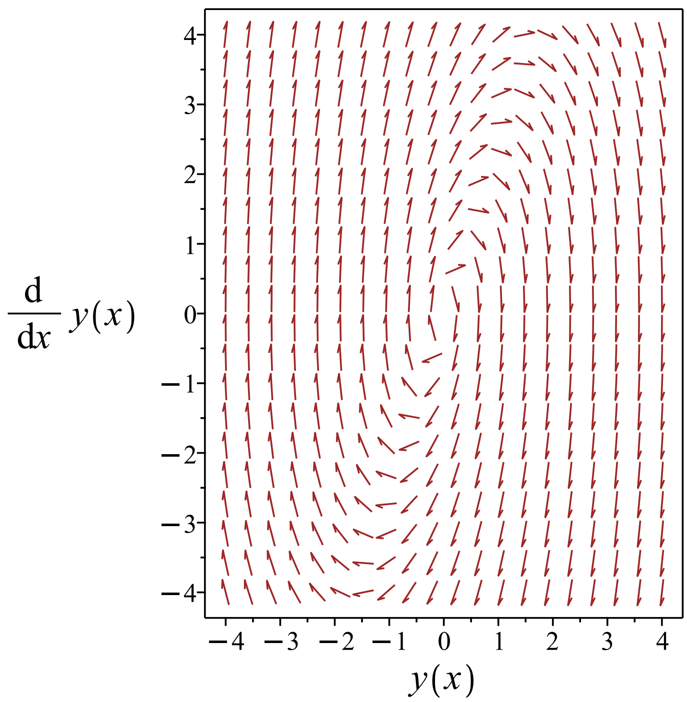


Figure 498: Slope field plot

Verification of solutions

$$
y = e^x(c_1 \cos(2x) + c_2 \sin(2x)) - \frac{e^x \cos(2x) \ln(\sec(2x) + \tan(2x))}{4}
$$

Verified OK.

12.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 439: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^x \\
&= z_1(\mathrm{e}^x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^x \cos(2x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x \cos(2x)) + c_2\left(\mathrm{e}^x \cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x \cos(2x)\, c_1 + \frac{\mathrm{e}^x \sin(2x)\, c_2}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^x \cos(2x) \\ y_2 &= \frac{\mathrm{e}^x \sin(2x)}{2} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^x \cos(2x) & \frac{\mathrm{e}^x \sin(2x)}{2} \\ \frac{d}{dx}\left(\mathrm{e}^x \cos(2x)\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^x \sin(2x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^x \cos(2x) & \frac{\mathrm{e}^x \sin(2x)}{2} \\ \mathrm{e}^x \cos(2x) - 2\,\mathrm{e}^x \sin(2x) & \frac{\mathrm{e}^x \sin(2x)}{2} + \mathrm{e}^x \cos(2x) \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^x \cos(2x)\right)\left(\frac{\mathrm{e}^x \sin(2x)}{2} + \mathrm{e}^x \cos(2x)\right) - \left(\frac{\mathrm{e}^x \sin(2x)}{2}\right)\left(\mathrm{e}^x \cos(2x) - 2\,\mathrm{e}^x \sin(2x)\right)$$

Which simplifies to

$$W = e^{2x} \sin(2x)^2 + e^{2x} \cos(2x)^2$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{2x} \sin(2x) \tan(2x)}{2}}{e^{2x}} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(2x) \tan(2x)}{2} dx$$

Hence

$$u_1 = \frac{\sin(2x)}{4} - \frac{\ln(\sec(2x) + \tan(2x))}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} \cos(2x) \tan(2x)}{e^{2x}} dx$$

Which simplifies to

$$u_2 = \int \sin(2x)\, dx$$

Hence

$$u_2 = -\frac{\cos(2x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\sin(2x)}{4} - \frac{\ln(\sec(2x) + \tan(2x))}{4}\right) e^x \cos(2x) - \frac{\cos(2x) e^x \sin(2x)}{4}$$

Which simplifies to

$$y_p(x) = -\frac{e^x \cos(2x) \ln(\sec(2x) + \tan(2x))}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^x \cos(2x) c_1 + \frac{e^x \sin(2x) c_2}{2}\right) + \left(-\frac{e^x \cos(2x) \ln(\sec(2x) + \tan(2x))}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = e^x \cos(2x) c_1 + \frac{e^x \sin(2x) c_2}{2} - \frac{e^x \cos(2x) \ln(\sec(2x) + \tan(2x))}{4} \tag{1}
$$

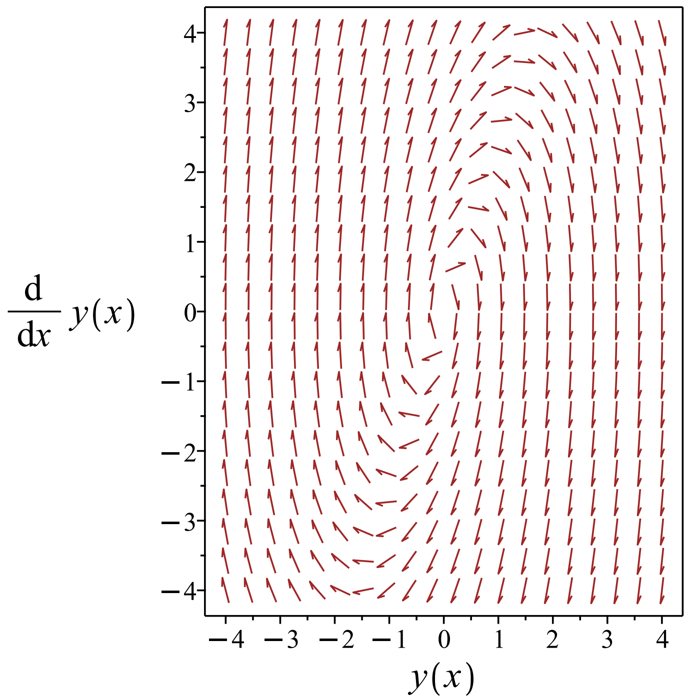


Figure 499: Slope field plot

Verification of solutions

$$
y = e^x \cos(2x) c_1 + \frac{e^x \sin(2x) c_2}{2} - \frac{e^x \cos(2x) \ln(\sec(2x) + \tan(2x))}{4}
$$

Verified OK.

12.8.3 Maple step by step solution

Let's solve

$y'' - 2y' + 5y = e^{x} \tan(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - 2\,\mathrm{I}, 1 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{x} \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x} \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{x} \cos(2x)\, c_1 + e^{x} \sin(2x)\, c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{x} \tan(2x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{x} \cos(2x) & e^{x} \sin(2x) \\ e^{x} \cos(2x) - 2 e^{x} \sin(2x) & e^{x} \sin(2x) + 2 e^{x} \cos(2x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2 e^{2x}$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{e^x \left(-\cos(2x) \left(\int \sin(2x) \tan(2x) dx\right) + \sin(2x) \left(\int \sin(2x) dx\right)\right)}{2}$$

- ∘ Compute integrals

$$y_p(x) = -\frac{e^x \cos(2x) \ln(\sec(2x) + \tan(2x))}{4}$$

- Substitute particular solution into general solution to ODE

$$y = e^x \sin(2x) c_2 - \frac{e^x \cos(2x) \ln(\sec(2x) + \tan(2x))}{4} + e^x \cos(2x) c_1$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+5*y(x)=exp(x)*tan(2*x),y(x), singsol=all)
```

$$y(x) = \frac{e^x (4c_2 \sin(2x) - \ln(\sec(2x) + \tan(2x)) \cos(2x) + 4 \cos(2x) c_1)}{4}$$

✓ Solution by Mathematica

Time used: 0.069 (sec). Leaf size: 42

```
DSolve[y''[x]-2*y'[x]+5*y[x]==Exp[x]*Tan[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{4} e^x (\cos(2x) \operatorname{arctanh}(\sin(2x)) - 4c_2 \cos(2x) + (1 - 4c_1) \sin(2x))$$

12.9 problem 9

Internal problem ID [11837]
Internal file name [`OUTPUT/11846_Thursday_April_11_2024_08_53_00_PM_2401944/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''+6y'+9y=\frac{\mathrm{e}^{-3x}}{x^3}}$$

12.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=1, B=6, C=9, f(x)=\frac{\mathrm{e}^{-3x}}{x^3}$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+6y'+9y=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda \, e^{\lambda x} + 9 \, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(6)^2 - (4)(1)(9)} \\ &= -3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$y = c_1 e^{-3x} + c_2 x \, e^{-3x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-3x} + c_2 x \, e^{-3x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^{-3x} \\ y_2 &= x \, e^{-3x}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & x\,\mathrm{e}^{-3x} \\ \frac{d}{dx}(\mathrm{e}^{-3x}) & \frac{d}{dx}(x\,\mathrm{e}^{-3x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & x\,\mathrm{e}^{-3x} \\ -3\,\mathrm{e}^{-3x} & \mathrm{e}^{-3x} - 3x\,\mathrm{e}^{-3x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-3x}\right)\left(\mathrm{e}^{-3x} - 3x\,\mathrm{e}^{-3x}\right) - \left(x\,\mathrm{e}^{-3x}\right)\left(-3\,\mathrm{e}^{-3x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-6x}$$

Which simplifies to

$$W = \mathrm{e}^{-6x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{-6x}}{x^2}}{\mathrm{e}^{-6x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{x^2}dx$$

Hence

$$u_1 = \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-6x}}{x^3}}{e^{-6x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^3}dx$$

Hence

$$u_2 = -\frac{1}{2x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^{-3x}}{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + c_2 x\, e^{-3x}\right) + \left(\frac{e^{-3x}}{2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-3x}(c_2 x + c_1) + \frac{e^{-3x}}{2x}$$

Summary

The solution(s) found are the following

$$y = e^{-3x}(c_2 x + c_1) + \frac{e^{-3x}}{2x} \tag{1}$$

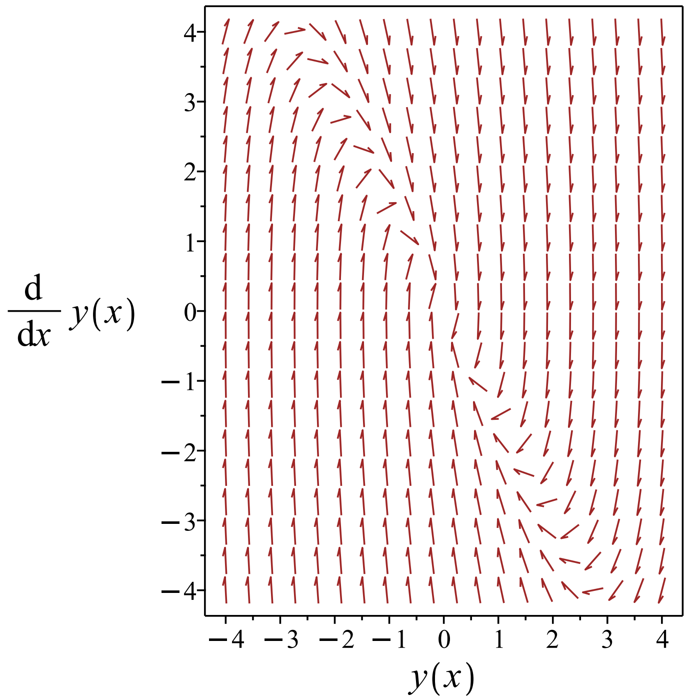


Figure 500: Slope field plot

Verification of solutions

$$y = e^{-3x}(c_2x + c_1) + \frac{e^{-3x}}{2x}$$

Verified OK.

12.9.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 6\,dx} \\ &= e^{3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \frac{e^{3x} e^{-3x}}{x^3} \\
(e^{3x} y)'' &= \frac{e^{3x} e^{-3x}}{x^3}
\end{aligned}
$$

Integrating once gives

$$
(e^{3x} y)' = -\frac{1}{2x^2} + c_1
$$

Integrating again gives

$$
(e^{3x} y) = c_1 x + \frac{1}{2x} + c_2
$$

Hence the solution is

$$
y = \frac{c_1 x + \frac{1}{2x} + c_2}{e^{3x}}
$$

Or

$$
y = c_1 x \, e^{-3x} + c_2 e^{-3x} + \frac{e^{-3x}}{2x}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x \, e^{-3x} + c_2 e^{-3x} + \frac{e^{-3x}}{2x} \tag{1}
$$

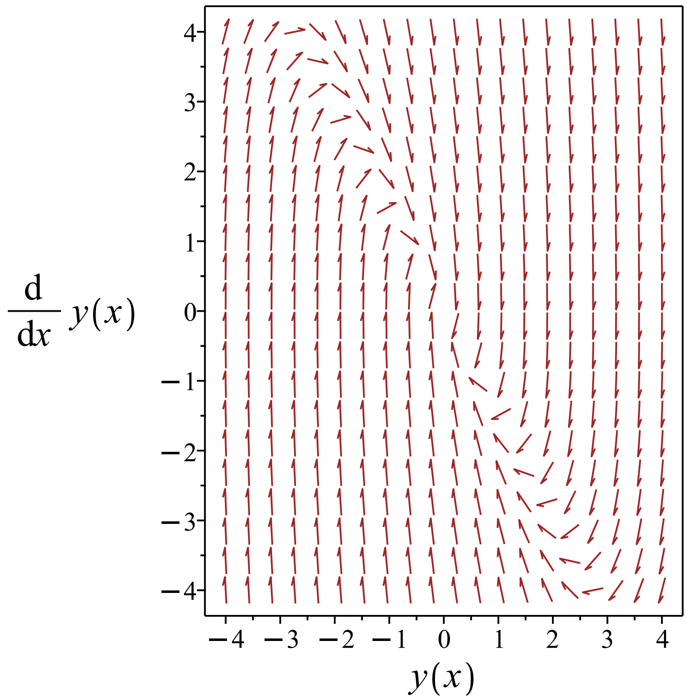


Figure 501: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x} + \frac{\mathrm{e}^{-3x}}{2x}$$

Verified OK.

12.9.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 441: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\
&= z_1 e^{-3x} \\
&= z_1 \left(\mathrm{e}^{-3x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-3x} \\
y_2 &= x\,\mathrm{e}^{-3x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-3x} & x\,e^{-3x} \\ \frac{d}{dx}\left(e^{-3x}\right) & \frac{d}{dx}\left(x\,e^{-3x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-3x} & x\,e^{-3x} \\ -3\,e^{-3x} & e^{-3x} - 3x\,e^{-3x} \end{vmatrix}$$

Therefore

$$W = \left(e^{-3x}\right)\left(e^{-3x} - 3x\,e^{-3x}\right) - \left(x\,e^{-3x}\right)\left(-3\,e^{-3x}\right)$$

Which simplifies to

$$W = e^{-6x}$$

Which simplifies to

$$W = e^{-6x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-6x}}{x^2}}{e^{-6x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{x^2}dx$$

Hence

$$u_1 = \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-6x}}{x^3}}{e^{-6x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^3} dx$$

Hence

$$u_2 = -\frac{1}{2x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^{-3x}}{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + c_2 x\, e^{-3x}\right) + \left(\frac{e^{-3x}}{2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-3x}(c_2 x + c_1) + \frac{e^{-3x}}{2x}$$

Summary

The solution(s) found are the following

$$y = e^{-3x}(c_2 x + c_1) + \frac{e^{-3x}}{2x} \tag{1}$$

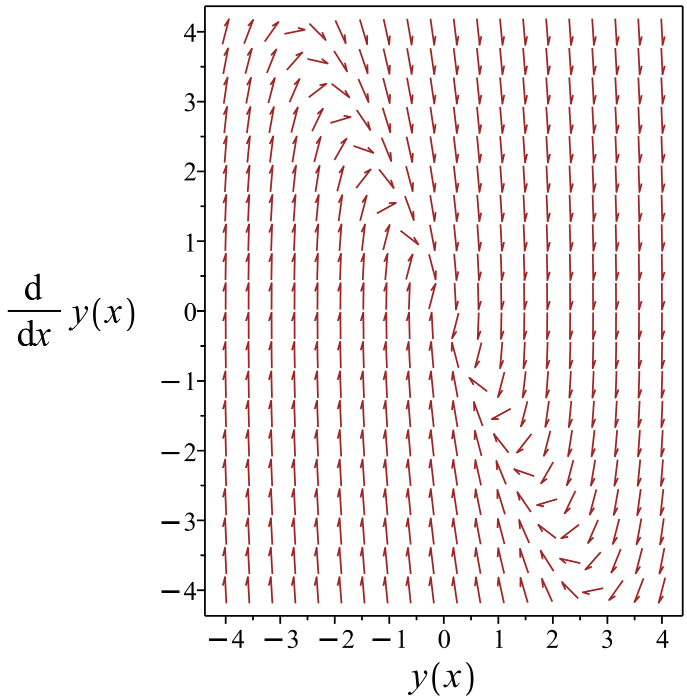


Figure 502: Slope field plot

Verification of solutions

$$y = e^{-3x}(c_2 x + c_1) + \frac{e^{-3x}}{2x}$$

Verified OK.

12.9.4 Maple step by step solution

Let's solve

$y'' + 6y' + 9y = \frac{e^{-3x}}{x^3}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r+3)^2 = 0$

- Root of the characteristic polynomial

$r = -3$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = \frac{\mathrm{e}^{-3x}}{x^3}\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & x\,\mathrm{e}^{-3x} \\ -3\,\mathrm{e}^{-3x} & \mathrm{e}^{-3x} - 3x\,\mathrm{e}^{-3x} \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-6x}$

- ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \mathrm{e}^{-3x}\left(-\left(\int \frac{1}{x^2}dx\right) + \left(\int \frac{1}{x^3}dx\right) x\right)$

- ◦ Compute integrals

  $y_p(x) = \frac{\mathrm{e}^{-3x}}{2x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x} + \frac{\mathrm{e}^{-3x}}{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+9*y(x)=exp(-3*x)/x^3,y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{-3x}(2c_1x^2 + 2c_2x + 1)}{2x}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 31

```
DSolve[y''[x]+6*y'[x]+9*y[x]==Exp[-3*x]/x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-3x}(2c_2x^2 + 2c_1x + 1)}{2x}$$

12.10 problem 10

Internal problem ID [11838]
Internal file name [`OUTPUT/11847_Thursday_April_11_2024_08_53_01_PM_44895357/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = x\,\mathrm{e}^{x} \ln(x)}$$

12.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = x\,\mathrm{e}^{x} \ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)} \\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^x + c_2 x\, e^x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^x c_1 + c_2 x\, e^x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^x \\ y_2 &= x\, e^x\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & x\,e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(x\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix}$$

Therefore

$$W = (e^x)(x\,e^x + e^x) - (x\,e^x)(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2 e^{2x} \ln(x)}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int x^2 \ln(x)\,dx$$

Hence

$$u_1 = -\frac{x^3 \ln(x)}{3} + \frac{x^3}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} x \ln(x)}{e^{2x}} dx$$

Which simplifies to

$$u_2 = \int x \ln(x) dx$$

Hence

$$u_2 = \frac{x^2 \ln(x)}{2} - \frac{x^2}{4}$$

Which simplifies to

$$u_1 = -\frac{x^3 (3 \ln(x) - 1)}{9}$$

$$u_2 = \frac{x^2 (-1 + 2 \ln(x))}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^3 (3 \ln(x) - 1) e^x}{9} + \frac{x^3 (-1 + 2 \ln(x)) e^x}{4}$$

Which simplifies to

$$y_p(x) = \frac{x^3 e^x (-5 + 6 \ln(x))}{36}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + c_2 x e^x) + \left(\frac{x^3 e^x (-5 + 6 \ln(x))}{36}\right) \end{aligned}$$

Which simplifies to

$$y = e^x (c_2 x + c_1) + \frac{x^3 e^x (-5 + 6 \ln(x))}{36}$$

Summary

The solution(s) found are the following

$$y = e^{x}(c_2x + c_1) + \frac{x^3e^x(-5 + 6\ln(x))}{36} \tag{1}$$

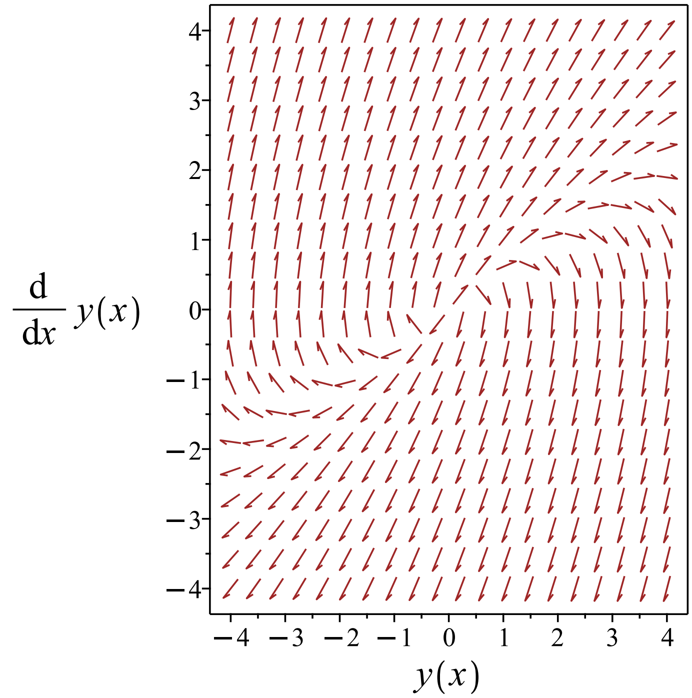


Figure 503: Slope field plot

Verification of solutions

$$y = e^{x}(c_2x + c_1) + \frac{x^3e^x(-5 + 6\ln(x))}{36}$$

Verified OK.

12.10.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{(p(x)^2 + p'(x))\,y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -2\, dx} \\ &= \mathrm{e}^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \mathrm{e}^{-x} x\, \mathrm{e}^{x} \ln(x) \\ \left(\mathrm{e}^{-x} y\right)'' &= \mathrm{e}^{-x} x\, \mathrm{e}^{x} \ln(x) \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-x} y\right)' = \frac{x^2(-1 + 2\ln(x))}{4} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-x} y\right) = -\frac{5x^3}{36} + \frac{x^3 \ln(x)}{6} + c_1 x + c_2$$

Hence the solution is

$$y = \frac{-\frac{5x^3}{36} + \frac{x^3 \ln(x)}{6} + c_1 x + c_2}{\mathrm{e}^{-x}}$$

Or

$$y = \frac{x^3 \mathrm{e}^{x} \ln(x)}{6} - \frac{5x^3 \mathrm{e}^{x}}{36} + c_1 x\, \mathrm{e}^{x} + c_2 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 \mathrm{e}^{x} \ln(x)}{6} - \frac{5x^3 \mathrm{e}^{x}}{36} + c_1 x\, \mathrm{e}^{x} + c_2 \mathrm{e}^{x} \tag{1}$$

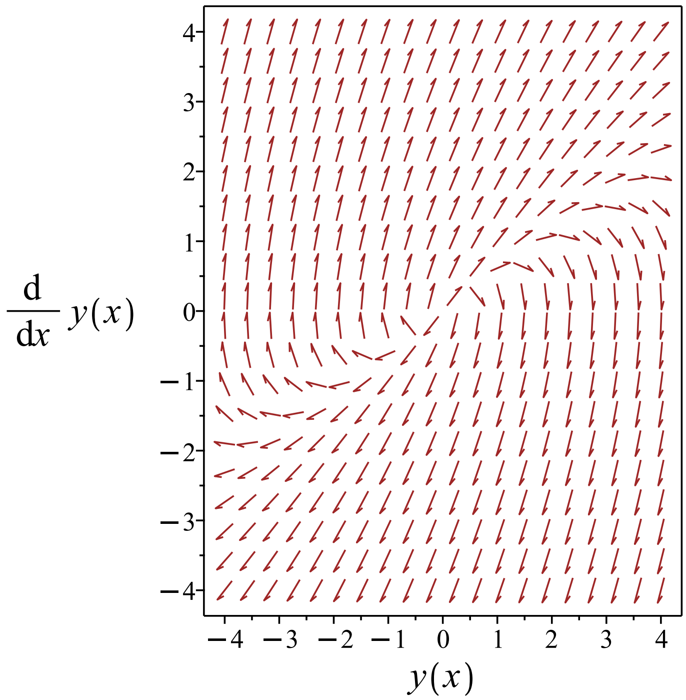


Figure 504: Slope field plot

Verification of solutions

$$y = \frac{x^3 e^x \ln(x)}{6} - \frac{5x^3 e^x}{36} + c_1 x \, e^x + c_2 e^x$$

Verified OK.

12.10.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 443: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x c_1 + c_2 x\,\mathrm{e}^x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^x \\
y_2 &= x\,\mathrm{e}^x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & x\,e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(x\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix}$$

Therefore

$$W = (e^x)(x\,e^x + e^x) - (x\,e^x)(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2 e^{2x} \ln(x)}{e^{2x}}\, dx$$

Which simplifies to

$$u_1 = -\int x^2 \ln(x)\, dx$$

Hence

$$u_1 = -\frac{x^3 \ln(x)}{3} + \frac{x^3}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} x \ln(x)}{e^{2x}}\, dx$$

Which simplifies to

$$u_2 = \int x \ln(x)\, dx$$

Hence

$$u_2 = \frac{x^2 \ln(x)}{2} - \frac{x^2}{4}$$

Which simplifies to

$$u_1 = -\frac{x^3(3\ln(x) - 1)}{9}$$
$$u_2 = \frac{x^2(-1 + 2\ln(x))}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^3(3\ln(x) - 1)\,\mathrm{e}^x}{9} + \frac{x^3(-1 + 2\ln(x))\,\mathrm{e}^x}{4}$$

Which simplifies to

$$y_p(x) = \frac{x^3 \mathrm{e}^x(-5 + 6\ln(x))}{36}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\mathrm{e}^x c_1 + c_2 x\,\mathrm{e}^x) + \left(\frac{x^3 \mathrm{e}^x(-5 + 6\ln(x))}{36}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^x(c_2 x + c_1) + \frac{x^3 \mathrm{e}^x(-5 + 6\ln(x))}{36}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^x(c_2 x + c_1) + \frac{x^3 \mathrm{e}^x(-5 + 6\ln(x))}{36} \tag{1}$$

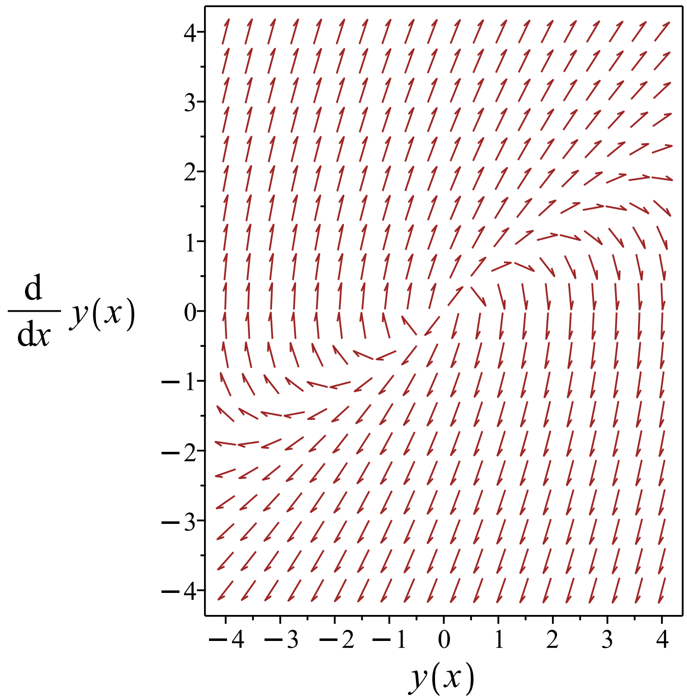


Figure 505: Slope field plot

Verification of solutions

$$y = e^{x}(c_2 x + c_1) + \frac{x^3 e^{x}(-5 + 6\ln(x))}{36}$$

Verified OK.

12.10.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = x\,e^{x} \ln(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

$r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x c_1 + c_2 x\,e^x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = x\,e^x \ln(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = e^x\left(-\left(\int x^2 \ln(x)\,dx\right) + \left(\int x \ln(x)\,dx\right) x\right)$

  - Compute integrals

    $y_p(x) = \frac{x^3 e^x (-5 + 6\ln(x))}{36}$

- Substitute particular solution into general solution to ODE

  $y = e^x c_1 + c_2 x\,e^x + \frac{x^3 e^x (-5 + 6\ln(x))}{36}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=x*exp(x)*ln(x),y(x), singsol=all)
```

$$y(x) = \frac{\left(\ln(x)\, x^3 - \frac{5x^3}{6} + 6c_1 x + 6c_2\right) e^x}{6}$$

✓ Solution by Mathematica
Time used: 0.028 (sec). Leaf size: 32

```
DSolve[y''[x]-2*y'[x]+y[x]==x*Exp[x]*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{36} e^x \left(x^3(6\log(x) - 5) + 36c_2 x + 36c_1\right)$$

12.11 problem 11

Internal problem ID [11839]
Internal file name [`OUTPUT/11848_Saturday_April_13_2024_01_12_48_AM_19754612/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \sec(x)\csc(x)}$$

12.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sec(x)\csc(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=0, C=1$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1, B=0, C=1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)\csc(x)}{1} dx$$

Which simplifies to

$$u_1 = -\int \sec(x)\, dx$$

Hence

$$u_1 = -\ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)\csc(x)}{1} dx$$

Which simplifies to

$$u_2 = \int \csc(x)\, dx$$

Hence

$$u_2 = -\ln(\csc(x) + \cot(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (-\ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) - \ln(\sec(x) + \tan(x)) \cos(x) - \ln(\csc(x) + \cot(x)) \sin(x) \tag{1}$$

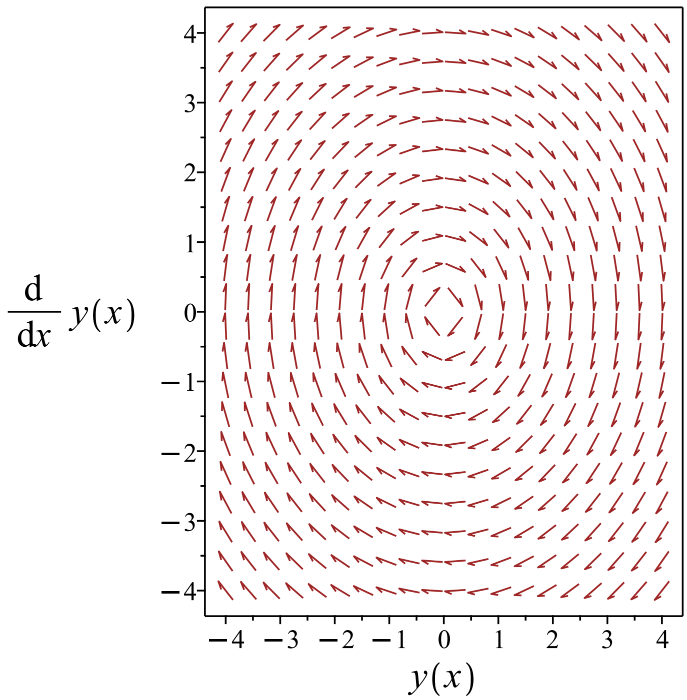


Figure 506: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \ln(\sec(x) + \tan(x)) \cos(x) - \ln(\csc(x) + \cot(x)) \sin(x)$$

Verified OK.

12.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 445: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\sec(x)\csc(x)}{1} dx$$

Which simplifies to

$$u_1 = -\int \sec(x) dx$$

Hence

$$u_1 = -\ln(\sec(x) + \tan(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\sec(x)\csc(x)}{1} dx$$

Which simplifies to

$$u_2 = \int \csc(x) dx$$

Hence

$$u_2 = -\ln(\csc(x) + \cot(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (-\ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(x) + c_2\sin(x) - \ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x) \tag{1}$$

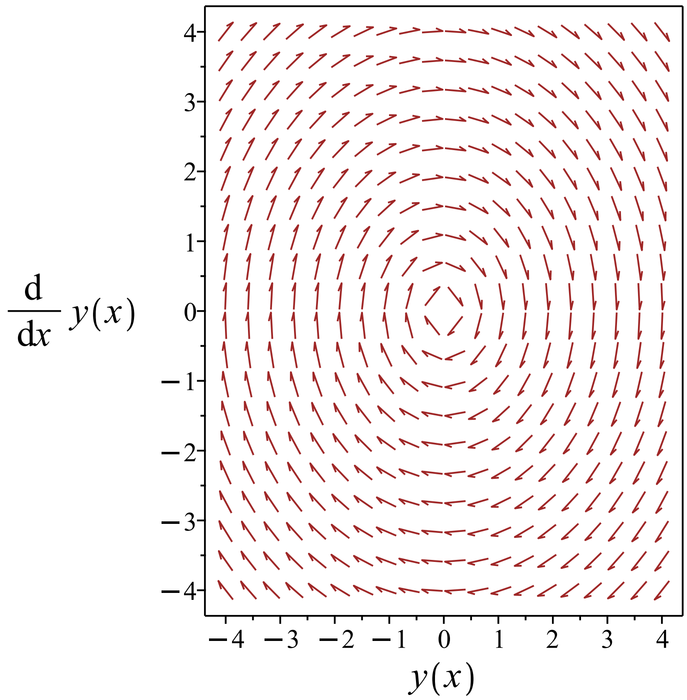


Figure 507: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) - \ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x)$$

Verified OK.

12.11.3 Maple step by step solution

Let's solve

$y'' + y = \sec(x)\csc(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sec(x)\csc(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x)\left(\int \sec(x)\,dx\right) + \sin(x)\left(\int \csc(x)\,dx\right)$

  ◦ Compute integrals

  $y_p(x) = -\ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) - \ln(\sec(x) + \tan(x))\cos(x) - \ln(\csc(x) + \cot(x))\sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$2)+y(x)=sec(x)*csc(x),y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + c_1 \cos(x) + \sin(x) \ln(\csc(x) - \cot(x)) - \cos(x) \ln(\sec(x) + \tan(x))$$

✓ Solution by Mathematica

Time used: 0.127 (sec). Leaf size: 30

```
DSolve[y''[x]+y[x]==Sec[x]*Csc[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sin(x)\operatorname{arctanh}(\cos(x)) + c_1 \cos(x) + c_2 \sin(x) + \cos(x) \left(-\coth^{-1}(\sin(x))\right)$$

12.12 problem 12

Internal problem ID [11840]
Internal file name [`OUTPUT/11849_Saturday_April_13_2024_01_12_50_AM_16619352/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = \tan(x)^3}$$

12.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \tan(x)^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\tan(x)^3}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\tan(x)^3\,dx$$

Hence

$$u_1 = -\frac{\sin(x)^5}{2\cos(x)^2} - \frac{\sin(x)^3}{2} - \frac{3\sin(x)}{2} + \frac{3\ln(\sec(x)+\tan(x))}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\tan(x)^3}{1}\,dx$$

Which simplifies to

$$u_2 = \int \sin(x)\tan(x)^2\,dx$$

Hence

$$u_2 = \frac{\sin(x)^4}{\cos(x)} + \left(2+\sin(x)^2\right)\cos(x)$$

Which simplifies to

$$\begin{aligned}
u_1 &= -\sin(x) + \frac{3\ln(\sec(x)+\tan(x))}{2} - \frac{\sec(x)\tan(x)}{2} \\
u_2 &= \cos(x) + \sec(x)
\end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\sin(x) + \frac{3\ln(\sec(x)+\tan(x))}{2} - \frac{\sec(x)\tan(x)}{2}\right)\cos(x) + (\cos(x)+\sec(x))\sin(x)$$

Which simplifies to

$$y_p(x) = \frac{3\ln(\sec(x)+\tan(x))\cos(x)}{2} + \frac{\tan(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + \left(\frac{3\ln(\sec(x)+\tan(x))\cos(x)}{2} + \frac{\tan(x)}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\cos(x) + c_2\sin(x) + \frac{3\ln(\sec(x)+\tan(x))\cos(x)}{2} + \frac{\tan(x)}{2} \tag{1}$$

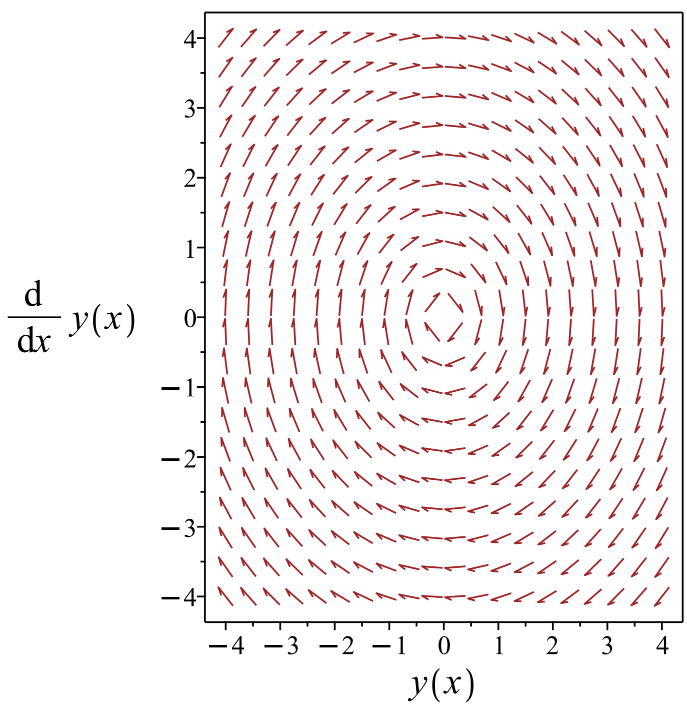


Figure 508: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{3\ln(\sec(x) + \tan(x))\cos(x)}{2} + \frac{\tan(x)}{2}$$

Verified OK.

12.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

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The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 447: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
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$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

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Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\tan(x)^3}{1}\, dx$$

Which simplifies to

$$u_1 = -\int \sin(x)\tan(x)^3\, dx$$

Hence

$$u_1 = -\frac{\sin(x)^5}{2\cos(x)^2} - \frac{\sin(x)^3}{2} - \frac{3\sin(x)}{2} + \frac{3\ln(\sec(x)+\tan(x))}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\tan(x)^3}{1}\, dx$$

Which simplifies to

$$u_2 = \int \sin(x)\tan(x)^2\, dx$$

Hence

$$u_2 = \frac{\sin(x)^4}{\cos(x)} + \left(2+\sin(x)^2\right)\cos(x)$$

Which simplifies to

$$u_1 = -\sin(x) + \frac{3\ln(\sec(x)+\tan(x))}{2} - \frac{\sec(x)\tan(x)}{2}$$

$$u_2 = \cos(x) + \sec(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\sin(x) + \frac{3\ln(\sec(x)+\tan(x))}{2} - \frac{\sec(x)\tan(x)}{2}\right)\cos(x) + (\cos(x)+\sec(x))\sin(x)$$

Which simplifies to

$$y_p(x) = \frac{3\ln(\sec(x)+\tan(x))\cos(x)}{2} + \frac{\tan(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + \left(\frac{3\ln(\sec(x)+\tan(x))\cos(x)}{2} + \frac{\tan(x)}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(x) + c_2\sin(x) + \frac{3\ln(\sec(x)+\tan(x))\cos(x)}{2} + \frac{\tan(x)}{2} \tag{1}$$

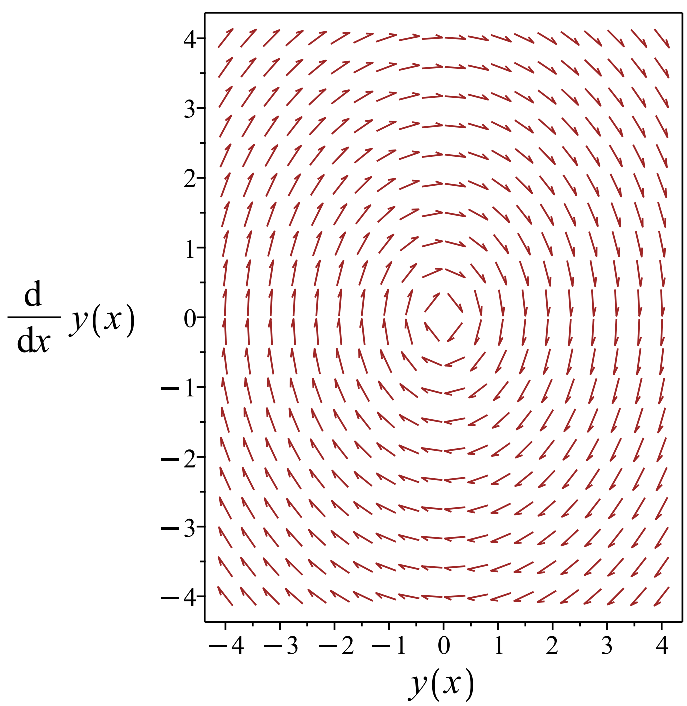

Figure 509: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{3 \ln(\sec(x) + \tan(x)) \cos(x)}{2} + \frac{\tan(x)}{2}$$

Verified OK.

12.12.3 Maple step by step solution

Let's solve

$y'' + y = \tan(x)^3$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \tan(x)^3\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x) \left(\int \sin(x) \tan(x)^3 \, dx\right) + \sin(x) \left(\int \sin(x) \tan(x)^2 \, dx\right)$

- Compute integrals

  $y_p(x) = \frac{3\ln(\sec(x)+\tan(x))\cos(x)}{2} + \frac{\tan(x)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \frac{3\ln(\sec(x)+\tan(x))\cos(x)}{2} + \frac{\tan(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+y(x)=tan(x)^3,y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + c_1 \cos(x) + \frac{\tan(x)}{2} + \frac{3\cos(x)\ln(\sec(x) + \tan(x))}{2}$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 39

```
DSolve[y''[x]+y[x]==Tan[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\sec(x)\left(3\cos^2(x)\text{arctanh}(\sin(x)) + \sin(x) + c_1\cos(2x) + c_2\sin(2x) + c_1\right)$$

12.13 problem 13

Internal problem ID [11841]
Internal file name [`OUTPUT/11850_Saturday_April_13_2024_01_12_51_AM_33949059/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = \frac{1}{e^{x} + 1}}$$

12.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = \frac{1}{e^{x}+1}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 3\lambda\, \mathrm{e}^{\lambda x} + 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)} \\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{-2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-2x} \\ \frac{d}{dx}(\mathrm{e}^{-x}) & \frac{d}{dx}(\mathrm{e}^{-2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-2x} \\ -\mathrm{e}^{-x} & -2\,\mathrm{e}^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(-2\,\mathrm{e}^{-2x}\right) - \left(\mathrm{e}^{-2x}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = -\mathrm{e}^{-x}\mathrm{e}^{-2x}$$

Which simplifies to

$$W = -\mathrm{e}^{-3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{-2x}}{\mathrm{e}^x+1}}{-\mathrm{e}^{-3x}} dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^x}{\mathrm{e}^x + 1} dx$$

Hence

$$u_1 = \ln\left(\mathrm{e}^x + 1\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{-x}}{\mathrm{e}^x+1}}{-\mathrm{e}^{-3x}} dx$$

Which simplifies to

$$u_2 = \int -\frac{\mathrm{e}^{2x}}{\mathrm{e}^x + 1} dx$$

Hence

$$u_2 = -\mathrm{e}^x + \ln\left(\mathrm{e}^x + 1\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln\left(\mathrm{e}^x + 1\right)\mathrm{e}^{-x} + \left(-\mathrm{e}^x + \ln\left(\mathrm{e}^x + 1\right)\right)\mathrm{e}^{-2x}$$

Which simplifies to

$$y_p(x) = \left(\mathrm{e}^{-2x} + \mathrm{e}^{-x}\right)\ln\left(\mathrm{e}^x + 1\right) - \mathrm{e}^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{-2x}\right) + \left(\left(\mathrm{e}^{-2x} + \mathrm{e}^{-x}\right)\ln\left(\mathrm{e}^x + 1\right) - \mathrm{e}^{-x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} + \left(e^{-2x} + e^{-x}\right) \ln\left(e^x + 1\right) - e^{-x} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} + \left(e^{-2x} + e^{-x}\right) \ln\left(e^x + 1\right) - e^{-x}$$

Verified OK.

12.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 449: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(\mathrm{e}^{x})\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+3y'+2y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1=\mathrm{e}^{-2x}$$
$$y_2=\mathrm{e}^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1=-\int\frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2=\int\frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W=\begin{vmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-x} \\ \frac{d}{dx}(\mathrm{e}^{-2x}) & \frac{d}{dx}(\mathrm{e}^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{vmatrix}$$

Therefore

$$W = \left(e^{-2x}\right)\left(-e^{-x}\right) - \left(e^{-x}\right)\left(-2\,e^{-2x}\right)$$

Which simplifies to

$$W = e^{-x}e^{-2x}$$

Which simplifies to

$$W = e^{-3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-x}}{e^x+1}}{e^{-3x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{2x}}{e^x + 1}dx$$

Hence

$$u_1 = -e^x + \ln\left(e^x + 1\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-2x}}{e^x+1}}{e^{-3x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{e^x}{e^x + 1}dx$$

Hence

$$u_2 = \ln\left(e^x + 1\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln\left(e^x + 1\right)e^{-x} + \left(-e^x + \ln\left(e^x + 1\right)\right)e^{-2x}$$

Which simplifies to

$$y_p(x) = \left(\mathrm{e}^{-2x} + \mathrm{e}^{-x}\right) \ln \left(\mathrm{e}^{x} + 1\right) - \mathrm{e}^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}\right) + \left(\left(\mathrm{e}^{-2x} + \mathrm{e}^{-x}\right) \ln \left(\mathrm{e}^{x} + 1\right) - \mathrm{e}^{-x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + \left(\mathrm{e}^{-2x} + \mathrm{e}^{-x}\right) \ln \left(\mathrm{e}^{x} + 1\right) - \mathrm{e}^{-x} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + \left(\mathrm{e}^{-2x} + \mathrm{e}^{-x}\right) \ln \left(\mathrm{e}^{x} + 1\right) - \mathrm{e}^{-x}$$

Verified OK.

12.13.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = \frac{1}{\mathrm{e}^{x}+1}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + y_p(x)$$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{1}{\mathrm{e}^x + 1}\right]$$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-x} \\ -2\,\mathrm{e}^{-2x} & -\mathrm{e}^{-x} \end{bmatrix}$$

  - Compute Wronskian

$$W(y_1(x), y_2(x)) = \mathrm{e}^{-3x}$$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\mathrm{e}^{-2x}\left(\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^x + 1} dx\right) + \mathrm{e}^{-x}\left(\int \frac{\mathrm{e}^x}{\mathrm{e}^x + 1} dx\right)$$

  - Compute integrals

$$y_p(x) = \left(\mathrm{e}^{-2x} + \mathrm{e}^{-x}\right) \ln\left(\mathrm{e}^x + 1\right) - \mathrm{e}^{-x}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + \left(\mathrm{e}^{-2x} + \mathrm{e}^{-x}\right) \ln\left(\mathrm{e}^x + 1\right) - \mathrm{e}^{-x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=1/(1+exp(x)),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-2x}(\ln(\mathrm{e}^{x}+1)(\mathrm{e}^{x}+1) - \ln(\mathrm{e}^{x})\,\mathrm{e}^{x} + (c_2 + x)\,\mathrm{e}^{x} - c_1)$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 34

```
DSolve[y''[x]+3*y'[x]+2*y[x]==1/(1+Exp[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}((e^{x}+1)\log(e^{x}+1) + (-1+c_2)e^{x} + c_1)$$

12.14 problem 14

Internal problem ID [11842]
Internal file name [`OUTPUT/11851_Saturday_April_13_2024_01_12_52_AM_47237680/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$y'' + 3y' + 2y = \frac{1}{e^{2x} + 1}$$

12.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = \frac{1}{e^{2x}+1}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{-2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-2x} \\ \frac{d}{dx}\left(\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(\mathrm{e}^{-2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-2x} \\ -\mathrm{e}^{-x} & -2\,\mathrm{e}^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(-2\,\mathrm{e}^{-2x}\right) - \left(\mathrm{e}^{-2x}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = -\mathrm{e}^{-x}\mathrm{e}^{-2x}$$

Which simplifies to

$$W = -\mathrm{e}^{-3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{-2x}}{\mathrm{e}^{2x}+1}}{-\mathrm{e}^{-3x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^{x}}{\mathrm{e}^{2x}+1}dx$$

Hence

$$u_1 = \arctan\left(\mathrm{e}^{x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{-x}}{\mathrm{e}^{2x}+1}}{-\mathrm{e}^{-3x}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}dx$$

Hence

$$u_2 = -\frac{\ln\left(\mathrm{e}^{2x}+1\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \arctan\left(\mathrm{e}^{x}\right)\mathrm{e}^{-x} - \frac{\ln\left(\mathrm{e}^{2x}+1\right)\mathrm{e}^{-2x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + c_2\mathrm{e}^{-2x}\right) + \left(\arctan\left(\mathrm{e}^{x}\right)\mathrm{e}^{-x} - \frac{\ln\left(\mathrm{e}^{2x}+1\right)\mathrm{e}^{-2x}}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
y = c_1 e^{-x} + c_2 e^{-2x} + \arctan\left(e^{x}\right) e^{-x} - \frac{\ln\left(e^{2x}+1\right) e^{-2x}}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 e^{-x} + c_2 e^{-2x} + \arctan\left(e^{x}\right) e^{-x} - \frac{\ln\left(e^{2x}+1\right) e^{-2x}}{2}
$$

Verified OK.

12.14.2 Solving using Kovacic algorithm

Writing the ode as

$$
y'' + 3y' + 2y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 3 \\
C &= 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 451: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(\mathrm{e}^{x})\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} + c_2 e^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-2x}$$
$$y_2 = e^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ \frac{d}{dx}(e^{-2x}) & \frac{d}{dx}(e^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{vmatrix}$$

Therefore

$$W = \left(e^{-2x}\right)\left(-e^{-x}\right) - \left(e^{-x}\right)\left(-2\,e^{-2x}\right)$$

Which simplifies to

$$W = e^{-x}e^{-2x}$$

Which simplifies to

$$W = e^{-3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-x}}{e^{2x}+1}}{e^{-3x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{2x}}{e^{2x}+1}dx$$

Hence

$$u_1 = -\frac{\ln\left(e^{2x}+1\right)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-2x}}{e^{2x}+1}}{e^{-3x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{e^{x}}{e^{2x}+1}dx$$

Hence

$$u_2 = \arctan\left(e^{x}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \arctan(e^x) e^{-x} - \frac{\ln(e^{2x}+1) e^{-2x}}{2}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= (c_1 e^{-2x} + c_2 e^{-x}) + \left(\arctan(e^x) e^{-x} - \frac{\ln(e^{2x}+1) e^{-2x}}{2}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{-x} + \arctan(e^x) e^{-x} - \frac{\ln(e^{2x}+1) e^{-2x}}{2} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-x} + \arctan(e^x) e^{-x} - \frac{\ln(e^{2x}+1) e^{-2x}}{2}$$

Verified OK.

12.14.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = \frac{1}{e^{2x}+1}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + y_p(x)$

- ☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{1}{\mathrm{e}^{2x}+1}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-x} \\ -2\,\mathrm{e}^{-2x} & -\mathrm{e}^{-x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-3x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-2x}\left(\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}dx\right) + \mathrm{e}^{-x}\left(\int \frac{\mathrm{e}^{x}}{\mathrm{e}^{2x}+1}dx\right)$

  - Compute integrals

    $y_p(x) = \arctan(\mathrm{e}^{x})\,\mathrm{e}^{-x} - \frac{\ln(\mathrm{e}^{2x}+1)\mathrm{e}^{-2x}}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + \arctan(\mathrm{e}^{x})\,\mathrm{e}^{-x} - \frac{\ln(\mathrm{e}^{2x}+1)\mathrm{e}^{-2x}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=1/(1+exp(2*x)),y(x), singsol=all)
```

$$y(x) = -\frac{(\ln(e^{2x}+1)\,e^{-x} + 2c_1 e^{-x} - 2\arctan(e^{x}) - 2c_2)\,e^{-x}}{2}$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 45

```
DSolve[y''[x]+3*y'[x]+2*y[x]==1/(1+Exp[2*x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-2x}\left(2e^{x}\arctan(e^{x}) - \log\left(e^{2x}+1\right) + 2(c_2 e^{x} + c_1)\right)$$

12.15 problem 15

Internal problem ID [11843]
Internal file name [`OUTPUT/11852_Saturday_April_13_2024_01_12_52_AM_53610816/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \frac{1}{1 + \sin(x)}}$$

12.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \frac{1}{1+\sin(x)}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(x)}{1+\sin(x)}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(x)}{1 + \sin(x)} dx$$

Hence

$$u_1 = -\frac{2}{\tan\left(\frac{x}{2}\right) + 1} - x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(x)}{1+\sin(x)}}{1} dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(x)}{1 + \sin(x)} dx$$

Hence

$$u_2 = \ln(1 + \sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{2}{\tan\left(\frac{x}{2}\right) + 1} - x\right) \cos(x) + \ln(1 + \sin(x)) \sin(x)$$

Which simplifies to

$$y_p(x) = \ln(1 + \sin(x)) \sin(x) + (-x - 1) \cos(x) + \sin(x) - 1$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + (\ln(1+\sin(x))\sin(x) + (-x-1)\cos(x) + \sin(x) - 1)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) + \ln(1+\sin(x))\sin(x) + (-x-1)\cos(x) + \sin(x) - 1 \quad (1)
$$

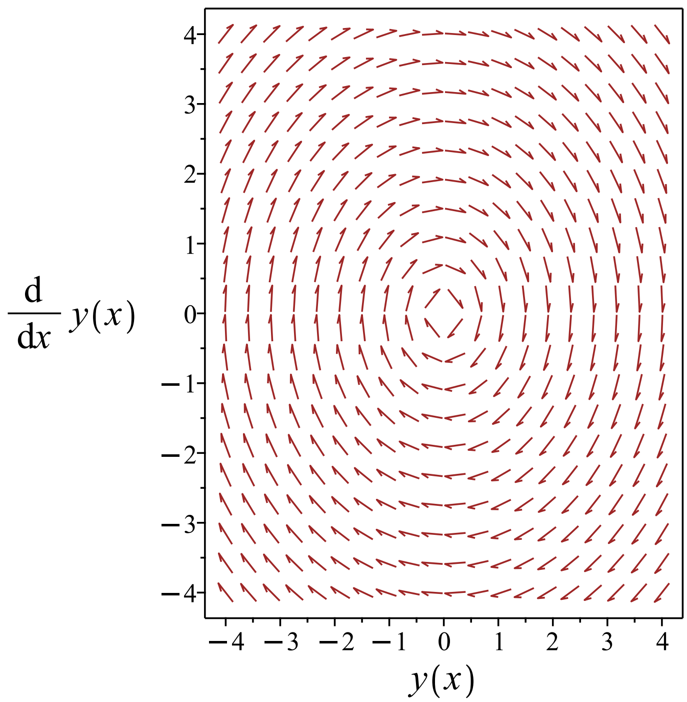

Figure 510: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) + \ln(1+\sin(x))\sin(x) + (-x-1)\cos(x) + \sin(x) - 1
$$

Verified OK.

12.15.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 453: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B=0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B=0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x)\,(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin(x)}{1+\sin(x)}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(x)}{1+\sin(x)} dx$$

Hence

$$u_1 = -\frac{2}{\tan\left(\frac{x}{2}\right)+1} - x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos(x)}{1+\sin(x)}}{1} dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(x)}{1+\sin(x)} dx$$

Hence

$$u_2 = \ln(1+\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{2}{\tan\left(\frac{x}{2}\right)+1} - x\right)\cos(x) + \ln(1+\sin(x))\sin(x)$$

Which simplifies to

$$y_p(x) = \ln(1+\sin(x))\sin(x) + (-x-1)\cos(x) + \sin(x) - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (\ln(1+\sin(x))\sin(x) + (-x-1)\cos(x) + \sin(x) - 1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(1 + \sin(x)) \sin(x) + (-x - 1) \cos(x) + \sin(x) - 1 \quad (1)$$

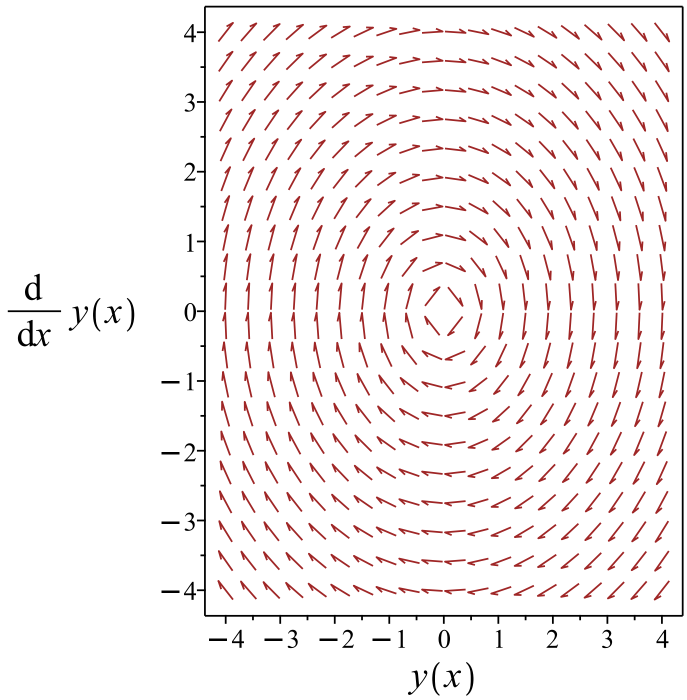

Figure 511: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(1 + \sin(x)) \sin(x) + (-x - 1) \cos(x) + \sin(x) - 1$$

Verified OK.

12.15.3 Maple step by step solution

Let's solve

$y'' + y = \frac{1}{1+\sin(x)}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{1}{1+\sin(x)}\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int \frac{\sin(x)}{1+\sin(x)} dx\right) + \sin(x) \left(\int \frac{\cos(x)}{1+\sin(x)} dx\right)$
  - Compute integrals

    $y_p(x) = \ln(1+\sin(x)) \sin(x) + (-x-1) \cos(x) + \sin(x) - 1$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \ln(1+\sin(x)) \sin(x) + (-x-1) \cos(x) + \sin(x) - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+y(x)=1/(1+sin(x)),y(x), singsol=all)
```

$$y(x) = \ln(1+\sin(x))\sin(x) + (-x + c_1 - 1)\cos(x) - 1 + (c_2+1)\sin(x)$$

✓ Solution by Mathematica

Time used: 0.188 (sec). Leaf size: 40

```
DSolve[y''[x]+y[x]==1/(1+Sin[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-x + 1 + c_1)\cos(x) + \sin(x)\left(2\log\left(\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right) + 1 + c_2\right) - 1$$

12.16 problem 16

Internal problem ID [11844]
Internal file name [`OUTPUT/11853_Saturday_April_13_2024_01_12_54_AM_97141954/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = \mathrm{e}^{x} \arcsin(x)}$$

12.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = \mathrm{e}^{x} \arcsin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)} \\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^x + c_2 x e^x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^x c_1 + c_2 x e^x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^x \\ y_2 &= x e^x\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & x\,e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(x\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix}$$

Therefore

$$W = (e^x)(x\,e^x + e^x) - (x\,e^x)(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\,e^{2x} \arcsin(x)}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int x \arcsin(x)\,dx$$

Hence

$$u_1 = -\frac{x^2 \arcsin(x)}{2} - \frac{\sqrt{-x^2+1}\,x}{4} + \frac{\arcsin(x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} \arcsin(x)}{e^{2x}} dx$$

Which simplifies to

$$u_2 = \int \arcsin(x)\, dx$$

Hence

$$u_2 = x \arcsin(x) + \sqrt{-x^2 + 1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{x^2 \arcsin(x)}{2} - \frac{\sqrt{-x^2+1}\, x}{4} + \frac{\arcsin(x)}{4}\right) e^x + \left(x \arcsin(x) + \sqrt{-x^2+1}\right) x\, e^x$$

Which simplifies to

$$y_p(x) = \frac{e^x \left(2x^2 \arcsin(x) + 3\sqrt{-x^2+1}\, x + \arcsin(x)\right)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + c_2 x\, e^x) + \left(\frac{e^x \left(2x^2 \arcsin(x) + 3\sqrt{-x^2+1}\, x + \arcsin(x)\right)}{4}\right) \end{aligned}$$

Which simplifies to

$$y = e^x (c_2 x + c_1) + \frac{e^x \left(2x^2 \arcsin(x) + 3\sqrt{-x^2+1}\, x + \arcsin(x)\right)}{4}$$

Summary

The solution(s) found are the following

$$y = e^x (c_2 x + c_1) + \frac{e^x \left(2x^2 \arcsin(x) + 3\sqrt{-x^2+1}\, x + \arcsin(x)\right)}{4} \tag{1}$$

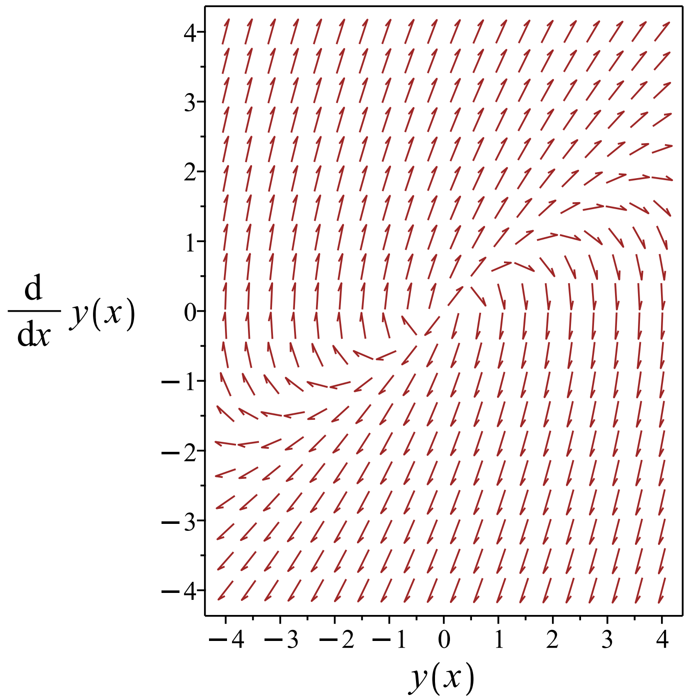

Figure 512: Slope field plot

Verification of solutions

$$y = e^{x}(c_2 x + c_1) + \frac{e^{x}\left(2x^2 \arcsin(x) + 3\sqrt{-x^2+1}\, x + \arcsin(x)\right)}{4}$$

Verified OK.

12.16.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -2\, dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = e^{-x}e^{x} \arcsin(x)$$
$$(e^{-x}y)'' = e^{-x}e^{x} \arcsin(x)$$

Integrating once gives

$$(e^{-x}y)' = x \arcsin(x) + \sqrt{-x^2+1} + c_1$$

Integrating again gives

$$(e^{-x}y) = c_1 x + \frac{3\sqrt{-x^2+1}\,x}{4} + \frac{\arcsin(x)}{4} + \frac{x^2 \arcsin(x)}{2} + c_2$$

Hence the solution is

$$y = \frac{c_1 x + \frac{3\sqrt{-x^2+1}\,x}{4} + \frac{\arcsin(x)}{4} + \frac{x^2 \arcsin(x)}{2} + c_2}{e^{-x}}$$

Or

$$y = \frac{x^2 \arcsin(x)\, e^{x}}{2} + c_1 x\, e^{x} + c_2 e^{x} + \frac{3\sqrt{-x^2+1}\,x\, e^{x}}{4} + \frac{\arcsin(x)\, e^{x}}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 \arcsin(x)\, e^{x}}{2} + c_1 x\, e^{x} + c_2 e^{x} + \frac{3\sqrt{-x^2+1}\,x\, e^{x}}{4} + \frac{\arcsin(x)\, e^{x}}{4} \tag{1}$$

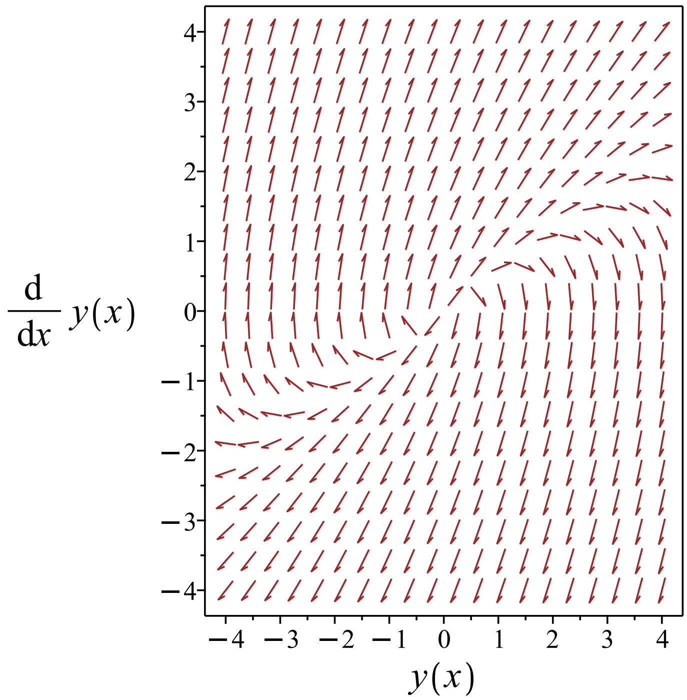

Figure 513: Slope field plot

Verification of solutions

$$y = \frac{x^2 \arcsin(x)\, e^x}{2} + c_1 x\, e^x + c_2 e^x + \frac{3\sqrt{-x^2+1}\, x\, e^x}{4} + \frac{\arcsin(x)\, e^x}{4}$$

Verified OK.

12.16.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 455: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1 (\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1y_1 + c_2y_2\\ &= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^x c_1 + c_2 x\,\mathrm{e}^x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \mathrm{e}^x\\ y_2 &= x\,\mathrm{e}^x\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{x} & x\,e^{x} \\ \frac{d}{dx}\left(e^{x}\right) & \frac{d}{dx}\left(x\,e^{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{x} & x\,e^{x} \\ e^{x} & x\,e^{x}+e^{x} \end{vmatrix}$$

Therefore

$$W = \left(e^{x}\right)\left(x\,e^{x}+e^{x}\right) - \left(x\,e^{x}\right)\left(e^{x}\right)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\,e^{2x} \arcsin(x)}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int x \arcsin(x)\,dx$$

Hence

$$u_1 = -\frac{x^2 \arcsin(x)}{2} - \frac{\sqrt{-x^2+1}\,x}{4} + \frac{\arcsin(x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} \arcsin(x)}{e^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int \arcsin(x)\, dx$$

Hence

$$u_2 = x \arcsin(x) + \sqrt{-x^2+1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{x^2 \arcsin(x)}{2} - \frac{\sqrt{-x^2+1}\, x}{4} + \frac{\arcsin(x)}{4}\right) e^x + \left(x \arcsin(x) + \sqrt{-x^2+1}\right) x\, e^x$$

Which simplifies to

$$y_p(x) = \frac{e^x \left(2x^2 \arcsin(x) + 3\sqrt{-x^2+1}\, x + \arcsin(x)\right)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x c_1 + c_2 x\, e^x) + \left(\frac{e^x \left(2x^2 \arcsin(x) + 3\sqrt{-x^2+1}\, x + \arcsin(x)\right)}{4}\right) \end{aligned}$$

Which simplifies to

$$y = e^x (c_2 x + c_1) + \frac{e^x \left(2x^2 \arcsin(x) + 3\sqrt{-x^2+1}\, x + \arcsin(x)\right)}{4}$$

Summary

The solution(s) found are the following

$$y = e^x (c_2 x + c_1) + \frac{e^x \left(2x^2 \arcsin(x) + 3\sqrt{-x^2+1}\, x + \arcsin(x)\right)}{4} \tag{1}$$

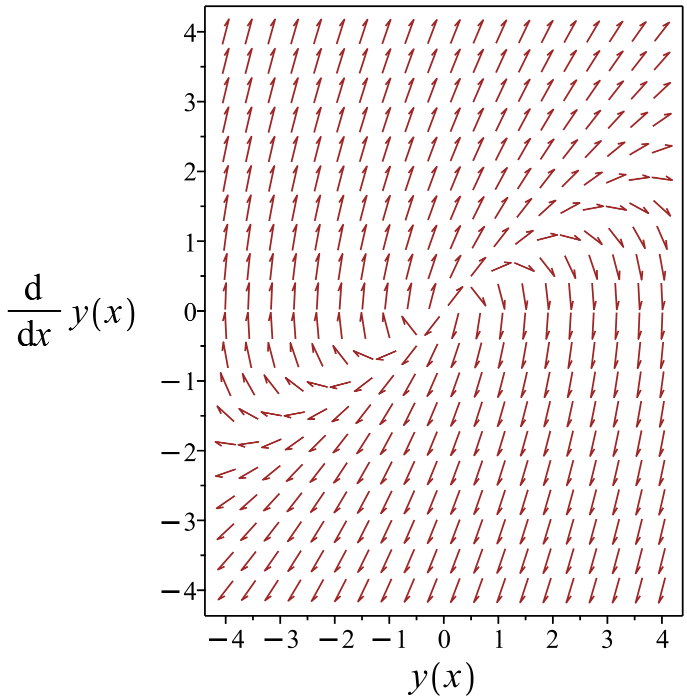

Figure 514: Slope field plot

Verification of solutions

$$y = e^x (c_2 x + c_1) + \frac{e^x \left(2x^2 \arcsin(x) + 3\sqrt{-x^2+1}\, x + \arcsin(x)\right)}{4}$$

Verified OK.

12.16.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = \arcsin(x)\, e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

$r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x c_1 + c_2 x\,e^x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \arcsin(x)\,e^x\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^x\left(-\left(\int x\arcsin(x)\,dx\right) + \left(\int \arcsin(x)\,dx\right)x\right)$

  ◦ Compute integrals

  $y_p(x) = \frac{e^x\left(2x^2\arcsin(x) + 3\sqrt{-x^2+1}\,x + \arcsin(x)\right)}{4}$

- Substitute particular solution into general solution to ODE

  $y = e^x c_1 + c_2 x\,e^x + \frac{e^x\left(2x^2\arcsin(x) + 3\sqrt{-x^2+1}\,x + \arcsin(x)\right)}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=exp(x)*arcsin(x),y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{x}\left(2x^{2}\arcsin\left(x\right)+3x\sqrt{-x^{2}+1}+4c_{1}x+\arcsin\left(x\right)+4c_{2}\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 45

```
DSolve[y''[x]-2*y'[x]+y[x]==Exp[x]*ArcSin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{1}{4}e^{x}\left(2x^{2}\arcsin(x)+\arcsin(x)+3\sqrt{1-x^{2}}x+4c_{2}x+4c_{1}\right)$$

12.17 problem 17

Internal problem ID [11845]
Internal file name [`OUTPUT/11854_Saturday_April_13_2024_01_12_55_AM_35752472/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = \frac{e^{-x}}{x}}$$

12.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = \frac{e^{-x}}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 3\lambda\,\mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \mathrm{e}^{-2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-2x} \\ \frac{d}{dx}\left(\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(\mathrm{e}^{-2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-2x} \\ -\mathrm{e}^{-x} & -2\,\mathrm{e}^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(-2\,\mathrm{e}^{-2x}\right) - \left(\mathrm{e}^{-2x}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = -\mathrm{e}^{-x}\mathrm{e}^{-2x}$$

Which simplifies to

$$W = -e^{-3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-2x}e^{-x}}{x}}{-e^{-3x}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{1}{x} dx$$

Hence

$$u_1 = \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-2x}}{x}}{-e^{-3x}}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{x}}{x} dx$$

Hence

$$u_2 = \text{expIntegral}_1(-x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x)\, e^{-x} + \text{expIntegral}_1(-x)\, e^{-2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left(\ln(x)\, e^{-x} + \text{expIntegral}_1(-x)\, e^{-2x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^{-2x} + \ln(x)\, e^{-x} + \text{expIntegral}_1(-x)\, e^{-2x} \tag{1}$$

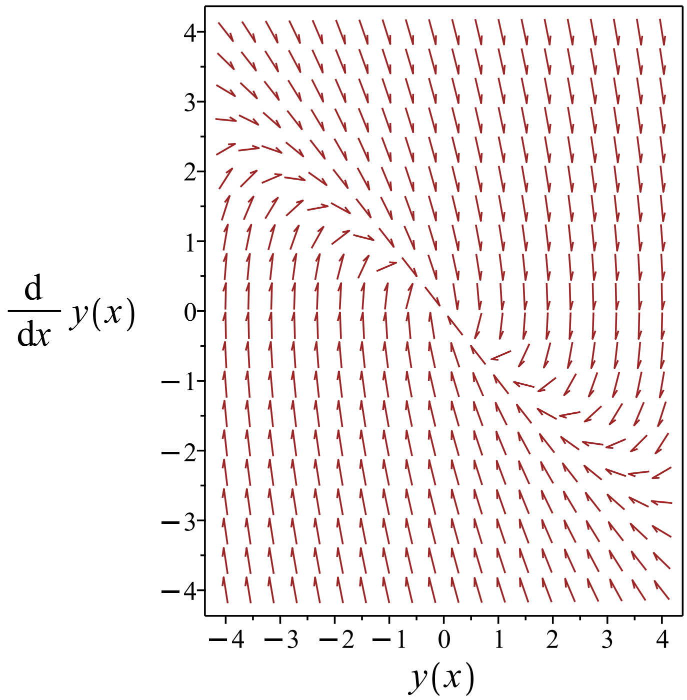

Figure 515: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^{-2x} + \ln(x) e^{-x} + \text{expIntegral}_1(-x) e^{-2x}$$

Verified OK.

12.17.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 457: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} (\mathrm{e}^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-2x} \\
y_2 &= \mathrm{e}^{-x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-x} \\ \frac{d}{dx}\left(\mathrm{e}^{-2x}\right) & \frac{d}{dx}\left(\mathrm{e}^{-x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-x} \\ -2\,\mathrm{e}^{-2x} & -\mathrm{e}^{-x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-2x}\right)\left(-\mathrm{e}^{-x}\right) - \left(\mathrm{e}^{-x}\right)\left(-2\,\mathrm{e}^{-2x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-x}\mathrm{e}^{-2x}$$

Which simplifies to

$$W = \mathrm{e}^{-3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{-2x}}{x}}{\mathrm{e}^{-3x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\mathrm{e}^{x}}{x}dx$$

Hence

$$u_1 = \mathrm{expIntegral}_1\left(-x\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{-2x}\mathrm{e}^{-x}}{x}}{\mathrm{e}^{-3x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x}dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x)\,\mathrm{e}^{-x} + \mathrm{expIntegral}_1(-x)\,\mathrm{e}^{-2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}\right) + \left(\ln(x)\,\mathrm{e}^{-x} + \mathrm{expIntegral}_1(-x)\,\mathrm{e}^{-2x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + \ln(x)\,\mathrm{e}^{-x} + \mathrm{expIntegral}_1(-x)\,\mathrm{e}^{-2x} \tag{1}$$

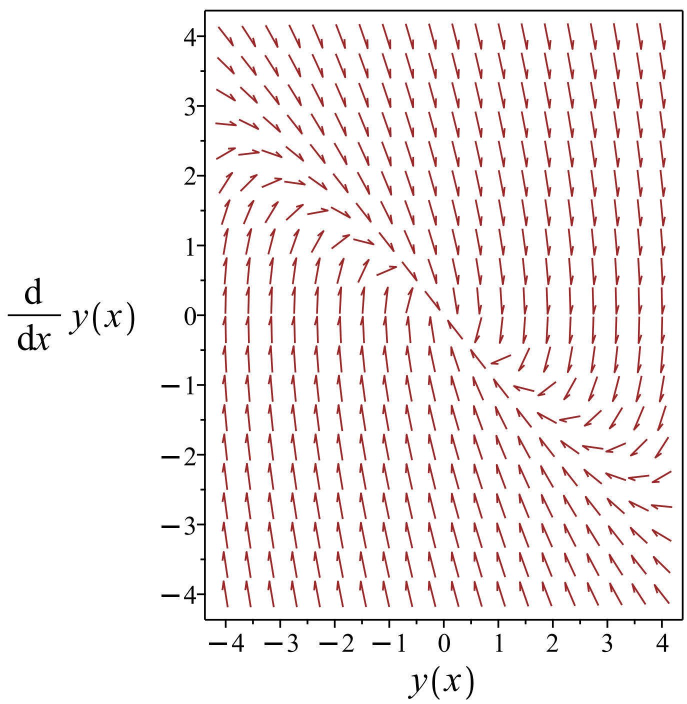

Figure 516: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + \ln(x)\,\mathrm{e}^{-x} + \mathrm{expIntegral}_1(-x)\,\mathrm{e}^{-2x}$$

Verified OK.

12.17.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = \frac{e^{-x}}{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ∘ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{e^{-x}}{x}\right]$

- ∘ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{bmatrix}$

- ∘ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-3x}$

- ∘ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^{-2x}\left(\int \frac{e^{x}}{x} dx\right) + e^{-x}\left(\int \frac{1}{x} dx\right)$

- Compute integrals

  $$y_p(x) = \ln(x)\,\mathrm{e}^{-x} + \mathrm{Ei}_1(-x)\,\mathrm{e}^{-2x}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-x} + \ln(x)\,\mathrm{e}^{-x} + \mathrm{Ei}_1(-x)\,\mathrm{e}^{-2x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=exp(-x)/x,y(x), singsol=all)
```

$$y(x) = \left(-\left(\int \mathrm{e}^{-x}(\mathrm{expIntegral}_1(-x) - c_1)\,dx\right) + c_2\right)\mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 30

```
DSolve[y''[x]+3*y'[x]+2*y[x]==Exp[-x]/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}(-\mathrm{ExpIntegralEi}(x) + e^x \log(x) + c_2e^x + c_1)$$

12.18 problem 18

Internal problem ID [11846]
Internal file name [`OUTPUT/11855_Saturday_April_13_2024_01_12_55_AM_46328560/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = x \ln(x)}$$

12.18.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = x \ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)} \\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^x + c_2 x e^x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^x c_1 + c_2 x e^x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^x \\ y_2 &= x e^x\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & x\,e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(x\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix}$$

Therefore

$$W = (e^x)(x\,e^x + e^x) - (x\,e^x)(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2 e^x \ln(x)}{e^{2x}}\, dx$$

Which simplifies to

$$u_1 = -\int x^2 \ln(x)\, e^{-x} dx$$

Hence

$$u_1 = -\left(-x^2 - 2x - 2\right) e^{-x} \ln(x) + 2\, \text{expIntegral}_1(x) + x\, e^{-x} + 3\, e^{-x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\,\mathrm{e}^{x} \ln \left(x \right)}{\mathrm{e}^{2 x}}d x$$

Which simplifies to

$$u_2 = \int x \ln \left(x \right) \mathrm{e}^{-x}d x$$

Hence

$$u_2 = \left(-x -1\right) \mathrm{e}^{-x} \ln \left(x \right)-\operatorname{expIntegral}_{1}\left(x \right)-\mathrm{e}^{-x}$$

Which simplifies to

$$\begin{aligned} u_1 &= \left(\left(x^{2}+2 x +2\right) \ln \left(x \right)+x +3\right) \mathrm{e}^{-x}+2 \operatorname{expIntegral}_{1}\left(x \right)\\ u_2 &= \left(-1+\ln \left(x \right) \left(-x -1\right)\right) \mathrm{e}^{-x}-\operatorname{expIntegral}_{1}\left(x \right) \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) &= \left(\left(\left(x^{2}+2 x +2\right) \ln \left(x \right)+x +3\right) \mathrm{e}^{-x}+2 \operatorname{expIntegral}_{1}\left(x \right)\right) \mathrm{e}^{x}\\ &\quad + \left(\left(-1+\ln \left(x \right) \left(-x -1\right)\right) \mathrm{e}^{-x}-\operatorname{expIntegral}_{1}\left(x \right)\right) x\,\mathrm{e}^{x} \end{aligned}$$

Which simplifies to

$$y_p(x) = -\mathrm{e}^{x} \left(x -2\right) \operatorname{expIntegral}_{1}\left(x \right)+3+\left(x +2\right) \ln \left(x \right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{x} c_{1} +c_{2} x\,\mathrm{e}^{x}\right) + \left(-\mathrm{e}^{x} \left(x -2\right) \operatorname{expIntegral}_{1}\left(x \right)+3+\left(x +2\right) \ln \left(x \right)\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{x} \left(c_{2} x +c_{1} \right)-\mathrm{e}^{x} \left(x -2\right) \operatorname{expIntegral}_{1}\left(x \right)+3+\left(x +2\right) \ln \left(x \right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \mathrm{e}^{x} \left(c_{2} x +c_{1} \right)-\mathrm{e}^{x} \left(x -2\right) \operatorname{expIntegral}_{1}\left(x \right)+3+\left(x +2\right) \ln \left(x \right) \tag{1}$$

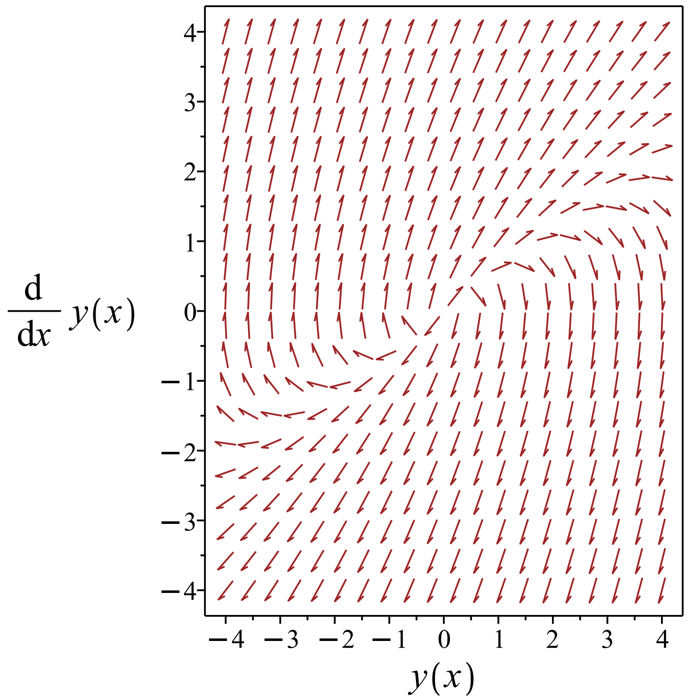

Figure 517: Slope field plot

Verification of solutions

$$y = e^x(c_2x + c_1) - e^x(x - 2)\,\text{expIntegral}_1(x) + 3 + (x + 2)\ln(x)$$

Verified OK.

12.18.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{(p(x)^2 + p'(x))\,y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = x\ln(x)\,e^{-x}$$
$$\left(e^{-x}y\right)'' = x\ln(x)\,e^{-x}$$

Integrating once gives

$$\left(e^{-x}y\right)' = (-1+\ln(x)(-x-1))\,e^{-x} - \operatorname{expIntegral}_1(x) + c_1$$

Integrating again gives

$$\left(e^{-x}y\right) = (3+(x+2)\ln(x))\,e^{-x} + (-x+2)\operatorname{expIntegral}_1(x) + c_1 x + c_2$$

Hence the solution is

$$y = \frac{(3+(x+2)\ln(x))\,e^{-x} + (-x+2)\operatorname{expIntegral}_1(x) + c_1 x + c_2}{e^{-x}}$$

Or

$$y = c_1 x\,e^{x} - x\operatorname{expIntegral}_1(x)\,e^{x} + c_2 e^{x} + x\ln(x) + 2\operatorname{expIntegral}_1(x)\,e^{x} + 2\ln(x) + 3$$

Summary

The solution(s) found are the following

$$y = c_1 x\,e^{x} - x\operatorname{expIntegral}_1(x)\,e^{x} + c_2 e^{x} + x\ln(x) + 2\operatorname{expIntegral}_1(x)\,e^{x} + 2\ln(x) + 3 \tag{1}$$

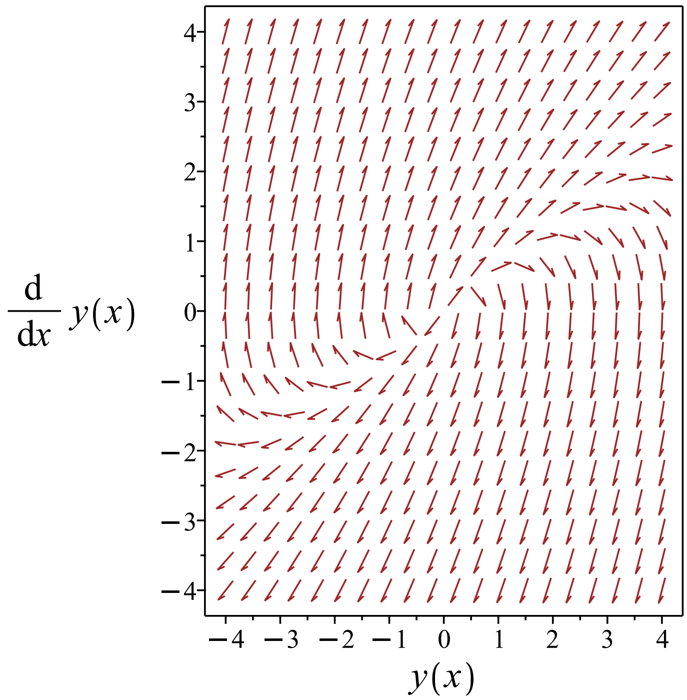

Figure 518: Slope field plot

Verification of solutions

$$y = c_1 x \, e^x - x \, \text{expIntegral}_1(x) \, e^x + c_2 e^x + x \ln(x) + 2 \, \text{expIntegral}_1(x) \, e^x + 2 \ln(x) + 3$$

Verified OK.

12.18.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 459: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1(\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{x} c_1 + c_2 x\,\mathrm{e}^{x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{x} \\
y_2 &= x\,\mathrm{e}^{x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & x\,e^x \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(x\,e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix}$$

Therefore

$$W = (e^x)(x\,e^x + e^x) - (x\,e^x)(e^x)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2 e^x \ln(x)}{e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int x^2 \ln(x)\,e^{-x} dx$$

Hence

$$u_1 = -\left(-x^2 - 2x - 2\right) e^{-x} \ln(x) + 2\ \mathrm{expIntegral}_1(x) + x\,e^{-x} + 3\,e^{-x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\,e^x \ln(x)}{e^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int x \ln(x)\,e^{-x} dx$$

Hence

$$u_2 = (-x-1)\,\mathrm{e}^{-x} \ln(x) - \mathrm{expIntegral}_1(x) - \mathrm{e}^{-x}$$

Which simplifies to

$$\begin{aligned}
u_1 &= \left(\left(x^2+2x+2\right)\ln(x)+x+3\right)\mathrm{e}^{-x} + 2\ \mathrm{expIntegral}_1(x)\\
u_2 &= (-1+\ln(x)(-x-1))\,\mathrm{e}^{-x} - \mathrm{expIntegral}_1(x)
\end{aligned}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned}
y_p(x) = {}&\left(\left(\left(x^2+2x+2\right)\ln(x)+x+3\right)\mathrm{e}^{-x} + 2\ \mathrm{expIntegral}_1(x)\right)\mathrm{e}^{x}\\
&+ \left((-1+\ln(x)(-x-1))\,\mathrm{e}^{-x} - \mathrm{expIntegral}_1(x)\right) x\,\mathrm{e}^{x}
\end{aligned}$$

Which simplifies to

$$y_p(x) = -\mathrm{e}^{x}(x-2)\,\mathrm{expIntegral}_1(x) + 3 + (x+2)\ln(x)$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p\\
&= \left(\mathrm{e}^{x} c_1 + c_2 x\,\mathrm{e}^{x}\right) + \left(-\mathrm{e}^{x}(x-2)\,\mathrm{expIntegral}_1(x) + 3 + (x+2)\ln(x)\right)
\end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{x}(c_2 x + c_1) - \mathrm{e}^{x}(x-2)\,\mathrm{expIntegral}_1(x) + 3 + (x+2)\ln(x)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x}(c_2 x + c_1) - \mathrm{e}^{x}(x-2)\,\mathrm{expIntegral}_1(x) + 3 + (x+2)\ln(x) \tag{1}$$

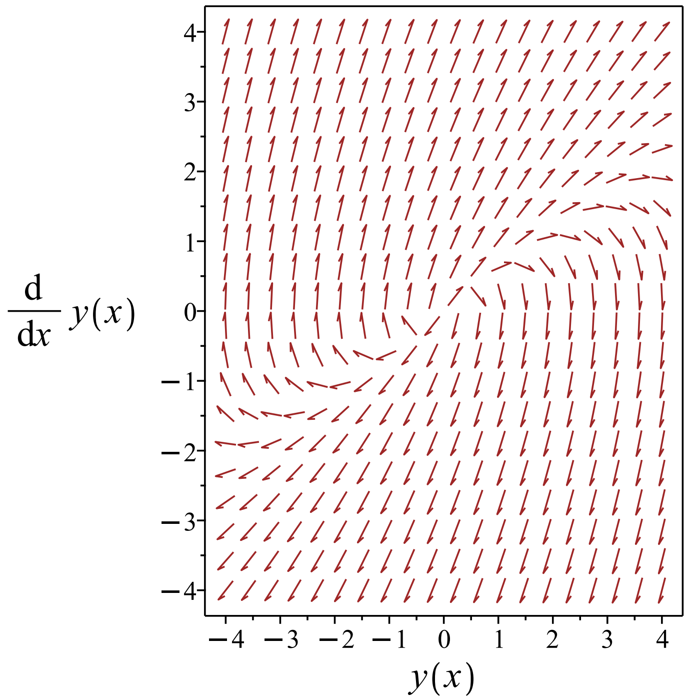

Figure 519: Slope field plot

Verification of solutions

$$y = e^x(c_2x + c_1) - e^x(x - 2)\,\text{expIntegral}_1(x) + 3 + (x + 2)\ln(x)$$

Verified OK.

12.18.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = x\ln(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)^2 = 0$

- Root of the characteristic polynomial

$r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{x} c_1 + c_2 x\,\mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\ln(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{x} & x\,\mathrm{e}^{x} \\ \mathrm{e}^{x} & x\,\mathrm{e}^{x} + \mathrm{e}^{x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \mathrm{e}^{x}\left(-\left(\int x^2 \ln(x)\,\mathrm{e}^{-x} dx\right) + \left(\int x \ln(x)\,\mathrm{e}^{-x} dx\right) x\right)$

  - Compute integrals

    $y_p(x) = -\mathrm{e}^{x}(x-2)\,\mathrm{Ei}_1(x) + 3 + (x+2)\ln(x)$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{x} c_1 + c_2 x\,\mathrm{e}^{x} - \mathrm{e}^{x}(x-2)\,\mathrm{Ei}_1(x) + 3 + (x+2)\ln(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=x*ln(x),y(x), singsol=all)
```

$$y(x) = -(x-2)\,e^{x}\,\mathrm{expIntegral}_1\,(x) + (c_1x + c_2)\,e^{x} + 3 + (x+2)\ln(x)$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 37

```
DSolve[y''[x]-2*y'[x]+y[x]==x*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(x-2)\,\mathrm{ExpIntegralEi}(-x) + (x+2)\log(x) + c_1e^x + c_2e^x x + 3$$

12.19 problem 19

Internal problem ID [11847]
Internal file name [`OUTPUT/11856_Saturday_April_13_2024_01_12_56_AM_67587560/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - 6y'x + 10y = 3x^4 + 6x^3}$$

12.19.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -6x, C = 10, f(x) = 3x^4 + 6x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 6y'x + 10y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 6xrx^{r-1} + 10x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 6r\,x^r + 10x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 6r + 10 = 0$$

Or

$$r^2 - 7r + 10 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 5$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2 x^5 + c_1 x^2$$

Next, we find the particular solution to the ODE

$$x^2 y'' - 6y'x + 10y = 3x^4 + 6x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = x^5$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^5 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^5) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^5 \\ 2x & 5x^4 \end{vmatrix}$$

Therefore

$$W = (x^2)(5x^4) - (x^5)(2x)$$

Which simplifies to

$$W = 3x^6$$

Which simplifies to

$$W = 3x^6$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5(3x^4 + 6x^3)}{3x^8}\, dx$$

Which simplifies to

$$u_1 = -\int (x+2)\, dx$$

Hence

$$u_1 = -\frac{1}{2}x^2 - 2x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(3x^4 + 6x^3)}{3x^8}\, dx$$

Which simplifies to

$$u_2 = \int \frac{x+2}{x^3} dx$$

Hence

$$u_2 = -\frac{1}{x^2} - \frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{1}{2}x^2 - 2x\right) x^2 + \left(-\frac{1}{x^2} - \frac{1}{x}\right) x^5$$

Which simplifies to

$$y_p(x) = -\frac{3x^3(x+2)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= -\frac{3}{2}x^4 - 3x^3 + c_2x^5 + c_1x^2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{3}{2}x^4 - 3x^3 + c_2x^5 + c_1x^2 \tag{1}$$

Verification of solutions

$$y = -\frac{3}{2}x^4 - 3x^3 + c_2x^5 + c_1x^2$$

Verified OK.

12.19.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 6y'x + 10y = 0$$

In normal form the ode

$$x^2y'' - 6y'x + 10y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{6}{x} \\ q(x) &= \frac{10}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{6}{x} dx\right)} dx \\
&= \int e^{6\ln(x)} \, dx \\
&= \int x^6 dx \\
&= \frac{x^7}{7} \qquad (6)
\end{aligned}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{10}{x^2}}{x^{12}} \\
&= \frac{10}{x^{14}} \qquad (7)
\end{aligned}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{10y(\tau)}{x^{14}} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{10}{x^{14}} = \frac{10}{49\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{10y(\tau)}{49\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
49\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 10y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
49\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 10\tau^r = 0
$$

Simplifying gives

$$49r(r-1)\,\tau^r + 0\,\tau^r + 10\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$49r(r-1) + 0 + 10 = 0$$

Or

$$49r^2 - 49r + 10 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{2}{7}$$
$$r_2 = \frac{5}{7}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{2}{7}} + c_2\tau^{\frac{5}{7}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 7^{\frac{5}{7}} (x^7)^{\frac{2}{7}}}{7} + \frac{c_2 7^{\frac{2}{7}} (x^7)^{\frac{5}{7}}}{7}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 7^{\frac{5}{7}} (x^7)^{\frac{2}{7}}}{7} + \frac{c_2 7^{\frac{2}{7}} (x^7)^{\frac{5}{7}}}{7}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x^7)^{\frac{2}{7}}$$
$$y_2 = (x^7)^{\frac{5}{7}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^7)^{\frac{2}{7}} & (x^7)^{\frac{5}{7}} \\ \frac{d}{dx}\left((x^7)^{\frac{2}{7}}\right) & \frac{d}{dx}\left((x^7)^{\frac{5}{7}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^7)^{\frac{2}{7}} & (x^7)^{\frac{5}{7}} \\ \frac{2x^6}{(x^7)^{\frac{5}{7}}} & \frac{5x^6}{(x^7)^{\frac{2}{7}}} \end{vmatrix}$$

Therefore

$$W = \left((x^7)^{\frac{2}{7}}\right)\left(\frac{5x^6}{(x^7)^{\frac{2}{7}}}\right) - \left((x^7)^{\frac{5}{7}}\right)\left(\frac{2x^6}{(x^7)^{\frac{5}{7}}}\right)$$

Which simplifies to

$$W = 3x^6$$

Which simplifies to

$$W = 3x^6$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^7)^{\frac{5}{7}}(3x^4 + 6x^3)}{3x^8}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^7)^{\frac{5}{7}}(x + 2)}{x^5} dx$$

Hence

$$u_1 = -\frac{(x + 4)(x^7)^{\frac{5}{7}}}{2x^4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^7)^{\frac{2}{7}}(3x^4 + 6x^3)}{3x^8}\, dx$$

Which simplifies to

$$u_2 = \int \frac{(x^7)^{\frac{2}{7}}(x + 2)}{x^5} dx$$

Hence

$$u_2 = -\frac{(1 + x)(x^7)^{\frac{2}{7}}}{x^4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^3(x + 4)}{2} - (1 + x)\, x^3$$

Which simplifies to

$$y_p(x) = -\frac{3}{2}x^4 - 3x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 7^{\frac{5}{7}}(x^7)^{\frac{2}{7}}}{7} + \frac{c_2 7^{\frac{2}{7}}(x^7)^{\frac{5}{7}}}{7}\right) + \left(-\frac{3}{2}x^4 - 3x^3\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 7^{\frac{5}{7}} (x^7)^{\frac{2}{7}}}{7} + \frac{c_2 7^{\frac{2}{7}} (x^7)^{\frac{5}{7}}}{7} - \frac{3x^4}{2} - 3x^3 \tag{1}$$

Verification of solutions

$$y = \frac{c_1 7^{\frac{5}{7}} (x^7)^{\frac{2}{7}}}{7} + \frac{c_2 7^{\frac{2}{7}} (x^7)^{\frac{5}{7}}}{7} - \frac{3x^4}{2} - 3x^3$$

Verified OK.

12.19.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -6x, C = 10, f(x) = 3x^4 + 6x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 6y'x + 10y = 0$$

In normal form the ode

$$x^2y'' - 6y'x + 10y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{6}{x}$$
$$q(x) = \frac{10}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{10}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{10}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{10}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{6}{x}\frac{\sqrt{10}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{10}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{7c\sqrt{10}}{10}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{7c\sqrt{10}\,\left(\frac{d}{d\tau}y(\tau)\right)}{10} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{7\sqrt{10}\,c\tau}{20}}\left(c_1\cosh\left(\frac{3\sqrt{10}\,c\tau}{20}\right) + ic_2\sinh\left(\frac{3\sqrt{10}\,c\tau}{20}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{10}\, \sqrt{\frac{1}{x^2}} dx}{c} \\
&= \frac{\sqrt{10}\, \sqrt{\frac{1}{x^2}}\, x \ln{(x)}}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{7}{2}} \left(c_1 \cosh\left(\frac{3\ln{(x)}}{2}\right) + i c_2 \sinh\left(\frac{3\ln{(x)}}{2}\right)\right)
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 6y'x + 10y = 3x^4 + 6x^3
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \left(x^7\right)^{\frac{2}{7}} \\
y_2 &= \left(x^7\right)^{\frac{5}{7}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^7)^{\frac{2}{7}} & (x^7)^{\frac{5}{7}} \\ \frac{d}{dx}\left((x^7)^{\frac{2}{7}}\right) & \frac{d}{dx}\left((x^7)^{\frac{5}{7}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^7)^{\frac{2}{7}} & (x^7)^{\frac{5}{7}} \\ \frac{2x^6}{(x^7)^{\frac{5}{7}}} & \frac{5x^6}{(x^7)^{\frac{2}{7}}} \end{vmatrix}$$

Therefore

$$W = \left((x^7)^{\frac{2}{7}}\right)\left(\frac{5x^6}{(x^7)^{\frac{2}{7}}}\right) - \left((x^7)^{\frac{5}{7}}\right)\left(\frac{2x^6}{(x^7)^{\frac{5}{7}}}\right)$$

Which simplifies to

$$W = 3x^6$$

Which simplifies to

$$W = 3x^6$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^7)^{\frac{5}{7}} (3x^4 + 6x^3)}{3x^8}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^7)^{\frac{5}{7}} (x + 2)}{x^5} dx$$

Hence

$$u_1 = -\frac{(x + 4)(x^7)^{\frac{5}{7}}}{2x^4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^7)^{\frac{2}{7}} (3x^4 + 6x^3)}{3x^8}\, dx$$

Which simplifies to

$$u_2 = \int \frac{(x^7)^{\frac{2}{7}} (x+2)}{x^5} dx$$

Hence

$$u_2 = -\frac{(1+x)(x^7)^{\frac{2}{7}}}{x^4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^3(x+4)}{2} - (1+x)\,x^3$$

Which simplifies to

$$y_p(x) = -\frac{3}{2}x^4 - 3x^3$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(x^{\frac{7}{2}}\left(c_1 \cosh\left(\frac{3\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{3\ln(x)}{2}\right)\right)\right) + \left(-\frac{3}{2}x^4 - 3x^3\right) \\
&= -\frac{3x^4}{2} - 3x^3 + x^{\frac{7}{2}}\left(c_1 \cosh\left(\frac{3\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{3\ln(x)}{2}\right)\right)
\end{aligned}$$

Which simplifies to

$$y = i \sinh\left(\frac{3\ln(x)}{2}\right) x^{\frac{7}{2}} c_2 + \cosh\left(\frac{3\ln(x)}{2}\right) x^{\frac{7}{2}} c_1 - \frac{3x^4}{2} - 3x^3$$

Summary

The solution(s) found are the following

$$y = i \sinh\left(\frac{3\ln(x)}{2}\right) x^{\frac{7}{2}} c_2 + \cosh\left(\frac{3\ln(x)}{2}\right) x^{\frac{7}{2}} c_1 - \frac{3x^4}{2} - 3x^3 \tag{1}$$

Verification of solutions

$$y = i \sinh\left(\frac{3\ln(x)}{2}\right) x^{\frac{7}{2}} c_2 + \cosh\left(\frac{3\ln(x)}{2}\right) x^{\frac{7}{2}} c_1 - \frac{3x^4}{2} - 3x^3$$

Verified OK.

12.19.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -6x, C = 10, f(x) = 3x^4 + 6x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 6y'x + 10y = 0$$

In normal form the ode

$$x^2y'' - 6y'x + 10y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{6}{x} \\ q(x) &= \frac{10}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{6n}{x^2} + \frac{10}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 5 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{4v'(x)}{x} &= 0 \\ v''(x) + \frac{4v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{4u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{4u}{x} \end{aligned}$$

Where $f(x) = -\frac{4}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{4}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{4}{x}\,dx \\ \ln(u) &= -4\ln(x) + c_1 \\ u &= \mathrm{e}^{-4\ln(x)+c_1} \\ &= \frac{c_1}{x^4} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{3x^3} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{3x^3} + c_2\right) x^5 \\
&= c_2 x^5 - \frac{1}{3} c_1 x^2
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 6y'x + 10y = 3x^4 + 6x^3
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= x^5
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x^2 & x^5 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^5) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} x^2 & x^5 \\ 2x & 5x^4 \end{vmatrix}$$

Therefore

$$W = \left(x^2\right)\left(5x^4\right) - \left(x^5\right)(2x)$$

Which simplifies to

$$W = 3x^6$$

Which simplifies to

$$W = 3x^6$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^5(3x^4 + 6x^3)}{3x^8}\,dx$$

Which simplifies to

$$u_1 = -\int (x+2)\,dx$$

Hence

$$u_1 = -\frac{1}{2}x^2 - 2x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(3x^4 + 6x^3)}{3x^8}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x+2}{x^3}dx$$

Hence

$$u_2 = -\frac{1}{x^2} - \frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{1}{2}x^2 - 2x\right)x^2 + \left(-\frac{1}{x^2} - \frac{1}{x}\right)x^5$$

Which simplifies to

$$y_p(x) = -\frac{3x^3(x+2)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{3x^3} + c_2\right)x^5\right) + \left(-\frac{3x^3(x+2)}{2}\right) \\ &= -\frac{3x^3(x+2)}{2} + \left(-\frac{c_1}{3x^3} + c_2\right)x^5 \end{aligned}$$

Which simplifies to

$$y = -\frac{x^2(-6c_2x^3 + 9x^2 + 2c_1 + 18x)}{6}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^2(-6c_2x^3 + 9x^2 + 2c_1 + 18x)}{6} \tag{1}$$

Verification of solutions

$$y = -\frac{x^2(-6c_2x^3 + 9x^2 + 2c_1 + 18x)}{6}$$

Verified OK.

12.19.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 6y'x + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -6x \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 461: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{2}{x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -1$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6x}{x^2}\,dx} \\
&= z_1 e^{3\ln(x)} \\
&= z_1\left(x^3\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6x}{x^2}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{6\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(\frac{x^3}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - 6y'x + 10y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x^2 + \frac{1}{3} c_2 x^5$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = \frac{x^5}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \frac{x^5}{3} \\ \frac{d}{dx}(x^2) & \frac{d}{dx}\left(\frac{x^5}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \frac{x^5}{3} \\ 2x & \frac{5x^4}{3} \end{vmatrix}$$

Therefore

$$W = (x^2)\left(\frac{5x^4}{3}\right) - \left(\frac{x^5}{3}\right)(2x)$$

Which simplifies to

$$W = x^6$$

Which simplifies to

$$W = x^6$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^5(3x^4+6x^3)}{3}}{x^8}\, dx$$

Which simplifies to

$$u_1 = -\int (x+2)\, dx$$

Hence

$$u_1 = -\frac{1}{2}x^2 - 2x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(3x^4+6x^3)}{x^8}\, dx$$

Which simplifies to

$$u_2 = \int \frac{3x+6}{x^3} dx$$

Hence

$$u_2 = -\frac{3}{x^2} - \frac{3}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{1}{2}x^2 - 2x\right)x^2 + \frac{\left(-\frac{3}{x^2} - \frac{3}{x}\right)x^5}{3}$$

Which simplifies to

$$y_p(x) = -\frac{3x^3(x+2)}{2}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1x^2 + \frac{1}{3}c_2x^5\right) + \left(-\frac{3x^3(x+2)}{2}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x^2 + \frac{c_2x^5}{3} - \frac{3x^3(x+2)}{2} \tag{1}$$

Verification of solutions

$$y = c_1x^2 + \frac{c_2x^5}{3} - \frac{3x^3(x+2)}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(x^2*diff(y(x),x$2)-6*x*diff(y(x),x)+10*y(x)=3*x^4+6*x^3,y(x), singsol=all)
```

$$y(x) = -\frac{3}{2}x^4 - 3x^3 + \frac{1}{3}c_1x^5 + c_2x^2$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 28

```
DSolve[x^2*y''[x]-6*x*y'[x]+10*y[x]==3*x^4+6*x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 x^5 - \frac{3}{2}(x+2)x^3 + c_1 x^2$$

12.20 problem 20

Internal problem ID [11848]
Internal file name [`OUTPUT/11857_Saturday_April_13_2024_01_12_59_AM_10005007/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "linear_second_order_ode_solved_by_an_integrating_factor", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(1+x)^2 y'' - 2(1+x) y' + 2y = 1}$$

12.20.1 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x) y' + \frac{(p(x)^2 + p'(x)) y}{2} = f(x)$$

Where $p(x) = -\frac{2}{1+x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -\frac{2}{1+x}\,dx} \\ &= \frac{1}{1+x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \frac{1}{(1+x)^3} \\ \left(\frac{y}{1+x}\right)'' &= \frac{1}{(1+x)^3} \end{aligned}$$

Integrating once gives

$$\left(\frac{y}{1+x}\right)' = -\frac{1}{2(1+x)^2} + c_1$$

Integrating again gives

$$\left(\frac{y}{1+x}\right) = c_1 x + \frac{1}{2+2x} + c_2$$

Hence the solution is

$$y = \frac{c_1 x + \frac{1}{2+2x} + c_2}{\frac{1}{1+x}}$$

Or

$$y = (x^2 + x) c_1 + (x+1) c_2 + \frac{x}{2x+2} + \frac{1}{2x+2}$$

Summary

The solution(s) found are the following

$$y = (x^2 + x) c_1 + (x+1) c_2 + \frac{x}{2x+2} + \frac{1}{2x+2} \tag{1}$$

Verification of solutions

$$y = (x^2 + x) c_1 + (x+1) c_2 + \frac{x}{2x+2} + \frac{1}{2x+2}$$

Verified OK.

12.20.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$(x+1)^2 y'' + (-2x-2) y' + 2y = 0$$

In normal form the ode

$$(x+1)^2 y'' + (-2x-2) y' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2}{x+1} \\ q(x) &= \frac{2}{(x+1)^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\ &= \int e^{-\left(\int -\frac{2}{x+1}dx\right)}dx \\ &= \int e^{2\ln(x+1)}\,dx \\ &= \int (x+1)^2\,dx \\ &= \frac{(x+1)^3}{3}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{2}{(x+1)^2}}{(x+1)^4} \\ &= \frac{2}{(x+1)^6}\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{(x+1)^6} = 0$$

But in terms of τ

$$\frac{2}{(x+1)^6} = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{3}} + c_2\tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}}\left((x+1)^3\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}}\left((x+1)^3\right)^{\frac{2}{3}}}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 3^{\frac{2}{3}}\left((x+1)^3\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}}\left((x+1)^3\right)^{\frac{2}{3}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^3 + 3x^2 + 3x + 1\right)^{\frac{1}{3}}$$

$$y_2 = \left(x^3 + 3x^2 + 3x + 1\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^3 + 3x^2 + 3x + 1\right)^{\frac{1}{3}} & \left(x^3 + 3x^2 + 3x + 1\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(x^3 + 3x^2 + 3x + 1\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(x^3 + 3x^2 + 3x + 1\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^3 + 3x^2 + 3x + 1\right)^{\frac{1}{3}} & \left(x^3 + 3x^2 + 3x + 1\right)^{\frac{2}{3}} \\ \frac{3x^2+6x+3}{3(x^3+3x^2+3x+1)^{\frac{2}{3}}} & \frac{2x^2+4x+2}{(x^3+3x^2+3x+1)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^3 + 3x^2 + 3x + 1\right)^{\frac{1}{3}}\right)\left(\frac{2x^2 + 4x + 2}{\left(x^3 + 3x^2 + 3x + 1\right)^{\frac{1}{3}}}\right) - \left(\left(x^3 + 3x^2 + 3x + 1\right)^{\frac{2}{3}}\right)\left(\frac{3x^2 + 6x + 3}{3\left(x^3 + 3x^2 + 3x + 1\right)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = x^2 + 2x + 1$$

Which simplifies to

$$W = (x + 1)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^3+3x^2+3x+1)^{\frac{2}{3}}}{(x+1)^4}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\left((x+1)^3\right)^{\frac{2}{3}}}{(x+1)^4} dx$$

Hence

$$u_1 = \frac{\left((x+1)^3\right)^{\frac{2}{3}}}{(x+1)^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^3+3x^2+3x+1)^{\frac{1}{3}}}{(x+1)^4}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\left((x+1)^3\right)^{\frac{1}{3}}}{(x+1)^4} dx$$

Hence

$$u_2 = -\frac{\left((x+1)^3\right)^{\frac{1}{3}}}{2\,(x+1)^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left((x+1)^3\right)^{\frac{2}{3}} (x^3+3x^2+3x+1)^{\frac{1}{3}}}{(x+1)^3} - \frac{\left((x+1)^3\right)^{\frac{1}{3}} (x^3+3x^2+3x+1)^{\frac{2}{3}}}{2\,(x+1)^3}$$

Which simplifies to

$$y_p(x) = \frac{1}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1 3^{\frac{2}{3}} \left((x+1)^3\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left((x+1)^3\right)^{\frac{2}{3}}}{3}\right) + \left(\frac{1}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 3^{\frac{2}{3}} \left((x+1)^3\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left((x+1)^3\right)^{\frac{2}{3}}}{3} + \frac{1}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 3^{\frac{2}{3}} \left((x+1)^3\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left((x+1)^3\right)^{\frac{2}{3}}}{3} + \frac{1}{2}
$$

Verified OK.

12.20.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = (x+1)^2, B = -2x - 2, C = 2, f(x) = 1$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
(x+1)^2 y'' + (-2x-2) y' + 2y = 0
$$

In normal form the ode

$$
(x+1)^2 y'' + (-2x-2) y' + 2y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x) y' + q(x) y = 0 \tag{2}
$$

Where

$$p(x) = -\frac{2}{x+1}$$
$$q(x) = \frac{2}{(x+1)^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$
$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{2}\,\sqrt{\frac{1}{(x+1)^2}}}{c} \\ \tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{(x+1)^2}}\,(x+1)^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{(x+1)^2}}(x+1)^3} - \frac{2}{x+1}\frac{\sqrt{2}\sqrt{\frac{1}{(x+1)^2}}}{c}}{\left(\frac{\sqrt{2}\sqrt{\frac{1}{(x+1)^2}}}{c}\right)^2} \\ &= -\frac{3c\sqrt{2}}{2}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{3c\sqrt{2}\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{\frac{3\sqrt{2}\, c\tau}{4}} \left(c_1 \cosh\left(\frac{\sqrt{2}\, c\tau}{4}\right) + i c_2 \sinh\left(\frac{\sqrt{2}\, c\tau}{4}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{2}\sqrt{\frac{1}{(x+1)^2}}dx}{c} \\
&= \frac{\sqrt{2}\sqrt{\frac{1}{(x+1)^2}}\,(x+1)\ln(x+1)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = (x+1)^{\frac{3}{2}} \left(c_1 \cosh\left(\frac{\ln(x+1)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x+1)}{2}\right)\right)$$

Now the particular solution to this ODE is found

$$(x+1)^2 y'' + (-2x-2)\, y' + 2y = 1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^3 + 3x^2 + 3x + 1\right)^{\frac{1}{3}}$$

$$y_2 = \left(x^3 + 3x^2 + 3x + 1\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^3+3x^2+3x+1)^{\frac{1}{3}} & (x^3+3x^2+3x+1)^{\frac{2}{3}} \\ \frac{d}{dx}\left((x^3+3x^2+3x+1)^{\frac{1}{3}}\right) & \frac{d}{dx}\left((x^3+3x^2+3x+1)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^3+3x^2+3x+1)^{\frac{1}{3}} & (x^3+3x^2+3x+1)^{\frac{2}{3}} \\ \frac{3x^2+6x+3}{3(x^3+3x^2+3x+1)^{\frac{2}{3}}} & \frac{2x^2+4x+2}{(x^3+3x^2+3x+1)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((x^3+3x^2+3x+1)^{\frac{1}{3}}\right)\left(\frac{2x^2+4x+2}{(x^3+3x^2+3x+1)^{\frac{1}{3}}}\right) - \left((x^3+3x^2+3x+1)^{\frac{2}{3}}\right)\left(\frac{3x^2+6x+3}{3(x^3+3x^2+3x+1)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = x^2+2x+1$$

Which simplifies to

$$W = (x+1)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^3+3x^2+3x+1)^{\frac{2}{3}}}{(x+1)^4}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\left((x+1)^3\right)^{\frac{2}{3}}}{(x+1)^4}dx$$

Hence

$$u_1 = \frac{\left((x+1)^3\right)^{\frac{2}{3}}}{(x+1)^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^3 + 3x^2 + 3x + 1)^{\frac{1}{3}}}{(x+1)^4}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\left((x+1)^3\right)^{\frac{1}{3}}}{(x+1)^4} dx$$

Hence

$$u_2 = -\frac{\left((x+1)^3\right)^{\frac{1}{3}}}{2\,(x+1)^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left((x+1)^3\right)^{\frac{2}{3}} (x^3 + 3x^2 + 3x + 1)^{\frac{1}{3}}}{(x+1)^3} - \frac{\left((x+1)^3\right)^{\frac{1}{3}} (x^3 + 3x^2 + 3x + 1)^{\frac{2}{3}}}{2\,(x+1)^3}$$

Which simplifies to

$$y_p(x) = \frac{1}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left((x+1)^{\frac{3}{2}} \left(c_1 \cosh\left(\frac{\ln(x+1)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x+1)}{2}\right)\right)\right) + \left(\frac{1}{2}\right) \\ &= \frac{1}{2} + (x+1)^{\frac{3}{2}} \left(c_1 \cosh\left(\frac{\ln(x+1)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x+1)}{2}\right)\right) \end{aligned}$$

Which simplifies to

$$y = \frac{1}{2} + (x+1)^{\frac{3}{2}} \left(c_1 \cosh\left(\frac{\ln(x+1)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x+1)}{2}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{1}{2} + (x+1)^{\frac{3}{2}} \left(c_1 \cosh\left(\frac{\ln(x+1)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x+1)}{2}\right)\right) \tag{1}$$

Verification of solutions

$$y = \frac{1}{2} + (x+1)^{\frac{3}{2}} \left(c_1 \cosh\left(\frac{\ln(x+1)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x+1)}{2}\right)\right)$$

Verified OK.

12.20.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= (x+1)^2 \\ B &= -2x - 2 \\ C &= 2 \\ F &= 1 \end{aligned}$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left((x+1)^2\right)(0) + (-2x-2)(-2) + (2)(-2x-2) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-2(x+1)^3 v'' + (0) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
-2(x+1)^3 u'(x) = 0
$$

Which is now solved for u. Integrating both sides gives

$$
\begin{aligned}
u(x) &= \int 0 \, \mathrm{d}x \\
&= c_1
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int c_1 \, \mathrm{d}x \\
&= c_1 x + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (-2x-2)(c_1 x + c_2) \\
&= -2(x+1)(c_1 x + c_2)
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = -2x - 2$$
$$y_2 = -2x^2 - 2x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -2x - 2 & -2x^2 - 2x \\ \frac{d}{dx}(-2x - 2) & \frac{d}{dx}(-2x^2 - 2x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -2x - 2 & -2x^2 - 2x \\ -2 & -4x - 2 \end{vmatrix}$$

Therefore

$$W = (-2x - 2)(-4x - 2) - (-2x^2 - 2x)(-2)$$

Which simplifies to

$$W = 4x^2 + 8x + 4$$

Which simplifies to

$$W = 4(x + 1)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2x^2 - 2x}{4(x + 1)^4} dx$$

Which simplifies to

$$u_1 = -\int -\frac{x}{2\left(x+1\right)^3} dx$$

Hence

$$u_1 = -\frac{1}{2\left(x+1\right)} + \frac{1}{4\left(x+1\right)^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-2x-2}{4\left(x+1\right)^4}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{1}{2\left(x+1\right)^3} dx$$

Hence

$$u_2 = \frac{1}{4\left(x+1\right)^2}$$

Which simplifies to

$$u_1 = \frac{-2x-1}{4\left(x+1\right)^2}$$
$$u_2 = \frac{1}{4\left(x+1\right)^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\left(-2x-1\right)\left(-2x-2\right)}{4\left(x+1\right)^2} + \frac{-2x^2-2x}{4\left(x+1\right)^2}$$

Which simplifies to

$$y_p(x) = \frac{1}{2}$$

Hence the complete solution is

$$\begin{aligned}
y(x) &= y_h + y_p \\
&= (-2(x+1)\,(c_1x + c_2)) + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - 2c_1x^2 + 2(-c_1 - c_2)\,x - 2c_2
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{2} - 2c_1x^2 + 2(-c_1 - c_2)\,x - 2c_2 \tag{1}$$

Verification of solutions

$$y = \frac{1}{2} - 2c_1x^2 + 2(-c_1 - c_2)\,x - 2c_2$$

Verified OK.

12.20.5 Solving using Kovacic algorithm

Writing the ode as

$$(x+1)^2\,y'' + (-2x-2)\,y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= (x+1)^2 \\
B &= -2x - 2 \\
C &= 2
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 462: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-2}{(x+1)^2}\,dx} \\ &= z_1 e^{\ln(x+1)} \\ &= z_1(x+1) \end{aligned}$$

Which simplifies to

$$y_1 = x + 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{(x+1)^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2\ln(x+1)}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x+1) + c_2(x + 1(x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(x+1)^2 y'' + (-2x-2)\, y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1(x+1) + c_2 x(x+1)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x+1$$
$$y_2 = x(x+1)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x+1 & x(x+1) \\ \frac{d}{dx}(x+1) & \frac{d}{dx}(x(x+1)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x+1 & x(x+1) \\ 1 & 2x+1 \end{vmatrix}$$

Therefore

$$W = (x+1)(2x+1) - (x(x+1))(1)$$

Which simplifies to

$$W = x^2 + 2x + 1$$

Which simplifies to

$$W = (x+1)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x(x+1)}{(x+1)^4}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x}{(x+1)^3}dx$$

Hence

$$u_1 = \frac{1}{x+1} - \frac{1}{2(x+1)^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x+1}{(x+1)^4}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{(x+1)^3}dx$$

Hence

$$u_2 = -\frac{1}{2(x+1)^2}$$

Which simplifies to

$$u_1 = \frac{2x+1}{2\left(x+1\right)^2}$$
$$u_2 = -\frac{1}{2\left(x+1\right)^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2x+1}{2x+2} - \frac{x}{2\left(x+1\right)}$$

Which simplifies to

$$y_p(x) = \frac{1}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1(x+1) + c_2x(x+1)) + \left(\frac{1}{2}\right) \end{aligned}$$

Which simplifies to

$$y = \left(x+1\right)\left(c_2x + c_1\right) + \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = \left(x+1\right)\left(c_2x + c_1\right) + \frac{1}{2} \tag{1}$$

Verification of solutions

$$y = \left(x+1\right)\left(c_2x + c_1\right) + \frac{1}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve((x+1)^2*diff(y(x),x$2)-2*(x+1)*diff(y(x),x)+2*y(x)=1,y(x), singsol=all)
```

$$y(x) = (1+x)^2 c_1 + c_2 x + c_2 + \frac{1}{2}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 23

```
DSolve[(x+1)^2*y''[x]-2*(x+1)*y'[x]+2*y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2(x+1)^2 + c_1(x+1) + \frac{1}{2}$$

12.21 problem 21

Internal problem ID [11849]
Internal file name [`OUTPUT/11858_Saturday_April_13_2024_01_13_01_AM_84043608/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{(x^2+2x)\,y''-2(x+1)\,y'+2y=(x+2)^2}$$

12.21.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=x^2+2x, B=-2x-2, C=2, f(x)=(x+2)^2$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$(x^2+2x)\,y''+(-2x-2)\,y'+2y=0$$

In normal form the ode

$$\left(x^2+2x\right)y''+\left(-2x-2\right)y'+2y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=\frac{-2x-2}{x\left(x+2\right)}\\ q(x)&=\frac{2}{x\left(x+2\right)}\end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right)v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right)v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{n(-2x-2)}{x^2\left(x+2\right)}+\frac{2}{x\left(x+2\right)}=0 \tag{5}$$

Solving (5) for n gives

$$n=2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(x)+\left(\frac{4}{x}+\frac{-2x-2}{x\left(x+2\right)}\right)v'(x)&=0\\ v''(x)+\frac{\left(2x+6\right)v'(x)}{x\left(x+2\right)}&=0\end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x)+\frac{(2x+6)\,u(x)}{x\,(x+2)}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{2u(x+3)}{x\,(x+2)}
\end{aligned}$$

Where $f(x)=-\frac{2(x+3)}{x(x+2)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2(x+3)}{x\,(x+2)}\,dx\\
\int\frac{1}{u}\,du &= \int-\frac{2(x+3)}{x\,(x+2)}\,dx\\
\ln(u) &= -3\ln(x)+\ln(x+2)+c_1\\
u &= \mathrm{e}^{-3\ln(x)+\ln(x+2)+c_1}\\
&= c_1\mathrm{e}^{-3\ln(x)+\ln(x+2)}
\end{aligned}$$

Which simplifies to

$$u(x)=c_1\left(\frac{1}{x^2}+\frac{2}{x^3}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x)\\
v(x) &= \int u(x)\,dx+c_2\\
&= c_1\left(-\frac{1}{x^2}-\frac{1}{x}\right)+c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n\\
&= \left(c_1\left(-\frac{1}{x^2}-\frac{1}{x}\right)+c_2\right)x^2\\
&= (-x-1)\,c_1+c_2x^2
\end{aligned}$$

Now the particular solution to this ODE is found

$$\left(x^2+2x\right) y''+(-2x-2)\, y'+2y=(x+2)^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^2 \\ y_2 &= -x-1 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1=-\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2=\int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W=\begin{vmatrix} x^2 & -x-1 \\ \frac{d}{dx}\left(x^2\right) & \frac{d}{dx}(-x-1) \end{vmatrix}$$

Which gives

$$W=\begin{vmatrix} x^2 & -x-1 \\ 2x & -1 \end{vmatrix}$$

Therefore

$$W=\left(x^2\right)(-1)-(-x-1)\,(2x)$$

Which simplifies to

$$W = x^2 + 2x$$

Which simplifies to

$$W = x^2 + 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(-x-1)(x+2)^2}{(x^2+2x)^2}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{-x-1}{x^2} dx$$

Hence

$$u_1 = -\frac{1}{x} + \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(x+2)^2}{(x^2+2x)^2}\,dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{1}{x} + \ln(x)\right) x^2 + (-x-1)\,x$$

Which simplifies to

$$y_p(x) = x(x\ln(x) - x - 2)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(c_1\left(-\frac{1}{x^2} - \frac{1}{x}\right) + c_2\right) x^2\right) + (x(x\ln(x) - x - 2)) \\
&= x(x\ln(x) - x - 2) + \left(c_1\left(-\frac{1}{x^2} - \frac{1}{x}\right) + c_2\right) x^2
\end{aligned}
$$

Which simplifies to

$$
y = x^2\ln(x) + c_2x^2 - c_1x - x^2 - c_1 - 2x
$$

Summary

The solution(s) found are the following

$$
y = x^2\ln(x) + c_2x^2 - c_1x - x^2 - c_1 - 2x \tag{1}
$$

Verification of solutions

$$
y = x^2\ln(x) + c_2x^2 - c_1x - x^2 - c_1 - 2x
$$

Verified OK.

12.21.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right)v' + (AB'' + BB' + CB)v &= 0
\end{aligned} \tag{1}
$$

If the term $AB''+BB'+CB$ is zero, then this method works and can be used to solve

$$ABv''+\left(2AB'+B^2\right)v'=0$$

By Using $u=v'$ which reduces the order of the above ode to one. The new ode is

$$ABu'+\left(2AB'+B^2\right)u=0$$

The above ode is first order ode which is solved for u. Now a new ode $v'=u$ is solved for v as first order ode. Then the final solution is obtain from $y=Bv$.

This method works only if the term $AB''+BB'+CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2+2x\\
B &= -2x-2\\
C &= 2\\
F &= (x+2)^2
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB''+BB'+CB &= \left(x^2+2x\right)(0)+(-2x-2)(-2)+(2)(-2x-2)\\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x(x+2)(x+1)v''+(4)v'=0$$

Now by applying $v'=u$ the above becomes

$$\left(-2x^3-6x^2-4x\right)u'(x)+4u(x)=0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= \frac{2u}{x\left(x^2+3x+2\right)}
\end{aligned}$$

Where $f(x)=\frac{2}{x(x^2+3x+2)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{2}{x\left(x^2+3x+2\right)}\,dx\\
\int\frac{1}{u}\,du &= \int\frac{2}{x\left(x^2+3x+2\right)}\,dx\\
\ln(u) &= -2\ln(x+1)+\ln(x)+\ln(x+2)+c_1\\
u &= \mathrm{e}^{-2\ln(x+1)+\ln(x)+\ln(x+2)+c_1}\\
&= c_1\mathrm{e}^{-2\ln(x+1)+\ln(x)+\ln(x+2)}
\end{aligned}$$

Which simplifies to

$$u(x) = c_1 \left(\frac{x^2}{(x+1)^2} + \frac{2x}{(x+1)^2} \right)$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= c_1 \left(\frac{x^2}{(x+1)^2} + \frac{2x}{(x+1)^2} \right) \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1 x(x+2)}{(x+1)^2} \, \mathrm{d}x \\ &= c_1 \left(x + \frac{1}{x+1} \right) + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (-2x-2) \left(c_1 \left(x + \frac{1}{x+1} \right) + c_2 \right) \\ &= \left(-2x^2 - 2x - 2 \right) c_1 - 2(x+1) \, c_2 \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= -2x - 2 \\ y_2 &= -2x^2 - 2x - 2 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -2x-2 & -2x^2-2x-2 \\ \frac{d}{dx}(-2x-2) & \frac{d}{dx}(-2x^2-2x-2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -2x-2 & -2x^2-2x-2 \\ -2 & -4x-2 \end{vmatrix}$$

Therefore

$$W = (-2x-2)(-4x-2) - \left(-2x^2-2x-2\right)(-2)$$

Which simplifies to

$$W = 4x^2 + 8x$$

Which simplifies to

$$W = 4x^2 + 8x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-2x^2-2x-2\right)(x+2)^2}{(x^2+2x)(4x^2+8x)}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{-x^2-x-1}{2x^2}dx$$

Hence

$$u_1 = \frac{x}{2} - \frac{1}{2x} + \frac{\ln(x)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(-2x-2)(x+2)^2}{(x^2+2x)(4x^2+8x)}\,dx$$

Which simplifies to

$$u_2 = \int \frac{-x-1}{2x^2} dx$$

Hence

$$u_2 = \frac{1}{2x} - \frac{\ln(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{x}{2} - \frac{1}{2x} + \frac{\ln(x)}{2}\right)(-2x-2) + \left(\frac{1}{2x} - \frac{\ln(x)}{2}\right)(-2x^2-2x-2)$$

Which simplifies to

$$y_p(x) = x(x\ln(x) - x - 2)$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(\left(-2x^2-2x-2\right)c_1 - 2(x+1)\,c_2\right) + \left(x(x\ln(x) - x - 2)\right) \\ &= x^2\ln(x) + (-2c_1-1)\,x^2 + (-2c_1-2c_2-2)\,x - 2c_1 - 2c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^2\ln(x) + (-2c_1-1)\,x^2 + (-2c_1-2c_2-2)\,x - 2c_1 - 2c_2 \tag{1}$$

Verification of solutions

$$y = x^2\ln(x) + (-2c_1-1)\,x^2 + (-2c_1-2c_2-2)\,x - 2c_1 - 2c_2$$

Verified OK.

12.21.3 Solving using Kovacic algorithm

Writing the ode as

$$\left(x^2+2x\right)y'' + (-2x-2)\,y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 + 2x \\
B &= -2x - 2 \\
C &= 2
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3}{\left(x^2 + 2x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3 \\
t &= \left(x^2 + 2x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3}{\left(x^2 + 2x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 463: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2} + \frac{3}{4(x+2)^2} + \frac{3}{4(x+2)} - \frac{3}{4x}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{(x^2 + 2x)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(x+2)} + \frac{3}{2x} + (-)(0) \\ &= -\frac{1}{2(x+2)} + \frac{3}{2x} \\ &= \frac{x+3}{x(x+2)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)(0) + \left(\left(\frac{1}{2(x+2)^2} - \frac{3}{2x^2}\right) + \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)^2 - \left(\frac{3}{(x^2+2x)^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right) dx} \\ &= \frac{x^{\frac{3}{2}}}{\sqrt{x+2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2x-2}{x^2+2x} \, dx} \\
&= z_1 e^{\frac{\ln(x(x+2))}{2}} \\
&= z_1 \left(\sqrt{x\,(x+2)}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{x^{\frac{3}{2}} \sqrt{x\,(x+2)}}{\sqrt{x+2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x^2+2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\ln(x(x+2))}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{-x-1}{x^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{3}{2}} \sqrt{x\,(x+2)}}{\sqrt{x+2}}\right) + c_2 \left(\frac{x^{\frac{3}{2}} \sqrt{x\,(x+2)}}{\sqrt{x+2}} \left(\frac{-x-1}{x^2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$\left(x^2+2x\right) y''+(-2x-2)\, y'+2y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1 x^{\frac{3}{2}} \sqrt{x\,(x+2)}}{\sqrt{x+2}} + \frac{c_2 \sqrt{x\,(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{x^{\frac{3}{2}} \sqrt{x\,(x+2)}}{\sqrt{x+2}}$$

$$y_2 = \frac{\sqrt{x\,(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{x^{\frac{3}{2}} \sqrt{x(x+2)}}{\sqrt{x+2}} & \frac{\sqrt{x(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}} \\ \frac{d}{dx}\left(\frac{x^{\frac{3}{2}} \sqrt{x(x+2)}}{\sqrt{x+2}}\right) & \frac{d}{dx}\left(\frac{\sqrt{x(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{x^{\frac{3}{2}}\sqrt{x(x+2)}}{\sqrt{x+2}} & \frac{\sqrt{x(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}} \\ \frac{3\sqrt{x}\,\sqrt{x(x+2)}}{2\sqrt{x+2}} - \frac{x^{\frac{3}{2}}\sqrt{x(x+2)}}{2(x+2)^{\frac{3}{2}}} + \frac{x^{\frac{3}{2}}(2x+2)}{2\sqrt{x+2}\,\sqrt{x(x+2)}} & -\frac{\sqrt{x(x+2)}\,(-x-1)}{2x^{\frac{3}{2}}\sqrt{x+2}} - \frac{\sqrt{x(x+2)}\,(-x-1)}{2\sqrt{x}\,(x+2)^{\frac{3}{2}}} + \frac{(-x-1)(2x+2)}{2\sqrt{x}\,\sqrt{x+2}\,\sqrt{x(x+2)}} - \frac{\sqrt{x(x+2)}}{\sqrt{x}\,\sqrt{x+2}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{x^{\frac{3}{2}}\sqrt{x\,(x+2)}}{\sqrt{x+2}}\right)\left(-\frac{\sqrt{x\,(x+2)}\,(-x-1)}{2x^{\frac{3}{2}}\sqrt{x+2}} - \frac{\sqrt{x\,(x+2)}\,(-x-1)}{2\sqrt{x}\,(x+2)^{\frac{3}{2}}} + \frac{(-x-1)\,(2x+2)}{2\sqrt{x}\,\sqrt{x+2}\,\sqrt{x\,(x+2)}} - \frac{\sqrt{x\,(x+2)}}{\sqrt{x}\,\sqrt{x+2}}\right) - \left(\frac{\sqrt{x\,(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}}\right)\left(\frac{3\sqrt{x}\,\sqrt{x\,(x+2)}}{2\sqrt{x+2}} - \frac{x^{\frac{3}{2}}\sqrt{x\,(x+2)}}{2\,(x+2)^{\frac{3}{2}}} + \frac{x^{\frac{3}{2}}(2x+2)}{2\sqrt{x+2}\,\sqrt{x\,(x+2)}}\right)$$

Which simplifies to

$$W = x(x+2)$$

Which simplifies to

$$W = x(x+2)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{(x+2)^{\frac{3}{2}}\sqrt{x(x+2)}\,(-x-1)}{\sqrt{x}}}{(x^2+2x)\,x\,(x+2)}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(-x-1)\sqrt{x\,(x+2)}}{\sqrt{x+2}\,x^{\frac{5}{2}}}dx$$

Hence

$$u_1 = \frac{\sqrt{x\,(x+2)}\,(x\ln(x)-1)}{\sqrt{x+2}\,x^{\frac{3}{2}}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{\frac{3}{2}}(x+2)^{\frac{3}{2}}\sqrt{x(x+2)}}{(x^2+2x)\,x\,(x+2)}dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{x(x+2)}}{\sqrt{x}\,\sqrt{x+2}}dx$$

Hence

$$u_2 = \frac{\sqrt{x}\,\sqrt{x(x+2)}}{\sqrt{x+2}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x(x\ln(x)-1) + (-x-1)\,x$$

Which simplifies to

$$y_p(x) = x(x\ln(x)-x-2)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x^{\frac{3}{2}}\sqrt{x(x+2)}}{\sqrt{x+2}} + \frac{c_2\sqrt{x(x+2)}\,(-x-1)}{\sqrt{x}\,\sqrt{x+2}}\right) + (x(x\ln(x)-x-2)) \end{aligned}$$

Which simplifies to

$$y = \frac{\sqrt{x(x+2)}\,((-x-1)\,c_2 + c_1x^2)}{\sqrt{x+2}\,\sqrt{x}} + x(x\ln(x)-x-2)$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{x(x+2)}\,((-x-1)\,c_2 + c_1x^2)}{\sqrt{x+2}\,\sqrt{x}} + x(x\ln(x)-x-2) \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{x (x+2)}\, ((-x-1)\, c_2 + c_1 x^2)}{\sqrt{x+2}\, \sqrt{x}} + x(x \ln (x) - x - 2)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve((x^2+2*x)*diff(y(x),x$2)-2*(x+1)*diff(y(x),x)+2*y(x)=(x+2)^2,y(x), singsol=all)
```

$$y(x) = \ln (x)\, x^2 + (c_2 - 1)\, x^2 + (-2 + c_1)\, x + c_1$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 31

```
DSolve[(x^2+2*x)*y''[x]-2*(x+1)*y'[x]+2*y[x]==(x+2)^2,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to x^2 \log(x) + (-1 + c_1)x^2 - (2 + c_2)x - c_2$$

12.22 problem 22

Internal problem ID [11850]
Internal file name [`OUTPUT/11859_Saturday_April_13_2024_01_13_03_AM_82095675/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - x(x+2)\,y' + (x+2)\,y = x^3}$$

12.22.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + \left(-x^2 - 2x\right) y' + (x+2)\,y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{-x^2 - 2x}{x^2}$$
$$q(x) = \frac{x + 2}{x^2}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{x+2}{x^2} - \frac{\left(\frac{-x^2-2x}{x^2}\right)'}{2} - \frac{\left(\frac{-x^2-2x}{x^2}\right)^2}{4} \\
&= \frac{x+2}{x^2} - \frac{\left(\frac{-2x-2}{x^2} - \frac{2(-x^2-2x)}{x^3}\right)}{2} - \frac{\left(\frac{(-x^2-2x)^2}{x^4}\right)}{4} \\
&= \frac{x+2}{x^2} - \left(\frac{-2x-2}{2x^2} - \frac{-x^2-2x}{x^3}\right) - \frac{(-x^2-2x)^2}{4x^4} \\
&= -\frac{1}{4}
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{-x^2-2x}{x^2}}{2}} \\
&= \mathrm{e}^{\frac{x}{2}} x
\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, \mathrm{e}^{\frac{x}{2}} x \tag{4}$$

Applying this change of variable to the original ode results in

$$\mathrm{e}^{\frac{x}{2}} (4v''(x) - v(x)) = 4$$

Which is now solved for $v(x)$ Simplyfing the ode gives

$$v''(x) - \frac{v(x)}{4} = \mathrm{e}^{-\frac{x}{2}}$$

This is second order non-homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = f(x)$$

Where $A = 1, B = 0, C = -\frac{1}{4}, f(x) = e^{-\frac{x}{2}}$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$v''(x) - \frac{v(x)}{4} = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -\frac{1}{4}$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \frac{e^{\lambda x}}{4} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \frac{1}{4} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -\frac{1}{4}$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)\left(-\frac{1}{4}\right)} \\
&= \pm\frac{1}{2}
\end{aligned}$$

Hence

$$\lambda_1 = +\frac{1}{2}$$

$$\lambda_2 = -\frac{1}{2}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{2}$$

Since roots are real and distinct, then the solution is

$$v(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$v(x) = c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{\left(-\frac{1}{2}\right)x}$$

Or

$$v(x) = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-\frac{x}{2}}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\mathrm{e}^{-\frac{x}{2}}\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}}, \mathrm{e}^{\frac{x}{2}}\right\}$$

Since $\mathrm{e}^{-\frac{x}{2}}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$\left[\left\{x\,\mathrm{e}^{-\frac{x}{2}}\right\}\right]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$v_p = A_1 x\,\mathrm{e}^{-\frac{x}{2}}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{-\frac{x}{2}} = e^{-\frac{x}{2}}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = -x\, e^{-\frac{x}{2}}$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}}\right) + \left(-x\, e^{-\frac{x}{2}}\right) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} - x\, e^{-\frac{x}{2}}\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = e^{\frac{x}{2}} x$$

Hence (7) becomes

$$y = \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} - x\, e^{-\frac{x}{2}}\right) e^{\frac{x}{2}} x$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}} - x\, e^{-\frac{x}{2}}\right) e^{\frac{x}{2}} x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x\,e^x$$
$$y_2 = e^{-\frac{x}{2}} e^{\frac{x}{2}} x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x\,e^x & e^{-\frac{x}{2}} e^{\frac{x}{2}} x \\ \frac{d}{dx}(x\,e^x) & \frac{d}{dx}\left(e^{-\frac{x}{2}} e^{\frac{x}{2}} x\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x\,e^x & e^{-\frac{x}{2}} e^{\frac{x}{2}} x \\ x\,e^x + e^x & e^{-\frac{x}{2}} e^{\frac{x}{2}} \end{vmatrix}$$

Therefore

$$W = (x\,e^x)\left(e^{-\frac{x}{2}} e^{\frac{x}{2}}\right) - \left(e^{-\frac{x}{2}} e^{\frac{x}{2}} x\right)(x\,e^x + e^x)$$

Which simplifies to

$$W = -e^{\frac{3x}{2}} e^{-\frac{x}{2}} x^2$$

Which simplifies to

$$W = -x^2 e^x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-\frac{x}{2}} e^{\frac{x}{2}} x^4}{-x^4 e^x}\, dx$$

Which simplifies to

$$u_1 = -\int -\mathrm{e}^{-x} dx$$

Hence

$$u_1 = -\mathrm{e}^{-x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4 \mathrm{e}^x}{-x^4 \mathrm{e}^x} dx$$

Which simplifies to

$$u_2 = \int (-1)\, dx$$

Hence

$$u_2 = -x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\mathrm{e}^{-x} \mathrm{e}^x x - x^2 \mathrm{e}^{-\frac{x}{2}} \mathrm{e}^{\frac{x}{2}}$$

Which simplifies to

$$y_p(x) = -x^2 - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}} - x\, \mathrm{e}^{-\frac{x}{2}}\right) \mathrm{e}^{\frac{x}{2}} x\right) + \left(-x^2 - x\right) \end{aligned}$$

Which simplifies to

$$y = x(\mathrm{e}^x c_1 + c_2 - x) - x^2 - x$$

Summary

The solution(s) found are the following

$$y = x(\mathrm{e}^x c_1 + c_2 - x) - x^2 - x \tag{1}$$

Verification of solutions

$$y = x(\mathrm{e}^x c_1 + c_2 - x) - x^2 - x$$

Verified OK.

12.22.2 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x^2 - 2x, C = x + 2, f(x) = x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + (-x^2 - 2x)\, y' + (x + 2)\, y = 0$$

In normal form the ode

$$x^2y'' + (-x^2 - 2x)\, y' + (x + 2)\, y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{-x - 2}{x} \\ q(x) &= \frac{x + 2}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(-x-2)}{x^2} + \frac{x+2}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} + \frac{-x-2}{x}\right) v'(x) &= 0 \\ v''(x) - v'(x) &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) - u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. Integrating both sides gives

$$\begin{aligned} \int \frac{1}{u} du &= x + c_1 \\ \ln(u) &= x + c_1 \\ u &= e^{x+c_1} \\ u &= e^x c_1 \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= e^x c_1 + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= (e^x c_1 + c_2)\, x \\ &= (e^x c_1 + c_2)\, x \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' + (-x^2 - 2x)\, y' + (x + 2)\, y = x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = x\,\mathrm{e}^x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x\,\mathrm{e}^x \\ \frac{d}{dx}(x) & \frac{d}{dx}(x\,\mathrm{e}^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x\,\mathrm{e}^x \\ 1 & x\,\mathrm{e}^x + \mathrm{e}^x \end{vmatrix}$$

Therefore

$$W = (x)\,(x\,\mathrm{e}^x + \mathrm{e}^x) - (x\,\mathrm{e}^x)\,(1)$$

Which simplifies to

$$W = x^2 e^x$$

Which simplifies to

$$W = x^2 e^x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^4 e^x}{x^4 e^x}\, dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^4}{x^4 e^x}\, dx$$

Which simplifies to

$$u_2 = \int e^{-x} dx$$

Hence

$$u_2 = -e^{-x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x^2 - e^{-x} e^x x$$

Which simplifies to

$$y_p(x) = -x^2 - x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= ((e^x c_1 + c_2)\, x) + (-x^2 - x) \\
&= -x^2 - x + (e^x c_1 + c_2)\, x
\end{aligned}
$$

Which simplifies to

$$y = x(e^x c_1 + c_2 - x - 1)$$

Summary

The solution(s) found are the following

$$y = x(e^x c_1 + c_2 - x - 1) \tag{1}$$

Verification of solutions

$$y = x(e^x c_1 + c_2 - x - 1)$$

Verified OK.

12.22.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + (-x^2 - 2x)\, y' + (x + 2)\, y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= -x^2 - 2x \\
C &= x + 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 464: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x^2-2x}{x^2}\,dx} \\ &= z_1 e^{\frac{x}{2}+\ln(x)} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}x\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{x+2\ln(x)}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2(x(e^x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' + \left(-x^2 - 2x\right) y' + (x + 2) y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 x + c_2 x \, e^x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= x \, e^x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x\,\mathrm{e}^{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}(x\,\mathrm{e}^{x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x\,\mathrm{e}^{x} \\ 1 & x\,\mathrm{e}^{x} + \mathrm{e}^{x} \end{vmatrix}$$

Therefore

$$W = (x)\left(x\,\mathrm{e}^{x} + \mathrm{e}^{x}\right) - \left(x\,\mathrm{e}^{x}\right)(1)$$

Which simplifies to

$$W = x^{2}\mathrm{e}^{x}$$

Which simplifies to

$$W = x^{2}\mathrm{e}^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{4}\mathrm{e}^{x}}{x^{4}\mathrm{e}^{x}}\,dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{4}}{x^{4}\mathrm{e}^{x}}\,dx$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{-x} dx$$

Hence

$$u_2 = -e^{-x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x^2 - e^{-x} e^x x$$

Which simplifies to

$$y_p(x) = -x^2 - x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x + c_2 x\, e^x) + \left(-x^2 - x\right) \end{aligned}$$

Which simplifies to

$$y = x(c_1 + c_2 e^x) - x^2 - x$$

Summary

The solution(s) found are the following

$$y = x(c_1 + c_2 e^x) - x^2 - x \tag{1}$$

Verification of solutions

$$y = x(c_1 + c_2 e^x) - x^2 - x$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)-x*(x+2)*diff(y(x),x)+(x+2)*y(x)=x^3,y(x), singsol=all)
```

$$y(x) = x(-x + c_1 e^x + c_2)$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]-x*(x+2)*y'[x]+(x+2)*y[x]==x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x(x - c_2 e^x + 1 - c_1)$$

12.23 problem 23

Internal problem ID [11851]
Internal file name [`OUTPUT/11860_Saturday_April_13_2024_01_13_04_AM_25656480/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x(x-2)\,y''-\left(x^2-2\right)y'+2y(x-1)=3x^2(x-2)^2\,\mathrm{e}^x}$$

12.23.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=x^2-2x, B=-x^2+2, C=2x-2, f(x)=3x^2(x-2)^2\,\mathrm{e}^x$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$\left(x^2-2x\right)y''+\left(-x^2+2\right)y'+(2x-2)\,y=0$$

In normal form the ode

$$\left(x^{2}-2x\right) y''+\left(-x^{2}+2\right) y'+\left(2x-2\right) y = 0 \tag{1}$$

Becomes

$$y''+p(x) y'+q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{-x^{2}+2}{x \left(x-2\right)}\\ q(x) &= \frac{2x-2}{x \left(x-2\right)} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x) x^{n}$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right) v'(x)+\left(\frac{n(n-1)}{x^{2}}+\frac{np}{x}+q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^{2}}+\frac{np}{x}+q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^{2}}+\frac{n(-x^{2}+2)}{x^{2} \left(x-2\right)}+\frac{2x-2}{x \left(x-2\right)} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x)+\left(\frac{4}{x}+\frac{-x^{2}+2}{x \left(x-2\right)}\right) v'(x) &= 0\\ v''(x)+\frac{\left(-x^{2}+4x-6\right) v'(x)}{x \left(x-2\right)} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x)+\frac{\left(-x^{2}+4x-6\right)u(x)}{x\left(x-2\right)}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= \frac{u(x^{2}-4x+6)}{x\left(x-2\right)}
\end{aligned}$$

Where $f(x)=\frac{x^{2}-4x+6}{x(x-2)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{x^{2}-4x+6}{x\left(x-2\right)}\,dx\\
\int \frac{1}{u}\,du &= \int \frac{x^{2}-4x+6}{x\left(x-2\right)}\,dx\\
\ln\left(u\right) &= x+\ln\left(x-2\right)-3\ln\left(x\right)+c_{1}\\
u &= \mathrm{e}^{x+\ln(x-2)-3\ln(x)+c_{1}}\\
&= c_{1}\mathrm{e}^{x+\ln(x-2)-3\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x)=c_{1}\left(\frac{\mathrm{e}^{x}}{x^{2}}-\frac{2\,\mathrm{e}^{x}}{x^{3}}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x)\\
v(x) &= \int u(x)\,dx+c_{2}\\
&= \frac{\mathrm{e}^{x}c_{1}}{x^{2}}+c_{2}
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^{n}\\
&= \left(\frac{\mathrm{e}^{x}c_{1}}{x^{2}}+c_{2}\right)x^{2}\\
&= c_{2}x^{2}+\mathrm{e}^{x}c_{1}
\end{aligned}$$

Now the particular solution to this ODE is found

$$\left(x^2-2x\right) y''+\left(-x^2+2\right) y'+\left(2x-2\right) y = 3x^2(x-2)^2 \, \mathrm{e}^x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = \mathrm{e}^x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \mathrm{e}^x \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\mathrm{e}^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \mathrm{e}^x \\ 2x & \mathrm{e}^x \end{vmatrix}$$

Therefore

$$W = \left(x^2\right)\left(\mathrm{e}^x\right) - \left(\mathrm{e}^x\right)\left(2x\right)$$

Which simplifies to

$$W = x^2 e^x - 2x\, e^x$$

Which simplifies to

$$W = e^x (x-2)\, x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{3\, e^{2x} (x-2)^2\, x^2}{(x^2 - 2x)\, e^x\, (x-2)\, x}\, dx$$

Which simplifies to

$$u_1 = -\int 3\, e^x dx$$

Hence

$$u_1 = -3\, e^x$$

And Eq. (3) becomes

$$u_2 = \int \frac{3x^4 (x-2)^2\, e^x}{(x^2 - 2x)\, e^x\, (x-2)\, x}\, dx$$

Which simplifies to

$$u_2 = \int 3x^2 dx$$

Hence

$$u_2 = x^3$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -3x^2 e^x + x^3 e^x$$

Which simplifies to

$$y_p(x) = e^x x^2 (x-3)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(\frac{e^x c_1}{x^2} + c_2\right) x^2\right) + \left(e^x x^2 (x-3)\right) \\
&= e^x x^2 (x-3) + \left(\frac{e^x c_1}{x^2} + c_2\right) x^2
\end{aligned}
$$

Which simplifies to

$$y = \left(x^3 - 3x^2 + c_1\right) e^x + c_2 x^2$$

Summary

The solution(s) found are the following

$$y = \left(x^3 - 3x^2 + c_1\right) e^x + c_2 x^2 \tag{1}$$

Verification of solutions

$$y = \left(x^3 - 3x^2 + c_1\right) e^x + c_2 x^2$$

Verified OK.

12.23.2 Solving using Kovacic algorithm

Writing the ode as

$$\left(x^2 - 2x\right) y'' + \left(-x^2 + 2\right) y' + (2x - 2) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 - 2x \\
B &= -x^2 + 2 \\
C &= 2x - 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^4 - 8x^3 + 24x^2 - 24x + 12 \\
t &= 4\left(x^2 - 2x\right)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 465: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 4 - 4\\ &= 0\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x^2 - 2x)^2$. There is a pole at $x = 2$ of order 2. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4x^2} - \frac{1}{4(x-2)} + \frac{3}{4(x-2)^2} - \frac{3}{4x}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = 2$ let b be the coefficient of $\frac{1}{(x-2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{2}{x^3} + \frac{11}{x^4} + \frac{42}{x^5} + \frac{132}{x^6} + \frac{348}{x^7} + \frac{711}{x^8} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{1}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2} \\
&= Q + \frac{R}{4x^4 - 16x^3 + 16x^2} \\
&= \left(\frac{1}{4}\right) + \left(\frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}\right) \\
&= \frac{1}{4} + \frac{-4x^3 + 20x^2 - 24x + 12}{4x^4 - 16x^3 + 16x^2}
\end{aligned}
$$

Since the degree of t is 4, then we see that the coefficient of the term x^3 in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$
\begin{aligned}
b &= (-1) - (0) \\
&= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^4 - 8x^3 + 24x^2 - 24x + 12}{4\left(x^2 - 2x\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{+} = -1$ then

$$\begin{aligned} d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{-}\right) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((-)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{-}}{x-c_2} \right) + (+)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{2x} - \frac{1}{2\,(x-2)} + \left(\frac{1}{2}\right) \\ &= -\frac{1}{2x} - \frac{1}{2\,(x-2)} + \frac{1}{2} \\ &= -\frac{1}{2x} - \frac{1}{2x-4} + \frac{1}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x} - \frac{1}{2\,(x-2)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2x^2} + \frac{1}{2\,(x-2)^2}\right) + \left(-\frac{1}{2x} - \frac{1}{2\,(x-2)} + \frac{1}{2}\right)^2 - \left(\frac{x^4 - 8x^3 +}{4(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - \frac{1}{2(x-2)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x}\,\sqrt{x-2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x^2+2}{x^2-2x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-2)}{2} + \frac{\ln(x)}{2}} \\
&= z_1 \left(\sqrt{x}\,\sqrt{x-2}\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\mathrm{e}^x \sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x^2+2}{x^2-2x} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x+\ln(x-2)+\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(-x^2 \mathrm{e}^{-x}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\mathrm{e}^x \sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}}\right) + c_2 \left(\frac{\mathrm{e}^x \sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}} \left(-x^2 \mathrm{e}^{-x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$\left(x^2 - 2x\right) y'' + \left(-x^2 + 2\right) y' + (2x - 2) y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1 e^x \sqrt{x}\, \sqrt{x-2}}{\sqrt{x\,(x-2)}} - \frac{c_2 x^{\frac{5}{2}} \sqrt{x-2}}{\sqrt{x\,(x-2)}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{e^x \sqrt{x}\, \sqrt{x-2}}{\sqrt{x\,(x-2)}}$$

$$y_2 = -\frac{x^{\frac{5}{2}} \sqrt{x-2}}{\sqrt{x\,(x-2)}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{e^x \sqrt{x}\,\sqrt{x-2}}{\sqrt{x(x-2)}} & -\frac{x^{\frac{5}{2}}\sqrt{x-2}}{\sqrt{x(x-2)}} \\ \frac{d}{dx}\left(\frac{e^x \sqrt{x}\,\sqrt{x-2}}{\sqrt{x(x-2)}}\right) & \frac{d}{dx}\left(-\frac{x^{\frac{5}{2}}\sqrt{x-2}}{\sqrt{x(x-2)}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{e^x \sqrt{x}\,\sqrt{x-2}}{\sqrt{x(x-2)}} & -\frac{x^{\frac{5}{2}}\sqrt{x-2}}{\sqrt{x(x-2)}} \\ \frac{e^x \sqrt{x}\,\sqrt{x-2}}{\sqrt{x(x-2)}} - \frac{e^x \sqrt{x}\,\sqrt{x-2}\,(2x-2)}{2(x(x-2))^{\frac{3}{2}}} + \frac{e^x \sqrt{x-2}}{2\sqrt{x(x-2)}\,\sqrt{x}} + \frac{e^x \sqrt{x}}{2\sqrt{x(x-2)}\,\sqrt{x-2}} & -\frac{5x^{\frac{3}{2}}\sqrt{x-2}}{2\sqrt{x(x-2)}} - \frac{x^{\frac{5}{2}}}{2\sqrt{x-2}\,\sqrt{x(x-2)}} + \frac{x^{\frac{5}{2}}\sqrt{x-2}\,(2x-}{2(x(x-2))^{\frac{3}{2}}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{e^x \sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}}\right)\left(-\frac{5x^{\frac{3}{2}}\sqrt{x-2}}{2\sqrt{x\,(x-2)}} - \frac{x^{\frac{5}{2}}}{2\sqrt{x-2}\,\sqrt{x\,(x-2)}} + \frac{x^{\frac{5}{2}}\sqrt{x-2}\,(2x-2)}{2\,(x\,(x-2))^{\frac{3}{2}}}\right) - \left(-\frac{x^{\frac{5}{2}}\sqrt{x-2}}{\sqrt{x\,(x-2)}}\right)\left(\frac{e^x \sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}} - \frac{e^x \sqrt{x}\,\sqrt{x-2}\,(2x-2)}{2\,(x\,(x-2))^{\frac{3}{2}}} + \frac{e^x \sqrt{x-2}}{2\sqrt{x\,(x-2)}\,\sqrt{x}} + \frac{e^x \sqrt{x}}{2\sqrt{x\,(x-2)}\,\sqrt{x-2}}\right)$$

Which simplifies to

$$W = e^x (x-2)\,x$$

Which simplifies to

$$W = e^x (x-2)\,x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{3x^{\frac{9}{2}}(x-2)^{\frac{5}{2}} e^x}{\sqrt{x(x-2)}}}{(x^2-2x)\,e^x\,(x-2)\,x}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{3x^{\frac{5}{2}}\sqrt{x-2}}{\sqrt{x\,(x-2)}}dx$$

Hence

$$u_1 = \frac{x^{\frac{7}{2}}\sqrt{x-2}}{\sqrt{x\,(x-2)}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{3\,\mathrm{e}^{2x}x^{\frac{5}{2}}(x-2)^{\frac{5}{2}}}{\sqrt{x(x-2)}}}{(x^2-2x)\,\mathrm{e}^x\,(x-2)\,x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{3\,\mathrm{e}^x\sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}}dx$$

Hence

$$u_2 = \frac{3\,\mathrm{e}^x\sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -3x^2\mathrm{e}^x + x^3\mathrm{e}^x$$

Which simplifies to

$$y_p(x) = \mathrm{e}^x x^2(x-3)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1\mathrm{e}^x\sqrt{x}\,\sqrt{x-2}}{\sqrt{x\,(x-2)}} - \frac{c_2x^{\frac{5}{2}}\sqrt{x-2}}{\sqrt{x\,(x-2)}}\right) + \left(\mathrm{e}^x x^2(x-3)\right) \end{aligned}$$

Which simplifies to

$$y = \frac{\sqrt{x}\,\sqrt{x-2}\,(-c_2x^2+\mathrm{e}^x c_1)}{\sqrt{x\,(x-2)}} + \mathrm{e}^x x^2(x-3)$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{x}\,\sqrt{x-2}\,(-c_2x^2 + \mathrm{e}^x c_1)}{\sqrt{x\,(x-2)}} + \mathrm{e}^x x^2(x-3) \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{x}\,\sqrt{x-2}\,(-c_2x^2 + \mathrm{e}^x c_1)}{\sqrt{x\,(x-2)}} + \mathrm{e}^x x^2(x-3)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*(x-2)*diff(y(x),x$2)-(x^2-2)*diff(y(x),x)+2*(x-1)*y(x)=3*x^2*(x-2)^2*exp(x),y(x), s
```

$$y(x) = (x^3 - 3x^2 + c_1)\,\mathrm{e}^x + c_2x^2$$

✓ Solution by Mathematica

Time used: 0.1 (sec). Leaf size: 27

```
DSolve[x*(x-2)*y''[x]-(x^2-2)*y'[x]+2*(x-1)*y[x]==3*x^2*(x-2)^2*Exp[x],y[x],x,IncludeSingula
```

$$y(x) \rightarrow c_2x^2 + e^x\left(x^3 - 3x^2 + c_1\right)$$

12.24 problem 24

Internal problem ID [11852]
Internal file name [`OUTPUT/11861_Saturday_April_13_2024_01_13_06_AM_59685877/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{(2x+1)(x+1)y''+2y'x-2y=(2x+1)^2}$$

12.24.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=2x^2+3x+1, B=2x, C=-2, f(x)=4\left(x+\frac{1}{2}\right)^2$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$y''(2x^2+3x+1)+2y'x-2y=0$$

In normal form the ode

$$y''(2x^2+3x+1)+2y'x-2y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2x}{2x^2+3x+1} \\ q(x) &= -\frac{2}{2x^2+3x+1} \end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right)v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right)v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{2n}{2x^2+3x+1}-\frac{2}{2x^2+3x+1}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x)+\left(\frac{2}{x}+\frac{2x}{2x^2+3x+1}\right)v'(x) &= 0 \\ v''(x)+\left(\frac{2}{x}+\frac{2x}{2x^2+3x+1}\right)v'(x) &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \left(\frac{2}{x} + \frac{2x}{2x^2 + 3x + 1}\right) u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{2u(3x^2 + 3x + 1)}{x\,(2x^2 + 3x + 1)}
\end{aligned}$$

Where $f(x) = -\frac{2(3x^2+3x+1)}{x(2x^2+3x+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{2(3x^2 + 3x + 1)}{x\,(2x^2 + 3x + 1)}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{2(3x^2 + 3x + 1)}{x\,(2x^2 + 3x + 1)}\, dx \\
\ln(u) &= -2\ln(x + 1) - 2\ln(x) + \ln(2x + 1) + c_1 \\
u &= \mathrm{e}^{-2\ln(x+1) - 2\ln(x) + \ln(2x+1) + c_1} \\
&= c_1 \mathrm{e}^{-2\ln(x+1) - 2\ln(x) + \ln(2x+1)}
\end{aligned}$$

Which simplifies to

$$u(x) = c_1 \left(\frac{2}{(x + 1)^2\, x} + \frac{1}{(x + 1)^2\, x^2}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= c_1 \left(\frac{1}{x + 1} - \frac{1}{x}\right) + c_2
\end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(c_1 \left(\frac{1}{x+1} - \frac{1}{x}\right) + c_2\right) x \\
&= \frac{(x^2+x)\, c_2 - c_1}{x+1}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
y''(2x^2+3x+1) + 2y'x - 2y = 4\left(x + \frac{1}{2}\right)^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \frac{x}{x+1} - 1
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x & \frac{x}{x+1} - 1 \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{x}{x+1} - 1\right) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} x & \frac{x}{x+1} - 1 \\ 1 & \frac{1}{x+1} - \frac{x}{(x+1)^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{1}{x+1} - \frac{x}{(x+1)^2}\right) - \left(\frac{x}{x+1} - 1\right)(1)$$

Which simplifies to

$$W = \frac{2x+1}{(x+1)^2}$$

Which simplifies to

$$W = \frac{2x+1}{(x+1)^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\left(\frac{x}{x+1} - 1\right)\left(x + \frac{1}{2}\right)^2}{\frac{(2x^2+3x+1)(2x+1)}{(x+1)^2}}\, dx$$

Which simplifies to

$$u_1 = -\int (-1)\, dx$$

Hence

$$u_1 = x$$

And Eq. (3) becomes

$$u_2 = \int \frac{4x\left(x + \frac{1}{2}\right)^2}{\frac{(2x^2+3x+1)(2x+1)}{(x+1)^2}}\, dx$$

Which simplifies to

$$u_2 = \int x(x+1)\, dx$$

Hence

$$u_2 = \frac{1}{3}x^3 + \frac{1}{2}x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x^2 + \left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\left(\frac{x}{x+1} - 1\right)$$

Which simplifies to

$$y_p(x) = \frac{x^2(4x+3)}{6x+6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1\left(\frac{1}{x+1} - \frac{1}{x}\right) + c_2\right)x\right) + \left(\frac{x^2(4x+3)}{6x+6}\right) \\ &= \frac{x^2(4x+3)}{6x+6} + \left(c_1\left(\frac{1}{x+1} - \frac{1}{x}\right) + c_2\right)x \end{aligned}$$

Which simplifies to

$$y = \frac{4x^3 + (6c_2+3)\,x^2 + 6c_2x - 6c_1}{6x+6}$$

Summary

The solution(s) found are the following

$$y = \frac{4x^3 + (6c_2+3)\,x^2 + 6c_2x - 6c_1}{6x+6} \tag{1}$$

Verification of solutions

$$y = \frac{4x^3 + (6c_2+3)\,x^2 + 6c_2x - 6c_1}{6x+6}$$

Verified OK.

12.24.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(y''\left(2x^2+3x+1\right)+2y'x-2y\right)dx = \int 4\left(x+\frac{1}{2}\right)^2 dx$$

$$\left(-2x-3\right)y+\left(2x^2+3x+1\right)y' = \frac{4\left(x+\frac{1}{2}\right)^3}{3}+c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3+2x}{2x^2+3x+1}$$
$$q(x) = \frac{8x^3+12x^2+6c_1+6x+1}{12x^2+18x+6}$$

Hence the ode is

$$y'-\frac{\left(3+2x\right)y}{2x^2+3x+1} = \frac{8x^3+12x^2+6c_1+6x+1}{12x^2+18x+6}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3+2x}{2x^2+3x+1}dx}$$
$$= e^{\ln(x+1)-2\ln(2x+1)}$$

Which simplifies to

$$\mu = \frac{x+1}{\left(2x+1\right)^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{8x^3+12x^2+6c_1+6x+1}{12x^2+18x+6}\right)$$
$$\frac{d}{dx}\left(\frac{\left(x+1\right)y}{\left(2x+1\right)^2}\right) = \left(\frac{x+1}{\left(2x+1\right)^2}\right)\left(\frac{8x^3+12x^2+6c_1+6x+1}{12x^2+18x+6}\right)$$
$$d\left(\frac{\left(x+1\right)y}{\left(2x+1\right)^2}\right) = \left(\frac{8x^3+12x^2+6c_1+6x+1}{6\left(2x+1\right)^3}\right)dx$$

Integrating gives

$$
\begin{aligned}
\frac{(x+1)\,y}{(2x+1)^2} &= \int \frac{8x^3+12x^2+6c_1+6x+1}{6\,(2x+1)^3}\,\mathrm{d}x \\
\frac{(x+1)\,y}{(2x+1)^2} &= \frac{x}{6}-\frac{c_1}{4\,(2x+1)^2}+c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{x+1}{(2x+1)^2}$ results in

$$
y = \frac{(2x+1)^2\left(\frac{x}{6}-\frac{c_1}{4(2x+1)^2}\right)}{x+1}+\frac{c_2(2x+1)^2}{x+1}
$$

which simplifies to

$$
y = \frac{8x^3+(48c_2+8)\,x^2+(48c_2+2)\,x-3c_1+12c_2}{12x+12}
$$

Summary
The solution(s) found are the following

$$
y = \frac{8x^3+(48c_2+8)\,x^2+(48c_2+2)\,x-3c_1+12c_2}{12x+12} \tag{1}
$$

Verification of solutions

$$
y = \frac{8x^3+(48c_2+8)\,x^2+(48c_2+2)\,x-3c_1+12c_2}{12x+12}
$$

Verified OK.

12.24.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay''+By'+Cy=F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v+v'B \\
y'' &= B''v+B'v'+v''B+v'B' \\
&= v''B+2v'+B'+B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= 2x^2 + 3x + 1 \\
B &= 2x \\
C &= -2 \\
F &= 4\left(x + \frac{1}{2}\right)^2
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(2x^2 + 3x + 1\right)(0) + (2x)(2) + (-2)(2x) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
4x^3 + 6x^2 + 2xv'' + \left(12x^2 + 12x + 4\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
\left(4x^3 + 6x^2 + 2x\right) u'(x) + 12u(x)\left(x^2 + x + \frac{1}{3}\right) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{2u(3x^2 + 3x + 1)}{x\left(2x^2 + 3x + 1\right)}
\end{aligned}
$$

Where $f(x) = -\frac{2(3x^2+3x+1)}{x(2x^2+3x+1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{2(3x^2+3x+1)}{x\left(2x^2+3x+1\right)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2(3x^2+3x+1)}{x\left(2x^2+3x+1\right)}\,dx \\
\ln\left(u\right) &= -2\ln\left(x+1\right) - 2\ln\left(x\right) + \ln\left(2x+1\right) + c_1 \\
u &= \mathrm{e}^{-2\ln(x+1)-2\ln(x)+\ln(2x+1)+c_1} \\
&= c_1\mathrm{e}^{-2\ln(x+1)-2\ln(x)+\ln(2x+1)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_1\left(\frac{2}{\left(x+1\right)^2 x} + \frac{1}{\left(x+1\right)^2 x^2}\right)
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1\left(\frac{2}{\left(x+1\right)^2 x} + \frac{1}{\left(x+1\right)^2 x^2}\right)
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1(2x+1)}{x^2\left(x+1\right)^2}\,\mathrm{d}x \\
&= c_1\left(\frac{1}{x+1} - \frac{1}{x}\right) + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (2x)\left(c_1\left(\frac{1}{x+1} - \frac{1}{x}\right) + c_2\right) \\
&= \frac{2c_2x^2 + 2c_2x - 2c_1}{x+1}
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x+1}$$

$$y_2 = \frac{2x^2}{x+1} + \frac{2x}{x+1}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{2x^2}{x+1} + \frac{2x}{x+1} \\ \frac{d}{dx}\left(\frac{1}{x+1}\right) & \frac{d}{dx}\left(\frac{2x^2}{x+1} + \frac{2x}{x+1}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{2x^2}{x+1} + \frac{2x}{x+1} \\ -\frac{1}{(x+1)^2} & -\frac{2x^2}{(x+1)^2} + \frac{4x}{x+1} + \frac{2}{x+1} - \frac{2x}{(x+1)^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x+1}\right)\left(-\frac{2x^2}{(x+1)^2} + \frac{4x}{x+1} + \frac{2}{x+1} - \frac{2x}{(x+1)^2}\right) - \left(\frac{2x^2}{x+1} + \frac{2x}{x+1}\right)\left(-\frac{1}{(x+1)^2}\right)$$

Which simplifies to

$$W = \frac{4x+2}{(x+1)^2}$$

Which simplifies to

$$W = \frac{4x+2}{(x+1)^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\left(\frac{2x^2}{x+1} + \frac{2x}{x+1}\right)\left(x+\frac{1}{2}\right)^2}{\frac{(2x^2+3x+1)(4x+2)}{(x+1)^2}} \, dx$$

Which simplifies to

$$u_1 = -\int x(x+1) \, dx$$

Hence

$$u_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\left(x+\frac{1}{2}\right)^2}{x+1}}{\frac{(2x^2+3x+1)(4x+2)}{(x+1)^2}} \, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{2} dx$$

Hence

$$u_2 = \frac{x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{3}x^3 - \frac{1}{2}x^2}{x+1} + \frac{x\left(\frac{2x^2}{x+1} + \frac{2x}{x+1}\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{x^2(4x+3)}{6x+6}$$

Hence the complete solution is

$$\begin{aligned}
y(x) &= y_h + y_p \\
&= \left(\frac{2c_2x^2 + 2c_2x - 2c_1}{x+1}\right) + \left(\frac{x^2(4x+3)}{6x+6}\right) \\
&= \frac{4x^3 + (12c_2+3)\,x^2 + 12c_2x - 12c_1}{6x+6}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{4x^3 + (12c_2+3)\,x^2 + 12c_2x - 12c_1}{6x+6} \tag{1}$$

Verification of solutions

$$y = \frac{4x^3 + (12c_2+3)\,x^2 + 12c_2x - 12c_1}{6x+6}$$

Verified OK.

12.24.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y''(2x^2+3x+1) + 2y'x - 2y = 4\left(x+\frac{1}{2}\right)^2$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(y''(2x^2+3x+1) + 2y'x - 2y\right) dx = \int 4\left(x+\frac{1}{2}\right)^2 dx$$

$$(-2x-3)\,y + (2x^2+3x+1)\,y' = \frac{4\left(x+\frac{1}{2}\right)^3}{3} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{3+2x}{2x^2+3x+1} \\
q(x) &= \frac{8x^3 + 12x^2 + 6c_1 + 6x + 1}{12x^2 + 18x + 6}
\end{aligned}$$

Hence the ode is

$$y' - \frac{(3+2x)\,y}{2x^2+3x+1} = \frac{8x^3+12x^2+6c_1+6x+1}{12x^2+18x+6}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{3+2x}{2x^2+3x+1}dx} \\
&= \mathrm{e}^{\ln(x+1)-2\ln(2x+1)}
\end{aligned}$$

Which simplifies to

$$\mu = \frac{x+1}{(2x+1)^2}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{8x^3+12x^2+6c_1+6x+1}{12x^2+18x+6}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(x+1)\,y}{(2x+1)^2}\right) &= \left(\frac{x+1}{(2x+1)^2}\right)\left(\frac{8x^3+12x^2+6c_1+6x+1}{12x^2+18x+6}\right) \\
\mathrm{d}\left(\frac{(x+1)\,y}{(2x+1)^2}\right) &= \left(\frac{8x^3+12x^2+6c_1+6x+1}{6\,(2x+1)^3}\right)\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{(x+1)\,y}{(2x+1)^2} &= \int \frac{8x^3+12x^2+6c_1+6x+1}{6\,(2x+1)^3}\,\mathrm{d}x \\
\frac{(x+1)\,y}{(2x+1)^2} &= \frac{x}{6} - \frac{c_1}{4\,(2x+1)^2} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{x+1}{(2x+1)^2}$ results in

$$y = \frac{(2x+1)^2\left(\frac{x}{6} - \frac{c_1}{4(2x+1)^2}\right)}{x+1} + \frac{c_2(2x+1)^2}{x+1}$$

which simplifies to

$$y = \frac{8x^3+(48c_2+8)\,x^2+(48c_2+2)\,x-3c_1+12c_2}{12x+12}$$

Summary

The solution(s) found are the following

$$y = \frac{8x^3+(48c_2+8)\,x^2+(48c_2+2)\,x-3c_1+12c_2}{12x+12} \tag{1}$$

Verification of solutions

$$y = \frac{8x^3 + (48c_2 + 8)\,x^2 + (48c_2 + 2)\,x - 3c_1 + 12c_2}{12x + 12}$$

Verified OK.

12.24.5 Solving using Kovacic algorithm

Writing the ode as

$$y''(2x^2 + 3x + 1) + 2y'x - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2x^2 + 3x + 1 \\ B &= 2x \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(2x + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= (2x + 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(2x+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 466: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x+1)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(x+\frac{1}{2}\right)^2}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(x+\frac{1}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{(2x+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(2x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x+\frac{1}{2}\right)} + (-)\left(0\right) \\ &= -\frac{1}{2\left(x+\frac{1}{2}\right)} \\ &= -\frac{1}{2x+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2\left(x+\frac{1}{2}\right)}\right)(0) + \left(\left(\frac{1}{2\left(x+\frac{1}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x+\frac{1}{2}\right)}\right)^2 - \left(\frac{3}{\left(2x+1\right)^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{2\left(x+\frac{1}{2}\right)} dx} \\
&= \frac{1}{\sqrt{2x+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{2x}{2x^2+3x+1} \, dx} \\
&= z_1 e^{-\ln(x+1)+\frac{\ln(2x+1)}{2}} \\
&= z_1 \left(\frac{\sqrt{2x+1}}{x+1}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x+1}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{2x^2+3x+1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-2\ln(x+1)+\ln(2x+1)}}{(y_1)^2} \, dx \\
&= y_1 (x(x+1))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x+1}\right) + c_2 \left(\frac{1}{x+1} (x(x+1))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y''\left(2x^2 + 3x + 1\right) + 2y'x - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x+1} + c_2 x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x+1}$$
$$y_2 = x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x+1} & x \\ \frac{d}{dx}\left(\frac{1}{x+1}\right) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x+1} & x \\ -\frac{1}{(x+1)^2} & 1 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x+1}\right)(1) - (x)\left(-\frac{1}{(x+1)^2}\right)$$

Which simplifies to

$$W = \frac{2x+1}{(x+1)^2}$$

Which simplifies to

$$W = \frac{2x+1}{(x+1)^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4x\left(x+\frac{1}{2}\right)^2}{\frac{(2x^2+3x+1)(2x+1)}{(x+1)^2}}\,dx$$

Which simplifies to

$$u_1 = -\int x(x+1)\,dx$$

Hence

$$u_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\left(x+\frac{1}{2}\right)^2}{x+1}}{\frac{(2x^2+3x+1)(2x+1)}{(x+1)^2}}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{1}{3}x^3 - \frac{1}{2}x^2}{x+1} + x^2$$

Which simplifies to

$$y_p(x) = \frac{x^2(4x+3)}{6x+6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x+1} + c_2 x\right) + \left(\frac{x^2(4x+3)}{6x+6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x+1} + c_2 x + \frac{x^2(4x+3)}{6x+6} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x+1} + c_2 x + \frac{x^2(4x+3)}{6x+6}$$

Verified OK.

12.24.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 2x^2 + 3x + 1 \\
q(x) &= 2x \\
r(x) &= -2 \\
s(x) &= 4\left(x + \frac{1}{2}\right)^2
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 4 \\
q'(x) &= 2
\end{aligned}$$

Therefore (1) becomes

$$4 - (2) + (-2) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$(-2x - 3)\, y + \left(2x^2 + 3x + 1\right) y' = \int 4\left(x + \frac{1}{2}\right)^2 dx$$

We now have a first order ode to solve which is

$$(-2x - 3)\, y + \left(2x^2 + 3x + 1\right) y' = \frac{4\left(x + \frac{1}{2}\right)^3}{3} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{3 + 2x}{2x^2 + 3x + 1} \\
q(x) &= \frac{8x^3 + 12x^2 + 6c_1 + 6x + 1}{12x^2 + 18x + 6}
\end{aligned}$$

Hence the ode is

$$y' - \frac{(3+2x)\, y}{2x^2+3x+1} = \frac{8x^3+12x^2+6c_1+6x+1}{12x^2+18x+6}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{3+2x}{2x^2+3x+1}dx} \\
&= \mathrm{e}^{\ln(x+1)-2\ln(2x+1)}
\end{aligned}$$

Which simplifies to

$$\mu = \frac{x+1}{(2x+1)^2}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{8x^3+12x^2+6c_1+6x+1}{12x^2+18x+6}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(x+1)\, y}{(2x+1)^2}\right) &= \left(\frac{x+1}{(2x+1)^2}\right)\left(\frac{8x^3+12x^2+6c_1+6x+1}{12x^2+18x+6}\right) \\
\mathrm{d}\left(\frac{(x+1)\, y}{(2x+1)^2}\right) &= \left(\frac{8x^3+12x^2+6c_1+6x+1}{6\,(2x+1)^3}\right)\, \mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{(x+1)\, y}{(2x+1)^2} &= \int \frac{8x^3+12x^2+6c_1+6x+1}{6\,(2x+1)^3}\, \mathrm{d}x \\
\frac{(x+1)\, y}{(2x+1)^2} &= \frac{x}{6} - \frac{c_1}{4\,(2x+1)^2} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{x+1}{(2x+1)^2}$ results in

$$y = \frac{(2x+1)^2 \left(\frac{x}{6} - \frac{c_1}{4(2x+1)^2}\right)}{x+1} + \frac{c_2(2x+1)^2}{x+1}$$

which simplifies to

$$y = \frac{8x^3+(48c_2+8)\, x^2+(48c_2+2)\, x-3c_1+12c_2}{12x+12}$$

Summary

The solution(s) found are the following

$$y = \frac{8x^3+(48c_2+8)\, x^2+(48c_2+2)\, x-3c_1+12c_2}{12x+12} \tag{1}$$

Verification of solutions

$$y = \frac{8x^3 + (48c_2 + 8)\, x^2 + (48c_2 + 2)\, x - 3c_1 + 12c_2}{12x + 12}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve((2*x+1)*(x+1)*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=(2*x+1)^2,y(x), singsol=all)
```

$$y(x) = \frac{4x^3 + (6c_1 + 24c_2 + 4)\, x^2 + (6c_1 + 24c_2 + 1)\, x + 6c_2}{6x + 6}$$

✓ Solution by Mathematica

Time used: 1.049 (sec). Leaf size: 72

```
DSolve[(2*x+1)*(x+1)*y''[x]+2*x*y'[x]-2*y[x]==(2*x+1)^2,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{\sqrt{-2x-1}(4x+3)x^2 - 6c_2(x+1)\sqrt{2x+1}x + 6c_1\sqrt{2x+1}}{6\sqrt{-2x-1}(x+1)}$$

12.25 problem 25

Internal problem ID [11853]
Internal file name [OUTPUT/11862_Saturday_April_13_2024_01_13_09_AM_37230/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{\sin(x)^2 y'' - 2\sin(x)\cos(x) y' + \left(\cos(x)^2 + 1\right) y = \sin(x)^3}$$

12.25.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$\sin(x)^2 y'' - y'\sin(2x) + \left(\cos(x)^2 + 1\right) y = 0$$

In normal form the given ode is written as

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -\frac{\sin(2x)}{\sin(x)^2} \\
q(x) &= \frac{\cos(x)^2+1}{\sin(x)^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{\cos(x)^2+1}{\sin(x)^2} - \frac{\left(-\frac{\sin(2x)}{\sin(x)^2}\right)'}{2} - \frac{\left(-\frac{\sin(2x)}{\sin(x)^2}\right)^2}{4} \\
&= \frac{\cos(x)^2+1}{\sin(x)^2} - \frac{\left(\frac{2\sin(2x)\cos(x)}{\sin(x)^3} - \frac{2\cos(2x)}{\sin(x)^2}\right)}{2} - \frac{\left(\frac{\sin(2x)^2}{\sin(x)^4}\right)}{4} \\
&= \frac{\cos(x)^2+1}{\sin(x)^2} - \left(\frac{\sin(2x)\cos(x)}{\sin(x)^3} - \frac{\cos(2x)}{\sin(x)^2}\right) - \frac{\sin(2x)^2}{4\sin(x)^4} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{\sin(2x)}{\sin(x)^2}}{2}} \\
&= \sin(x)
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\sin(x) \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(x) = 1
$$

Which is now solved for $v(x)$ The ODE can be written as

$$
v''(x) = 1
$$

Integrating once gives

$$v'(x) = x + c_1$$

Integrating again gives

$$v(x) = \frac{x^2}{2} + c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(\frac{1}{2}x^2 + c_1 x + c_2\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \sin(x)$$

Hence (7) becomes

$$y = \left(\frac{1}{2}x^2 + c_1 x + c_2\right) \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \left(\frac{1}{2}x^2 + c_1 x + c_2\right) \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \sin(x) \\ y_2 &= x \sin(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sin(x) & x\sin(x) \\ \frac{d}{dx}(\sin(x)) & \frac{d}{dx}(x\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sin(x) & x\sin(x) \\ \cos(x) & \sin(x) + \cos(x)\,x \end{vmatrix}$$

Therefore

$$W = (\sin(x))(\sin(x) + \cos(x)\,x) - (x\sin(x))(\cos(x))$$

Which simplifies to

$$W = \sin(x)^2$$

Which simplifies to

$$W = \sin(x)^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\sin(x)^4}{\sin(x)^4}\,dx$$

Which simplifies to

$$u_1 = -\int x dx$$

Hence

$$u_1 = -\frac{x^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sin(x)^4}{\sin(x)^4}\,dx$$

Which simplifies to

$$u_2 = \int 1dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sin(x)\,x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(\frac{1}{2}x^2 + c_1x + c_2\right)\sin(x)\right) + \left(\frac{\sin(x)\,x^2}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(\frac{1}{2}x^2 + c_1x + c_2\right)\sin(x) + \frac{\sin(x)\,x^2}{2} \tag{1}$$

Verification of solutions

$$y = \left(\frac{1}{2}x^2 + c_1x + c_2\right)\sin(x) + \frac{\sin(x)\,x^2}{2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
-> Try solving first the homogeneous part of the ODE
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 17

```
dsolve(sin(x)^2*diff(y(x),x$2)-2*sin(x)*cos(x)*diff(y(x),x)+(cos(x)^2+1)*y(x)=sin(x)^3,y(x),
```

$$y(x) = \sin(x)\left(c_2 + c_1 x + \frac{1}{2}x^2\right)$$

✓ Solution by Mathematica
Time used: 0.092 (sec). Leaf size: 24

```
DSolve[Sin[x]^2*y''[x]-2*Sin[x]*Cos[x]*y'[x]+(Cos[x]^2+1)*y[x]==Sin[x]^3,y[x],x,IncludeSingu
```

$$y(x) \to \frac{1}{2}\left(x^2 + 2c_2 x + 2c_1\right)\sin(x)$$

12.26 problem 26

Internal problem ID [11854]
Internal file name [OUTPUT/11863_Saturday_April_13_2024_01_13_10_AM_74535894/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.4. Variation of parameters. Exercises page 162
Problem number: 26.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - 3y'' - y' + 3y = x^2 e^x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 3y'' - y' + 3y = 0$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$
$$\lambda_3 = -1$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^{x} + c_3 e^{3x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{x}$$
$$y_3 = e^{3x}$$

Now the particular solution to the given ODE is found

$$y''' - 3y'' - y' + 3y = x^2 e^{x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^{x}, x^2 e^{x}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{-x}, e^{3x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x e^{x}, x^2 e^{x}, x^3 e^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^{x} + A_2 x^2 e^{x} + A_3 x^3 e^{x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_2 x e^{x} - 12A_3 x^2 e^{x} - 4A_1 e^{x} + 6A_3 e^{x} = x^2 e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = 0, A_3 = -\frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^x}{8} - \frac{x^3 e^x}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^x + c_3 e^{3x}\right) + \left(-\frac{x\,e^x}{8} - \frac{x^3 e^x}{12}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^x + c_3 e^{3x} - \frac{x\,e^x}{8} - \frac{x^3 e^x}{12} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^x + c_3 e^{3x} - \frac{x\,e^x}{8} - \frac{x^3 e^x}{12}$$

Verified OK.

12.26.1 Maple step by step solution

Let's solve

$y''' - 3y'' - y' + 3y = x^2 e^x$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = x^2 e^x + 3y_3(x) + y_2(x) - 3y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x^2 e^x + 3y_3(x) + y_2(x) - 3y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 1 & 3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x^2 e^x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x^2 e^x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & 1 & 3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]\right], \left[1, \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]\right], \left[3, \left[\begin{array}{c}\frac{1}{9} \\ \frac{1}{3} \\ 1\end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c}1 \\ -1 \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_2 = \mathrm{e}^{x} \cdot \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c}\frac{1}{9} \\ \frac{1}{3} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = \mathrm{e}^{3x} \cdot \left[\begin{array}{c}\frac{1}{9} \\ \frac{1}{3} \\ 1\end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

  $\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \overrightarrow{y}_p(x)$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(x) = \begin{bmatrix} e^{-x} & e^{x} & \frac{e^{3x}}{9} \\ -e^{-x} & e^{x} & \frac{e^{3x}}{3} \\ e^{-x} & e^{x} & e^{3x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

  $$\Phi(x) = \begin{bmatrix} e^{-x} & e^{x} & \frac{e^{3x}}{9} \\ -e^{-x} & e^{x} & \frac{e^{3x}}{3} \\ e^{-x} & e^{x} & e^{3x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & 1 & \frac{1}{9} \\ -1 & 1 & \frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

  $$\Phi(x) = \begin{bmatrix} \frac{3e^{-x}}{8} + \frac{3e^{x}}{4} - \frac{e^{3x}}{8} & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{e^{-x}}{8} - \frac{e^{x}}{4} + \frac{e^{3x}}{8} \\ -\frac{3e^{-x}}{8} + \frac{3e^{x}}{4} - \frac{3e^{3x}}{8} & \frac{e^{-x}}{2} + \frac{e^{x}}{2} & -\frac{e^{-x}}{8} - \frac{e^{x}}{4} + \frac{3e^{3x}}{8} \\ \frac{3e^{-x}}{8} + \frac{3e^{x}}{4} - \frac{9e^{3x}}{8} & -\frac{e^{-x}}{2} + \frac{e^{x}}{2} & \frac{e^{-x}}{8} - \frac{e^{x}}{4} + \frac{9e^{3x}}{8} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot\vec{v}'(x)=\vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x)=\frac{1}{\Phi(x)}\cdot\vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x)=\int_0^x\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x)=\Phi(x)\cdot\left(\int_0^x\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x)=\left[\begin{array}{c}-\frac{e^{-x}}{32}+\frac{e^{3x}}{32}+\frac{(-2x^3-3x)e^x}{24}\\ \frac{e^{-x}}{32}+\frac{3e^{3x}}{32}+\frac{(-2x^3-6x^2-3x-3)e^x}{24}\\ -\frac{e^{-x}}{32}+\frac{9e^{3x}}{32}+\frac{(-2x^3-12x^2-15x-6)e^x}{24}\end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3+\left[\begin{array}{c}-\frac{e^{-x}}{32}+\frac{e^{3x}}{32}+\frac{(-2x^3-3x)e^x}{24}\\ \frac{e^{-x}}{32}+\frac{3e^{3x}}{32}+\frac{(-2x^3-6x^2-3x-3)e^x}{24}\\ -\frac{e^{-x}}{32}+\frac{9e^{3x}}{32}+\frac{(-2x^3-12x^2-15x-6)e^x}{24}\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=\frac{(-1+32c_1)e^{-x}}{32}+\frac{(32c_3+9)e^{3x}}{288}-\frac{e^x\left(x^3+\frac{3}{2}x-12c_2\right)}{12}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$3)-3*diff(y(x),x$2)-diff(y(x),x)+3*y(x)=x^2*exp(x),y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + c_3 e^{3x} - \frac{\left(x^3 + \frac{3}{2}x - 12c_1\right) e^x}{12}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 41

```
DSolve[y'''[x]-3*y''[x]-y'[x]+3*y[x]==x^2*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x \left(-\frac{x^3}{12} - \frac{x}{8} + c_2\right) + c_1 e^{-x} + c_3 e^{3x}$$

13 Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169

13.1 problem 1

Internal problem ID [11855]
Internal file name [`OUTPUT/11864_Saturday_April_13_2024_01_13_10_AM_73858456/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - 3y'x + 3y = 0}$$

13.1.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 3x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 3 = 0$$

Or

$$r^2 - 4r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2 x^3 + c_1 x$$

Summary
The solution(s) found are the following

$$y = c_2 x^3 + c_1 x \tag{1}$$

Verification of solutions

$$y = c_2 x^3 + c_1 x$$

Verified OK.

13.1.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - 3y'x + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)}dx \\ &= \int e^{3\ln(x)}\,dx \\ &= \int x^3dx \\ &= \frac{x^4}{4} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{3}{x^2}}{x^6} \\ &= \frac{3}{x^8} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{x^8} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{3}{x^8} = \frac{3}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 3\tau^r = 0$$

Simplifying gives

$$16r(r-1)\tau^r + 0\,\tau^r + 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$16r(r-1) + 0 + 3 = 0$$

Or

$$16r^2 - 16r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{4} \\
r_2 &= \frac{3}{4}
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$
y(\tau) = c_1 y_1 + c_2 y_2
$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$
y(\tau) = c_1 \tau^{\frac{1}{4}} + c_2 \tau^{\frac{3}{4}}
$$

The above solution is now transformed back to y using (6) which results in

$$
y = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}} \left(c_2 \sqrt{x^4} + 2c_1\right)}{4}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}} \left(c_2 \sqrt{x^4} + 2c_1\right)}{4} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}} \left(c_2 \sqrt{x^4} + 2c_1\right)}{4}
$$

Verified OK.

13.1.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$
x^2 y'' - 3y'x + 3y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= \frac{3}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}\\ \tau'' &= -\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2}\\ &= \frac{-\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2}\\ &= -\frac{4c\sqrt{3}}{3}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) - \frac{4c\sqrt{3}\,\left(\frac{d}{d\tau}y(\tau)\right)}{3} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{\frac{2\sqrt{3}\,c\tau}{3}}\left(c_1\cosh\left(\frac{\sqrt{3}\,c\tau}{3}\right) + ic_2\sinh\left(\frac{\sqrt{3}\,c\tau}{3}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{3}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}\, x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{x((ic_2 + c_1)\, x^2 - ic_2 + c_1)}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x((ic_2 + c_1)\, x^2 - ic_2 + c_1)}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{x((ic_2 + c_1)\, x^2 - ic_2 + c_1)}{2}
$$

Verified OK.

13.1.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$
x^2y'' - 3y'x + 3y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= \frac{3}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{3}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u} \, du &= -\frac{3}{x} \, dx \\ \int \frac{1}{u} \, du &= \int -\frac{3}{x} \, dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= e^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x^3 \\
&= c_2 x^3 - \frac{1}{2} c_1 x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x^3 \tag{1}$$

Verification of solutions

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x^3$$

Verified OK.

13.1.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B+2v'B'+B''v)+B(B'v+v'B)+CBv &= 0\\
ABv''+\left(2AB'+B^2\right)v'+\left(AB''+BB'+CB\right)v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB''+BB'+CB$ is zero, then this method works and can be used to solve

$$
ABv''+\left(2AB'+B^2\right)v'=0
$$

By Using $u=v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu'+\left(2AB'+B^2\right)u=0
$$

The above ode is first order ode which is solved for u. Now a new ode $v'=u$ is solved for v as first order ode. Then the final solution is obtain from $y=Bv$.

This method works only if the term $AB''+BB'+CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x^2\\
B &= -3x\\
C &= 3\\
F &= 0
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB''+BB'+CB &= \left(x^2\right)(0)+(-3x)(-3)+(3)(-3x)\\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-3x^3v''+\left(3x^2\right)v'=0
$$

Now by applying $v'=u$ the above becomes

$$
-3x^2(u'(x)\,x-u(x))=0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= \frac{u}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{1}{x}\,dx \\
\ln(u) &= \ln(x) + c_1 \\
u &= \mathrm{e}^{\ln(x)+c_1} \\
&= c_1 x
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1 x
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int c_1 x \, \mathrm{d}x \\
&= \frac{c_1 x^2}{2} + c_2
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (-3x)\left(\frac{c_1 x^2}{2} + c_2\right) \\
&= -\frac{3x(c_1 x^2 + 2c_2)}{2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{3x(c_1 x^2 + 2c_2)}{2} \tag{1}
$$

Verification of solutions

$$
y = -\frac{3x(c_1 x^2 + 2c_2)}{2}
$$

Verified OK.

13.1.6 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 3y'x + 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 468: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}} \\
&= z_1\left(x^{\frac{3}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + \frac{1}{2} c_2 x^3 \tag{1}
$$

Verification of solutions

$$
y = c_1 x + \frac{1}{2} c_2 x^3
$$

Verified OK.

13.1.7 Maple step by step solution

Let's solve

$y''x^2 - 3y'x + 3y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{3y'}{x} - \frac{3y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{3y'}{x} + \frac{3y}{x^2} = 0$

- Multiply by denominators of the ODE

$y''x^2 - 3y'x + 3y = 0$

- Make a change of variables

$t = \ln(x)$

□ Substitute the change of variables back into the ODE

 ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

 $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

 ◦ Compute derivative

 $y' = \frac{\frac{d}{dt}y(t)}{x}$

 ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

 $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

 ◦ Compute derivative

 $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^2 - 3\frac{d}{dt}y(t) + 3y(t) = 0$

- Simplify

$\frac{d^2}{dt^2}y(t) - 4\frac{d}{dt}y(t) + 3y(t) = 0$

- Characteristic polynomial of ODE

$r^2 - 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-3)=0$
- Roots of the characteristic polynomial

  $r=(1,3)$
- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^t$
- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{3t}$
- General solution of the ODE

  $y(t)=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y(t)=c_1\mathrm{e}^t+c_2\mathrm{e}^{3t}$
- Change variables back using $t=\ln(x)$

  $y=c_2x^3+c_1x$
- Simplify

  $y=x(c_2x^2+c_1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x)=x(c_2x^2+c_1)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 16

```
DSolve[x^2*y''[x]-3*x*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(c_2x^2 + c_1)$$

13.2 problem 2

Internal problem ID [11856]
Internal file name [`OUTPUT/11865_Saturday_April_13_2024_01_13_12_AM_22950223/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' + y'x - 4y = 0}$$

13.2.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1)+r-4=0$$

Or

$$r^2-4=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1=-2$$
$$r_2=2$$

Since the roots are real and distinct, then the general solution is

$$y=c_1y_1+c_2y_2$$

Where $y_1=x^{r_1}$ and $y_2=x^{r_2}$. Hence

$$y=\frac{c_1}{x^2}+c_2x^2$$

Summary

The solution(s) found are the following

$$y=\frac{c_1}{x^2}+c_2x^2 \tag{1}$$

Verification of solutions

$$y=\frac{c_1}{x^2}+c_2x^2$$

Verified OK.

13.2.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y''+y'x-4y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=\frac{1}{x}$$
$$q(x)=-\frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\
&= \int e^{-\ln(x)}\,dx \\
&= \int \frac{1}{x}dx \\
&= \ln(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{4}{x^2}}{\frac{1}{x^2}} \\
&= -4
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 4y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} - 4\, e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-4)} \\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2 \\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\ y(\tau) &= c_1 e^{(2)\tau} + c_2 e^{(-2)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{2\tau} + c_2 e^{-2\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^4 + c_2}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^4 + c_2}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^4 + c_2}{x^2}$$

Verified OK.

13.2.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' + y'x - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{2\sqrt{-\frac{1}{x^2}}}{c} \\
\tau'' &= \frac{2}{c\sqrt{-\frac{1}{x^2}}\, x^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{2}{c\sqrt{-\frac{1}{x^2}}\, x^3} + \frac{1}{x}\frac{2\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\
&= 0
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int 2\sqrt{-\frac{1}{x^2}} dx}{c} \\
&= \frac{2\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cosh(2\ln(x)) + i c_2 \sinh(2\ln(x))$$

Summary
The solution(s) found are the following

$$y = c_1 \cosh\left(2\ln\left(x\right)\right) + ic_2 \sinh\left(2\ln\left(x\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cosh\left(2\ln\left(x\right)\right) + ic_2 \sinh\left(2\ln\left(x\right)\right)$$

Verified OK.

13.2.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + y'x - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \frac{5v'(x)}{x} &= 0 \\
v''(x) + \frac{5v'(x)}{x} &= 0 \qquad (7)
\end{aligned}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \frac{5u(x)}{x} = 0 \qquad (8)
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{5u}{x}
\end{aligned}
$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{5}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{5}{x}\, dx \\
\ln(u) &= -5\ln(x) + c_1 \\
u &= e^{-5\ln(x)+c_1} \\
&= \frac{c_1}{x^5}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{4x^4} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{4x^4} + c_2\right) x^2 \\
&= \frac{4c_2x^4 - c_1}{4x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{4x^4} + c_2\right) x^2 \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{4x^4} + c_2\right) x^2
$$

Verified OK.

13.2.5 Solving using Kovacic algorithm

Writing the ode as

$$
x^2y'' + y'x - 4y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= x \\
C &= -4
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 470: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + (-)\,(0) \\
&= -\frac{3}{2x} \\
&= -\frac{3}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x}\right)(0) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x}\right)^2 - \left(\frac{15}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{3}{2x} dx} \\ &= \frac{1}{x^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^4}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1y_1 + c_2y_2 \\ &= c_1\left(\frac{1}{x^2}\right) + c_2\left(\frac{1}{x^2}\left(\frac{x^4}{4}\right)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^2} + \frac{c_2x^2}{4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^2} + \frac{c_2x^2}{4}$$

Verified OK.

13.2.6 Maple step by step solution

Let's solve

$y''x^2 + y'x - 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + \frac{4y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{4y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 + y'x - 4y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$
- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$
- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^2 + \frac{d}{dt}y(t) - 4y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 4y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 - 4 = 0$
- Factor the characteristic polynomial

  $(r-2)(r+2) = 0$
- Roots of the characteristic polynomial

  $r = (-2, 2)$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-2t}$
- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{2t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{2t}$
- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^2} + c_2 x^2$
- Simplify

  $y = \frac{c_1}{x^2} + c_2 x^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^4 c_1 + c_2}{x^2}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]+x*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^4 + c_1}{x^2}$$

13.3 problem 3

Internal problem ID [11857]
Internal file name [`OUTPUT/11866_Saturday_April_13_2024_01_13_13_AM_53292417/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{4x^2y'' - 4y'x + 3y = 0}$$

13.3.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$4x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 3x^r = 0$$

Simplifying gives

$$4r(r-1)\,x^r - 4r\,x^r + 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$4r(r-1) - 4r + 3 = 0$$

Or

$$4r^2 - 8r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{3}{2}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_1\sqrt{x} + c_2x^{\frac{3}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x} + c_2x^{\frac{3}{2}} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x} + c_2x^{\frac{3}{2}}$$

Verified OK.

13.3.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -\frac{1}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2} \int p\, dx} \\ &= e^{\int -\frac{1}{x}\, dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\frac{y}{\sqrt{x}}\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\frac{y}{\sqrt{x}}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{\sqrt{x}}\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\frac{1}{\sqrt{x}}}$$

Or

$$y = c_1 x^{\frac{3}{2}} + c_2 \sqrt{x}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{3}{2}} + c_2 \sqrt{x} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}} + c_2 \sqrt{x}$$

Verified OK.

13.3.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$4x^2y'' - 4y'x + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{3}{4x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)}dx \\ &= \int e^{\ln(x)}\, dx \\ &= \int x dx \\ &= \frac{x^2}{2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{3}{4x^2}}{x^2} \\
&= \frac{3}{4x^4}
\end{aligned}
\tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{4x^4} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{3}{4x^4} = \frac{3}{16\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{16\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 3y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 3\tau^r = 0
$$

Simplifying gives

$$
16r(r-1)\,\tau^r + 0\,\tau^r + 3\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
16r(r-1) + 0 + 3 = 0
$$

Or

$$
16r^2 - 16r + 3 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{4} \\
r_2 &= \frac{3}{4}
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$
y(\tau) = c_1 y_1 + c_2 y_2
$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$
y(\tau) = c_1 \tau^{\frac{1}{4}} + c_2 \tau^{\frac{3}{4}}
$$

The above solution is now transformed back to y using (6) which results in

$$
y = \frac{c_1 2^{\frac{3}{4}} (x^2)^{\frac{1}{4}}}{2} + \frac{c_2 2^{\frac{1}{4}} (x^2)^{\frac{3}{4}}}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 2^{\frac{3}{4}} (x^2)^{\frac{1}{4}}}{2} + \frac{c_2 2^{\frac{1}{4}} (x^2)^{\frac{3}{4}}}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 2^{\frac{3}{4}} (x^2)^{\frac{1}{4}}}{2} + \frac{c_2 2^{\frac{1}{4}} (x^2)^{\frac{3}{4}}}{2}
$$

Verified OK.

13.3.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$
4x^2 y'' - 4y'x + 3y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{3}{4x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{\sqrt{3}\sqrt{\frac{1}{x^2}}}{2c}\\ \tau'' &= -\frac{\sqrt{3}}{2c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2}\\ &= \frac{-\frac{\sqrt{3}}{2c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{1}{x}\frac{\sqrt{3}\sqrt{\frac{1}{x^2}}}{2c}}{\left(\frac{\sqrt{3}\sqrt{\frac{1}{x^2}}}{2c}\right)^2}\\ &= -\frac{4c\sqrt{3}}{3}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) - \frac{4c\sqrt{3}\left(\frac{d}{d\tau}y(\tau)\right)}{3} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{2\sqrt{3}\,c\tau}{3}}\left(c_1\cosh\left(\frac{\sqrt{3}\,c\tau}{3}\right) + ic_2\sinh\left(\frac{\sqrt{3}\,c\tau}{3}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{2}dx}{c} \\
&= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}\, x \ln(x)}{2c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Summary

The solution(s) found are the following

$$
y = x\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) \tag{1}
$$

Verification of solutions

$$
y = x\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Verified OK.

13.3.5 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{3}{4x^2}
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{3}{4x^2} - \frac{\left(-\frac{1}{x}\right)'}{2} - \frac{\left(-\frac{1}{x}\right)^2}{4} \\
&= \frac{3}{4x^2} - \frac{\left(\frac{1}{x^2}\right)}{2} - \frac{\left(\frac{1}{x^2}\right)}{4} \\
&= \frac{3}{4x^2} - \left(\frac{1}{2x^2}\right) - \frac{1}{4x^2} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{1}{x}}{2}} \\
&= \sqrt{x}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, \sqrt{x} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
4x^{\frac{5}{2}} v''(x) = 0
$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$
v(x) = c_1 x + c_2
$$

Now that $v(x)$ is known, then

$$
\begin{aligned}
y &= v(x)\, z(x) \\
&= (c_1 x + c_2)\, (z(x))
\end{aligned} \tag{7}
$$

But from (5)

$$
z(x) = \sqrt{x}
$$

Hence (7) becomes

$$y = (c_1 x + c_2) \sqrt{x}$$

Summary

The solution(s) found are the following

$$y = (c_1 x + c_2) \sqrt{x} \tag{1}$$

Verification of solutions

$$y = (c_1 x + c_2) \sqrt{x}$$

Verified OK.

13.3.6 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$4x^2 y'' - 4y' x + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{3}{4x^2}$$

Applying change of variables on the depndent variable $y = v(x) x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x^2} + \frac{3}{4x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = \frac{3}{2} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{2}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{2}{x}\, dx \\ \ln(u) &= -2\ln(x) + c_1 \\ u &= \mathrm{e}^{-2\ln(x) + c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1}{x} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{x} + c_2\right) x^{\frac{3}{2}} \\
&= (c_2 x - c_1)\sqrt{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{x} + c_2\right) x^{\frac{3}{2}} \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{x} + c_2\right) x^{\frac{3}{2}}
$$

Verified OK.

13.3.7 Solving using Kovacic algorithm

Writing the ode as

$$
4x^2 y'' - 4y'x + 3y = 0 \tag{1}
$$
$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 4x^2 \\
B &= -4x \\
C &= 3
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 472: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}(x)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1\sqrt{x} + c_2x^{\frac{3}{2}} \tag{1}$$

Verification of solutions

$$y = c_1\sqrt{x} + c_2x^{\frac{3}{2}}$$

Verified OK.

13.3.8 Maple step by step solution

Let's solve

$4y''x^2 - 4y'x + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} - \frac{3y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{3y}{4x^2} = 0$

- Multiply by denominators of the ODE

  $4y''x^2 - 4y'x + 3y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$$4\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)x^2-4\frac{d}{dt}y(t)+3y(t)=0$$

- Simplify

$$4\frac{d^2}{dt^2}y(t)-8\frac{d}{dt}y(t)+3y(t)=0$$

- Isolate 2nd derivative

$$\frac{d^2}{dt^2}y(t)=2\frac{d}{dt}y(t)-\frac{3y(t)}{4}$$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^2}{dt^2}y(t)-2\frac{d}{dt}y(t)+\frac{3y(t)}{4}=0$$

- Characteristic polynomial of ODE

$$r^2-2r+\frac{3}{4}=0$$

- Factor the characteristic polynomial

$$\frac{(2r-1)(2r-3)}{4}=0$$

- Roots of the characteristic polynomial

$$r=\left(\frac{1}{2},\frac{3}{2}\right)$$

- 1st solution of the ODE

$$y_1(t)=\mathrm{e}^{\frac{t}{2}}$$

- 2nd solution of the ODE

$$y_2(t)=\mathrm{e}^{\frac{3t}{2}}$$

- General solution of the ODE

$$y(t)=c_1y_1(t)+c_2y_2(t)$$

- Substitute in solutions

$$y(t)=c_1\mathrm{e}^{\frac{t}{2}}+c_2\mathrm{e}^{\frac{3t}{2}}$$

- Change variables back using $t=\ln(x)$

$$y=c_1\sqrt{x}+c_2x^{\frac{3}{2}}$$

- Simplify

$$y=\sqrt{x}\,(c_2x+c_1)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x}\,(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 18

```
DSolve[4*x^2*y''[x]-4*x*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \sqrt{x}(c_2x + c_1)$$

13.4 problem 4

Internal problem ID [11858]
Internal file name [`OUTPUT/11867_Saturday_April_13_2024_01_13_15_AM_60726113/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 3y'x + 4y = 0}$$

13.4.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 2$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Summary
The solution(s) found are the following

$$y = c_1 x^2 + c_2 x^2 \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Verified OK.

13.4.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)}dx \\ &= \int e^{3\ln(x)}\,dx \\ &= \int x^3 dx \\ &= \frac{x^4}{4}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{4}{x^2}}{x^6} \\ &= \frac{4}{x^8}\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{x^8} &= 0\end{aligned}$$

But in terms of τ

$$\frac{4}{x^8} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{(c_2\ln(x^4) - 2c_2\ln(2) + c_1)\sqrt{x^4}}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2} \tag{1}$$

Verification of solutions

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2}$$

Verified OK.

13.4.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \\
&= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\,\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln{(x)}}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 x^2
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2 \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2
$$

Verified OK.

13.4.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= e^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= (c_1 \ln(x) + c_2)\,x^2 \\ &= (c_1 \ln(x) + c_2)\,x^2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 \ln(x) + c_2)\,x^2 \tag{1}$$

Verification of solutions

$$y = (c_1 \ln(x) + c_2)\,x^2$$

Verified OK.

13.4.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 3y'x + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 474: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}} \\
&= z_1\left(x^{\frac{3}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(\ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2 + c_2 x^2 \ln(x) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2 + c_2 x^2 \ln(x)
$$

Verified OK.

13.4.6 Maple step by step solution

Let's solve

$y''x^2 - 3y'x + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{x} - \frac{4y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{x} + \frac{4y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 - 3y'x + 4y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - 3\frac{d}{dt}y(t) + 4y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 4\frac{d}{dt}y(t) + 4y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$
- Root of the characteristic polynomial

  $r = 2$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{2t}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{2t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{2t} + c_2 t\,\mathrm{e}^{2t}$
- Change variables back using $t = \ln(x)$

  $y = c_1 x^2 + c_2 x^2 \ln(x)$
- Simplify

  $y = x^2(c_1 + \ln(x)\, c_2)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^2(c_1 + c_2 \ln(x))$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(2c_2\log(x) + c_1)$$

13.5 problem 5

Internal problem ID [11859]
Internal file name [`OUTPUT/11868_Saturday_April_13_2024_01_13_16_AM_62882544/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' + y'x + 4y = 0}$$

13.5.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1)+r+4=0$$

Or

$$r^2+4=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -2i$$
$$r_2 = 2i$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = 0$ and $\beta = -2$. Hence the solution becomes

$$\begin{aligned}
y &= c_1 x^{r_1} + c_2 x^{r_2} \\
&= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\
&= x^{\alpha}\left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\
&= x^{\alpha}\left(c_1 e^{\ln(x^{i\beta})} + c_2 e^{\ln(x^{-i\beta})}\right) \\
&= x^{\alpha}\left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right)
\end{aligned}$$

Using the values for $\alpha = 0, \beta = -2$, the above becomes

$$y = x^0\left(c_1 e^{-2i\ln(x)} + c_2 e^{2i\ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

Verified OK.

13.5.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\left(\int p(x)dx\right)}dx \\ &= \int e^{-\left(\int \frac{1}{x}dx\right)}dx \\ &= \int e^{-\ln(x)}\,dx \\ &= \int \frac{1}{x}dx \\ &= \ln(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{x^2}}{\frac{1}{x^2}} \\
&= 4
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 4y(\tau) &= 0
\end{aligned}
$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Ay''(\tau) + By'(\tau) + Cy(\tau) = 0
$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda\tau} + 4\,\mathrm{e}^{\lambda\tau} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$
\lambda^2 + 4 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\
&= \pm 2i
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +2i \\
\lambda_2 &= -2i
\end{aligned}
$$

Which simplifies to

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

Verified OK.

13.5.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' + y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$
$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\ &= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

Verified OK.

13.5.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= \frac{1}{x} \\ q(x) &= \frac{4}{x^2}\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{4i}{x} + \frac{1}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(1+4i)\,v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+4i)\,u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{(-1-4i)\,u}{x} \end{aligned}$$

Where $f(x) = \frac{-1-4i}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{-1-4i}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{-1-4i}{x}\,dx \\
\ln(u) &= (-1-4i)\ln(x) + c_1 \\
u &= \mathrm{e}^{(-1-4i)\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{(-1-4i)\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-4i}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= \frac{i c_1 x^{-4i}}{4} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(\frac{i c_1 x^{-4i}}{4} + c_2\right) x^{2i} \\
&= x^{2i} c_2 + \frac{i x^{-2i} c_1}{4}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(\frac{i c_1 x^{-4i}}{4} + c_2\right) x^{2i} \tag{1}$$

Verification of solutions

$$y = \left(\frac{i c_1 x^{-4i}}{4} + c_2\right) x^{2i}$$

Verified OK.

13.5.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + y'x + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-17}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -17 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{17}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 476: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{17}{4x^2}
$$

For the pole at $x=0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b=-\frac{17}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}+2i \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}-2i
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{17}{4x^2}
$$

Since the $\gcd(s,t)=1$. This gives $b=-\frac{17}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{1}{2}+2i \\
\alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = \frac{1}{2}-2i
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{17}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}+2i$	$\frac{1}{2}-2i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+2i$	$\frac{1}{2}-2i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2} - 2i$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - 2i - \left(\frac{1}{2} - 2i\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{1}{2} - 2i}{x} + (-)(0) \\
&= \frac{\frac{1}{2} - 2i}{x} \\
&= \frac{\frac{1}{2} - 2i}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - 2i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + 2i}{x^2}\right) + \left(\frac{\frac{1}{2} - 2i}{x}\right)^2 - \left(-\frac{17}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{\frac{1}{2} - 2i}{x} dx} \\
&= x^{\frac{1}{2} - 2i}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{-2i}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i x^{4i}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-2i}\right) + c_2 \left(x^{-2i} \left(-\frac{i x^{4i}}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x^{-2i} c_1 - \frac{i c_2 x^{2i}}{4} \tag{1}
$$

Verification of solutions

$$
y = x^{-2i} c_1 - \frac{i c_2 x^{2i}}{4}
$$

Verified OK.

13.5.6 Maple step by step solution

Let's solve

$y''x^2 + y'x + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{4y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{4y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 + y'x + 4y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 + \frac{d}{dt}y(t) + 4y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 4y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = \cos(2t)$

- 2nd solution of the ODE

  $y_2(t) = \sin(2t)$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \cos(2t) + c_2 \sin(2t)$

- Change variables back using $t = \ln(x)$

  $y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(2\ln(x)) + c_2 \cos(2\ln(x))$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]+x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(2\log(x)) + c_2 \sin(2\log(x))$$

13.6 problem 6

Internal problem ID [11860]
Internal file name [`OUTPUT/11869_Saturday_April_13_2024_01_13_17_AM_55471947/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - 3y'x + 13y = 0}$$

13.6.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 13x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 13x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 13 = 0$$

Or

$$r^2 - 4r + 13 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 - 3i \\ r_2 &= 2 + 3i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 2$ and $\beta = -3$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha + i\beta} + c_2 x^{\alpha - i\beta} \\ &= x^{\alpha}\left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^{\alpha}\left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^{\alpha}\left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = 2, \beta = -3$, the above becomes

$$y = x^2\left(c_1 e^{-3i \ln(x)} + c_2 e^{3i \ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = x^2(c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x)))$$

Summary

The solution(s) found are the following

$$y = x^2(c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))) \tag{1}$$

Verification of solutions

$$y = x^2(c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x)))$$

Verified OK.

13.6.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - 3y'x + 13y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{13}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}\,dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)}\,dx \\ &= \int e^{3\ln(x)}\,dx \\ &= \int x^3 dx \\ &= \frac{x^4}{4} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{13}{x^2}}{x^6} \\
&= \frac{13}{x^8}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{13y(\tau)}{x^8} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{13}{x^8} = \frac{13}{16\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{13y(\tau)}{16\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 13y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 13\tau^r = 0
$$

Simplifying gives

$$
16r(r-1)\,\tau^r + 0\,\tau^r + 13\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
16r(r-1) + 0 + 13 = 0
$$

Or

$$
16r^2 - 16r + 13 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - \frac{3i}{4}$$
$$r_2 = \frac{1}{2} + \frac{3i}{4}$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{3}{4}$. Hence the solution becomes

$$\begin{aligned}
y(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\
&= c_1 \tau^{\alpha + i\beta} + c_2 \tau^{\alpha - i\beta} \\
&= \tau^{\alpha} \left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta}\right) \\
&= \tau^{\alpha} \left(c_1 e^{\ln\left(\tau^{i\beta}\right)} + c_2 e^{\ln\left(\tau^{-i\beta}\right)}\right) \\
&= \tau^{\alpha} \left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)}\right)
\end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{3}{4}$, the above becomes

$$y(\tau) = \tau^{\frac{1}{2}} \left(c_1 e^{-\frac{3i \ln(\tau)}{4}} + c_2 e^{\frac{3i \ln(\tau)}{4}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y(\tau) = \sqrt{\tau}\left(c_1 \cos\left(\frac{3\ln(\tau)}{4}\right) + c_2 \sin\left(\frac{3\ln(\tau)}{4}\right)\right)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\left(c_1 \cos\left(-\frac{3\ln(2)}{2} + 3\ln(x)\right) + c_2 \sin\left(-\frac{3\ln(2)}{2} + 3\ln(x)\right)\right) x^2}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_1 \cos\left(-\frac{3\ln(2)}{2} + 3\ln(x)\right) + c_2 \sin\left(-\frac{3\ln(2)}{2} + 3\ln(x)\right)\right) x^2}{2} \tag{1}$$

Verification of solutions

$$y = \frac{\left(c_1 \cos\left(-\frac{3\ln(2)}{2} + 3\ln(x)\right) + c_2 \sin\left(-\frac{3\ln(2)}{2} + 3\ln(x)\right)\right) x^2}{2}$$

Verified OK.

13.6.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' - 3y'x + 13y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{13}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{13}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{13}}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{13}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\frac{\sqrt{13}\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{13}\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{4c\sqrt{13}}{13}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{4c\sqrt{13}\,\left(\frac{d}{d\tau}y(\tau)\right)}{13} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{2\sqrt{13}\,c\tau}{13}}\left(c_1\cos\left(\frac{3\sqrt{13}\,c\tau}{13}\right) + c_2\sin\left(\frac{3\sqrt{13}\,c\tau}{13}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{13}\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{13}\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^2(c_1\cos(3\ln(x)) + c_2\sin(3\ln(x)))
$$

Summary

The solution(s) found are the following

$$
y = x^2(c_1\cos(3\ln(x)) + c_2\sin(3\ln(x))) \qquad (1)
$$

Verification of solutions

$$
y = x^2(c_1\cos(3\ln(x)) + c_2\sin(3\ln(x)))
$$

Verified OK.

13.6.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 3y'x + 13y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{13}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{13}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 + 3i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{4+6i}{x} - \frac{3}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(1+6i)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+6i)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{(-1-6i)\, u}{x}
\end{aligned}$$

Where $f(x) = \frac{-1-6i}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= \frac{-1-6i}{x}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{-1-6i}{x}\, dx \\
\ln(u) &= (-1-6i)\ln(x) + c_1 \\
u &= \mathrm{e}^{(-1-6i)\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{(-1-6i)\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-6i}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= \frac{i c_1 x^{-6i}}{6} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= \left(\frac{i c_1 x^{-6i}}{6} + c_2\right) x^{2+3i} \\
&= c_2 x^{2+3i} + \frac{i c_1 x^{2-3i}}{6}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(\frac{i c_1 x^{-6i}}{6}+c_2\right) x^{2+3i} \tag{1}$$

Verification of solutions

$$y = \left(\frac{i c_1 x^{-6i}}{6}+c_2\right) x^{2+3i}$$

Verified OK.

13.6.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 3y'x + 13y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-37}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -37$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{37}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 478: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{37}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{37}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 3i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 3i
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{37}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{37}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 3i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 3i
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{37}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + 3i$	$\frac{1}{2} - 3i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}+3i$	$\frac{1}{2}-3i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2} - 3i$ then

$$\begin{aligned}
d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - 3i - \left(\frac{1}{2} - 3i\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2}-3i}{x} + (-)\,(0) \\
&= \frac{\frac{1}{2}-3i}{x} \\
&= \frac{\frac{1}{2}-3i}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{\frac{1}{2}-3i}{x}\right)(0)+\left(\left(\frac{-\frac{1}{2}+3i}{x^2}\right)+\left(\frac{\frac{1}{2}-3i}{x}\right)^2-\left(-\frac{37}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x)&=pe^{\int \omega\,dx}\\
&=e^{\int \frac{\frac{1}{2}-3i}{x}dx}\\
&=x^{\frac{1}{2}-3i}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1&=z_1e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\
&=z_1e^{-\int \frac{1}{2}\frac{-3x}{x^2}\,dx}\\
&=z_1e^{\frac{3\ln(x)}{2}}\\
&=z_1\left(x^{\frac{3}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1=x^{2-3i}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2&=y_1\int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx\\
&=y_1\int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx\\
&=y_1\left(-\frac{ix^{6i}}{6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{2-3i}\right) + c_2 \left(x^{2-3i} \left(-\frac{i x^{6i}}{6}\right)\right)\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x^{2-3i} - \frac{i c_2 x^{2+3i}}{6} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 x^{2-3i} - \frac{i c_2 x^{2+3i}}{6}$$

Verified OK.

13.6.6 Maple step by step solution

Let's solve

$y''x^2 - 3y'x + 13y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{x} - \frac{13y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{x} + \frac{13y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 - 3y'x + 13y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^2 - 3\frac{d}{dt}y(t) + 13y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 4\frac{d}{dt}y(t) + 13y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 4r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - 3\,\mathrm{I}, 2 + 3\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{2t} \cos(3t)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{2t} \sin(3t)$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{2t} \cos(3t) + c_2 \mathrm{e}^{2t} \sin(3t)$

- Change variables back using $t = \ln(x)$

  $y = c_1 x^2 \cos(3\ln(x)) + c_2 x^2 \sin(3\ln(x))$

- Simplify

  $y = x^2 (c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x)))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+13*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^2(c_1 \sin(3\ln(x)) + c_2 \cos(3\ln(x)))$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 26

```
DSolve[x^2*y''[x]-3*x*y'[x]+13*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(c_2 \cos(3\log(x)) + c_1 \sin(3\log(x)))$$

13.7 problem 7

Internal problem ID [11861]
Internal file name [`OUTPUT/11870_Saturday_April_13_2024_01_13_19_AM_34740163/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{3x^2y'' - 4y'x + 2y = 0}$$

13.7.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$3x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$3r(r-1)\,x^r - 4r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$3r(r-1) - 4r + 2 = 0$$

Or

$$3r^2 - 7r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = \frac{1}{3}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_1 x^2 + c_2 x^{\frac{1}{3}}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_2 x^{\frac{1}{3}} \tag{1}$$

Verification of solutions

$$y = c_1 x^2 + c_2 x^{\frac{1}{3}}$$

Verified OK.

13.7.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$3x^2 y'' - 4y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{3x}$$
$$q(x) = \frac{2}{3x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{4}{3x}dx\right)}dx \\
&= \int e^{\frac{4\ln(x)}{3}}\,dx \\
&= \int x^{\frac{4}{3}}dx \\
&= \frac{3x^{\frac{7}{3}}}{7}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{2}{3x^2}}{x^{\frac{8}{3}}} \\
&= \frac{2}{3x^{\frac{14}{3}}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{3x^{\frac{14}{3}}} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{2}{3x^{\frac{14}{3}}} = \frac{6}{49\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{49\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$49\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$49\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0$$

Simplifying gives

$$49r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$49r(r-1) + 0 + 6 = 0$$

Or

$$49r^2 - 49r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{7}$$
$$r_2 = \frac{6}{7}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{7}} + c_2\tau^{\frac{6}{7}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{1}{7}} 7^{\frac{6}{7}} \left(x^{\frac{7}{3}}\right)^{\frac{1}{7}}}{7} + \frac{c_2 3^{\frac{6}{7}} 7^{\frac{1}{7}} \left(x^{\frac{7}{3}}\right)^{\frac{6}{7}}}{7}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 3^{\frac{1}{7}} 7^{\frac{6}{7}} \left(x^{\frac{7}{3}}\right)^{\frac{1}{7}}}{7} + \frac{c_2 3^{\frac{6}{7}} 7^{\frac{1}{7}} \left(x^{\frac{7}{3}}\right)^{\frac{6}{7}}}{7} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 3^{\frac{1}{7}} 7^{\frac{6}{7}} \left(x^{\frac{7}{3}}\right)^{\frac{1}{7}}}{7} + \frac{c_2 3^{\frac{6}{7}} 7^{\frac{1}{7}} \left(x^{\frac{7}{3}}\right)^{\frac{6}{7}}}{7}$$

Verified OK.

13.7.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$3x^2 y'' - 4y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{3x} \\ q(x) &= \frac{2}{3x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{3c} \\ \tau'' &= -\frac{\sqrt{6}}{3c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{6}}{3c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{3x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{3c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{3c}\right)^2} \\ &= -\frac{7c\sqrt{6}}{6} \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{7c\sqrt{6}\left(\frac{d}{d\tau}y(\tau)\right)}{6} + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{7\sqrt{6}\,c\tau}{12}}\left(c_1\cosh\left(\frac{5\sqrt{6}\,c\tau}{12}\right) + ic_2\sinh\left(\frac{5\sqrt{6}\,c\tau}{12}\right)\right)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{3}dx}{c} \\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{3c} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = x^{\frac{7}{6}} \left(c_1 \cosh\left(\frac{5\ln(x)}{6}\right) + ic_2 \sinh\left(\frac{5\ln(x)}{6}\right)\right)$$

Summary

The solution(s) found are the following

$$y = x^{\frac{7}{6}} \left(c_1 \cosh\left(\frac{5\ln(x)}{6}\right) + ic_2 \sinh\left(\frac{5\ln(x)}{6}\right)\right) \tag{1}$$

Verification of solutions

$$y = x^{\frac{7}{6}} \left(c_1 \cosh\left(\frac{5\ln(x)}{6}\right) + ic_2 \sinh\left(\frac{5\ln(x)}{6}\right)\right)$$

Verified OK.

13.7.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$3x^2y'' - 4y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{3x}$$
$$q(x) = \frac{2}{3x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{3x^2} + \frac{2}{3x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{8v'(x)}{3x} &= 0 \\ v''(x) + \frac{8v'(x)}{3x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{8u(x)}{3x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{8u}{3x} \end{aligned}$$

Where $f(x) = -\frac{8}{3x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{8}{3x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{8}{3x}\, dx \\ \ln(u) &= -\frac{8\ln(x)}{3} + c_1 \\ u &= \mathrm{e}^{-\frac{8\ln(x)}{3} + c_1} \\ &= \frac{c_1}{x^{\frac{8}{3}}} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{3c_1}{5x^{\frac{5}{3}}} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{3c_1}{5x^{\frac{5}{3}}} + c_2\right) x^2 \\
&= c_2 x^2 - \frac{3x^{\frac{1}{3}} c_1}{5}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{3c_1}{5x^{\frac{5}{3}}} + c_2\right) x^2 \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{3c_1}{5x^{\frac{5}{3}}} + c_2\right) x^2
$$

Verified OK.

13.7.5 Solving using Kovacic algorithm

Writing the ode as

$$
3x^2 y'' - 4y'x + 2y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 3x^2 \\
B &= -4x \\
C &= 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{9x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 9x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4}{9x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 480: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 9x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{4}{9x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition

of r given above. Therefore $b = \frac{4}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{4}{3} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{3}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{4}{9x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{4}{9}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{4}{3} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{3}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{4}{9x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{4}{3}$	$-\frac{1}{3}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{4}{3}$	$-\frac{1}{3}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{1}{3}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{3} - \left(-\frac{1}{3}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{3x} + (-)(0) \\
&= -\frac{1}{3x} \\
&= -\frac{1}{3x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{3x}\right)(0) + \left(\left(\frac{1}{3x^2}\right) + \left(-\frac{1}{3x}\right)^2 - \left(\frac{4}{9x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{1}{3x} dx} \\
&= \frac{1}{x^{\frac{1}{3}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{3x^2}\,dx} \\
&= z_1 e^{\frac{2\ln(x)}{3}} \\
&= z_1 \left(x^{\frac{2}{3}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{\frac{1}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{3x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{4\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{3x^{\frac{5}{3}}}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{\frac{1}{3}}\right) + c_2\left(x^{\frac{1}{3}}\left(\frac{3x^{\frac{5}{3}}}{5}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = x^{\frac{1}{3}} c_1 + \frac{3c_2 x^2}{5} \tag{1}
$$

Verification of solutions

$$y = x^{\frac{1}{3}} c_1 + \frac{3 c_2 x^2}{5}$$

Verified OK.

13.7.6 Maple step by step solution

Let's solve

$3y'' x^2 - 4y' x + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{3x} - \frac{2y}{3x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{3x} + \frac{2y}{3x^2} = 0$

- Multiply by denominators of the ODE

  $3y'' x^2 - 4y' x + 2y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $3\left(\frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}\right) x^2 - 4 \frac{d}{dt} y(t) + 2y(t) = 0$

- Simplify

  $3\frac{d^2}{dt^2}y(t) - 7\frac{d}{dt}y(t) + 2y(t) = 0$
- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = \frac{7\frac{d}{dt}y(t)}{3} - \frac{2y(t)}{3}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - \frac{7\frac{d}{dt}y(t)}{3} + \frac{2y(t)}{3} = 0$
- Characteristic polynomial of ODE

  $r^2 - \frac{7}{3}r + \frac{2}{3} = 0$
- Factor the characteristic polynomial

  $\frac{(3r-1)(r-2)}{3} = 0$
- Roots of the characteristic polynomial

  $r = \left(2, \frac{1}{3}\right)$
- 1st solution of the ODE

  $y_1(t) = e^{2t}$
- 2nd solution of the ODE

  $y_2(t) = e^{\frac{t}{3}}$
- General solution of the ODE

  $y(t) = c_1y_1(t) + c_2y_2(t)$
- Substitute in solutions

  $y(t) = c_1e^{2t} + c_2e^{\frac{t}{3}}$
- Change variables back using $t = \ln(x)$

  $y = c_1x^2 + c_2x^{\frac{1}{3}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(3*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1x^2 + c_2x^{\frac{1}{3}}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 20

```
DSolve[3*x^2*y''[x]-4*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2x^2 + c_1\sqrt[3]{x}$$

13.8 problem 8

Internal problem ID [11862]
Internal file name [`OUTPUT/11871_Saturday_April_13_2024_01_13_20_AM_491032/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' + y'x + 9y = 0}$$

13.8.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} + 9x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r + 9x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r + 9 = 0$$

Or

$$r^2 + 9 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -3i \\ r_2 &= 3i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 0$ and $\beta = -3$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\ &= x^{\alpha}\left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^{\alpha}\left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^{\alpha}\left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = 0, \beta = -3$, the above becomes

$$y = x^0\left(c_1 e^{-3i\ln(x)} + c_2 e^{3i\ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = c_1 \cos\left(3\ln\left(x\right)\right) + c_2 \sin\left(3\ln\left(x\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(3\ln\left(x\right)\right) + c_2 \sin\left(3\ln\left(x\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(3\ln\left(x\right)\right) + c_2 \sin\left(3\ln\left(x\right)\right)$$

Verified OK.

13.8.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' + y'x + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{9}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\ &= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)} dx \\ &= \int e^{-\ln(x)}\, dx \\ &= \int \frac{1}{x} dx \\ &= \ln(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{9}{x^2}}{\frac{1}{x^2}} \\ &= 9 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2} y(\tau) + 9y(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + 9\, e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)} \\ &= \pm 3i \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +3i \\ \lambda_2 &= -3i \end{aligned}$$

Which simplifies to

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(3\tau) + c_2 \sin(3\tau))$$

Or

$$y(\tau) = c_1 \cos(3\tau) + c_2 \sin(3\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$$

Verified OK.

13.8.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' + y'x + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{9}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$
$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{3\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{3}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{3}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{3\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{3\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int 3\sqrt{\frac{1}{x^2}}dx}{c} \\ &= \frac{3\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x)) \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$$

Verified OK.

13.8.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + y'x + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= \frac{1}{x} \\ q(x) &= \frac{9}{x^2}\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} + \frac{9}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{6i}{x} + \frac{1}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(1+6i)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+6i)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{(-1-6i)\, u}{x} \end{aligned}$$

Where $f(x) = \frac{-1-6i}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{-1-6i}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{-1-6i}{x}\,dx \\
\ln(u) &= (-1-6i)\ln(x) + c_1 \\
u &= \mathrm{e}^{(-1-6i)\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{(-1-6i)\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_1 x^{-6i}}{x}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= \frac{i c_1 x^{-6i}}{6} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(\frac{i c_1 x^{-6i}}{6} + c_2\right) x^{3i} \\
&= x^{3i} c_2 + \frac{i x^{-3i} c_1}{6}
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \left(\frac{i c_1 x^{-6i}}{6} + c_2\right) x^{3i} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \left(\frac{i c_1 x^{-6i}}{6} + c_2\right) x^{3i}
$$

Verified OK.

13.8.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + y'x + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-37}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -37 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{37}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 482: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{37}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{37}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 3i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 3i
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{37}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{37}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 3i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 3i
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{37}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + 3i$	$\frac{1}{2} - 3i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + 3i$	$\frac{1}{2} - 3i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2} - 3i$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - 3i - \left(\frac{1}{2} - 3i\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - 3i}{x} + (-)\,(0) \\
&= \frac{\frac{1}{2} - 3i}{x} \\
&= \frac{\frac{1}{2} - 3i}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - 3i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + 3i}{x^2}\right) + \left(\frac{\frac{1}{2} - 3i}{x}\right)^2 - \left(-\frac{37}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2} - 3i}{x} dx} \\
&= x^{\frac{1}{2} - 3i}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{-3i}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i x^{6i}}{6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-3i}\right) + c_2 \left(x^{-3i}\left(-\frac{i x^{6i}}{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x^{-3i} c_1 - \frac{i c_2 x^{3i}}{6} \tag{1}
$$

Verification of solutions

$$
y = x^{-3i} c_1 - \frac{i c_2 x^{3i}}{6}
$$

Verified OK.

13.8.6 Maple step by step solution

Let's solve

$y''x^2 + y'x + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{9y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{9y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 + y'x + 9y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ○ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  ○ Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ○ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ○ Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 + \frac{d}{dt}y(t) + 9y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 9y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$
- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$
- 1st solution of the ODE

  $y_1(t) = \cos(3t)$
- 2nd solution of the ODE

  $y_2(t) = \sin(3t)$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 \cos(3t) + c_2 \sin(3t)$
- Change variables back using $t = \ln(x)$

  $y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+9*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(3\ln(x)) + c_2 \cos(3\ln(x))$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]+x*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(3\log(x)) + c_2 \sin(3\log(x))$$

13.9 problem 9

Internal problem ID [11863]
Internal file name [`OUTPUT/11872_Saturday_April_13_2024_01_13_22_AM_68095640/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{9x^2y'' + 3y'x + y = 0}$$

13.9.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$9x^2(r(r-1))x^{r-2} + 3xrx^{r-1} + x^r = 0$$

Simplifying gives

$$9r(r-1)\,x^r + 3r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$9r(r-1) + 3r + 1 = 0$$

Or

$$9r^2 - 6r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{1}{3}$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = x^{\frac{1}{3}} c_1 + c_2 x^{\frac{1}{3}} \ln(x)$$

Summary

The solution(s) found are the following

$$y = x^{\frac{1}{3}} c_1 + c_2 x^{\frac{1}{3}} \ln(x) \tag{1}$$

Verification of solutions

$$y = x^{\frac{1}{3}} c_1 + c_2 x^{\frac{1}{3}} \ln(x)$$

Verified OK.

13.9.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$9x^2 y'' + 3y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{3x}$$
$$q(x) = \frac{1}{9x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{3x}dx\right)}dx \\
&= \int e^{-\frac{\ln(x)}{3}}\,dx \\
&= \int \frac{1}{x^{\frac{1}{3}}}dx \\
&= \frac{3x^{\frac{2}{3}}}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{9x^2}}{\frac{1}{x^{\frac{2}{3}}}} \\
&= \frac{1}{9x^{\frac{4}{3}}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{9x^{\frac{4}{3}}} = 0$$

But in terms of τ

$$\frac{1}{9x^{\frac{4}{3}}} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{6}\,x^{\frac{1}{3}}(2\ln(x)\,c_2 - 3c_2\ln(2) + 3c_2\ln(3) + 3c_1)}{6}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{6}\, x^{\frac{1}{3}} \left(2 \ln \left(x \right) c_{2}-3 c_{2} \ln \left(2\right)+3 c_{2} \ln \left(3\right)+3 c_{1}\right)}{6} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{6}\, x^{\frac{1}{3}} \left(2 \ln \left(x \right) c_{2}-3 c_{2} \ln \left(2\right)+3 c_{2} \ln \left(3\right)+3 c_{1}\right)}{6}$$

Verified OK.

13.9.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$9x^2y''+3y'x+y = 0 \tag{1}$$

Becomes

$$y''+p(x)\, y'+q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{3x}\\ q(x) &= \frac{1}{9x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\left(x\right)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{\sqrt{\frac{1}{x^2}}}{3c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{1}{3c\sqrt{\frac{1}{x^2}}\, x^3}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{3c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{1}{3x}\frac{\sqrt{\frac{1}{x^2}}}{3c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{3c}\right)^2} \\
&= -2c
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{c\tau} c_1$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \frac{\sqrt{\frac{1}{x^2}}}{3}dx}{c} \\
&= \frac{\sqrt{\frac{1}{x^2}}\,x\ln(x)}{3c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = x^{\frac{1}{3}} c_1$$

Summary

The solution(s) found are the following

$$y = x^{\frac{1}{3}} c_1 \qquad (1)$$

Verification of solutions

$$y = x^{\frac{1}{3}} c_1$$

Verified OK.

13.9.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$9x^2y'' + 3y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{3x} \\ q(x) &= \frac{1}{9x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{3x^2} + \frac{1}{9x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = \frac{1}{3} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= (c_1 \ln(x) + c_2)\, x^{\frac{1}{3}} \\ &= (c_1 \ln(x) + c_2)\, x^{\frac{1}{3}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 \ln(x) + c_2)\, x^{\frac{1}{3}} \tag{1}$$

Verification of solutions

$$y = (c_1 \ln(x) + c_2)\, x^{\frac{1}{3}}$$

Verified OK.

13.9.5 Solving using Kovacic algorithm

Writing the ode as

$$9x^2y'' + 3y'x + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9x^2 \\ B &= 3x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 484: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{9x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{6}} \\
&= z_1 \left(\frac{1}{x^{\frac{1}{6}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{\frac{1}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{9x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{\ln(x)}{3}}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{\frac{1}{3}}\right) + c_2\left(x^{\frac{1}{3}}(\ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x^{\frac{1}{3}} c_1 + c_2 x^{\frac{1}{3}} \ln(x) \tag{1}
$$

Verification of solutions

$$
y = x^{\frac{1}{3}} c_1 + c_2 x^{\frac{1}{3}} \ln(x)
$$

Verified OK.

13.9.6 Maple step by step solution

Let's solve

$9y''x^2 + 3y'x + y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = -\frac{y'}{3x} - \frac{y}{9x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{y'}{3x} + \frac{y}{9x^2} = 0$

- Multiply by denominators of the ODE

$9y''x^2 + 3y'x + y = 0$

- Make a change of variables

$t = \ln(x)$

- □ Substitute the change of variables back into the ODE
  - ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $9\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 + 3\frac{d}{dt}y(t) + y(t) = 0$

- Simplify

$9\frac{d^2}{dt^2}y(t) - 6\frac{d}{dt}y(t) + y(t) = 0$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t) = \frac{2\frac{d}{dt}y(t)}{3} - \frac{y(t)}{9}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - \frac{2\frac{d}{dt}y(t)}{3} + \frac{y(t)}{9} = 0$
- Characteristic polynomial of ODE

  $r^2 - \frac{2}{3}r + \frac{1}{9} = 0$
- Factor the characteristic polynomial

  $\frac{(3r-1)^2}{9} = 0$
- Root of the characteristic polynomial

  $r = \frac{1}{3}$
- 1st solution of the ODE

  $y_1(t) = e^{\frac{t}{3}}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^{\frac{t}{3}}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 e^{\frac{t}{3}} + c_2 t\,e^{\frac{t}{3}}$
- Change variables back using $t = \ln(x)$

  $y = x^{\frac{1}{3}} c_1 + c_2 x^{\frac{1}{3}} \ln(x)$
- Simplify

  $y = (c_1 + \ln(x)\, c_2)\, x^{\frac{1}{3}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(9*x^2*diff(y(x),x$2)+3*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_1 + c_2 \ln(x))\, x^{\frac{1}{3}}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 24

```
DSolve[9*x^2*y''[x]+3*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}\sqrt[3]{x}(c_2 \log(x) + 3c_1)$$

13.10 problem 10

Internal problem ID [11864]
Internal file name [`OUTPUT/11873_Saturday_April_13_2024_01_13_23_AM_97187487/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - 5y'x + 10y = 0}$$

13.10.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 5xrx^{r-1} + 10x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 5r\,x^r + 10x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 5r + 10 = 0$$

Or

$$r^2 - 6r + 10 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 3 - i \\ r_2 &= 3 + i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 3$ and $\beta = -1$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\ &= x^\alpha \left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^\alpha \left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^\alpha \left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = 3, \beta = -1$, the above becomes

$$y = x^3 \left(c_1 e^{-i\ln(x)} + c_2 e^{i\ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = x^3 (c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)))$$

Summary

The solution(s) found are the following

$$y = x^3 (c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))) \tag{1}$$

Verification of solutions

$$y = x^3 (c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)))$$

Verified OK.

13.10.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - 5y'x + 10y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= \frac{10}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{5}{x}dx\right)}dx \\ &= \int e^{5\ln(x)}\, dx \\ &= \int x^5dx \\ &= \frac{x^6}{6} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{10}{x^2}}{x^{10}} \\ &= \frac{10}{x^{12}} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{10y(\tau)}{x^{12}} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{10}{x^{12}} = \frac{5}{18\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{18\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$18\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 5y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$18\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 5\tau^r = 0$$

Simplifying gives

$$18r(r-1)\,\tau^r + 0\,\tau^r + 5\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$18r(r-1) + 0 + 5 = 0$$

Or

$$18r^2 - 18r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - \frac{i}{6}$$
$$r_2 = \frac{1}{2} + \frac{i}{6}$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{1}{6}$. Hence the solution becomes

$$\begin{aligned} y(\tau) &= c_1\tau^{r_1} + c_2\tau^{r_2} \\ &= c_1\tau^{\alpha+i\beta} + c_2\tau^{\alpha-i\beta} \\ &= \tau^{\alpha}\left(c_1\tau^{i\beta} + c_2\tau^{-i\beta}\right) \\ &= \tau^{\alpha}\left(c_1 e^{\ln(\tau^{i\beta})} + c_2 e^{\ln(\tau^{-i\beta})}\right) \\ &= \tau^{\alpha}\left(c_1 e^{i(\beta\ln\tau)} + c_2 e^{-i(\beta\ln\tau)}\right) \end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{1}{6}$, the above becomes

$$y(\tau) = \tau^{\frac{1}{2}}\left(c_1 e^{-\frac{i\ln(\tau)}{6}} + c_2 e^{\frac{i\ln(\tau)}{6}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y(\tau) = \sqrt{\tau}\left(c_1 \cos\left(\frac{\ln(\tau)}{6}\right) + c_2 \sin\left(\frac{\ln(\tau)}{6}\right)\right)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{6}\,\sqrt{x^6}\left(c_1 \cos\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln(x^6)}{6}\right) + c_2 \sin\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln(x^6)}{6}\right)\right)}{6}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{6}\,\sqrt{x^6}\left(c_1 \cos\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln(x^6)}{6}\right) + c_2 \sin\left(-\frac{\ln(2)}{6} - \frac{\ln(3)}{6} + \frac{\ln(x^6)}{6}\right)\right)}{6} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{6}\,\sqrt{x^6}\left(c_1 \cos\left(-\frac{\ln(2)}{6}-\frac{\ln(3)}{6}+\frac{\ln(x^6)}{6}\right)+c_2 \sin\left(-\frac{\ln(2)}{6}-\frac{\ln(3)}{6}+\frac{\ln(x^6)}{6}\right)\right)}{6}$$

Verified OK.

13.10.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' - 5y'x + 10y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= \frac{10}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{10}\,\sqrt{\frac{1}{x^2}}}{c} \qquad (6) \\ \tau'' &= -\frac{\sqrt{10}}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{10}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{5}{x}\,\frac{\sqrt{10}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{10}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{3c\sqrt{10}}{5}
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{3c\sqrt{10}\,\left(\frac{d}{d\tau}y(\tau)\right)}{5} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{3\sqrt{10}\,c\tau}{10}}\left(c_1 \cos\left(\frac{\sqrt{10}\,c\tau}{10}\right) + c_2 \sin\left(\frac{\sqrt{10}\,c\tau}{10}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{10}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{10}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = x^3(c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)))$$

Summary

The solution(s) found are the following

$$y = x^3(c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))) \qquad (1)$$

Verification of solutions

$$y = x^3(c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)))$$

Verified OK.

13.10.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 5y'x + 10y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= \frac{10}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{5n}{x^2} + \frac{10}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 + i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{6+2i}{x} - \frac{5}{x}\right)v'(x) &= 0 \\ v''(x) + \frac{(1+2i)\,v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+2i)\,u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= \frac{(-1-2i)\,u}{x}
\end{aligned}$$

Where $f(x)=\frac{-1-2i}{x}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{-1-2i}{x}\,dx\\
\int \frac{1}{u}\,du &= \int \frac{-1-2i}{x}\,dx\\
\ln(u) &= (-1-2i)\ln(x) + c_1\\
u &= \mathrm{e}^{(-1-2i)\ln(x)+c_1}\\
&= c_1\mathrm{e}^{(-1-2i)\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-2i}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x)\\
v(x) &= \int u(x)\,dx + c_2\\
&= \frac{i x^{-2i} c_1}{2} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n\\
&= \left(\frac{i x^{-2i} c_1}{2} + c_2\right) x^{3+i}\\
&= c_2 x^{3+i} + \frac{i c_1 x^{3-i}}{2}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(\frac{i x^{-2 i} c_1}{2}+c_2\right) x^{3+i} \tag{1}$$

Verification of solutions

$$y = \left(\frac{i x^{-2 i} c_1}{2}+c_2\right) x^{3+i}$$

Verified OK.

13.10.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 5y'x + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -5x \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -5 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{5}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 486: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+i$	$\frac{1}{2}-i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - i$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - i - \left(\frac{1}{2} - i\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2}-i}{x} + (-)\,(0) \\ &= \frac{\frac{1}{2}-i}{x} \\ &= \frac{\frac{1}{2}-i}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + i}{x^2}\right) + \left(\frac{\frac{1}{2} - i}{x}\right)^2 - \left(-\frac{5}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{\frac{1}{2} - i}{x} dx} \\ &= x^{\frac{1}{2} - i} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-5x}{x^2} \, dx} \\ &= z_1 e^{\frac{5\ln(x)}{2}} \\ &= z_1\left(x^{\frac{5}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^{3-i}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-5x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{5\ln(x)}}{(y_1)^2} \, dx \\ &= y_1\left(-\frac{i x^{2i}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{3-i}\right) + c_2 \left(x^{3-i} \left(-\frac{i x^{2i}}{2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{3-i} - \frac{i c_2 x^{3+i}}{2} \tag{1}$$

Verification of solutions

$$y = c_1 x^{3-i} - \frac{i c_2 x^{3+i}}{2}$$

Verified OK.

13.10.6 Maple step by step solution

Let's solve

$y'' x^2 - 5y' x + 10y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{5y'}{x} - \frac{10y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{5y'}{x} + \frac{10y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y'' x^2 - 5y' x + 10y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

- ○ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$
- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$
- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^2 - 5\frac{d}{dt}y(t) + 10y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 6\frac{d}{dt}y(t) + 10y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 - 6r + 10 = 0$
- Use quadratic formula to solve for r

  $r = \frac{6 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (3 - \mathrm{I}, 3 + \mathrm{I})$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{3t} \cos(t)$
- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{3t} \sin(t)$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{3t} \cos(t) + c_2 \mathrm{e}^{3t} \sin(t)$
- Change variables back using $t = \ln(x)$

  $y = c_1 x^3 \cos(\ln(x)) + c_2 x^3 \sin(\ln(x))$
- Simplify

  $y = x^3 (c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)-5*x*diff(y(x),x)+10*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^3(c_1 \sin(\ln(x)) + \cos(\ln(x)) c_2)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]-5*x*y'[x]+10*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3(c_2 \cos(\log(x)) + c_1 \sin(\log(x)))$$

13.11 problem 11

Internal problem ID [11865]
Internal file name [`OUTPUT/11874_Saturday_April_13_2024_01_13_24_AM_53887226/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 11.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

`[[_3rd_order, _with_linear_symmetries]]`

$$\boxed{x^3y''' - 3x^2y'' + 6y'x - 6y = 0}$$

This is Euler ODE of higher order. Let $y = x^{\lambda}$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' - 3x^2y'' + 6y'x - 6y = 0$$

gives

$$6x\lambda\, x^{\lambda-1} - 3x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} - 6x^{\lambda} = 0$$

Which simplifies to

$$6\lambda\, x^{\lambda} - 3\lambda(\lambda-1)\, x^{\lambda} + \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda} - 6x^{\lambda} = 0$$

And since $x^{\lambda} \neq 0$ then dividing through by x^{λ}, the above becomes

$$6\lambda - 3\lambda(\lambda-1) + \lambda(\lambda-1)(\lambda-2) - 6 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

This table summarises the result

root	multiplicity	type of root
1	1	real root
2	1	real root
3	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_3 x^3 + c_2 x^2 + c_1 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = x$$
$$y_2 = x^2$$
$$y_3 = x^3$$

Summary

The solution(s) found are the following

$$y = c_3 x^3 + c_2 x^2 + c_1 x \tag{1}$$

Verification of solutions

$$y = c_3 x^3 + c_2 x^2 + c_1 x$$

Verified OK.

13.11.1 Maple step by step solution

Let's solve

$x^3 y''' - 3y'' x^2 + 6y' x - 6y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{6y}{x^3} + \frac{3(y'' x - 2y')}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{3y''}{x} + \frac{6y'}{x^2} - \frac{6y}{x^3} = 0$

- Multiply by denominators of the ODE

  $x^3 y''' - 3y'' x^2 + 6y' x - 6y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  ◦ Compute derivative

  $y' = \frac{\frac{d}{dt} y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  ◦ Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  ◦ Calculate the 3rd derivative of y with respect to x , using the chain rule

  $y''' = \left(\frac{d^3}{dt^3} y(t)\right) t'(x)^3 + 3t'(x) t''(x) \left(\frac{d^2}{dt^2} y(t)\right) + t'''(x) \left(\frac{d}{dt} y(t)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) - 3\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 + 6\frac{d}{dt}y(t) - 6y(t) = 0$

- Simplify

  $\frac{d^3}{dt^3}y(t) - 6\frac{d^2}{dt^2}y(t) + 11\frac{d}{dt}y(t) - 6y(t) = 0$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = 6y_3(t) - 11y_2(t) + 6y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 6y_3(t) - 11y_2(t) + 6y_1(t)\right]$

- Define vector

  $\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$

- System to solve

  $\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} \cdot \vec{y}(t)$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$

- Rewrite the system as

  $\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{y}_1 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{y}_2 = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c}\frac{1}{9} \\ \frac{1}{3} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_3 = e^{3t} \cdot \left[\begin{array}{c}\frac{1}{9} \\ \frac{1}{3} \\ 1\end{array}\right]$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{y} = c_1 e^{t} \cdot \left[\begin{array}{c}1 \\ 1 \\ 1\end{array}\right] + c_2 e^{2t} \cdot \left[\begin{array}{c}\frac{1}{4} \\ \frac{1}{2} \\ 1\end{array}\right] + c_3 e^{3t} \cdot \left[\begin{array}{c}\frac{1}{9} \\ \frac{1}{3} \\ 1\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y(t) = c_1 e^{t} + \frac{c_2 e^{2t}}{4} + \frac{c_3 e^{3t}}{9}$$

- Change variables back using $t = \ln(x)$

$$y = c_1 x + \frac{1}{4}c_2 x^2 + \frac{1}{9}c_3 x^3$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x^3*diff(y(x),x$3)-3*x^2*diff(y(x),x$2)+6*x*diff(y(x),x)-6*y(x)=0,y(x), singsol=all)
```

$$y(x) = x\left(c_3x^2 + c_2x + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 19

```
DSolve[x^3*y'''[x]-3*x^2*y''[x]+6*x*y'[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(x(c_3x + c_2) + c_1)$$

13.12 problem 12

Internal problem ID [11866]
Internal file name [`OUTPUT/11875_Saturday_April_13_2024_01_13_25_AM_90650534/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 12.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _fully, _exact, _linear]]
```

$$\boxed{x^3y''' + 2x^2y'' - 10y'x - 8y = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' + 2x^2y'' - 10y'x - 8y = 0$$

gives

$$-10x\lambda\, x^{\lambda-1} + 2x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} - 8x^\lambda = 0$$

Which simplifies to

$$-10\lambda\, x^\lambda + 2\lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda - 8x^\lambda = 0$$

And since $x^{\lambda} \neq 0$ then dividing through by x^{λ}, the above becomes

$$-10\lambda + 2\lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) - 8 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - \lambda^2 - 10\lambda - 8 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 4$$
$$\lambda_2 = -2$$
$$\lambda_3 = -1$$

This table summarises the result

root	multiplicity	type of root
-1	1	real root
-2	1	real root
4	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^{\lambda}$ basis solution. Each real root of multiplicty two, generates $c_1 x^{\lambda}$ and $c_2 x^{\lambda} \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^{\lambda}$ and $c_2 x^{\lambda} \ln(x)$ and $c_3 x^{\lambda} \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x) x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2 x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} + c_3 x^4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{1}{x^2}$$
$$y_3 = x^4$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} + c_3 x^4 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2}{x^2} + c_3 x^4$$

Verified OK.

13.12.1 Maple step by step solution

Let's solve

$x^3 y''' + 2y'' x^2 - 10y' x - 8y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = \frac{8y}{x^3} - \frac{2(y''x - 5y')}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' + \frac{2y''}{x} - \frac{10y'}{x^2} - \frac{8y}{x^3} = 0$

- Multiply by denominators of the ODE

  $x^3 y''' + 2y'' x^2 - 10y' x - 8y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  ◦ Compute derivative

$$y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$$

- Calculate the 3rd derivative of y with respect to x , using the chain rule

$$y''' = \left(\frac{d^3}{dt^3}y(t)\right)t'(x)^3 + 3t'(x)\,t''(x)\left(\frac{d^2}{dt^2}y(t)\right) + t'''(x)\left(\frac{d}{dt}y(t)\right)$$

- Compute derivative

$$y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$$

Substitute the change of variables back into the ODE

$$x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) + 2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - 10\frac{d}{dt}y(t) - 8y(t) = 0$$

- Simplify

$$\frac{d^3}{dt^3}y(t) - \frac{d^2}{dt^2}y(t) - 10\frac{d}{dt}y(t) - 8y(t) = 0$$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

$$y_1(t) = y(t)$$

- Define new variable $y_2(t)$

$$y_2(t) = \frac{d}{dt}y(t)$$

- Define new variable $y_3(t)$

$$y_3(t) = \frac{d^2}{dt^2}y(t)$$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

$$\frac{d}{dt}y_3(t) = y_3(t) + 10y_2(t) + 8y_1(t)$$

Convert linear ODE into a system of first order ODEs

$$\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = y_3(t) + 10y_2(t) + 8y_1(t)\right]$$

- Define vector

$$\vec{y}(t) = \left[\begin{array}{c} y_1(t) \\ y_2(t) \\ y_3(t) \end{array}\right]$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 10 & 1 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & 10 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{4t} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2t} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{-t} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{4t} \cdot \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y(t) = \frac{(c_3 e^{6t} + 16 c_2 e^{t} + 4 c_1) e^{-2t}}{16}$$

- Change variables back using $t = \ln(x)$

$$y = \frac{c_3 x^6 + 16 c_2 x + 4 c_1}{16 x^2}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(x^3*diff(y(x),x$3)+2*x^2*diff(y(x),x$2)-10*x*diff(y(x),x)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1x^6 + c_2x + c_3}{x^2}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 22

```
DSolve[x^3*y'''[x]+2*x^2*y''[x]-10*x*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{c_3x^6 + c_2x + c_1}{x^2}$$

13.13 problem 13

Internal problem ID [11867]
Internal file name [`OUTPUT/11876_Saturday_April_13_2024_01_13_25_AM_68307060/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 13.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y''' - x^2y'' - 6y'x + 18y = 0}$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^3y''' - x^2y'' - 6y'x + 18y = 0$$

gives

$$-6x\lambda\, x^{\lambda-1} - x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 18x^\lambda = 0$$

Which simplifies to

$$-6\lambda\, x^\lambda - \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 18x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$
-6\lambda -\lambda(\lambda -1)+\lambda(\lambda -1)(\lambda -2)+18 = 0
$$

Simplifying gives the characteristic equation as

$$
(\lambda +2)(\lambda -3)^{2} = 0
$$

Solving the above gives the following roots

$$
\lambda_1 = -2 \\
\lambda_2 = 3 \\
\lambda_3 = 3
$$

This table summarises the result

root	multiplicity	type of root
-2	1	real root
3	2	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^{\lambda}$ basis solution. Each real root of multiplicty two, generates $c_1 x^{\lambda}$ and $c_2 x^{\lambda} \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^{\lambda}$ and $c_2 x^{\lambda} \ln(x)$ and $c_3 x^{\lambda} \ln(x)^{2}$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^{2} x^{\alpha}(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$
y = \frac{c_1}{x^{2}}+c_2 x^{3}+c_3 \ln(x)\, x^{3}
$$

The fundamental set of solutions for the homogeneous solution are the following

$$
y_1 = \frac{1}{x^{2}} \\
y_2 = x^{3} \\
y_3 = x^{3} \ln(x)
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^{2}}+c_2 x^{3}+c_3 \ln(x)\, x^{3} \tag{1}
$$

Verification of solutions

$$y = \frac{c_1}{x^2} + c_2 x^3 + c_3 \ln(x)\, x^3$$

Verified OK.

13.13.1 Maple step by step solution

Let's solve

$x^3 y''' - y'' x^2 - 6y' x + 18y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- Isolate 3rd derivative

  $y''' = -\frac{18y}{x^3} + \frac{y'' x + 6y'}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''' - \frac{y''}{x} - \frac{6y'}{x^2} + \frac{18y}{x^3} = 0$

- Multiply by denominators of the ODE

  $x^3 y''' - y'' x^2 - 6y' x + 18y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  ◦ Calculate the 3rd derivative of y with respect to x , using the chain rule

    $y''' = \left(\frac{d^3}{dt^3} y(t)\right) t'(x)^3 + 3t'(x)\, t''(x) \left(\frac{d^2}{dt^2} y(t)\right) + t'''(x) \left(\frac{d}{dt} y(t)\right)$

- Compute derivative

  $y''' = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE

  $x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) - \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - 6\frac{d}{dt}y(t) + 18y(t) = 0$

- Simplify

  $\frac{d^3}{dt^3}y(t) - 4\frac{d^2}{dt^2}y(t) - 3\frac{d}{dt}y(t) + 18y(t) = 0$

☐ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(t)$

  $y_1(t) = y(t)$

- Define new variable $y_2(t)$

  $y_2(t) = \frac{d}{dt}y(t)$

- Define new variable $y_3(t)$

  $y_3(t) = \frac{d^2}{dt^2}y(t)$

- Isolate for $\frac{d}{dt}y_3(t)$ using original ODE

  $\frac{d}{dt}y_3(t) = 4y_3(t) + 3y_2(t) - 18y_1(t)$

  Convert linear ODE into a system of first order ODEs

  $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 4y_3(t) + 3y_2(t) - 18y_1(t)\right]$

- Define vector

  $$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

- System to solve

  $$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & 3 & 4 \end{bmatrix} \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & 3 & 4 \end{bmatrix}$$

- Rewrite the system as

  $\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{y}_1 = e^{-2t} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

  $$\left[3, \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 3

  $$\vec{y}_2(t) = e^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, an

$$\overrightarrow{y}_3(t) = \mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Note that the t multiplying $\overrightarrow{v}$ makes this solution linearly independent to the 1st solution obta
- Substitute $\overrightarrow{y}_3(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \left(\mathrm{e}^{\lambda t}A\right)\cdot\left(t\overrightarrow{v} + \overrightarrow{p}\right)$$

- Use the fact that $\overrightarrow{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\overrightarrow{v} + \overrightarrow{p}\right) + \mathrm{e}^{\lambda t}\overrightarrow{v} = \mathrm{e}^{\lambda t}\left(\lambda t\overrightarrow{v} + A\cdot\overrightarrow{p}\right)$$

- Simplify equation

$$\lambda\overrightarrow{p} + \overrightarrow{v} = A\cdot\overrightarrow{p}$$

- Make use of the identity matrix I

$$(\lambda\cdot I)\cdot\overrightarrow{p} + \overrightarrow{v} = A\cdot\overrightarrow{p}$$

- Condition $\overrightarrow{p}$ must meet for $\overrightarrow{y}_3(t)$ to be a solution to the homogeneous system

$$(A - \lambda\cdot I)\cdot\overrightarrow{p} = \overrightarrow{v}$$

- Choose $\overrightarrow{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

$$\left(\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & 3 & 4 \end{array}\right] - 3\cdot\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right)\cdot\overrightarrow{p} = \left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

$$\overrightarrow{p} = \left[\begin{array}{c} -\frac{1}{27} \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 3

$$\overrightarrow{y}_3(t) = \mathrm{e}^{3t}\cdot\left(t\cdot\left[\begin{array}{c} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{array}\right] + \left[\begin{array}{c} -\frac{1}{27} \\ 0 \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

$$\overrightarrow{y} = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2(t) + c_3\overrightarrow{y}_3(t)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2t} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + c_3 e^{3t} \cdot \left(t \cdot \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{27} \\ 0 \\ 0 \end{bmatrix} \right)$$

- First component of the vector is the solution to the ODE

  $y(t) = \frac{e^{-2t}\left(\left(\left(t-\frac{1}{3}\right)c_3+c_2\right)e^{5t}+\frac{9c_1}{4}\right)}{9}$

- Change variables back using $t = \ln(x)$

  $y = \frac{\left(\left(\ln(x)-\frac{1}{3}\right)c_3+c_2\right)x^5+\frac{9c_1}{4}}{9x^2}$

- Simplify

  $y = \frac{c_3 \ln(x) x^3}{9} + \frac{c_2 x^3}{9} - \frac{c_3 x^3}{27} + \frac{c_1}{4x^2}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(x^3*diff(y(x),x$3)-x^2*diff(y(x),x$2)-6*x*diff(y(x),x)+18*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_3 x^5 \ln(x) + c_2 x^5 + c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 26

```
DSolve[x^3*y'''[x]-x^2*y''[x]-6*x*y'[x]+18*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^5 + c_3 x^5 \log(x) + c_1}{x^2}$$

13.14 problem 14

Internal problem ID [11868]
Internal file name [`OUTPUT/11877_Saturday_April_13_2024_01_13_26_AM_73878653/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 4y'x + 6y = 4x - 6}$$

13.14.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 6, f(x) = 4x - 6$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 4y'x + 6y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^3 + c_1x^2$$

Next, we find the particular solution to the ODE

$$x^2y'' - 4y'x + 6y = 4x - 6$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x^2)(3x^2) - (x^3)(2x)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3(4x-6)}{x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{4x-6}{x^3} dx$$

Hence

$$u_1 = -\frac{3}{x^2} + \frac{4}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(4x-6)}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \frac{4x-6}{x^4} dx$$

Hence

$$u_2 = -\frac{2}{x^2} + \frac{2}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{3}{x^2} + \frac{4}{x}\right) x^2 + \left(-\frac{2}{x^2} + \frac{2}{x^3}\right) x^3$$

Which simplifies to

$$y_p(x) = 2x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_2 x^3 + c_1 x^2 + 2x - 1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2x^3 + c_1x^2 + 2x - 1 \tag{1}$$

Verification of solutions

$$y = c_2x^3 + c_1x^2 + 2x - 1$$

Verified OK.

13.14.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -\frac{4}{x}\,dx} \\ &= \frac{1}{x^2} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \frac{4x-6}{x^4} \\ \left(\frac{y}{x^2}\right)'' &= \frac{4x-6}{x^4} \end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x^2}\right)' = \frac{-2x+2}{x^3} + c_1$$

Integrating again gives

$$\left(\frac{y}{x^2}\right) = \frac{c_1x^3 + 2x - 1}{x^2} + c_2$$

Hence the solution is

$$y = \frac{\frac{c_1x^3+2x-1}{x^2} + c_2}{\frac{1}{x^2}}$$

Or

$$y = c_1 x^3 + c_2 x^2 + 2x - 1$$

Summary
The solution(s) found are the following

$$y = c_1 x^3 + c_2 x^2 + 2x - 1 \tag{1}$$

Verification of solutions

$$y = c_1 x^3 + c_2 x^2 + 2x - 1$$

Verified OK.

13.14.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2 y'' - 4y'x + 6y = 0$$

In normal form the ode

$$x^2 y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{4}{x}dx\right)}dx \\
&= \int e^{4\ln(x)}\,dx \\
&= \int x^4 dx \\
&= \frac{x^5}{5}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{6}{x^2}}{x^8} \\
&= \frac{6}{x^{10}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^{10}} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{6}{x^{10}} = \frac{6}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 6 = 0$$

Or

$$25r^2 - 25r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{2}{5}$$
$$r_2 = \frac{3}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{2}{5}} + c_2\tau^{\frac{3}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{3}{5}}\left(x^5\right)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}}\left(x^5\right)^{\frac{3}{5}}}{5}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x^5)^{\frac{2}{5}}$$
$$y_2 = (x^5)^{\frac{3}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^5)^{\frac{2}{5}} & (x^5)^{\frac{3}{5}} \\ \frac{d}{dx}\left((x^5)^{\frac{2}{5}}\right) & \frac{d}{dx}\left((x^5)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^5)^{\frac{2}{5}} & (x^5)^{\frac{3}{5}} \\ \frac{2x^4}{(x^5)^{\frac{3}{5}}} & \frac{3x^4}{(x^5)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^5\right)^{\frac{2}{5}}\right)\left(\frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}}\right) - \left(\left(x^5\right)^{\frac{3}{5}}\right)\left(\frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x^5\right)^{\frac{3}{5}}\left(4x-6\right)}{x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(x^5\right)^{\frac{3}{5}}\left(4x-6\right)}{x^6} dx$$

Hence

$$u_1 = \frac{\left(x^5\right)^{\frac{3}{5}}\left(-3+4x\right)}{x^5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(x^5\right)^{\frac{2}{5}}\left(4x-6\right)}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\left(x^5\right)^{\frac{2}{5}}\left(4x-6\right)}{x^6} dx$$

Hence

$$u_2 = -\frac{2\left(x^5\right)^{\frac{2}{5}}\left(x-1\right)}{x^5}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 2x - 1$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(\frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}\right) + (2x - 1)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} + 2x - 1 \tag{1}$$

Verification of solutions

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} + 2x - 1$$

Verified OK.

13.14.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 6, f(x) = 4x - 6$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 4y'x + 6y = 0$$

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2}\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{5c\sqrt{6}}{6}
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{5c\sqrt{6}\left(\frac{d}{d\tau}y(\tau)\right)}{6} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\,c\tau}{12}}\left(c_1 \cosh\left(\frac{\sqrt{6}\,c\tau}{12}\right) + ic_2 \sinh\left(\frac{\sqrt{6}\,c\tau}{12}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{6}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{5}{2}}\left(c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Now the particular solution to this ODE is found

$$
x^2y'' - 4y'x + 6y = 4x - 6
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \left(x^5\right)^{\frac{2}{5}} \\
y_2 &= \left(x^5\right)^{\frac{3}{5}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^5)^{\frac{2}{5}} & (x^5)^{\frac{3}{5}} \\ \frac{d}{dx}\left((x^5)^{\frac{2}{5}}\right) & \frac{d}{dx}\left((x^5)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^5)^{\frac{2}{5}} & (x^5)^{\frac{3}{5}} \\ \frac{2x^4}{(x^5)^{\frac{3}{5}}} & \frac{3x^4}{(x^5)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left((x^5)^{\frac{2}{5}}\right)\left(\frac{3x^4}{(x^5)^{\frac{2}{5}}}\right) - \left((x^5)^{\frac{3}{5}}\right)\left(\frac{2x^4}{(x^5)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}}(4x-6)}{x^6}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}}(4x-6)}{x^6}dx$$

Hence

$$u_1 = \frac{(x^5)^{\frac{3}{5}}(-3+4x)}{x^5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}}(4x-6)}{x^6}\,dx$$

Which simplifies to

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}} (4x-6)}{x^6} dx$$

Hence

$$u_2 = -\frac{2(x^5)^{\frac{2}{5}} (x-1)}{x^5}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 2x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)\right) + (2x-1) \\ &= 2x - 1 + x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) \end{aligned}$$

Which simplifies to

$$y = i x^{\frac{5}{2}} \sinh\left(\frac{\ln(x)}{2}\right) c_2 + x^{\frac{5}{2}} \cosh\left(\frac{\ln(x)}{2}\right) c_1 + 2x - 1$$

Summary

The solution(s) found are the following

$$y = i x^{\frac{5}{2}} \sinh\left(\frac{\ln(x)}{2}\right) c_2 + x^{\frac{5}{2}} \cosh\left(\frac{\ln(x)}{2}\right) c_1 + 2x - 1 \tag{1}$$

Verification of solutions

$$y = i x^{\frac{5}{2}} \sinh\left(\frac{\ln(x)}{2}\right) c_2 + x^{\frac{5}{2}} \cosh\left(\frac{\ln(x)}{2}\right) c_1 + 2x - 1$$

Verified OK.

13.14.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 4y'x + 6y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\ &= \frac{6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2}dx\right)} \\ &= e^{-\int \frac{-\frac{4}{x}}{2}} \\ &= x^2 \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x^2 \tag{4}$$

Applying this change of variable to the original ode results in

$$x^4 v''(x) = 4x - 6$$

Which is now solved for $v(x)$ Simplyfing the ode gives

$$v''(x) = \frac{4x - 6}{x^4}$$

Integrating once gives

$$v'(x) = -\frac{2}{x^2} + \frac{2}{x^3} + c_1$$

Integrating again gives

$$v(x) = \frac{2}{x} - \frac{1}{x^2} + c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 x + \frac{2}{x} - \frac{1}{x^2} + c_2\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = \left(c_1 x + \frac{2}{x} - \frac{1}{x^2} + c_2\right) x^2$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 x + \frac{2}{x} - \frac{1}{x^2} + c_2\right) x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^5\right)^{\frac{2}{5}}$$

$$y_2 = \left(x^5\right)^{\frac{3}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{d}{dx}\left(\left(x^5\right)^{\frac{2}{5}}\right) & \frac{d}{dx}\left(\left(x^5\right)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}} & \frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^5\right)^{\frac{2}{5}}\right)\left(\frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}}\right) - \left(\left(x^5\right)^{\frac{3}{5}}\right)\left(\frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}} (4x-6)}{x^6} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}} (4x-6)}{x^6} dx$$

Hence

$$u_1 = \frac{(x^5)^{\frac{3}{5}} (-3+4x)}{x^5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}} (4x-6)}{x^6} \, dx$$

Which simplifies to

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}} (4x-6)}{x^6} dx$$

Hence

$$u_2 = -\frac{2(x^5)^{\frac{2}{5}} (x-1)}{x^5}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 2x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1 x + \frac{2}{x} - \frac{1}{x^2} + c_2\right) x^2\right) + (2x - 1) \end{aligned}$$

Which simplifies to

$$y = c_1 x^3 + c_2 x^2 + 4x - 2$$

Summary
The solution(s) found are the following

$$y = c_1x^3 + c_2x^2 + 4x - 2 \tag{1}$$

Verification of solutions

$$y = c_1x^3 + c_2x^2 + 4x - 2$$

Verified OK.

13.14.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 6, f(x) = 4x - 6$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 4y'x + 6y = 0$$

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{6}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\ \ln(u) &= -2\ln(x) + c_1 \\ u &= e^{-2\ln(x)+c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}v'(x) &= u(x)\\ v(x) &= \int u(x)\, dx + c_2\\ &= -\frac{c_1}{x} + c_2\end{aligned}$$

Hence

$$\begin{aligned}y &= v(x)\, x^n\\ &= \left(-\frac{c_1}{x} + c_2\right) x^3\\ &= x^2(c_2 x - c_1)\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' - 4y'x + 6y = 4x - 6$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= x^2\\ y_2 &= x^3\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x^2)(3x^2) - (x^3)(2x)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3(4x-6)}{x^6}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{4x-6}{x^3}dx$$

Hence

$$u_1 = -\frac{3}{x^2} + \frac{4}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(4x-6)}{x^6}\,dx$$

Which simplifies to

$$u_2 = \int \frac{4x - 6}{x^4} dx$$

Hence

$$u_2 = -\frac{2}{x^2} + \frac{2}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{3}{x^2} + \frac{4}{x}\right) x^2 + \left(-\frac{2}{x^2} + \frac{2}{x^3}\right) x^3$$

Which simplifies to

$$y_p(x) = 2x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{x} + c_2\right) x^3\right) + (2x - 1) \\ &= 2x - 1 + \left(-\frac{c_1}{x} + c_2\right) x^3 \end{aligned}$$

Which simplifies to

$$y = c_2 x^3 - c_1 x^2 + 2x - 1$$

Summary

The solution(s) found are the following

$$y = c_2 x^3 - c_1 x^2 + 2x - 1 \tag{1}$$

Verification of solutions

$$y = c_2 x^3 - c_1 x^2 + 2x - 1$$

Verified OK.

13.14.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 491: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{2\ln(x)} \\
&= z_1 \left(x^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{\left(y_1\right)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2\right) + c_2 \left(x^2(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' - 4y'x + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_2 x^3 + c_1 x^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x^2)(3x^2) - (x^3)(2x)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3(4x-6)}{x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{4x-6}{x^3} dx$$

Hence

$$u_1 = -\frac{3}{x^2} + \frac{4}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(4x-6)}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \frac{4x-6}{x^4} dx$$

Hence

$$u_2 = -\frac{2}{x^2} + \frac{2}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{3}{x^2} + \frac{4}{x}\right) x^2 + \left(-\frac{2}{x^2} + \frac{2}{x^3}\right) x^3$$

Which simplifies to

$$y_p(x) = 2x - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_2 x^3 + c_1 x^2\right) + (2x - 1) \end{aligned}$$

Which simplifies to

$$y = x^2(c_2x + c_1) + 2x - 1$$

Summary

The solution(s) found are the following

$$y = x^2(c_2x + c_1) + 2x - 1 \tag{1}$$

Verification of solutions

$$y = x^2(c_2x + c_1) + 2x - 1$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=4*x-6,y(x), singsol=all)
```

$$y(x) = c_1x^3 + c_2x^2 + 2x - 1$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]-4*x*y'[x]+6*y[x]==4*x-6,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2x^3 + c_1x^2 + 2x - 1$$

13.15 problem 15

Internal problem ID [11869]
Internal file name [`OUTPUT/11878_Saturday_April_13_2024_01_13_28_AM_92244890/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 5y'x + 8y = 2x^3}$$

13.15.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -5x, C = 8, f(x) = 2x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 5y'x + 8y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 5xrx^{r-1} + 8x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 5r\,x^r + 8x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 5r + 8 = 0$$

Or

$$r^2 - 6r + 8 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 4$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2 x^4 + c_1 x^2$$

Next, we find the particular solution to the ODE

$$x^2 y'' - 5y'x + 8y = 2x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = x^4$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^4 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^4) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{vmatrix}$$

Therefore

$$W = (x^2)(4x^3) - (x^4)(2x)$$

Which simplifies to

$$W = 2x^5$$

Which simplifies to

$$W = 2x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2x^7}{2x^7}\,dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^5}{2x^7}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2} dx$$

Hence

$$u_2 = -\frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x^2\left(c_2x^2 + c_1 - 2x\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x^2\left(c_2x^2 + c_1 - 2x\right) \tag{1}$$

Verification of solutions

$$y = x^2\left(c_2x^2 + c_1 - 2x\right)$$

Verified OK.

13.15.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 5y'x + 8y = 0$$

In normal form the ode

$$x^2y'' - 5y'x + 8y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= \frac{8}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}\,dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{5}{x}dx\right)}\,dx \\ &= \int e^{5\ln(x)}\,dx \\ &= \int x^5 dx \\ &= \frac{x^6}{6} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{8}{x^2}}{x^{10}} \\
&= \frac{8}{x^{12}}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{8y(\tau)}{x^{12}} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{8}{x^{12}} = \frac{2}{9\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0
$$

Simplifying gives

$$
9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
9r(r-1) + 0 + 2 = 0
$$

Or

$$
9r^2 - 9r + 2 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{1}{3}} + c_2 \tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 6^{\frac{2}{3}} (x^6)^{\frac{1}{3}}}{6} + \frac{c_2 6^{\frac{1}{3}} (x^6)^{\frac{2}{3}}}{6}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 6^{\frac{2}{3}} (x^6)^{\frac{1}{3}}}{6} + \frac{c_2 6^{\frac{1}{3}} (x^6)^{\frac{2}{3}}}{6}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x^6)^{\frac{1}{3}}$$
$$y_2 = (x^6)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^6)^{\frac{1}{3}} & (x^6)^{\frac{2}{3}} \\ \frac{d}{dx}\left((x^6)^{\frac{1}{3}}\right) & \frac{d}{dx}\left((x^6)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^6)^{\frac{1}{3}} & (x^6)^{\frac{2}{3}} \\ \frac{2x^5}{(x^6)^{\frac{2}{3}}} & \frac{4x^5}{(x^6)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((x^6)^{\frac{1}{3}}\right)\left(\frac{4x^5}{(x^6)^{\frac{1}{3}}}\right) - \left((x^6)^{\frac{2}{3}}\right)\left(\frac{2x^5}{(x^6)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = 2x^5$$

Which simplifies to

$$W = 2x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2(x^6)^{\frac{2}{3}}\, x^3}{2x^7}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^6)^{\frac{2}{3}}}{x^4} dx$$

Hence

$$u_1 = -\frac{(x^6)^{\frac{2}{3}}}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2(x^6)^{\frac{1}{3}}\, x^3}{2x^7}\, dx$$

Which simplifies to

$$u_2 = \int \frac{(x^6)^{\frac{1}{3}}}{x^4} dx$$

Hence

$$u_2 = -\frac{(x^6)^{\frac{1}{3}}}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 6^{\frac{2}{3}} (x^6)^{\frac{1}{3}}}{6} + \frac{c_2 6^{\frac{1}{3}} (x^6)^{\frac{2}{3}}}{6}\right) + (-2x^3) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 6^{\frac{2}{3}} (x^6)^{\frac{1}{3}}}{6} + \frac{c_2 6^{\frac{1}{3}} (x^6)^{\frac{2}{3}}}{6} - 2x^3 \tag{1}$$

Verification of solutions

$$y = \frac{c_1 6^{\frac{2}{3}} (x^6)^{\frac{1}{3}}}{6} + \frac{c_2 6^{\frac{1}{3}} (x^6)^{\frac{2}{3}}}{6} - 2x^3$$

Verified OK.

13.15.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -5x, C = 8, f(x) = 2x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$x^2y''-5y'x+8y=0$$

In normal form the ode

$$x^2y''-5y'x+8y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=-\frac{5}{x}\\ q(x)&=\frac{8}{x^2}\end{aligned}$$

Applying change of variables $\tau=g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau)=\frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1=c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau'&=\frac{1}{c}\sqrt{q}\\ &=\frac{2\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c} \qquad (6)\\ \tau''&=-\frac{2\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \\ &= \frac{-\frac{2\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{5}{x}\,\frac{2\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{3c\sqrt{2}}{2} \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{3c\sqrt{2}\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1 \cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2 \sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int 2\sqrt{2}\,\sqrt{\frac{1}{x^2}}dx}{c} \\ &= \frac{2\sqrt{2}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{((ic_2 + c_1)\,x^2 - ic_2 + c_1)\,x^2}{2}$$

Now the particular solution to this ODE is found

$$x^2y'' - 5y'x + 8y = 2x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x^6)^{\frac{1}{3}}$$
$$y_2 = (x^6)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^6)^{\frac{1}{3}} & (x^6)^{\frac{2}{3}} \\ \frac{d}{dx}\left((x^6)^{\frac{1}{3}}\right) & \frac{d}{dx}\left((x^6)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^6)^{\frac{1}{3}} & (x^6)^{\frac{2}{3}} \\ \frac{2x^5}{(x^6)^{\frac{2}{3}}} & \frac{4x^5}{(x^6)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((x^6)^{\frac{1}{3}}\right)\left(\frac{4x^5}{(x^6)^{\frac{1}{3}}}\right) - \left((x^6)^{\frac{2}{3}}\right)\left(\frac{2x^5}{(x^6)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = 2x^5$$

Which simplifies to

$$W = 2x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\left(x^6\right)^{\frac{2}{3}} x^3}{2x^7} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(x^6\right)^{\frac{2}{3}}}{x^4} dx$$

Hence

$$u_1 = -\frac{\left(x^6\right)^{\frac{2}{3}}}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\left(x^6\right)^{\frac{1}{3}} x^3}{2x^7} \, dx$$

Which simplifies to

$$u_2 = \int \frac{\left(x^6\right)^{\frac{1}{3}}}{x^4} dx$$

Hence

$$u_2 = -\frac{\left(x^6\right)^{\frac{1}{3}}}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{\left(\left(ic_2 + c_1\right) x^2 - ic_2 + c_1\right) x^2}{2}\right) + \left(-2x^3\right) \\ &= -2x^3 + \frac{\left(\left(ic_2 + c_1\right) x^2 - ic_2 + c_1\right) x^2}{2} \end{aligned}$$

Which simplifies to

$$y = \frac{x^2((ic_2 + c_1) x^2 - 4x - ic_2 + c_1)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2((ic_2 + c_1) x^2 - 4x - ic_2 + c_1)}{2} \tag{1}$$

Verification of solutions

$$y = \frac{x^2((ic_2 + c_1) x^2 - 4x - ic_2 + c_1)}{2}$$

Verified OK.

13.15.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -5x, C = 8, f(x) = 2x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 5y'x + 8y = 0$$

In normal form the ode

$$x^2y'' - 5y'x + 8y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{5}{x}$$
$$q(x) = \frac{8}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{5n}{x^2} + \frac{8}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 4 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u} \, du &= -\frac{3}{x} \, dx \\
\int \frac{1}{u} \, du &= \int -\frac{3}{x} \, dx \\
\ln(u) &= -3\ln(x) + c_1 \\
u &= \mathrm{e}^{-3\ln(x)+c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x) \, dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x) \, x^n \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x^4 \\
&= c_2 x^4 - \frac{1}{2} c_1 x^2
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 5y'x + 8y = 2x^3
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= x^4
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^4 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^4) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{vmatrix}$$

Therefore

$$W = (x^2)(4x^3) - (x^4)(2x)$$

Which simplifies to

$$W = 2x^5$$

Which simplifies to

$$W = 2x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2x^7}{2x^7} \, dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^5}{2x^7}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2} dx$$

Hence

$$u_2 = -\frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x^4\right) + \left(-2x^3\right) \\ &= -2x^3 + \left(-\frac{c_1}{2x^2} + c_2\right) x^4 \end{aligned}$$

Which simplifies to

$$y = -\frac{x^2(-2c_2x^2 + c_1 + 4x)}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^2(-2c_2x^2 + c_1 + 4x)}{2} \tag{1}$$

Verification of solutions

$$y = -\frac{x^2(-2c_2x^2 + c_1 + 4x)}{2}$$

Verified OK.

13.15.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 5y'x + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -5x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 492: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x}{x^2}\, dx} \\
&= z_1 e^{\frac{5\ln(x)}{2}} \\
&= z_1 \left(x^{\frac{5}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{5\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2\right) + c_2 \left(x^2 \left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' - 5y'x + 8y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x^2 + \frac{1}{2} c_2 x^4$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = \frac{x^4}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \frac{x^4}{2} \\ \frac{d}{dx}(x^2) & \frac{d}{dx}\left(\frac{x^4}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \frac{x^4}{2} \\ 2x & 2x^3 \end{vmatrix}$$

Therefore

$$W = (x^2)(2x^3) - \left(\frac{x^4}{2}\right)(2x)$$

Which simplifies to

$$W = x^5$$

Which simplifies to

$$W = x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^7}{x^7}\, dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^5}{x^7}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x^2} dx$$

Hence

$$u_2 = -\frac{2}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2x^3$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 x^2 + \frac{1}{2} c_2 x^4\right) + \left(-2x^3\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2 + \frac{1}{2} c_2 x^4 - 2x^3 \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2 + \frac{1}{2} c_2 x^4 - 2x^3
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve(x^2*diff(y(x),x$2)-5*x*diff(y(x),x)+8*y(x)=2*x^3,y(x), singsol=all)
```

$$
y(x) = x^2 \left(c_2 x^2 + c_1 - 2x\right)
$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 21

```
DSolve[x^2*y''[x]-5*x*y'[x]+8*y[x]==2*x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(c_2x^2 - 2x + c_1)$$

13.16 problem 16

Internal problem ID [11870]
Internal file name [`OUTPUT/11879_Saturday_April_13_2024_01_13_30_AM_60665861/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

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[[_2nd_order, _exact, _linear, _nonhomogeneous]]
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$$\boxed{x^2y'' + 4y'x + 2y = 4\ln(x)}$$

13.16.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 2, f(x) = 4\ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 4y'x + 2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 4xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 4r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 4r + 2 = 0$$

Or

$$r^2 + 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -2$$
$$r_2 = -1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + \frac{c_2}{x}$$

Next, we find the particular solution to the ODE

$$x^2y'' + 4y'x + 2y = 4\ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ -\frac{2}{x^3} & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\ln(x)}{x}}{\frac{1}{x^2}}\, dx$$

Which simplifies to

$$u_1 = -\int 4x \ln (x)\, dx$$

Hence

$$u_1 = -2x^2 \ln (x) + x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\ln(x)}{x^2}}{\frac{1}{x^2}}\, dx$$

Which simplifies to

$$u_2 = \int 4 \ln (x)\, dx$$

Hence

$$u_2 = 4x \ln (x) - 4x$$

Which simplifies to

$$u_1 = x^2(1 - 2\ln (x))$$
$$u_2 = 4x \ln (x) - 4x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 1 - 2\ln (x) + \frac{4x \ln (x) - 4x}{x}$$

Which simplifies to

$$y_p(x) = -3 + 2\ln (x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= -3 + 2\ln(x) + \frac{c_1}{x^2} + \frac{c_2}{x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = -3 + 2\ln(x) + \frac{c_1}{x^2} + \frac{c_2}{x} \tag{1}
$$

Verification of solutions

$$
y = -3 + 2\ln(x) + \frac{c_1}{x^2} + \frac{c_2}{x}
$$

Verified OK.

13.16.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$
y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)
$$

Where $p(x) = \frac{4}{x}$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int \frac{4}{x}\,dx} \\
&= x^2
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 4\ln(x) \\
\left(x^2 y\right)'' &= 4\ln(x)
\end{aligned}
$$

Integrating once gives

$$
\left(x^2 y\right)' = 4x\ln(x) - 4x + c_1
$$

Integrating again gives

$$
\left(x^2 y\right) = x(2x\ln(x) + c_1 - 3x) + c_2
$$

Hence the solution is

$$y = \frac{x(2x \ln(x) + c_1 - 3x) + c_2}{x^2}$$

Or

$$y = 2\ln(x) + \frac{c_1}{x} + \frac{c_2}{x^2} - 3$$

<u>Summary</u>
The solution(s) found are the following

$$y = 2\ln(x) + \frac{c_1}{x} + \frac{c_2}{x^2} - 3 \tag{1}$$

<u>Verification of solutions</u>

$$y = 2\ln(x) + \frac{c_1}{x} + \frac{c_2}{x^2} - 3$$

Verified OK.

13.16.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 4y'x + 2y = 0$$

In normal form the ode

$$x^2y'' + 4y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{4}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int e^{-\left(\int p(x)dx\right)}\,dx \\ &= \int e^{-\left(\int \frac{4}{x}dx\right)}\,dx \\ &= \int e^{-4\ln(x)}\,dx \\ &= \int \frac{1}{x^4}dx \\ &= -\frac{1}{3x^3}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{2}{x^2}}{\frac{1}{x^8}} \\ &= 2x^6\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + 2x^6 y(\tau) &= 0\end{aligned}$$

But in terms of τ

$$2x^6 = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{3}} + c_2\tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}}\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}}\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}$$

$$y_2 = \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4} & \frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right)\left(\frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4}\right) - \left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right)\left(\frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln(x)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_1 = -\int 4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln(x)\, x^2 dx$$

Hence

$$u_1 = -4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3 \ln(x) + 4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \ln(x)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_2 = \int 4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \ln(x)\, x^2 dx$$

Hence

$$u_2 = 2\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3 \ln(x) - \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3$$

Which simplifies to

$$u_1 = -4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3(\ln(x) - 1)$$

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3(-1 + 2\ln(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -3 + 2\ln(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3}\right) + (-3 + 2\ln(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3} - 3 + 2\ln(x) \tag{1}$$

Verification of solutions

$$y = \frac{c_1 3^{\frac{2}{3}} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}}{3} - 3 + 2\ln(x)$$

Verified OK.

13.16.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 2, f(x) = 4\ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 4y'x + 2y = 0$$

In normal form the ode

$$x^2y'' + 4y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{4}{x}\frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{2}\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= \frac{3c\sqrt{2}}{2}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{3c\sqrt{2}\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1\cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2\sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{2}\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{2}\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}
$$

Now the particular solution to this ODE is found

$$
x^2y'' + 4y'x + 2y = 4\ln(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}$$

$$y_2 = \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4} & \frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right)\left(\frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4}\right) - \left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right)\left(\frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln(x)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_1 = -\int 4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln(x)\, x^2 dx$$

Hence

$$u_1 = -4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3 \ln(x) + 4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \ln(x)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_2 = \int 4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \ln(x)\, x^2 dx$$

Hence

$$u_2 = 2\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3 \ln(x) - \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3$$

Which simplifies to

$$u_1 = -4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3(\ln(x) - 1)$$

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3(-1 + 2\ln(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -3 + 2\ln(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}\right) + (-3 + 2\ln(x)) \\ &= -3 + 2\ln(x) + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}} \end{aligned}$$

Which simplifies to

$$y = -3 + 2\ln(x) + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = -3 + 2\ln(x) + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}} \tag{1}$$

Verification of solutions

$$y = -3 + 2\ln(x) + \frac{c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)}{x^{\frac{3}{2}}}$$

Verified OK.

13.16.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 4y'x + 2y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(\frac{4}{x}\right)'}{2} - \frac{\left(\frac{4}{x}\right)^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(-\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\ &= \frac{2}{x^2} - \left(-\frac{2}{x^2}\right) - \frac{4}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{\frac{4}{x}}{2}} \\ &= \frac{1}{x^2} \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{x^2} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) = 4 \ln(x)$$

Which is now solved for $v(x)$ Integrating once gives

$$v'(x) = 4x\ln(x) - 4x + c_1$$

Integrating again gives

$$v(x) = 2x^2\ln(x) - 3x^2 + c_1x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(-3x^2 + c_1x + 2x^2\ln(x) + c_2\right)(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x^2}$$

Hence (7) becomes

$$y = \frac{-3x^2 + c_1x + 2x^2\ln(x) + c_2}{x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{-3x^2 + c_1x + 2x^2\ln(x) + c_2}{x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}}$$

$$y_2 = \left(-\frac{1}{x^3}\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} & \left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \\ \frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4} & \frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4} \end{vmatrix}$$

Therefore

$$W = \left(\left(-\frac{1}{x^3}\right)^{\frac{1}{3}}\right)\left(\frac{2}{\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^4}\right) - \left(\left(-\frac{1}{x^3}\right)^{\frac{2}{3}}\right)\left(\frac{1}{\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln(x)}{\frac{1}{x^2}}\, dx$$

Which simplifies to

$$u_1 = -\int 4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} \ln(x)\, x^2 dx$$

Hence

$$u_1 = -4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3 \ln(x) + 4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \ln(x)}{\frac{1}{x^2}} dx$$

Which simplifies to

$$u_2 = \int 4\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} \ln(x)\, x^2 dx$$

Hence

$$u_2 = 2\left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3 \ln(x) - \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3$$

Which simplifies to

$$u_1 = -4\left(-\frac{1}{x^3}\right)^{\frac{2}{3}} x^3(\ln(x) - 1)$$

$$u_2 = \left(-\frac{1}{x^3}\right)^{\frac{1}{3}} x^3(-1 + 2\ln(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -3 + 2\ln(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{-3x^2 + c_1 x + 2x^2 \ln(x) + c_2}{x^2}\right) + (-3 + 2\ln(x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{-3x^2 + c_1 x + 2x^2 \ln(x) + c_2}{x^2} - 3 + 2\ln(x) \tag{1}$$

Verification of solutions

$$y = \frac{-3x^2 + c_1 x + 2x^2 \ln(x) + c_2}{x^2} - 3 + 2\ln(x)$$

Verified OK.

13.16.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 4x, C = 2, f(x) = 4\ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 4y'x + 2y = 0$$

In normal form the ode

$$x^2y'' + 4y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{4}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{4n}{x^2} + \frac{2}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \frac{2v'(x)}{x} = 0$$
$$v''(x) + \frac{2v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{2}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{2}{x}\, dx \\ \ln(u) &= -2\ln(x) + c_1 \\ u &= \mathrm{e}^{-2\ln(x)+c_1} \\ &= \frac{c_1}{x^2} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{x} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \frac{-\frac{c_1}{x} + c_2}{x} \\
&= \frac{c_2 x - c_1}{x^2}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + 4y' x + 2y = 4 \ln(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x^2} \\
y_2 &= \frac{1}{x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ -\frac{2}{x^3} & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\ln(x)}{x}}{\frac{1}{x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int 4x \ln(x)\,dx$$

Hence

$$u_1 = -2x^2 \ln(x) + x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\ln(x)}{x^2}}{\frac{1}{x^2}}\,dx$$

Which simplifies to

$$u_2 = \int 4\ln(x)\,dx$$

Hence

$$u_2 = 4x\ln(x) - 4x$$

Which simplifies to

$$u_1 = x^2(1 - 2\ln(x))$$
$$u_2 = 4x\ln(x) - 4x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 1 - 2\ln(x) + \frac{4x\ln(x) - 4x}{x}$$

Which simplifies to

$$y_p(x) = -3 + 2\ln(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{-\frac{c_1}{x} + c_2}{x}\right) + (-3 + 2\ln(x)) \\ &= -3 + 2\ln(x) + \frac{-\frac{c_1}{x} + c_2}{x} \end{aligned}$$

Which simplifies to

$$y = -3 + 2\ln(x) + \frac{-\frac{c_1}{x} + c_2}{x}$$

Summary
The solution(s) found are the following

$$y = -3 + 2\ln(x) + \frac{-\frac{c_1}{x} + c_2}{x} \tag{1}$$

Verification of solutions

$$y = -3 + 2\ln(x) + \frac{-\frac{c_1}{x} + c_2}{x}$$

Verified OK.

13.16.7 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2 y'' + 4y'x + 2y)\, dx = \int 4\ln(x)\, dx$$
$$x^2 y' + 2yx = 4x \ln(x) - 4x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{4x\ln(x) - 4x + c_1}{x^2}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{4x\ln(x) - 4x + c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{x} dx}$$
$$= x^2$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{4x\ln(x) - 4x + c_1}{x^2}\right)$$
$$\frac{d}{dx}(x^2 y) = (x^2)\left(\frac{4x\ln(x) - 4x + c_1}{x^2}\right)$$
$$d(x^2 y) = (4x\ln(x) - 4x + c_1)\, dx$$

Integrating gives

$$x^2 y = \int 4x\ln(x) - 4x + c_1\, dx$$
$$x^2 y = -3x^2 + c_1 x + 2x^2 \ln(x) + c_2$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{-3x^2 + c_1 x + 2x^2 \ln(x)}{x^2} + \frac{c_2}{x^2}$$

which simplifies to

$$y = \frac{-3x^2 + c_1 x + 2x^2 \ln(x) + c_2}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-3x^2 + c_1 x + 2x^2 \ln(x) + c_2}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{-3x^2 + c_1 x + 2x^2 \ln(x) + c_2}{x^2}$$

Verified OK.

13.16.8 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2 y'' + 4y'x + 2y = 4\ln(x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2 y'' + 4y'x + 2y)\, dx = \int 4\ln(x)\, dx$$
$$x^2 y' + 2yx = 4x\ln(x) - 4x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{4x\ln(x) - 4x + c_1}{x^2}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{4x\ln(x) - 4x + c_1}{x^2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{2}{x} dx} \\
&= x^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{4x\ln(x) - 4x + c_1}{x^2}\right) \\
\frac{d}{dx}(x^2 y) &= (x^2)\left(\frac{4x\ln(x) - 4x + c_1}{x^2}\right) \\
d(x^2 y) &= (4x\ln(x) - 4x + c_1)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^2 y &= \int 4x\ln(x) - 4x + c_1 \, dx \\
x^2 y &= -3x^2 + c_1 x + 2x^2 \ln(x) + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$
y = \frac{-3x^2 + c_1 x + 2x^2\ln(x)}{x^2} + \frac{c_2}{x^2}
$$

which simplifies to

$$
y = \frac{-3x^2 + c_1 x + 2x^2\ln(x) + c_2}{x^2}
$$

Summary
The solution(s) found are the following

$$
y = \frac{-3x^2 + c_1 x + 2x^2\ln(x) + c_2}{x^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{-3x^2 + c_1 x + 2x^2\ln(x) + c_2}{x^2}
$$

Verified OK.

13.16.9 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 4y'x + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 4x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 493: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x}{x^2}\,dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-2\ln(x)} \\
&= z_1 \left(\frac{1}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^2}\right) + c_2 \left(\frac{1}{x^2}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' + 4y'x + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1}{x^2} + \frac{c_2}{x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$

$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{1}{x} \\ -\frac{2}{x^3} & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\ln(x)}{x}}{\frac{1}{x^2}}\,dx$$

Which simplifies to

$$u_1 = -\int 4x\ln(x)\,dx$$

Hence

$$u_1 = -2x^2\ln(x) + x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\ln(x)}{x^2}}{\frac{1}{x^2}}\,dx$$

Which simplifies to

$$u_2 = \int 4\ln(x)\,dx$$

Hence

$$u_2 = 4x\ln(x) - 4x$$

Which simplifies to

$$u_1 = x^2(1 - 2\ln(x))$$
$$u_2 = 4x\ln(x) - 4x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 1 - 2\ln(x) + \frac{4x\ln(x) - 4x}{x}$$

Which simplifies to

$$y_p(x) = -3 + 2\ln(x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x^2} + \frac{c_2}{x}\right) + (-3 + 2\ln(x))
\end{aligned}
$$

Which simplifies to

$$
y = \frac{c_2 x + c_1}{x^2} - 3 + 2\ln(x)
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_2 x + c_1}{x^2} - 3 + 2\ln(x) \tag{1}
$$

Verification of solutions

$$
y = \frac{c_2 x + c_1}{x^2} - 3 + 2\ln(x)
$$

Verified OK.

13.16.10 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= x^2 \\
q(x) &= 4x \\
r(x) &= 2 \\
s(x) &= 4\ln(x)
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 4
\end{aligned}
$$

Therefore (1) becomes

$$2-(4)+(2)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$x^2y'+2yx=\int 4\ln\left(x\right)\;dx$$

We now have a first order ode to solve which is

$$x^2y'+2yx=4x\ln\left(x\right)-4x+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}p(x)&=\frac{2}{x}\\ q(x)&=\frac{4x\ln\left(x\right)-4x+c_1}{x^2}\end{aligned}$$

Hence the ode is

$$y'+\frac{2y}{x}=\frac{4x\ln\left(x\right)-4x+c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu&=\mathrm{e}^{\int\frac{2}{x}dx}\\ &=x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=(\mu)\left(\frac{4x\ln\left(x\right)-4x+c_1}{x^2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}(x^2y)&=(x^2)\left(\frac{4x\ln\left(x\right)-4x+c_1}{x^2}\right)\\ \mathrm{d}(x^2y)&=(4x\ln\left(x\right)-4x+c_1)\;\mathrm{d}x\end{aligned}$$

Integrating gives

$$x^2y = \int 4x\ln(x) - 4x + c_1\, \mathrm{d}x$$
$$x^2y = -3x^2 + c_1x + 2x^2\ln(x) + c_2$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{-3x^2 + c_1x + 2x^2\ln(x)}{x^2} + \frac{c_2}{x^2}$$

which simplifies to

$$y = \frac{-3x^2 + c_1x + 2x^2\ln(x) + c_2}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-3x^2 + c_1x + 2x^2\ln(x) + c_2}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{-3x^2 + c_1x + 2x^2\ln(x) + c_2}{x^2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)+4*x*diff(y(x),x)+2*y(x)=4*ln(x),y(x), singsol=all)
```

$$y(x) = 2\ln(x) + \frac{c_1}{x} - 3 + \frac{c_2}{x^2}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 23

```
DSolve[x^2*y''[x]+4*x*y'[x]+2*y[x]==4*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x^2} + 2\log(x) + \frac{c_2}{x} - 3$$

13.17 problem 17

Internal problem ID [11871]
Internal file name [`OUTPUT/11880_Saturday_April_13_2024_01_13_34_AM_96256138/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y'x + 4y = 2x\ln(x)}$$

13.17.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 4, f(x) = 2x\ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y'x + 4y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r + 4 = 0$$

Or

$$r^2 + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2i \\ r_2 &= 2i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 0$ and $\beta = -2$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\ &= x^\alpha \left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^\alpha \left(c_1 e^{\ln(x^{i\beta})} + c_2 e^{\ln(x^{-i\beta})}\right) \\ &= x^\alpha \left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = 0, \beta = -2$, the above becomes

$$y = x^0 \left(c_1 e^{-2i \ln(x)} + c_2 e^{2i \ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

Next, we find the particular solution to the ODE

$$x^2y'' + y'x + 4y = 2x\ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \cos(2\ln(x))\\ y_2 &= -\sin(2\ln(x))\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2\ln(x)) & -\sin(2\ln(x)) \\ \frac{d}{dx}(\cos(2\ln(x))) & \frac{d}{dx}(-\sin(2\ln(x))) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2\ln(x)) & -\sin(2\ln(x)) \\ -\frac{2\sin(2\ln(x))}{x} & -\frac{2\cos(2\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\cos(2\ln(x)))\left(-\frac{2\cos(2\ln(x))}{x}\right) - (-\sin(2\ln(x)))\left(-\frac{2\sin(2\ln(x))}{x}\right)$$

Which simplifies to

$$W = -\frac{2\left(\cos\left(2\ln\left(x\right)\right)^{2}+\sin\left(2\ln\left(x\right)\right)^{2}\right)}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2\sin\left(2\ln\left(x\right)\right) x\ln\left(x\right)}{-2x}\, dx$$

Which simplifies to

$$u_1 = -\int \sin\left(2\ln\left(x\right)\right)\ln\left(x\right) dx$$

Hence

$$u_1 = -\left(-\frac{2\ln\left(x\right)}{5}+\frac{4}{25}\right) x\cos\left(2\ln\left(x\right)\right) - \left(\frac{\ln\left(x\right)}{5}+\frac{3}{25}\right) x\sin\left(2\ln\left(x\right)\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\cos\left(2\ln\left(x\right)\right) x\ln\left(x\right)}{-2x}\, dx$$

Which simplifies to

$$u_2 = \int -\cos\left(2\ln\left(x\right)\right)\ln\left(x\right) dx$$

Hence

$$u_2 = -\left(\frac{\ln\left(x\right)}{5}+\frac{3}{25}\right) x\cos\left(2\ln\left(x\right)\right) + \left(-\frac{2\ln\left(x\right)}{5}+\frac{4}{25}\right) x\sin\left(2\ln\left(x\right)\right)$$

Which simplifies to

$$u_1 = \left(-\frac{2\left(\frac{2}{5}-\ln\left(x\right)\right)\cos\left(2\ln\left(x\right)\right)}{5} - \frac{\left(\frac{3}{5}+\ln\left(x\right)\right)\sin\left(2\ln\left(x\right)\right)}{5}\right) x$$

$$u_2 = \left(-\frac{\left(\frac{3}{5}+\ln\left(x\right)\right)\cos\left(2\ln\left(x\right)\right)}{5} + \frac{2\left(\frac{2}{5}-\ln\left(x\right)\right)\sin\left(2\ln\left(x\right)\right)}{5}\right) x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{2\left(\frac{2}{5}-\ln(x)\right)\cos(2\ln(x))}{5} - \frac{\left(\frac{3}{5}+\ln(x)\right)\sin(2\ln(x))}{5}\right)x\cos(2\ln(x)) - \left(-\frac{\left(\frac{3}{5}+\ln(x)\right)\cos(2\ln(x))}{5} + \frac{2\left(\frac{2}{5}-\ln(x)\right)\sin(2\ln(x))}{5}\right)x\sin(2\ln(x))$$

Which simplifies to

$$y_p(x) = \frac{2x\ln(x)}{5} - \frac{4x}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{2x\ln(x)}{5} - \frac{4x}{25} + c_1\cos(2\ln(x)) + c_2\sin(2\ln(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{2x\ln(x)}{5} - \frac{4x}{25} + c_1\cos(2\ln(x)) + c_2\sin(2\ln(x)) \tag{1}$$

Verification of solutions

$$y = \frac{2x\ln(x)}{5} - \frac{4x}{25} + c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))$$

Verified OK.

13.17.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + y'x + 4y = 0$$

In normal form the ode

$$x^2y'' + y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{4}{x^2}
\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x) dx\right)} dx \\
&= \int e^{-\left(\int \frac{1}{x} dx\right)} dx \\
&= \int e^{-\ln(x)}\, dx \\
&= \int \frac{1}{x} dx \\
&= \ln(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{x^2}}{\frac{1}{x^2}} \\
&= 4
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + 4y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} + 4\,\mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\lambda_1 = +2i$$
$$\lambda_2 = -2i$$

Which simplifies to

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^{0}(c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sin(\ln(x))\cos(\ln(x))$$

$$y_2 = 2\cos(\ln(x))^2 - 1$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sin(\ln(x))\cos(\ln(x)) & 2\cos(\ln(x))^2 - 1 \\ \frac{d}{dx}(\sin(\ln(x))\cos(\ln(x))) & \frac{d}{dx}\left(2\cos(\ln(x))^2 - 1\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sin(\ln(x))\cos(\ln(x)) & 2\cos(\ln(x))^2 - 1 \\ \frac{\cos(\ln(x))^2}{x} - \frac{\sin(\ln(x))^2}{x} & -\frac{4\cos(\ln(x))\sin(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\sin(\ln(x))\cos(\ln(x)))\left(-\frac{4\cos(\ln(x))\sin(\ln(x))}{x}\right) - \left(2\cos(\ln(x))^2 - 1\right)\left(\frac{\cos(\ln(x))^2}{x} - \frac{\sin(\ln(x))^2}{x}\right)$$

Which simplifies to

$$W = -\frac{2\sin(\ln(x))^2\cos(\ln(x))^2 + 2\cos(\ln(x))^4 + \sin(\ln(x))^2 - \cos(\ln(x))^2}{x}$$

Which simplifies to

$$W = -\frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\left(2\cos(\ln(x))^2 - 1\right)x\ln(x)}{-x}\,dx$$

Which simplifies to

$$u_1 = -\int -2\cos(2\ln(x))\ln(x)\,dx$$

Hence

$$u_1 = 2\left(\frac{\ln(x)}{5} + \frac{3}{25}\right)x\cos(2\ln(x)) - 2\left(-\frac{2\ln(x)}{5} + \frac{4}{25}\right)x\sin(2\ln(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\sin(\ln(x))\cos(\ln(x))\, x\ln(x)}{-x}\, dx$$

Which simplifies to

$$u_2 = \int -\sin(2\ln(x))\ln(x)\, dx$$

Hence

$$u_2 = -\left(-\frac{2\ln(x)}{5} + \frac{4}{25}\right) x\cos(2\ln(x)) - \left(\frac{\ln(x)}{5} + \frac{3}{25}\right) x\sin(2\ln(x))$$

Which simplifies to

$$u_1 = \left(\frac{2\left(\frac{3}{5} + \ln(x)\right)\cos(2\ln(x))}{5} - \frac{4\left(\frac{2}{5} - \ln(x)\right)\sin(2\ln(x))}{5}\right) x$$

$$u_2 = \left(-\frac{2\left(\frac{2}{5} - \ln(x)\right)\cos(2\ln(x))}{5} - \frac{\left(\frac{3}{5} + \ln(x)\right)\sin(2\ln(x))}{5}\right) x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{2\left(\frac{3}{5} + \ln(x)\right)\cos(2\ln(x))}{5} - \frac{4\left(\frac{2}{5} - \ln(x)\right)\sin(2\ln(x))}{5}\right) x\sin(\ln(x))\cos(\ln(x)) + \left(-\frac{2\left(\frac{2}{5} - \ln(x)\right)\cos(2\ln(x))}{5} - \frac{\left(\frac{3}{5} + \ln(x)\right)\sin(2\ln(x))}{5}\right) x\left(2\cos(\ln(x))^2 - 1\right)$$

Which simplifies to

$$y_p(x) = \frac{2x\ln(x)}{5} - \frac{4x}{25}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))) + \left(\frac{2x\ln(x)}{5} - \frac{4x}{25}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{2x\ln(x)}{5} - \frac{4x}{25} + c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)) \tag{1}
$$

Verification of solutions

$$
y = \frac{2x\ln(x)}{5} - \frac{4x}{25} + c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))
$$

Verified OK.

13.17.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^2, B = x, C = 4, f(x) = 2x\ln(x)$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
x^2y'' + y'x + 4y = 0
$$

In normal form the ode

$$
x^2y'' + y'x + 4y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{4}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))$$

Now the particular solution to this ODE is found

$$x^2y'' + y'x + 4y = 2x\ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \sin(\ln(x))\cos(\ln(x)) \\
y_2 &= 2\cos(\ln(x))^2 - 1
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sin(\ln(x))\cos(\ln(x)) & 2\cos(\ln(x))^2 - 1 \\ \frac{d}{dx}(\sin(\ln(x))\cos(\ln(x))) & \frac{d}{dx}\left(2\cos(\ln(x))^2 - 1\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sin(\ln(x))\cos(\ln(x)) & 2\cos(\ln(x))^2 - 1 \\ \frac{\cos(\ln(x))^2}{x} - \frac{\sin(\ln(x))^2}{x} & -\frac{4\cos(\ln(x))\sin(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\sin(\ln(x))\cos(\ln(x)))\left(-\frac{4\cos(\ln(x))\sin(\ln(x))}{x}\right) - \left(2\cos(\ln(x))^2 - 1\right)\left(\frac{\cos(\ln(x))^2}{x} - \frac{\sin(\ln(x))^2}{x}\right)$$

Which simplifies to

$$W = -\frac{2\sin(\ln(x))^2\cos(\ln(x))^2 + 2\cos(\ln(x))^4 + \sin(\ln(x))^2 - \cos(\ln(x))^2}{x}$$

Which simplifies to

$$W = -\frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\left(2\cos(\ln(x))^2 - 1\right)x\ln(x)}{-x}\,dx$$

Which simplifies to

$$u_1 = -\int -2\cos(2\ln(x))\ln(x)\,dx$$

Hence

$$u_1 = 2\left(\frac{\ln(x)}{5} + \frac{3}{25}\right)x\cos(2\ln(x)) - 2\left(-\frac{2\ln(x)}{5} + \frac{4}{25}\right)x\sin(2\ln(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{2 \sin (\ln (x)) \cos (\ln (x)) x \ln (x)}{-x} dx$$

Which simplifies to

$$u_2 = \int -\sin (2 \ln (x)) \ln (x) dx$$

Hence

$$u_2 = -\left(-\frac{2 \ln (x)}{5} + \frac{4}{25}\right) x \cos (2 \ln (x)) - \left(\frac{\ln (x)}{5} + \frac{3}{25}\right) x \sin (2 \ln (x))$$

Which simplifies to

$$u_1 = \left(\frac{2\left(\frac{3}{5} + \ln (x)\right) \cos (2 \ln (x))}{5} - \frac{4\left(\frac{2}{5} - \ln (x)\right) \sin (2 \ln (x))}{5}\right) x$$
$$u_2 = \left(-\frac{2\left(\frac{2}{5} - \ln (x)\right) \cos (2 \ln (x))}{5} - \frac{\left(\frac{3}{5} + \ln (x)\right) \sin (2 \ln (x))}{5}\right) x$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left(\frac{2\left(\frac{3}{5} + \ln (x)\right) \cos (2 \ln (x))}{5} \right. \\ &\left. - \frac{4\left(\frac{2}{5} - \ln (x)\right) \sin (2 \ln (x))}{5}\right) x \sin (\ln (x)) \cos (\ln (x)) \\ &+ \left(-\frac{2\left(\frac{2}{5} - \ln (x)\right) \cos (2 \ln (x))}{5} - \frac{\left(\frac{3}{5} + \ln (x)\right) \sin (2 \ln (x))}{5}\right) x\left(2 \cos (\ln (x))^2 \right. \\ &\left. - 1\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{2x \ln (x)}{5} - \frac{4x}{25}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))) + \left(\frac{2x\ln(x)}{5} - \frac{4x}{25}\right) \\
&= \frac{2x\ln(x)}{5} - \frac{4x}{25} + c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))
\end{aligned}$$

Which simplifies to

$$y = \frac{2x\ln(x)}{5} - \frac{4x}{25} + c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

Summary
The solution(s) found are the following

$$y = \frac{2x\ln(x)}{5} - \frac{4x}{25} + c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)) \tag{1}$$

Verification of solutions

$$y = \frac{2x\ln(x)}{5} - \frac{4x}{25} + c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

Verified OK.

13.17.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 4, f(x) = 2x\ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y'x + 4y = 0$$

In normal form the ode

$$x^2y'' + y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{4i}{x} + \frac{1}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(1+4i)\,v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+4i)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{(-1-4i)\, u}{x} \end{aligned}$$

Where $f(x) = \frac{-1-4i}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{-1-4i}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{-1-4i}{x}\, dx \\ \ln(u) &= (-1-4i)\ln(x) + c_1 \\ u &= e^{(-1-4i)\ln(x)+c_1} \\ &= c_1 e^{(-1-4i)\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-4i}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= \frac{i c_1 x^{-4i}}{4} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(\frac{i c_1 x^{-4i}}{4} + c_2\right) x^{2i} \\ &= x^{2i} c_2 + \frac{i x^{-2i} c_1}{4} \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' + y'x + 4y = 2x\ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{2i}$$
$$y_2 = x^{2i}x^{-4i}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{2i} & x^{2i}x^{-4i} \\ \frac{d}{dx}(x^{2i}) & \frac{d}{dx}(x^{2i}x^{-4i}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{2i} & x^{2i}x^{-4i} \\ \frac{2ix^{2i}}{x} & -\frac{2ix^{2i}x^{-4i}}{x} \end{vmatrix}$$

Therefore

$$W = \left(x^{2i}\right)\left(-\frac{2ix^{2i}x^{-4i}}{x}\right) - \left(x^{2i}x^{-4i}\right)\left(\frac{2ix^{2i}}{x}\right)$$

Which simplifies to

$$W = -\frac{4i x^{4i} x^{-4i}}{x}$$

Which simplifies to

$$W = -\frac{4i}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\left(\int \frac{2x^{2i} x^{-4i} x \ln(x)}{-4ix}\, dx\right)$$

Which simplifies to

$$u_1 = -\left(\int \frac{i x^{-2i} \ln(x)}{2} dx\right)$$

Hence

$$u_1 = -\frac{(4 + 3i + (-10 + 5i) \ln(x))\, x^{1-2i}}{50}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^{2i} x \ln(x)}{-4ix}\, dx$$

Which simplifies to

$$u_2 = \int \frac{i x^{2i} \ln(x)}{2} dx$$

Hence

$$u_2 = \frac{(-4 + 3i + (10 + 5i) \ln(x))\, x^{1+2i}}{50}$$

Which simplifies to

$$u_1 = -\frac{x^{1-2i} \left(\frac{4}{5} + \frac{3i}{5} + (-2 + i) \ln(x)\right)}{10}$$

$$u_2 = \frac{(-4 + 3i + (10 + 5i) \ln(x))\, x^{1+2i}}{50}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^{1-2i}\left(\frac{4}{5}+\frac{3i}{5}+(-2+i)\ln(x)\right)x^{2i}}{10} + \frac{(-4+3i+(10+5i)\ln(x))\,x^{1+2i}x^{2i}x^{-4i}}{50}$$

Which simplifies to

$$y_p(x) = \frac{2x\ln(x)}{5} - \frac{4x}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(\frac{ic_1x^{-4i}}{4} + c_2\right)x^{2i}\right) + \left(\frac{2x\ln(x)}{5} - \frac{4x}{25}\right) \\ &= \frac{2x\ln(x)}{5} - \frac{4x}{25} + \left(\frac{ic_1x^{-4i}}{4} + c_2\right)x^{2i} \end{aligned}$$

Which simplifies to

$$y = \frac{2x\ln(x)}{5} + x^{2i}c_2 - \frac{4x}{25} + \frac{ix^{-2i}c_1}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{2x\ln(x)}{5} + x^{2i}c_2 - \frac{4x}{25} + \frac{ix^{-2i}c_1}{4} \tag{1}$$

Verification of solutions

$$y = \frac{2x\ln(x)}{5} + x^{2i}c_2 - \frac{4x}{25} + \frac{ix^{-2i}c_1}{4}$$

Verified OK.

13.17.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + y'x + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-17}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -17 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{17}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 494: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{17}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{17}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 2i \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 2i
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{17}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{17}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 2i \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 2i
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{17}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2} + 2i$	$\frac{1}{2} - 2i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2} + 2i$	$\frac{1}{2} - 2i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2} - 2i$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - 2i - \left(\frac{1}{2} - 2i\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2} - 2i}{x} + (-)\,(0) \\ &= \frac{\frac{1}{2} - 2i}{x} \\ &= \frac{\frac{1}{2} - 2i}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{\frac{1}{2} - 2i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + 2i}{x^2}\right) + \left(\frac{\frac{1}{2} - 2i}{x}\right)^2 - \left(-\frac{17}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{\frac{1}{2} - 2i}{x} dx} \\ &= x^{\frac{1}{2} - 2i} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{-2i}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i x^{4i}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-2i}\right) + c_2 \left(x^{-2i}\left(-\frac{i x^{4i}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^2y''+y'x+4y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = x^{-2i}c_1 - \frac{ic_2x^{2i}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{-2i}$$
$$y_2 = -\frac{ix^{2i}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{-2i} & -\frac{ix^{2i}}{4} \\ \frac{d}{dx}(x^{-2i}) & \frac{d}{dx}\left(-\frac{ix^{2i}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{-2i} & -\frac{i x^{2i}}{4} \\ -\frac{2i x^{-2i}}{x} & \frac{x^{2i}}{2x} \end{vmatrix}$$

Therefore

$$W = \left(x^{-2i}\right)\left(\frac{x^{2i}}{2x}\right) - \left(-\frac{i x^{2i}}{4}\right)\left(-\frac{2i x^{-2i}}{x}\right)$$

Which simplifies to

$$W = \frac{x^{-2i} x^{2i}}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{i x^{2i} x \ln(x)}{2}}{x} \, dx$$

Which simplifies to

$$u_1 = -\int -\frac{i x^{2i} \ln(x)}{2} dx$$

Hence

$$u_1 = \frac{x^{1+2i}\left(-\frac{4}{5} + \frac{3i}{5} + (2+i)\ln(x)\right)}{10}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x \ln(x)\, x^{-2i}}{x} \, dx$$

Which simplifies to

$$u_2 = \int 2x^{-2i} \ln(x) \, dx$$

Hence

$$u_2 = \frac{2x^{1-2i}(3 - 4i + (5+10i)\ln(x))}{25}$$

Which simplifies to

$$u_1 = \frac{(-4 + 3i + (10 + 5i)\ln(x))\, x^{1+2i}}{50}$$

$$u_2 = \frac{2x^{1-2i}(3 - 4i + (5 + 10i)\ln(x))}{25}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(-4 + 3i + (10 + 5i)\ln(x))\, x^{1+2i} x^{-2i}}{50} - \frac{i x^{1-2i}(3 - 4i + (5 + 10i)\ln(x))\, x^{2i}}{50}$$

Which simplifies to

$$y_p(x) = \frac{2x\ln(x)}{5} - \frac{4x}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^{-2i}c_1 - \frac{i c_2 x^{2i}}{4}\right) + \left(\frac{2x\ln(x)}{5} - \frac{4x}{25}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^{-2i}c_1 - \frac{i c_2 x^{2i}}{4} + \frac{2x\ln(x)}{5} - \frac{4x}{25} \tag{1}$$

Verification of solutions

$$y = x^{-2i}c_1 - \frac{i c_2 x^{2i}}{4} + \frac{2x\ln(x)}{5} - \frac{4x}{25}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+4*y(x)=2*x*ln(x),y(x), singsol=all)
```

$$y(x) = \sin\left(2\ln\left(x\right)\right) c_2 + \cos\left(2\ln\left(x\right)\right) c_1 + \frac{2\ln\left(x\right) x}{5} - \frac{4x}{25}$$

✓ Solution by Mathematica

Time used: 0.11 (sec). Leaf size: 33

```
DSolve[x^2*y''[x]+x*y'[x]+4*y[x]==2*x*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{25}x(5\log(x) - 2) + c_1\cos(2\log(x)) + c_2\sin(2\log(x))$$

13.18 problem 18

Internal problem ID [11872]
Internal file name [`OUTPUT/11881_Saturday_April_13_2024_01_13_37_AM_23588908/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x^2y'' + y'x + 4y = 4\sin(\ln(x))}$$

13.18.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 4, f(x) = 4\sin(\ln(x))$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y'x + 4y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r + 4 = 0$$

Or

$$r^2 + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2i \\ r_2 &= 2i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 0$ and $\beta = -2$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\ &= x^\alpha \left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^\alpha \left(c_1 e^{\ln(x^{i\beta})} + c_2 e^{\ln(x^{-i\beta})}\right) \\ &= x^\alpha \left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = 0, \beta = -2$, the above becomes

$$y = x^0 \left(c_1 e^{-2i\ln(x)} + c_2 e^{2i\ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

Next, we find the particular solution to the ODE

$$x^2y'' + y'x + 4y = 4\sin(\ln(x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2\ln(x))$$
$$y_2 = -\sin(2\ln(x))$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2\ln(x)) & -\sin(2\ln(x)) \\ \frac{d}{dx}(\cos(2\ln(x))) & \frac{d}{dx}(-\sin(2\ln(x))) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2\ln(x)) & -\sin(2\ln(x)) \\ -\frac{2\sin(2\ln(x))}{x} & -\frac{2\cos(2\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\cos(2\ln(x)))\left(-\frac{2\cos(2\ln(x))}{x}\right) - (-\sin(2\ln(x)))\left(-\frac{2\sin(2\ln(x))}{x}\right)$$

Which simplifies to

$$W = -\frac{2\left(\cos\left(2\ln\left(x\right)\right)^2 + \sin\left(2\ln\left(x\right)\right)^2\right)}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-4\sin\left(2\ln\left(x\right)\right)\sin\left(\ln\left(x\right)\right)}{-2x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\cos\left(\ln\left(x\right)\right) - \cos\left(3\ln\left(x\right)\right)}{x}dx$$

Hence

$$u_1 = \frac{\sin\left(3\ln\left(x\right)\right)}{3} - \sin\left(\ln\left(x\right)\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\cos\left(2\ln\left(x\right)\right)\sin\left(\ln\left(x\right)\right)}{-2x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{-\sin\left(3\ln\left(x\right)\right) + \sin\left(\ln\left(x\right)\right)}{x}dx$$

Hence

$$u_2 = \frac{\cos\left(3\ln\left(x\right)\right)}{3} - \cos\left(\ln\left(x\right)\right)$$

Which simplifies to

$$u_1 = -\frac{4\sin\left(\ln\left(x\right)\right)^3}{3}$$

$$u_2 = \frac{4\cos\left(\ln\left(x\right)\right)^3}{3} - 2\cos\left(\ln\left(x\right)\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{4 \sin(\ln(x))^3 \cos(2\ln(x))}{3} - \left(\frac{4 \cos(\ln(x))^3}{3} - 2\cos(\ln(x))\right) \sin(2\ln(x))$$

Which simplifies to

$$y_p(x) = \frac{4 \sin(\ln(x))}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{4 \sin(\ln(x))}{3} + c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{4 \sin(\ln(x))}{3} + c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)) \tag{1}$$

Verification of solutions

$$y = \frac{4 \sin(\ln(x))}{3} + c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

Verified OK.

13.18.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + y'x + 4y = 0$$

In normal form the ode

$$x^2y'' + y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\ &= \int e^{-\ln(x)}\,dx \\ &= \int \frac{1}{x}dx \\ &= \ln(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{4}{x^2}}{\frac{1}{x^2}} \\ &= 4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 4y(\tau) &= 0
\end{aligned}
$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Ay''(\tau) + By'(\tau) + Cy(\tau) = 0
$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda\tau} + 4\,\mathrm{e}^{\lambda\tau} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$
\lambda^2 + 4 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\
&= \pm 2i
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +2i \\
\lambda_2 &= -2i
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 2i \\
\lambda_2 &= -2i
\end{aligned}
$$

Since roots are complex conjugate of each others, then let the roots be

$$
\lambda_{1,2} = \alpha \pm i\beta
$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sin(\ln(x))\cos(\ln(x))$$
$$y_2 = 2\cos(\ln(x))^2 - 1$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sin(\ln(x))\cos(\ln(x)) & 2\cos(\ln(x))^2 - 1 \\ \frac{d}{dx}(\sin(\ln(x))\cos(\ln(x))) & \frac{d}{dx}\left(2\cos(\ln(x))^2 - 1\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sin(\ln(x))\cos(\ln(x)) & 2\cos(\ln(x))^2 - 1 \\ \frac{\cos(\ln(x))^2}{x} - \frac{\sin(\ln(x))^2}{x} & -\frac{4\cos(\ln(x))\sin(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & (\sin(\ln(x))\cos(\ln(x)))\left(-\frac{4\cos(\ln(x))\sin(\ln(x))}{x}\right) \\ & - \left(2\cos(\ln(x))^2 - 1\right)\left(\frac{\cos(\ln(x))^2}{x} - \frac{\sin(\ln(x))^2}{x}\right) \end{aligned}$$

Which simplifies to

$$W = -\frac{2\sin(\ln(x))^2\cos(\ln(x))^2 + 2\cos(\ln(x))^4 + \sin(\ln(x))^2 - \cos(\ln(x))^2}{x}$$

Which simplifies to

$$W = -\frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\left(2\cos(\ln(x))^2 - 1\right)\sin(\ln(x))}{-x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{-8\sin(\ln(x))\cos(\ln(x))^2 + 4\sin(\ln(x))}{x}dx$$

Hence

$$u_1 = -\frac{8\cos(\ln(x))^3}{3} + 4\cos(\ln(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{4 \sin(\ln(x))^2 \cos(\ln(x))}{-x} dx$$

Which simplifies to

$$u_2 = \int -\frac{4 \sin(\ln(x))^2 \cos(\ln(x))}{x} dx$$

Hence

$$u_2 = -\frac{4 \sin(\ln(x))^3}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{8 \cos(\ln(x))^3}{3} + 4 \cos(\ln(x))\right) \sin(\ln(x)) \cos(\ln(x)) \\ - \frac{4 \sin(\ln(x))^3 \left(2 \cos(\ln(x))^2 - 1\right)}{3}$$

Which simplifies to

$$y_p(x) = \frac{4 \sin(\ln(x))}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))) + \left(\frac{4 \sin(\ln(x))}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{4 \sin(\ln(x))}{3} + c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x)) \tag{1}$$

Verification of solutions

$$y = \frac{4 \sin(\ln(x))}{3} + c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))$$

Verified OK.

13.18.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 4, f(x) = 4\sin(\ln(x))$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y'x + 4y = 0$$

In normal form the ode

$$x^2y'' + y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + y'x + 4y = 4\sin(\ln(x))
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sin(\ln(x))\cos(\ln(x))$$
$$y_2 = 2\cos(\ln(x))^2 - 1$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sin(\ln(x))\cos(\ln(x)) & 2\cos(\ln(x))^2 - 1 \\ \frac{d}{dx}(\sin(\ln(x))\cos(\ln(x))) & \frac{d}{dx}(2\cos(\ln(x))^2 - 1) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sin(\ln(x))\cos(\ln(x)) & 2\cos(\ln(x))^2 - 1 \\ \frac{\cos(\ln(x))^2}{x} - \frac{\sin(\ln(x))^2}{x} & -\frac{4\cos(\ln(x))\sin(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\sin(\ln(x))\cos(\ln(x)))\left(-\frac{4\cos(\ln(x))\sin(\ln(x))}{x}\right) - \left(2\cos(\ln(x))^2 - 1\right)\left(\frac{\cos(\ln(x))^2}{x} - \frac{\sin(\ln(x))^2}{x}\right)$$

Which simplifies to

$$W = -\frac{2\sin(\ln(x))^2\cos(\ln(x))^2 + 2\cos(\ln(x))^4 + \sin(\ln(x))^2 - \cos(\ln(x))^2}{x}$$

Which simplifies to

$$W = -\frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4(2\cos(\ln(x))^2 - 1)\sin(\ln(x))}{-x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{-8\sin(\ln(x))\cos(\ln(x))^2 + 4\sin(\ln(x))}{x}dx$$

Hence

$$u_1 = -\frac{8\cos(\ln(x))^3}{3} + 4\cos(\ln(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\sin(\ln(x))^2\cos(\ln(x))}{-x}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{4\sin(\ln(x))^2\cos(\ln(x))}{x}dx$$

Hence

$$u_2 = -\frac{4\sin(\ln(x))^3}{3}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left(-\frac{8\cos(\ln(x))^3}{3} + 4\cos(\ln(x))\right)\sin(\ln(x))\cos(\ln(x)) \\ &- \frac{4\sin(\ln(x))^3\left(2\cos(\ln(x))^2 - 1\right)}{3} \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{4\sin(\ln(x))}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))) + \left(\frac{4\sin(\ln(x))}{3}\right) \\ &= \frac{4\sin(\ln(x))}{3} + c_1\cos(2\ln(x)) + c_2\sin(2\ln(x)) \end{aligned}$$

Which simplifies to

$$y = \frac{4\sin(\ln(x))}{3} + c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))$$

Summary
The solution(s) found are the following

$$y = \frac{4\sin(\ln(x))}{3} + c_1\cos(2\ln(x)) + c_2\sin(2\ln(x)) \tag{1}$$

Verification of solutions

$$y = \frac{4\sin(\ln(x))}{3} + c_1\cos(2\ln(x)) + c_2\sin(2\ln(x))$$

Verified OK.

13.18.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 4, f(x) = 4\sin(\ln(x))$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y'x + 4y = 0$$

In normal form the ode

$$x^2y'' + y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{4i}{x} + \frac{1}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(1+4i)\,v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x)+\frac{(1+4i)\,u(x)}{x}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{(-1-4i)\,u}{x}\end{aligned}$$

Where $f(x)=\frac{-1-4i}{x}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{-1-4i}{x}\,dx\\ \int\frac{1}{u}\,du &= \int\frac{-1-4i}{x}\,dx\\ \ln(u) &= (-1-4i)\ln(x)+c_1\\ u &= \mathrm{e}^{(-1-4i)\ln(x)+c_1}\\ &= c_1\mathrm{e}^{(-1-4i)\ln(x)}\end{aligned}$$

Which simplifies to

$$u(x)=\frac{c_1x^{-4i}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned}v'(x) &= u(x)\\ v(x) &= \int u(x)\,dx+c_2\\ &= \frac{ic_1x^{-4i}}{4}+c_2\end{aligned}$$

Hence

$$\begin{aligned}y &= v(x)\,x^n\\ &= \left(\frac{ic_1x^{-4i}}{4}+c_2\right)x^{2i}\\ &= x^{2i}c_2+\frac{ix^{-2i}c_1}{4}\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' + y'x + 4y = 4\sin(\ln(x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{2i}$$
$$y_2 = x^{2i}x^{-4i}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{2i} & x^{2i}x^{-4i} \\ \frac{d}{dx}(x^{2i}) & \frac{d}{dx}(x^{2i}x^{-4i}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{2i} & x^{2i}x^{-4i} \\ \frac{2ix^{2i}}{x} & -\frac{2ix^{2i}x^{-4i}}{x} \end{vmatrix}$$

Therefore

$$W = (x^{2i})\left(-\frac{2ix^{2i}x^{-4i}}{x}\right) - (x^{2i}x^{-4i})\left(\frac{2ix^{2i}}{x}\right)$$

Which simplifies to

$$W = -\frac{4i x^{4i} x^{-4i}}{x}$$

Which simplifies to

$$W = -\frac{4i}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4x^{2i} x^{-4i} \sin(\ln(x))}{-4ix} dx$$

Which simplifies to

$$u_1 = -\int i x^{-1-2i} \sin(\ln(x)) dx$$

Hence

$$u_1 = undefined$$

And Eq. (3) becomes

$$u_2 = \int \frac{4x^{2i} \sin(\ln(x))}{-4ix} dx$$

Which simplifies to

$$u_2 = \int i x^{-1+2i} \sin(\ln(x)) dx$$

Hence

$$u_2 = undefined$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined } x^{2i} + \text{undefined } x^{2i} x^{-4i}$$

Which simplifies to

$$y_p(x) = \text{undefined} \left(x^{2i} + x^{-2i}\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(\frac{ic_1 x^{-4i}}{4} + c_2\right) x^{2i}\right) + \left(\text{undefined}\left(x^{2i} + x^{-2i}\right)\right) \\
&= \text{undefined}\left(x^{2i} + x^{-2i}\right) + \left(\frac{ic_1 x^{-4i}}{4} + c_2\right) x^{2i}
\end{aligned}
$$

Which simplifies to

$$
y = \frac{(ic_1 + \text{undefined})\, x^{-2i}}{4} + (\text{undefined} + c_2)\, x^{2i}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(ic_1 + \text{undefined})\, x^{-2i}}{4} + (\text{undefined} + c_2)\, x^{2i} \tag{1}
$$

Verification of solutions

$$
y = \frac{(ic_1 + \text{undefined})\, x^{-2i}}{4} + (\text{undefined} + c_2)\, x^{2i}
$$

Verified OK.

13.18.5 Solving using Kovacic algorithm

Writing the ode as

$$
x^2 y'' + y' x + 4y = 0 \tag{1}
$$
$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= x \\
C &= 4
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-17}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -17 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{17}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 495: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{17}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{17}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 2i\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 2i\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{17}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{17}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 2i\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 2i\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{17}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}+2i$	$\frac{1}{2}-2i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+2i$	$\frac{1}{2}-2i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - 2i$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - 2i - \left(\frac{1}{2} - 2i\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2} - 2i}{x} + (-)(0) \\ &= \frac{\frac{1}{2} - 2i}{x} \\ &= \frac{\frac{1}{2} - 2i}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - 2i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + 2i}{x^2}\right) + \left(\frac{\frac{1}{2} - 2i}{x}\right)^2 - \left(-\frac{17}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \frac{\frac{1}{2} - 2i}{x} dx} \\ &= x^{\frac{1}{2} - 2i} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1\left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^{-2i}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{i x^{4i}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-2i}\right) + c_2 \left(x^{-2i} \left(-\frac{i x^{4i}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' + y' x + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = x^{-2i} c_1 - \frac{i c_2 x^{2i}}{4}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{-2i}$$

$$y_2 = -\frac{i x^{2i}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{-2i} & -\frac{i x^{2i}}{4} \\ \frac{d}{dx}\left(x^{-2i}\right) & \frac{d}{dx}\left(-\frac{i x^{2i}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{-2i} & -\frac{i x^{2i}}{4} \\ -\frac{2i x^{-2i}}{x} & \frac{x^{2i}}{2x} \end{vmatrix}$$

Therefore

$$W = \left(x^{-2i}\right)\left(\frac{x^{2i}}{2x}\right) - \left(-\frac{i x^{2i}}{4}\right)\left(-\frac{2i x^{-2i}}{x}\right)$$

Which simplifies to

$$W = \frac{x^{-2i} x^{2i}}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-ix^{2i} \sin(\ln(x))}{x} dx$$

Which simplifies to

$$u_1 = -\int -ix^{-1+2i} \sin(\ln(x)) dx$$

Hence

$$u_1 = undefined$$

And Eq. (3) becomes

$$u_2 = \int \frac{4x^{-2i} \sin(\ln(x))}{x} dx$$

Which simplifies to

$$u_2 = \int 4x^{-1-2i} \sin(\ln(x)) dx$$

Hence

$$u_2 = undefined$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined } x^{-2i} - i \text{ undefined } x^{2i}$$

Which simplifies to

$$y_p(x) = (ix^{2i} + x^{-2i}) \text{ undefined}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^{-2i}c_1 - \frac{ic_2x^{2i}}{4}\right) + \left((ix^{2i} + x^{-2i}) \text{ undefined}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^{-2i}c_1 - \frac{ic_2x^{2i}}{4} + (ix^{2i} + x^{-2i}) \text{ undefined} \tag{1}$$

Verification of solutions

$$y = x^{-2i}c_1 - \frac{ic_2x^{2i}}{4} + \left(ix^{2i} + x^{-2i}\right) \text{undefined}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+4*y(x)=4*sin(ln(x)),y(x), singsol=all)
```

$$y(x) = \sin\left(2\ln\left(x\right)\right)c_2 + \cos\left(2\ln\left(x\right)\right)c_1 + \frac{4\sin\left(\ln\left(x\right)\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.176 (sec). Leaf size: 29

```
DSolve[x^2*y''[x]+x*y'[x]+4*y[x]==4*Sin[Log[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4}{3}\sin(\log(x)) + c_1\cos(2\log(x)) + c_2\sin(2\log(x))$$

13.19 problem 19

Internal problem ID [11873]
Internal file name [`OUTPUT/11882_Saturday_April_13_2024_01_13_53_AM_23825213/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 19.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y''' - x^2y'' + 2y'x - 2y = x^3}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' - x^2y'' + 2y'x - 2y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' - x^2y'' + 2y'x - 2y = x^3$$

gives

$$2x\lambda\, x^{\lambda-1} - x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} - 2x^\lambda = 0$$

Which simplifies to

$$2\lambda\, x^\lambda - \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda - 2x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$2\lambda - \lambda(\lambda-1) + \lambda(\lambda-1)\,(\lambda-2) - 2 = 0$$

Simplifying gives the characteristic equation as

$$(\lambda-2)\,(\lambda-1)^2 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$
$$\lambda_3 = 1$$

This table summarises the result

root	multiplicity	type of root
1	2	real root
2	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha (c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha (c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_1 x + c_2 \ln(x)\, x + c_3 x^2$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= x \\
y_2 &= x \ln(x) \\
y_3 &= x^2
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
x^3 y''' - x^2 y'' + 2y' x - 2y = x^3
$$

Let the particular solution be

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$
U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx
$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$
W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}
$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$
W = \begin{bmatrix} x & x \ln(x) & x^2 \\ 1 & \ln(x) + 1 & 2x \\ 0 & \frac{1}{x} & 2 \end{bmatrix}
$$

$$
|W| = x
$$

The determinant simplifies to

$$
|W| = x
$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} x\ln(x) & x^2 \\ \ln(x)+1 & 2x \end{bmatrix} \\ &= x^2(\ln(x) - 1) \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix} \\ &= x^2 \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} x & x\ln(x) \\ 1 & \ln(x)+1 \end{bmatrix} \\ &= x \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2 \int \frac{(x^3)\,(x^2(\ln(x)-1))}{(x^3)\,(x)}\,dx \\ &= \int \frac{x^5(\ln(x)-1)}{x^4}\,dx \\ &= \int (x(\ln(x)-1))\,dx \\ &= \frac{x^2\ln(x)}{2} - \frac{3x^2}{4} \end{aligned}$$

$$\begin{aligned} U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\ &= (-1)^1 \int \frac{(x^3)\,(x^2)}{(x^3)\,(x)}\,dx \\ &= -\int \frac{x^5}{x^4}\,dx \\ &= -\int (x)\,dx \\ &= -\frac{x^2}{2} \end{aligned}$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(x^3)\,(x)}{(x^3)\,(x)}\,dx \\
&= \int \frac{x^4}{x^4}\,dx \\
&= \int (1)\,dx \\
&= x
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{x^2 \ln (x)}{2} - \frac{3x^2}{4}\right)(x) \\
&+ \left(-\frac{x^2}{2}\right)(x \ln (x)) \\
&+ (x)\,(x^2)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{x^3}{4}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 x + c_2 \ln (x)\,x + c_3 x^2) + \left(\frac{x^3}{4}\right)
\end{aligned}
$$

Which simplifies to

$$
y = x(\ln (x)\,c_2 + c_3 x + c_1) + \frac{x^3}{4}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = x(\ln (x)\,c_2 + c_3 x + c_1) + \frac{x^3}{4} \tag{1}
$$

Verification of solutions

$$y = x(\ln(x) c_2 + c_3 x + c_1) + \frac{x^3}{4}$$

Verified OK.

13.19.1 Maple step by step solution

Let's solve

$x^3 y''' - y'' x^2 + 2y' x - 2y = x^3$

- Highest derivative means the order of the ODE is 3

y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(x^3*diff(y(x),x$3)-x^2*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=x^3,y(x), singsol=all)
```

$$y(x) = \frac{x(4c_3 \ln(x) + 4c_2 x + x^2 + 4c_1)}{4}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 29

```
DSolve[x^3*y'''[x]-x^2*y''[x]+2*x*y'[x]-2*y[x]==x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}x\left(x^2 + 4c_3x + 4c_2\log(x) + 4c_1\right)$$

13.20 problem 20

Internal problem ID [11874]
Internal file name [`OUTPUT/11883_Saturday_April_13_2024_01_13_54_AM_33597945/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 2y'x - 10y = 0}$$

With initial conditions

$$[y(1) = 5, y'(1) = 4]$$

13.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= -\frac{10}{x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{2y'}{x} - \frac{10y}{x^2} = 0$$

The domain of $p(x) = -\frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -\frac{10}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

13.20.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 2xrx^{r-1} - 10x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 2r\,x^r - 10x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 2r - 10 = 0$$

Or

$$r^2 - 3r - 10 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= -2 \\
r_2 &= 5
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$
y = c_1 y_1 + c_2 y_2
$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$
y = \frac{c_1}{x^2} + c_2 x^5
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_1}{x^2} + c_2 x^5 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 1$ in the above gives

$$
5 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\frac{2c_1}{x^3} + 5c_2 x^4
$$

substituting $y' = 4$ and $x = 1$ in the above gives

$$
4 = -2c_1 + 5c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= 2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{2x^7 + 3}{x^2}
$$

Summary

The solution(s) found are the following

$$y = \frac{2x^7 + 3}{x^2} \tag{1}$$

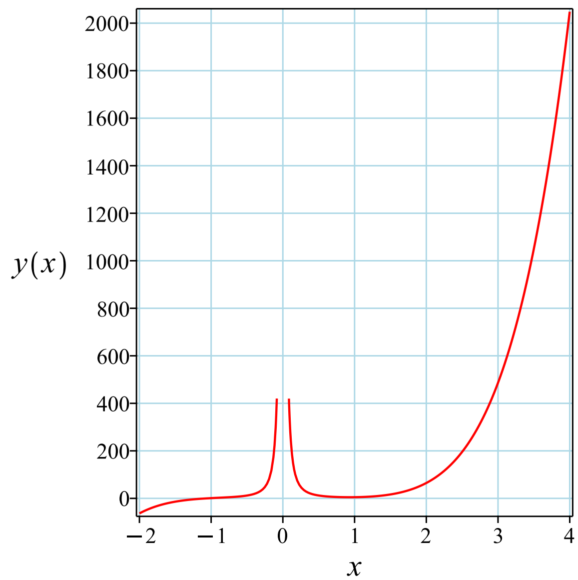

Figure 520: Solution plot

Verification of solutions

$$y = \frac{2x^7 + 3}{x^2}$$

Verified OK.

13.20.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - 2y'x - 10y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = -\frac{10}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\left(x\right)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{2}{x}dx\right)}dx \\
&= \int e^{2\ln(x)}\,dx \\
&= \int x^2 dx \\
&= \frac{x^3}{3}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'\left(x\right)^2} \\
&= \frac{-\frac{10}{x^2}}{x^4} \\
&= -\frac{10}{x^6}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{10y(\tau)}{x^6} &= 0
\end{aligned}$$

But in terms of τ

$$-\frac{10}{x^6} = -\frac{10}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{10y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 10y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 10\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r - 10\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 - 10 = 0$$

Or

$$9r^2 - 9r - 10 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{2}{3}$$
$$r_2 = \frac{5}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau^{\frac{2}{3}}} + c_2\tau^{\frac{5}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{3^{\frac{1}{3}}\left(c_2x^6(x^3)^{\frac{1}{3}} + 9c_1 3^{\frac{1}{3}}\right)}{9\,(x^3)^{\frac{2}{3}}}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{3^{\frac{1}{3}} \left(c_2 x^6 (x^3)^{\frac{1}{3}} + 9 c_1 3^{\frac{1}{3}}\right)}{9 \left(x^3\right)^{\frac{2}{3}}} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 1$ in the above gives

$$5 = 3^{\frac{2}{3}} c_1 + \frac{3^{\frac{1}{3}} c_2}{9} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{3^{\frac{1}{3}} \left(6 c_2 x^5 (x^3)^{\frac{1}{3}} + \frac{c_2 x^8}{(x^3)^{\frac{2}{3}}}\right)}{9 \left(x^3\right)^{\frac{2}{3}}} - \frac{2\, 3^{\frac{1}{3}} \left(c_2 x^6 (x^3)^{\frac{1}{3}} + 9 c_1 3^{\frac{1}{3}}\right) x^2}{9 \left(x^3\right)^{\frac{5}{3}}}$$

substituting $y' = 4$ and $x = 1$ in the above gives

$$4 = -2\, 3^{\frac{2}{3}} c_1 + \frac{5\, 3^{\frac{1}{3}} c_2}{9} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3^{\frac{1}{3}}$$
$$c_2 = 6\, 3^{\frac{2}{3}}$$

Substituting these values back in above solution results in

$$y = \frac{2x^6 (x^3)^{\frac{1}{3}} + 3}{(x^3)^{\frac{2}{3}}}$$

Summary

The solution(s) found are the following

$$y = \frac{2x^6 (x^3)^{\frac{1}{3}} + 3}{(x^3)^{\frac{2}{3}}} \tag{1}$$

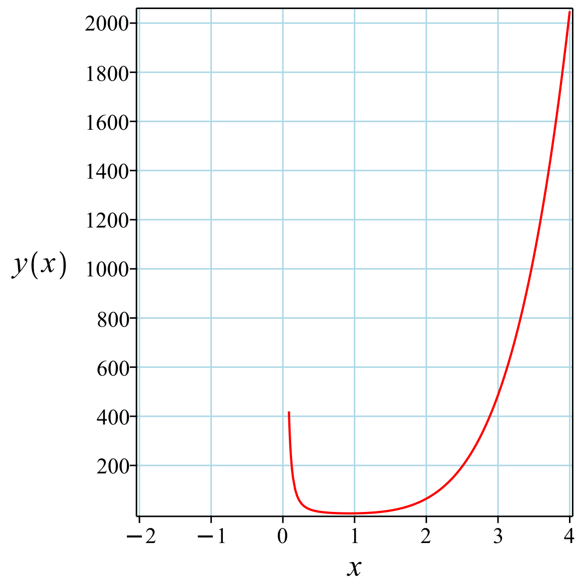

Figure 521: Solution plot

Verification of solutions

$$y = \frac{2x^6 (x^3)^{\frac{1}{3}} + 3}{(x^3)^{\frac{2}{3}}}$$

Verified OK.

13.20.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' - 2y'x - 10y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = -\frac{10}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{2n}{x^2} - \frac{10}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 5 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{8v'(x)}{x} &= 0 \\ v''(x) + \frac{8v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{8u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{8u}{x} \end{aligned}$$

Where $f(x) = -\frac{8}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{8}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{8}{x}\,dx \\ \ln(u) &= -8\ln(x) + c_1 \\ u &= e^{-8\ln(x)+c_1} \\ &= \frac{c_1}{x^8} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{7x^7} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{7x^7} + c_2\right) x^5 \\
&= \frac{7c_2x^7 - c_1}{7x^2}
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \left(-\frac{c_1}{7x^7} + c_2\right) x^5 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 1$ in the above gives

$$
5 = -\frac{c_1}{7} + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{c_1}{x^3} + 5\left(-\frac{c_1}{7x^7} + c_2\right) x^4
$$

substituting $y' = 4$ and $x = 1$ in the above gives

$$
4 = \frac{2c_1}{7} + 5c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -21 \\
c_2 &= 2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{2x^7 + 3}{x^2}
$$

Summary
The solution(s) found are the following

$$y = \frac{2x^7 + 3}{x^2} \tag{1}$$

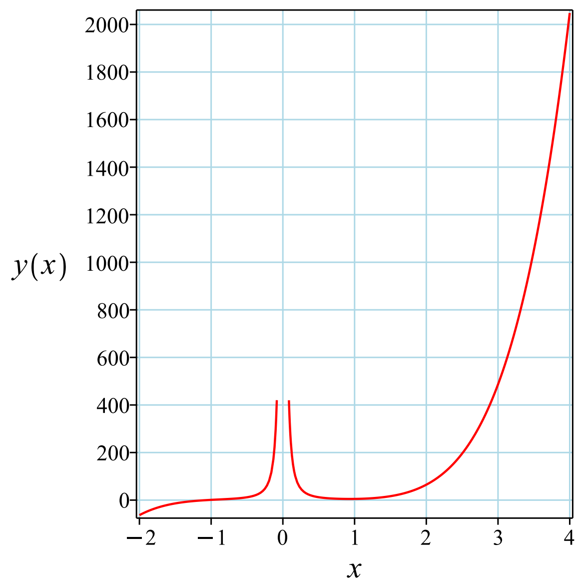

Figure 522: Solution plot

Verification of solutions

$$y = \frac{2x^7 + 3}{x^2}$$

Verified OK.

13.20.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 2y'x - 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= -10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{12}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 12 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{12}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 497: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{12}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition

of r given above. Therefore $b = 12$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{12}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 12$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -3 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{12}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	4	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -3$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\ &= -3 - (-3) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{3}{x} + (-)\,(0) \\ &= -\frac{3}{x} \\ &= -\frac{3}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{3}{x}\right)(0) + \left(\left(\frac{3}{x^2}\right) + \left(-\frac{3}{x}\right)^2 - \left(\frac{12}{x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{3}{x} dx} \\ &= \frac{1}{x^3} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^7}{7}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^2}\right) + c_2\left(\frac{1}{x^2}\left(\frac{x^7}{7}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_1}{x^2} + \frac{c_2 x^5}{7} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 1$ in the above gives

$$5 = c_1 + \frac{c_2}{7} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{2c_1}{x^3} + \frac{5c_2x^4}{7}$$

substituting $y' = 4$ and $x = 1$ in the above gives

$$4 = -2c_1 + \frac{5c_2}{7} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 14$$

Substituting these values back in above solution results in

$$y = \frac{2x^7 + 3}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{2x^7 + 3}{x^2} \tag{1}$$

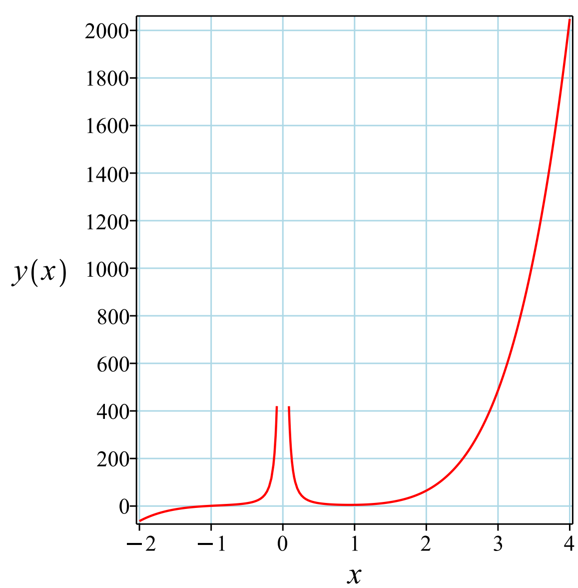

Figure 523: Solution plot

Verification of solutions

$$y = \frac{2x^7 + 3}{x^2}$$

Verified OK.

13.20.6 Maple step by step solution

Let's solve

$$\left[y''x^2 - 2y'x - 10y = 0, y(1) = 5, y'\Big|_{\{x=1\}} = 4\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y'}{x} + \frac{10y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} - \frac{10y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 - 2y'x - 10y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)x^2-2\frac{d}{dt}y(t)-10y(t)=0$$

- Simplify

  $\frac{d^2}{dt^2}y(t)-3\frac{d}{dt}y(t)-10y(t)=0$
- Characteristic polynomial of ODE

  $r^2-3r-10=0$
- Factor the characteristic polynomial

  $(r+2)(r-5)=0$
- Roots of the characteristic polynomial

  $r=(-2,5)$
- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-2t}$
- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{5t}$
- General solution of the ODE

  $y(t)=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y(t)=c_1\mathrm{e}^{-2t}+c_2\mathrm{e}^{5t}$
- Change variables back using $t=\ln(x)$

  $y=\frac{c_1}{x^2}+c_2x^5$
- Simplify

  $y=\frac{c_1}{x^2}+c_2x^5$

☐ Check validity of solution $y=\frac{c_1}{x^2}+c_2x^5$

  - Use initial condition $y(1)=5$

    $5=c_1+c_2$
  - Compute derivative of the solution

    $y'=-\frac{2c_1}{x^3}+5c_2x^4$
  - Use the initial condition $y'\Big|_{\{x=1\}}=4$

    $4=-2c_1+5c_2$

- Solve for c_1 and c_2

  $\{c_1 = 3, c_2 = 2\}$

- Substitute constant values into general solution and simplify

  $y = \frac{3}{x^2} + 2x^5$

- Solution to the IVP

  $y = \frac{3}{x^2} + 2x^5$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([x^2*diff(y(x),x$2)-2*x*diff(y(x),x)-10*y(x)=0,y(1) = 5, D(y)(1) = 4],y(x), singsol=a
```

$$y(x) = 2x^5 + \frac{3}{x^2}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 16

```
DSolve[{x^2*y''[x]-2*x*y'[x]-10*y[x]==0,{y[1]==5,y'[1]==4}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{2x^7 + 3}{x^2}$$

13.21 problem 21

Internal problem ID [11875]
Internal file name [`OUTPUT/11884_Saturday_April_13_2024_01_13_56_AM_26227055/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 4y'x + 6y = 0}$$

With initial conditions

$$[y(2) = 0, y'(2) = 4]$$

13.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{4y'}{x} + \frac{6y}{x^2} = 0$$

The domain of $p(x) = -\frac{4}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = \frac{6}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

13.21.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 6 = 0$$

Or

$$r^2 - 5r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= 2 \\
r_2 &= 3
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2 x^3 + c_1 x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x^3 + c_1 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$0 = 8c_2 + 4c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_2 x^2 + 2c_1 x$$

substituting $y' = 4$ and $x = 2$ in the above gives

$$4 = 12c_2 + 4c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -2 \\
c_2 &= 1
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = x^3 - 2x^2$$

Which simplifies to

$$y = (x - 2)\, x^2$$

Summary
The solution(s) found are the following

$$y = (x-2)\,x^2 \tag{1}$$

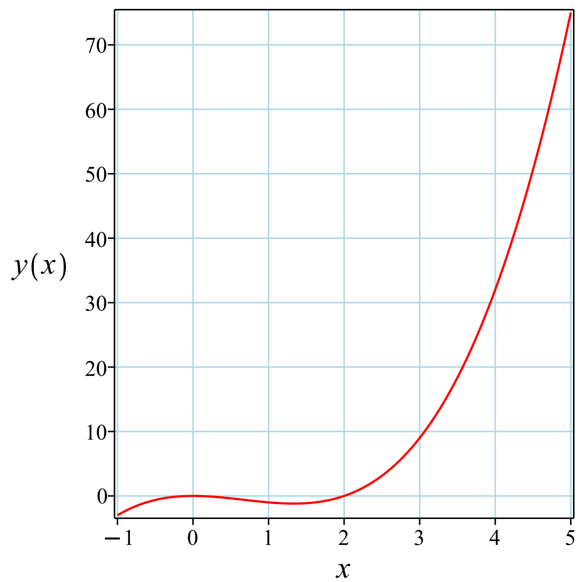

Figure 524: Solution plot

Verification of solutions

$$y = (x-2)\,x^2$$

Verified OK.

13.21.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -\frac{4}{x}\,dx} \\ &= \frac{1}{x^2} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 0$$
$$\left(\frac{y}{x^2}\right)'' = 0$$

Integrating once gives

$$\left(\frac{y}{x^2}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{x^2}\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\frac{1}{x^2}}$$

Or

$$y = c_1 x^3 + c_2 x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x^3 + c_2 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$0 = 8c_1 + 4c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1 x^2 + 2c_2 x$$

substituting $y' = 4$ and $x = 2$ in the above gives

$$4 = 12c_1 + 4c_2 \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$y = x^3 - 2x^2$$

Which simplifies to

$$y = (x - 2) x^2$$

Summary

The solution(s) found are the following

$$y = (x - 2) x^2 \tag{1}$$

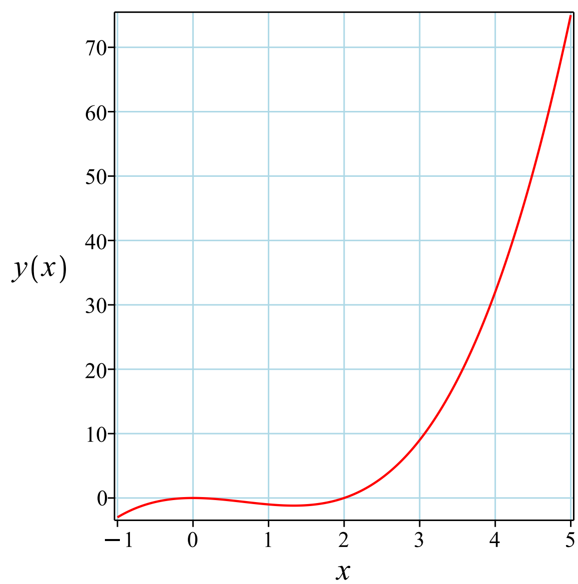

Figure 525: Solution plot

Verification of solutions

$$y = (x - 2) x^2$$

Verified OK.

13.21.4 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{4}{x}dx\right)}dx \\ &= \int e^{4\ln(x)}\, dx \\ &= \int x^4 dx \\ &= \frac{x^5}{5} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{6}{x^2}}{x^8} \\ &= \frac{6}{x^{10}} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^{10}} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{6}{x^{10}} = \frac{6}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 6 = 0$$

Or

$$25r^2 - 25r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{2}{5}$$
$$r_2 = \frac{3}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{2}{5}} + c_2 \tau^{\frac{3}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$0 = \frac{4\, 5^{\frac{2}{5}} \left(5^{\frac{1}{5}} c_1 + 2c_2\right)}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2c_1 5^{\frac{3}{5}} x^4}{5 (x^5)^{\frac{3}{5}}} + \frac{3c_2 5^{\frac{2}{5}} x^4}{5 (x^5)^{\frac{2}{5}}}$$

substituting $y' = 4$ and $x = 2$ in the above gives

$$4 = \frac{4\left(5^{\frac{1}{5}} c_1 + 3c_2\right) 5^{\frac{2}{5}}}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2\,5^{\frac{2}{5}}$$
$$c_2 = 5^{\frac{3}{5}}$$

Substituting these values back in above solution results in

$$y = (x^5)^{\frac{3}{5}} - 2(x^5)^{\frac{2}{5}}$$

Summary

The solution(s) found are the following

$$y = (x^5)^{\frac{3}{5}} - 2(x^5)^{\frac{2}{5}} \tag{1}$$

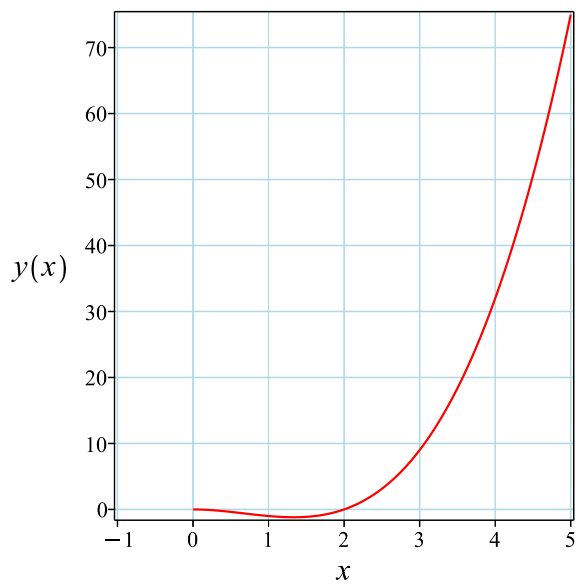

Figure 526: Solution plot

Verification of solutions

$$y = (x^5)^{\frac{3}{5}} - 2(x^5)^{\frac{2}{5}}$$

Verified OK.

13.21.5 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{5c\sqrt{6}}{6} \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) - \frac{5c\sqrt{6}\left(\frac{d}{d\tau} y(\tau)\right)}{6} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\, c\tau}{12}} \left(c_1 \cosh\left(\frac{\sqrt{6}\, c\tau}{12}\right) + i c_2 \sinh\left(\frac{\sqrt{6}\, c\tau}{12}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{6}\sqrt{\frac{1}{x^2}} dx}{c} \\
&= \frac{\sqrt{6}\sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$
0 = 2ic_2 + 6c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{5x^{\frac{3}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)}{2} + x^{\frac{5}{2}} \left(\frac{c_1 \sinh\left(\frac{\ln(x)}{2}\right)}{2x} + \frac{i c_2 \cosh\left(\frac{\ln(x)}{2}\right)}{2x}\right)
$$

substituting $y' = 4$ and $x = 2$ in the above gives

$$4 = 4ic_2 + 8c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = -3i$$

Substituting these values back in above solution results in

$$y = 3x^{\frac{5}{2}} \sinh\left(\frac{\ln(x)}{2}\right) - x^{\frac{5}{2}} \cosh\left(\frac{\ln(x)}{2}\right)$$

Which simplifies to

$$y = -\left(\cosh\left(\frac{\ln(x)}{2}\right) - 3\sinh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{5}{2}}$$

Summary

The solution(s) found are the following

$$y = -\left(\cosh\left(\frac{\ln(x)}{2}\right) - 3\sinh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{5}{2}} \tag{1}$$

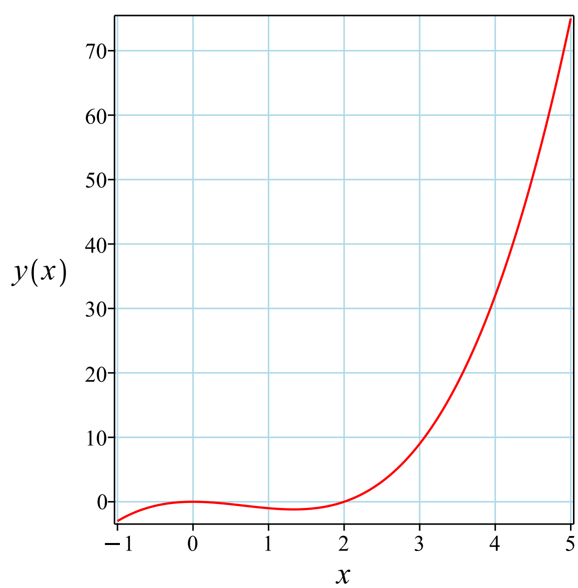

Figure 527: Solution plot

Verification of solutions

$$y=-\left(\cosh\left(\frac{\ln(x)}{2}\right)-3\sinh\left(\frac{\ln(x)}{2}\right)\right)x^{\frac{5}{2}}$$

Verified OK.

13.21.6 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x}\\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q-\frac{p'}{2}-\frac{p^2}{4}\\ &= \frac{6}{x^2}-\frac{\left(-\frac{4}{x}\right)'}{2}-\frac{\left(-\frac{4}{x}\right)^2}{4}\\ &= \frac{6}{x^2}-\frac{\left(\frac{4}{x^2}\right)}{2}-\frac{\left(\frac{16}{x^2}\right)}{4}\\ &= \frac{6}{x^2}-\left(\frac{2}{x^2}\right)-\frac{4}{x^2}\\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y=v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2}dx\right)}\\ &= e^{-\int \frac{-\frac{4}{x}}{2}}\\ &= x^2 \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x^2 \tag{4}$$

Applying this change of variable to the original ode results in

$$x^4 v''(x) = 0$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x + c_2)\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = (c_1 x + c_2)\, x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (c_1 x + c_2)\, x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$0 = 8c_1 + 4c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 x^2 + 2(c_1 x + c_2)\, x$$

substituting $y' = 4$ and $x = 2$ in the above gives

$$4 = 12c_1 + 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$y = (x-2)\, x^2$$

Summary

The solution(s) found are the following

$$y = (x-2)\, x^2 \tag{1}$$

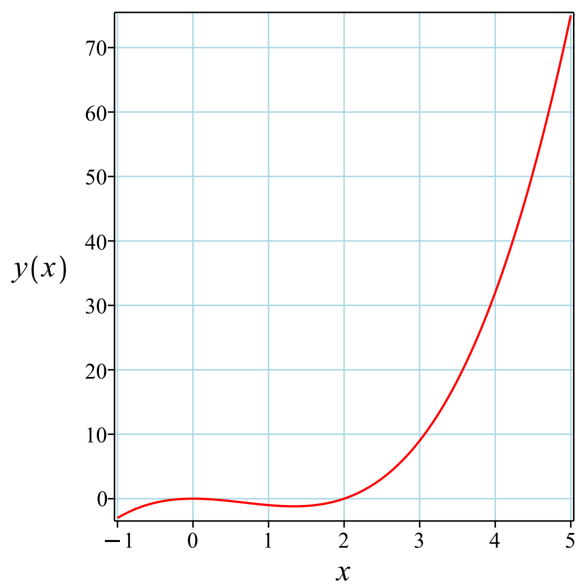

Figure 528: Solution plot

Verification of solutions

$$y = (x-2)\, x^2$$

Verified OK.

13.21.7 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{2u}{x}
\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{2}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{2}{x}\, dx \\
\ln(u) &= -2\ln(x) + c_1 \\
u &= \mathrm{e}^{-2\ln(x)+c_1} \\
&= \frac{c_1}{x^2}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{x} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{x} + c_2\right) x^3 \\
&= (c_2 x - c_1)\, x^2
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \left(-\frac{c_1}{x} + c_2\right) x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$0 = -4c_1 + 8c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1x + 3\left(-\frac{c_1}{x} + c_2\right)x^2$$

substituting $y' = 4$ and $x = 2$ in the above gives

$$4 = -4c_1 + 12c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = (x - 2)\,x^2$$

Summary

The solution(s) found are the following

$$y = (x - 2)\,x^2 \tag{1}$$

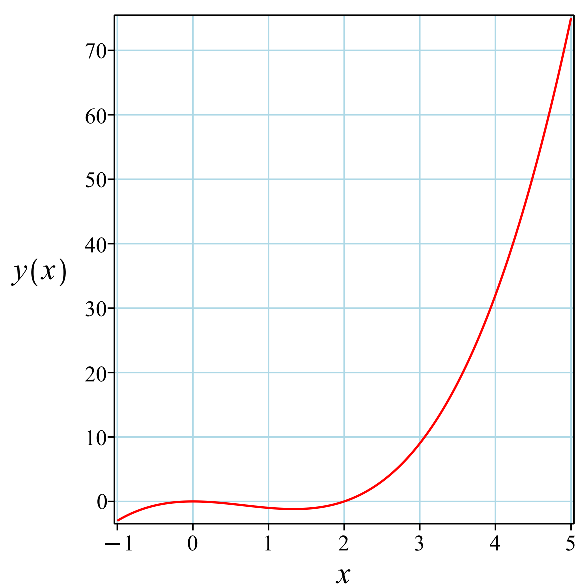

Figure 529: Solution plot

Verification of solutions

$$y = (x - 2)\, x^2$$

Verified OK.

13.21.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 499: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\
&= z_1 e^{2\ln(x)} \\
&= z_1\left(x^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(x)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_2 x^3 + c_1 x^2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$
0 = 8c_2 + 4c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = 3c_2x^2 + 2c_1x$$

substituting $y' = 4$ and $x = 2$ in the above gives

$$4 = 12c_2 + 4c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = x^3 - 2x^2$$

Which simplifies to

$$y = (x - 2)\, x^2$$

Summary

The solution(s) found are the following

$$y = (x - 2)\, x^2 \tag{1}$$

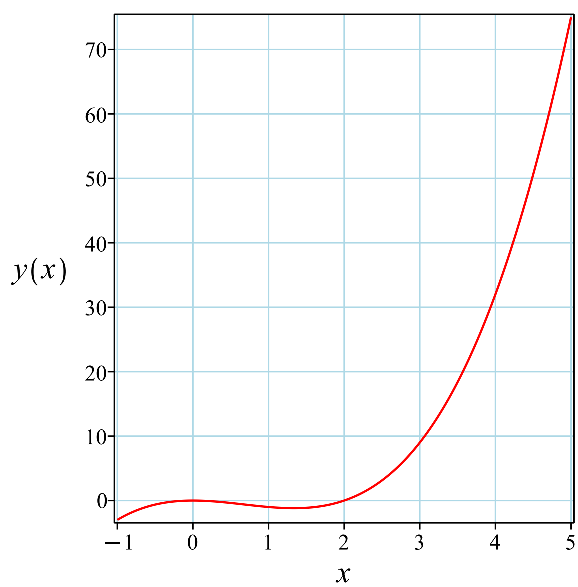

Figure 530: Solution plot

Verification of solutions

$$y = (x - 2)\, x^2$$

Verified OK.

13.21.9 Maple step by step solution

Let's solve

$$\left[y''x^2-4y'x+6y=0, y(2)=0, y'\Big|_{\{x=2\}}=4\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{4y'}{x}-\frac{6y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{4y'}{x}+\frac{6y}{x^2}=0$

- Multiply by denominators of the ODE

  $y''x^2-4y'x+6y=0$

- Make a change of variables

  $t=\ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y'=\left(\frac{d}{dt}y(t)\right)t'(x)$

  ◦ Compute derivative

    $y'=\frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y''=\left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2+t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y''=\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)x^2-4\frac{d}{dt}y(t)+6y(t)=0$

- Simplify

  $\frac{d^2}{dt^2}y(t)-5\frac{d}{dt}y(t)+6y(t)=0$

- Characteristic polynomial of ODE

$r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the ODE

  $y_1(t) = e^{2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{3t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{2t} + c_2 e^{3t}$

- Change variables back using $t = \ln(x)$

  $y = c_2 x^3 + c_1 x^2$

- Simplify

  $y = x^2(c_2 x + c_1)$

□ Check validity of solution $y = x^2(c_2 x + c_1)$

  - Use initial condition $y(2) = 0$

    $0 = 8c_2 + 4c_1$

  - Compute derivative of the solution

    $y' = 2x(c_2 x + c_1) + c_2 x^2$

  - Use the initial condition $y'\Big|_{\{x=2\}} = 4$

    $4 = 12c_2 + 4c_1$

  - Solve for c_1 and c_2

    $\{c_1 = -2, c_2 = 1\}$

  - Substitute constant values into general solution and simplify

    $y = (x-2)\, x^2$

- Solution to the IVP

$$y = (x-2)\,x^2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 11

```
dsolve([x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=0,y(2) = 0, D(y)(2) = 4],y(x), singsol=al
```

$$y(x) = x^2(x-2)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 12

```
DSolve[{x^2*y''[x]-4*x*y'[x]+6*y[x]==0,{y[2]==0,y'[2]==4}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to (x-2)x^2$$

13.22 problem 22

Internal problem ID [11876]
Internal file name [`OUTPUT/11885_Saturday_April_13_2024_01_13_58_AM_6549318/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2y'' + 5y'x + 3y = 0}$$

With initial conditions

$$[y(1) = 1, y'(1) = -5]$$

13.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{5}{x} \\ q(x) &= \frac{3}{x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{5y'}{x} + \frac{3y}{x^2} = 0$$

The domain of $p(x) = \frac{5}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{3}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

13.22.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 5xrx^{r-1} + 3x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 5r\,x^r + 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 5r + 3 = 0$$

Or

$$r^2 + 4r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -3$$
$$r_2 = -1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^3} + \frac{c_2}{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x^3} + \frac{c_2}{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{3c_1}{x^4} - \frac{c_2}{x^2}$$

substituting $y' = -5$ and $x = 1$ in the above gives

$$-5 = -3c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = -\frac{x^2 - 2}{x^3}$$

Summary
The solution(s) found are the following

$$y = -\frac{x^2 - 2}{x^3} \tag{1}$$

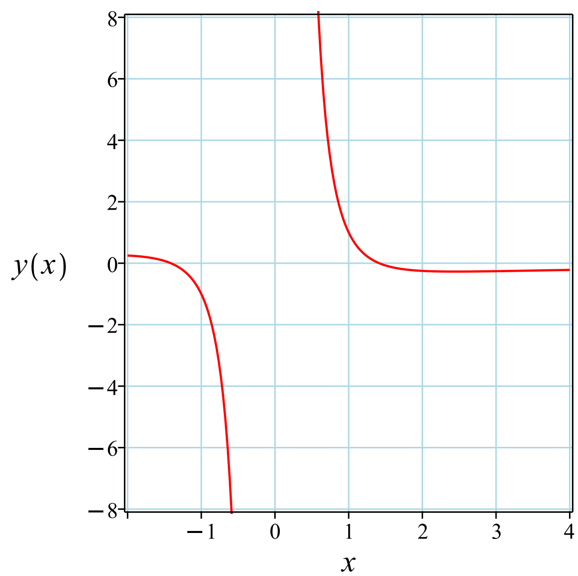

Figure 531: Solution plot

Verification of solutions

$$y = -\frac{x^2 - 2}{x^3}$$

Verified OK.

13.22.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' + 5y'x + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{5}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{5}{x}dx\right)}dx \\
&= \int e^{-5\ln(x)}\,dx \\
&= \int \frac{1}{x^5}dx \\
&= -\frac{1}{4x^4} \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{3}{x^2}}{\frac{1}{x^{10}}} \\
&= 3x^8 \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 3x^8 y(\tau) &= 0
\end{aligned}$$

But in terms of τ

$$3x^8 = \frac{3}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 3\tau^r = 0$$

Simplifying gives

$$16r(r-1)\,\tau^r + 0\,\tau^r + 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$16r(r-1) + 0 + 3 = 0$$

Or

$$16r^2 - 16r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{4}$$
$$r_2 = \frac{3}{4}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{4}} + c_2\tau^{\frac{3}{4}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\left(-\frac{1}{x^4}\right)^{\frac{1}{4}}\left(c_2\sqrt{-\frac{1}{x^4}} + 2c_1\right)}{4}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{\sqrt{2}\left(-\frac{1}{x^4}\right)^{\frac{1}{4}}\left(c_2\sqrt{-\frac{1}{x^4}}+2c_1\right)}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = \left(\frac{1}{2}+\frac{i}{2}\right)c_1 + \left(-\frac{1}{4}+\frac{i}{4}\right)c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\sqrt{2}\left(c_2\sqrt{-\frac{1}{x^4}}+2c_1\right)}{4\left(-\frac{1}{x^4}\right)^{\frac{3}{4}}x^5} + \frac{\sqrt{2}\,c_2}{2\left(-\frac{1}{x^4}\right)^{\frac{1}{4}}x^5}$$

substituting $y' = -5$ and $x = 1$ in the above gives

$$-5 = \left(-\frac{1}{2}-\frac{i}{2}\right)c_1 + \left(\frac{3}{4}-\frac{3i}{4}\right)c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -1+i \\ c_2 &= -4-4i \end{aligned}$$

Substituting these values back in above solution results in

$$y = -i\sqrt{2}\left(-\frac{1}{x^4}\right)^{\frac{3}{4}} + \frac{i\sqrt{2}\left(-\frac{1}{x^4}\right)^{\frac{1}{4}}}{2} - \sqrt{2}\left(-\frac{1}{x^4}\right)^{\frac{3}{4}} - \frac{\sqrt{2}\left(-\frac{1}{x^4}\right)^{\frac{1}{4}}}{2}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\left((1+i)\sqrt{-\frac{1}{x^4}}+\frac{1}{2}-\frac{i}{2}\right)\sqrt{2}\left(-\frac{1}{x^4}\right)^{\frac{1}{4}} \tag{1}$$

<u>Verification of solutions</u>

$$y = -\left((1+i)\sqrt{-\frac{1}{x^4}}+\frac{1}{2}-\frac{i}{2}\right)\sqrt{2}\left(-\frac{1}{x^4}\right)^{\frac{1}{4}}$$

Verified OK.

13.22.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' + 5y'x + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{5}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{5}{x}\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= \frac{4c\sqrt{3}}{3} \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + \frac{4c\sqrt{3}\left(\frac{d}{d\tau} y(\tau)\right)}{3} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-\frac{2\sqrt{3}\, c\tau}{3}} \left(c_1 \cosh\left(\frac{\sqrt{3}\, c\tau}{3}\right) + i c_2 \sinh\left(\frac{\sqrt{3}\, c\tau}{3}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dx \\
&= \frac{\int \sqrt{3}\, \sqrt{\frac{1}{x^2}} dx}{c} \\
&= \frac{\sqrt{3}\, \sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{(i c_2 + c_1)\, x^2 - i c_2 + c_1}{2x^3}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{(i c_2 + c_1)\, x^2 - i c_2 + c_1}{2x^3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{i c_2 + c_1}{x^2} - \frac{3\left((i c_2 + c_1)\, x^2 - i c_2 + c_1\right)}{2x^4}
$$

substituting $y' = -5$ and $x = 1$ in the above gives

$$-5 = ic_2 - 2c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 3i$$

Substituting these values back in above solution results in

$$y = \frac{-x^2 + 2}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{-x^2 + 2}{x^3} \tag{1}$$

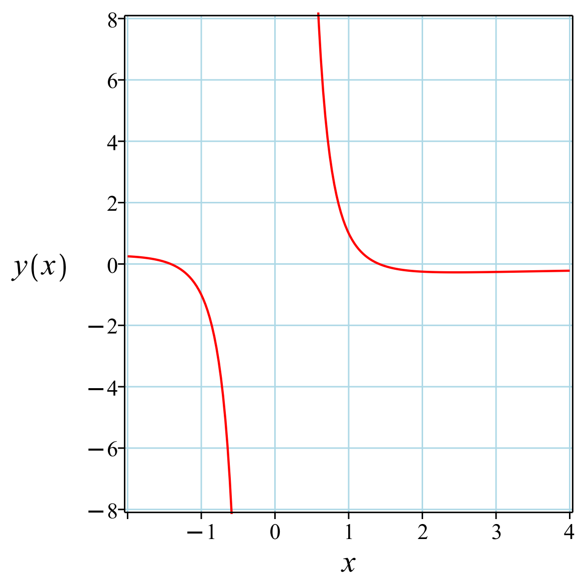

Figure 532: Solution plot

Verification of solutions

$$y = \frac{-x^2 + 2}{x^3}$$

Verified OK.

13.22.5 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + 5y'x + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{5}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{5n}{x^2} + \frac{3}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{3u}{x}
\end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{3}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\
\ln(u) &= -3\ln(x) + c_1 \\
u &= e^{-3\ln(x)+c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= \frac{-\frac{c_1}{2x^2} + c_2}{x} \\
&= \frac{2c_2x^2 - c_1}{2x^3}
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{-\frac{c_1}{2x^2} + c_2}{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = -\frac{c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1}{x^4} - \frac{-\frac{c_1}{2x^2} + c_2}{x^2}$$

substituting $y' = -5$ and $x = 1$ in the above gives

$$-5 = \frac{3c_1}{2} - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -4$$
$$c_2 = -1$$

Substituting these values back in above solution results in

$$y = -\frac{x^2 - 2}{x^3}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^2 - 2}{x^3} \tag{1}$$

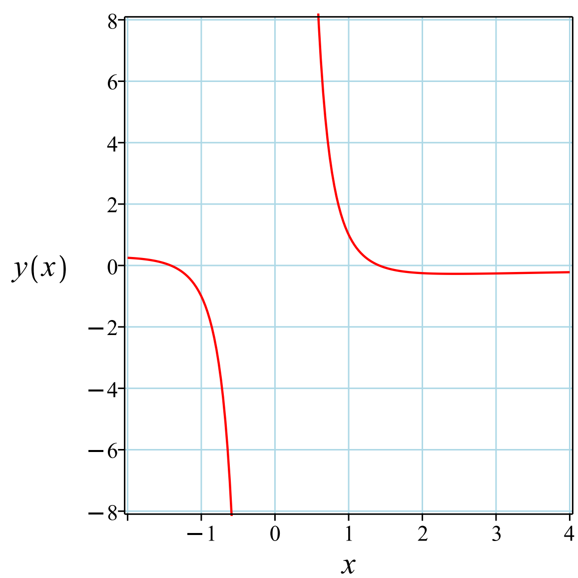

Figure 533: Solution plot

Verification of solutions

$$y = -\frac{x^2 - 2}{x^3}$$

Verified OK.

13.22.6 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x^2 y'' + 5y'x + 3y\right) dx = 0$$
$$x^2 y' + 3yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{3}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' + \frac{3y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{3}{x} dx}$$
$$= x^3$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x^2}\right)$$
$$\frac{d}{dx}\left(y\,x^3\right) = \left(x^3\right)\left(\frac{c_1}{x^2}\right)$$
$$d\left(y\,x^3\right) = (c_1 x)\ dx$$

Integrating gives

$$y\,x^3 = \int c_1 x \,dx$$
$$y\,x^3 = \frac{c_1 x^2}{2} + c_2$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$y = \frac{c_1}{2x} + \frac{c_2}{x^3}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{2x} + \frac{c_2}{x^3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_2 + \frac{c_1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{2x^2} - \frac{3c_2}{x^4}$$

substituting $y' = -5$ and $x = 1$ in the above gives

$$-5 = -\frac{c_1}{2} - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = -\frac{x^2 - 2}{x^3}$$

Summary
The solution(s) found are the following

$$y = -\frac{x^2 - 2}{x^3} \tag{1}$$

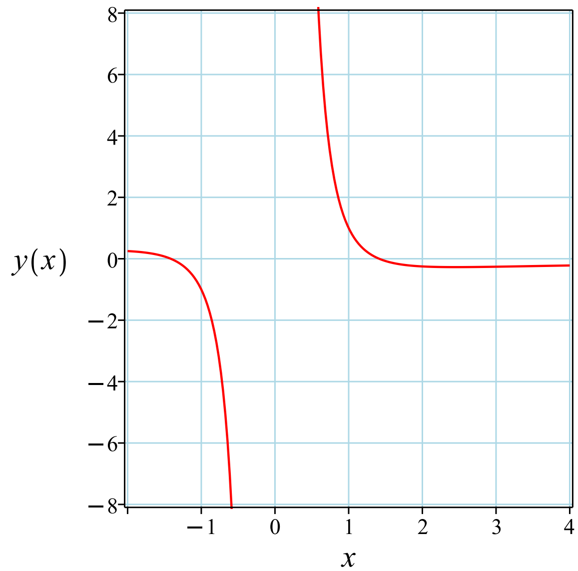

Figure 534: Solution plot

Verification of solutions

$$y = -\frac{x^2 - 2}{x^3}$$

Verified OK.

13.22.7 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' + 5y'x + 3y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(x^2y'' + 5y'x + 3y\right) dx = 0$$
$$x^2y' + 3yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \frac{3}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y' + \frac{3y}{x} = \frac{c_1}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{3}{x} dx} \\
&= x^3
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\
\frac{d}{dx}\left(y\, x^3\right) &= \left(x^3\right)\left(\frac{c_1}{x^2}\right) \\
d\left(y\, x^3\right) &= (c_1 x)\ dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\, x^3 &= \int c_1 x\, dx \\
y\, x^3 &= \frac{c_1 x^2}{2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$
y = \frac{c_1}{2x} + \frac{c_2}{x^3}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_1}{2x} + \frac{c_2}{x^3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$
1 = c_2 + \frac{c_1}{2} \tag{1A}
$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{2x^2} - \frac{3c_2}{x^4}$$

substituting $y' = -5$ and $x = 1$ in the above gives

$$-5 = -\frac{c_1}{2} - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = -\frac{x^2 - 2}{x^3}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^2 - 2}{x^3} \tag{1}$$

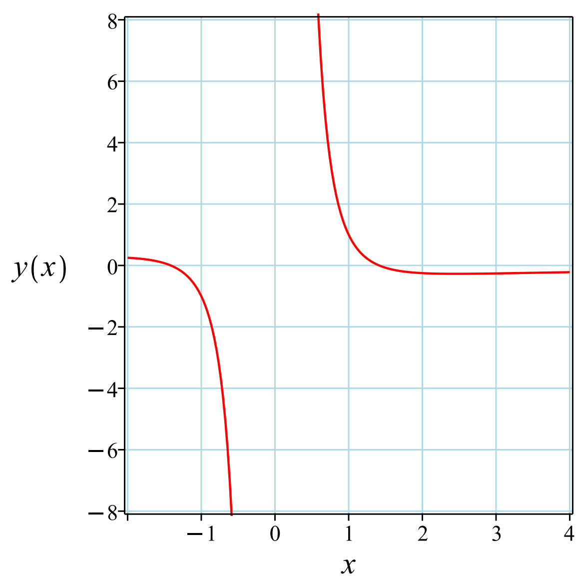

Figure 535: Solution plot

Verification of solutions

$$y = -\frac{x^2 - 2}{x^3}$$

Verified OK.

13.22.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 5y'x + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 5x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 501: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5x}{x^2}\,dx} \\
&= z_1 e^{-\frac{5\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{\frac{5}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{5x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^3}\right) + c_2\left(\frac{1}{x^3}\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_1}{x^3} + \frac{c_2}{2x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 + \frac{c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{3c_1}{x^4} - \frac{c_2}{2x^2}$$

substituting $y' = -5$ and $x = 1$ in the above gives

$$-5 = -3c_1 - \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$y = -\frac{x^2 - 2}{x^3}$$

Summary
The solution(s) found are the following

$$y = -\frac{x^2 - 2}{x^3} \tag{1}$$

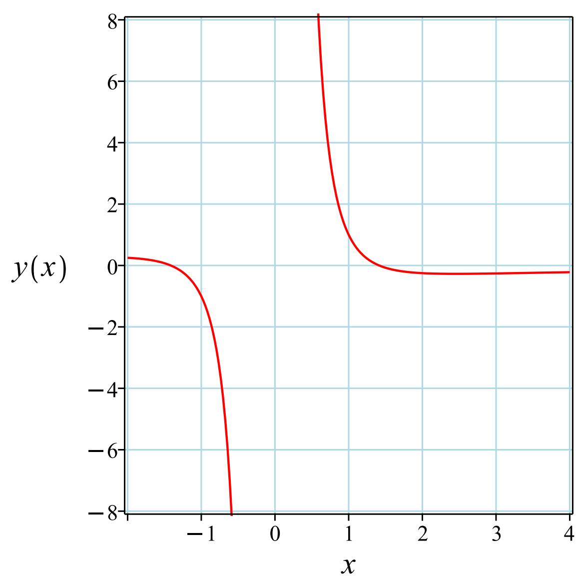

Figure 536: Solution plot

Verification of solutions

$$y = -\frac{x^2 - 2}{x^3}$$

Verified OK.

13.22.9 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= x^2 \\
q(x) &= 5x \\
r(x) &= 3 \\
s(x) &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 5
\end{aligned}$$

Therefore (1) becomes

$$2 - (5) + (3) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$x^2 y' + 3yx = c_1$$

We now have a first order ode to solve which is

$$x^2y' + 3yx = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{c_1}{x^2} \end{aligned}$$

Hence the ode is

$$y' + \frac{3y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{3}{x}dx} \\ &= x^3 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\ \frac{d}{dx}(y\,x^3) &= (x^3)\left(\frac{c_1}{x^2}\right) \\ d(y\,x^3) &= (c_1 x)\ dx \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,x^3 &= \int c_1 x\,dx \\ y\,x^3 &= \frac{c_1 x^2}{2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$y = \frac{c_1}{2x} + \frac{c_2}{x^3}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{2x} + \frac{c_2}{x^3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_2 + \frac{c_1}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{2x^2} - \frac{3c_2}{x^4}$$

substituting $y' = -5$ and $x = 1$ in the above gives

$$-5 = -\frac{c_1}{2} - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 2$$

Substituting these values back in above solution results in

$$y = -\frac{x^2 - 2}{x^3}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^2 - 2}{x^3} \tag{1}$$

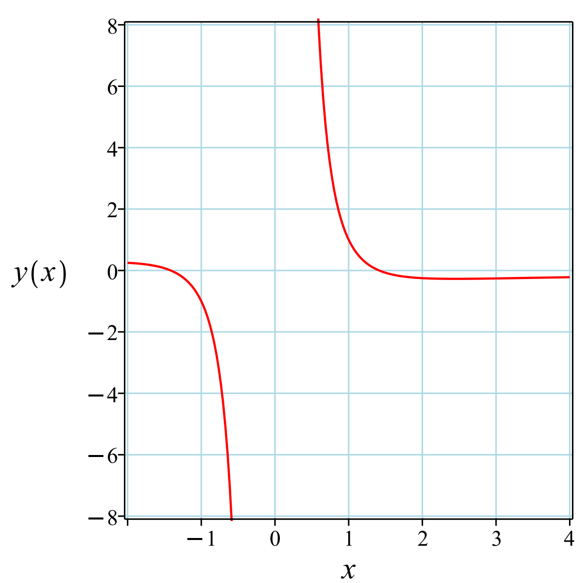

Figure 537: Solution plot

Verification of solutions

$$y = -\frac{x^2-2}{x^3}$$

Verified OK.

13.22.10 Maple step by step solution

Let's solve

$\left[y''x^2+5y'x+3y=0, y(1)=1, y'\Big|_{\{x=1\}}=-5\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{5y'}{x}-\frac{3y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{5y'}{x}+\frac{3y}{x^2}=0$

- Multiply by denominators of the ODE

  $y''x^2+5y'x+3y=0$

- Make a change of variables

  $t=\ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y'=\left(\frac{d}{dt}y(t)\right)t'(x)$

  ◦ Compute derivative

    $y'=\frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y''=\left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2+t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y''=\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)x^2+5\frac{d}{dt}y(t)+3y(t)=0$$

- Simplify

  $\frac{d^2}{dt^2}y(t)+4\frac{d}{dt}y(t)+3y(t)=0$
- Characteristic polynomial of ODE

  $r^2+4r+3=0$
- Factor the characteristic polynomial

  $(r+3)(r+1)=0$
- Roots of the characteristic polynomial

  $r=(-3,-1)$
- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-3t}$
- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{-t}$
- General solution of the ODE

  $y(t)=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y(t)=c_1\mathrm{e}^{-3t}+c_2\mathrm{e}^{-t}$
- Change variables back using $t=\ln(x)$

  $y=\frac{c_1}{x^3}+\frac{c_2}{x}$
- Simplify

  $y=\frac{c_1}{x^3}+\frac{c_2}{x}$

☐ Check validity of solution $y=\frac{c_1}{x^3}+\frac{c_2}{x}$

- Use initial condition $y(1)=1$

  $1=c_1+c_2$
- Compute derivative of the solution

  $y'=-\frac{3c_1}{x^4}-\frac{c_2}{x^2}$
- Use the initial condition $y'\Big|_{\{x=1\}}=-5$

  $-5=-3c_1-c_2$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = -1\}$
- Substitute constant values into general solution and simplify

  $y = \frac{-x^2+2}{x^3}$

- Solution to the IVP

  $y = \frac{-x^2+2}{x^3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([x^2*diff(y(x),x$2)+5*x*diff(y(x),x)+3*y(x)=0,y(1) = 1, D(y)(1) = -5],y(x), singsol=a
```

$$y(x) = \frac{-x^2+2}{x^3}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 16

```
DSolve[{x^2*y''[x]+5*x*y'[x]+3*y[x]==0,{y[1]==1,y'[1]==-5}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{2-x^2}{x^3}$$

13.23 problem 23

Internal problem ID [11877]
Internal file name [`OUTPUT/11886_Saturday_April_13_2024_01_14_01_AM_52727080/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - 2y = 4x - 8}$$

With initial conditions

$$[y(1) = 4, y'(1) = -1]$$

13.23.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -\frac{2}{x^2} \\ F &= \frac{4x-8}{x^2} \end{aligned}$$

Hence the ode is

$$y'' - \frac{2y}{x^2} = \frac{4x-8}{x^2}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -\frac{2}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. The domain of $F = \frac{4x-8}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

13.23.2 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 0, C = -2, f(x) = 4x - 8$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 0rx^{r-1} - 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 0\,x^r - 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 0 - 2 = 0$$

Or

$$r^2 - r - 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2x^2$$

Next, we find the particular solution to the ODE

$$x^2y'' - 2y = 4x - 8$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x^2 \\ -\frac{1}{x^2} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(2x) - (x^2)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = 3$$

Which simplifies to

$$W = 3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2(4x - 8)}{3x^2}\,dx$$

Which simplifies to

$$u_1 = -\int \left(\frac{4x}{3} - \frac{8}{3}\right) dx$$

Hence

$$u_1 = -\frac{2}{3}x^2 + \frac{8}{3}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4x-8}{x}}{3x^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{4x-8}{3x^3} dx$$

Hence

$$u_2 = \frac{4}{3x^2} - \frac{4}{3x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{2}{3}x^2 + \frac{8}{3}x}{x} + \left(\frac{4}{3x^2} - \frac{4}{3x}\right) x^2$$

Which simplifies to

$$y_p(x) = -2x + 4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= -2x + 4 + \frac{c_1}{x} + c_2 x^2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -2x + 4 + \frac{c_1}{x} + c_2 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 1$ in the above gives

$$4 = c_1 + c_2 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2 - \frac{c_1}{x^2} + 2c_2x$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = -2 - c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x} \tag{1}$$

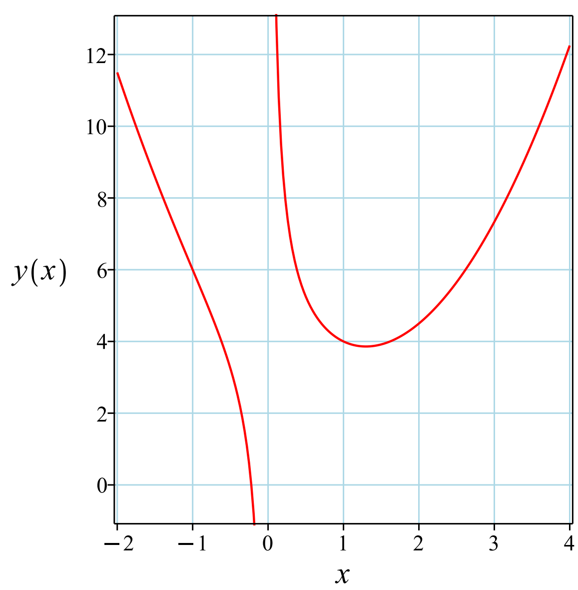

Figure 538: Solution plot

Verification of solutions

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x}$$

Verified OK.

13.23.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' - 2y)\,dx = \int (4x - 8)\,dx$$

$$x^2y' - 2yx = 2x^2 - 8x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{2}{x}\\ q(x) &= \frac{2x^2 + c_1 - 8x}{x^2}\end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{2x^2 + c_1 - 8x}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{2}{x}dx}\\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{2x^2 + c_1 - 8x}{x^2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{2x^2 + c_1 - 8x}{x^2}\right)\\ \mathrm{d}\left(\frac{y}{x^2}\right) &= \left(\frac{2x^2 + c_1 - 8x}{x^4}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2} &= \int \frac{2x^2 + c_1 - 8x}{x^4}\,\mathrm{d}x\\ \frac{y}{x^2} &= \frac{4}{x^2} - \frac{c_1}{3x^3} - \frac{2}{x} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2 \left(\frac{4}{x^2} - \frac{c_1}{3x^3} - \frac{2}{x}\right) + c_2 x^2$$

which simplifies to

$$y = \frac{3c_2x^3 - 6x^2 - c_1 + 12x}{3x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{3c_2x^3 - 6x^2 - c_1 + 12x}{3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 1$ in the above gives

$$4 = -\frac{c_1}{3} + c_2 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{9c_2x^2 - 12x + 12}{3x} - \frac{3c_2x^3 - 6x^2 - c_1 + 12x}{3x^2}$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = \frac{c_1}{3} + 2c_2 - 2 \tag{2A}$$

Equations $\{1\text{A}, 2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -3$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x} \tag{1}$$

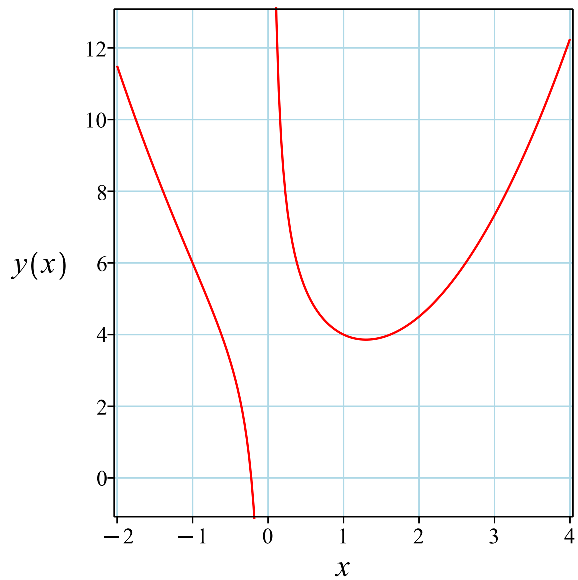

Figure 539: Solution plot

Verification of solutions

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x}$$

Verified OK.

13.23.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2 y'' - 2y = 4x - 8$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2 y'' - 2y)\, dx = \int (4x - 8)\, dx$$

$$x^2 y' - 2yx = 2x^2 - 8x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{2x^2 + c_1 - 8x}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{2y}{x} = \frac{2x^2 + c_1 - 8x}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{2}{x}dx} \\
&= \frac{1}{x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{2x^2 + c_1 - 8x}{x^2}\right) \\
\frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{2x^2 + c_1 - 8x}{x^2}\right) \\
d\left(\frac{y}{x^2}\right) &= \left(\frac{2x^2 + c_1 - 8x}{x^4}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x^2} &= \int \frac{2x^2 + c_1 - 8x}{x^4} dx \\
\frac{y}{x^2} &= \frac{4}{x^2} - \frac{c_1}{3x^3} - \frac{2}{x} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$
y = x^2\left(\frac{4}{x^2} - \frac{c_1}{3x^3} - \frac{2}{x}\right) + c_2 x^2
$$

which simplifies to

$$
y = \frac{3c_2 x^3 - 6x^2 - c_1 + 12x}{3x}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{3c_2 x^3 - 6x^2 - c_1 + 12x}{3x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 1$ in the above gives

$$4 = -\frac{c_1}{3} + c_2 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{9c_2x^2 - 12x + 12}{3x} - \frac{3c_2x^3 - 6x^2 - c_1 + 12x}{3x^2}$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = \frac{c_1}{3} + 2c_2 - 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -3$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x} \tag{1}$$

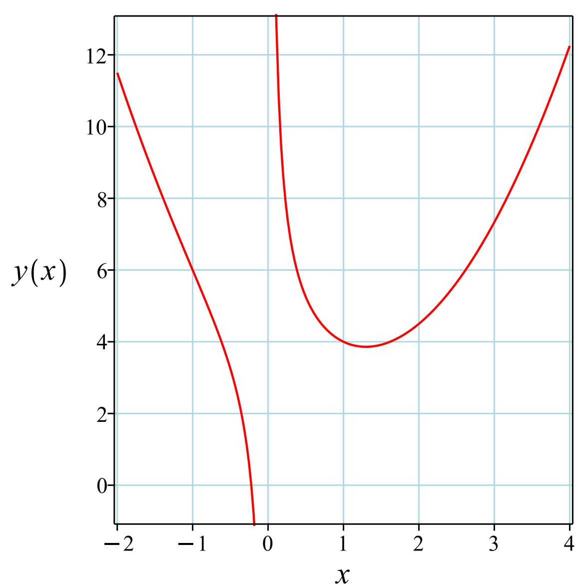

Figure 540: Solution plot

Verification of solutions

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x}$$

Verified OK.

13.23.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 503: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = -1$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \frac{1}{x} \int \frac{1}{\frac{1}{x^2}} \, dx \\ &= \frac{1}{x} \left(\frac{x^3}{3} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x} \right) + c_2 \left(\frac{1}{x} \left(\frac{x^3}{3} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x} + \frac{c_2x^2}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{x^2}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x^2}{3} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x^2}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x^2}{3} \\ -\frac{1}{x^2} & \frac{2x}{3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{2x}{3}\right) - \left(\frac{x^2}{3}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^2(4x-8)}{3}}{x^2}\,dx$$

Which simplifies to

$$u_1 = -\int \left(\frac{4x}{3} - \frac{8}{3}\right) dx$$

Hence

$$u_1 = -\frac{2}{3}x^2 + \frac{8}{3}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4x-8}{x}}{x^2}\,dx$$

Which simplifies to

$$u_2 = \int \frac{4x - 8}{x^3} dx$$

Hence

$$u_2 = \frac{4}{x^2} - \frac{4}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{2}{3}x^2 + \frac{8}{3}x}{x} + \frac{\left(\frac{4}{x^2} - \frac{4}{x}\right) x^2}{3}$$

Which simplifies to

$$y_p(x) = -2x + 4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2 x^2}{3}\right) + (-2x + 4) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x} + \frac{c_2 x^2}{3} - 2x + 4 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 1$ in the above gives

$$4 = c_1 + \frac{c_2}{3} + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1}{x^2} + \frac{2c_2 x}{3} - 2$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = -c_1 + \frac{2c_2}{3} - 2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x} \tag{1}$$

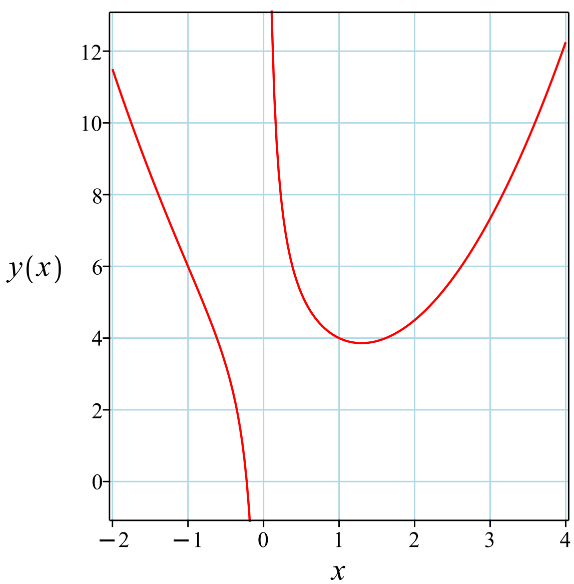

Figure 541: Solution plot

Verification of solutions

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x}$$

Verified OK.

13.23.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)$$

is exact if

$$p''(x)-q'(x)+r(x)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x)&=x^2\\ q(x)&=0\\ r(x)&=-2\\ s(x)&=4x-8\end{aligned}$$

Hence

$$\begin{aligned}p''(x)&=2\\ q'(x)&=0\end{aligned}$$

Therefore (1) becomes

$$2-(0)+(-2)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+\left(q(x)-p'(x)\right)y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$x^2y'-2yx=\int 4x-8\,dx$$

We now have a first order ode to solve which is

$$x^2y'-2yx=2x^2+c_1-8x$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{2x^2 + c_1 - 8x}{x^2} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{2x^2 + c_1 - 8x}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{2x^2 + c_1 - 8x}{x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{2x^2 + c_1 - 8x}{x^2}\right) \\ \mathrm{d}\left(\frac{y}{x^2}\right) &= \left(\frac{2x^2 + c_1 - 8x}{x^4}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x^2} &= \int \frac{2x^2 + c_1 - 8x}{x^4}\,\mathrm{d}x \\ \frac{y}{x^2} &= \frac{4}{x^2} - \frac{c_1}{3x^3} - \frac{2}{x} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(\frac{4}{x^2} - \frac{c_1}{3x^3} - \frac{2}{x}\right) + c_2 x^2$$

which simplifies to

$$y = \frac{3c_2x^3 - 6x^2 - c_1 + 12x}{3x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{3c_2 x^3 - 6x^2 - c_1 + 12x}{3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 1$ in the above gives

$$4 = -\frac{c_1}{3} + c_2 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{9c_2 x^2 - 12x + 12}{3x} - \frac{3c_2 x^3 - 6x^2 - c_1 + 12x}{3x^2}$$

substituting $y' = -1$ and $x = 1$ in the above gives

$$-1 = \frac{c_1}{3} + 2c_2 - 2 \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -3$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x} \tag{1}$$

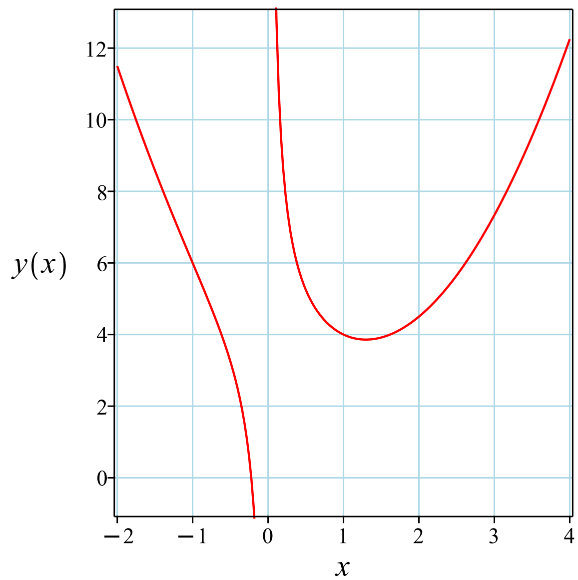

Figure 542: Solution plot

Verification of solutions

$$y = \frac{x^3 - 2x^2 + 4x + 1}{x}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([x^2*diff(y(x),x$2)-2*y(x)=4*x-8,y(1) = 4, D(y)(1) = -1],y(x), singsol=all)
```

$$y(x) = x^2 + 4 - 2x + \frac{1}{x}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 16

```
DSolve[{x^2*y''[x]-2*y[x]==4*x-8,{y[1]==4,y'[1]==-1}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to x^2 - 2x + \frac{1}{x} + 4$$

13.24 problem 24

Internal problem ID [11878]
Internal file name [`OUTPUT/11887_Saturday_April_13_2024_01_14_03_AM_90411466/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 4y'x + 4y = -6x^3 + 4x^2}$$

With initial conditions

$$[y(2) = 4, y'(2) = -1]$$

13.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned}p(x) &= -\frac{4}{x}\\ q(x) &= \frac{4}{x^2}\\ F &= \frac{-6x^3 + 4x^2}{x^2}\end{aligned}$$

Hence the ode is

$$y'' - \frac{4y'}{x} + \frac{4y}{x^2} = \frac{-6x^3 + 4x^2}{x^2}$$

The domain of $p(x) = -\frac{4}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = \frac{4}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is also inside this domain. The domain of $F = \frac{-6x^3+4x^2}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

13.24.2 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 4, f(x) = -6x^3 + 4x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 4y'x + 4y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 4xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 4r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 4r + 4 = 0$$

Or

$$r^2 - 5r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 4$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^4 + c_1x$$

Next, we find the particular solution to the ODE

$$x^2y'' - 4y'x + 4y = -6x^3 + 4x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = x^4$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^4 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^4) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^4 \\ 1 & 4x^3 \end{vmatrix}$$

Therefore

$$W = (x)\left(4x^3\right) - \left(x^4\right)(1)$$

Which simplifies to

$$W = 3x^4$$

Which simplifies to

$$W = 3x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^4(-6x^3 + 4x^2)}{3x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \left(-2x + \frac{4}{3}\right) dx$$

Hence

$$u_1 = x^2 - \frac{4}{3}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(-6x^3 + 4x^2)}{3x^6} dx$$

Which simplifies to

$$u_2 = \int \frac{-6x + 4}{3x^3} dx$$

Hence

$$u_2 = -\frac{2}{3x^2} + \frac{2}{x}$$

Which simplifies to

$$u_1 = x^2 - \frac{4}{3}x$$
$$u_2 = \frac{6x - 2}{3x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(x^2 - \frac{4}{3}x\right) x + \frac{(6x - 2)\, x^2}{3}$$

Which simplifies to

$$y_p(x) = 3x^3 - 2x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x\left(c_2 x^3 + 3x^2 + c_1 - 2x\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = x(c_2x^3 + 3x^2 + c_1 - 2x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = 16c_2 + 16 + 2c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2x^3 + 3x^2 + c_1 - 2x + x(3c_2x^2 + 6x - 2)$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = 32c_2 + 28 + c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{5}{3} \\ c_2 &= -\frac{23}{24} \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{x(23x^3 - 72x^2 + 48x - 40)}{24}$$

Summary

The solution(s) found are the following

$$y = -\frac{x(23x^3 - 72x^2 + 48x - 40)}{24} \tag{1}$$

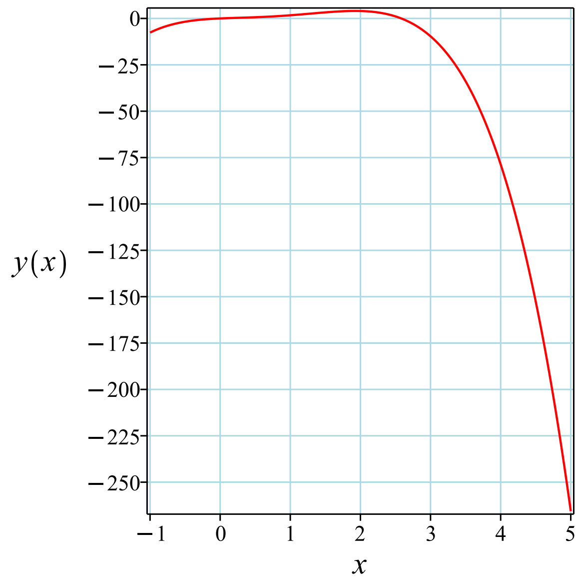

Figure 543: Solution plot

Verification of solutions

$$y = -\frac{x(23x^3 - 72x^2 + 48x - 40)}{24}$$

Verified OK.

13.24.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 4y'x + 4y = 0$$

In normal form the ode

$$x^2y'' - 4y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$
$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\
&= \int e^{-\left(\int -\frac{4}{x}dx\right)}dx \\
&= \int e^{4\ln(x)}\,dx \\
&= \int x^4 dx \\
&= \frac{x^5}{5}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{x^2}}{x^8} \\
&= \frac{4}{x^{10}}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{x^{10}} = 0$$

But in terms of τ

$$\frac{4}{x^{10}} = \frac{4}{25\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{25\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 4y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 4\tau^r = 0$$

Simplifying gives

$$25r(r-1)\,\tau^r + 0\,\tau^r + 4\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$25r(r-1) + 0 + 4 = 0$$

Or

$$25r^2 - 25r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{5}$$
$$r_2 = \frac{4}{5}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{5}} + c_2\tau^{\frac{4}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 5^{\frac{4}{5}} (x^5)^{\frac{1}{5}}}{5} + \frac{c_2 5^{\frac{1}{5}} (x^5)^{\frac{4}{5}}}{5}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 5^{\frac{4}{5}} (x^5)^{\frac{1}{5}}}{5} + \frac{c_2 5^{\frac{1}{5}} (x^5)^{\frac{4}{5}}}{5}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x^5)^{\frac{1}{5}}$$

$$y_2 = (x^5)^{\frac{4}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^5)^{\frac{1}{5}} & (x^5)^{\frac{4}{5}} \\ \frac{d}{dx}\left((x^5)^{\frac{1}{5}}\right) & \frac{d}{dx}\left((x^5)^{\frac{4}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^5)^{\frac{1}{5}} & (x^5)^{\frac{4}{5}} \\ \frac{x^4}{(x^5)^{\frac{4}{5}}} & \frac{4x^4}{(x^5)^{\frac{1}{5}}} \end{vmatrix}$$

Therefore

$$W = \left((x^5)^{\frac{1}{5}}\right)\left(\frac{4x^4}{(x^5)^{\frac{1}{5}}}\right) - \left((x^5)^{\frac{4}{5}}\right)\left(\frac{x^4}{(x^5)^{\frac{4}{5}}}\right)$$

Which simplifies to

$$W = 3x^4$$

Which simplifies to

$$W = 3x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^5)^{\frac{4}{5}}(-6x^3 + 4x^2)}{3x^6}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{2(x^5)^{\frac{4}{5}}(3x-2)}{3x^4}dx$$

Hence

$$u_1 = \frac{(3x-4)(x^5)^{\frac{4}{5}}}{3x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^5)^{\frac{1}{5}}(-6x^3 + 4x^2)}{3x^6}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{2(x^5)^{\frac{1}{5}}(3x-2)}{3x^4}dx$$

Hence

$$u_2 = \frac{2(3x-1)\left(x^5\right)^{\frac{1}{5}}}{3x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^2(3x-4)}{3} + \frac{2(3x-1)\,x^2}{3}$$

Which simplifies to

$$y_p(x) = 3x^3 - 2x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 5^{\frac{4}{5}}\left(x^5\right)^{\frac{1}{5}}}{5} + \frac{c_2 5^{\frac{1}{5}}\left(x^5\right)^{\frac{4}{5}}}{5}\right) + \left(3x^3 - 2x^2\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 5^{\frac{4}{5}}\left(x^5\right)^{\frac{1}{5}}}{5} + \frac{c_2 5^{\frac{1}{5}}\left(x^5\right)^{\frac{4}{5}}}{5} + 3x^3 - 2x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = \frac{2c_1 5^{\frac{4}{5}}}{5} + \frac{16c_2 5^{\frac{1}{5}}}{5} + 16 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 5^{\frac{4}{5}} x^4}{5\left(x^5\right)^{\frac{4}{5}}} + \frac{4c_2 5^{\frac{1}{5}} x^4}{5\left(x^5\right)^{\frac{1}{5}}} + 9x^2 - 4x$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = \frac{c_1 5^{\frac{4}{5}}}{5} + \frac{32c_2 5^{\frac{1}{5}}}{5} + 28 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{5\,5^{\frac{1}{5}}}{3}$$
$$c_2 = -\frac{23\,5^{\frac{4}{5}}}{24}$$

Substituting these values back in above solution results in

$$y = \frac{5(x^5)^{\frac{1}{5}}}{3} - \frac{23(x^5)^{\frac{4}{5}}}{24} + 3x^3 - 2x^2$$

Summary

The solution(s) found are the following

$$y = \frac{5(x^5)^{\frac{1}{5}}}{3} - \frac{23(x^5)^{\frac{4}{5}}}{24} + 3x^3 - 2x^2 \tag{1}$$

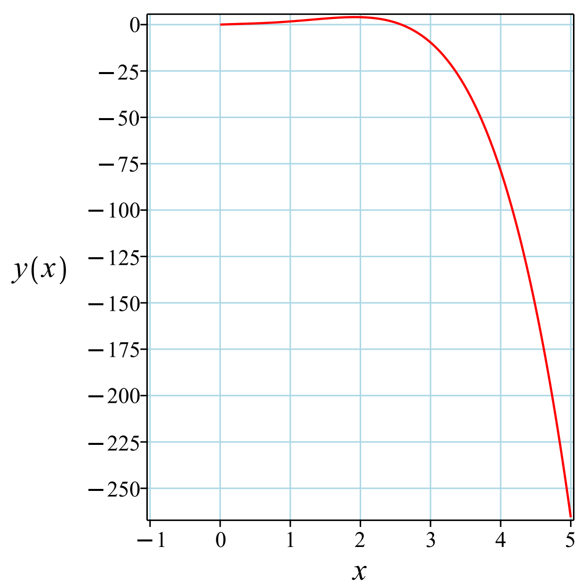

Figure 544: Solution plot

Verification of solutions

$$y = \frac{5(x^5)^{\frac{1}{5}}}{3} - \frac{23(x^5)^{\frac{4}{5}}}{24} + 3x^3 - 2x^2$$

Verified OK.

13.24.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 4, f(x) = -6x^3 + 4x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 4y'x + 4y = 0$$

In normal form the ode

$$x^2y'' - 4y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \qquad (6) \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{5c}{2}
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{5c\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{\frac{5c\tau}{4}}\left(c_1\cosh\left(\frac{3c\tau}{4}\right) + ic_2\sinh\left(\frac{3c\tau}{4}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = x^{\frac{5}{2}}\left(c_1\cosh\left(\frac{3\ln(x)}{2}\right) + ic_2\sinh\left(\frac{3\ln(x)}{2}\right)\right)$$

Now the particular solution to this ODE is found

$$x^2y'' - 4y'x + 4y = -6x^3 + 4x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^5\right)^{\frac{1}{5}}$$
$$y_2 = \left(x^5\right)^{\frac{4}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{1}{5}} & \left(x^5\right)^{\frac{4}{5}} \\ \frac{d}{dx}\left(\left(x^5\right)^{\frac{1}{5}}\right) & \frac{d}{dx}\left(\left(x^5\right)^{\frac{4}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{1}{5}} & \left(x^5\right)^{\frac{4}{5}} \\ \frac{x^4}{\left(x^5\right)^{\frac{4}{5}}} & \frac{4x^4}{\left(x^5\right)^{\frac{1}{5}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^5\right)^{\frac{1}{5}}\right)\left(\frac{4x^4}{\left(x^5\right)^{\frac{1}{5}}}\right) - \left(\left(x^5\right)^{\frac{4}{5}}\right)\left(\frac{x^4}{\left(x^5\right)^{\frac{4}{5}}}\right)$$

Which simplifies to

$$W = 3x^4$$

Which simplifies to

$$W = 3x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^5)^{\frac{4}{5}} (-6x^3 + 4x^2)}{3x^6}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{2(x^5)^{\frac{4}{5}} (3x-2)}{3x^4} dx$$

Hence

$$u_1 = \frac{(3x-4)\,(x^5)^{\frac{4}{5}}}{3x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^5)^{\frac{1}{5}} (-6x^3 + 4x^2)}{3x^6}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{2(x^5)^{\frac{1}{5}} (3x-2)}{3x^4} dx$$

Hence

$$u_2 = \frac{2(3x-1)\,(x^5)^{\frac{1}{5}}}{3x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^2(3x-4)}{3} + \frac{2(3x-1)\,x^2}{3}$$

Which simplifies to

$$y_p(x) = 3x^3 - 2x^2$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{3\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{3\ln(x)}{2}\right)\right)\right) + (3x^3 - 2x^2) \\
&= 3x^3 - 2x^2 + x^{\frac{5}{2}} \left(c_1 \cosh\left(\frac{3\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{3\ln(x)}{2}\right)\right)
\end{aligned}
$$

Which simplifies to

$$
y = i \sinh\left(\frac{3\ln(x)}{2}\right) x^{\frac{5}{2}} c_2 + \cosh\left(\frac{3\ln(x)}{2}\right) x^{\frac{5}{2}} c_1 + 3x^3 - 2x^2
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = i \sinh\left(\frac{3\ln(x)}{2}\right) x^{\frac{5}{2}} c_2 + \cosh\left(\frac{3\ln(x)}{2}\right) x^{\frac{5}{2}} c_1 + 3x^3 - 2x^2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$
4 = 7ic_2 + 9c_1 + 16 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{3ix^{\frac{3}{2}} \cosh\left(\frac{3\ln(x)}{2}\right) c_2}{2} + \frac{5i \sinh\left(\frac{3\ln(x)}{2}\right) x^{\frac{3}{2}} c_2}{2} + \frac{3x^{\frac{3}{2}} \sinh\left(\frac{3\ln(x)}{2}\right) c_1}{2} + \frac{5 \cosh\left(\frac{3\ln(x)}{2}\right) x^{\frac{3}{2}} c_1}{2} + 9x^2 - 4
$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$
-1 = \frac{31ic_2}{2} + \frac{33c_1}{2} + 28 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{17}{24} \\
c_2 &= \frac{21i}{8}
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = -\frac{21 \sinh\left(\frac{3\ln(x)}{2}\right) x^{\frac{5}{2}}}{8} + \frac{17 \cosh\left(\frac{3\ln(x)}{2}\right) x^{\frac{5}{2}}}{24} + 3x^3 - 2x^2$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{21 \sinh\left(\frac{3\ln(x)}{2}\right) x^{\frac{5}{2}}}{8} + \frac{17 \cosh\left(\frac{3\ln(x)}{2}\right) x^{\frac{5}{2}}}{24} + 3x^3 - 2x^2 \tag{1}$$

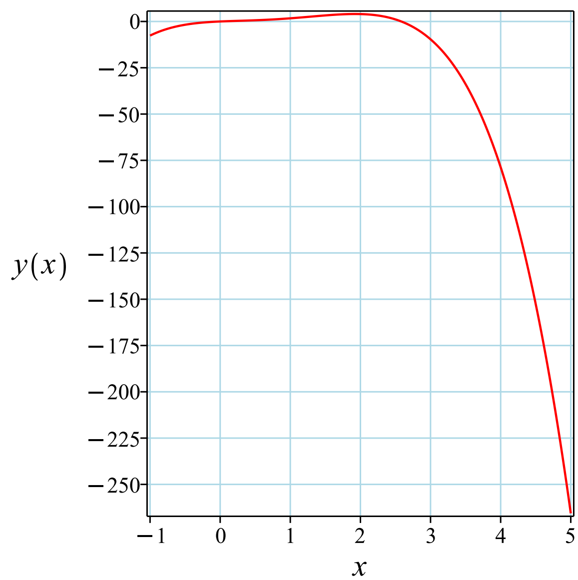

Figure 545: Solution plot

<u>Verification of solutions</u>

$$y = -\frac{21 \sinh\left(\frac{3\ln(x)}{2}\right) x^{\frac{5}{2}}}{8} + \frac{17 \cosh\left(\frac{3\ln(x)}{2}\right) x^{\frac{5}{2}}}{24} + 3x^3 - 2x^2$$

Verified OK.

13.24.5 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 4, f(x) = -6x^3 + 4x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 4y'x + 4y = 0$$

In normal form the ode

$$x^2y'' - 4y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 4 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{4v'(x)}{x} &= 0 \\ v''(x) + \frac{4v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{4u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{4u}{x} \end{aligned}$$

Where $f(x) = -\frac{4}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{4}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{4}{x}\, dx \\ \ln(u) &= -4\ln(x) + c_1 \\ u &= e^{-4\ln(x)+c_1} \\ &= \frac{c_1}{x^4} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1}{3x^3} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{3x^3} + c_2\right) x^4 \\
&= c_2 x^4 - \frac{1}{3} c_1 x
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 4y'x + 4y = -6x^3 + 4x^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= x^4
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x & x^4 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^4) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} x & x^4 \\ 1 & 4x^3 \end{vmatrix}$$

Therefore

$$W = (x)\left(4x^3\right) - \left(x^4\right)(1)$$

Which simplifies to

$$W = 3x^4$$

Which simplifies to

$$W = 3x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^4(-6x^3 + 4x^2)}{3x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \left(-2x + \frac{4}{3}\right) dx$$

Hence

$$u_1 = x^2 - \frac{4}{3}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(-6x^3 + 4x^2)}{3x^6}\, dx$$

Which simplifies to

$$u_2 = \int \frac{-6x + 4}{3x^3} dx$$

Hence

$$u_2 = -\frac{2}{3x^2} + \frac{2}{x}$$

Which simplifies to

$$u_1 = x^2 - \frac{4}{3}x$$
$$u_2 = \frac{6x - 2}{3x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(x^2 - \frac{4}{3}x\right)x + \frac{(6x - 2)\,x^2}{3}$$

Which simplifies to

$$y_p(x) = 3x^3 - 2x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{3x^3} + c_2\right)x^4\right) + \left(3x^3 - 2x^2\right) \\ &= 3x^3 - 2x^2 + \left(-\frac{c_1}{3x^3} + c_2\right)x^4 \end{aligned}$$

Which simplifies to

$$y = -\frac{x(-3c_2x^3 - 9x^2 + c_1 + 6x)}{3}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{x(-3c_2x^3 - 9x^2 + c_1 + 6x)}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = 16c_2 + 16 - \frac{2c_1}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2x^3 + 3x^2 - \frac{c_1}{3} - 2x - \frac{x(-9c_2x^2 - 18x + 6)}{3}$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = 32c_2 + 28 - \frac{c_1}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -5 \\ c_2 &= -\frac{23}{24} \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{x(23x^3 - 72x^2 + 48x - 40)}{24}$$

Summary

The solution(s) found are the following

$$y = -\frac{x(23x^3 - 72x^2 + 48x - 40)}{24} \tag{1}$$

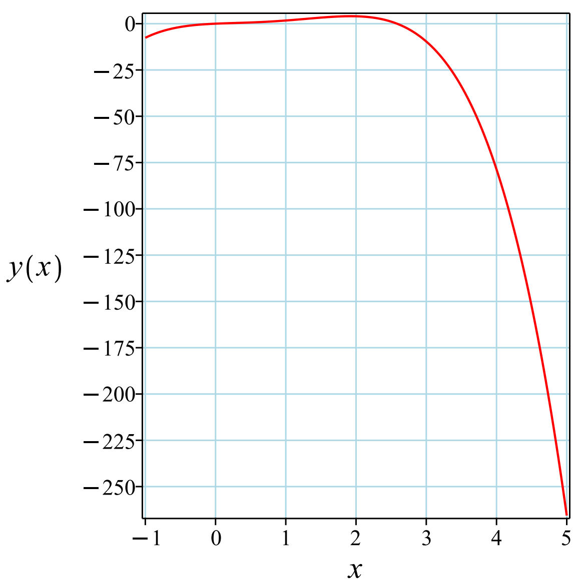

Figure 546: Solution plot

Verification of solutions

$$y = -\frac{x(23x^3 - 72x^2 + 48x - 40)}{24}$$

Verified OK.

13.24.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 4 \\ F &= -6x^3 + 4x^2 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2\right)(0) + (-4x)(-4) + (4)(-4x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-4x^3v'' + (8x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$-4x^2(u'(x)\, x - 2u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{2u}{x} \end{aligned}$$

Where $f(x) = \frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{2}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{2}{x}\, dx \\ \ln(u) &= 2\ln(x) + c_1 \\ u &= \mathrm{e}^{2\ln(x)+c_1} \\ &= c_1x^2 \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= c_1x^2 \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int c_1x^2\, \mathrm{d}x \\ &= \frac{c_1x^3}{3} + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (-4x)\left(\frac{c_1x^3}{3} + c_2\right) \\ &= -\frac{4x(c_1x^3 + 3c_2)}{3} \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= x^4 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^4 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^4) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^4 \\ 1 & 4x^3 \end{vmatrix}$$

Therefore

$$W = (x)\left(4x^3\right) - \left(x^4\right)(1)$$

Which simplifies to

$$W = 3x^4$$

Which simplifies to

$$W = 3x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^4(-6x^3 + 4x^2)}{3x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \left(-2x + \frac{4}{3}\right) dx$$

Hence

$$u_1 = x^2 - \frac{4}{3}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(-6x^3 + 4x^2)}{3x^6}\, dx$$

Which simplifies to

$$u_2 = \int \frac{-6x + 4}{3x^3} dx$$

Hence

$$u_2 = -\frac{2}{3x^2} + \frac{2}{x}$$

Which simplifies to

$$u_1 = x^2 - \frac{4}{3}x$$

$$u_2 = \frac{6x - 2}{3x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(x^2 - \frac{4}{3}x\right) x + \frac{(6x - 2)\, x^2}{3}$$

Which simplifies to

$$y_p(x) = 3x^3 - 2x^2$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-\frac{4x(c_1x^3 + 3c_2)}{3}\right) + (3x^3 - 2x^2) \\ &= -\frac{4}{3}c_1x^4 - 4c_2x + 3x^3 - 2x^2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{4}{3}c_1x^4 - 4c_2x + 3x^3 - 2x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = -\frac{64c_1}{3} - 8c_2 + 16 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{16}{3}c_1x^3 - 4c_2 + 9x^2 - 4x$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = -\frac{128c_1}{3} - 4c_2 + 28 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{23}{32} \\ c_2 &= -\frac{5}{12} \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{23}{24}x^4 + 3x^3 - 2x^2 + \frac{5}{3}x$$

Summary

The solution(s) found are the following

$$y = -\frac{23}{24}x^4 + 3x^3 - 2x^2 + \frac{5}{3}x \tag{1}$$

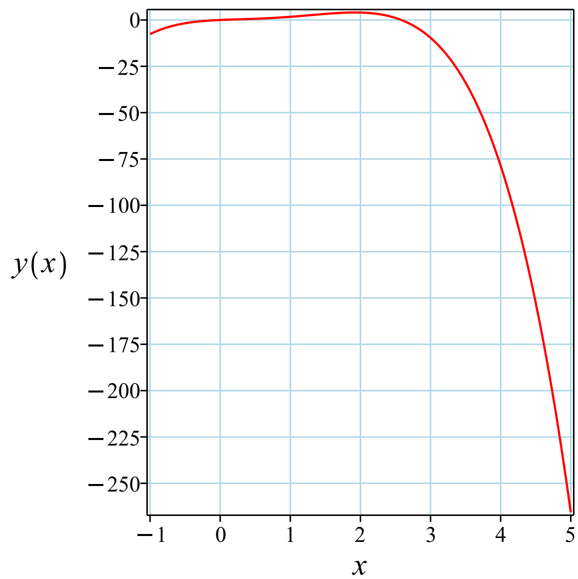

Figure 547: Solution plot

Verification of solutions

$$y = -\frac{23}{24}x^4 + 3x^3 - 2x^2 + \frac{5}{3}x$$

Verified OK.

13.24.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 4y'x + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 504: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition

of r given above. Therefore $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{2}{x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -1$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{x} + (-)\,(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\
&= z_1 e^{2\ln(x)} \\
&= z_1 \left(x^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^3}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{x^3}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - 4y'x + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x + \frac{1}{3} c_2 x^4$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = \frac{x^4}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{x^4}{3} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{x^4}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{x^4}{3} \\ 1 & \frac{4x^3}{3} \end{vmatrix}$$

Therefore

$$W = (x)\left(\frac{4x^3}{3}\right) - \left(\frac{x^4}{3}\right)(1)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^4(-6x^3+4x^2)}{3}}{x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \left(-2x + \frac{4}{3}\right) dx$$

Hence

$$u_1 = x^2 - \frac{4}{3}x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(-6x^3 + 4x^2)}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \frac{-6x + 4}{x^3} dx$$

Hence

$$u_2 = -\frac{2}{x^2} + \frac{6}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(x^2 - \frac{4}{3}x\right)x + \frac{\left(-\frac{2}{x^2} + \frac{6}{x}\right)x^4}{3}$$

Which simplifies to

$$y_p(x) = 3x^3 - 2x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x + \frac{1}{3} c_2 x^4\right) + (3x^3 - 2x^2) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x + \frac{1}{3} c_2 x^4 + 3x^3 - 2x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 2$ in the above gives

$$4 = 2c_1 + \frac{16c_2}{3} + 16 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 + \frac{4}{3} c_2 x^3 + 9x^2 - 4x$$

substituting $y' = -1$ and $x = 2$ in the above gives

$$-1 = c_1 + \frac{32c_2}{3} + 28 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{5}{3} \\ c_2 &= -\frac{23}{8} \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{23}{24} x^4 + 3x^3 - 2x^2 + \frac{5}{3} x$$

Summary

The solution(s) found are the following

$$y = -\frac{23}{24}x^4 + 3x^3 - 2x^2 + \frac{5}{3}x \tag{1}$$

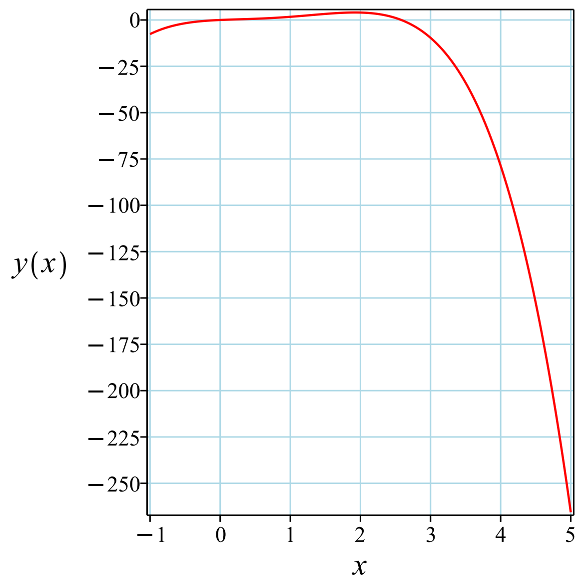

Figure 548: Solution plot

Verification of solutions

$$y = -\frac{23}{24}x^4 + 3x^3 - 2x^2 + \frac{5}{3}x$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+4*y(x)=4*x^2-6*x^3,y(2) = 4, D(y)(2) = -1],y(x),
```

$$y(x) = -\frac{23}{24}x^4 + 3x^3 - 2x^2 + \frac{5}{3}x$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 28

```
DSolve[{x^2*y''[x]-4*x*y'[x]+4*y[x]==4*x^2-6*x^3,{y[2]==4,y'[2]==-1}},y[x],x,IncludeSingular
```

$$y(x) \to -\frac{23x^4}{24} + 3x^3 - 2x^2 + \frac{5x}{3}$$

13.25 problem 25

Internal problem ID [11879]
Internal file name [`OUTPUT/11888_Saturday_April_13_2024_01_14_07_AM_57165022/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + 2y'x - 6y = 10x^2}$$

With initial conditions

$$[y(1) = 1, y'(1) = -6]$$

13.25.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = -\frac{6}{x^2}$$
$$F = 10$$

Hence the ode is

$$y'' + \frac{2y'}{x} - \frac{6y}{x^2} = 10$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -\frac{6}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. The domain of $F = 10$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

13.25.2 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 2x, C = -6, f(x) = 10x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + 2y'x - 6y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 2xrx^{r-1} - 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 2r\,x^r - 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 2r - 6 = 0$$

Or

$$r^2 + r - 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -3$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^3} + c_2 x^2$$

Next, we find the particular solution to the ODE

$$x^2 y'' + 2y' x - 6y = 10x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^3}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^3} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^3}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^3} & x^2 \\ -\frac{3}{x^4} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^3}\right)(2x) - (x^2)\left(-\frac{3}{x^4}\right)$$

Which simplifies to

$$W = \frac{5}{x^2}$$

Which simplifies to

$$W = \frac{5}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{10x^4}{5}\, dx$$

Which simplifies to

$$u_1 = -\int 2x^4 dx$$

Hence

$$u_1 = -\frac{2x^5}{5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{10}{x}}{5}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x} dx$$

Hence

$$u_2 = 2 \ln (x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{2x^2}{5} + 2x^2 \ln (x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= -\frac{2x^2}{5} + 2x^2 \ln (x) + \frac{c_1}{x^3} + c_2 x^2 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{2x^2}{5} + 2x^2 \ln (x) + \frac{c_1}{x^3} + c_2 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 + c_2 - \frac{2}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{6x}{5} + 4x \ln (x) - \frac{3c_1}{x^4} + 2c_2 x$$

substituting $y' = -6$ and $x = 1$ in the above gives

$$-6 = -3c_1 + 2c_2 + \frac{6}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 2 \\ c_2 &= -\frac{3}{5} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{2x^5 \ln(x) - x^5 + 2}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{2x^5 \ln(x) - x^5 + 2}{x^3} \tag{1}$$

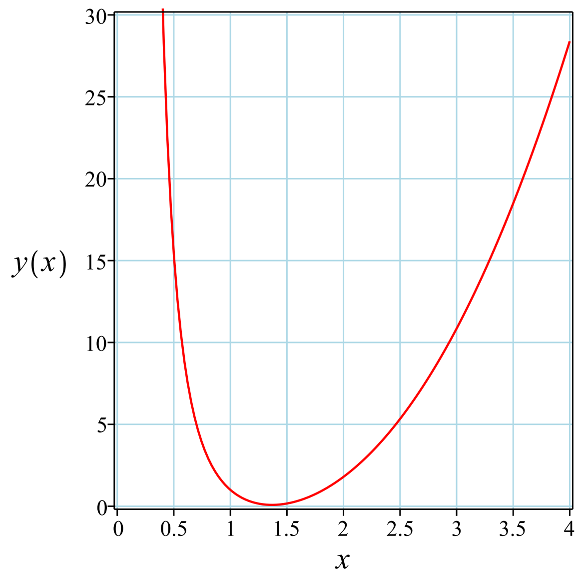

Figure 549: Solution plot

Verification of solutions

$$y = \frac{2x^5 \ln(x) - x^5 + 2}{x^3}$$

Verified OK.

13.25.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + 2y'x - 6y = 0$$

In normal form the ode

$$x^2y'' + 2y'x - 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= -\frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\ &= \int \mathrm{e}^{-\left(\int \frac{2}{x}dx\right)} dx \\ &= \int e^{-2\ln(x)}\, dx \\ &= \int \frac{1}{x^2} dx \\ &= -\frac{1}{x} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{6}{x^2}}{\frac{1}{x^4}} \\
&= -6x^2 \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 6x^2 y(\tau) &= 0
\end{aligned}$$

But in terms of τ

$$-6x^2 = -\frac{6}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{6y(\tau)}{\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 6\tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r - 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 - 6 = 0$$

Or

$$r^2 - r - 6 = 0 \qquad (1)$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= -2 \\
r_2 &= 3
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$
y(\tau) = c_1 y_1 + c_2 y_2
$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$
y(\tau) = \frac{c_1}{\tau^2} + c_2 \tau^3
$$

The above solution is now transformed back to y using (6) which results in

$$
y = \frac{c_1 x^5 - c_2}{x^3}
$$

Therefore the homogeneous solution y_h is

$$
y_h = \frac{c_1 x^5 - c_2}{x^3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x^3} \\
y_2 &= x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^3} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^3}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^3} & x^2 \\ -\frac{3}{x^4} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^3}\right)(2x) - (x^2)\left(-\frac{3}{x^4}\right)$$

Which simplifies to

$$W = \frac{5}{x^2}$$

Which simplifies to

$$W = \frac{5}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{10x^4}{5}\, dx$$

Which simplifies to

$$u_1 = -\int 2x^4 dx$$

Hence

$$u_1 = -\frac{2x^5}{5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{10}{x}}{5}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x} dx$$

Hence

$$u_2 = 2\ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{2x^2}{5} + 2x^2 \ln(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x^5 - c_2}{x^3}\right) + \left(-\frac{2x^2}{5} + 2x^2 \ln(x)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 x^5 - c_2}{x^3} - \frac{2x^2}{5} + 2x^2 \ln(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 - c_2 - \frac{2}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 5c_1 x - \frac{3(c_1 x^5 - c_2)}{x^4} + \frac{6x}{5} + 4x \ln(x)$$

substituting $y' = -6$ and $x = 1$ in the above gives

$$-6 = 2c_1 + 3c_2 + \frac{6}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{3}{5} \\ c_2 &= -2 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{2x^5 \ln(x) - x^5 + 2}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{2x^5 \ln(x) - x^5 + 2}{x^3} \tag{1}$$

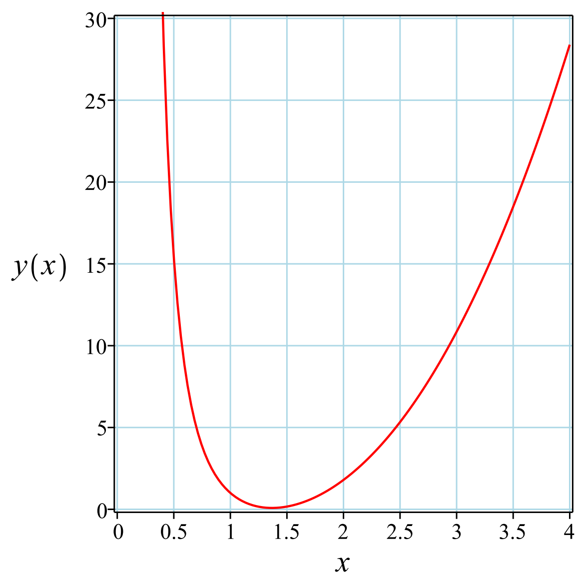

Figure 550: Solution plot

Verification of solutions

$$y = \frac{2x^5 \ln(x) - x^5 + 2}{x^3}$$

Verified OK.

13.25.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 2x, C = -6, f(x) = 10x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$x^2y''+2y'x-6y=0$$

In normal form the ode

$$x^2y''+2y'x-6y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=\frac{2}{x}\\ q(x)&=-\frac{6}{x^2}\end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right)v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right)v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}+\frac{2n}{x^2}-\frac{6}{x^2}=0 \tag{5}$$

Solving (5) for n gives

$$n=2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(x)+\frac{6v'(x)}{x}&=0\\ v''(x)+\frac{6v'(x)}{x}&=0\end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{6u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{6u}{x}
\end{aligned}$$

Where $f(x) = -\frac{6}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{6}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{6}{x}\,dx \\
\ln(u) &= -6\ln(x) + c_1 \\
u &= \mathrm{e}^{-6\ln(x)+c_1} \\
&= \frac{c_1}{x^6}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{5x^5} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{5x^5} + c_2\right) x^2 \\
&= \frac{5c_2x^5 - c_1}{5x^3}
\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' + 2y'x - 6y = 10x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^3}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^3} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^3}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^3} & x^2 \\ -\frac{3}{x^4} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^3}\right)(2x) - \left(x^2\right)\left(-\frac{3}{x^4}\right)$$

Which simplifies to

$$W = \frac{5}{x^2}$$

Which simplifies to

$$W = \frac{5}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\left(\int \frac{10x^4}{5}dx\right)$$

Which simplifies to

$$u_1 = -\left(\int 2x^4 dx\right)$$

Hence

$$u_1 = -\frac{2x^5}{5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{10}{x}}{5}dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x}dx$$

Hence

$$u_2 = 2\ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{2x^2}{5} + 2x^2\ln(x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(-\frac{c_1}{5x^5} + c_2\right) x^2\right) + \left(-\frac{2x^2}{5} + 2x^2 \ln(x)\right) \\
&= -\frac{2x^2}{5} + 2x^2 \ln(x) + \left(-\frac{c_1}{5x^5} + c_2\right) x^2
\end{aligned}
$$

Which simplifies to

$$
y = \frac{10x^5 \ln(x) + 5c_2x^5 - 2x^5 - c_1}{5x^3}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{10x^5 \ln(x) + 5c_2x^5 - 2x^5 - c_1}{5x^3} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$
1 = -\frac{c_1}{5} + c_2 - \frac{2}{5} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{50x^4 \ln(x) + 25c_2x^4}{5x^3} - \frac{3(10x^5 \ln(x) + 5c_2x^5 - 2x^5 - c_1)}{5x^4}
$$

substituting $y' = -6$ and $x = 1$ in the above gives

$$
-6 = \frac{3c_1}{5} + 2c_2 + \frac{6}{5} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -10 \\
c_2 &= -\frac{3}{5}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{2x^5 \ln(x) - x^5 + 2}{x^3}
$$

Summary
The solution(s) found are the following

$$y = \frac{2x^5 \ln(x) - x^5 + 2}{x^3} \tag{1}$$

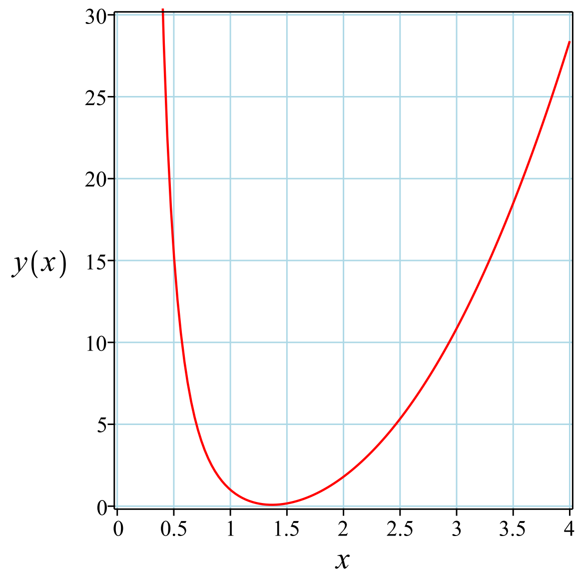

Figure 551: Solution plot

Verification of solutions

$$y = \frac{2x^5 \ln(x) - x^5 + 2}{x^3}$$

Verified OK.

13.25.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 2y'x - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 505: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition

of r given above. Therefore $b = 6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{6}{x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -2$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -2 - (-2) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{2}{x} + (-)(0) \\ &= -\frac{2}{x} \\ &= -\frac{2}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{2}{x}\right)(0) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x}\right)^2 - \left(\frac{6}{x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{2}{x} dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^5}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^3}\right) + c_2 \left(\frac{1}{x^3}\left(\frac{x^5}{5}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^2y''+2y'x-6y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=\frac{c_1}{x^3}+\frac{c_2x^2}{5}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1,u_2 to be determined, and y_1,y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1=\frac{1}{x^3}$$
$$y_2=\frac{x^2}{5}$$

In the Variation of parameters u_1,u_2 are found using

$$u_1=-\int\frac{y_2f(x)}{aW(x)} \tag{2}$$
$$u_2=\int\frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}$. Hence

$$W=\begin{vmatrix}\frac{1}{x^3} & \frac{x^2}{5}\\ \frac{d}{dx}\left(\frac{1}{x^3}\right) & \frac{d}{dx}\left(\frac{x^2}{5}\right)\end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^3} & \frac{x^2}{5} \\ -\frac{3}{x^4} & \frac{2x}{5} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^3}\right)\left(\frac{2x}{5}\right) - \left(\frac{x^2}{5}\right)\left(-\frac{3}{x^4}\right)$$

Which simplifies to

$$W = \frac{1}{x^2}$$

Which simplifies to

$$W = \frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2x^4}{1}\,dx$$

Which simplifies to

$$u_1 = -\int 2x^4 dx$$

Hence

$$u_1 = -\frac{2x^5}{5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{10}{x}}{1}\,dx$$

Which simplifies to

$$u_2 = \int \frac{10}{x} dx$$

Hence

$$u_2 = 10 \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{2x^2}{5} + 2x^2 \ln(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x^3} + \frac{c_2 x^2}{5}\right) + \left(-\frac{2x^2}{5} + 2x^2 \ln(x)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x^3} + \frac{c_2 x^2}{5} - \frac{2x^2}{5} + 2x^2 \ln(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 1$ in the above gives

$$1 = c_1 + \frac{c_2}{5} - \frac{2}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{3c_1}{x^4} + \frac{2c_2 x}{5} + \frac{6x}{5} + 4x \ln(x)$$

substituting $y' = -6$ and $x = 1$ in the above gives

$$-6 = -3c_1 + \frac{2c_2}{5} + \frac{6}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 2 \\ c_2 &= -3 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{2x^5 \ln(x) - x^5 + 2}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{2x^5 \ln(x) - x^5 + 2}{x^3} \tag{1}$$

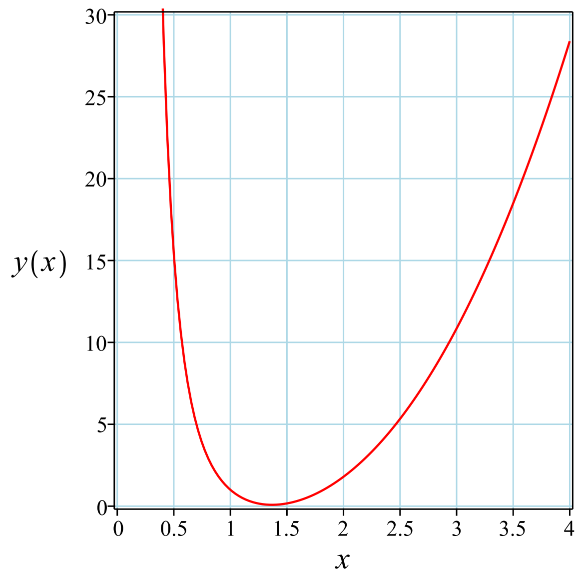

Figure 552: Solution plot

Verification of solutions

$$y = \frac{2x^5 \ln(x) - x^5 + 2}{x^3}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve([x^2*diff(y(x),x$2)+2*x*diff(y(x),x)-6*y(x)=10*x^2,y(1) = 1, D(y)(1) = -6],y(x), sing
```

$$y(x) = \frac{2x^5 \ln (x) - x^5 + 2}{x^3}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 23

```
DSolve[{x^2*y''[x]+2*x*y'[x]-6*y[x]==10*x^2,{y[1]==1,y'[1]==-6}},y[x],x,IncludeSingularSolut
```

$$y(x) \to \frac{-x^5 + 2x^5 \log(x) + 2}{x^3}$$

13.26 problem 26

Internal problem ID [11880]
Internal file name [`OUTPUT/11889_Saturday_April_13_2024_01_14_09_AM_33480691/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 5y'x + 8y = 2x^3}$$

With initial conditions

$$[y(2) = 0, y'(2) = -8]$$

13.26.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= \frac{8}{x^2} \\ F &= 2x \end{aligned}$$

Hence the ode is

$$y'' - \frac{5y'}{x} + \frac{8y}{x^2} = 2x$$

The domain of $p(x) = -\frac{5}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = \frac{8}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is also inside this domain. The domain of $F = 2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

13.26.2 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -5x, C = 8, f(x) = 2x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 5y'x + 8y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 5xrx^{r-1} + 8x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 5r\,x^r + 8x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 5r + 8 = 0$$

Or

$$r^2 - 6r + 8 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 4 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^4 + c_1x^2$$

Next, we find the particular solution to the ODE

$$x^2y'' - 5y'x + 8y = 2x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^2 \\ y_2 &= x^4 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^4 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^4) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{vmatrix}$$

Therefore

$$W = (x^2)(4x^3) - (x^4)(2x)$$

Which simplifies to

$$W = 2x^5$$

Which simplifies to

$$W = 2x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2x^7}{2x^7}\, dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^5}{2x^7}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2} dx$$

Hence

$$u_2 = -\frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x^2 \left(c_2 x^2 + c_1 - 2x\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = x^2 \left(c_2 x^2 + c_1 - 2x\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$0 = 16c_2 + 4c_1 - 16 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2x\left(c_2 x^2 + c_1 - 2x\right) + x^2(2c_2 x - 2)$$

substituting $y' = -8$ and $x = 2$ in the above gives

$$-8 = 32c_2 + 4c_1 - 24 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = -2(x-2)\,x^2$$

Summary

The solution(s) found are the following

$$y = -2(x-2)\,x^2 \tag{1}$$

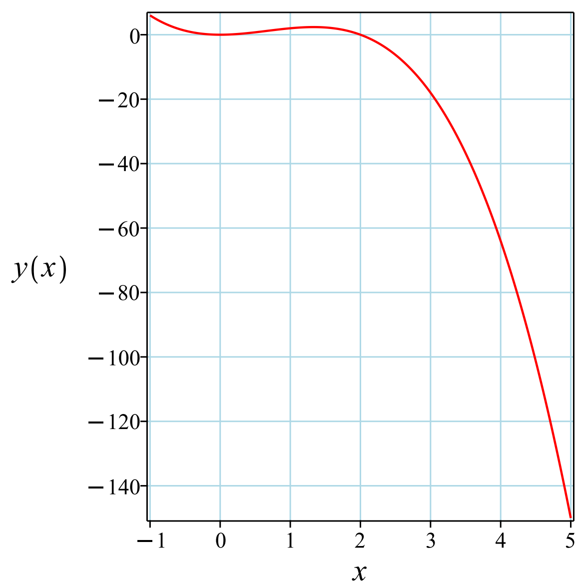

Figure 553: Solution plot

Verification of solutions

$$y = -2(x-2)\,x^2$$

Verified OK.

13.26.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 5y'x + 8y = 0$$

In normal form the ode

$$x^2y'' - 5y'x + 8y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= \frac{8}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{5}{x}dx\right)} dx \\ &= \int e^{5\ln(x)}\, dx \\ &= \int x^5 dx \\ &= \frac{x^6}{6} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{8}{x^2}}{x^{10}} \\ &= \frac{8}{x^{12}} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{8y(\tau)}{x^{12}} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{8}{x^{12}} = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^{r}+0\,\tau^{r}+2\tau^{r}=0$$

Since $\tau^{r}\neq 0$ then dividing throughout by τ^{r} gives

$$9r(r-1)+0+2=0$$

Or

$$9r^{2}-9r+2=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1=\frac{1}{3}$$
$$r_2=\frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau)=c_1y_1+c_2y_2$$

Where $y_1=\tau^{r_1}$ and $y_2=\tau^{r_2}$. Hence

$$y(\tau)=c_1\tau^{\frac{1}{3}}+c_2\tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y=\frac{c_1 6^{\frac{2}{3}}\left(x^{6}\right)^{\frac{1}{3}}}{6}+\frac{c_2 6^{\frac{1}{3}}\left(x^{6}\right)^{\frac{2}{3}}}{6}$$

Therefore the homogeneous solution y_h is

$$y_h=\frac{c_1 6^{\frac{2}{3}}\left(x^{6}\right)^{\frac{1}{3}}}{6}+\frac{c_2 6^{\frac{1}{3}}\left(x^{6}\right)^{\frac{2}{3}}}{6}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^6\right)^{\frac{1}{3}}$$

$$y_2 = \left(x^6\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^6\right)^{\frac{1}{3}} & \left(x^6\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(x^6\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(x^6\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^6\right)^{\frac{1}{3}} & \left(x^6\right)^{\frac{2}{3}} \\ \frac{2x^5}{\left(x^6\right)^{\frac{2}{3}}} & \frac{4x^5}{\left(x^6\right)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^6\right)^{\frac{1}{3}}\right)\left(\frac{4x^5}{\left(x^6\right)^{\frac{1}{3}}}\right) - \left(\left(x^6\right)^{\frac{2}{3}}\right)\left(\frac{2x^5}{\left(x^6\right)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = 2x^5$$

Which simplifies to

$$W = 2x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2(x^6)^{\frac{2}{3}} x^3}{2x^7} dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^6)^{\frac{2}{3}}}{x^4} dx$$

Hence

$$u_1 = -\frac{(x^6)^{\frac{2}{3}}}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2(x^6)^{\frac{1}{3}} x^3}{2x^7} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^6)^{\frac{1}{3}}}{x^4} dx$$

Hence

$$u_2 = -\frac{(x^6)^{\frac{1}{3}}}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 6^{\frac{2}{3}} (x^6)^{\frac{1}{3}}}{6} + \frac{c_2 6^{\frac{1}{3}} (x^6)^{\frac{2}{3}}}{6}\right) + (-2x^3) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 6^{\frac{2}{3}} (x^6)^{\frac{1}{3}}}{6} + \frac{c_2 6^{\frac{1}{3}} (x^6)^{\frac{2}{3}}}{6} - 2x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$0 = \frac{2c_1 6^{\frac{2}{3}}}{3} + \frac{8c_2 6^{\frac{1}{3}}}{3} - 16 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 6^{\frac{2}{3}} x^5}{3 (x^6)^{\frac{2}{3}}} + \frac{2c_2 6^{\frac{1}{3}} x^5}{3 (x^6)^{\frac{1}{3}}} - 6x^2$$

substituting $y' = -8$ and $x = 2$ in the above gives

$$-8 = \frac{2c_1 6^{\frac{2}{3}}}{3} + \frac{16c_2 6^{\frac{1}{3}}}{3} - 24 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4\,6^{\frac{1}{3}}$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = 4(x^6)^{\frac{1}{3}} - 2x^3$$

Summary

The solution(s) found are the following

$$y = 4(x^6)^{\frac{1}{3}} - 2x^3 \tag{1}$$

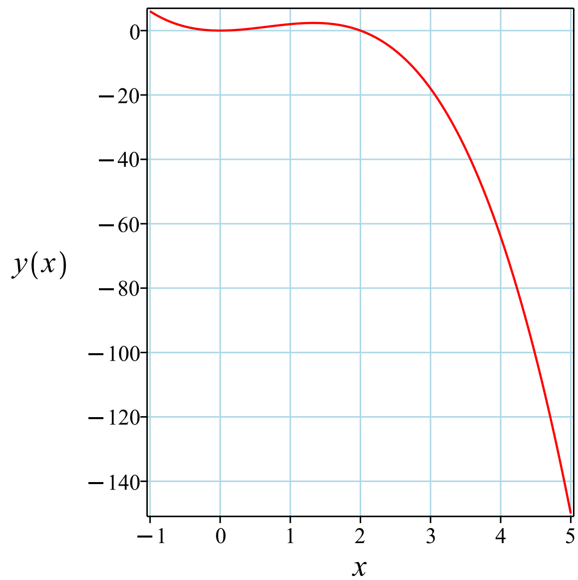

Figure 554: Solution plot

Verification of solutions

$$y = 4\left(x^{6}\right)^{\frac{1}{3}} - 2x^{3}$$

Verified OK.

13.26.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -5x, C = 8, f(x) = 2x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 5y'x + 8y = 0$$

In normal form the ode

$$x^2y'' - 5y'x + 8y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= \frac{8}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{2}\, \sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\, x^3} - \frac{5}{x}\frac{2\sqrt{2}\, \sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{2}\, \sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{3c\sqrt{2}}{2} \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) - \frac{3c\sqrt{2}\left(\frac{d}{d\tau} y(\tau)\right)}{2} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = e^{\frac{3\sqrt{2}\,c\tau}{4}} \left(c_1 \cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + i c_2 \sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{2}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{2}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{x^2((ic_2 + c_1)\,x^2 - ic_2 + c_1)}{2}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 5y'x + 8y = 2x^3
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \left(x^6\right)^{\frac{1}{3}} \\
y_2 &= \left(x^6\right)^{\frac{2}{3}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^6)^{\frac{1}{3}} & (x^6)^{\frac{2}{3}} \\ \frac{d}{dx}\left((x^6)^{\frac{1}{3}}\right) & \frac{d}{dx}\left((x^6)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^6)^{\frac{1}{3}} & (x^6)^{\frac{2}{3}} \\ \frac{2x^5}{(x^6)^{\frac{2}{3}}} & \frac{4x^5}{(x^6)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((x^6)^{\frac{1}{3}}\right)\left(\frac{4x^5}{(x^6)^{\frac{1}{3}}}\right) - \left((x^6)^{\frac{2}{3}}\right)\left(\frac{2x^5}{(x^6)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = 2x^5$$

Which simplifies to

$$W = 2x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2(x^6)^{\frac{2}{3}}\, x^3}{2x^7}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^6)^{\frac{2}{3}}}{x^4} dx$$

Hence

$$u_1 = -\frac{\left(x^6\right)^{\frac{2}{3}}}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\left(x^6\right)^{\frac{1}{3}} x^3}{2x^7}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\left(x^6\right)^{\frac{1}{3}}}{x^4} dx$$

Hence

$$u_2 = -\frac{\left(x^6\right)^{\frac{1}{3}}}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{x^2((ic_2 + c_1)\, x^2 - ic_2 + c_1)}{2}\right) + \left(-2x^3\right) \\ &= -2x^3 + \frac{x^2((ic_2 + c_1)\, x^2 - ic_2 + c_1)}{2} \end{aligned}$$

Which simplifies to

$$y = \frac{x^2((ic_2 + c_1)\, x^2 - 4x - ic_2 + c_1)}{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{x^2((ic_2 + c_1)\, x^2 - 4x - ic_2 + c_1)}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$0 = 6ic_2 + 10c_1 - 16 \tag{1A}$$

Taking derivative of the solution gives

$$y' = x\left((ic_2 + c_1)\, x^2 - 4x - ic_2 + c_1\right) + \frac{x^2(2(ic_2 + c_1)\, x - 4)}{2}$$

substituting $y' = -8$ and $x = 2$ in the above gives

$$-8 = 14ic_2 + 18c_1 - 24 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = 4i$$

Substituting these values back in above solution results in

$$y = -2x^3 + 4x^2$$

Summary
The solution(s) found are the following

$$y = -2x^3 + 4x^2 \tag{1}$$

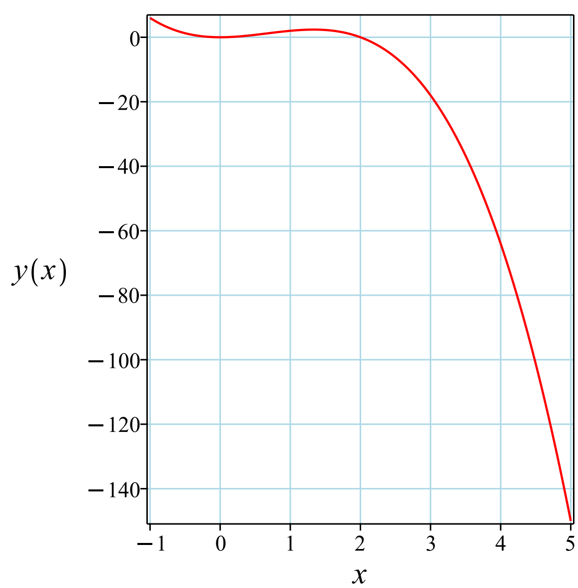

Figure 555: Solution plot

Verification of solutions

$$y = -2x^3 + 4x^2$$

Verified OK.

13.26.5 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -5x, C = 8, f(x) = 2x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 5y'x + 8y = 0$$

In normal form the ode

$$x^2y'' - 5y'x + 8y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= \frac{8}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{5n}{x^2} + \frac{8}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 4 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{3}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{3}{x}\,dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x^4 \\
&= c_2 x^4 - \frac{1}{2} c_1 x^2
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 5y'x + 8y = 2x^3
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= x^4
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^4 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^4) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{vmatrix}$$

Therefore

$$W = (x^2)(4x^3) - (x^4)(2x)$$

Which simplifies to

$$W = 2x^5$$

Which simplifies to

$$W = 2x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2x^7}{2x^7}\,dx$$

Which simplifies to

$$u_1 = -\int 1dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^5}{2x^7}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2}dx$$

Hence

$$u_2 = -\frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x^4\right) + \left(-2x^3\right) \\ &= -2x^3 + \left(-\frac{c_1}{2x^2} + c_2\right) x^4 \end{aligned}$$

Which simplifies to

$$y = -\frac{x^2(-2c_2x^2 + c_1 + 4x)}{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{x^2(-2c_2x^2 + c_1 + 4x)}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$0 = 16c_2 - 2c_1 - 16 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -x\left(-2c_2x^2 + c_1 + 4x\right) - \frac{x^2(-4c_2x + 4)}{2}$$

substituting $y' = -8$ and $x = 2$ in the above gives

$$-8 = 32c_2 - 2c_1 - 24 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -8$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = -2(x-2)\,x^2$$

Summary
The solution(s) found are the following

$$y = -2(x-2)\,x^2 \tag{1}$$

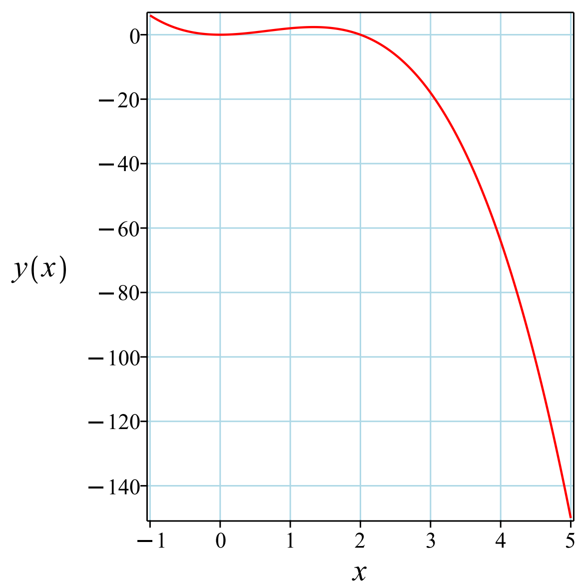

Figure 556: Solution plot

Verification of solutions

$$y = -2(x-2)\,x^2$$

Verified OK.

13.26.6 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 5y'x + 8y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -5x \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 506: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{2x} + (-)\,(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x}{x^2}\,dx} \\
&= z_1 e^{\frac{5\ln(x)}{2}} \\
&= z_1\left(x^{\frac{5}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - 5y'x + 8y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x^2 + \frac{1}{2} c_2 x^4$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = \frac{x^4}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \frac{x^4}{2} \\ \frac{d}{dx}(x^2) & \frac{d}{dx}\left(\frac{x^4}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \frac{x^4}{2} \\ 2x & 2x^3 \end{vmatrix}$$

Therefore

$$W = (x^2)(2x^3) - \left(\frac{x^4}{2}\right)(2x)$$

Which simplifies to

$$W = x^5$$

Which simplifies to

$$W = x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^7}{x^7}\, dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^5}{x^7}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x^2} dx$$

Hence

$$u_2 = -\frac{2}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2x^3$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 x^2 + \frac{1}{2} c_2 x^4\right) + \left(-2x^3\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 x^2 + \frac{1}{2} c_2 x^4 - 2x^3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 2$ in the above gives

$$
0 = 4c_1 + 8c_2 - 16 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 2c_2 x^3 + 2c_1 x - 6x^2
$$

substituting $y' = -8$ and $x = 2$ in the above gives

$$
-8 = 16c_2 + 4c_1 - 24 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 4 \\
c_2 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -2x^3 + 4x^2
$$

Summary

The solution(s) found are the following

$$
y = -2x^3 + 4x^2 \tag{1}
$$

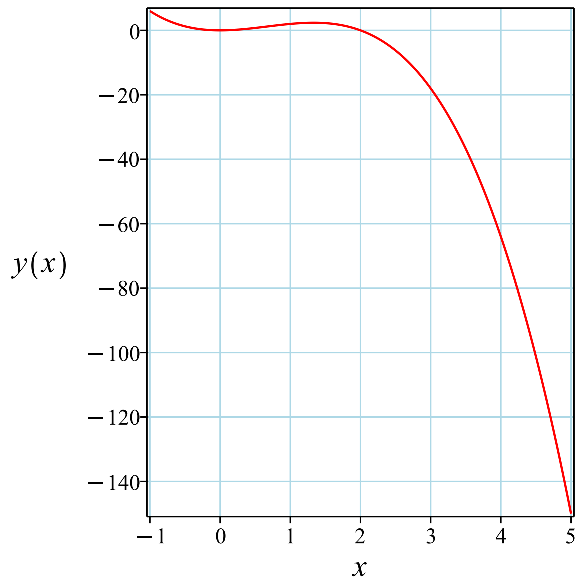

Figure 557: Solution plot

Verification of solutions

$$y = -2x^3 + 4x^2$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([x^2*diff(y(x),x$2)-5*x*diff(y(x),x)+8*y(x)=2*x^3,y(2) = 0, D(y)(2) = -8],y(x), sings
```

$$y(x) = -2x^3 + 4x^2$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 13

```
DSolve[{x^2*y''[x]-5*x*y'[x]+8*y[x]==2*x^3,{y[2]==0,y'[2]==-8}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to -2(x-2)x^2$$

13.27 problem 27

Internal problem ID [11881]
Internal file name [`OUTPUT/11890_Saturday_April_13_2024_01_14_12_AM_12077989/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' - 6y = \ln(x)}$$

With initial conditions

$$\left[y(1) = \frac{1}{6}, y'(1) = -\frac{1}{6}\right]$$

13.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned}
p(x) &= 0 \\
q(x) &= -\frac{6}{x^2} \\
F &= \frac{\ln(x)}{x^2}
\end{aligned}$$

Hence the ode is

$$y'' - \frac{6y}{x^2} = \frac{\ln(x)}{x^2}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -\frac{6}{x^2}$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. The domain of $F = \frac{\ln(x)}{x^2}$ is

$$\{0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

13.27.2 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 0, C = -6, f(x) = \ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 6y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 0rx^{r-1} - 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 0\,x^r - 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 0 - 6 = 0$$

Or

$$r^2 - r - 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2 \\ r_2 &= 3 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + c_2 x^3$$

Next, we find the particular solution to the ODE

$$x^2 y'' - 6y = \ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \frac{1}{x^2} \\ y_2 &= x^3 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^3 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^3 \\ -\frac{2}{x^3} & 3x^2 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(3x^2) - (x^3)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = 5$$

Which simplifies to

$$W = 5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3 \ln(x)}{5x^2}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x \ln(x)}{5} dx$$

Hence

$$u_1 = -\frac{x^2 \ln(x)}{10} + \frac{x^2}{20}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\ln(x)}{x^2}}{5x^2}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)}{5x^4} dx$$

Hence

$$u_2 = -\frac{\ln(x)}{15x^3} - \frac{1}{45x^3}$$

Which simplifies to

$$u_1 = -\frac{x^2(-1 + 2\ln(x))}{20}$$
$$u_2 = \frac{-1 - 3\ln(x)}{45x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{1}{36} - \frac{\ln(x)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{1}{36} - \frac{\ln(x)}{6} + \frac{c_1}{x^2} + c_2 x^3 \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{1}{36} - \frac{\ln(x)}{6} + \frac{c_1}{x^2} + c_2 x^3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{6}$ and $x = 1$ in the above gives

$$\frac{1}{6} = c_1 + c_2 + \frac{1}{36} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{1}{6x} - \frac{2c_1}{x^3} + 3c_2 x^2$$

substituting $y' = -\frac{1}{6}$ and $x = 1$ in the above gives

$$-\frac{1}{6} = -2c_1 + 3c_2 - \frac{1}{6} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{12}$$
$$c_2 = \frac{1}{18}$$

Substituting these values back in above solution results in

$$y = \frac{2x^5 - 6x^2 \ln(x) + x^2 + 3}{36x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{2x^5 - 6x^2 \ln(x) + x^2 + 3}{36x^2} \tag{1}$$

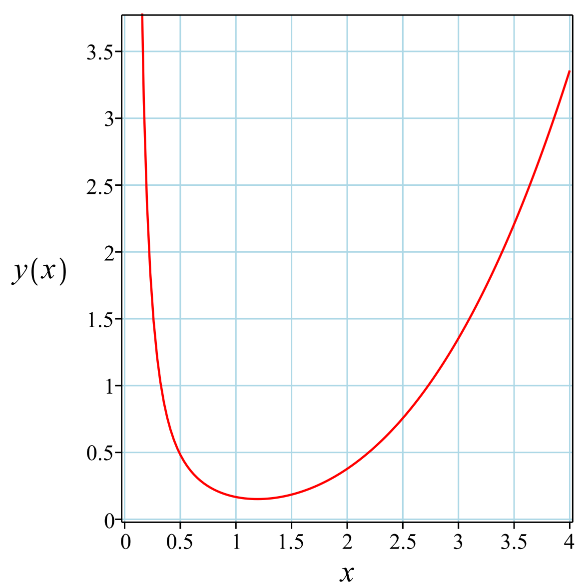

Figure 558: Solution plot

Verification of solutions

$$y = \frac{2x^5 - 6x^2 \ln(x) + x^2 + 3}{36x^2}$$

Verified OK.

13.27.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 6y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 507: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{6}{x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = 6$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{6}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -2$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -2 - (-2) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + (-)\,(0) \\
&= -\frac{2}{x} \\
&= -\frac{2}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{2}{x}\right)(0) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x}\right)^2 - \left(\frac{6}{x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{2}{x} dx} \\
&= \frac{1}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \frac{1}{x^2} \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \frac{1}{x^2} \int \frac{1}{\frac{1}{x^4}}\,dx \\ &= \frac{1}{x^2}\left(\frac{x^5}{5}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\frac{1}{x^2}\right) + c_2\left(\frac{1}{x^2}\left(\frac{x^5}{5}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^2y''-6y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=\frac{c_1}{x^2}+\frac{c_2x^3}{5}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1,u_2 to be determined, and y_1,y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1=\frac{1}{x^2}$$
$$y_2=\frac{x^3}{5}$$

In the Variation of parameters u_1,u_2 are found using

$$u_1=-\int\frac{y_2f(x)}{aW(x)} \tag{2}$$
$$u_2=\int\frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}$. Hence

$$W=\begin{vmatrix}\frac{1}{x^2} & \frac{x^3}{5}\\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{x^3}{5}\right)\end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{x^3}{5} \\ -\frac{2}{x^3} & \frac{3x^2}{5} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(\frac{3x^2}{5}\right) - \left(\frac{x^3}{5}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^3 \ln(x)}{5}}{x^2}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x \ln(x)}{5} dx$$

Hence

$$u_1 = -\frac{x^2 \ln(x)}{10} + \frac{x^2}{20}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\ln(x)}{x^2}}{x^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)}{x^4} dx$$

Hence

$$u_2 = -\frac{\ln(x)}{3x^3} - \frac{1}{9x^3}$$

Which simplifies to

$$
\begin{aligned}
u_1 &= -\frac{x^2(-1+2\ln(x))}{20} \\
u_2 &= \frac{-1-3\ln(x)}{9x^3}
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
y_p(x) = \frac{1}{36} - \frac{\ln(x)}{6}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1}{x^2} + \frac{c_2 x^3}{5}\right) + \left(\frac{1}{36} - \frac{\ln(x)}{6}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_1}{x^2} + \frac{c_2 x^3}{5} + \frac{1}{36} - \frac{\ln(x)}{6} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = \frac{1}{6}$ and $x = 1$ in the above gives

$$
\frac{1}{6} = c_1 + \frac{c_2}{5} + \frac{1}{36} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -\frac{2c_1}{x^3} + \frac{3c_2 x^2}{5} - \frac{1}{6x}
$$

substituting $y' = -\frac{1}{6}$ and $x = 1$ in the above gives

$$
-\frac{1}{6} = -2c_1 + \frac{3c_2}{5} - \frac{1}{6} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{1}{12} \\
c_2 &= \frac{5}{18}
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = \frac{2x^5 - 6x^2 \ln(x) + x^2 + 3}{36x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{2x^5 - 6x^2 \ln(x) + x^2 + 3}{36x^2} \tag{1}$$

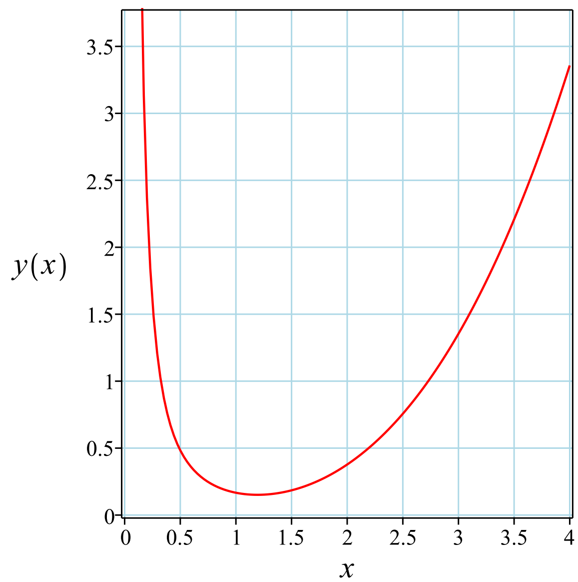

Figure 559: Solution plot

Verification of solutions

$$y = \frac{2x^5 - 6x^2 \ln(x) + x^2 + 3}{36x^2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve([x^2*diff(y(x),x$2)-6*y(x)=ln(x),y(1) = 1/6, D(y)(1) = -1/6],y(x), singsol=all)
```

$$y(x) = \frac{1}{12x^2} + \frac{x^3}{18} - \frac{\ln(x)}{6} + \frac{1}{36}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 29

```
DSolve[{x^2*y''[x]-6*y[x]==Log[x],{y[1]==1/6,y'[1]==-1/6}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{2x^5 + x^2 - 6x^2 \log(x) + 3}{36x^2}$$

13.28 problem 28

Internal problem ID [11882]
Internal file name [`OUTPUT/11891_Saturday_April_13_2024_01_14_13_AM_67873546/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x+2)^2 y'' - (x+2) y' - 3y = 0}$$

13.28.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left((x+2)^2 y'' + (-x-2) y' - 3y\right) dx = 0$$
$$(-3x-6) y + \left(x^2+4x+4\right) y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3}{x+2} \\
q(x) &= \frac{c_1}{(x+2)^2}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{3y}{x+2} = \frac{c_1}{(x+2)^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{3}{x+2}dx} \\
&= \frac{1}{(x+2)^3}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{(x+2)^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{(x+2)^3}\right) &= \left(\frac{1}{(x+2)^3}\right)\left(\frac{c_1}{(x+2)^2}\right) \\
\mathrm{d}\left(\frac{y}{(x+2)^3}\right) &= \left(\frac{c_1}{(x+2)^5}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{(x+2)^3} &= \int \frac{c_1}{(x+2)^5}\, \mathrm{d}x \\
\frac{y}{(x+2)^3} &= -\frac{c_1}{4(x+2)^4} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(x+2)^3}$ results in

$$
y = -\frac{c_1}{4(x+2)} + c_2(x+2)^3
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = -\frac{c_1}{4(x+2)} + c_2(x+2)^3 \tag{1}
$$

<u>Verification of solutions</u>

$$
y = -\frac{c_1}{4(x+2)} + c_2(x+2)^3
$$

Verified OK.

13.28.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(x+2)^2 y'' + (-x-2) y' - 3y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left((x+2)^2 y'' + (-x-2) y' - 3y\right) dx = 0$$
$$(-3x-6) y + \left(x^2+4x+4\right) y' = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{x+2}$$
$$q(x) = \frac{c_1}{(x+2)^2}$$

Hence the ode is

$$y' - \frac{3y}{x+2} = \frac{c_1}{(x+2)^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{x+2}dx}$$
$$= \frac{1}{(x+2)^3}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{(x+2)^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{(x+2)^3}\right) = \left(\frac{1}{(x+2)^3}\right)\left(\frac{c_1}{(x+2)^2}\right)$$
$$d\left(\frac{y}{(x+2)^3}\right) = \left(\frac{c_1}{(x+2)^5}\right) dx$$

Integrating gives

$$\frac{y}{(x+2)^3} = \int \frac{c_1}{(x+2)^5} \,\mathrm{d}x$$
$$\frac{y}{(x+2)^3} = -\frac{c_1}{4(x+2)^4} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(x+2)^3}$ results in

$$y = -\frac{c_1}{4(x+2)} + c_2(x+2)^3$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{4(x+2)} + c_2(x+2)^3 \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{4(x+2)} + c_2(x+2)^3$$

Verified OK.

13.28.3 Solving using Kovacic algorithm

Writing the ode as

$$(x+2)^2 y'' + (-x-2) y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x+2)^2 \\ B &= -x-2 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4\left(x+2\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 15$$
$$t = 4(x+2)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4\left(x+2\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 508: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+2)^2$. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4\left(x+2\right)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{4\left(x+2\right)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{4\left(x+2\right)^{2}}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-2	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = -\frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2\left(x+2\right)} + (-)\left(0\right) \\ &= -\frac{3}{2\left(x+2\right)} \\ &= -\frac{3}{2\left(x+2\right)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2\left(x+2\right)}\right)(0) + \left(\left(\frac{3}{2\left(x+2\right)^2}\right) + \left(-\frac{3}{2\left(x+2\right)}\right)^2 - \left(\frac{15}{4\left(x+2\right)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int -\frac{3}{2(x+2)} dx} \\ &= \frac{1}{\left(x+2\right)^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x-2}{(x+2)^2}\, dx} \\ &= z_1 e^{\frac{\ln(x+2)}{2}} \\ &= z_1\left(\sqrt{x+2}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x+2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-x-2}{(x+2)^2} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{\ln(x+2)}}{(y_1)^2} dx$$
$$= y_1 \left(\frac{(x+2)^4}{4} \right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{1}{x+2} \right) + c_2 \left(\frac{1}{x+2} \left(\frac{(x+2)^4}{4} \right) \right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x+2} + \frac{c_2(x+2)^3}{4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x+2} + \frac{c_2(x+2)^3}{4}$$

Verified OK.

13.28.4 Solving as exact linear second order ode ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$p(x) = (x+2)^2$$
$$q(x) = -x - 2$$
$$r(x) = -3$$
$$s(x) = 0$$

Hence

$$p''(x) = 2$$
$$q'(x) = -1$$

Therefore (1) becomes

$$2 - (-1) + (-3) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$(x+2)^2\, y' + (-3x-6)\, y = c_1$$

We now have a first order ode to solve which is

$$(x+2)^2\, y' + (-3x-6)\, y = c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{x+2}$$
$$q(x) = \frac{c_1}{(x+2)^2}$$

Hence the ode is

$$y' - \frac{3y}{x+2} = \frac{c_1}{(x+2)^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{x+2}dx}$$
$$= \frac{1}{(x+2)^3}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{(x+2)^2}\right)$$

$$\frac{d}{dx}\left(\frac{y}{(x+2)^3}\right) = \left(\frac{1}{(x+2)^3}\right)\left(\frac{c_1}{(x+2)^2}\right)$$

$$d\left(\frac{y}{(x+2)^3}\right) = \left(\frac{c_1}{(x+2)^5}\right)\, dx$$

Integrating gives

$$\frac{y}{(x+2)^3} = \int \frac{c_1}{(x+2)^5}\, dx$$

$$\frac{y}{(x+2)^3} = -\frac{c_1}{4\,(x+2)^4} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(x+2)^3}$ results in

$$y = -\frac{c_1}{4\,(x+2)} + c_2(x+2)^3$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{4\,(x+2)} + c_2(x+2)^3 \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{4\,(x+2)} + c_2(x+2)^3$$

Verified OK.

13.28.5 Maple step by step solution

Let's solve

$(x+2)^2\, y'' + (-x-2)\, y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x+2} + \frac{3y}{(x+2)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{y'}{x+2} - \frac{3y}{(x+2)^2} = 0$

☐ Check to see if $x_0 = -2$ is a regular singular point

  - Define functions

  $\left[P_2(x) = -\frac{1}{x+2}, P_3(x) = -\frac{3}{(x+2)^2}\right]$

  - $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

  $\left((x+2) \cdot P_2(x)\right) \bigg|_{x=-2} = -1$

  - $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

  $\left((x+2)^2 \cdot P_3(x)\right) \bigg|_{x=-2} = -3$

  - $x = -2$is a regular singular point

  Check to see if $x_0 = -2$ is a regular singular point

  $x_0 = -2$

- Multiply by denominators

$(x+2)^2 y'' + (-x-2) y' - 3y = 0$

- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

$u^2 \left(\frac{d^2}{du^2} y(u)\right) - u\left(\frac{d}{du} y(u)\right) - 3y(u) = 0$

- Assume series solution for $y(u)$

$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite DE with series expansions

  - Convert $u \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r) u^{k+r}$

  - Convert $u^2 \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion

  $u^2 \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) u^{k+r}$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} a_k (k+r+1)(k+r-3) u^{k+r} = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+1)(k-3) = 0$
- Recursion relation that defines series solution to ODE

  $a_k = 0$
- Recursion relation for $r = 0$

  $a_k = 0$
- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_k = 0\right]$
- Revert the change of variables $u = x + 2$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+2)^k, a_k = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve((x+2)^2*diff(y(x),x$2)-(x+2)*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 + c_2(x+2)^4}{x+2}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 22

```
DSolve[(x+2)^2*y''[x]-(x+2)*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to c_1(x+2)^3+\frac{c_2}{x+2}$$

13.29 problem 29

Internal problem ID [11883]
Internal file name [`OUTPUT/11892_Saturday_April_13_2024_01_14_14_AM_480238/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 4, Section 4.5. The Cauchy-Euler Equation. Exercises page 169
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_x_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x-3)^2 y'' - 6(2x-3) y' + 12y = 0}$$

13.29.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$4\left(x-\frac{3}{2}\right)^2 y'' + (-12x+18) y' + 12y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$p(x) = \frac{6}{-2x+3}$$
$$q(x) = \frac{12}{(2x-3)^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{6}{-2x+3}dx\right)}dx \\
&= \int e^{3\ln(-2x+3)}\,dx \\
&= \int (-2x+3)^3\,dx \\
&= -\frac{(-2x+3)^4}{8}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{12}{(2x-3)^2}}{(-2x+3)^6} \\
&= \frac{12}{(2x-3)^8}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{12y(\tau)}{(2x-3)^8} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{12}{(2x-3)^8} = \frac{3}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 3\tau^r = 0$$

Simplifying gives

$$16r(r-1)\,\tau^r + 0\,\tau^r + 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$16r(r-1) + 0 + 3 = 0$$

Or

$$16r^2 - 16r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{4}$$
$$r_2 = \frac{3}{4}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{4}} + c_2\tau^{\frac{3}{4}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{2^{\frac{1}{4}}\left(-(2x-3)^4\right)^{\frac{1}{4}}\left(c_2\sqrt{2}\,\sqrt{-(2x-3)^4} + 4c_1\right)}{8}$$

Summary

The solution(s) found are the following

$$y = \frac{2^{\frac{1}{4}} \left(-(2x-3)^4\right)^{\frac{1}{4}} \left(c_2 \sqrt{2}\, \sqrt{-\left(2x-3\right)^4}+4c_1\right)}{8} \tag{1}$$

Verification of solutions

$$y = \frac{2^{\frac{1}{4}} \left(-(2x-3)^4\right)^{\frac{1}{4}} \left(c_2 \sqrt{2}\, \sqrt{-\left(2x-3\right)^4}+4c_1\right)}{8}$$

Verified OK.

13.29.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= 4\left(x-\frac{3}{2}\right)^{2} \\
B &= -12x+18 \\
C &= 12 \\
F &= 0
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(4\left(x-\frac{3}{2}\right)^{2}\right)(0) + (-12x+18)(-12) + (12)(-12x+18) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-6(2x-3)^{3} v'' + \left(12(2x-3)^{2}\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
-6(2x-3)^{3} u'(x) + 12u(x)(2x-3)^{2} = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{2u}{2x-3}
\end{aligned}
$$

Where $f(x) = \frac{2}{2x-3}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{2}{2x-3}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{2}{2x-3}\,dx \\
\ln(u) &= \ln(2x-3) + c_1 \\
u &= \mathrm{e}^{\ln(2x-3)+c_1} \\
&= c_1(2x-3)
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1(2x-3)
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int c_1(2x-3)\, \mathrm{d}x \\ &= c_1\left(x^2-3x\right)+c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (-12x+18)\left(c_1\left(x^2-3x\right)+c_2\right) \\ &= -12\left(c_1x^2-3c_1x+c_2\right)\left(x-\frac{3}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -12\left(c_1x^2-3c_1x+c_2\right)\left(x-\frac{3}{2}\right) \tag{1}$$

Verification of solutions

$$y = -12\left(c_1x^2-3c_1x+c_2\right)\left(x-\frac{3}{2}\right)$$

Verified OK.

13.29.3 Solving using Kovacic algorithm

Writing the ode as

$$4\left(x-\frac{3}{2}\right)^2 y'' + (-12x+18)\, y' + 12y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4\left(x-\frac{3}{2}\right)^2 \\ B &= -12x+18 \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(2x-3)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= (2x-3)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(2x-3)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 510: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (2x-3)^2$. There is a pole at $x = \frac{3}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(x - \frac{3}{2}\right)^2}$$

For the pole at $x = \frac{3}{2}$ let b be the coefficient of $\frac{1}{\left(x-\frac{3}{2}\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{(2x-3)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(2x-3)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$\frac{3}{2}$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\left(x - \frac{3}{2}\right)} + (-)\,(0) \\ &= -\frac{1}{2\left(x - \frac{3}{2}\right)} \\ &= -\frac{1}{2x - 3} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x-\frac{3}{2}\right)}\right)(0) + \left(\left(\frac{1}{2\left(x-\frac{3}{2}\right)^2}\right) + \left(-\frac{1}{2\left(x-\frac{3}{2}\right)}\right)^2 - \left(\frac{3}{(2x-3)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{2\left(x-\frac{3}{2}\right)} dx} \\
&= \frac{1}{\sqrt{2x-3}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-12x+18}{4\left(x-\frac{3}{2}\right)^2} dx} \\
&= z_1 e^{\frac{3\ln(2x-3)}{2}} \\
&= z_1\left((2x-3)^{\frac{3}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = 2x - 3$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-12x+18}{4\left(x-\frac{3}{2}\right)^2}dx}}{\left(y_1\right)^2} dx \\
&= y_1 \int \frac{e^{3\ln(2x-3)}}{\left(y_1\right)^2} dx \\
&= y_1\left(x^2-3x\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(2x-3) + c_2\left(2x-3\left(x^2-3x\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(2x-3) + c_2\left(2x^3-9x^2+9x\right) \tag{1}
$$

Verification of solutions

$$
y = c_1(2x-3) + c_2\left(2x^3-9x^2+9x\right)
$$

Verified OK.

13.29.4 Maple step by step solution

Let's solve

$4\left(x-\frac{3}{2}\right)^2 y'' + (-12x+18)\,y' + 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{6y'}{2x-3} - \frac{12y}{(2x-3)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{6y'}{2x-3} + \frac{12y}{(2x-3)^2} = 0$

□ Check to see if $x_0 = \frac{3}{2}$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{6}{2x-3}, P_3(x) = \frac{12}{(2x-3)^2}\right]$

- $\left(x-\frac{3}{2}\right) \cdot P_2(x)$ is analytic at $x = \frac{3}{2}$

  $\left(\left(x-\frac{3}{2}\right) \cdot P_2(x)\right)\Bigg|_{x=\frac{3}{2}} = -3$

- $\left(x-\frac{3}{2}\right)^2 \cdot P_3(x)$ is analytic at $x = \frac{3}{2}$

  $\left(\left(x-\frac{3}{2}\right)^2 \cdot P_3(x)\right)\Bigg|_{x=\frac{3}{2}} = 3$

- $x = \frac{3}{2}$is a regular singular point

  Check to see if $x_0 = \frac{3}{2}$ is a regular singular point

  $x_0 = \frac{3}{2}$

- Multiply by denominators

  $(2x-3)^2 y'' + (-12x+18) y' + 12y = 0$

- Change variables using $x = u + \frac{3}{2}$ so that the regular singular point is at $u = 0$

  $4u^2\left(\frac{d^2}{du^2}y(u)\right) - 12u\left(\frac{d}{du}y(u)\right) + 12y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite DE with series expansions

- Convert $u \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r}$

- Convert $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

  $u^2 \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r}$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} 4a_k(k+r-1)\,(k+r-3)\, u^{k+r} = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

$$4a_k(k-1)(k-3) = 0$$

- Recursion relation that defines series solution to ODE

$$a_k = 0$$

- Recursion relation for $r = 0$

$$a_k = 0$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_k = 0\right]$$

- Revert the change of variables $u = x - \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x - \frac{3}{2}\right)^k, a_k = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve((2*x-3)^2*diff(y(x),x$2)-6*(2*x-3)*diff(y(x),x)+12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(x - \frac{3}{2}\right)\left(c_1 + c_2\left(x - \frac{3}{2}\right)^2\right)$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 24

```
DSolve[(2*x-3)^2*y''[x]-6*(2*x-3)*y'[x]+12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2(3-2x)^3 + c_1(3-2x)$$

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14.1 problem 1

Internal problem ID [11884]
Internal file name [`OUTPUT/11893_Saturday_April_13_2024_01_14_15_AM_40632902/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' + y'x + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (783) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (784) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y'x - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= x^2 y' + yx - 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -y'x^3 - x^2 y + 5y'x + 3y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 9x^2 + 8\right) y' + xy\left(x^2 - 7\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 + 14x^3 - 33x\right) y' - y\left(x^4 - 12x^2 + 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -2y'(0) \\
F_2 &= 3y(0) \\
F_3 &= 8y'(0) \\
F_4 &= -15y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\,a_{n+2}(n+1) + na_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{3} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) a_0 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) y'(0) + O\left(x^6\right)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O\left(x^6\right)$$

Verified OK.

14.1.1 Maple step by step solution

Let's solve

$y'' = -y'x - y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- > k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)\left(a_{k+2}(k+2) + a_k\right) = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{15}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica
Time used: 0.007 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]+x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{15} - \frac{x^3}{3} + x\right) + c_1\left(\frac{x^4}{8} - \frac{x^2}{2} + 1\right)$$

14.2 problem 2

Internal problem ID [11885]
Internal file name [OUTPUT/11894_Saturday_April_13_2024_01_14_16_AM_16203133/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 8y'x - 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (786)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (787)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -8y'x + 4y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 64x^2 y' - 32yx - 4y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 128\left(-4x^3 + x\right) y' + 16y\left(16x^2 - 3\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(4096x^4 - 2304x^2 + 80\right) y' - 2048x\left(x^2 - \frac{1}{2}\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-32768x^5 + 32768x^3 - 4224x\right) y' + \left(16384x^4 - 15360x^2 + 1344\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 4y(0) \\
F_1 &= -4y'(0) \\
F_2 &= -48y(0) \\
F_3 &= 80y'(0) \\
F_4 &= 1344y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + 2x^2 - 2x^4 + \frac{28}{15}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{2}{3}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -8\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 4\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} 8n\, x^n a_n\right) + \sum_{n=0}^{\infty} (-4 a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} 8n\, x^n a_n\right) + \sum_{n=0}^{\infty} (-4 a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 4a_0 = 0$$

$$a_2 = 2a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) + 8na_n - 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{4a_n(2n-1)}{(n+2)(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 12a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -2a_0$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 20a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{2a_1}{3}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 28a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{28a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 36a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{4a_1}{7}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + 2a_0 x^2 - \frac{2}{3} a_1 x^3 - 2a_0 x^4 + \frac{2}{3} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(-2x^4 + 2x^2 + 1\right) a_0 + \left(x - \frac{2}{3}x^3 + \frac{2}{3}x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(-2x^4 + 2x^2 + 1\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{2}{3}x^5\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 + 2x^2 - 2x^4 + \frac{28}{15}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{2}{3}x^5\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(-2x^4 + 2x^2 + 1\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{2}{3}x^5\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(1 + 2x^2 - 2x^4 + \frac{28}{15}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{2}{3}x^5\right) y'(0) + O\left(x^6\right)$$

Verified OK.

$$y = \left(-2x^4 + 2x^2 + 1\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{2}{3}x^5\right) c_2 + O\left(x^6\right)$$

Verified OK.

14.2.1 Maple step by step solution

Let's solve

$y'' = -8y'x + 4y$

- Highest derivative means the order of the ODE is 2

y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + 8y'x - 4y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)\, x^k$

Rewrite DE with series expansions

$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) + 4a_k(2k-1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

$(k^2 + 3k + 2)\, a_{k+2} + 8a_k k - 4a_k = 0$

- Recursion relation that defines the series solution to the ODE

$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{4a_k(2k-1)}{k^2+3k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+8*x*diff(y(x),x)-4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(-2x^4 + 2x^2 + 1\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{2}{3}x^5\right) D(y)(0) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 38

```
AsymptoticDSolveValue[y''[x]+8*x*y'[x]-4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{2x^5}{3} - \frac{2x^3}{3} + x\right) + c_1\left(-2x^4 + 2x^2 + 1\right)$$

14.3 problem 3

Internal problem ID [11886]
Internal file name [`OUTPUT/11895_Saturday_April_13_2024_01_14_17_AM_52170285/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x + \left(2x^2 + 1\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (789)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (790)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
F_0 &= -2x^2 y - y' x - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 2yx^3 - x^2 y' - 3yx - 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(3x^3 - 3x\right) y' + y\left(2x^4 + 11x^2 - 1\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(-x^4 + 23x^2 - 4\right) y' - 6xy\left(x^4 - \frac{11}{6}x^2 - \frac{25}{6}\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-5x^5 - 16x^3 + 75x\right) y' + 2\left(x^6 - \frac{75}{2}x^4 + 9x^2 + \frac{29}{2}\right) y
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -2y'(0) \\
F_2 &= -y(0) \\
F_3 &= -4y'(0) \\
F_4 &= 29y(0)
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{29}{720}x^6\right) y(0) + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) y'(0) + O\left(x^6\right)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -2x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 2x^{n+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=0}^{\infty} 2x^{n+2} a_n = \sum_{n=2}^{\infty} 2a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + na_n + 2a_{n-2} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{na_n + a_n + 2a_{n-2}}{(n+2)\,(n+1)} \\ &= -\frac{a_n}{n+2} - \frac{2a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_2 + 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{30}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 5a_4 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{29a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{13a_1}{630}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{3} a_1 x^3 - \frac{1}{24} a_0 x^4 - \frac{1}{30} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) a_0 + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{29}{720}x^6\right) y(0) + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1-\frac{1}{2}x^{2}-\frac{1}{24}x^{4}+\frac{29}{720}x^{6}\right)y(0)+\left(x-\frac{1}{3}x^{3}-\frac{1}{30}x^{5}\right)y'(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1-\frac{1}{2}x^{2}-\frac{1}{24}x^{4}\right)c_{1}+\left(x-\frac{1}{3}x^{3}-\frac{1}{30}x^{5}\right)c_{2}+O\left(x^{6}\right)$$

Verified OK.

14.3.1 Maple step by step solution

Let's solve

$y'' = -2x^{2}y - y'x - y$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \left(-2x^{2}-1\right)y - y'x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x + \left(2x^{2}+1\right)y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_{k}x^{k}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^{m}\cdot y$ to series expansion for $m = 0..2$

    $x^{m}\cdot y = \sum_{k=\max(0,-m)}^{\infty} a_{k}x^{k+m}$

  ◦ Shift index using $k- >k-m$

    $x^{m}\cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m}x^{k}$

  ◦ Convert $x\cdot y'$ to series expansion

    $x\cdot y' = \sum_{k=0}^{\infty} a_{k}k\,x^{k}$

  ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + a_0 + (6a_3 + 2a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+1) + 2a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + a_0 = 0, 6a_3 + 2a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + a_k k + a_k + 2a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + a_{k+2}(k+2) + a_{k+2} + 2a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{k a_{k+2} + 2a_k + 3a_{k+2}}{k^2 + 7k + 12}, a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{3}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+(2*x^2+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4\right) y(0) + \left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]+x*y'[x]+(2*x^2+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{x^5}{30} - \frac{x^3}{3} + x\right) + c_1\left(-\frac{x^4}{24} - \frac{x^2}{2} + 1\right)$$

14.4 problem 4

Internal problem ID [11887]
Internal file name [`OUTPUT/11896_Saturday_April_13_2024_01_14_18_AM_24910550/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x + \left(x^2 - 4\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y_0'}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{792}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{793}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n\big|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -x^2 y - y' x + 4y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 3y' + \left(x^3 - 6x\right) y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^3 - 9x\right) y' + 6y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(-x^4 + 12x^2 - 3\right) y' + \left(-x^5 + 13x^3 - 36x\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-3x^3 - 9x\right) y' + y\left(x^6 - 21x^4 + 90x^2 - 48\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 4y(0) \\
F_1 &= 3y'(0) \\
F_2 &= 6y(0) \\
F_3 &= -3y'(0) \\
F_4 &= -48y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + 2x^2 + \frac{1}{4}x^4 - \frac{1}{15}x^6\right) y(0) + \left(x + \frac{1}{2}x^3 - \frac{1}{40}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x^2\left(\sum_{n=0}^{\infty} a_n x^n\right) - \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 4\left(\sum_{n=0}^{\infty} a_n x^n\right) \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n\right) + \sum_{n=0}^{\infty} (-4 a_n x^n) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) + \sum_{n=0}^{\infty} (-4 a_n x^n) = 0 \quad (3)$$

$n=0$ gives

$$2a_2 - 4a_0 = 0$$

$$a_2 = 2a_0$$

$n = 1$ gives

$$6a_3 - 3a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_1}{2}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + na_n + a_{n-2} - 4a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{na_n - 4a_n + a_{n-2}}{(n+2)\,(n+1)} \\ &= -\frac{(n-4)\, a_n}{(n+2)\,(n+1)} - \frac{a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 2a_2 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{4}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{19a_1}{1680}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + 2a_0 x^2 + \frac{1}{2} a_1 x^3 + \frac{1}{4} a_0 x^4 - \frac{1}{40} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + 2x^2 + \frac{1}{4}x^4\right) a_0 + \left(x + \frac{1}{2}x^3 - \frac{1}{40}x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + 2x^2 + \frac{1}{4}x^4\right) c_1 + \left(x + \frac{1}{2}x^3 - \frac{1}{40}x^5\right) c_2 + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = \left(1 + 2x^2 + \frac{1}{4}x^4 - \frac{1}{15}x^6\right) y(0) + \left(x + \frac{1}{2}x^3 - \frac{1}{40}x^5\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(1 + 2x^2 + \frac{1}{4}x^4\right) c_1 + \left(x + \frac{1}{2}x^3 - \frac{1}{40}x^5\right) c_2 + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \left(1+2x^2+\frac{1}{4}x^4-\frac{1}{15}x^6\right)y(0)+\left(x+\frac{1}{2}x^3-\frac{1}{40}x^5\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1+2x^2+\frac{1}{4}x^4\right)c_1+\left(x+\frac{1}{2}x^3-\frac{1}{40}x^5\right)c_2+O(x^6)$$

Verified OK.

14.4.1 Maple step by step solution

Let's solve

$y'' = -x^2y - y'x + 4y$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-x^2+4)\,y - y'x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x + (x^2-4)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

☐ Rewrite ODE with series expansions

- ∘ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ∘ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ∘ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

- ∘ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

Rewrite ODE with series expansions

$$2a_2 - 4a_0 + (6a_3 - 3a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) + a_k(k-4) + a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 - 4a_0 = 0, 6a_3 - 3a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = 2a_0, a_3 = \frac{a_1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + a_k k - 4a_k + a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + a_{k+2}(k+2) - 4a_{k+2} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{k a_{k+2} + a_k - 2a_{k+2}}{k^2 + 7k + 12}, a_2 = 2a_0, a_3 = \frac{a_1}{2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+(x^2-4)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + 2x^2 + \frac{1}{4}x^4\right) y(0) + \left(x + \frac{1}{2}x^3 - \frac{1}{40}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[y''[x]+x*y'[x]+(x^2-4)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{x^5}{40} + \frac{x^3}{2} + x\right) + c_1\left(\frac{x^4}{4} + 2x^2 + 1\right)$$

14.5 problem 5

Internal problem ID [11888]
Internal file name [OUTPUT/11897_Saturday_April_13_2024_01_14_19_AM_78059033/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x + (3x + 2)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{795}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{796}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -y'x - 3yx - 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^2 - 3x - 3\right) y' + 3\left(x^2 + \frac{2}{3}x - 1\right) y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(-x^3 + 6x^2 + 7x - 6\right) y' - 3\left(x^3 - \frac{7}{3}x^2 - 7x - \frac{8}{3}\right) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 9x^3 - 3x^2 + 39x + 15\right) y' + 3\left(x^4 - \frac{16}{3}x^3 - 14x^2 + 6x + 11\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-x^5 + 12x^4 - 9x^3 - 108x^2 - 3x + 72\right) y' - 3\left(x^5 - \frac{25}{3}x^4 - 13x^3 + 53x^2 + 69x + 4\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= -3y'(0) - 3y(0) \\
F_2 &= -6y'(0) + 8y(0) \\
F_3 &= 15y'(0) + 33y(0) \\
F_4 &= 72y'(0) - 12y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = & \left(1 - x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{11}{40}x^5 - \frac{1}{60}x^6\right) y(0) \\
& + \left(x - \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 + \frac{1}{10}x^6\right) y'(0) + O(x^6)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x - 3\left(\sum_{n=0}^{\infty} a_n x^n\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n \\
\sum_{n=0}^{\infty} 3x^{1+n} a_n &= \sum_{n=1}^{\infty} 3a_{n-1} x^n
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(1+n)\, x^{n}\right)+\left(\sum_{n=1}^{\infty}n\, x^{n} a_{n}\right)+\left(\sum_{n=1}^{\infty}3 a_{n-1} x^{n}\right)+\left(\sum_{n=0}^{\infty}2 a_{n} x^{n}\right)=0 \quad (3)$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) + n a_n + 3a_{n-1} + 2a_n = 0 \quad (4)$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n a_n + 2a_n + 3a_{n-1}}{(n+2)\,(1+n)} \\ &= -\frac{a_n}{1+n} - \frac{3a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \quad (5)$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 3a_1 + 3a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{2} - \frac{a_0}{2}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 + 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{3} - \frac{a_1}{4}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 5a_3 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{8} + \frac{11a_0}{40}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 6a_4 + 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{60} + \frac{a_1}{10}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 7a_5 + 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{336} - \frac{39a_0}{560}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - a_0 x^2 + \left(-\frac{a_1}{2} - \frac{a_0}{2}\right) x^3 + \left(\frac{a_0}{3} - \frac{a_1}{4}\right) x^4 + \left(\frac{a_1}{8} + \frac{11a_0}{40}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{11}{40}x^5\right) a_0 + \left(x - \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{11}{40}x^5\right) c_1 + \left(x - \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &\left(1 - x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{11}{40}x^5 - \frac{1}{60}x^6\right) y(0) \\ &+ \left(x - \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 + \frac{1}{10}x^6\right) y'(0) + O(x^6) \end{aligned} \tag{1}$$

$$y = \left(1 - x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{11}{40}x^5\right) c_1 + \left(x - \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$\begin{aligned} y = &\left(1 - x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{11}{40}x^5 - \frac{1}{60}x^6\right) y(0) \\ &+ \left(x - \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 + \frac{1}{10}x^6\right) y'(0) + O(x^6) \end{aligned}$$

Verified OK.

$$y = \left(1 - x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{11}{40}x^5\right) c_1 + \left(x - \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5\right) c_2 + O(x^6)$$

Verified OK.

14.5.1 Maple step by step solution

Let's solve

$y'' = -y'x - 3yx - 2y$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-3x - 2)\, y - y'x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x + (3x + 2)\, y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k + 2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + 2a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(k+2) + 3a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

$$2a_2 + 2a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + a_k k + 2a_k + 3a_{k-1} = 0$$

- Shift index using $k- >k + 1$

$$\left((k+1)^2 + 3k + 5\right) a_{k+3} + a_{k+1}(k+1) + 2a_{k+1} + 3a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{k a_{k+1} + 3a_k + 3a_{k+1}}{k^2 + 5k + 6}, 2a_2 + 2a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(diff(y(x),x$2)+x*diff(y(x),x)+(3*x+2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{11}{40}x^5\right) y(0) + \left(x - \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 61

```
AsymptoticDSolveValue[y''[x]+x*y'[x]+(3*x+2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{8} - \frac{x^4}{4} - \frac{x^3}{2} + x\right) + c_1\left(\frac{11x^5}{40} + \frac{x^4}{3} - \frac{x^3}{2} - x^2 + 1\right)$$

14.6 problem 6

Internal problem ID [11889]
Internal file name [`OUTPUT/11898_Saturday_April_13_2024_01_14_20_AM_85922392/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y'x + (3x - 2)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (798) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (799) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -3yx + y'x + 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \left(x^2 - 3x + 3\right) y' - 3\left(x^2 - \frac{2}{3}x + 1\right) y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^3 - 6x^2 + 7x - 6\right) y' - 3y\left(x^3 - \frac{11}{3}x^2 + 7x - \frac{8}{3}\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 9x^3 + 21x^2 - 39x + 15\right) y' - 3y\left(x^4 - \frac{20}{3}x^3 + 14x^2 - 18x + 11\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^5 - 12x^4 + 45x^3 - 108x^2 + 111x - 72\right) y' - 3\left(x^5 - \frac{29}{3}x^4 + 31x^3 - 73x^2 + 69x - 28\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= 3y'(0) - 3y(0) \\
F_2 &= -6y'(0) + 8y(0) \\
F_3 &= 15y'(0) - 33y(0) \\
F_4 &= -72y'(0) + 84y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 + x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \frac{11}{40}x^5 + \frac{7}{60}x^6\right) y(0) \\ + \left(x + \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{10}x^6\right) y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -3\left(\sum_{n=0}^{\infty} a_n x^n\right) x + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 3x^{1+n} a_n\right) + \sum_{n=0}^{\infty} (-2 a_n x^n) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(1+n) x^n$$

$$\sum_{n=0}^{\infty} 3x^{1+n} a_n = \sum_{n=1}^{\infty} 3a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^{n}\right)+\sum_{n =1}^{\infty} \left(-n\, x^{n} a_{n}\right)+\left(\sum_{n=1}^{\infty} 3 a_{n-1} x^{n}\right)+\sum_{n =0}^{\infty} \left(-2 a_{n} x^{n}\right) = 0 \tag{3}$$

$n = 0$ gives

$$2 a_{2}-2 a_{0} = 0$$

$$a_{2} = a_{0}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n)-n a_{n}+3 a_{n-1}-2 a_{n} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n a_{n}+2 a_{n}-3 a_{n-1}}{(n+2)\,(1+n)} \\ &= \frac{a_{n}}{1+n}-\frac{3 a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6 a_{3}-3 a_{1}+3 a_{0} = 0$$

Which after substituting the earlier terms found becomes

$$a_{3} = \frac{a_{1}}{2}-\frac{a_{0}}{2}$$

For $n = 2$ the recurrence equation gives

$$12 a_{4}-4 a_{2}+3 a_{1} = 0$$

Which after substituting the earlier terms found becomes

$$a_{4} = \frac{a_{0}}{3}-\frac{a_{1}}{4}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 5a_3 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{8} - \frac{11a_0}{40}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 6a_4 + 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{7a_0}{60} - \frac{a_1}{10}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 7a_5 + 3a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{13a_1}{336} - \frac{39a_0}{560}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + a_0 x^2 + \left(\frac{a_1}{2} - \frac{a_0}{2}\right) x^3 + \left(\frac{a_0}{3} - \frac{a_1}{4}\right) x^4 + \left(\frac{a_1}{8} - \frac{11a_0}{40}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \frac{11}{40}x^5\right) a_0 + \left(x + \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1+x^{2}-\frac{1}{2}x^{3}+\frac{1}{3}x^{4}-\frac{11}{40}x^{5}\right) c_{1}+\left(x+\frac{1}{2}x^{3}-\frac{1}{4}x^{4}+\frac{1}{8}x^{5}\right) c_{2}+O\left(x^{6}\right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &\left(1+x^{2}-\frac{1}{2}x^{3}+\frac{1}{3}x^{4}-\frac{11}{40}x^{5}+\frac{7}{60}x^{6}\right) y(0) \\ &+\left(x+\frac{1}{2}x^{3}-\frac{1}{4}x^{4}+\frac{1}{8}x^{5}-\frac{1}{10}x^{6}\right) y'(0)+O\left(x^{6}\right) \end{aligned} \tag{1}$$

$$y = \left(1+x^{2}-\frac{1}{2}x^{3}+\frac{1}{3}x^{4}-\frac{11}{40}x^{5}\right) c_{1}+\left(x+\frac{1}{2}x^{3}-\frac{1}{4}x^{4}+\frac{1}{8}x^{5}\right) c_{2}+O\left(x^{6}\right) \tag{2}$$

Verification of solutions

$$\begin{aligned} y = &\left(1+x^{2}-\frac{1}{2}x^{3}+\frac{1}{3}x^{4}-\frac{11}{40}x^{5}+\frac{7}{60}x^{6}\right) y(0) \\ &+\left(x+\frac{1}{2}x^{3}-\frac{1}{4}x^{4}+\frac{1}{8}x^{5}-\frac{1}{10}x^{6}\right) y'(0)+O\left(x^{6}\right) \end{aligned}$$

Verified OK.

$$y = \left(1+x^{2}-\frac{1}{2}x^{3}+\frac{1}{3}x^{4}-\frac{11}{40}x^{5}\right) c_{1}+\left(x+\frac{1}{2}x^{3}-\frac{1}{4}x^{4}+\frac{1}{8}x^{5}\right) c_{2}+O\left(x^{6}\right)$$

Verified OK.

14.6.1 Maple step by step solution

Let's solve

$y'' = -3yx + y'x + 2y$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-3x+2)\,y + y'x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y'x + (3x-2)\,y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$$x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$$

- Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k + 2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 - 2a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(k+2) + 3a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

$$2a_2 - 2a_0 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - a_k k - 2a_k + 3a_{k-1} = 0$$

- Shift index using $k- >k + 1$

$$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_{k+1}(k+1) - 2a_{k+1} + 3a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k a_{k+1} - 3a_k + 3a_{k+1}}{k^2 + 5k + 6}, 2a_2 - 2a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form could result into a too large expression - returning special functi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
Order:=6;
dsolve(diff(y(x),x$2)-x*diff(y(x),x)+(3*x-2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 - \frac{11}{40}x^5\right) y(0) + \left(x + \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 59

```
AsymptoticDSolveValue[y''[x]-x*y'[x]+(3*x-2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{8}-\frac{x^4}{4}+\frac{x^3}{2}+x\right)+c_1\left(-\frac{11x^5}{40}+\frac{x^4}{3}-\frac{x^3}{2}+x^2+1\right)$$

14.7 problem 7

Internal problem ID [11890]
Internal file name [`OUTPUT/11899_Saturday_April_13_2024_01_14_21_AM_32507608/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''+y'x+yx=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{801}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{802}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{x(y' + y)}{x^2 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(-x^3 + 2x^2 - x - 1)\, y' + y(2x^2 - 1)}{(x^2 + 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(4x^4 - 6x^3 + 2x^2 + 9x - 2)\, y' + xy(x^3 - 6x^2 + x + 9)}{(x^2 + 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^6 - 18x^5 + 26x^4 + 9x^3 - 71x^2 + 27x + 9)\, y' - 6y\left(x^5 - 4x^4 + \frac{1}{3}x^3 + 12x^2 - \frac{2}{3}x - \frac{3}{2}\right)}{(x^2 + 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-9x^7 + 96x^6 - 132x^5 - 192x^4 + 603x^3 - 252x^2 - 219x + 36)\, y' - y(x^7 - 36x^6 + 122x^5 + 29x^4 - }{(x^2 + 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y'(0) - y(0) \\
F_2 &= -2y'(0) \\
F_3 &= 9y(0) + 9y'(0) \\
F_4 &= 4y(0) + 36y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{1}{180}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{3}{40}x^5 + \frac{1}{20}x^6\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2+1)\,y''+y'x+yx=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''=\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)x+\left(\sum_{n=0}^{\infty} a_n x^n\right)x=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\left(\sum_{n=0}^{\infty} x^{1+n} a_n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}=\sum_{n=0}^{\infty} (n+2)\,a_{n+2}(1+n)\,x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n=\sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\left(\sum_{n=0}^{\infty} (n+2)\,a_{n+2}(1+n)\,x^n\right)\\&+\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\left(\sum_{n=1}^{\infty} a_{n-1} x^n\right)=0\end{aligned} \tag{3}$$

$n = 1$ gives

$$6a_3 + a_1 + a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{6} - \frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$na_n(n-1) + (n+2)\, a_{n+2}(1+n) + na_n + a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_n + a_{n-1}}{(n+2)\,(1+n)} \\ &= -\frac{n^2 a_n}{(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$4a_2 + 12a_4 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$9a_3 + 20a_5 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{3a_0}{40} + \frac{3a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$16a_4 + 30a_6 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_1}{20} + \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$25a_5 + 42a_7 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{5a_0}{112} - \frac{43a_1}{1008}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(-\frac{a_0}{6} - \frac{a_1}{6}\right) x^3 - \frac{a_1 x^4}{12} + \left(\frac{3a_0}{40} + \frac{3a_1}{40}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{6}x^3 + \frac{3}{40}x^5\right) a_0 + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{3}{40}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{6}x^3 + \frac{3}{40}x^5\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{3}{40}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &\left(1 - \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{1}{180}x^6\right) y(0) \\ &+ \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{3}{40}x^5 + \frac{1}{20}x^6\right) y'(0) + O(x^6) \end{aligned} \tag{1}$$

$$y = \left(1 - \frac{1}{6}x^3 + \frac{3}{40}x^5\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{3}{40}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{1}{180}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{3}{40}x^5 + \frac{1}{20}x^6\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{6}x^3 + \frac{3}{40}x^5\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{3}{40}x^5\right) c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
Order:=6;
dsolve((x^2+1)*diff(y(x),x$2)+x*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{6}x^3 + \frac{3}{40}x^5\right) y(0) + \left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{3}{40}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 49

```
AsymptoticDSolveValue[(x^2+1)*y''[x]+x*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{3x^5}{40} - \frac{x^3}{6} + 1\right) + c_2\left(\frac{3x^5}{40} - \frac{x^4}{12} - \frac{x^3}{6} + x\right)$$

14.8 problem 8

Internal problem ID [11891]
Internal file name [`OUTPUT/11900_Saturday_April_13_2024_01_14_21_AM_74726995/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\,y''-(3x-2)\,y'+2yx=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy_0'+\frac{x^2}{2}f|_{x_0,y_0,y_0'}+\frac{x^3}{3!}f'|_{x_0,y_0,y_0'}+\cdots\\
&= y_0+xy_0'+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (804)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (805)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{3y'x - 2yx - 2y'}{x-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(7x-3)\, y' + (-6x-2)\, y}{x-1} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(15x-4)\, y' + (-14x-8)\, y}{x-1} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(31x-5)\, y' + (-30x-22)\, y}{x-1} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(63x-6)\, y' + (-62x-52)\, y}{x-1}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y'(0) \\
F_1 &= 2y(0) + 3y'(0) \\
F_2 &= 8y(0) + 4y'(0) \\
F_3 &= 22y(0) + 5y'(0) \\
F_4 &= 52y(0) + 6y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1+\frac{1}{3}x^3+\frac{1}{3}x^4+\frac{11}{60}x^5+\frac{13}{180}x^6\right)y(0) + \left(x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4+\frac{1}{24}x^5+\frac{1}{120}x^6\right)y'(0)+O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x-1)\,y''+(-3x+2)\,y'+2yx = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x-1)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+(-3x+2)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+2\left(\sum_{n=0}^{\infty} a_n x^n\right)x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\,x^{n-1} a_n(n-1)\right)+\sum_{n=2}^{\infty}\left(-n(n-1)\,a_n x^{n-2}\right) + \sum_{n=1}^{\infty}(-3n a_n x^n)+\left(\sum_{n=1}^{\infty} 2n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} 2x^{1+n}a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\,x^{n-1} a_n(n-1) = \sum_{n=1}^{\infty} (1+n)\,a_{1+n} n\,x^n$$

$$\sum_{n=2}^{\infty} \left(-n(n-1)\,a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\,a_{n+2}(1+n)\,x^n\right)$$

$$\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 2(1+n)\,a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} 2x^{1+n} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (1+n)\,a_{1+n} n\,x^n\right) + \sum_{n=0}^{\infty} \left(-(n+2)\,a_{n+2}(1+n)\,x^n\right) + \sum_{n=1}^{\infty} (-3n a_n x^n) + \left(\sum_{n=0}^{\infty} 2(1+n)\,a_{1+n} x^n\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-2a_2 + 2a_1 = 0$$

$$a_2 = a_1$$

For $1 \leq n$, the recurrence equation is

$$(1+n)\,a_{1+n} n - (n+2)\,a_{n+2}(1+n) - 3n a_n + 2(1+n)\,a_{1+n} + 2a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n^2 a_{1+n} - 3n a_n + 3n a_{1+n} + 2a_{1+n} + 2a_{n-1}}{(n+2)\,(1+n)} \\ &= -\frac{3n a_n}{(n+2)\,(1+n)} + \frac{(n^2+3n+2)\,a_{1+n}}{(n+2)\,(1+n)} + \frac{2a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_2 - 6a_3 - 3a_1 + 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{2} + \frac{a_0}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_3 - 12a_4 - 6a_2 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{6} + \frac{a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_4 - 20a_5 - 9a_3 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{24} + \frac{11a_0}{60}$$

For $n = 4$ the recurrence equation gives

$$30a_5 - 30a_6 - 12a_4 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_1}{120} + \frac{13a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_6 - 42a_7 - 15a_5 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{720} + \frac{19a_0}{840}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x + a_1 x^2 + \left(\frac{a_1}{2} + \frac{a_0}{3}\right) x^3 + \left(\frac{a_1}{6} + \frac{a_0}{3}\right) x^4 + \left(\frac{a_1}{24} + \frac{11a_0}{60}\right) x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 + \frac{1}{3}x^3 + \frac{1}{3}x^4 + \frac{11}{60}x^5\right) a_0 + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) a_1 + O(x^6) \quad (3)
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 + \frac{1}{3}x^3 + \frac{1}{3}x^4 + \frac{11}{60}x^5\right) c_1 + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) c_2 + O(x^6)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1 + \frac{1}{3}x^3 + \frac{1}{3}x^4 + \frac{11}{60}x^5 + \frac{13}{180}x^6\right) y(0) \\
& + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \frac{1}{120}x^6\right) y'(0) + O(x^6)
\end{aligned} \quad (1)
$$

$$
y = \left(1 + \frac{1}{3}x^3 + \frac{1}{3}x^4 + \frac{11}{60}x^5\right) c_1 + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) c_2 + O(x^6) \quad (2)
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1 + \frac{1}{3}x^3 + \frac{1}{3}x^4 + \frac{11}{60}x^5 + \frac{13}{180}x^6\right) y(0) \\
& + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \frac{1}{120}x^6\right) y'(0) + O(x^6)
\end{aligned}
$$

Verified OK.

$$
y = \left(1 + \frac{1}{3}x^3 + \frac{1}{3}x^4 + \frac{11}{60}x^5\right) c_1 + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) c_2 + O(x^6)
$$

Verified OK.

14.8.1 Maple step by step solution

Let's solve

$(x-1)\, y''+(-3x+2)\, y'+2yx=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{2xy}{x-1}+\frac{(3x-2)y'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''-\frac{(3x-2)y'}{x-1}+\frac{2xy}{x-1}=0$

□ Check to see if $x_0=1$ is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{3x-2}{x-1}, P_3(x)=\frac{2x}{x-1}\right]$

- ◦ $(x-1)\cdot P_2(x)$ is analytic at $x=1$

$\left((x-1)\cdot P_2(x)\right)\Big|_{x=1}=-1$

- ◦ $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

$\left((x-1)^2\cdot P_3(x)\right)\Big|_{x=1}=0$

- ◦ $x=1$is a regular singular point

Check to see if $x_0=1$ is a regular singular point

$x_0=1$

- Multiply by denominators

$(x-1)\, y''+(-3x+2)\, y'+2yx=0$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(-3u-1)\left(\frac{d}{du}y(u)\right)+(2u+2)\, y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, u^{-1+r} + (a_1(1+r)\,(-1+r) - a_0(-2+3r))\, u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)\,(k+r-1)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term must be 0

$$a_1(1+r)\,(-1+r) - a_0(-2+3r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)\,(k+r-1) + a_k(-3k-3r+2) + 2a_{k-1} = 0$$

- Shift index using $k- >k+1$

$$a_{k+2}(k+2+r)\,(k+r) + a_{k+1}(-3k-1-3r) + 2a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{3k a_{k+1} + 3r a_{k+1} - 2a_k + a_{k+1}}{(k+2+r)(k+r)}$$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{3k a_{k+1} - 2a_k + a_{k+1}}{(k+2)k}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = \frac{3k a_{k+1} - 2a_k + a_{k+1}}{(k+2)k}$

- Recursion relation for $r = 2$

  $a_{k+2} = \frac{3k a_{k+1} - 2a_k + 7a_{k+1}}{(k+4)(k+2)}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+2} = \frac{3k a_{k+1} - 2a_k + 7a_{k+1}}{(k+4)(k+2)}, 3a_1 - 4a_0 = 0\right]$

- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+2} = \frac{3k a_{k+1} - 2a_k + 7a_{k+1}}{(k+4)(k+2)}, 3a_1 - 4a_0 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
Order:=6;
dsolve((x-1)*diff(y(x),x$2)-(3*x-2)*diff(y(x),x)+2*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{3}x^3 + \frac{1}{3}x^4 + \frac{11}{60}x^5\right) y(0) + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 59

```
AsymptoticDSolveValue[(x-1)*y''[x]-(3*x-2)*y'[x]+2*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1\left(\frac{11x^5}{60}+\frac{x^4}{3}+\frac{x^3}{3}+1\right)+c_2\left(\frac{x^5}{24}+\frac{x^4}{6}+\frac{x^3}{2}+x^2+x\right)$$

14.9 problem 9

Internal problem ID [11892]
Internal file name [`OUTPUT/11901_Saturday_April_13_2024_01_14_22_AM_46985098/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^{3}-1\right) y''+x^{2} y'+y x=0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (807) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (808) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{x(y'x+y)}{x^3-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{x^4 y' + 3y x^3 + 3y'x + y}{(x^3-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{-10y x^5 - 24y'x^3 - 18x^2 y - 4y'}{(x^3-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= -\frac{10x((x^7 - 16x^4 - 13x)\, y' - 4y(x^6 + 5x^3 + 1))}{(x^3-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(90x^{10} - 1040x^7 - 2390x^4 - 300x)\, y' - 190\left(x^9 + \frac{204}{19}x^6 + \frac{137}{19}x^3 + \frac{4}{19}\right) y}{(x^3-1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= y(0) \\
F_2 &= 4y'(0) \\
F_3 &= 0 \\
F_4 &= 40y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{6}x^3 + \frac{1}{18}x^6\right) y(0) + \left(x + \frac{1}{6}x^4\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(x^3 - 1\right) y'' + x^2 y' + yx = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(x^3 - 1\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + x^2 \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, x^{1+n} a_n (n-1)\right) + \sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{1+n} a_n (n-1) = \sum_{n=3}^{\infty} (n-1)\, a_{n-1} (n-2)\, x^n$$
$$\sum_{n=2}^{\infty} \left(-n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2} (1+n)\, x^n\right)$$
$$\sum_{n=1}^{\infty} n\, x^{1+n} a_n = \sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^n$$
$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=3}^{\infty} (n-1)\, a_{n-1}(n-2)\, x^{n}\right)+\sum_{n\, =0}^{\infty} \left(-(n+2)\, a_{n+2}(1+n)\, x^{n}\right) \\ +\left(\sum_{n=2}^{\infty} (n-1)\, a_{n-1} x^{n}\right)+\left(\sum_{n=1}^{\infty} a_{n-1} x^{n}\right) = 0 \tag{3}$$

$n = 1$ gives

$$-6a_{3}+a_{0} = 0$$

Which after substituting earlier equations, simplifies to

$$a_{3} = \frac{a_{0}}{6}$$

$n = 2$ gives

$$-12a_{4}+2a_{1} = 0$$

Which after substituting earlier equations, simplifies to

$$a_{4} = \frac{a_{1}}{6}$$

For $3 \le n$, the recurrence equation is

$$(n-1)\, a_{n-1}(n-2)-(n+2)\, a_{n+2}(1+n)+(n-1)\, a_{n-1}+a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_{n-1}(n^{2}-2n+2)}{(n+2)\, (1+n)} \tag{5}$$

For $n = 3$ the recurrence equation gives

$$5a_{2}-20a_{5} = 0$$

Which after substituting the earlier terms found becomes

$$a_{5} = 0$$

For $n = 4$ the recurrence equation gives

$$10a_3 - 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{18}$$

For $n = 5$ the recurrence equation gives

$$17a_4 - 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{17a_1}{252}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{6} a_0 x^3 + \frac{1}{6} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{x^3}{6}\right) a_0 + \left(x + \frac{1}{6} x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{x^3}{6}\right) c_1 + \left(x + \frac{1}{6} x^4\right) c_2 + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(1 + \frac{1}{6} x^3 + \frac{1}{18} x^6\right) y(0) + \left(x + \frac{1}{6} x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{x^3}{6}\right) c_1 + \left(x + \frac{1}{6} x^4\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1+\frac{1}{6}x^3+\frac{1}{18}x^6\right)y(0)+\left(x+\frac{1}{6}x^4\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1+\frac{x^3}{6}\right)c_1+\left(x+\frac{1}{6}x^4\right)c_2+O(x^6)$$

Verified OK.

14.9.1 Maple step by step solution

Let's solve

$(x^3-1)\,y''+x^2y'+yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{x^2y'}{x^3-1}-\frac{xy}{x^3-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{x^2y'}{x^3-1}+\frac{xy}{x^3-1}=0$

☐ Check to see if x_0 is a regular singular point

  ○ Define functions

  $\left[P_2(x)=\frac{x^2}{x^3-1}, P_3(x)=\frac{x}{x^3-1}\right]$

  ○ $(x-1)\cdot P_2(x)$ is analytic at $x=1$

  $\left.((x-1)\cdot P_2(x))\right|_{x=1}=\frac{1}{3}$

  ○ $(x-1)^2\cdot P_3(x)$ is analytic at $x=1$

  $\left.((x-1)^2\cdot P_3(x))\right|_{x=1}=0$

  ○ $x=1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=1$

- Multiply by denominators

$$\left(x^3-1\right) y''+x^2 y'+y x=0$$

- Change variables using $x=u+1$ so that the regular singular point is at $u=0$

$$\left(u^3+3u^2+3u\right)\left(\frac{d^2}{du^2}y(u)\right)+\left(u^2+2u+1\right)\left(\frac{d}{du}y(u)\right)+\left(u+1\right)y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

☐ Rewrite ODE with series expansions

  ◦ Convert $u^m \cdot y(u)$ to series expansion for $m=0..1$

$$u^m \cdot y(u)=\sum_{k=0}^{\infty} a_k u^{k+r+m}$$

  ◦ Shift index using $k->k-m$

$$u^m \cdot y(u)=\sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

  ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

  ◦ Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

  ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

  ◦ Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+3r)\,u^{-1+r}+\left(a_1(1+r)(1+3r)+a_0\left(3r^2-r+1\right)\right)u^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)(3k+1\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+3r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\left\{0,\frac{2}{3}\right\}$$

- Each term must be 0

  $a_1(1+r)(1+3r)+a_0(3r^2-r+1)=0$

- Each term in the series must be 0, giving the recursion relation

  $(3a_k+a_{k-1}+3a_{k+1})k^2+((6a_k+2a_{k-1}+6a_{k+1})r-a_k-2a_{k-1}+4a_{k+1})k+(3a_k+a_{k-1}+3a_$

- Shift index using $k->k+1$

  $(3a_{k+1}+a_k+3a_{k+2})(k+1)^2+((6a_{k+1}+2a_k+6a_{k+2})r-a_{k+1}-2a_k+4a_{k+2})(k+1)+(3a_{k+}$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{k^2a_k+3k^2a_{k+1}+2kra_k+6kra_{k+1}+r^2a_k+3r^2a_{k+1}+5ka_{k+1}+5ra_{k+1}+a_k+3a_{k+1}}{3k^2+6kr+3r^2+10k+10r+8}$

- Recursion relation for $r=0$

  $a_{k+2}=-\frac{k^2a_k+3k^2a_{k+1}+5ka_{k+1}+a_k+3a_{k+1}}{3k^2+10k+8}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^k, a_{k+2}=-\frac{k^2a_k+3k^2a_{k+1}+5ka_{k+1}+a_k+3a_{k+1}}{3k^2+10k+8}, a_1+a_0=0\right]$

- Revert the change of variables $u=x-1$

  $\left[y=\sum_{k=0}^{\infty}a_k(x-1)^k, a_{k+2}=-\frac{k^2a_k+3k^2a_{k+1}+5ka_{k+1}+a_k+3a_{k+1}}{3k^2+10k+8}, a_1+a_0=0\right]$

- Recursion relation for $r=\frac{2}{3}$

  $a_{k+2}=-\frac{k^2a_k+3k^2a_{k+1}+\frac{4}{3}ka_k+9ka_{k+1}+\frac{13}{9}a_k+\frac{23}{3}a_{k+1}}{3k^2+14k+16}$

- Solution for $r=\frac{2}{3}$

  $\left[y(u)=\sum_{k=0}^{\infty}a_k u^{k+\frac{2}{3}}, a_{k+2}=-\frac{k^2a_k+3k^2a_{k+1}+\frac{4}{3}ka_k+9ka_{k+1}+\frac{13}{9}a_k+\frac{23}{3}a_{k+1}}{3k^2+14k+16}, 5a_1+\frac{5a_0}{3}=0\right]$

- Revert the change of variables $u=x-1$

  $\left[y=\sum_{k=0}^{\infty}a_k(x-1)^{k+\frac{2}{3}}, a_{k+2}=-\frac{k^2a_k+3k^2a_{k+1}+\frac{4}{3}ka_k+9ka_{k+1}+\frac{13}{9}a_k+\frac{23}{3}a_{k+1}}{3k^2+14k+16}, 5a_1+\frac{5a_0}{3}=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k(x-1)^k\right)+\left(\sum_{k=0}^{\infty}b_k(x-1)^{k+\frac{2}{3}}\right), a_{k+2}=-\frac{k^2a_k+3k^2a_{1+k}+5ka_{1+k}+a_k+3a_{1+k}}{3k^2+10k+8}, a_1+a_0=0\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
Order:=6;
dsolve((x^3-1)*diff(y(x),x$2)+x^2*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{x^3}{6}\right) y(0) + \left(x + \frac{1}{6}x^4\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[(x^3-1)*y''[x]+x^2*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^4}{6} + x\right) + c_1\left(\frac{x^3}{6} + 1\right)$$

14.10 problem 10

Internal problem ID [11893]
Internal file name [OUTPUT/11902_Saturday_April_13_2024_01_14_23_AM_37826173/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{(x+3)\,y''+(x+2)\,y'+y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (810) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (811) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') &&(4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f &&(5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 &&(6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
F_0 &= -\frac{y'x + 2y' + y}{x+3} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{y'x + y}{x+3} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-x^2 - x + 6)\, y' - y(x+1)}{(x+3)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x^3 + 2x^2 - 13x - 30)\, y' + y(x^2 + 2x - 7)}{(x+3)^3} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{-(x+3)\,(x^3 - 21x - 30)\, y' - y(x^3 + 3x^2 - 15x - 57)}{(x+3)^4}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= -\frac{y(0)}{3} - \frac{2y'(0)}{3} \\
F_1 &= \frac{y(0)}{3} \\
F_2 &= -\frac{y(0)}{9} + \frac{2y'(0)}{3} \\
F_3 &= -\frac{7y(0)}{27} - \frac{10y'(0)}{9} \\
F_4 &= \frac{19y(0)}{27} + \frac{10y'(0)}{9}
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1-\frac{1}{6}x^2+\frac{1}{18}x^3-\frac{1}{216}x^4-\frac{7}{3240}x^5+\frac{19}{19440}x^6\right)y(0) + \left(x-\frac{1}{3}x^2+\frac{1}{36}x^4-\frac{1}{108}x^5+\frac{1}{648}x^6\right)y'(0)+O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x+3)\,y''+(x+2)\,y'+y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x+3)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+(x+2)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\,x^{n-1} a_n (n-1)\right)+\left(\sum_{n=2}^{\infty} 3n(n-1)\,a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right)+\left(\sum_{n=1}^{\infty} 2n a_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, x^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^{n}$$

$$\sum_{n=2}^{\infty} 3n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(n+1)\, x^{n}$$

$$\sum_{n=1}^{\infty} 2n a_n x^{n-1} = \sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} x^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, x^{n}\right) + \left(\sum_{n=0}^{\infty} 3(n+2)\, a_{n+2}(n+1)\, x^{n}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n}\right) + \left(\sum_{n=0}^{\infty} 2(n+1)\, a_{n+1} x^{n}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n}\right) = 0 \tag{3}$$

$n = 0$ gives

$$6a_2 + 2a_1 + a_0 = 0$$

$$a_2 = -\frac{a_0}{6} - \frac{a_1}{3}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n + 3(n+2)\, a_{n+2}(n+1) + n a_n + 2(n+1)\, a_{n+1} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n a_{n+1} + a_n + 2a_{n+1}}{3\,(n+2)} \\ &= -\frac{a_n}{3\,(n+2)} - \frac{a_{n+1}}{3} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_2 + 18a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{18}$$

For $n = 2$ the recurrence equation gives

$$12a_3 + 36a_4 + 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{216} + \frac{a_1}{36}$$

For $n = 3$ the recurrence equation gives

$$20a_4 + 60a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{7a_0}{3240} - \frac{a_1}{108}$$

For $n = 4$ the recurrence equation gives

$$30a_5 + 90a_6 + 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{19a_0}{19440} + \frac{a_1}{648}$$

For $n = 5$ the recurrence equation gives

$$42a_6 + 126a_7 + 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{13a_0}{58320} - \frac{a_1}{13608}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x + \left(-\frac{a_0}{6} - \frac{a_1}{3}\right) x^2 + \frac{a_0 x^3}{18} + \left(-\frac{a_0}{216} + \frac{a_1}{36}\right) x^4 + \left(-\frac{7a_0}{3240} - \frac{a_1}{108}\right) x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 - \frac{1}{6}x^2 + \frac{1}{18}x^3 - \frac{1}{216}x^4 - \frac{7}{3240}x^5\right) a_0 + \left(x - \frac{1}{3}x^2 + \frac{1}{36}x^4 - \frac{1}{108}x^5\right) a_1 + O\left(x^6\right) \tag{3}
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 - \frac{1}{6}x^2 + \frac{1}{18}x^3 - \frac{1}{216}x^4 - \frac{7}{3240}x^5\right) c_1 + \left(x - \frac{1}{3}x^2 + \frac{1}{36}x^4 - \frac{1}{108}x^5\right) c_2 + O\left(x^6\right)
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &\left(1 - \frac{1}{6}x^2 + \frac{1}{18}x^3 - \frac{1}{216}x^4 - \frac{7}{3240}x^5 + \frac{19}{19440}x^6\right) y(0) \\
&+ \left(x - \frac{1}{3}x^2 + \frac{1}{36}x^4 - \frac{1}{108}x^5 + \frac{1}{648}x^6\right) y'(0) + O\left(x^6\right)
\end{aligned} \tag{1}
$$

$$
y = \left(1 - \frac{1}{6}x^2 + \frac{1}{18}x^3 - \frac{1}{216}x^4 - \frac{7}{3240}x^5\right) c_1 + \left(x - \frac{1}{3}x^2 + \frac{1}{36}x^4 - \frac{1}{108}x^5\right) c_2 + O\left(x^6\right) \tag{2}
$$

Verification of solutions

$$
\begin{aligned}
y = &\left(1 - \frac{1}{6}x^2 + \frac{1}{18}x^3 - \frac{1}{216}x^4 - \frac{7}{3240}x^5 + \frac{19}{19440}x^6\right) y(0) \\
&+ \left(x - \frac{1}{3}x^2 + \frac{1}{36}x^4 - \frac{1}{108}x^5 + \frac{1}{648}x^6\right) y'(0) + O\left(x^6\right)
\end{aligned}
$$

Verified OK.

$$
y = \left(1 - \frac{1}{6}x^2 + \frac{1}{18}x^3 - \frac{1}{216}x^4 - \frac{7}{3240}x^5\right) c_1 + \left(x - \frac{1}{3}x^2 + \frac{1}{36}x^4 - \frac{1}{108}x^5\right) c_2 + O\left(x^6\right)
$$

Verified OK.

14.10.1 Maple step by step solution

Let's solve

$(x+3)\, y''+(x+2)\, y'+y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{y}{x+3}-\frac{(x+2)y'}{x+3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{(x+2)y'}{x+3}+\frac{y}{x+3}=0$

□ Check to see if $x_0=-3$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x+2}{x+3}, P_3(x)=\frac{1}{x+3}\right]$

  ◦ $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

  $\left((x+3)\cdot P_2(x)\right)\bigg|_{x=-3}=-1$

  ◦ $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

  $\left((x+3)^2\cdot P_3(x)\right)\bigg|_{x=-3}=0$

  ◦ $x=-3$is a regular singular point

Check to see if $x_0=-3$ is a regular singular point

$x_0=-3$

- Multiply by denominators

$(x+3)\, y''+(x+2)\, y'+y=0$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

$u\left(\frac{d^2}{du^2}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)+y(u)=0$

- Assume series solution for $y(u)$

$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

$$u^{m}\cdot \left(\frac{d}{du}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) u^{k +r -1+m}$$

- Shift index using $k->k +1-m$

$$u^{m}\cdot \left(\frac{d}{du}y \left(u \right)\right)=\sum_{k =-1+m}^{\infty}a_{k +1-m} \left(k +1-m +r \right) u^{k +r}$$

- Convert $u \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)$ to series expansion

$$u \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =0}^{\infty}a_{k} \left(k +r \right) \left(k +r -1\right) u^{k +r -1}$$

- Shift index using $k->k +1$

$$u \cdot \left(\frac{d^{2}}{d u^{2}}y \left(u \right)\right)=\sum_{k =-1}^{\infty}a_{k +1} \left(k +1+r \right) \left(k +r \right) u^{k +r}$$

Rewrite ODE with series expansions

$$a_{0} r \left(-2+r \right) u^{-1+r}+\left(\sum_{k =0}^{\infty}\left(a_{k +1} \left(k +1+r \right) \left(k +r -1\right)+a_{k} \left(k +1+r \right)\right) u^{k +r}\right)=0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$r \left(-2+r \right)=0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k +1+r \right) \left(a_{k +1} \left(k +r -1\right)+a_{k}\right)=0$$

- Recursion relation that defines series solution to ODE

$$a_{k +1}=-\frac{a_{k}}{k +r -1}$$

- Recursion relation for $r =0$

$$a_{k +1}=-\frac{a_{k}}{k -1}$$

- Series not valid for $r =0$, division by 0 in the recursion relation at $k =1$

$$a_{k +1}=-\frac{a_{k}}{k -1}$$

- Recursion relation for $r =2$

$$a_{k +1}=-\frac{a_{k}}{k +1}$$

- Solution for $r =2$

$$\left[y \left(u \right)=\sum_{k =0}^{\infty}a_{k} u^{k +2}, a_{k +1}=-\frac{a_{k}}{k +1}\right]$$

- Revert the change of variables $u = x + 3$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+3)^{k+2}, a_{k+1} = -\frac{a_k}{k+1}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve((x+3)*diff(y(x),x$2)+(x+2)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{6}x^2 + \frac{1}{18}x^3 - \frac{1}{216}x^4 - \frac{7}{3240}x^5\right) y(0) + \left(x - \frac{1}{3}x^2 + \frac{1}{36}x^4 - \frac{1}{108}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 63

```
AsymptoticDSolveValue[(x+3)*y''[x]+(x+2)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(-\frac{x^5}{108} + \frac{x^4}{36} - \frac{x^2}{3} + x\right) + c_1 \left(-\frac{7x^5}{3240} - \frac{x^4}{216} + \frac{x^3}{18} - \frac{x^2}{6} + 1\right)$$

14.11 problem 11

Internal problem ID [11894]
Internal file name [`OUTPUT/11903_Saturday_April_13_2024_01_14_24_AM_94254307/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact linear second order ode", "second_order_integrable_as_is", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{y'' - y'x - y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

14.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -x \\ q(x) &= -1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - y'x - y = 0$$

The domain of $p(x) = -x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{813}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{814}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= y'x + y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= x^2 y' + yx + 2y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(x^3 + 5x\right) y' + y\left(x^2 + 3\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 + 9x^2 + 8\right) y' + xy\left(x^2 + 7\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(x^5 + 14x^3 + 33x\right) y' + y\left(x^4 + 12x^2 + 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = 1$ and $y'(0) = 0$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 0 \\
F_2 &= 3 \\
F_3 &= 0 \\
F_4 &= 15
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O\left(x^6\right)
$$

$$
y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-n\, x^n a_n) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$\left(n+2\right) a_{n+2}(n+1) - n a_n - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n}{n+2} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 5a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{48}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{105}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{3} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{1}{15} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) a_0 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x + \frac{1}{3}x^3 + \frac{1}{15}x^5\right) c_2 + O(x^6)$$

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^6) \tag{1}$$

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + O(x^6)$$

Verified OK.

$$y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)$$

Verified OK.

14.11.2 Maple step by step solution

Let's solve

$\left[y'' = y'x + y, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y'x - y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k(k+1)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k+1)(a_{k+2}(k+2)-a_k)=0$

- Recursion relation that defines the series solution to the ODE

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+2}=\frac{a_k}{k+2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)-x*diff(y(x),x)-y(x)=0,y(0) = 1, D(y)(0) = 0],y(x),type='series',x=0);
```

$$y(x)=1+\frac{1}{2}x^2+\frac{1}{8}x^4+\mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{y''[x]-x*y'[x]-y[x]==0,{y[0]==1,y'[0]==0}},y[x],{x,0,5}]
```

$$y(x)\to\frac{x^4}{8}+\frac{x^2}{2}+1$$

14.12 problem 12

Internal problem ID [11895]
Internal file name [`OUTPUT/11904_Saturday_April_13_2024_01_14_26_AM_92243548/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y'x - 2y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 1]$$

With the expansion point for the power series method at $x = 0$.

14.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= x \\ q(x) &= -2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y'x - 2y = 0$$

The domain of $p(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{816}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{817}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -y'x + 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= x^2y' - 2yx + y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= -x\left(x^2+1\right)y' + 2x^2y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \left(x^4-1\right)y' + \left(-2x^3+2x\right)y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= -\left(x^2-3\right)x\left(\left(x^2+1\right)y' - 2yx\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=0$ and $y'(0)=1$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 1 \\
F_2 &= 0 \\
F_3 &= -1 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = x + \frac{x^3}{6} - \frac{x^5}{120} + O\left(x^6\right)
$$

$$
y = x + \frac{x^3}{6} - \frac{x^5}{120} + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \sum_{n=0}^{\infty} (-2 a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=1}^{\infty} n\, x^n a_n\right) + \sum_{n=0}^{\infty} (-2 a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 2a_0 = 0$$

$$a_2 = a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n - 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n-2)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 3a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{1680}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + a_0 x^2 + \frac{1}{6} a_1 x^3 - \frac{1}{120} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(x^2 + 1\right) a_0 + \left(x + \frac{1}{6}x^3 - \frac{1}{120}x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(x^2 + 1\right) c_1 + \left(x + \frac{1}{6}x^3 - \frac{1}{120}x^5\right) c_2 + O\left(x^6\right)$$

$$y = x + \frac{x^3}{6} - \frac{x^5}{120} + O\left(x^6\right)$$

Summary

The solution(s) found are the following

$$y = x + \frac{x^3}{6} - \frac{x^5}{120} + O\left(x^6\right) \tag{1}$$

$$y = x + \frac{x^3}{6} - \frac{x^5}{120} + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = x + \frac{x^3}{6} - \frac{x^5}{120} + O(x^6)$$

Verified OK.

$$y = x + \frac{x^3}{6} - \frac{x^5}{120} + O(x^6)$$

Verified OK.

14.12.2 Maple step by step solution

Let's solve

$\left[y'' = -y'x + 2y, y(0) = 0, y'\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y'x - 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite DE with series expansions

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$

  ◦ Shift index using $k- >k+2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$

  Rewrite DE with series expansions

  $\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) + a_k(k-2)\right) x^k = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2) a_{k+2} + a_k(k - 2) = 0$

- Recursion relation; series terminates at $k = 2$

  $a_{k+2} = -\frac{a_k(k-2)}{k^2+3k+2}$

- Apply recursion relation for $k = 0$

  $a_2 = a_0$

- Terminating series solution of the ODE. Use reduction of order to find the second linearly ind

  $y = A_2x^2 + A_1x + a_0$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve([diff(y(x),x$2)+x*diff(y(x),x)-2*y(x)=0,y(0) = 0, D(y)(0) = 1],y(x),type='series',x=0
```

$$y(x) = x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 19

```
AsymptoticDSolveValue[{y''[x]+x*y'[x]-2*y[x]==0,{y[0]==0,y'[0]==1}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{x^5}{120} + \frac{x^3}{6} + x$$

14.13 problem 13

Internal problem ID [11896]
Internal file name [`OUTPUT/11905_Saturday_April_13_2024_01_14_28_AM_24219364/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''+y'x+2yx=0}$$

With initial conditions

$$[y(0)=2, y'(0)=3]$$

With the expansion point for the power series method at $x=0$.

14.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(x)y'+q(x)y=F$$

Where here

$$\begin{aligned} p(x) &= \frac{x}{x^2+1} \\ q(x) &= \frac{2x}{x^2+1} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{xy'}{x^2+1} + \frac{2xy}{x^2+1} = 0$$

The domain of $p(x) = \frac{x}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{2x}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (819) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (820) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned} F_0 &= f(x, y, y') \qquad (4) \\ F_1 &= \frac{df}{dx} \\ &= \frac{dF_0}{dx} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5) \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\ F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\ &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\ &\vdots \\ F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6) \end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{x(2y+y')}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
&= \frac{\left(-2x^3+2x^2-2x-1\right)y' + \left(4x^2-2\right)y}{\left(x^2+1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
&= \frac{\left(8x^4-6x^3+4x^2+9x-4\right)y' + 4xy\left(x^3-3x^2+x+\frac{9}{2}\right)}{\left(x^2+1\right)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
&= \frac{\left(4x^6-36x^5+32x^4+18x^3-68x^2+54x+9\right)y' - 24\left(x^5-2x^4+\frac{1}{3}x^3+6x^2-\frac{2}{3}x-\frac{3}{4}\right)y}{\left(x^2+1\right)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
&= \frac{\left(-36x^7+192x^6-168x^5-384x^4+612x^3-504x^2-201x+72\right)y' - 8\left(x^7-18x^6+32x^5+\frac{29}{2}x^4-\right.}{\left(x^2+1\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=2$ and $y'(0)=3$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -7 \\
F_2 &= -12 \\
F_3 &= 63 \\
F_4 &= 248
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = 3x + 2 - \frac{7x^3}{6} - \frac{x^4}{2} + \frac{21x^5}{40} + \frac{31x^6}{90} + O\left(x^6\right)
$$

$$y = 3x + 2 - \frac{7x^3}{6} - \frac{x^4}{2} + \frac{21x^5}{40} + \frac{31x^6}{90} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2 + 1)\, y'' + y'x + 2yx = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=0}^{\infty} 2x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=0}^{\infty} 2x^{1+n} a_n = \sum_{n=1}^{\infty} 2a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=1}^{\infty} n a_n x^n\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^n\right) = 0 \tag{3}$$

$n = 1$ gives

$$6a_3 + a_1 + 2a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{3} - \frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + (n+2)\, a_{n+2}(1+n) + n a_n + 2a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_n + 2a_{n-1}}{(n+2)\,(1+n)} \\ &= -\frac{n^2 a_n}{(n+2)\,(1+n)} - \frac{2a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$4a_2 + 12a_4 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{6}$$

For $n = 3$ the recurrence equation gives

$$9a_3 + 20a_5 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{3a_0}{20} + \frac{3a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$16a_4 + 30a_6 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_1}{10} + \frac{a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$25a_5 + 42a_7 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{5a_0}{56} - \frac{37a_1}{1008}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(-\frac{a_0}{3} - \frac{a_1}{6}\right) x^3 - \frac{a_1 x^4}{6} + \left(\frac{3a_0}{20} + \frac{3a_1}{40}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{3}x^3 + \frac{3}{20}x^5\right) a_0 + \left(x - \frac{1}{6}x^3 - \frac{1}{6}x^4 + \frac{3}{40}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{3}x^3 + \frac{3}{20}x^5\right) c_1 + \left(x - \frac{1}{6}x^3 - \frac{1}{6}x^4 + \frac{3}{40}x^5\right) c_2 + O(x^6)$$

$$y = 2 - \frac{7x^3}{6} + \frac{21x^5}{40} + 3x - \frac{x^4}{2} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = 3x + 2 - \frac{7x^3}{6} - \frac{x^4}{2} + \frac{21x^5}{40} + \frac{31x^6}{90} + O(x^6) \tag{1}$$

$$y = 2 - \frac{7x^3}{6} + \frac{21x^5}{40} + 3x - \frac{x^4}{2} + O(x^6) \tag{2}$$

Verification of solutions

$$y = 3x + 2 - \frac{7x^3}{6} - \frac{x^4}{2} + \frac{21x^5}{40} + \frac{31x^6}{90} + O(x^6)$$

Verified OK.

$$y = 2 - \frac{7x^3}{6} + \frac{21x^5}{40} + 3x - \frac{x^4}{2} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c =
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=6;
dsolve([(x^2+1)*diff(y(x),x$2)+x*diff(y(x),x)+2*x*y(x)=0,y(0) = 2, D(y)(0) = 3],y(x),type='s
```

$$y(x) = 2 + 3x - \frac{7}{6}x^3 - \frac{1}{2}x^4 + \frac{21}{40}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 29

```
AsymptoticDSolveValue[{(x^2+1)*y''[x]+x*y'[x]+2*x*y[x]==0,{y[0]==2,y'[0]==3}},y[x],{x,0,5}]
```

$$y(x) \to \frac{21x^5}{40} - \frac{x^4}{2} - \frac{7x^3}{6} + 3x + 2$$

14.14 problem 14

Internal problem ID [11897]
Internal file name [`OUTPUT/11906_Saturday_April_13_2024_01_14_29_AM_59679552/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(2x^2-3)\, y''-2y'x+y=0}$$

With initial conditions

$$[y(0)=-1, y'(0)=5]$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots \\
&= y_0 + xy_0' + \frac{x^2}{2}\,f|_{x_0,y_0,y_0'} + \frac{x^3}{3!}\,f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (822)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (823)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y'_0} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{2y'x - y}{2x^2 - 3}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
&= \frac{-2x^2 y' + 2yx - 3y'}{(2x^2 - 3)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1\\
&= \frac{8(x^3 + 3x)\, y' + (-10x^2 - 3)\, y}{(2x^2 - 3)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2\\
&= \frac{(-52x^4 - 240x^2 - 63)\, y' + 72xy(x^2 + 1)}{(2x^2 - 3)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3\\
&= \frac{(456x^5 + 2952x^3 + 2106x)\, y' + (-668x^4 - 1416x^2 - 153)\, y}{(2x^2 - 3)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = -1$ and $y'(0) = 5$ gives

$$
\begin{aligned}
F_0 &= -\frac{1}{3}\\
F_1 &= -\frac{5}{3}\\
F_2 &= -\frac{1}{9}\\
F_3 &= -\frac{35}{9}\\
F_4 &= -\frac{17}{27}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = 5x - 1 - \frac{x^2}{6} - \frac{5x^3}{18} - \frac{x^4}{216} - \frac{7x^5}{216} - \frac{17x^6}{19440} + O(x^6)$$

$$y = 5x - 1 - \frac{x^2}{6} - \frac{5x^3}{18} - \frac{x^4}{216} - \frac{7x^5}{216} - \frac{17x^6}{19440} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(2x^2 - 3\right) y'' - 2y'x + y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(2x^2 - 3\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) x + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \sum_{n=2}^{\infty} \left(-3n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-2n a_n x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} \left(-3n(n-1)\, a_n x^{n-2}\right) = \sum_{n=0}^{\infty} \left(-3(n+2)\, a_{n+2}(n+1)\, x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} 2x^n a_n n(n-1)\right) + \sum_{n=0}^{\infty} \left(-3(n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=1}^{\infty} (-2n a_n x^n) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$-6a_2 + a_0 = 0$$

$$a_2 = \frac{a_0}{6}$$

$n = 1$ gives

$$-18a_3 - a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{18}$$

For $2 \leq n$, the recurrence equation is

$$2na_n(n-1) - 3(n+2) a_{n+2}(n+1) - 2na_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(2n^2 - 4n + 1)}{3(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$a_2 - 36a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{216}$$

For $n = 3$ the recurrence equation gives

$$7a_3 - 60a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{7a_1}{1080}$$

For $n = 4$ the recurrence equation gives

$$17a_4 - 90a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{17a_0}{19440}$$

For $n = 5$ the recurrence equation gives

$$31a_5 - 126a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{31a_1}{19440}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{6} a_0 x^2 - \frac{1}{18} a_1 x^3 + \frac{1}{216} a_0 x^4 - \frac{7}{1080} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{6}x^2 + \frac{1}{216}x^4\right) a_0 + \left(x - \frac{1}{18}x^3 - \frac{7}{1080}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x=0$ the solution above becomes

$$y = \left(1+\frac{1}{6}x^2+\frac{1}{216}x^4\right)c_1+\left(x-\frac{1}{18}x^3-\frac{7}{1080}x^5\right)c_2+O(x^6)$$

$$y = -1-\frac{x^2}{6}-\frac{x^4}{216}+5x-\frac{5x^3}{18}-\frac{7x^5}{216}+O(x^6)$$

Summary
The solution(s) found are the following

$$y = 5x-1-\frac{x^2}{6}-\frac{5x^3}{18}-\frac{x^4}{216}-\frac{7x^5}{216}-\frac{17x^6}{19440}+O(x^6) \tag{1}$$

$$y = -1-\frac{x^2}{6}-\frac{x^4}{216}+5x-\frac{5x^3}{18}-\frac{7x^5}{216}+O(x^6) \tag{2}$$

Verification of solutions

$$y = 5x-1-\frac{x^2}{6}-\frac{5x^3}{18}-\frac{x^4}{216}-\frac{7x^5}{216}-\frac{17x^6}{19440}+O(x^6)$$

Verified OK.

$$y = -1-\frac{x^2}{6}-\frac{x^4}{216}+5x-\frac{5x^3}{18}-\frac{7x^5}{216}+O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
Order:=6;
dsolve([(2*x^2-3)*diff(y(x),x$2)-2*x*diff(y(x),x)+y(x)=0,y(0) = -1, D(y)(0) = 5],y(x),type='
```

$$y(x) = -1 + 5x - \frac{1}{6}x^2 - \frac{5}{18}x^3 - \frac{1}{216}x^4 - \frac{7}{216}x^5 + \mathrm{O}\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 36

```
AsymptoticDSolveValue[{(2*x^2-3)*y''[x]-2*x*y'[x]+y[x]==0,{y[0]==-1,y'[0]==5}},y[x],{x,0,5}]
```

$$y(x) \to -\frac{7x^5}{216} - \frac{x^4}{216} - \frac{5x^3}{18} - \frac{x^2}{6} + 5x - 1$$

14.15 problem 15

Internal problem ID [11898]
Internal file name [`OUTPUT/11907_Saturday_April_13_2024_01_14_31_AM_97603693/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' + y'x + y = 0}$$

With the expansion point for the power series method at $x = 1$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(\frac{d^2}{dt^2}y(t)\right)(1+t)^2 + \left(\frac{d}{dt}y(t)\right)(1+t) + y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (825) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (826) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{\left(\frac{d}{dt}y(t)\right)t + \frac{d}{dt}y(t) + y(t)}{(1+t)^2} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{\left(\frac{d}{dt}y(t)\right)(1+t) + 3y(t)}{(1+t)^3} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= -\frac{10y(t)}{(1+t)^4} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{(-10t-10)\left(\frac{d}{dt}y(t)\right) + 40y(t)}{(1+t)^5} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{(90t+90)\left(\frac{d}{dt}y(t)\right) - 190y(t)}{(1+t)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y'(0) - y(0) \\
F_1 &= y'(0) + 3y(0) \\
F_2 &= -10y(0) \\
F_3 &= 40y(0) - 10y'(0) \\
F_4 &= -190y(0) + 90y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{2}t^3 - \frac{5}{12}t^4 + \frac{1}{3}t^5 - \frac{19}{72}t^6\right) y(0) + \left(t - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{12}t^5 + \frac{1}{8}t^6\right) y'(0) + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(t^2 + 2t + 1)\left(\frac{d^2}{dt^2}y(t)\right) + \left(\frac{d}{dt}y(t)\right)(1+t) + y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

Substituting the above back into the ode gives

$$(t^2 + 2t + 1)\left(\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)(1+t) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 2n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \quad (2)$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 2(n+1)\, a_{n+1} n\, t^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, t^n$$

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 2(n+1)\, a_{n+1} n\, t^n\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, t^n\right) \\ + \left(\sum_{n=0}^{\infty} (n+1)\, a_{n+1} t^n\right) + \left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_1 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2} - \frac{a_1}{2}$$

$n = 1$ gives

$$6a_2 + 6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{2} + \frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) + 2(n+1)\, a_{n+1} n + (n+2)\, a_{n+2} (n+1) + (n+1)\, a_{n+1} + n a_n + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{n^2 a_n + 2n^2 a_{n+1} + 3n a_{n+1} + a_n + a_{n+1}}{(n+2)(n+1)} \\
&= -\frac{(n^2+1)\, a_n}{(n+2)(n+1)} - \frac{(2n^2+3n+1)\, a_{n+1}}{(n+2)(n+1)}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$5a_2 + 15a_3 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{5a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$10a_3 + 28a_4 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{3} - \frac{a_1}{12}$$

For $n = 4$ the recurrence equation gives

$$17a_4 + 45a_5 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{19a_0}{72} + \frac{a_1}{8}$$

For $n = 5$ the recurrence equation gives

$$26a_5 + 66a_6 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{5a_0}{24} - \frac{73a_1}{504}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y(t) = a_0 + a_1 t + \left(-\frac{a_0}{2} - \frac{a_1}{2}\right) t^2 + \left(\frac{a_0}{2} + \frac{a_1}{6}\right) t^3 - \frac{5 a_0 t^4}{12} + \left(\frac{a_0}{3} - \frac{a_1}{12}\right) t^5 + \dots
$$

Collecting terms, the solution becomes

$$
y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{2}t^3 - \frac{5}{12}t^4 + \frac{1}{3}t^5\right) a_0 + \left(t - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{12}t^5\right) a_1 + O(t^6) \quad (3)
$$

At $t = 0$ the solution above becomes

$$
y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{2}t^3 - \frac{5}{12}t^4 + \frac{1}{3}t^5\right) c_1 + \left(t - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{12}t^5\right) c_2 + O(t^6)
$$

Replacing t in the above with the original independent variable $x$susing $t = x-1$ results in

$$
\begin{aligned}
y = &\left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{2} - \frac{5(x-1)^4}{12} + \frac{(x-1)^5}{3} - \frac{19(x-1)^6}{72}\right) y(1) \\
&+ \left(x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^5}{12} + \frac{(x-1)^6}{8}\right) y'(1) + O\left((x-1)^6\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = &\left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{2} - \frac{5(x-1)^4}{12} + \frac{(x-1)^5}{3} - \frac{19(x-1)^6}{72}\right) y(1) \\
&+ \left(x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^5}{12} + \frac{(x-1)^6}{8}\right) y'(1) + O\left((x-1)^6\right)
\end{aligned} \quad (1)
$$

Verification of solutions

$$y = \left(1-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{2}-\frac{5(x-1)^4}{12}+\frac{(x-1)^5}{3}-\frac{19(x-1)^6}{72}\right)y(1)$$
$$+\left(x-1-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{6}-\frac{(x-1)^5}{12}+\frac{(x-1)^6}{8}\right)y'(1)+O\left((x-1)^6\right)$$

Verified OK.

14.15.1 Maple step by step solution

Let's solve

$y''x^2 + y'x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $y''x^2 + y'x + y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$$\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 + \frac{d}{dt}y(t) + y(t) = 0$$

- Simplify

$$\frac{d^2}{dt^2}y(t) + y(t) = 0$$

- Characteristic polynomial of ODE

$$r^2 + 1 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{0 \pm (\sqrt{-4})}{2}$$

- Roots of the characteristic polynomial

$$r = (-\mathrm{I}, \mathrm{I})$$

- 1st solution of the ODE

$$y_1(t) = \cos(t)$$

- 2nd solution of the ODE

$$y_2(t) = \sin(t)$$

- General solution of the ODE

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

- Substitute in solutions

$$y(t) = c_1 \cos(t) + c_2 \sin(t)$$

- Change variables back using $t = \ln(x)$

$$y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 49

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1 - \frac{(-1+x)^2}{2} + \frac{(-1+x)^3}{2} - \frac{5(-1+x)^4}{12} + \frac{(-1+x)^5}{3}\right) y(1) + \left(-1 + x - \frac{(-1+x)^2}{2} + \frac{(-1+x)^3}{6} - \frac{(-1+x)^5}{12}\right) D(y)(1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 78

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+y[x]==0,y[x],{x,1,5}]
```

$$y(x) \to c_1\left(\frac{1}{3}(x-1)^5 - \frac{5}{12}(x-1)^4 + \frac{1}{2}(x-1)^3 - \frac{1}{2}(x-1)^2 + 1\right) + c_2\left(-\frac{1}{12}(x-1)^5 + \frac{1}{6}(x-1)^3 - \frac{1}{2}(x-1)^2 + x - 1\right)$$

14.16 problem 16

Internal problem ID [11899]
Internal file name [`OUTPUT/11908_Saturday_April_13_2024_01_14_32_AM_9598584/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_2", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + 3y'x - y = 0}$$

With the expansion point for the power series method at $x = 1$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(\frac{d^2}{dt^2}y(t)\right)(1+t)^2 + 3\left(\frac{d}{dt}y(t)\right)(1+t) - y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (828) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (829) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{3\left(\frac{d}{dt}y(t)\right)t + 3\frac{d}{dt}y(t) - y(t)}{(1+t)^2} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{13\left(\frac{d}{dt}y(t)\right)(1+t) - 5y(t)}{(1+t)^3} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{(-70t - 70)\left(\frac{d}{dt}y(t)\right) + 28y(t)}{(1+t)^4} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{(448t + 448)\left(\frac{d}{dt}y(t)\right) - 182y(t)}{(1+t)^5} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{(-3318t - 3318)\left(\frac{d}{dt}y(t)\right) + 1358y(t)}{(1+t)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) - 3y'(0) \\
F_1 &= -5y(0) + 13y'(0) \\
F_2 &= 28y(0) - 70y'(0) \\
F_3 &= -182y(0) + 448y'(0) \\
F_4 &= 1358y(0) - 3318y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(t) = &\left(1+\frac{1}{2}t^2-\frac{5}{6}t^3+\frac{7}{6}t^4-\frac{91}{60}t^5+\frac{679}{360}t^6\right)y(0)\\
&+\left(t-\frac{3}{2}t^2+\frac{13}{6}t^3-\frac{35}{12}t^4+\frac{56}{15}t^5-\frac{553}{120}t^6\right)y'(0)+O(t^6)
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
(t^2+2t+1)\left(\frac{d^2}{dt^2}y(t)\right)+(3t+3)\left(\frac{d}{dt}y(t)\right)-y(t)=0
$$

Let the solution be represented as power series of the form

$$
y(t)=\sum_{n=0}^{\infty}a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty}n a_n t^{n-1}\\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(t^2+2t+1)\left(\sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}\right)+(3t+3)\left(\sum_{n=1}^{\infty}n a_n t^{n-1}\right)-\left(\sum_{n=0}^{\infty}a_n t^n\right)=0 \quad (1)
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty}t^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty}2n\,t^{n-1}a_n(n-1)\right)+\left(\sum_{n=2}^{\infty}n(n-1)\,a_n t^{n-2}\right)\\
&+\left(\sum_{n=1}^{\infty}3n a_n t^n\right)+\left(\sum_{n=1}^{\infty}3n a_n t^{n-1}\right)+\sum_{n=0}^{\infty}(-a_n t^n)=0
\end{aligned} \quad (2)
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} 2n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} 2(n+1)\, a_{n+1} n\, t^{n}$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, t^{n}$$

$$\sum_{n=1}^{\infty} 3n a_n t^{n-1} = \sum_{n=0}^{\infty} 3(n+1)\, a_{n+1} t^{n}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} t^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 2(n+1)\, a_{n+1} n\, t^n\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, t^n\right) + \left(\sum_{n=1}^{\infty} 3n a_n t^n\right) + \left(\sum_{n=0}^{\infty} 3(n+1)\, a_{n+1} t^n\right) + \sum_{n=0}^{\infty} (-a_n t^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + 3a_1 - a_0 = 0$$

$$a_2 = \frac{a_0}{2} - \frac{3a_1}{2}$$

$n = 1$ gives

$$10a_2 + 6a_3 + 2a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{5a_0}{6} + \frac{13a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) + 2(n+1)\, a_{n+1} n + (n+2)\, a_{n+2} (n+1) + 3n a_n + 3(n+1)\, a_{n+1} - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{n^2 a_n + 2n^2 a_{n+1} + 2n a_n + 5n a_{n+1} - a_n + 3a_{n+1}}{(n+2)(n+1)} \\
&= -\frac{(n^2+2n-1) a_n}{(n+2)(n+1)} - \frac{(2n^2+5n+3) a_{n+1}}{(n+2)(n+1)}
\end{aligned} \tag{5}
$$

For $n = 2$ the recurrence equation gives

$$7a_2 + 21a_3 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{7a_0}{6} - \frac{35a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$14a_3 + 36a_4 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{91a_0}{60} + \frac{56a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$23a_4 + 55a_5 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{679a_0}{360} - \frac{553a_1}{120}$$

For $n = 5$ the recurrence equation gives

$$34a_5 + 78a_6 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{91a_0}{40} + \frac{1993a_1}{360}$$

And so on. Therefore the solution is

$$\begin{aligned}y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned}y(t) = a_0 + a_1 t + \left(\frac{a_0}{2} - \frac{3a_1}{2}\right) t^2 + \left(-\frac{5a_0}{6} + \frac{13a_1}{6}\right) t^3 \\ + \left(\frac{7a_0}{6} - \frac{35a_1}{12}\right) t^4 + \left(-\frac{91a_0}{60} + \frac{56a_1}{15}\right) t^5 + \dots\end{aligned}$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + \frac{1}{2}t^2 - \frac{5}{6}t^3 + \frac{7}{6}t^4 - \frac{91}{60}t^5\right) a_0 + \left(t - \frac{3}{2}t^2 + \frac{13}{6}t^3 - \frac{35}{12}t^4 + \frac{56}{15}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + \frac{1}{2}t^2 - \frac{5}{6}t^3 + \frac{7}{6}t^4 - \frac{91}{60}t^5\right) c_1 + \left(t - \frac{3}{2}t^2 + \frac{13}{6}t^3 - \frac{35}{12}t^4 + \frac{56}{15}t^5\right) c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$using $t = x-1$ results in

$$\begin{aligned}y = &\left(1 + \frac{(x-1)^2}{2} - \frac{5(x-1)^3}{6} + \frac{7(x-1)^4}{6} - \frac{91(x-1)^5}{60} + \frac{679(x-1)^6}{360}\right) y(1) \\ &+ \left(x - 1 - \frac{3(x-1)^2}{2} + \frac{13(x-1)^3}{6} - \frac{35(x-1)^4}{12} + \frac{56(x-1)^5}{15} - \frac{553(x-1)^6}{120}\right) y'(1) \\ &+ O\left((x-1)^6\right)\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned}y = &\left(1 + \frac{(x-1)^2}{2} - \frac{5(x-1)^3}{6} + \frac{7(x-1)^4}{6} - \frac{91(x-1)^5}{60} + \frac{679(x-1)^6}{360}\right) y(1) \\ &+ \left(x - 1 - \frac{3(x-1)^2}{2} + \frac{13(x-1)^3}{6} - \frac{35(x-1)^4}{12} + \frac{56(x-1)^5}{15} \right. \\ &\left. - \frac{553(x-1)^6}{120}\right) y'(1) + O\left((x-1)^6\right)\end{aligned} \quad (1)$$

Verification of solutions

$$y = \left(1+\frac{(x-1)^{2}}{2}-\frac{5(x-1)^{3}}{6}+\frac{7(x-1)^{4}}{6}-\frac{91(x-1)^{5}}{60}+\frac{679(x-1)^{6}}{360}\right) y(1) + \left(x-1-\frac{3(x-1)^{2}}{2}+\frac{13(x-1)^{3}}{6}-\frac{35(x-1)^{4}}{12}+\frac{56(x-1)^{5}}{15}-\frac{553(x-1)^{6}}{120}\right) y'(1) + O\left((x-1)^{6}\right)$$

Verified OK.

14.16.1 Maple step by step solution

Let's solve

$y''x^2+3y'x-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y'}{x}+\frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{x}-\frac{y}{x^2}=0$

- Multiply by denominators of the ODE

  $y''x^2+3y'x-y=0$

- Make a change of variables

  $t=\ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y'=\left(\frac{d}{dt}y(t)\right)t'(x)$

  ◦ Compute derivative

    $y'=\frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y''=\left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2+t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

$y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 + 3\frac{d}{dt}y(t) - y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) - y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + 2r - 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm\left(\sqrt{8}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\sqrt{2} - 1, \sqrt{2} - 1\right)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{\left(-\sqrt{2}-1\right)t}$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{\left(\sqrt{2}-1\right)t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{\left(-\sqrt{2}-1\right)t} + c_2 \mathrm{e}^{\left(\sqrt{2}-1\right)t}$

- Change variables back using $t = \ln(x)$

  $y = c_1 \mathrm{e}^{\left(-\sqrt{2}-1\right)\ln(x)} + c_2 \mathrm{e}^{\left(\sqrt{2}-1\right)\ln(x)}$

- Simplify

  $y = \frac{c_1 x^{-\sqrt{2}}}{x} + \frac{c_2 x^{\sqrt{2}}}{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 54

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+3*x*diff(y(x),x)-y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1+\frac{(-1+x)^2}{2}-\frac{5(-1+x)^3}{6}+\frac{7(-1+x)^4}{6}-\frac{91(-1+x)^5}{60}\right)y(1) + \left(-1+x-\frac{3(-1+x)^2}{2}+\frac{13(-1+x)^3}{6}-\frac{35(-1+x)^4}{12}+\frac{56(-1+x)^5}{15}\right)D(y)(1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 87

```
AsymptoticDSolveValue[x^2*y''[x]+3*x*y'[x]-y[x]==0,y[x],{x,1,5}]
```

$$y(x) \to c_1\left(-\frac{91}{60}(x-1)^5+\frac{7}{6}(x-1)^4-\frac{5}{6}(x-1)^3+\frac{1}{2}(x-1)^2+1\right) + c_2\left(\frac{56}{15}(x-1)^5-\frac{35}{12}(x-1)^4+\frac{13}{6}(x-1)^3-\frac{3}{2}(x-1)^2+x-1\right)$$

14.17 problem 17

Internal problem ID [11900]
Internal file name [`OUTPUT/11909_Saturday_April_13_2024_01_14_33_AM_53322424/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y' + 2y = 0}$$

With initial conditions

$$[y(1) = 2, y'(1) = 4]$$

With the expansion point for the power series method at $x = 1$.

14.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{2}{x}$$
$$F = 0$$

Hence the ode is

$$y'' + \frac{y'}{x} + \frac{2y}{x} = 0$$

The domain of $p(x) = \frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\left(\frac{d^2}{dt^2}y(t)\right)(1+t) + \frac{d}{dt}y(t) + 2y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. With initial conditions now becoming

$$\begin{aligned} y(0) &= 2 \\ y'(0) &= 4 \end{aligned}$$

The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (831) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (832) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{\frac{d}{dt}y(t)+2y(t)}{1+t} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t}+\frac{\partial F_0}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_0}{\partial \frac{d}{dt}y\left(t\right)}F_0 \\
&= \frac{-2\left(\frac{d}{dt}y(t)\right)t+4y(t)}{\left(1+t\right)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t}+\frac{\partial F_1}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_1}{\partial \frac{d}{dt}y\left(t\right)}F_1 \\
&= \frac{(8t+2)\left(\frac{d}{dt}y(t)\right)+4y(t)\left(-2+t\right)}{\left(1+t\right)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t}+\frac{\partial F_2}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_2}{\partial \frac{d}{dt}y\left(t\right)}F_2 \\
&= \frac{(4t^2-28t-8)\left(\frac{d}{dt}y(t)\right)-24y(t)\left(t-1\right)}{\left(1+t\right)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t}+\frac{\partial F_3}{\partial y}\frac{d}{dt}y(t)+\frac{\partial F_3}{\partial \frac{d}{dt}y\left(t\right)}F_3 \\
&= \frac{(-36t^2+120t+36)\left(\frac{d}{dt}y(t)\right)-8y(t)\left(t^2-16t+13\right)}{\left(1+t\right)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t=0$ and $y(0)=2$ and $y'(0)=4$ gives

$$
\begin{aligned}
F_0 &= -8 \\
F_1 &= 8 \\
F_2 &= -8 \\
F_3 &= 16 \\
F_4 &= -64
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = -4t^2 + 4t + 2 + \frac{4t^3}{3} - \frac{t^4}{3} + \frac{2t^5}{15} - \frac{4t^6}{45} + O(t^6)$$

$$y(t) = -4t^2 + 4t + 2 + \frac{4t^3}{3} - \frac{t^4}{3} + \frac{2t^5}{15} - \frac{4t^6}{45} + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)(1+t) + \frac{d}{dt}y(t) + 2y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)(1+t) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the

power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, t^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, t^n$$

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, t^n\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, t^n\right) + \left(\sum_{n=0}^{\infty} (n+1)\, a_{n+1} t^n\right) + \left(\sum_{n=0}^{\infty} 2 a_n t^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_1 + 2a_0 = 0$$

$$a_2 = -a_0 - \frac{a_1}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_{n+1} n + (n+2)\, a_{n+2} (n+1) + (n+1)\, a_{n+1} + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_{n+1} + 2n a_{n+1} + 2a_n + a_{n+1}}{(n+2)\,(n+1)} \\ &= -\frac{2a_n}{(n+2)\,(n+1)} - \frac{(n^2+2n+1)\, a_{n+1}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$4a_2 + 6a_3 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{2a_0}{3}$$

For $n = 2$ the recurrence equation gives

$$9a_3 + 12a_4 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{3} + \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$16a_4 + 20a_5 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{5} - \frac{a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$25a_5 + 30a_6 + 2a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{13a_0}{90} + \frac{a_1}{20}$$

For $n = 5$ the recurrence equation gives

$$36a_6 + 42a_7 + 2a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{4a_0}{35} - \frac{5a_1}{126}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y(t) = a_0 + a_1 t + \left(-a_0 - \frac{a_1}{2}\right) t^2 + \frac{2a_0 t^3}{3} + \left(-\frac{a_0}{3} + \frac{a_1}{12}\right) t^4 + \left(\frac{a_0}{5} - \frac{a_1}{15}\right) t^5 + \dots
$$

Collecting terms, the solution becomes

$$
y(t) = \left(1 - t^2 + \frac{2}{3}t^3 - \frac{1}{3}t^4 + \frac{1}{5}t^5\right) a_0 + \left(t - \frac{1}{2}t^2 + \frac{1}{12}t^4 - \frac{1}{15}t^5\right) a_1 + O(t^6) \quad (3)
$$

At $t = 0$ the solution above becomes

$$
y(t) = \left(1 - t^2 + \frac{2}{3}t^3 - \frac{1}{3}t^4 + \frac{1}{5}t^5\right) c_1 + \left(t - \frac{1}{2}t^2 + \frac{1}{12}t^4 - \frac{1}{15}t^5\right) c_2 + O(t^6)
$$

$$
y(t) = 2 - 4t^2 + \frac{4t^3}{3} - \frac{t^4}{3} + \frac{2t^5}{15} + 4t + O(t^6)
$$

Replacing t in the above with the original independent variable $x$using $t = x - 1$ results in

$$
y = -4(x-1)^2 + 4x - 2 + \frac{4(x-1)^3}{3} - \frac{(x-1)^4}{3} + \frac{2(x-1)^5}{15} - \frac{4(x-1)^6}{45} + O\left((x-1)^6\right)
$$

Summary

The solution(s) found are the following

$$
y = -4(x-1)^2 + 4x - 2 + \frac{4(x-1)^3}{3} - \frac{(x-1)^4}{3} + \frac{2(x-1)^5}{15} - \frac{4(x-1)^6}{45} + O\left((x-1)^6\right) \quad (1)
$$

Verification of solutions

$$
y = -4(x-1)^2 + 4x - 2 + \frac{4(x-1)^3}{3} - \frac{(x-1)^4}{3} + \frac{2(x-1)^5}{15} - \frac{4(x-1)^6}{45} + O\left((x-1)^6\right)
$$

Verified OK.

14.17.2 Maple step by step solution

Let's solve

$$\left[y''x+y'+2y=0, y(1)=2, y'\Big|_{\{x=1\}}=4\right]$$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=-\frac{2y}{x}-\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''+\frac{y'}{x}+\frac{2y}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{2}{x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\Big|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\Big|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

$y''x+y'+2y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

  $y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$

- Shift index using $k->k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + 2a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1)^2 + 2a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{2a_k}{(k+1)^2}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = -\frac{2a_k}{(k+1)^2}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
Order:=6;
dsolve([x*diff(y(x),x$2)+diff(y(x),x)+2*y(x)=0,y(1) = 2, D(y)(1) = 4],y(x),type='series',x=1
```

$$y(x) = 2 + 4(-1+x) - 4(-1+x)^2 + \frac{4}{3}(-1+x)^3 - \frac{1}{3}(-1+x)^4 + \frac{2}{15}(-1+x)^5 + O\left((-1+x)^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 44

```
AsymptoticDSolveValue[{x*y''[x]+y'[x]+2*y[x]==0,{y[1]==2,y'[1]==4}},y[x],{x,1,5}]
```

$$y(x) \to \frac{2}{15}(x-1)^5 - \frac{1}{3}(x-1)^4 + \frac{4}{3}(x-1)^3 - 4(x-1)^2 + 4(x-1) + 2$$

14.18 problem 18

Internal problem ID [11901]
Internal file name [`OUTPUT/11910_Saturday_April_13_2024_01_14_35_AM_36640186/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.1. Exercises page 232
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{\left(-x^{2}+1\right) y''-2y'x+n(n+1)\, y=0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (834)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (835)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{yn^2 + yn - 2y'x}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(n^2x^2 + n\,x^2 - n^2 + 6x^2 - n + 2)\,y' - 4ynx(n+1)}{(x^2-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(x-1)\,(-8((n^2+n+3)\,x^2 - n^2 - n + 3)\,xy' + yn(n+1)\,((n^2+n+18)\,x^2 - n^2 - n + 6))\,(x+1)}{(x^2-1)^4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(((n^4 + 2n^3 + 59n^2 + 58n + 120)\,x^4 + (-2n^4 - 4n^3 - 46n^2 - 44n + 240)\,x^2 + n^4 + 2n^3 - 13n^2 - 1}{(x^2-1)^5} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(-18\left(\left(n^4 + 2n^3 + \frac{77}{3}n^2 + \frac{74}{3}n + 40\right)x^4 - 2(n^2+n+10)\left(n^2+n-\frac{20}{3}\right)x^2 + n^4 + 2n^3 - 17n^2 - 18n\right.\right.}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0)\,n(n+1) \\
F_1 &= -y'(0)\,n^2 - y'(0)\,n + 2y'(0) \\
F_2 &= y(0)\,n^4 + 2y(0)\,n^3 - 5y(0)\,n^2 - 6y(0)\,n \\
F_3 &= y'(0)\,n^4 + 2y'(0)\,n^3 - 13y'(0)\,n^2 - 14y'(0)\,n + 24y'(0) \\
F_4 &= -y(0)\,n^6 - 3y(0)\,n^5 + 23y(0)\,n^4 + 51y(0)\,n^3 - 94y(0)\,n^2 - 120y(0)\,n
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1-\frac{1}{2}n^{2}x^{2}-\frac{1}{2}n\,x^{2}+\frac{1}{24}n^{4}x^{4}+\frac{1}{12}x^{4}n^{3}-\frac{5}{24}n^{2}x^{4}-\frac{1}{4}n\,x^{4}-\frac{1}{720}x^{6}n^{6}-\frac{1}{240}n^{5}x^{6}\right.$$
$$\left.+\frac{23}{720}n^{4}x^{6}+\frac{17}{240}n^{3}x^{6}-\frac{47}{360}n^{2}x^{6}-\frac{1}{6}x^{6}n\right) y(0)$$
$$+\left(x-\frac{1}{6}n^{2}x^{3}-\frac{1}{6}n\,x^{3}+\frac{1}{3}x^{3}+\frac{1}{120}x^{5}n^{4}+\frac{1}{60}x^{5}n^{3}-\frac{13}{120}x^{5}n^{2}-\frac{7}{60}x^{5}n+\frac{1}{5}x^{5}\right) y'(0)$$
$$+O\left(x^{6}\right)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(-x^{2}+1\right) y''-2y'x+\left(n^{2}+n\right) y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_{n} x^{n}$$

Then

$$y' = \sum_{n=1}^{\infty} n a_{n} x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}$$

Substituting the above back into the ode gives

$$\left(-x^{2}+1\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)-2\left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right) x+\left(n^{2}+n\right)\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=2}^{\infty}\left(-x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right) \tag{2}$$
$$+\sum_{n=1}^{\infty}\left(-2n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty}\left(n^{2}+n\right) a_{n} x^{n}\right) = 0$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} \left(-x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} \left(-2n a_n x^n\right) + \left(\sum_{n=0}^{\infty} \left(n^2+n\right) a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 n(n+1) = 0$$

$$a_2 = -\frac{1}{2} a_0 n^2 - \frac{1}{2} a_0 n$$

$n = 1$ gives

$$6a_3 - 2a_1 + a_1 n(n+1) = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{1}{6} a_1 n^2 - \frac{1}{6} a_1 n + \frac{1}{3} a_1$$

For $2 \leq n$, the recurrence equation is

$$-n a_n (n-1) + (n+2)\, a_{n+2}(n+1) - 2n a_n + a_n n(n+1) = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n \left(n^2 - n^2 + n - n\right)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-6a_2 + 12a_4 + a_2 n(n+1) = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{5}{24}a_0 n^2 - \frac{1}{4}a_0 n + \frac{1}{24}a_0 n^4 + \frac{1}{12}a_0 n^3$$

For $n = 3$ the recurrence equation gives

$$-12a_3 + 20a_5 + a_3 n(n+1) = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{13}{120}a_1 n^2 - \frac{7}{60}a_1 n + \frac{1}{5}a_1 + \frac{1}{120}a_1 n^4 + \frac{1}{60}a_1 n^3$$

For $n = 4$ the recurrence equation gives

$$-20a_4 + 30a_6 + a_4 n(n+1) = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{47}{360}a_0 n^2 - \frac{1}{6}a_0 n + \frac{23}{720}a_0 n^4 + \frac{17}{240}a_0 n^3 - \frac{1}{720}a_0 n^6 - \frac{1}{240}a_0 n^5$$

For $n = 5$ the recurrence equation gives

$$-30a_5 + 42a_7 + a_5 n(n+1) = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{5}{63}a_1 n^2 - \frac{37}{420}a_1 n + \frac{1}{7}a_1 + \frac{41}{5040}a_1 n^4 + \frac{29}{1680}a_1 n^3 - \frac{1}{5040}a_1 n^6 - \frac{1}{1680}a_1 n^5$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y = {} & a_0 + a_1 x + \left(-\frac{1}{2} a_0 n^2 - \frac{1}{2} a_0 n\right) x^2 + \left(-\frac{1}{6} a_1 n^2 - \frac{1}{6} a_1 n + \frac{1}{3} a_1\right) x^3 \\
& + \left(-\frac{5}{24} a_0 n^2 - \frac{1}{4} a_0 n + \frac{1}{24} a_0 n^4 + \frac{1}{12} a_0 n^3\right) x^4 \\
& + \left(-\frac{13}{120} a_1 n^2 - \frac{7}{60} a_1 n + \frac{1}{5} a_1 + \frac{1}{120} a_1 n^4 + \frac{1}{60} a_1 n^3\right) x^5 + \dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y = {} & \left(1 + \left(-\frac{1}{2} n^2 - \frac{1}{2} n\right) x^2 + \left(-\frac{5}{24} n^2 - \frac{1}{4} n + \frac{1}{24} n^4 + \frac{1}{12} n^3\right) x^4\right) a_0 \\
& + \left(x + \left(-\frac{1}{6} n^2 - \frac{1}{6} n + \frac{1}{3}\right) x^3 + \left(-\frac{13}{120} n^2 - \frac{7}{60} n + \frac{1}{5} + \frac{1}{120} n^4 + \frac{1}{60} n^3\right) x^5\right) a_1 + O\left(x^6\right)
\end{aligned}
\tag{3}
$$

At $x = 0$ the solution above becomes

$$
\begin{aligned}
y = {} & \left(1 + \left(-\frac{1}{2} n^2 - \frac{1}{2} n\right) x^2 + \left(-\frac{5}{24} n^2 - \frac{1}{4} n + \frac{1}{24} n^4 + \frac{1}{12} n^3\right) x^4\right) c_1 \\
& + \left(x + \left(-\frac{1}{6} n^2 - \frac{1}{6} n + \frac{1}{3}\right) x^3 + \left(-\frac{13}{120} n^2 - \frac{7}{60} n + \frac{1}{5} + \frac{1}{120} n^4 + \frac{1}{60} n^3\right) x^5\right) c_2 + O\left(x^6\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1 - \frac{1}{2} n^2 x^2 - \frac{1}{2} n\, x^2 + \frac{1}{24} n^4 x^4 + \frac{1}{12} x^4 n^3 - \frac{5}{24} n^2 x^4 - \frac{1}{4} n\, x^4 - \frac{1}{720} x^6 n^6 - \frac{1}{240} n^5 x^6 \right. \\
& \left. + \frac{23}{720} n^4 x^6 + \frac{17}{240} n^3 x^6 - \frac{47}{360} n^2 x^6 - \frac{1}{6} x^6 n\right) y(0) + \left(x - \frac{1}{6} n^2 x^3 - \frac{1}{6} n\, x^3 + \frac{1}{3} x^3 \right. \\
& \left. + \frac{1}{120} x^5 n^4 + \frac{1}{60} x^5 n^3 - \frac{13}{120} x^5 n^2 - \frac{7}{60} x^5 n + \frac{1}{5} x^5\right) y'(0) + O\left(x^6\right)
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
y = {} & \left(1 + \left(-\frac{1}{2} n^2 - \frac{1}{2} n\right) x^2 + \left(-\frac{5}{24} n^2 - \frac{1}{4} n + \frac{1}{24} n^4 + \frac{1}{12} n^3\right) x^4\right) c_1 \\
& + \left(x + \left(-\frac{1}{6} n^2 - \frac{1}{6} n + \frac{1}{3}\right) x^3 + \left(-\frac{13}{120} n^2 - \frac{7}{60} n + \frac{1}{5} + \frac{1}{120} n^4 + \frac{1}{60} n^3\right) x^5\right) c_2 \\
& + O\left(x^6\right)
\end{aligned}
\tag{2}
$$

Verification of solutions

$$y = \left(1-\frac{1}{2}n^{2}x^{2}-\frac{1}{2}n\,x^{2}+\frac{1}{24}n^{4}x^{4}+\frac{1}{12}x^{4}n^{3}-\frac{5}{24}n^{2}x^{4}-\frac{1}{4}n\,x^{4}-\frac{1}{720}x^{6}n^{6}-\frac{1}{240}n^{5}x^{6}+\frac{23}{720}n^{4}x^{6}+\frac{17}{240}n^{3}x^{6}-\frac{47}{360}n^{2}x^{6}-\frac{1}{6}x^{6}n\right)y(0)+\left(x-\frac{1}{6}n^{2}x^{3}-\frac{1}{6}n\,x^{3}+\frac{1}{3}x^{3}+\frac{1}{120}x^{5}n^{4}+\frac{1}{60}x^{5}n^{3}-\frac{13}{120}x^{5}n^{2}-\frac{7}{60}x^{5}n+\frac{1}{5}x^{5}\right)y'(0)+O(x^{6})$$

Verified OK.

$$y = \left(1+\left(-\frac{1}{2}n^{2}-\frac{1}{2}n\right)x^{2}+\left(-\frac{5}{24}n^{2}-\frac{1}{4}n+\frac{1}{24}n^{4}+\frac{1}{12}n^{3}\right)x^{4}\right)c_{1}+\left(x+\left(-\frac{1}{6}n^{2}-\frac{1}{6}n+\frac{1}{3}\right)x^{3}+\left(-\frac{13}{120}n^{2}-\frac{7}{60}n+\frac{1}{5}+\frac{1}{120}n^{4}+\frac{1}{60}n^{3}\right)x^{5}\right)c_{2}+O(x^{6})$$

Verified OK.

14.18.1 Maple step by step solution

Let's solve

$(-x^{2}+1)\,y''-2y'x+(n^{2}+n)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{n(n+1)y}{x^{2}-1}-\frac{2xy'}{x^{2}-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2xy'}{x^{2}-1}-\frac{n(n+1)y}{x^{2}-1}=0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_{2}(x)=\frac{2x}{x^{2}-1}, P_{3}(x)=-\frac{n(n+1)}{x^{2}-1}\right]$

- ◦ $(x+1)\cdot P_{2}(x)$ is analytic at $x=-1$

  $\left((x+1)\cdot P_{2}(x)\right)\Big|_{x=-1}=1$

- ◦ $(x+1)^{2}\cdot P_{3}(x)$ is analytic at $x=-1$

$$\left(\left(x+1\right)^{2}\cdot P_{3}(x)\right)\bigg|_{x=-1}=0$$

- $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $y''(x^{2}-1)+2y'x-n(n+1)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^{2}-2u)\left(\frac{d^{2}}{du^{2}}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)+(-n^{2}-n)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^{m}\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,u^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $u^{m}\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- Convert $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)$ to series expansion for $m=1..2$

  $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=0}^{\infty}a_{k}(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^{m}\cdot\left(\frac{d^{2}}{du^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_{0}r^{2}u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^{2}+a_{k}(r+1+n+k)\,(r-n+k)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^{2}=0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(1+n+k)(-n+k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}\right]$

- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = \frac{a_k(1+n+k)(-n+k)}{2(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 101

```
Order:=6;
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+n*(n+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{n(n+1)\,x^2}{2} + \frac{n(n^3+2n^2-5n-6)\,x^4}{24}\right) y(0) + \left(x - \frac{(n^2+n-2)\,x^3}{6} + \frac{(n^4+2n^3-13n^2-14n+24)\,x^5}{120}\right) D(y)\,(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 120

```
AsymptoticDSolveValue[(1-x^2)*y''[x]-2*x*y'[x]+n*(n+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{1}{120}(n^2+n)^2 x^5 + \frac{7}{60}(-n^2-n)\,x^5 + \frac{1}{6}(-n^2-n)\,x^3 + \frac{x^5}{5} + \frac{x^3}{3} + x\right) + c_1\left(\frac{1}{24}(n^2+n)^2 x^4 + \frac{1}{4}(-n^2-n)\,x^4 + \frac{1}{2}(-n^2-n)\,x^2 + 1\right)$$

15 Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251

15.1 problem 1

Internal problem ID [11902]
Internal file name [`OUTPUT/11911_Saturday_April_13_2024_01_14_37_AM_31651573/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2 - 3x)\, y'' + (x + 2)\, y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^2 - 3x)\, y'' + (x + 2)\, y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x + 2}{x\,(x - 3)}$$
$$q(x) = \frac{1}{x\,(x - 3)}$$

Table 527: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+2}{x(x-3)}$	
singularity	type
$x = 0$	"regular"
$x = 3$	"regular"

$q(x) = \frac{1}{x(x-3)}$	
singularity	type
$x = 0$	"regular"
$x = 3$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 3, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x(x-3) + (x+2)\,y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x(x-3) + (x+2)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r-1} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} x^{n+r} a_n(n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r-1} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$-3x^{n+r-1} a_n(n+r)(n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$-3x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(-3x^{-1+r} r(-1+r)+2r\,x^{-1+r}\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(-3r^2+5r\right) x^{-1+r}=0$$

Since the above is true for all x then the indicial equation becomes

$$-3r^2+5r=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=\frac{5}{3}$$
$$r_2=0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(-3r^2+5r\right) x^{-1+r}=0$$

Solving for r gives the roots of the indicial equation as Since $r_1-r_2=\frac{5}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x)=x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x)=x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x)=\sum_{n=0}^{\infty} a_n x^{n+\frac{5}{3}}$$
$$y_2(x)=\sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n=0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0=1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned}&a_{n-1}(n+r-1)(n+r-2)-3a_n(n+r)(n+r-1)\\&+a_{n-1}(n+r-1)+2a_n(n+r)+a_{n-1}=0\end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n^2 + 2nr + r^2 - 2n - 2r + 2)}{3n^2 + 6nr + 3r^2 - 5n - 5r} \tag{4}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_n = \frac{a_{n-1}(9n^2 + 12n + 13)}{27n^2 + 45n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{5}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{r^2 + 1}{3r^2 + r - 2}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_1 = \frac{17}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2+1}{3r^2+r-2}$	$\frac{17}{36}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r^2 + 1)(r^2 + 2r + 2)}{9r^4 + 24r^3 + 7r^2 - 12r - 4}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_2 = \frac{1241}{7128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2+1}{3r^2+r-2}$	$\frac{17}{36}$
a_2	$\frac{(r^2+1)(r^2+2r+2)}{9r^4+24r^3+7r^2-12r-4}$	$\frac{1241}{7128}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)}{27r^6+189r^5+441r^4+343r^3-84r^2-196r-48}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_3 = \frac{80665}{1347192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2+1}{3r^2+r-2}$	$\frac{17}{36}$
a_2	$\frac{(r^2+1)(r^2+2r+2)}{9r^4+24r^3+7r^2-12r-4}$	$\frac{1241}{7128}$
a_3	$\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)}{27r^6+189r^5+441r^4+343r^3-84r^2-196r-48}$	$\frac{80665}{1347192}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}{81r^8+1080r^7+5670r^6+14700r^5+18613r^4+7420r^3-6220r^2-6400r-1344}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_4 = \frac{972725}{48498912}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2+1}{3r^2+r-2}$	$\frac{17}{36}$
a_2	$\frac{(r^2+1)(r^2+2r+2)}{9r^4+24r^3+7r^2-12r-4}$	$\frac{1241}{7128}$
a_3	$\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)}{27r^6+189r^5+441r^4+343r^3-84r^2-196r-48}$	$\frac{80665}{1347192}$
a_4	$\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}{81r^8+1080r^7+5670r^6+14700r^5+18613r^4+7420r^3-6220r^2-6400r-1344}$	$\frac{972725}{48498912}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)}{(81r^8+1080r^7+5670r^6+14700r^5+18613r^4+7420r^3-6220r^2-6400r-1344)(3r^2+25r+50)}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_5 = \frac{5797441}{872980416}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r^2+1}{3r^2+r-2}$	$\frac{17}{36}$
a_2	$\frac{(r^2+1)(r^2+2r+2)}{9r^4+24r^3+7r^2-12r-4}$	$\frac{1241}{7128}$
a_3	$\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)}{27r^6+189r^5+441r^4+343r^3-84r^2-196r-48}$	$\frac{80665}{1347192}$
a_4	$\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}{81r^8+1080r^7+5670r^6+14700r^5+18613r^4+7420r^3-6220r^2-6400r-1344}$	$\frac{972725}{48498912}$
a_5	$\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)}{(81r^8+1080r^7+5670r^6+14700r^5+18613r^4+7420r^3-6220r^2-6400r-1344)(3r^2+25r+50)}$	$\frac{5797441}{872980416}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{5}{3}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{5}{3}}\left(1 + \frac{17x}{36} + \frac{1241x^2}{7128} + \frac{80665x^3}{1347192} + \frac{972725x^4}{48498912} + \frac{5797441x^5}{872980416} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} b_{n-1}(n+r-1)(n+r-2) - 3b_n(n+r)(n+r-1) \\ + b_{n-1}(n+r-1) + 2(n+r)\, b_n + b_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}(n^2 + 2nr + r^2 - 2n - 2r + 2)}{3n^2 + 6nr + 3r^2 - 5n - 5r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-1}(n^2 - 2n + 2)}{n(3n - 5)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{r^2 + 1}{3r^2 + r - 2}$$

Which for the root $r = 0$ becomes

$$b_1 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2+1}{3r^2+r-2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{(r^2 + 1)(r^2 + 2r + 2)}{9r^4 + 24r^3 + 7r^2 - 12r - 4}$$

Which for the root $r = 0$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2+1}{3r^2+r-2}$	$-\frac{1}{2}$
b_2	$\frac{(r^2+1)(r^2+2r+2)}{9r^4+24r^3+7r^2-12r-4}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)}{27r^6+189r^5+441r^4+343r^3-84r^2-196r-48}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{5}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2+1}{3r^2+r-2}$	$-\frac{1}{2}$
b_2	$\frac{(r^2+1)(r^2+2r+2)}{9r^4+24r^3+7r^2-12r-4}$	$-\frac{1}{2}$
b_3	$\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)}{27r^6+189r^5+441r^4+343r^3-84r^2-196r-48}$	$-\frac{5}{24}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}{81r^8+1080r^7+5670r^6+14700r^5+18613r^4+7420r^3-6220r^2-6400r-1344}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{25}{336}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2+1}{3r^2+r-2}$	$-\frac{1}{2}$
b_2	$\frac{(r^2+1)(r^2+2r+2)}{9r^4+24r^3+7r^2-12r-4}$	$-\frac{1}{2}$
b_3	$\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)}{27r^6+189r^5+441r^4+343r^3-84r^2-196r-48}$	$-\frac{5}{24}$
b_4	$\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}{81r^8+1080r^7+5670r^6+14700r^5+18613r^4+7420r^3-6220r^2-6400r-1344}$	$-\frac{25}{336}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)}{(81r^8+1080r^7+5670r^6+14700r^5+18613r^4+7420r^3-6220r^2-6400r-1344)(3r^2+25r+50)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{17}{672}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r^2+1}{3r^2+r-2}$	$-\frac{1}{2}$
b_2	$\frac{(r^2+1)(r^2+2r+2)}{9r^4+24r^3+7r^2-12r-4}$	$-\frac{1}{2}$
b_3	$\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)}{27r^6+189r^5+441r^4+343r^3-84r^2-196r-48}$	$-\frac{5}{24}$
b_4	$\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}{81r^8+1080r^7+5670r^6+14700r^5+18613r^4+7420r^3-6220r^2-6400r-1344}$	$-\frac{25}{336}$
b_5	$\frac{(r^2+1)(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)}{(81r^8+1080r^7+5670r^6+14700r^5+18613r^4+7420r^3-6220r^2-6400r-1344)(3r^2+25r+50)}$	$-\frac{17}{672}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - \frac{x}{2} - \frac{x^2}{2} - \frac{5x^3}{24} - \frac{25x^4}{336} - \frac{17x^5}{672} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{5}{3}} \left(1+\frac{17x}{36}+\frac{1241x^2}{7128}+\frac{80665x^3}{1347192}+\frac{972725x^4}{48498912}+\frac{5797441x^5}{872980416}+O\left(x^6\right)\right) \\
&+ c_2 \left(1-\frac{x}{2}-\frac{x^2}{2}-\frac{5x^3}{24}-\frac{25x^4}{336}-\frac{17x^5}{672}+O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{5}{3}} \left(1+\frac{17x}{36}+\frac{1241x^2}{7128}+\frac{80665x^3}{1347192}+\frac{972725x^4}{48498912}+\frac{5797441x^5}{872980416}+O\left(x^6\right)\right) \\
&+ c_2 \left(1-\frac{x}{2}-\frac{x^2}{2}-\frac{5x^3}{24}-\frac{25x^4}{336}-\frac{17x^5}{672}+O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{5}{3}} \left(1+\frac{17x}{36}+\frac{1241x^2}{7128}+\frac{80665x^3}{1347192}+\frac{972725x^4}{48498912}+\frac{5797441x^5}{872980416}+O\left(x^6\right)\right) \\
&+ c_2 \left(1-\frac{x}{2}-\frac{x^2}{2}-\frac{5x^3}{24}-\frac{25x^4}{336}-\frac{17x^5}{672}+O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\frac{5}{3}} \left(1+\frac{17x}{36}+\frac{1241x^2}{7128}+\frac{80665x^3}{1347192}+\frac{972725x^4}{48498912}+\frac{5797441x^5}{872980416}+O\left(x^6\right)\right) \\
&+ c_2 \left(1-\frac{x}{2}-\frac{x^2}{2}-\frac{5x^3}{24}-\frac{25x^4}{336}-\frac{17x^5}{672}+O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

15.1.1 Maple step by step solution

Let's solve

$y''x(x-3)+(x+2)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{x(x-3)}-\frac{(x+2)y'}{x(x-3)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(x+2)y'}{x(x-3)}+\frac{y}{x(x-3)}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x+2}{x(x-3)}, P_3(x)=\frac{1}{x(x-3)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-\frac{2}{3}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x(x-3)+(x+2)\,y'+y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(-5+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(3k-2+3r) + a_k(k^2+2kr+r^2+1)\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-5+3r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{5}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-3\left(k-\frac{2}{3}+r\right)(k+1+r)\, a_{k+1} + a_k(k^2+2kr+r^2+1) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k^2+2kr+r^2+1)}{(3k-2+3r)(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k(k^2+1)}{(3k-2)(k+1)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(k^2+1)}{(3k-2)(k+1)}\right]$$

- Recursion relation for $r = \frac{5}{3}$

  $$a_{k+1} = \frac{a_k\left(k^2+\frac{10}{3}k+\frac{34}{9}\right)}{(3k+3)\left(k+\frac{8}{3}\right)}$$

- Solution for $r = \frac{5}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{3}}, a_{k+1} = \frac{a_k\left(k^2+\frac{10}{3}k+\frac{34}{9}\right)}{(3k+3)\left(k+\frac{8}{3}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{3}}\right), a_{1+k}=\frac{a_k\left(k^2+1\right)}{(3k-2)(1+k)}, b_{1+k}=\frac{b_k\left(k^2+\frac{10}{3}k+\frac{34}{9}\right)}{(3k+3)\left(k+\frac{8}{3}\right)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      <- heuristic approach successful
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 44

```
Order:=6;
dsolve((x^2-3*x)*diff(y(x),x$2)+(x+2)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1x^{\frac{5}{3}}\left(1+\frac{17}{36}x+\frac{1241}{7128}x^2+\frac{80665}{1347192}x^3+\frac{972725}{48498912}x^4+\frac{5797441}{872980416}x^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(1-\frac{1}{2}x-\frac{1}{2}x^2-\frac{5}{24}x^3-\frac{25}{336}x^4-\frac{17}{672}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 85

```
AsymptoticDSolveValue[(x^2-3*x)*y''[x]+(x+2)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{17x^5}{672} - \frac{25x^4}{336} - \frac{5x^3}{24} - \frac{x^2}{2} - \frac{x}{2} + 1\right) + c_1\left(\frac{5797441x^5}{872980416} + \frac{972725x^4}{48498912} + \frac{80665x^3}{1347192} + \frac{1241x^2}{7128} + \frac{17x}{36} + 1\right)x^{5/3}$$

15.2 problem 2

Internal problem ID [11903]
Internal file name [OUTPUT/11912_Saturday_April_13_2024_10_26_13_PM_19754612/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Complex roots"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^3+x^2)\,y''+(x^2-2x)\,y'+4y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^3+x^2)\,y''+(x^2-2x)\,y'+4y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{x-2}{x\,(x+1)}$$
$$q(x)=\frac{4}{x^2\,(x+1)}$$

Table 529: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x-2}{x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{4}{x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x^2(x+1) + (x^2 - 2x)\, y' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(x+1) + (x^2 - 2x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \tag{2A}$$
$$+ \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) \tag{2B}$$
$$+ \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 2x^{n+r} a_n (n+r) + 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 2x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 2x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 3r + 4\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 3r + 4 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned}
r_1 &= \frac{3}{2} + \frac{i\sqrt{7}}{2} \\
r_2 &= \frac{3}{2} - \frac{i\sqrt{7}}{2}
\end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 3r + 4\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}$$

Or

$$\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}+\frac{i\sqrt{7}}{2}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{3}{2}-\frac{i\sqrt{7}}{2}}
\end{aligned}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned}
&a_{n-1}(n + r - 1)(n + r - 2) + a_n(n + r)(n + r - 1) \\
&+ a_{n-1}(n + r - 1) - 2a_n(n + r) + 4a_n = 0
\end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n^2 + 2nr + r^2 - 2n - 2r + 1)}{n^2 + 2nr + r^2 - 3n - 3r + 4} \tag{4}$$

Which for the root $r = \frac{3}{2} + \frac{i\sqrt{7}}{2}$ becomes

$$a_n = -\frac{a_{n-1}\left(\left(in + \frac{1}{2}i\right)\sqrt{7} + n^2 + n - \frac{3}{2}\right)}{n\left(i\sqrt{7} + n\right)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2} + \frac{i\sqrt{7}}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{r^2}{r^2 - r + 2}$$

Which for the root $r = \frac{3}{2} + \frac{i\sqrt{7}}{2}$ becomes

$$a_1 = -\frac{\left(3 + i\sqrt{7}\right)^2}{4 + 4i\sqrt{7}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{r^2-r+2}$	$-\frac{\left(3+i\sqrt{7}\right)^2}{4+4i\sqrt{7}}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2(r+1)^2}{(r^2 - r + 2)(r^2 + r + 2)}$$

Which for the root $r = \frac{3}{2} + \frac{i\sqrt{7}}{2}$ becomes

$$a_2 = \frac{-4\sqrt{7} - 12i}{\left(-\sqrt{7} + i\right)\left(i\sqrt{7} + 2\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{r^2-r+2}$	$-\frac{\left(3+i\sqrt{7}\right)^2}{4+4i\sqrt{7}}$
a_2	$\frac{r^2(r+1)^2}{(r^2-r+2)(r^2+r+2)}$	$\frac{-4\sqrt{7}-12i}{\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{r^2(r+1)^2 (r+2)^2}{(r^2-r+2)(r^2+r+2)(r^2+3r+4)}$$

Which for the root $r = \frac{3}{2} + \frac{i\sqrt{7}}{2}$ becomes

$$a_3 = \frac{7\left(3i-\sqrt{7}\right)^2 \left(i\sqrt{7}+5\right)^2}{576i\sqrt{7}-960}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{r^2-r+2}$	$-\frac{\left(3+i\sqrt{7}\right)^2}{4+4i\sqrt{7}}$
a_2	$\frac{r^2(r+1)^2}{(r^2-r+2)(r^2+r+2)}$	$\frac{-4\sqrt{7}-12i}{\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)}$
a_3	$-\frac{r^2(r+1)^2(r+2)^2}{(r^2-r+2)(r^2+r+2)(r^2+3r+4)}$	$\frac{7\left(3i-\sqrt{7}\right)^2\left(i\sqrt{7}+5\right)^2}{576i\sqrt{7}-960}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^2(r+1)^2 (r+2)^2 (r+3)^2}{(r^2-r+2)(r^2+r+2)(r^2+3r+4)(r^2+5r+8)}$$

Which for the root $r = \frac{3}{2} + \frac{i\sqrt{7}}{2}$ becomes

$$a_4 = \frac{\left(3i-\sqrt{7}\right)\left(i\sqrt{7}+7\right)^2 \left(i\sqrt{7}+5\right)^2 \left(9i\sqrt{7}+37\right)}{3072\left(i\sqrt{7}+4\right)\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{r^2-r+2}$	$-\frac{\left(3+i\sqrt{7}\right)^2}{4+4i\sqrt{7}}$
a_2	$\frac{r^2(r+1)^2}{(r^2-r+2)(r^2+r+2)}$	$\frac{-4\sqrt{7}-12i}{\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)}$
a_3	$-\frac{r^2(r+1)^2(r+2)^2}{(r^2-r+2)(r^2+r+2)(r^2+3r+4)}$	$\frac{7\left(3i-\sqrt{7}\right)^2\left(i\sqrt{7}+5\right)^2}{576i\sqrt{7}-960}$
a_4	$\frac{r^2(r+1)^2(r+2)^2(r+3)^2}{(r^2-r+2)(r^2+r+2)(r^2+3r+4)(r^2+5r+8)}$	$\frac{\left(3i-\sqrt{7}\right)\left(i\sqrt{7}+7\right)^2\left(i\sqrt{7}+5\right)^2\left(9i\sqrt{7}+37\right)}{3072\left(i\sqrt{7}+4\right)\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2}{(r^2-r+2)(r^2+r+2)(r^2+3r+4)(r^2+5r+8)(r^2+7r+14)}$$

Which for the root $r = \frac{3}{2} + \frac{i\sqrt{7}}{2}$ becomes

$$a_5 = \frac{11\left(3i-\sqrt{7}\right)\left(i\sqrt{7}+7\right)^2\left(i\sqrt{7}+5\right)\left(i\sqrt{7}+\frac{57}{11}\right)\left(i\sqrt{7}+9\right)^2}{61440\left(i\sqrt{7}+4\right)\left(\sqrt{7}-i\right)\left(i\sqrt{7}+2\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r^2}{r^2-r+2}$	$-\frac{\left(3+i\sqrt{7}\right)^2}{4+4i\sqrt{7}}$
a_2	$\frac{r^2(r+1)^2}{(r^2-r+2)(r^2+r+2)}$	$\frac{-4\sqrt{7}-12i}{\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)}$
a_3	$-\frac{r^2(r+1)^2(r+2)^2}{(r^2-r+2)(r^2+r+2)(r^2+3r+4)}$	$\frac{7\left(3i-\sqrt{7}\right)^2\left(i\sqrt{7}+5\right)^2}{576i\sqrt{7}-960}$
a_4	$\frac{r^2(r+1)^2(r+2)^2(r+3)^2}{(r^2-r+2)(r^2+r+2)(r^2+3r+4)(r^2+5r+8)}$	$\frac{\left(3i-\sqrt{7}\right)\left(i\sqrt{7}+7\right)^2\left(i\sqrt{7}+5\right)^2\left(9i\sqrt{7}+37\right)}{3072\left(i\sqrt{7}+4\right)\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)}$
a_5	$-\frac{r^2(r+1)^2(r+2)^2(r+3)^2(r+4)^2}{(r^2-r+2)(r^2+r+2)(r^2+3r+4)(r^2+5r+8)(r^2+7r+14)}$	$\frac{11\left(3i-\sqrt{7}\right)\left(i\sqrt{7}+7\right)^2\left(i\sqrt{7}+5\right)\left(i\sqrt{7}+\frac{57}{11}\right)\left(i\sqrt{7}+9\right)^2}{61440\left(i\sqrt{7}+4\right)\left(\sqrt{7}-i\right)\left(i\sqrt{7}+2\right)}$

Using the above table, then the solution $y_1(x)$ is

$$y_1(x) = x^{\frac{3}{2}+\frac{i\sqrt{7}}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right)$$
$$= x^{\frac{3}{2}+\frac{i\sqrt{7}}{2}}\left(1 - \frac{\left(3+i\sqrt{7}\right)^2 x}{4+4i\sqrt{7}} + \frac{\left(-4\sqrt{7}-12i\right)x^2}{\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)} + \frac{7\left(3i-\sqrt{7}\right)^2\left(i\sqrt{7}+5\right)^2 x^3}{576i\sqrt{7}-960} + \frac{\left(3i-\sqrt{7}\right)\left(i\sqrt{7}+7\right)^2\left(i\sqrt{7}+5\right)^2\left(9i\sqrt{7}+37\right)x^4}{3072\left(i\sqrt{7}+4\right)\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)} + \frac{11\left(3i-\sqrt{7}\right)\left(i\sqrt{7}+7\right)^2\left(i\sqrt{7}+5\right)\left(i\sqrt{7}+\frac{57}{11}\right)\left(i\sqrt{7}+9\right)^2 x^5}{61440\left(i\sqrt{7}+4\right)\left(\sqrt{7}-i\right)\left(i\sqrt{7}+2\right)} + O\left(x^6\right)\right)$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$y_2(x) = x^{\frac{3}{2}-\frac{i\sqrt{7}}{2}}\left(1 - \frac{\left(3-i\sqrt{7}\right)^2 x}{4-4i\sqrt{7}} + \frac{\left(-4\sqrt{7}+12i\right)x^2}{\left(-\sqrt{7}-i\right)\left(-i\sqrt{7}+2\right)} + \frac{7\left(-3i-\sqrt{7}\right)^2\left(-i\sqrt{7}+5\right)^2 x^3}{-576i\sqrt{7}-960} + \frac{\left(-3i-\sqrt{7}\right)\left(-i\sqrt{7}+7\right)^2\left(-i\sqrt{7}+5\right)^2\left(-9i\sqrt{7}+37\right)x^4}{3072\left(-i\sqrt{7}+4\right)\left(-\sqrt{7}-i\right)\left(-i\sqrt{7}+2\right)} + \frac{11\left(-3i-\sqrt{7}\right)\left(-i\sqrt{7}+7\right)^2\left(-i\sqrt{7}+5\right)\left(-i\sqrt{7}+\frac{57}{11}\right)\left(-i\sqrt{7}+9\right)^2 x^5}{61440\left(-i\sqrt{7}+4\right)\left(\sqrt{7}+i\right)\left(-i\sqrt{7}+2\right)} + O\left(x^6\right)\right)$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$
\begin{aligned}
&= c_1 x^{\frac{3}{2}+\frac{i\sqrt{7}}{2}} \left(1-\frac{\left(3+i\sqrt{7}\right)^{2} x}{4+4i\sqrt{7}}+\frac{\left(-4\sqrt{7}-12i\right) x^{2}}{\left(-\sqrt{7}+i\right) \left(i\sqrt{7}+2\right)}+\frac{7\left(3i-\sqrt{7}\right)^{2} \left(i\sqrt{7}+5\right)^{2} x^{3}}{576i\sqrt{7}-960}\right. \\
&\quad + \frac{\left(3i-\sqrt{7}\right) \left(i\sqrt{7}+7\right)^{2} \left(i\sqrt{7}+5\right)^{2} \left(9i\sqrt{7}+37\right) x^{4}}{3072 \left(i\sqrt{7}+4\right) \left(-\sqrt{7}+i\right) \left(i\sqrt{7}+2\right)} \\
&\quad \left. + \frac{11\left(3i-\sqrt{7}\right) \left(i\sqrt{7}+7\right)^{2} \left(i\sqrt{7}+5\right) \left(i\sqrt{7}+\frac{57}{11}\right) \left(i\sqrt{7}+9\right)^{2} x^{5}}{61440 \left(i\sqrt{7}+4\right) \left(\sqrt{7}-i\right) \left(i\sqrt{7}+2\right)}+O\left(x^{6}\right)\right) \\
&\quad + c_2 x^{\frac{3}{2}-\frac{i\sqrt{7}}{2}} \left(1-\frac{\left(3-i\sqrt{7}\right)^{2} x}{4-4i\sqrt{7}}+\frac{\left(-4\sqrt{7}+12i\right) x^{2}}{\left(-\sqrt{7}-i\right) \left(-i\sqrt{7}+2\right)}\right. \\
&\quad + \frac{7\left(-3i-\sqrt{7}\right)^{2} \left(-i\sqrt{7}+5\right)^{2} x^{3}}{-576i\sqrt{7}-960} \\
&\quad + \frac{\left(-3i-\sqrt{7}\right) \left(-i\sqrt{7}+7\right)^{2} \left(-i\sqrt{7}+5\right)^{2} \left(-9i\sqrt{7}+37\right) x^{4}}{3072 \left(-i\sqrt{7}+4\right) \left(-\sqrt{7}-i\right) \left(-i\sqrt{7}+2\right)} \\
&\quad + \frac{11\left(-3i-\sqrt{7}\right) \left(-i\sqrt{7}+7\right)^{2} \left(-i\sqrt{7}+5\right) \left(-i\sqrt{7}+\frac{57}{11}\right) \left(-i\sqrt{7}+9\right)^{2} x^{5}}{61440 \left(-i\sqrt{7}+4\right) \left(\sqrt{7}+i\right) \left(-i\sqrt{7}+2\right)} \\
&\quad \left. + O\left(x^{6}\right)\right)
\end{aligned}
$$

Hence the final solution is

$$y = y_h$$

$$= c_1 x^{\frac{3}{2}+\frac{i\sqrt{7}}{2}} \left(1 - \frac{\left(3+i\sqrt{7}\right)^2 x}{4+4i\sqrt{7}} + \frac{\left(-4\sqrt{7}-12i\right) x^2}{\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)} + \frac{7\left(3i-\sqrt{7}\right)^2 \left(i\sqrt{7}+5\right)^2 x^3}{576i\sqrt{7}-960} + \frac{\left(3i-\sqrt{7}\right)\left(i\sqrt{7}+7\right)^2 \left(i\sqrt{7}+5\right)^2 \left(9i\sqrt{7}+37\right) x^4}{3072\left(i\sqrt{7}+4\right)\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)} + \frac{11\left(3i-\sqrt{7}\right)\left(i\sqrt{7}+7\right)^2 \left(i\sqrt{7}+5\right)\left(i\sqrt{7}+\frac{57}{11}\right)\left(i\sqrt{7}+9\right)^2 x^5}{61440\left(i\sqrt{7}+4\right)\left(\sqrt{7}-i\right)\left(i\sqrt{7}+2\right)} + O\left(x^6\right)\right) + c_2 x^{\frac{3}{2}-\frac{i\sqrt{7}}{2}} \left(1 - \frac{\left(3-i\sqrt{7}\right)^2 x}{4-4i\sqrt{7}} + \frac{\left(-4\sqrt{7}+12i\right) x^2}{\left(-\sqrt{7}-i\right)\left(-i\sqrt{7}+2\right)} + \frac{7\left(-3i-\sqrt{7}\right)^2 \left(-i\sqrt{7}+5\right)^2 x^3}{-576i\sqrt{7}-960} + \frac{\left(-3i-\sqrt{7}\right)\left(-i\sqrt{7}+7\right)^2 \left(-i\sqrt{7}+5\right)^2 \left(-9i\sqrt{7}+37\right) x^4}{3072\left(-i\sqrt{7}+4\right)\left(-\sqrt{7}-i\right)\left(-i\sqrt{7}+2\right)} + \frac{11\left(-3i-\sqrt{7}\right)\left(-i\sqrt{7}+7\right)^2 \left(-i\sqrt{7}+5\right)\left(-i\sqrt{7}+\frac{57}{11}\right)\left(-i\sqrt{7}+9\right)^2 x^5}{61440\left(-i\sqrt{7}+4\right)\left(\sqrt{7}+i\right)\left(-i\sqrt{7}+2\right)} + O\left(x^6\right)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{3}{2}+\frac{i\sqrt{7}}{2}} \left(1 - \frac{\left(3+i\sqrt{7}\right)^2 x}{4+4i\sqrt{7}} + \frac{\left(-4\sqrt{7}-12i\right) x^2}{\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)} + \frac{7\left(3i-\sqrt{7}\right)^2 \left(i\sqrt{7}+5\right)^2 x^3}{576i\sqrt{7}-960} + \frac{\left(3i-\sqrt{7}\right)\left(i\sqrt{7}+7\right)^2 \left(i\sqrt{7}+5\right)^2 \left(9i\sqrt{7}+37\right) x^4}{3072\left(i\sqrt{7}+4\right)\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)} + \frac{11\left(3i-\sqrt{7}\right)\left(i\sqrt{7}+7\right)^2 \left(i\sqrt{7}+5\right)\left(i\sqrt{7}+\frac{57}{11}\right)\left(i\sqrt{7}+9\right)^2 x^5}{61440\left(i\sqrt{7}+4\right)\left(\sqrt{7}-i\right)\left(i\sqrt{7}+2\right)} + O\left(x^6\right)\right) + c_2 x^{\frac{3}{2}-\frac{i\sqrt{7}}{2}} \left(1 - \frac{\left(3-i\sqrt{7}\right)^2 x}{4-4i\sqrt{7}} + \frac{\left(-4\sqrt{7}+12i\right) x^2}{\left(-\sqrt{7}-i\right)\left(-i\sqrt{7}+2\right)} + \frac{7\left(-3i-\sqrt{7}\right)^2 \left(-i\sqrt{7}+5\right)^2 x^3}{-576i\sqrt{7}-960} + \frac{\left(-3i-\sqrt{7}\right)\left(-i\sqrt{7}+7\right)^2 \left(-i\sqrt{7}+5\right)^2 \left(-9i\sqrt{7}+37\right) x^4}{3072\left(-i\sqrt{7}+4\right)\left(-\sqrt{7}-i\right)\left(-i\sqrt{7}+2\right)} + \frac{11\left(-3i-\sqrt{7}\right)\left(-i\sqrt{7}+7\right)^2 \left(-i\sqrt{7}+5\right)\left(-i\sqrt{7}+\frac{57}{11}\right)\left(-i\sqrt{7}+9\right)^2 x^5}{61440\left(-i\sqrt{7}+4\right)\left(\sqrt{7}+i\right)\left(-i\sqrt{7}+2\right)} + O\left(x^6\right)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}+\frac{i\sqrt{7}}{2}} \left(1 - \frac{\left(3+i\sqrt{7}\right)^2 x}{4+4i\sqrt{7}} + \frac{\left(-4\sqrt{7}-12i\right) x^2}{\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)} + \frac{7\left(3i-\sqrt{7}\right)^2 \left(i\sqrt{7}+5\right)^2 x^3}{576i\sqrt{7}-960} + \frac{\left(3i-\sqrt{7}\right)\left(i\sqrt{7}+7\right)^2 \left(i\sqrt{7}+5\right)^2 \left(9i\sqrt{7}+37\right) x^4}{3072\left(i\sqrt{7}+4\right)\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)} + \frac{11\left(3i-\sqrt{7}\right)\left(i\sqrt{7}+7\right)^2 \left(i\sqrt{7}+5\right)\left(i\sqrt{7}+\frac{57}{11}\right)\left(i\sqrt{7}+9\right)^2 x^5}{61440\left(i\sqrt{7}+4\right)\left(\sqrt{7}-i\right)\left(i\sqrt{7}+2\right)} + O(x^6)\right)$$
$$+ c_2 x^{\frac{3}{2}-\frac{i\sqrt{7}}{2}} \left(1 - \frac{\left(3-i\sqrt{7}\right)^2 x}{4-4i\sqrt{7}} + \frac{\left(-4\sqrt{7}+12i\right) x^2}{\left(-\sqrt{7}-i\right)\left(-i\sqrt{7}+2\right)} + \frac{7\left(-3i-\sqrt{7}\right)^2 \left(-i\sqrt{7}+5\right)^2 x^3}{-576i\sqrt{7}-960} + \frac{\left(-3i-\sqrt{7}\right)\left(-i\sqrt{7}+7\right)^2 \left(-i\sqrt{7}+5\right)^2 \left(-9i\sqrt{7}+37\right) x^4}{3072\left(-i\sqrt{7}+4\right)\left(-\sqrt{7}-i\right)\left(-i\sqrt{7}+2\right)} + \frac{11\left(-3i-\sqrt{7}\right)\left(-i\sqrt{7}+7\right)^2 \left(-i\sqrt{7}+5\right)\left(-i\sqrt{7}+\frac{57}{11}\right)\left(-i\sqrt{7}+9\right)^2 x^5}{61440\left(-i\sqrt{7}+4\right)\left(\sqrt{7}+i\right)\left(-i\sqrt{7}+2\right)} + O(x^6)\right)$$

Verified OK.

15.2.1 Maple step by step solution

Let's solve

$y''x^2(x+1) + (x^2-2x)\,y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{4y}{x^2(x+1)} - \frac{(x-2)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x-2)y'}{x(x+1)} + \frac{4y}{x^2(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{x-2}{x(x+1)}, P_3(x) = \frac{4}{x^2(x+1)}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$$\left((x+1)\cdot P_2(x)\right)\bigg|_{x=-1} = 3$$

- $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((x+1)^2\cdot P_3(x)\right)\bigg|_{x=-1} = 0$$

- $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-1$

- Multiply by denominators

$y''x^2(x+1)+x(x-2)\,y'+4y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

$(u^3-2u^2+u)\left(\frac{d^2}{du^2}y(u)\right)+(u^2-4u+3)\left(\frac{d}{du}y(u)\right)+4y(u)=0$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m\cdot\left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..2$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m\cdot\left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..3$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m\cdot\left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$a_0r(2+r)\,u^{-1+r}+\left(a_1(1+r)\,(3+r)-2a_0(2+r)\,(-1+r)\right)u^r+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)\,(k+3+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 0\}$

- Each term must be 0

  $a_1(1+r)(3+r) - 2a_0(2+r)(-1+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+3+r) - 2a_k(k+r+2)(k+r-1) + a_{k-1}(k+r-1)^2 = 0$

- Shift index using $k- > k+1$

  $a_{k+2}(k+r+2)(k+4+r) - 2a_{k+1}(k+3+r)(k+r) + a_k(k+r)^2 = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} + 2kra_k - 4kra_{k+1} + r^2a_k - 2r^2a_{k+1} - 6ka_{k+1} - 6ra_{k+1}}{(k+r+2)(k+4+r)}$

- Recursion relation for $r = -2$; series terminates at $k = 2$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 4ka_k + 2ka_{k+1} + 4a_k + 4a_{k+1}}{k(k+2)}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 0$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 4ka_k + 2ka_{k+1} + 4a_k + 4a_{k+1}}{k(k+2)}$

- Recursion relation for $r = 0$; series terminates at $k = 0$

  $a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 6ka_{k+1}}{(k+2)(k+4)}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 6ka_{k+1}}{(k+2)(k+4)}, 3a_1 + 4a_0 = 0\right]$

- Revert the change of variables $u = x+1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+2} = -\frac{k^2a_k - 2k^2a_{k+1} - 6ka_{k+1}}{(k+2)(k+4)}, 3a_1 + 4a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 1227

```
Order:=6;
dsolve((x^3+x^2)*diff(y(x),x$2)+(x^2-2*x)*diff(y(x),x)+4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x^{\frac{3}{2}}\left(c_2 x^{\frac{i\sqrt{7}}{2}}\left(1+\frac{3\sqrt{7}-i}{-2\sqrt{7}+2i}x+\frac{-4\sqrt{7}-12i}{\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)}x^2 + \frac{224}{3}\frac{1}{\left(\sqrt{7}-2i\right)\left(-\sqrt{7}+i\right)\left(3+i\sqrt{7}\right)}x^3 + \frac{84\sqrt{7}-\frac{1036i}{3}}{\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)\left(3+i\sqrt{7}\right)\left(4+i\sqrt{7}\right)}x^4 + \frac{\frac{2576i\sqrt{7}}{3}+\frac{6608}{5}}{\left(-4i+\sqrt{7}\right)\left(-\sqrt{7}+i\right)\left(i\sqrt{7}+2\right)\left(3+i\sqrt{7}\right)\left(i\sqrt{7}+5\right)}x^5+\mathrm{O}\left(x^6\right)\right) + c_1 x^{-\frac{i\sqrt{7}}{2}}\left(1+\frac{-3\sqrt{7}-i}{2\sqrt{7}+2i}x+\frac{12+4i\sqrt{7}}{5+3i\sqrt{7}}x^2 + \frac{224}{3}\frac{1}{\left(i\sqrt{7}-2\right)\left(\sqrt{7}+3i\right)\left(\sqrt{7}+i\right)}x^3+\frac{63i\sqrt{7}-259}{15i\sqrt{7}-129}x^4 + \frac{-1239i-805\sqrt{7}}{675i+255\sqrt{7}}x^5+\mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 5834

```
AsymptoticDSolveValue[(x^3+x^2)*y''[x]+(x^2-2*x)*y'[x]+4*y[x]==0,y[x],{x,0,5}]
```

Too large to display

15.3 problem 3

Internal problem ID [11904]
Internal file name [OUTPUT/11913_Saturday_April_13_2024_10_26_15_PM_27176372/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{\left(x^4-2x^3+x^2\right)y''+2(x-1)\,y'+x^2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^4-2x^3+x^2\right)y''+(2x-2)\,y'+x^2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}
p(x) &= \frac{2}{x^2\,(x-1)} \\
q(x) &= \frac{1}{(x-1)^2}
\end{aligned}$$

Table 531: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x^2(x-1)}$	
singularity	type
$x = 0$	"irregular"
$x = 1$	"regular"

$q(x) = \frac{1}{(x-1)^2}$	
singularity	type
$x = 1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[1, \infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

15.3.1 Maple step by step solution

Let's solve

$y''x^2(x-1)^2 + (2x-2)\, y' + x^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{(x-1)x^2} - \frac{y}{(x-1)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{(x-1)x^2} + \frac{y}{(x-1)^2} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{2}{x^2(x-1)}, P_3(x) = \frac{1}{(x-1)^2}\right]$

- ◦ $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

  $\left((x-1) \cdot P_2(x)\right)\Big|_{x=1} = 2$

- $(x-1)^2 \cdot P_3(x)$ is analytic at $x=1$

  $\left((x-1)^2 \cdot P_3(x)\right)\Big|_{x=1} = 1$

- $x=1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 1$

- Multiply by denominators

  $y''x^2(x-1)^2 + (2x-2)\,y' + x^2 y = 0$

- Change variables using $x = u+1$ so that the regular singular point is at $u=0$

  $(u^4 + 2u^3 + u^2)\left(\frac{d^2}{du^2}y(u)\right) + 2u\left(\frac{d}{du}y(u)\right) + (u^2 + 2u + 1)\,y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 0..2$

  $u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$

- Shift index using $k->k-m$

  $u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$

- Convert $u \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion

  $u \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..4$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $a_0(r^2+r+1)\,u^r + \left((r^2+3r+3)\,a_1 + 2a_0(r^2-r+1)\right) u^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k^2+2kr+r^2+k+r+\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 + r + 1 = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right\}$

- Each term must be 0

  $(r^2 + 3r + 3)\, a_1 + 2a_0(r^2 - r + 1) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = -\frac{2a_0(r^2-r+1)}{r^2+3r+3}$

- Each term in the series must be 0, giving the recursion relation

  $(a_k + a_{k-2} + 2a_{k-1})\, k^2 + ((2a_k + 2a_{k-2} + 4a_{k-1})\, r + a_k - 5a_{k-2} - 6a_{k-1})\, k + (a_k + a_{k-2} + 2a_{k-1}$

- Shift index using $k- >k + 2$

  $(a_{k+2} + a_k + 2a_{k+1})\,(k+2)^2 + ((2a_{k+2} + 2a_k + 4a_{k+1})\, r + a_{k+2} - 5a_k - 6a_{k+1})\,(k+2) + (a_{k+2}$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{k^2a_k+2k^2a_{k+1}+2kra_k+4kra_{k+1}+r^2a_k+2r^2a_{k+1}-ka_k+2ka_{k+1}-ra_k+2ra_{k+1}+a_k+2a_{k+1}}{k^2+2kr+r^2+5k+5r+7}$

- Recursion relation for $r = -\frac{1}{2} - \frac{I\sqrt{3}}{2}$

  $a_{k+2} = -\frac{k^2a_k+2k^2a_{k+1}+2k\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)a_k+4k\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)a_{k+1}+\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2a_k+2\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2a_{k+1}-ka_k+2ka_{k+1}-\left(-\frac{1}{2}-\right.}{k^2+2k\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)+\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2+5k+\frac{9}{2}-\frac{5I\sqrt{3}}{2}}$

- Solution for $r = -\frac{1}{2} - \frac{I\sqrt{3}}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}-\frac{I\sqrt{3}}{2}}, a_{k+2} = -\frac{k^2a_k+2k^2a_{k+1}+2k\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)a_k+4k\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)a_{k+1}+\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2a_k+2\left(-\frac{1}{2}-\right.}{k^2+2k\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)+\left(-\frac{1}{2}-\right.}\right.$

- Revert the change of variables $u = x - 1$

  $\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k-\frac{1}{2}-\frac{I\sqrt{3}}{2}}, a_{k+2} = -\frac{k^2a_k+2k^2a_{k+1}+2k\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)a_k+4k\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)a_{k+1}+\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)^2a_k+2\left(-\right.}{k^2+2k\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}\right)+\left(-\right.}\right.$

- Recursion relation for $r = -\frac{1}{2} + \frac{I\sqrt{3}}{2}$

  $a_{k+2} = -\frac{k^2a_k+2k^2a_{k+1}+2k\left(-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)a_k+4k\left(-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)a_{k+1}+\left(-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)^2a_k+2\left(-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)^2a_{k+1}-ka_k+2ka_{k+1}-\left(-\frac{1}{2}+\right.}{k^2+2k\left(-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)+\left(-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)^2+5k+\frac{9}{2}+\frac{5I\sqrt{3}}{2}}$

- Solution for $r = -\frac{1}{2} + \frac{I\sqrt{3}}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{1}{2}+\frac{I\sqrt{3}}{2}}, a_{k+2} = -\frac{k^2a_k+2k^2a_{k+1}+2k\left(-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)a_k+4k\left(-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)a_{k+1}+\left(-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)^2a_k+2\left(-\frac{1}{2}+\right.}{k^2+2k\left(-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)+\left(-\frac{1}{2}+\right.}\right.$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}}, a_{k+2} = -\frac{k^2 a_k + 2k^2 a_{k+1} + 2k\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)a_k + 4k\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)a_{k+1} + \left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2 a_k + 2\left(-\frac{1}{2}\right.}{k^2 + 2k\left(-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right) + \left(-\frac{1}{2}\right.}\right.$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^{k-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k-\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}}\right), a_{k+2} = -\frac{k^2 a_k + 2k^2 a_{1+k} + 2k\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)a_k + 4k}{}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a = 0, e <> 0, c <>
```

✗ Solution by Maple

```
Order:=6;
dsolve((x^4-2*x^3+x^2)*diff(y(x),x$2)+2*(x-1)*diff(y(x),x)+x^2*y(x)=0,y(x),type='series',x=0
```

No solution found

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 71

```
AsymptoticDSolveValue[(x^4-2*x^3+x^2)*y''[x]+2*(x-1)*y'[x]+x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{3x^5}{10} + \frac{x^4}{4} + \frac{x^3}{6} + 1\right) + c_2 e^{-2/x}\left(-\frac{429x^5}{5} + \frac{91x^4}{4} - \frac{31x^3}{6} + 3x^2 + 1\right)x^4$$

15.4 problem 4

Internal problem ID [11905]
Internal file name [OUTPUT/11914_Saturday_April_13_2024_10_26_15_PM_30461767/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{(x^5 + x^4 - 6x^3)\, y'' + x^2 y' + y(x-2) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^5 + x^4 - 6x^3)\, y'' + x^2 y' + y(x-2) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x\,(x^2 + x - 6)} \\ q(x) &= \frac{1}{x^3\,(x+3)} \end{aligned}$$

Table 533: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x(x^2+x-6)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"
$x = 2$	"regular"

$q(x) = \frac{1}{x^3(x+3)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-3, 2, \infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

15.4.1 Maple step by step solution

Let's solve

$(x^5 + x^4 - 6x^3)\, y'' + x^2 y' + y(x-2) = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x^3(x+3)} - \frac{y'}{x(x^2+x-6)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x(x^2+x-6)} + \frac{y}{x^3(x+3)} = 0$

☐ Check to see if x_0 is a regular singular point

- ∘ Define functions

  $\left[P_2(x) = \frac{1}{x(x^2+x-6)}, P_3(x) = \frac{1}{x^3(x+3)}\right]$

- ∘ $(x+3) \cdot P_2(x)$ is analytic at $x = -3$

$$\left((x+3)\cdot P_2(x)\right)\bigg|_{x=-3} = \frac{1}{15}$$

- $(x+3)^2 \cdot P_3(x)$ is analytic at $x=-3$

$$\left((x+3)^2 \cdot P_3(x)\right)\bigg|_{x=-3} = 0$$

- $x=-3$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -3$

- Multiply by denominators

$$y'' x^3(x^2+x-6)(x+3) + y' x^2(x+3) + y(x^2+x-6) = 0$$

- Change variables using $x = u-3$ so that the regular singular point is at $u=0$

$$(u^6 - 14u^5 + 72u^4 - 162u^3 + 135u^2)\left(\frac{d^2}{du^2}y(u)\right) + (u^3 - 6u^2 + 9u)\left(\frac{d}{du}y(u)\right) + (u^2 - 5u)\,y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot y(u)$ to series expansion for $m = 1..2$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- >k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 2..6$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$u^{m} \cdot \left(\frac{d^{2}}{du^{2}}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$9a_{0}r(-14+15r)\, u^{r} + (9a_{1}(1+r)(1+15r) - a_{0}(162r^{2}-156r+5))\, u^{1+r} + (9a_{2}(2+r)(16+15$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$9r(-14+15r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{14}{15}\right\}$$

- The coefficients of each power of u must be 0

  $$[9a_{1}(1+r)(1+15r) - a_{0}(162r^{2}-156r+5) = 0, 9a_{2}(2+r)(16+15r) - a_{1}(162r^{2}+168r+11$$

- Solve for the dependent coefficient(s)

  $$\left\{a_{1} = \frac{a_{0}(162r^{2}-156r+5)}{9(15r^{2}+16r+1)}, a_{2} = \frac{a_{0}(16524r^{4}+1161r^{3}-14175r^{2}-381r+46)}{81(225r^{4}+930r^{3}+1231r^{2}+558r+32)}, a_{3} = \frac{a_{0}(1357398r^{6}+4208274r^{5}+2090772r^{4}-31}{729(3375r^{6}+31050r^{5}+110070r^{4}+18}\right.$$

- Each term in the series must be 0, giving the recursion relation

  $$(135a_{k} + a_{k-4} - 14a_{k-3} + 72a_{k-2} - 162a_{k-1})\, k^{2} + (2(135a_{k} + a_{k-4} - 14a_{k-3} + 72a_{k-2} - 162a_{k-}$$

- Shift index using $k- > k+4$

  $$(135a_{k+4} + a_{k} - 14a_{k+1} + 72a_{k+2} - 162a_{k+3})\,(k+4)^{2} + (2(135a_{k+4} + a_{k} - 14a_{k+1} + 72a_{k+2} - 1$$

- Recursion relation that defines series solution to ODE

  $$a_{k+4} = -\frac{k^{2}a_{k} - 14k^{2}a_{k+1} + 72k^{2}a_{k+2} - 162k^{2}a_{k+3} + 2kra_{k} - 28kra_{k+1} + 144kra_{k+2} - 324kra_{k+3} + r^{2}a_{k} - 14r^{2}a_{k+1} + 72r^{2}a_{k+2} - 16}{9(15k^{2}+30kr+15r^{2}+106}$$

- Recursion relation for $r = 0$

  $$a_{k+4} = -\frac{k^{2}a_{k} - 14k^{2}a_{k+1} + 72k^{2}a_{k+2} - 162k^{2}a_{k+3} - ka_{k} - 14ka_{k+1} + 217ka_{k+2} - 816ka_{k+3} + 147a_{k+2} - 995a_{k+3}}{9(15k^{2}+106k+184)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_{k}u^{k}, a_{k+4} = -\frac{k^{2}a_{k} - 14k^{2}a_{k+1} + 72k^{2}a_{k+2} - 162k^{2}a_{k+3} - ka_{k} - 14ka_{k+1} + 217ka_{k+2} - 816ka_{k+3} + 147a_{k+2} - 99}{9(15k^{2}+106k+184)}\right.$$

- Revert the change of variables $u = x+3$

  $$\left[y = \sum_{k=0}^{\infty} a_{k}(x+3)^{k}, a_{k+4} = -\frac{k^{2}a_{k} - 14k^{2}a_{k+1} + 72k^{2}a_{k+2} - 162k^{2}a_{k+3} - ka_{k} - 14ka_{k+1} + 217ka_{k+2} - 816ka_{k+3} + 147a_{k+2}}{9(15k^{2}+106k+184)}\right.$$

- Recursion relation for $r = \frac{14}{15}$

  $$a_{k+4} = -\frac{k^{2}a_{k} - 14k^{2}a_{k+1} + 72k^{2}a_{k+2} - 162k^{2}a_{k+3} + \frac{13}{15}ka_{k} - \frac{602}{15}ka_{k+1} + \frac{1757}{5}ka_{k+2} - \frac{5592}{5}ka_{k+3} - \frac{14}{225}a_{k} - \frac{5684}{225}a_{k+1} + \frac{30919}{75}a_{k+2}}{9(15k^{2}+134k+296)}$$

- Solution for $r = \frac{14}{15}$

$$\left[y(u)=\sum_{k=0}^{\infty} a_k u^{k+\frac{14}{15}}, a_{k+4}=-\frac{k^2 a_k-14k^2 a_{k+1}+72k^2 a_{k+2}-162k^2 a_{k+3}+\frac{13}{15} k a_k-\frac{602}{15} k a_{k+1}+\frac{1757}{5} k a_{k+2}-\frac{5592}{5} k a_{k+3}-\frac{1}{2}}{9(15k^2+134k+296)}\right.$$

- Revert the change of variables $u=x+3$

$$\left[y=\sum_{k=0}^{\infty} a_k (x+3)^{k+\frac{14}{15}}, a_{k+4}=-\frac{k^2 a_k-14k^2 a_{k+1}+72k^2 a_{k+2}-162k^2 a_{k+3}+\frac{13}{15} k a_k-\frac{602}{15} k a_{k+1}+\frac{1757}{5} k a_{k+2}-\frac{5592}{5} k a_{k+}}{9(15k^2+134k+296)}\right.$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k (x+3)^k\right)+\left(\sum_{k=0}^{\infty} b_k (x+3)^{k+\frac{14}{15}}\right), a_{k+4}=-\frac{k^2 a_k-14k^2 a_{1+k}+72k^2 a_{k+2}-162k^2 a_{k+3}-k a_k-14k a}{9(15k^2+106}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <> 0, c <
```

✗ Solution by Maple

```
Order:=6;
dsolve((x^5+x^4-6*x^3)*diff(y(x),x$2)+x^2*diff(y(x),x)+(x-2)*y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.226 (sec). Leaf size: 282

```
AsymptoticDSolveValue[(x^5+x^4-6*x^3)*y''[x]+x^2*y'[x]+(x-2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 e^{-\frac{2i}{\sqrt{3}\sqrt{x}}} x^{5/6} \left(-\frac{70670717962217 i x^{9/2}}{8463329722368\sqrt{3}} + \frac{454703707 i x^{7/2}}{544195584\sqrt{3}} - \frac{287057 i x^{5/2}}{1679616\sqrt{3}} + \frac{22 i x^{3/2}}{243\sqrt{3}} + \frac{28128149072197063 x^5}{1523399350026240} - \frac{222818846149 x^4}{156728328192} + \frac{35197783 x^3}{181398528} - \frac{14123 x^2}{279936} + \frac{17x}{216} - \frac{7i\sqrt{x}}{6\sqrt{3}} + 1 \right) + c_2 e^{\frac{2i}{\sqrt{3}\sqrt{x}}} x^{5/6} \left(\frac{70670717962217 i x^{9/2}}{8463329722368\sqrt{3}} - \frac{454703707 i x^{7/2}}{544195584\sqrt{3}} + \frac{287057 i x^{5/2}}{1679616\sqrt{3}} - \frac{22 i x^{3/2}}{243\sqrt{3}} + \frac{28128149072197}{152339935002}\right.$$

15.5 problem 5

Internal problem ID [11906]
Internal file name [`OUTPUT/11915_Saturday_April_13_2024_10_26_16_PM_34342693/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' + y'x + y(x^2 - 1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + y'x + y(x^2 - 1) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{x^2 - 1}{2x^2}$$

Table 535: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + y'x + y(x^2 - 1) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2 + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) (x^2 - 1) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - n - r - 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{n(2n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{2r^2 + 7r + 5}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{14}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+7r+5}$	$-\frac{1}{14}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+7r+5}$	$-\frac{1}{14}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 44r^3 + 169r^2 + 264r + 135}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{616}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+7r+5}$	$-\frac{1}{14}$
a_3	0	0
a_4	$\frac{1}{4r^4+44r^3+169r^2+264r+135}$	$\frac{1}{616}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{2r^2+7r+5}$	$-\frac{1}{14}$
a_3	0	0
a_4	$\frac{1}{4r^4+44r^3+169r^2+264r+135}$	$\frac{1}{616}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{x^2}{14} + \frac{x^4}{616} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - n - r - 1} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{b_{n-2}}{n(2n-3)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{2r^2 + 7r + 5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+7r+5}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+7r+5}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 44r^3 + 169r^2 + 264r + 135}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{40}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+7r+5}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{4r^4+44r^3+169r^2+264r+135}$	$\frac{1}{40}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{2r^2+7r+5}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{4r^4+44r^3+169r^2+264r+135}$	$\frac{1}{40}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 - \frac{x^2}{14} + \frac{x^4}{616} + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 - \frac{x^2}{14} + \frac{x^4}{616} + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\left(1 - \frac{x^2}{14} + \frac{x^4}{616} + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1 x\left(1 - \frac{x^2}{14} + \frac{x^4}{616} + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{40} + O(x^6)\right)}{\sqrt{x}}$$

Verified OK.

15.5.1 Maple step by step solution

Let's solve

$2y''x^2 + y'x + y(x^2 - 1) = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{(x^2-1)y}{2x^2} - \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2x} + \frac{(x^2-1)y}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{2x}, P_3(x) = \frac{x^2-1}{2x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{1}{2}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x^2 + y'x + y(x^2 - 1) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r}$

  ◦ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+2r)(-1+r)\, x^r + a_1(3+2r)\, r\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)(k+r-1) + a_{k-2}\right) x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(1+2r)(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{1, -\frac{1}{2}\right\}$$

- Each term must be 0

$$a_1(3+2r)\, r = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(k+r-1)\left(k+r+\frac{1}{2}\right) a_k + a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$2(k+1+r)\left(k+\frac{5}{2}+r\right) a_{k+2} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{(k+1+r)(2k+5+2r)}$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{a_k}{(k+2)(2k+7)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)(2k+7)}, a_1 = 0\right]$$

- Recursion relation for $r = -\frac{1}{2}$

$$a_{k+2} = -\frac{a_k}{\left(k+\frac{1}{2}\right)(2k+4)}$$

- Solution for $r = -\frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{a_k}{\left(k+\frac{1}{2}\right)(2k+4)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{1+k}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\frac{1}{2}}\right), a_{k+2}=-\frac{a_k}{(k+2)(2k+7)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+\frac{1}{2})(2k+4)}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\frac{c_1\left(1-\frac{1}{2}x^2+\frac{1}{40}x^4+\mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}+c_2x\left(1-\frac{1}{14}x^2+\frac{1}{616}x^4+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 48

```
AsymptoticDSolveValue[2*x^2*y''[x]+x*y'[x]+(x^2-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1x\left(\frac{x^4}{616}-\frac{x^2}{14}+1\right)+\frac{c_2\left(\frac{x^4}{40}-\frac{x^2}{2}+1\right)}{\sqrt{x}}$$

15.6 problem 6

Internal problem ID [11907]
Internal file name [OUTPUT/11916_Saturday_April_13_2024_10_26_17_PM_79930221/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' + y'x + \left(2x^2 - 3\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + y'x + \left(2x^2 - 3\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= \frac{1}{2x} \\ q(x) &= \frac{2x^2 - 3}{2x^2} \end{aligned}$$

Table 537: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2x^2-3}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + y'x + (2x^2 - 3)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2 + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x + (2x^2 - 3)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-3 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n = \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - 3 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + x^r a_0 r - 3 a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) + x^r r - 3x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - r - 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r - 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - r - 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{5}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + a_n(n+r) + 2a_{n-2} - 3a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-2}}{2n^2 + 4nr + 2r^2 - n - r - 3} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = -\frac{2a_{n-2}}{n(2n+5)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{2}{2r^2 + 7r + 3}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = -\frac{1}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{2r^2+7r+3}$	$-\frac{1}{9}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{2r^2+7r+3}$	$-\frac{1}{9}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4}{(2r^2 + 7r + 3)(2r^2 + 15r + 25)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{1}{234}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{2r^2+7r+3}$	$-\frac{1}{9}$
a_3	0	0
a_4	$\frac{4}{(2r^2+7r+3)(2r^2+15r+25)}$	$\frac{1}{234}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{2}{2r^2+7r+3}$	$-\frac{1}{9}$
a_3	0	0
a_4	$\frac{4}{(2r^2+7r+3)(2r^2+15r+25)}$	$\frac{1}{234}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 - \frac{x^2}{9} + \frac{x^4}{234} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + b_n(n+r) + 2b_{n-2} - 3b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{2b_{n-2}}{2n^2 + 4nr + 2r^2 - n - r - 3} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{2b_{n-2}}{n(2n-5)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{2}{2r^2 + 7r + 3}$$

Which for the root $r = -1$ becomes

$$b_2 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{2}{2r^2+7r+3}$	1

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{2}{2r^2+7r+3}$	1
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{4}{(2r^2 + 7r + 3)(2r^2 + 15r + 25)}$$

Which for the root $r = -1$ becomes

$$b_4 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{2}{2r^2+7r+3}$	1
b_3	0	0
b_4	$\frac{4}{(2r^2+7r+3)(2r^2+15r+25)}$	$-\frac{1}{6}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{2}{2r^2+7r+3}$	1
b_3	0	0
b_4	$\frac{4}{(2r^2+7r+3)(2r^2+15r+25)}$	$-\frac{1}{6}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= x^{\frac{3}{2}}\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right) \\ &= \frac{1 + x^2 - \frac{x^4}{6} + O(x^6)}{x}\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x^{\frac{3}{2}}\left(1 - \frac{x^2}{9} + \frac{x^4}{234} + O(x^6)\right) + \frac{c_2\left(1 + x^2 - \frac{x^4}{6} + O(x^6)\right)}{x}\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h \\ &= c_1x^{\frac{3}{2}}\left(1 - \frac{x^2}{9} + \frac{x^4}{234} + O(x^6)\right) + \frac{c_2\left(1 + x^2 - \frac{x^4}{6} + O(x^6)\right)}{x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x^{\frac{3}{2}}\left(1 - \frac{x^2}{9} + \frac{x^4}{234} + O(x^6)\right) + \frac{c_2\left(1 + x^2 - \frac{x^4}{6} + O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1x^{\frac{3}{2}}\left(1 - \frac{x^2}{9} + \frac{x^4}{234} + O(x^6)\right) + \frac{c_2\left(1 + x^2 - \frac{x^4}{6} + O(x^6)\right)}{x}$$

Verified OK.

15.6.1 Maple step by step solution

Let's solve

$2y''x^2 + y'x + (2x^2 - 3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$$y'' = -\frac{(2x^2-3)y}{2x^2} - \frac{y'}{2x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{y'}{2x} + \frac{(2x^2-3)y}{2x^2} = 0$$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$$\left[P_2(x) = \frac{1}{2x}, P_3(x) = \frac{2x^2-3}{2x^2}\right]$$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = \frac{1}{2}$$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{3}{2}$$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$2y''x^2 + y'x + (2x^2 - 3)\, y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- ◦ Shift index using $k- >k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- ◦ Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$$

- ◦ Convert $x^2 \cdot y''$ to series expansion

$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(1+r)(-3+2r)\, x^r + a_1(2+r)(-1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)(2k+2r-3) + 2a_{k-2}\right)\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-1, \frac{3}{2}\right\}$

- Each term must be 0

  $a_1(2+r)(-1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r-\frac{3}{2}\right)(k+r+1)\, a_k + 2a_{k-2} = 0$

- Shift index using $k- >k+2$

  $2\left(k+\frac{1}{2}+r\right)(k+3+r)\, a_{k+2} + 2a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{2a_k}{(2k+1+2r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{2a_k}{(2k-1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{2a_k}{(2k-1)(k+2)}, a_1 = 0\right]$

- Recursion relation for $r = \frac{3}{2}$

  $a_{k+2} = -\frac{2a_k}{(2k+4)\left(k+\frac{9}{2}\right)}$

- Solution for $r = \frac{3}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+2} = -\frac{2a_k}{(2k+4)\left(k+\frac{9}{2}\right)}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+2} = -\frac{2a_k}{(2k-1)(k+2)}, a_1 = 0, b_{k+2} = -\frac{2b_k}{(2k+4)(k+\frac{9}{2})}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)+x*diff(y(x),x)+(2*x^2-3)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_2 x^{\frac{5}{2}} \left(1 - \frac{1}{9}x^2 + \frac{1}{234}x^4 + \mathrm{O}\left(x^6\right)\right) + c_1 \left(1 + x^2 - \frac{1}{6}x^4 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 46

```
AsymptoticDSolveValue[2*x^2*y''[x]+x*y'[x]+(2*x^2-3)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to \frac{c_2 \left(-\frac{x^4}{6} + x^2 + 1\right)}{x} + c_1 \left(\frac{x^4}{234} - \frac{x^2}{9} + 1\right) x^{3/2}$$

15.7 problem 7

Internal problem ID [11908]
Internal file name [`OUTPUT/11917_Saturday_April_13_2024_10_26_18_PM_80034732/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - y'x + \left(x^2 + \frac{8}{9}\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - y'x + \left(x^2 + \frac{8}{9}\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{9x^2 + 8}{9x^2}$$

Table 539: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{9x^2+8}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - y'x + \left(x^2 + \frac{8}{9}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) x^2 - \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) x + \left(x^2 + \frac{8}{9}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{8 a_n x^{n+r}}{9}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} \frac{8 a_n x^{n+r}}{9}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + \frac{8 a_n x^{n+r}}{9} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + \frac{8 a_0 x^r}{9} = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + \frac{8 x^r}{9}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(9r^2 - 18r + 8)\, x^r}{9} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 2r + \frac{8}{9} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{4}{3}$$
$$r_2 = \frac{2}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(9r^2 - 18r + 8) x^r}{9} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{2}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{4}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{2}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) - a_n(n + r) + a_{n-2} + \frac{8a_n}{9} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{9a_{n-2}}{9n^2 + 18nr + 9r^2 - 18n - 18r + 8} \tag{4}$$

Which for the root $r = \frac{4}{3}$ becomes

$$a_n = -\frac{3a_{n-2}}{n(3n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{4}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{9}{9r^2 + 18r + 8}$$

Which for the root $r = \frac{4}{3}$ becomes

$$a_2 = -\frac{3}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+18r+8}$	$-\frac{3}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+18r+8}$	$-\frac{3}{16}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{(9r^2 + 18r + 8)(9r^2 + 54r + 80)}$$

Which for the root $r = \frac{4}{3}$ becomes

$$a_4 = \frac{9}{896}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+18r+8}$	$-\frac{3}{16}$
a_3	0	0
a_4	$\frac{81}{(9r^2+18r+8)(9r^2+54r+80)}$	$\frac{9}{896}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+18r+8}$	$-\frac{3}{16}$
a_3	0	0
a_4	$\frac{81}{(9r^2+18r+8)(9r^2+54r+80)}$	$\frac{9}{896}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^{\frac{4}{3}} (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\
&= x^{\frac{4}{3}} \left(1 - \frac{3x^2}{16} + \frac{9x^4}{896} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - b_n(n+r) + b_{n-2} + \frac{8b_n}{9} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{9b_{n-2}}{9n^2 + 18nr + 9r^2 - 18n - 18r + 8} \tag{4}$$

Which for the root $r = \frac{2}{3}$ becomes

$$b_n = -\frac{3b_{n-2}}{n(3n-2)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{2}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{9}{9r^2 + 18r + 8}$$

Which for the root $r = \frac{2}{3}$ becomes

$$b_2 = -\frac{3}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+18r+8}$	$-\frac{3}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+18r+8}$	$-\frac{3}{8}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81}{(9r^2 + 18r + 8)(9r^2 + 54r + 80)}$$

Which for the root $r = \frac{2}{3}$ becomes

$$b_4 = \frac{9}{320}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+18r+8}$	$-\frac{3}{8}$
b_3	0	0
b_4	$\frac{81}{(9r^2+18r+8)(9r^2+54r+80)}$	$\frac{9}{320}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+18r+8}$	$-\frac{3}{8}$
b_3	0	0
b_4	$\frac{81}{(9r^2+18r+8)(9r^2+54r+80)}$	$\frac{9}{320}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{4}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{\frac{2}{3}}\left(1 - \frac{3x^2}{8} + \frac{9x^4}{320} + O\left(x^6\right)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{4}{3}}\left(1 - \frac{3x^2}{16} + \frac{9x^4}{896} + O\left(x^6\right)\right) + c_2 x^{\frac{2}{3}}\left(1 - \frac{3x^2}{8} + \frac{9x^4}{320} + O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{4}{3}}\left(1 - \frac{3x^2}{16} + \frac{9x^4}{896} + O\left(x^6\right)\right) + c_2 x^{\frac{2}{3}}\left(1 - \frac{3x^2}{8} + \frac{9x^4}{320} + O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{4}{3}}\left(1 - \frac{3x^2}{16} + \frac{9x^4}{896} + O\left(x^6\right)\right) + c_2 x^{\frac{2}{3}}\left(1 - \frac{3x^2}{8} + \frac{9x^4}{320} + O\left(x^6\right)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{4}{3}}\left(1 - \frac{3x^2}{16} + \frac{9x^4}{896} + O\left(x^6\right)\right) + c_2 x^{\frac{2}{3}}\left(1 - \frac{3x^2}{8} + \frac{9x^4}{320} + O\left(x^6\right)\right)
$$

Verified OK.

15.7.1 Maple step by step solution

Let's solve

$y''x^2 - y'x + \left(x^2 + \frac{8}{9}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(9x^2+8)y}{9x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{(9x^2+8)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{9x^2+8}{9x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{8}{9}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $9y''x^2 - 9y'x + (9x^2 + 8) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m} \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+3r)\,(-4+3r)\, x^{r} + a_1(1+3r)\,(-1+3r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(3k+3r-2)\,(3k+3r-4)+9\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+3r)\,(-4+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{2}{3}, \frac{4}{3}\right\}$

- Each term must be 0

  $a_1(1+3r)\,(-1+3r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $9\left(k-\frac{2}{3}+r\right)\left(k+r-\frac{4}{3}\right) a_k + 9a_{k-2} = 0$

- Shift index using $k->k+2$

  $9\left(k+\frac{4}{3}+r\right)\left(k+\frac{2}{3}+r\right) a_{k+2} + 9a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{9a_k}{(3k+4+3r)(3k+2+3r)}$

- Recursion relation for $r = \frac{2}{3}$

  $a_{k+2} = -\frac{9a_k}{(3k+6)(3k+4)}$

- Solution for $r = \frac{2}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{2}{3}}, a_{k+2} = -\frac{9a_k}{(3k+6)(3k+4)}, a_1 = 0\right]$

- Recursion relation for $r = \frac{4}{3}$

  $a_{k+2} = -\frac{9a_k}{(3k+8)(3k+6)}$

- Solution for $r = \frac{4}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{4}{3}}, a_{k+2} = -\frac{9a_k}{(3k+8)(3k+6)}, a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{2}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{4}{3}}\right), a_{k+2} = -\frac{9a_k}{(3k+6)(3k+4)}, a_1 = 0, b_{k+2} = -\frac{9b_k}{(3k+8)(3k+6)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+(x^2+8/9)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{2}{3}} \left(1 - \frac{3}{8}x^2 + \frac{9}{320}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 x^{\frac{4}{3}} \left(1 - \frac{3}{16}x^2 + \frac{9}{896}x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 52

```
AsymptoticDSolveValue[x^2*y''[x]-x*y'[x]+(x^2+8/9)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{9x^4}{896} - \frac{3x^2}{16} + 1 \right) x^{4/3} + c_2 \left(\frac{9x^4}{320} - \frac{3x^2}{8} + 1 \right) x^{2/3}$$

15.8 problem 8

Internal problem ID [11909]
Internal file name [`OUTPUT/11918_Saturday_April_13_2024_10_26_19_PM_67683918/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - y'x + \left(2x^2 + \frac{5}{9}\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - y'x + \left(2x^2 + \frac{5}{9}\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{18x^2 + 5}{9x^2} \end{aligned}$$

Table 541: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{18x^2+5}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - y'x + \left(2x^2 + \frac{5}{9}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2 - \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x + \left(2x^2 + \frac{5}{9}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{5a_n x^{n+r}}{9}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{n+r+2} a_n = \sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} \frac{5a_n x^{n+r}}{9}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + \frac{5a_n x^{n+r}}{9} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + \frac{5a_0 x^r}{9} = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + \frac{5x^r}{9}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(9r^2 - 18r + 5)\, x^r}{9} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 2r + \frac{5}{9} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{5}{3}$$
$$r_2 = \frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(9r^2 - 18r + 5)\, x^r}{9} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{5}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_n(n+r) + 2a_{n-2} + \frac{5a_n}{9} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{18a_{n-2}}{9n^2 + 18nr + 9r^2 - 18n - 18r + 5} \tag{4}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_n = -\frac{6a_{n-2}}{n\,(3n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{5}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{18}{9r^2 + 18r + 5}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_2 = -\frac{3}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{18}{9r^2+18r+5}$	$-\frac{3}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{18}{9r^2+18r+5}$	$-\frac{3}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{324}{(9r^2 + 18r + 5)(9r^2 + 54r + 77)}$$

Which for the root $r = \frac{5}{3}$ becomes

$$a_4 = \frac{9}{320}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{18}{9r^2+18r+5}$	$-\frac{3}{10}$
a_3	0	0
a_4	$\frac{324}{(9r^2+18r+5)(9r^2+54r+77)}$	$\frac{9}{320}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{18}{9r^2+18r+5}$	$-\frac{3}{10}$
a_3	0	0
a_4	$\frac{324}{(9r^2+18r+5)(9r^2+54r+77)}$	$\frac{9}{320}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{5}{3}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x^{\frac{5}{3}}\left(1 - \frac{3x^2}{10} + \frac{9x^4}{320} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - b_n(n+r) + 2b_{n-2} + \frac{5b_n}{9} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{18b_{n-2}}{9n^2 + 18nr + 9r^2 - 18n - 18r + 5} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_n = -\frac{6b_{n-2}}{n(3n-4)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{18}{9r^2 + 18r + 5}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_2 = -\frac{3}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{18}{9r^2+18r+5}$	$-\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{18}{9r^2+18r+5}$	$-\frac{3}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{324}{(9r^2 + 18r + 5)(9r^2 + 54r + 77)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$b_4 = \frac{9}{32}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{18}{9r^2+18r+5}$	$-\frac{3}{2}$
b_3	0	0
b_4	$\frac{324}{(9r^2+18r+5)(9r^2+54r+77)}$	$\frac{9}{32}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{18}{9r^2+18r+5}$	$-\frac{3}{2}$
b_3	0	0
b_4	$\frac{324}{(9r^2+18r+5)(9r^2+54r+77)}$	$\frac{9}{32}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{5}{3}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{\frac{1}{3}}\left(1 - \frac{3x^2}{2} + \frac{9x^4}{32} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{5}{3}}\left(1 - \frac{3x^2}{10} + \frac{9x^4}{320} + O(x^6)\right) + c_2 x^{\frac{1}{3}}\left(1 - \frac{3x^2}{2} + \frac{9x^4}{32} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{5}{3}}\left(1 - \frac{3x^2}{10} + \frac{9x^4}{320} + O(x^6)\right) + c_2 x^{\frac{1}{3}}\left(1 - \frac{3x^2}{2} + \frac{9x^4}{32} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{\frac{5}{3}}\left(1 - \frac{3x^2}{10} + \frac{9x^4}{320} + O(x^6)\right) + c_2 x^{\frac{1}{3}}\left(1 - \frac{3x^2}{2} + \frac{9x^4}{32} + O(x^6)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{\frac{5}{3}}\left(1 - \frac{3x^2}{10} + \frac{9x^4}{320} + O(x^6)\right) + c_2 x^{\frac{1}{3}}\left(1 - \frac{3x^2}{2} + \frac{9x^4}{32} + O(x^6)\right)
$$

Verified OK.

15.8.1 Maple step by step solution

Let's solve

$y''x^2 - y'x + \left(2x^2 + \frac{5}{9}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(18x^2+5)y}{9x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{(18x^2+5)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{18x^2+5}{9x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{5}{9}$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $9y''x^2 - 9y'x + (18x^2 + 5) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_{k}(k+r)\, x^{k+r}$
- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(-1+3r)\,(-5+3r)\, x^{r}+a_{1}(2+3r)\,(-2+3r)\, x^{1+r}+\left(\sum_{k=2}^{\infty} (a_{k}(3k+3r-1)\,(3k+3r-5)+1$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(-1+3r)\,(-5+3r)=0$
- Values of r that satisfy the indicial equation

  $r\in\left\{\frac{1}{3}, \frac{5}{3}\right\}$
- Each term must be 0

  $a_{1}(2+3r)\,(-2+3r)=0$
- Solve for the dependent coefficient(s)

  $a_{1}=0$
- Each term in the series must be 0, giving the recursion relation

  $9\left(k+r-\frac{1}{3}\right)\left(k+r-\frac{5}{3}\right) a_{k}+18a_{k-2}=0$
- Shift index using $k->k+2$

  $9\left(k+\frac{5}{3}+r\right)\left(k+\frac{1}{3}+r\right) a_{k+2}+18a_{k}=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{18a_{k}}{(3k+5+3r)(3k+1+3r)}$
- Recursion relation for $r=\frac{1}{3}$

  $a_{k+2}=-\frac{18a_{k}}{(3k+6)(3k+2)}$
- Solution for $r=\frac{1}{3}$

  $\left[y=\sum_{k=0}^{\infty} a_{k} x^{k+\frac{1}{3}}, a_{k+2}=-\frac{18a_{k}}{(3k+6)(3k+2)}, a_{1}=0\right]$

- Recursion relation for $r = \frac{5}{3}$

  $a_{k+2} = -\frac{18a_k}{(3k+10)(3k+6)}$
- Solution for $r = \frac{5}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{5}{3}}, a_{k+2} = -\frac{18a_k}{(3k+10)(3k+6)}, a_1 = 0\right]$$
- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{3}}\right), a_{k+2} = -\frac{18a_k}{(3k+6)(3k+2)}, a_1 = 0, b_{k+2} = -\frac{18b_k}{(3k+10)(3k+6)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+(2*x^2+5/9)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^{\frac{1}{3}} \left(1 - \frac{3}{2}x^2 + \frac{9}{32}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 x^{\frac{5}{3}} \left(1 - \frac{3}{10}x^2 + \frac{9}{320}x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 52

```
AsymptoticDSolveValue[x^2*y''[x]-x*y'[x]+(2*x^2+5/9)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{9x^4}{32} - \frac{3x^2}{2} + 1\right)\sqrt[3]{x} + c_1\left(\frac{9x^4}{320} - \frac{3x^2}{10} + 1\right)x^{5/3}$$

15.9 problem 9

Internal problem ID [11910]
Internal file name [`OUTPUT/11919_Saturday_April_13_2024_10_26_19_PM_61469210/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y'x + \left(x^2 - \frac{1}{9}\right)y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + y'x + \left(x^2 - \frac{1}{9}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{9x^2 - 1}{9x^2}$$

Table 543: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{9x^2-1}{9x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + y'x + \left(x^2 - \frac{1}{9}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x^2 + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) x + \left(x^2 - \frac{1}{9}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) \\ +\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right)+\sum_{n=0}^{\infty}\left(-\frac{a_n x^{n+r}}{9}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n=\sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) \\ +\left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-\frac{a_n x^{n+r}}{9}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1)+x^{n+r} a_n(n+r)-\frac{a_n x^{n+r}}{9}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)+x^r a_0 r-\frac{a_0 x^r}{9}=0$$

Or

$$\left(x^r r(-1+r)+x^r r-\frac{x^r}{9}\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{\left(9 r^2-1\right) x^r}{9}=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{1}{9} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -\frac{1}{3}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(9r^2 - 1)\, x^r}{9} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{2}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{3}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)\,(n + r - 1) + a_n(n + r) + a_{n-2} - \frac{a_n}{9} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{9a_{n-2}}{9n^2 + 18nr + 9r^2 - 1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = -\frac{3a_{n-2}}{n(3n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{9}{9r^2 + 36r + 35}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = -\frac{3}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{16}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{(9r^2 + 36r + 35)(9r^2 + 72r + 143)}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{9}{896}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{16}$
a_3	0	0
a_4	$\frac{81}{(9r^2+36r+35)(9r^2+72r+143)}$	$\frac{9}{896}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{16}$
a_3	0	0
a_4	$\frac{81}{(9r^2+36r+35)(9r^2+72r+143)}$	$\frac{9}{896}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{3}}\left(1 - \frac{3x^2}{16} + \frac{9x^4}{896} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - \frac{b_n}{9} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{9b_{n-2}}{9n^2 + 18nr + 9r^2 - 1} \tag{4}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_n = -\frac{3b_{n-2}}{n(3n-2)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{9}{9r^2 + 36r + 35}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_2 = -\frac{3}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{8}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{81}{(9r^2 + 36r + 35)(9r^2 + 72r + 143)}$$

Which for the root $r = -\frac{1}{3}$ becomes

$$b_4 = \frac{9}{320}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{8}$
b_3	0	0
b_4	$\frac{81}{(9r^2+36r+35)(9r^2+72r+143)}$	$\frac{9}{320}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{9}{9r^2+36r+35}$	$-\frac{3}{8}$
b_3	0	0
b_4	$\frac{81}{(9r^2+36r+35)(9r^2+72r+143)}$	$\frac{9}{320}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^{\frac{1}{3}}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6\dots\right) \\
&= \frac{1-\frac{3x^2}{8}+\frac{9x^4}{320}+O(x^6)}{x^{\frac{1}{3}}}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x) \\
&= c_1x^{\frac{1}{3}}\left(1-\frac{3x^2}{16}+\frac{9x^4}{896}+O(x^6)\right)+\frac{c_2\left(1-\frac{3x^2}{8}+\frac{9x^4}{320}+O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1x^{\frac{1}{3}}\left(1-\frac{3x^2}{16}+\frac{9x^4}{896}+O(x^6)\right)+\frac{c_2\left(1-\frac{3x^2}{8}+\frac{9x^4}{320}+O(x^6)\right)}{x^{\frac{1}{3}}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x^{\frac{1}{3}}\left(1-\frac{3x^2}{16}+\frac{9x^4}{896}+O(x^6)\right)+\frac{c_2\left(1-\frac{3x^2}{8}+\frac{9x^4}{320}+O(x^6)\right)}{x^{\frac{1}{3}}} \tag{1}$$

Verification of solutions

$$y = c_1x^{\frac{1}{3}}\left(1-\frac{3x^2}{16}+\frac{9x^4}{896}+O(x^6)\right)+\frac{c_2\left(1-\frac{3x^2}{8}+\frac{9x^4}{320}+O(x^6)\right)}{x^{\frac{1}{3}}}$$

Verified OK.

15.9.1 Maple step by step solution

Let's solve

$y''x^2 + y'x + \left(x^2 - \frac{1}{9}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(9x^2-1)y}{9x^2} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(9x^2-1)y}{9x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{9x^2-1}{9x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = 1$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = -\frac{1}{9}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $9y''x^2 + 9y'x + (9x^2 - 1)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k-m$

  $$x^{m}\cdot y=\sum_{k=m}^{\infty} a_{k-m}x^{k+r}$$

- Convert $x\cdot y'$ to series expansion

  $$x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$$

- Convert $x^2\cdot y''$ to series expansion

  $$x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+3r)\,(-1+3r)\, x^{r}+a_1(4+3r)\,(2+3r)\, x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_k(3k+3r+1)\,(3k+3r-1)+9a_{k-\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+3r)\,(-1+3r)=0$$

- Values of r that satisfy the indicial equation

  $$r\in\left\{-\frac{1}{3}, \frac{1}{3}\right\}$$

- Each term must be 0

  $$a_1(4+3r)\,(2+3r)=0$$

- Solve for the dependent coefficient(s)

  $$a_1=0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_k(3k+3r+1)\,(3k+3r-1)+9a_{k-2}=0$$

- Shift index using $k- >k+2$

  $$a_{k+2}(3k+7+3r)\,(3k+5+3r)+9a_k=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2}=-\frac{9a_k}{(3k+7+3r)(3k+5+3r)}$$

- Recursion relation for $r=-\frac{1}{3}$

  $$a_{k+2}=-\frac{9a_k}{(3k+6)(3k+4)}$$

- Solution for $r=-\frac{1}{3}$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}, a_{k+2}=-\frac{9a_k}{(3k+6)(3k+4)}, a_1=0\right]$$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = -\frac{9a_k}{(3k+8)(3k+6)}$

- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = -\frac{9a_k}{(3k+8)(3k+6)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

  $$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{3}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = -\frac{9a_k}{(3k+6)(3k+4)}, a_1 = 0, b_{k+2} = -\frac{9b_k}{(3k+8)(3k+6)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/9)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{x^{\frac{2}{3}} \left(1 - \frac{3}{16}x^2 + \frac{9}{896}x^4 + \mathrm{O}\left(x^6\right)\right) c_2 + \left(1 - \frac{3}{8}x^2 + \frac{9}{320}x^4 + \mathrm{O}\left(x^6\right)\right) c_1}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 52

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-1/9)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x}\left(\frac{9x^4}{896} - \frac{3x^2}{16} + 1\right) + \frac{c_2\left(\frac{9x^4}{320} - \frac{3x^2}{8} + 1\right)}{\sqrt[3]{x}}$$

15.10 problem 10

Internal problem ID [11911]
Internal file name [`OUTPUT/11920_Saturday_April_13_2024_10_26_20_PM_53141073/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{2xy'' + y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{1}{x}$$

Table 545: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2a_n x^{n+r} = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-1+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-1+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n + r)(n + r - 1) + a_n(n + r) + 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{2a_{n-1}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2}{2r^2 + 3r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+3r+1}$	$-\frac{2}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{2}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+3r+1}$	$-\frac{2}{3}$
a_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{15}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{4}{315}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+3r+1}$	$-\frac{2}{3}$
a_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{15}$
a_3	$-\frac{8}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{4}{315}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{16r^8 + 288r^7 + 2184r^6 + 9072r^5 + 22449r^4 + 33642r^3 + 29531r^2 + 13698r + 2520}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{2}{2835}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+3r+1}$	$-\frac{2}{3}$
a_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{15}$
a_3	$-\frac{8}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{4}{315}$
a_4	$\frac{16}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{2}{2835}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{32r^{10} + 880r^9 + 10560r^8 + 72600r^7 + 315546r^6 + 902055r^5 + 1708465r^4 + 2102375r^3 + 1594197r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{4}{155925}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{2r^2+3r+1}$	$-\frac{2}{3}$
a_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{15}$
a_3	$-\frac{8}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{4}{315}$
a_4	$\frac{16}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{2}{2835}$
a_5	$-\frac{32}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{4}{155925}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\,(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots) \\
&= \sqrt{x}\left(1 - \frac{2x}{3} + \frac{2x^2}{15} - \frac{4x^3}{315} + \frac{2x^4}{2835} - \frac{4x^5}{155925} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) + (n+r)\,b_n + 2b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{2b_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = -\frac{2b_{n-1}}{n\,(2n-1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{2}{2r^2 + 3r + 1}
$$

Which for the root $r = 0$ becomes

$$
b_1 = -2
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+3r+1}$	-2

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{2}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+3r+1}$	-2
b_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{3}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{8}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{4}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+3r+1}$	-2
b_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{3}$
b_3	$-\frac{8}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{4}{45}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{16r^8 + 288r^7 + 2184r^6 + 9072r^5 + 22449r^4 + 33642r^3 + 29531r^2 + 13698r + 2520}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{2}{315}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+3r+1}$	-2
b_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{3}$
b_3	$-\frac{8}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{4}{45}$
b_4	$\frac{16}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{2}{315}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{32}{32r^{10} + 880r^9 + 10560r^8 + 72600r^7 + 315546r^6 + 902055r^5 + 1708465r^4 + 2102375r^3 + 1594197r}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{4}{14175}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{2}{2r^2+3r+1}$	-2
b_2	$\frac{4}{4r^4+20r^3+35r^2+25r+6}$	$\frac{2}{3}$
b_3	$-\frac{8}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$-\frac{4}{45}$
b_4	$\frac{16}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{2}{315}$
b_5	$-\frac{32}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$-\frac{4}{14175}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - 2x + \frac{2x^2}{3} - \frac{4x^3}{45} + \frac{2x^4}{315} - \frac{4x^5}{14175} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x} \left(1 - \frac{2x}{3} + \frac{2x^2}{15} - \frac{4x^3}{315} + \frac{2x^4}{2835} - \frac{4x^5}{155925} + O(x^6)\right) \\ &\quad + c_2 \left(1 - 2x + \frac{2x^2}{3} - \frac{4x^3}{45} + \frac{2x^4}{315} - \frac{4x^5}{14175} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1-\frac{2x}{3}+\frac{2x^2}{15}-\frac{4x^3}{315}+\frac{2x^4}{2835}-\frac{4x^5}{155925}+O\left(x^6\right)\right) \\
&\quad + c_2\left(1-2x+\frac{2x^2}{3}-\frac{4x^3}{45}+\frac{2x^4}{315}-\frac{4x^5}{14175}+O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1\sqrt{x}\left(1-\frac{2x}{3}+\frac{2x^2}{15}-\frac{4x^3}{315}+\frac{2x^4}{2835}-\frac{4x^5}{155925}+O\left(x^6\right)\right) \\
+ c_2\left(1-2x+\frac{2x^2}{3}-\frac{4x^3}{45}+\frac{2x^4}{315}-\frac{4x^5}{14175}+O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1\sqrt{x}\left(1-\frac{2x}{3}+\frac{2x^2}{15}-\frac{4x^3}{315}+\frac{2x^4}{2835}-\frac{4x^5}{155925}+O\left(x^6\right)\right) \\
+ c_2\left(1-2x+\frac{2x^2}{3}-\frac{4x^3}{45}+\frac{2x^4}{315}-\frac{4x^5}{14175}+O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

15.10.1 Maple step by step solution

Let's solve

$2y''x + y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{2x} - \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2x} + \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{2x}, P_3(x) = \frac{1}{x}\right]$

- $x \cdot P_2(x)$ is analytic at $x=0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = \frac{1}{2}$

- $x^2 \cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

- $x=0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2y''x + y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert y' to series expansion

  $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k->k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(2k+1+2r) + 2a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r+\frac{1}{2}\right)(k+1+r)\, a_{k+1}+2a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{2a_k}{(2k+1+2r)(k+1+r)}$

- Recursion relation for $r=0$

  $a_{k+1}=-\frac{2a_k}{(2k+1)(k+1)}$

- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=-\frac{2a_k}{(2k+1)(k+1)}\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+1}=-\frac{2a_k}{(2k+2)\left(k+\frac{3}{2}\right)}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1}=-\frac{2a_k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{1+k}=-\frac{2a_k}{(2k+1)(1+k)}, b_{1+k}=-\frac{2b_k}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 44

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)+diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1-\frac{2}{3}x+\frac{2}{15}x^2-\frac{4}{315}x^3+\frac{2}{2835}x^4-\frac{4}{155925}x^5+\mathrm{O}\left(x^6\right)\right) + c_2\left(1-2x+\frac{2}{3}x^2-\frac{4}{45}x^3+\frac{2}{315}x^4-\frac{4}{14175}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 83

```
AsymptoticDSolveValue[2*x*y''[x]+y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(-\frac{4x^5}{155925}+\frac{2x^4}{2835}-\frac{4x^3}{315}+\frac{2x^2}{15}-\frac{2x}{3}+1\right) + c_2\left(-\frac{4x^5}{14175}+\frac{2x^4}{315}-\frac{4x^3}{45}+\frac{2x^2}{3}-2x+1\right)$$

15.11 problem 11

Internal problem ID [11912]
Internal file name [`OUTPUT/11921_Saturday_April_13_2024_10_26_21_PM_33879010/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3xy'' - (x-2)\,y' - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$3xy'' + (-x+2)\,y' - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{x-2}{3x} \\ q(x) &= -\frac{2}{3x} \end{aligned}$$

Table 547: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-2}{3x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{2}{3x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$3xy'' + (-x + 2)\,y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$3\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x + (-x+2)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 3x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 3x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1) x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$3x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(3x^{-1+r} r(-1+r) + 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-1+3r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-1 + 3r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$3a_n(n + r)(n + r - 1) - a_{n-1}(n + r - 1) + 2a_n(n + r) - 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n + r + 1)}{3n^2 + 6nr + 3r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = \frac{3na_{n-1} + 4a_{n-1}}{9n^2 + 3n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2 + r}{3r^2 + 5r + 2}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = \frac{7}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{3r^2+5r+2}$	$\frac{7}{12}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3 + r}{9r^3 + 30r^2 + 31r + 10}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{5}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{3r^2+5r+2}$	$\frac{7}{12}$
a_2	$\frac{3+r}{9r^3+30r^2+31r+10}$	$\frac{5}{36}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{4 + r}{27r^4 + 162r^3 + 333r^2 + 278r + 80}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = \frac{13}{648}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{3r^2+5r+2}$	$\frac{7}{12}$
a_2	$\frac{3+r}{9r^3+30r^2+31r+10}$	$\frac{5}{36}$
a_3	$\frac{4+r}{27r^4+162r^3+333r^2+278r+80}$	$\frac{13}{648}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5+r}{81r^5 + 783r^4 + 2781r^3 + 4497r^2 + 3298r + 880}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{1}{486}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{3r^2+5r+2}$	$\frac{7}{12}$
a_2	$\frac{3+r}{9r^3+30r^2+31r+10}$	$\frac{5}{36}$
a_3	$\frac{4+r}{27r^4+162r^3+333r^2+278r+80}$	$\frac{13}{648}$
a_4	$\frac{5+r}{81r^5+783r^4+2781r^3+4497r^2+3298r+880}$	$\frac{1}{486}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{6+r}{243r^6 + 3483r^5 + 19305r^4 + 52425r^3 + 72852r^2 + 48812r + 12320}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = \frac{19}{116640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2+r}{3r^2+5r+2}$	$\frac{7}{12}$
a_2	$\frac{3+r}{9r^3+30r^2+31r+10}$	$\frac{5}{36}$
a_3	$\frac{4+r}{27r^4+162r^3+333r^2+278r+80}$	$\frac{13}{648}$
a_4	$\frac{5+r}{81r^5+783r^4+2781r^3+4497r^2+3298r+880}$	$\frac{1}{486}$
a_5	$\frac{6+r}{243r^6+3483r^5+19305r^4+52425r^3+72852r^2+48812r+12320}$	$\frac{19}{116640}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{3}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x^{\frac{1}{3}}\left(1 + \frac{7x}{12} + \frac{5x^2}{36} + \frac{13x^3}{648} + \frac{x^4}{486} + \frac{19x^5}{116640} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
3b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) + 2(n+r)b_n - 2b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{b_{n-1}(n+r+1)}{3n^2 + 6nr + 3r^2 - n - r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = \frac{b_{n-1}(n+1)}{n(3n-1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{2 + r}{3r^2 + 5r + 2}$$

Which for the root $r = 0$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2+r}{3r^2+5r+2}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{3 + r}{9r^3 + 30r^2 + 31r + 10}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{3}{10}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2+r}{3r^2+5r+2}$	1
b_2	$\frac{3+r}{9r^3+30r^2+31r+10}$	$\frac{3}{10}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{4 + r}{27r^4 + 162r^3 + 333r^2 + 278r + 80}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{20}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2+r}{3r^2+5r+2}$	1
b_2	$\frac{3+r}{9r^3+30r^2+31r+10}$	$\frac{3}{10}$
b_3	$\frac{4+r}{27r^4+162r^3+333r^2+278r+80}$	$\frac{1}{20}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{5+r}{81r^5 + 783r^4 + 2781r^3 + 4497r^2 + 3298r + 880}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{176}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2+r}{3r^2+5r+2}$	1
b_2	$\frac{3+r}{9r^3+30r^2+31r+10}$	$\frac{3}{10}$
b_3	$\frac{4+r}{27r^4+162r^3+333r^2+278r+80}$	$\frac{1}{20}$
b_4	$\frac{5+r}{81r^5+783r^4+2781r^3+4497r^2+3298r+880}$	$\frac{1}{176}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{6+r}{243r^6 + 3483r^5 + 19305r^4 + 52425r^3 + 72852r^2 + 48812r + 12320}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{3}{6160}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2+r}{3r^2+5r+2}$	1
b_2	$\frac{3+r}{9r^3+30r^2+31r+10}$	$\frac{3}{10}$
b_3	$\frac{4+r}{27r^4+162r^3+333r^2+278r+80}$	$\frac{1}{20}$
b_4	$\frac{5+r}{81r^5+783r^4+2781r^3+4497r^2+3298r+880}$	$\frac{1}{176}$
b_5	$\frac{6+r}{243r^6+3483r^5+19305r^4+52425r^3+72852r^2+48812r+12320}$	$\frac{3}{6160}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + x + \frac{3x^2}{10} + \frac{x^3}{20} + \frac{x^4}{176} + \frac{3x^5}{6160} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{3}} \left(1 + \frac{7x}{12} + \frac{5x^2}{36} + \frac{13x^3}{648} + \frac{x^4}{486} + \frac{19x^5}{116640} + O(x^6)\right) \\ &\quad + c_2 \left(1 + x + \frac{3x^2}{10} + \frac{x^3}{20} + \frac{x^4}{176} + \frac{3x^5}{6160} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{1}{3}} \left(1 + \frac{7x}{12} + \frac{5x^2}{36} + \frac{13x^3}{648} + \frac{x^4}{486} + \frac{19x^5}{116640} + O(x^6)\right) \\ &\quad + c_2 \left(1 + x + \frac{3x^2}{10} + \frac{x^3}{20} + \frac{x^4}{176} + \frac{3x^5}{6160} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x^{\frac{1}{3}} \left(1 + \frac{7x}{12} + \frac{5x^2}{36} + \frac{13x^3}{648} + \frac{x^4}{486} + \frac{19x^5}{116640} + O(x^6)\right) \\ &\quad + c_2 \left(1 + x + \frac{3x^2}{10} + \frac{x^3}{20} + \frac{x^4}{176} + \frac{3x^5}{6160} + O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1 x^{\frac{1}{3}} \left(1 + \frac{7x}{12} + \frac{5x^2}{36} + \frac{13x^3}{648} + \frac{x^4}{486} + \frac{19x^5}{116640} + O(x^6)\right) \\ &\quad + c_2 \left(1 + x + \frac{3x^2}{10} + \frac{x^3}{20} + \frac{x^4}{176} + \frac{3x^5}{6160} + O(x^6)\right) \end{aligned}$$

Verified OK.

15.11.1 Maple step by step solution

Let's solve

$3y''x+(-x+2)\,y'-2y=0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y''=\frac{2y}{3x}+\frac{(x-2)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y''-\frac{(x-2)y'}{3x}-\frac{2y}{3x}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

$\left[P_2(x)=-\frac{x-2}{3x},P_3(x)=-\frac{2}{3x}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\Big|_{x=0}=\frac{2}{3}$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$

- ◦ $x=0$is a regular singular point

Check to see if $x_0=0$ is a regular singular point

$x_0=0$

- Multiply by denominators

$3y''x+(-x+2)\,y'-2y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

$x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(-1+3r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(3k+2+3r) - a_k(k+r+2)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-1+3r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{3}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$3\left(k+\frac{2}{3}+r\right)(k+1+r)\, a_{k+1} - a_k(k+r+2) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+r+2)}{(3k+2+3r)(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k(k+2)}{(3k+2)(k+1)}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k(k+2)}{(3k+2)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{3}$

  $$a_{k+1} = \frac{a_k\left(k+\frac{7}{3}\right)}{(3k+3)\left(k+\frac{4}{3}\right)}$$

- Solution for $r = \frac{1}{3}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+1} = \frac{a_k\left(k+\frac{7}{3}\right)}{(3k+3)\left(k+\frac{4}{3}\right)}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{1+k}=\frac{a_k(k+2)}{(3k+2)(1+k)}, b_{1+k}=\frac{b_k\left(k+\frac{7}{3}\right)}{(3k+3)\left(k+\frac{4}{3}\right)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      Solution using Kummer functions still has integrals. Trying a hypergeometric solution.
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - returning wi
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 44

```
Order:=6;
dsolve(3*x*diff(y(x),x$2)-(x-2)*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1 x^{\frac{1}{3}}\left(1+\frac{7}{12}x+\frac{5}{36}x^2+\frac{13}{648}x^3+\frac{1}{486}x^4+\frac{19}{116640}x^5+\mathrm{O}\left(x^6\right)\right)+c_2\left(1+x+\frac{3}{10}x^2+\frac{1}{20}x^3+\frac{1}{176}x^4+\frac{3}{6160}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 81

```
AsymptoticDSolveValue[3*x*y''[x]-(x-2)*y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \sqrt[3]{x} \left(\frac{19x^5}{116640} + \frac{x^4}{486} + \frac{13x^3}{648} + \frac{5x^2}{36} + \frac{7x}{12} + 1 \right) + c_2 \left(\frac{3x^5}{6160} + \frac{x^4}{176} + \frac{x^3}{20} + \frac{3x^2}{10} + x + 1 \right)$$

15.12 problem 12

Internal problem ID [11913]
Internal file name [`OUTPUT/11922_Saturday_April_13_2024_10_26_22_PM_18318295/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{xy'' + 2y' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 2y' + yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = 1$$

Table 549: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 2y' + yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r) (n+r-1) a_n x^{n+r-2}\right) x + 2\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \le n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + 2a_n(n+r) + a_{n-2} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
a_n = -\frac{a_{n-2}}{n(1+n)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 5r + 6}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 + 14r^3 + 71r^2 + 154r + 120}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\rightarrow r_2} 0 &= \lim_{r\rightarrow -1} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + 2(n+r) b_n + b_{n-2} = 0 \tag{4}
$$

Which for for the root $r = -1$ becomes

$$
b_n(n-1)(n-2) + 2(n-1) b_n + b_{n-2} = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{b_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{5}
$$

Which for the root $r = -1$ becomes

$$
b_n = -\frac{b_{n-2}}{n^2 - n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{r^2 + 5r + 6}$$

Which for the root $r = -1$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 5r + 6)(r^2 + 9r + 20)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x}
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1-\frac{x^2}{6}+\frac{x^4}{120}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(1-\frac{x^2}{6}+\frac{x^4}{120}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1-\frac{x^2}{6}+\frac{x^4}{120}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)}{x}
$$

Verified OK.

15.12.1 Maple step by step solution

Let's solve

$y''x+2y'+yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y'}{x}-y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2y'}{x}+y=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2}{x}, P_3(x)=1\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=2$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + 2y' + yx = 0$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

  - Shift index using $k- >k - 1$

    $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

  - Convert y' to series expansion

    $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

  - Shift index using $k- >k + 1$

    $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$

  - Convert $x \cdot y''$ to series expansion

    $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

  - Shift index using $k- >k + 1$

    $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r) + a_{k-1} = 0$

- Shift index using $k->k+1$

  $a_{k+2}(k+2+r)(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(1+k)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 32

```
Order:=6;
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 \left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 42

```
AsymptoticDSolveValue[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^3}{24} - \frac{x}{2} + \frac{1}{x}\right) + c_2 \left(\frac{x^4}{120} - \frac{x^2}{6} + 1\right)$$

15.13 problem 13

Internal problem ID [11914]
Internal file name [`OUTPUT/11923_Saturday_April_13_2024_10_26_23_PM_34633178/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y'x + \left(x^2 - \frac{1}{4}\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + y'x + \left(x^2 - \frac{1}{4}\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{4x^2 - 1}{4x^2}$$

Table 551: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^2-1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + y'x + \left(x^2 - \frac{1}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x^2 + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) x + \left(x^2 - \frac{1}{4}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{4}\right) = 0 \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{4}\right) = 0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - \frac{a_n x^{n+r}}{4} = 0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r) + x^r a_0 r - \frac{a_0 x^r}{4} = 0
$$

Or

$$
\left(x^r r(-1+r) + x^r r - \frac{x^r}{4}\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\frac{(4r^2-1)\, x^r}{4} = 0
$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{1}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 - 1)\, x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - \frac{a_n}{4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-2}}{4n^2 + 8nr + 4r^2 - 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{4}{4r^2 + 16r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} 0 &= \lim_{r \to -\frac{1}{2}} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - \frac{b_n}{4} = 0 \tag{4}
$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$b_n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)+b_n\left(n-\frac{1}{2}\right)+b_{n-2}-\frac{b_n}{4}=0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-2}}{4n^2+8nr+4r^2-1} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{4b_{n-2}}{4n^2-4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{4}{4r^2+16r+15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{\sqrt{x}} \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{\sqrt{x}}$$

Verified OK.

15.13.1 Maple step by step solution

Let's solve

$y'' x^2 + y' x + \left(x^2 - \frac{1}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(4x^2-1)y}{4x^2} - \frac{y'}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(4x^2-1)y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

  $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{4x^2-1}{4x^2}\right]$
- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = 1$
- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{1}{4}$
- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4y''x^2 + 4y'x + (4x^2 - 1)\, y = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- ○ Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)(-1+2r) x^r + a_1(3+2r)(1+2r) x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2) r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.047 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x^2-1/4)*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\frac{c_1\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4+\mathrm{O}\left(x^6\right)\right)x+c_2\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+\mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

✓ Solution by Mathematica
Time used: 0.01 (sec). Leaf size: 58

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x^2-1/4)*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{x^{7/2}}{24}-\frac{x^{3/2}}{2}+\frac{1}{\sqrt{x}}\right)+c_2\left(\frac{x^{9/2}}{120}-\frac{x^{5/2}}{6}+\sqrt{x}\right)$$

15.14 problem 14

Internal problem ID [11915]
Internal file name [`OUTPUT/11924_Saturday_April_13_2024_10_26_24_PM_10607916/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(x^4 + x\right)y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^4 + x\right)y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^3 + 1}{x}$$
$$q(x) = -\frac{1}{x^2}$$

Table 553: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^3+1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -\frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (x^4 + x) y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) x^2 + (x^4 + x) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+3} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+3} a_n (n+r) = \sum_{n=3}^{\infty} a_{n-3}(n-3+r)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=3}^{\infty} a_{n-3}(n-3+r)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = 0$$

For $3 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-3}(n-3+r) + a_n(n+r) - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-3}(n-3+r)}{n^2+2nr+r^2-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-3}(n-2)}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{r}{r^2+6r+8}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$-\frac{r}{r^2+6r+8}$	$-\frac{1}{15}$

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$-\frac{r}{r^2+6r+8}$	$-\frac{1}{15}$
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$-\frac{r}{r^2+6r+8}$	$-\frac{1}{15}$
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x\left(1 - \frac{x^3}{15} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r\to r_2} 0 &= \lim_{r\to -1} 0 \\ &= 0\end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1}\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

Substituting $n = 2$ in Eq(3) gives

$$b_2 = 0$$

For $3 \le n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_{n-3}(n-3+r) + b_n(n+r) - b_n = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n-1)(n-2) + b_{n-3}(n-4) + b_n(n-1) - b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-3}(n-3+r)}{n^2+2nr+r^2-1} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{b_{n-3}(n-4)}{n^2-2n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{r}{r^2 + 6r + 8}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$-\frac{r}{r^2+6r+8}$	$\frac{1}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$-\frac{r}{r^2+6r+8}$	$\frac{1}{3}$
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$-\frac{r}{r^2+6r+8}$	$\frac{1}{3}$
b_4	0	0
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 + \frac{x^3}{3} + O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1 - \frac{x^3}{15} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{x^3}{3} + O(x^6)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1 - \frac{x^3}{15} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{x^3}{3} + O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x \left(1 - \frac{x^3}{15} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{x^3}{3} + O(x^6)\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1 x \left(1 - \frac{x^3}{15} + O(x^6)\right) + \frac{c_2 \left(1 + \frac{x^3}{3} + O(x^6)\right)}{x}
$$

Verified OK.

15.14.1 Maple step by step solution

Let's solve

$y''x^2 + (x^4 + x)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{y}{x^2} - \frac{(x^3+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{(x^3+1)y'}{x} - \frac{y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$\left[P_2(x) = \frac{x^3+1}{x}, P_3(x) = -\frac{1}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -1$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$y''x^2 + x(x^3 + 1)\, y' - y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y'$ to series expansion for $m = 1..4$

$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+r)\,(-1+r)\, x^r + a_1(2+r)\, r\, x^{1+r} + a_2(3+r)\,(1+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (a_k(k+r+1)\,(k+r-$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$

- The coefficients of each power of x must be 0

  $[a_1(2+r)\, r = 0, a_2(3+r)\,(1+r) = 0]$

- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-1) + a_{k-3}(k-3+r) = 0$

- Shift index using $k->k+3$

  $a_{k+3}(k+4+r)\,(k+2+r) + a_k(k+r) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+3} = -\frac{a_k(k+r)}{(k+4+r)(k+2+r)}$

- Recursion relation for $r = -1$

  $a_{k+3} = -\frac{a_k(k-1)}{(k+3)(k+1)}$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+3} = -\frac{a_k(k-1)}{(k+3)(k+1)}, a_1 = 0, a_2 = 0\right]$$

- Recursion relation for $r = 1$

  $a_{k+3} = -\frac{a_k(k+1)}{(k+5)(k+3)}$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+3} = -\frac{a_k(k+1)}{(k+5)(k+3)}, a_1 = 0, a_2 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{1+k}\right), a_{k+3} = -\frac{a_k(k-1)}{(k+3)(1+k)}, a_1 = 0, a_2 = 0, b_{k+3} = -\frac{b_k(1+k)}{(k+5)(k+3)}, b_1 =\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 29

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+(x^4+x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x\left(1 - \frac{1}{15}x^3 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(-2 - \frac{2}{3}x^3 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 30

```
AsymptoticDSolveValue[x^2*y''[x]+(x^4+x)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x - \frac{x^4}{15}\right) + c_1\left(\frac{x^2}{3} + \frac{1}{x}\right)$$

15.15 problem 15

Internal problem ID [11916]
Internal file name [`OUTPUT/11925_Saturday_April_13_2024_10_26_24_PM_69209561/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' - (x^2 + 2) y' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (-x^2 - 2) y' + yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x^2 + 2}{x}$$
$$q(x) = 1$$

Table 555: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x^2+2}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (-x^2 - 2)\, y' + yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) x + (-x^2 - 2) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n (n+r)\right) = \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2}(n+r-2)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-3+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^3\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-2}(n+r-2) - 2a_n(n+r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{n+r} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = \frac{a_{n-2}}{n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2+r}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{1}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2+r}$	$\frac{1}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2+r}$	$\frac{1}{5}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(2+r)(4+r)}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{1}{35}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2+r}$	$\frac{1}{5}$
a_3	0	0
a_4	$\frac{1}{(2+r)(4+r)}$	$\frac{1}{35}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2+r}$	$\frac{1}{5}$
a_3	0	0
a_4	$\frac{1}{(2+r)(4+r)}$	$\frac{1}{35}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^3\left(1 + \frac{x^2}{5} + \frac{x^4}{35} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} 0 &= \lim_{r \to 0} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - b_{n-2}(n+r-2) - 2(n+r)b_n + b_{n-2} = 0 \tag{4}$$

Which for for the root $r = 0$ becomes

$$b_n n(n-1) - b_{n-2}(n-2) - 2nb_n + b_{n-2} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{n+r} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-2}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2+r}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2+r}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2+r}$	$\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(2+r)(4+r)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2+r}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{(2+r)(4+r)}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2+r}$	$\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{(2+r)(4+r)}$	$\frac{1}{8}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + \frac{x^2}{2} + \frac{x^4}{8} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^3 \left(1 + \frac{x^2}{5} + \frac{x^4}{35} + O(x^6)\right) + c_2 \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^3 \left(1 + \frac{x^2}{5} + \frac{x^4}{35} + O(x^6)\right) + c_2 \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^3 \left(1 + \frac{x^2}{5} + \frac{x^4}{35} + O(x^6)\right) + c_2 \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^3 \left(1 + \frac{x^2}{5} + \frac{x^4}{35} + O(x^6)\right) + c_2 \left(1 + \frac{x^2}{2} + \frac{x^4}{8} + O(x^6)\right)$$

Verified OK.

15.15.1 Maple step by step solution

Let's solve

$y''x + (-x^2 - 2)\, y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2+2)y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x^2+2)y'}{x} + y = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{x^2+2}{x}, P_3(x) = 1\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -2$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + (-x^2 - 2)\, y' + yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$

- Shift index using $k- >k - 1$

  $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..2$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k- >k + 1 - m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$

- Shift index using $k- >k + 1$

$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0 r(-3+r)\, x^{-1+r} + a_1(1+r)(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)(k-2+r) - a_{k-1}(k-2+r\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$
- Each term must be 0

  $a_1(1+r)(-2+r) = 0$
- Each term in the series must be 0, giving the recursion relation

  $(k-2+r)(a_{k+1}(k+r+1) - a_{k-1}) = 0$
- Shift index using $k- >k+1$

  $(k+r-1)(a_{k+2}(k+2+r) - a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{a_k}{k+2+r}$
- Recursion relation for $r = 0$

  $a_{k+2} = \frac{a_k}{k+2}$
- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{a_k}{k+2}, -2a_1 = 0\right]$
- Recursion relation for $r = 3$

  $a_{k+2} = \frac{a_k}{k+5}$
- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = \frac{a_k}{k+5}, 4a_1 = 0\right]$
- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2} = \frac{a_k}{k+2}, -2a_1 = 0, b_{k+2} = \frac{b_k}{k+5}, 4b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 32

```
Order:=6;
dsolve(x*diff(y(x),x$2)-(x^2+2)*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^3 \left(1 + \frac{1}{5}x^2 + \frac{1}{35}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(12 + 6x^2 + \frac{3}{2}x^4 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 44

```
AsymptoticDSolveValue[x*y''[x]-(x^2+2)*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^4}{8} + \frac{x^2}{2} + 1\right) + c_2 \left(\frac{x^7}{35} + \frac{x^5}{5} + x^3\right)$$

15.16 problem 16

Internal problem ID [11917]
Internal file name [`OUTPUT/11926_Saturday_April_13_2024_10_26_25_PM_53530479/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x^2y' - 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + x^2y' - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 1$$
$$q(x) = -\frac{2}{x^2}$$

Table 557: Table $p(x), q(x)$ singularites.

$p(x) = 1$	
singularity	type

$q(x) = -\frac{2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + x^2y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x^2 + x^2 \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)\,(n+r-1) - 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 2a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^2\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) - 2a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-1}(n+r-1)}{n^2 + 2nr + r^2 - n - r - 2} \tag{4}
$$

Which for the root $r = 2$ becomes

$$
a_n = -\frac{a_{n-1}(1+n)}{n(n+3)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{r}{r^2 + r - 2}$$

Which for the root $r = 2$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1 + r}{r^3 + 4r^2 + r - 6}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{3}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
a_2	$\frac{1+r}{r^3+4r^2+r-6}$	$\frac{3}{20}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{r^3 + 6r^2 + 5r - 12}$$

Which for the root $r = 2$ becomes

$$a_3 = -\frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
a_2	$\frac{1+r}{r^3+4r^2+r-6}$	$\frac{3}{20}$
a_3	$-\frac{1}{r^3+6r^2+5r-12}$	$-\frac{1}{30}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 + 10r^3 + 27r^2 + 2r - 40}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{1}{168}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
a_2	$\frac{1+r}{r^3+4r^2+r-6}$	$\frac{3}{20}$
a_3	$-\frac{1}{r^3+6r^2+5r-12}$	$-\frac{1}{30}$
a_4	$\frac{1}{r^4+10r^3+27r^2+2r-40}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(r^2 + 9r + 18)(r^3 + 6r^2 + 3r - 10)}$$

Which for the root $r = 2$ becomes

$$a_5 = -\frac{1}{1120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
a_2	$\frac{1+r}{r^3+4r^2+r-6}$	$\frac{3}{20}$
a_3	$-\frac{1}{r^3+6r^2+5r-12}$	$-\frac{1}{30}$
a_4	$\frac{1}{r^4+10r^3+27r^2+2r-40}$	$\frac{1}{168}$
a_5	$-\frac{1}{(r^2+9r+18)(r^3+6r^2+3r-10)}$	$-\frac{1}{1120}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^2\left(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \frac{x^5}{1120} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_3 \\
&= -\frac{1}{r^3 + 6r^2 + 5r - 12}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} -\frac{1}{r^3 + 6r^2 + 5r - 12} &= \lim_{r\to -1} -\frac{1}{r^3 + 6r^2 + 5r - 12} \\
&= \frac{1}{12}
\end{aligned}$$

The limit is $\frac{1}{12}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) - 2b_n = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n-1)(n-2) + b_{n-1}(n-2) - 2b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n+r-1)}{n^2 + 2nr + r^2 - n - r - 2} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{b_{n-1}(n-2)}{n^2 - 3n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{r}{r^2 + r - 2}$$

Which for the root $r = -1$ becomes

$$b_1 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1+r}{(r^2+r-2)(r+3)}$$

Which for the root $r = -1$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
b_2	$\frac{1+r}{r^3+4r^2+r-6}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{(r+4)(r+3)(-1+r)}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{1}{12}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
b_2	$\frac{1+r}{r^3+4r^2+r-6}$	0
b_3	$-\frac{1}{(r+4)(r+3)(-1+r)}$	$\frac{1}{12}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r+4)(-1+r)(r^2+7r+10)}$$

Which for the root $r = -1$ becomes

$$b_4 = -\frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
b_2	$\frac{1+r}{r^3+4r^2+r-6}$	0
b_3	$-\frac{1}{(r+4)(r+3)(-1+r)}$	$\frac{1}{12}$
b_4	$\frac{1}{(5+r)(2+r)(r+4)(-1+r)}$	$-\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{(-1+r)\,(r^2+7r+10)\,(r^2+9r+18)}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{1}{80}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{r^2+r-2}$	$-\frac{1}{2}$
b_2	$\frac{1+r}{r^3+4r^2+r-6}$	0
b_3	$-\frac{1}{(r+4)(r+3)(-1+r)}$	$\frac{1}{12}$
b_4	$\frac{1}{(5+r)(2+r)(r+4)(-1+r)}$	$-\frac{1}{24}$
b_5	$-\frac{1}{(r+6)(r+3)(5+r)(-1+r)(2+r)}$	$\frac{1}{80}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^2\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{x}{2} + \frac{x^3}{12} - \frac{x^4}{24} + \frac{x^5}{80} + O(x^6)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2\left(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \frac{x^5}{1120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x}{2} + \frac{x^3}{12} - \frac{x^4}{24} + \frac{x^5}{80} + O(x^6)\right)}{x}
\end{aligned}$$

Hence the final solution is

$$y = y_h$$

$$= c_1 x^2 \left(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \frac{x^5}{1120} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x}{2} + \frac{x^3}{12} - \frac{x^4}{24} + \frac{x^5}{80} + O(x^6)\right)}{x}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 \left(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \frac{x^5}{1120} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x}{2} + \frac{x^3}{12} - \frac{x^4}{24} + \frac{x^5}{80} + O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1 x^2 \left(1 - \frac{x}{2} + \frac{3x^2}{20} - \frac{x^3}{30} + \frac{x^4}{168} - \frac{x^5}{1120} + O(x^6)\right) + \frac{c_2 \left(1 - \frac{x}{2} + \frac{x^3}{12} - \frac{x^4}{24} + \frac{x^5}{80} + O(x^6)\right)}{x}$$

Verified OK.

15.16.1 Maple step by step solution

Let's solve

$y''x^2 + x^2y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' + \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' - \frac{2y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 1, P_3(x) = -\frac{2}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = 0$$

- ∘ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = -2$$

- ∘ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$y''x^2 + x^2y' - 2y = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ∘ Convert $x^2 \cdot y'$ to series expansion

$$x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$$

- ∘ Shift index using $k- > k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- ∘ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(1+r)\,(-2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-2) + a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(1+r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation

$r \in \{-1, 2\}$

- Each term in the series must be 0, giving the recursion relation

$a_k(k+r+1)\,(k+r-2) + a_{k-1}(k-1+r) = 0$

- Shift index using $k->k+1$

  $a_{k+1}(k+2+r)(k-1+r)+a_k(k+r)=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k(k+r)}{(k+2+r)(k-1+r)}$
- Recursion relation for $r=-1$; series terminates at $k=1$

  $a_{k+1}=-\frac{a_k(k-1)}{(k+1)(k-2)}$
- Apply recursion relation for $k=0$

  $a_1=-\frac{a_0}{2}$
- Terminating series solution of the ODE for $r=-1$. Use reduction of order to find the second

  $y=a_0\cdot\left(-\frac{x}{2}+1\right)$
- Recursion relation for $r=2$

  $a_{k+1}=-\frac{a_k(k+2)}{(k+4)(k+1)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty}a_kx^{k+2}, a_{k+1}=-\frac{a_k(k+2)}{(k+4)(k+1)}\right]$
- Combine solutions and rename parameters

  $\left[y=a_0\cdot\left(-\frac{x}{2}+1\right)+\left(\sum_{k=0}^{\infty}b_kx^{k+2}\right), b_{1+k}=-\frac{b_k(k+2)}{(k+4)(1+k)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x^2*diff(y(x),x)-2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^2 \left(1 - \frac{1}{2}x + \frac{3}{20}x^2 - \frac{1}{30}x^3 + \frac{1}{168}x^4 - \frac{1}{1120}x^5 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(12 - 6x + x^3 - \frac{1}{2}x^4 + \frac{3}{20}x^5 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 63

```
AsymptoticDSolveValue[x^2*y''[x]+x^2*y'[x]-2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{x^3}{24} + \frac{x^2}{12} + \frac{1}{x} - \frac{1}{2}\right) + c_2 \left(\frac{x^6}{168} - \frac{x^5}{30} + \frac{3x^4}{20} - \frac{x^3}{2} + x^2\right)$$

15.17 problem 17

Internal problem ID [11918]
Internal file name [OUTPUT/11927_Saturday_April_13_2024_10_26_26_PM_76951881/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(2x^2-x\right)y''+(2x-2)\,y'+\left(-2x^2+3x-2\right)y=0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^2-x\right)y''+(2x-2)\,y'+\left(-2x^2+3x-2\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{2x-2}{x\,(2x-1)}$$
$$q(x)=-\frac{2x^2-3x+2}{x\,(2x-1)}$$

Table 559: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2x-2}{x(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

$q(x) = -\frac{2x^2-3x+2}{x(2x-1)}$	
singularity	type
$x = 0$	"regular"
$x = \frac{1}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \frac{1}{2}\right]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x(2x-1) + (2x-2)\,y' + \left(-2x^2 + 3x - 2\right)y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x(2x-1) + (2x-2)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + \left(-2x^2+3x-2\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r-1} a_n (n+r)(n+r-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-2(n+r) a_n x^{n+r-1}\right) \\
&+ \sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1) &= \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1} \\
\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\, x^{n+r-1} \\
\sum_{n=0}^{\infty} \left(-2x^{n+r+2} a_n\right) &= \sum_{n=3}^{\infty} \left(-2a_{n-3} x^{n+r-1}\right) \\
\sum_{n=0}^{\infty} 3x^{1+n+r} a_n &= \sum_{n=2}^{\infty} 3a_{n-2} x^{n+r-1} \\
\sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) &= \sum_{n=1}^{\infty} \left(-2a_{n-1} x^{n+r-1}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r-1}a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\,x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-2(n+r)\,a_n x^{n+r-1}\right) + \sum_{n=3}^{\infty} \left(-2a_{n-3}x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} 3a_{n-2}x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-2a_{n-1}x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$-x^{n+r-1}a_n(n+r)\,(n+r-1) - 2(n+r)\,a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$-x^{-1+r}a_0 r(-1+r) - 2r a_0 x^{-1+r} = 0$$

Or

$$\left(-x^{-1+r}r(-1+r) - 2r\,x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\,x^{-1+r}(-1-r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$-r(1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(-1-r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = Cy_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-2 + 2r}{r + 2}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{4r^2 - 4r + 3}{r^2 + 5r + 6}$$

For $3 \leq n$ the recursive equation is

$$\begin{aligned} &2a_{n-1}(n + r - 1)(n + r - 2) - a_n(n + r)(n + r - 1) \\ &+ 2a_{n-1}(n + r - 1) - 2a_n(n + r) - 2a_{n-3} + 3a_{n-2} - 2a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{2n^2 a_{n-1} + 4nr a_{n-1} + 2r^2 a_{n-1} - 4n a_{n-1} - 4r a_{n-1} - 2a_{n-3} + 3a_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{2n^2 a_{n-1} - 4n a_{n-1} - 2a_{n-3} + 3a_{n-2}}{n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{r+2}$	-1
a_2	$\frac{4r^2-4r+3}{r^2+5r+6}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{8r^3 + 2r - 4}{(r+4)(r+3)(r+2)}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{r+2}$	-1
a_2	$\frac{4r^2-4r+3}{r^2+5r+6}$	$\frac{1}{2}$
a_3	$\frac{8r^3+2r-4}{(r+4)(r+3)(r+2)}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16r^4 + 32r^3 + 12r^2 - 20r + 5}{(r^2+5r+6)(r^2+9r+20)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{r+2}$	-1
a_2	$\frac{4r^2-4r+3}{r^2+5r+6}$	$\frac{1}{2}$
a_3	$\frac{8r^3+2r-4}{(r+4)(r+3)(r+2)}$	$-\frac{1}{6}$
a_4	$\frac{16r^4+32r^3+12r^2-20r+5}{(r^2+5r+6)(r^2+9r+20)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{32r^5 + 160r^4 + 232r^3 + 8r^2 - 78r - 6}{(r+6)(r+5)(r+4)(r+3)(r+2)}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2+2r}{r+2}$	-1
a_2	$\frac{4r^2-4r+3}{r^2+5r+6}$	$\frac{1}{2}$
a_3	$\frac{8r^3+2r-4}{(r+4)(r+3)(r+2)}$	$-\frac{1}{6}$
a_4	$\frac{16r^4+32r^3+12r^2-20r+5}{(r^2+5r+6)(r^2+9r+20)}$	$\frac{1}{24}$
a_5	$\frac{32r^5+160r^4+232r^3+8r^2-78r-6}{(r+6)(r+5)(r+4)(r+3)(r+2)}$	$-\frac{1}{120}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{-2+2r}{r+2} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{-2+2r}{r+2} &= \lim_{r\to -1} \frac{-2+2r}{r+2} \\ &= -4 \end{aligned}$$

The limit is -4. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = \frac{-2+2r}{r+2}$$

Substituting $n = 2$ in Eq(3) gives

$$b_2 = \frac{4r^2-4r+3}{r^2+5r+6}$$

For $3 \leq n$ the recursive equation is

$$\begin{aligned} &2b_{n-1}(n+r-1)(n+r-2) - b_n(n+r)(n+r-1) \\ &+ 2b_{n-1}(n+r-1) - 2(n+r)b_n - 2b_{n-3} + 3b_{n-2} - 2b_{n-1} = 0 \end{aligned} \tag{4}$$

Which for for the root $r = -1$ becomes

$$\begin{aligned} &2b_{n-1}(n-2)(n-3) - b_n(n-1)(n-2) + 2b_{n-1}(n-2) \\ &- 2(n-1)b_n - 2b_{n-3} + 3b_{n-2} - 2b_{n-1} = 0 \end{aligned} \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{2n^2 b_{n-1} + 4nr b_{n-1} + 2r^2 b_{n-1} - 4n b_{n-1} - 4r b_{n-1} - 2b_{n-3} + 3b_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{2n^2 b_{n-1} - 8n b_{n-1} - 2b_{n-3} + 3b_{n-2} + 6b_{n-1}}{n^2 - n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+2r}{r+2}$	-4
b_2	$\frac{4r^2-4r+3}{r^2+5r+6}$	$\frac{11}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{8r^3 + 2r - 4}{(r^2 + 7r + 12)(r + 2)}$$

Which for the root $r = -1$ becomes

$$b_3 = -\frac{7}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+2r}{r+2}$	-4
b_2	$\frac{4r^2-4r+3}{r^2+5r+6}$	$\frac{11}{2}$
b_3	$\frac{8r^3+2r-4}{(r+4)(r+3)(r+2)}$	$-\frac{7}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16r^4 + 32r^3 + 12r^2 - 20r + 5}{(r + 3)(r + 2)(r^2 + 9r + 20)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{7}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+2r}{r+2}$	-4
b_2	$\frac{4r^2-4r+3}{r^2+5r+6}$	$\frac{11}{2}$
b_3	$\frac{8r^3+2r-4}{(r+4)(r+3)(r+2)}$	$-\frac{7}{3}$
b_4	$\frac{16r^4+32r^3+12r^2-20r+5}{(r+5)(r+4)(r+3)(r+2)}$	$\frac{7}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{32r^5 + 160r^4 + 232r^3 + 8r^2 - 78r - 6}{(r^2 + 7r + 12)(r + 2)(r^2 + 11r + 30)}$$

Which for the root $r = -1$ becomes

$$b_5 = -\frac{1}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2+2r}{r+2}$	-4
b_2	$\frac{4r^2-4r+3}{r^2+5r+6}$	$\frac{11}{2}$
b_3	$\frac{8r^3+2r-4}{(r+4)(r+3)(r+2)}$	$-\frac{7}{3}$
b_4	$\frac{16r^4+32r^3+12r^2-20r+5}{(r+5)(r+4)(r+3)(r+2)}$	$\frac{7}{8}$
b_5	$\frac{32r^5+160r^4+232r^3+8r^2-78r-6}{(r+6)(r+5)(r+4)(r+3)(r+2)}$	$-\frac{1}{5}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right) \\
&= \frac{1 - 4x + \frac{11x^2}{2} - \frac{7x^3}{3} + \frac{7x^4}{8} - \frac{x^5}{5} + O(x^6)}{x}
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$= c_1 \left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O(x^6)\right)+\frac{c_2 \left(1-4x+\frac{11x^2}{2}-\frac{7x^3}{3}+\frac{7x^4}{8}-\frac{x^5}{5}+O(x^6)\right)}{x}$$

Hence the final solution is

$$y = y_h$$

$$= c_1 \left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O(x^6)\right)+\frac{c_2 \left(1-4x+\frac{11x^2}{2}-\frac{7x^3}{3}+\frac{7x^4}{8}-\frac{x^5}{5}+O(x^6)\right)}{x}$$

Summary
The solution(s) found are the following

$$y = c_1 \left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O(x^6)\right) + \frac{c_2 \left(1-4x+\frac{11x^2}{2}-\frac{7x^3}{3}+\frac{7x^4}{8}-\frac{x^5}{5}+O(x^6)\right)}{x} \tag{1}$$

Verification of solutions

$$y = c_1 \left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O(x^6)\right) + \frac{c_2 \left(1-4x+\frac{11x^2}{2}-\frac{7x^3}{3}+\frac{7x^4}{8}-\frac{x^5}{5}+O(x^6)\right)}{x}$$

Verified OK.

15.17.1 Maple step by step solution

Let's solve

$y''x(2x-1) + (2x-2)\, y' + (-2x^2+3x-2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(2x^2-3x+2)y}{x(2x-1)} - \frac{2(x-1)y'}{x(2x-1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2(x-1)y'}{x(2x-1)}-\frac{(2x^2-3x+2)y}{x(2x-1)}=0$

☐ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{2(x-1)}{x(2x-1)}, P_3(x)=-\frac{2x^2-3x+2}{x(2x-1)}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\bigg|_{x=0}=2$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x(2x-1)+(2x-2)y'+(-2x^2+3x-2)y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty}a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

  ◦ Convert $x^m\cdot y'$ to series expansion for $m=0..1$

  $x^m\cdot y'=\sum_{k=0}^{\infty}a_k(k+r)\,x^{k+r-1+m}$

  ◦ Shift index using $k->k+1-m$

  $x^m\cdot y'=\sum_{k=-1+m}^{\infty}a_{k+1-m}(k+1-m+r)\,x^{k+r}$

- Convert $x^m \cdot y''$ to series expansion for $m = 1..2$

  $$x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0 r(1+r)\, x^{-1+r} + (-a_1(1+r)(2+r) + 2a_0(1+r)(-1+r))\, x^r + (-a_2(2+r)(3+r) + 2a_1($$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(1+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{-1, 0\}$$

- The coefficients of each power of x must be 0

  $$[-a_1(1+r)(2+r) + 2a_0(1+r)(-1+r) = 0, -a_2(2+r)(3+r) + 2a_1(2+r)\, r + 3a_0 = 0]$$

- Solve for the dependent coefficient(s)

  $$\left\{a_1 = \frac{2a_0(-1+r)}{2+r}, a_2 = \frac{a_0(4r^2-4r+3)}{r^2+5r+6}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-a_{k+1}(k+r+1)(k+2+r) + 2k^2a_k + 4kra_k + 2r^2a_k - 2a_k - 2a_{k-2} + 3a_{k-1} = 0$$

- Shift index using $k- >k+2$

  $$-a_{k+3}(k+3+r)(k+4+r) + 2(k+2)^2\, a_{k+2} + 4(k+2)\, ra_{k+2} + 2r^2a_{k+2} - 2a_{k+2} - 2a_k + 3a_{k+}$$

- Recursion relation that defines series solution to ODE

  $$a_{k+3} = \frac{2k^2a_{k+2} + 4kra_{k+2} + 2r^2a_{k+2} + 8ka_{k+2} + 8ra_{k+2} - 2a_k + 3a_{k+1} + 6a_{k+2}}{(k+3+r)(k+4+r)}$$

- Recursion relation for $r = -1$

  $$a_{k+3} = \frac{2k^2a_{k+2} + 4ka_{k+2} - 2a_k + 3a_{k+1}}{(k+2)(k+3)}$$

- Solution for $r = -1$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+3} = \frac{2k^2a_{k+2} + 4ka_{k+2} - 2a_k + 3a_{k+1}}{(k+2)(k+3)}, a_1 = -4a_0, a_2 = \frac{11a_0}{2}\right]$$

- Recursion relation for $r = 0$

  $$a_{k+3} = \frac{2k^2a_{k+2} + 8ka_{k+2} - 2a_k + 3a_{k+1} + 6a_{k+2}}{(k+3)(k+4)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{2k^2 a_{k+2} + 8k a_{k+2} - 2a_k + 3a_{k+1} + 6a_{k+2}}{(k+3)(k+4)}, a_1 = -a_0, a_2 = \frac{a_0}{2}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+3} = \frac{2k^2 a_{k+2} + 4k a_{k+2} - 2a_k + 3a_{1+k}}{(k+2)(k+3)}, a_1 = -4a_0, a_2 = \frac{11a_0}{2}, b_{k+3} = \right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 44

```
Order:=6;
dsolve((2*x^2-x)*diff(y(x),x$2)+(2*x-2)*diff(y(x),x)+(-2*x^2+3*x-2)*y(x)=0,y(x),type='series
```

$$y(x) = c_1\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + O(x^6)\right) + \frac{c_2\left(1 - 2x + \frac{7}{2}x^2 - \frac{4}{3}x^3 + \frac{13}{24}x^4 - \frac{7}{60}x^5 + O(x^6)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 60

```
AsymptoticDSolveValue[(2*x^2-x)*y''[x]+(2*x-2)*y'[x]+(-2*x^2+3*x-2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1\left(\frac{7x^3}{8} - \frac{7x^2}{3} + \frac{11x}{2} + \frac{1}{x} - 4\right) + c_2\left(\frac{x^4}{24} - \frac{x^3}{6} + \frac{x^2}{2} - x + 1\right)$$

15.18 problem 18

Internal problem ID [11919]
Internal file name [`OUTPUT/11928_Saturday_April_13_2024_10_26_27_PM_76458216/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - y'x + \frac{3y}{4} = 0}$$

With the expansion point for the power series method at $x = 0$.

The ode can be written as

$$4x^2y'' - 4y'x + 3y = 0$$

Which shows it is a Euler ODE. The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4x^2y'' - 4y'x + 3y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{3}{4x^2}$$

Table 561: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4x^2y'' - 4y'x + 3y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&4\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}\right)x^2 \\
&-4\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)x+3\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0
\end{aligned}
\tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty}4x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-4x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}3a_n x^{n+r}\right)=0 \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\left(\sum_{n=0}^{\infty}4x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-4x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}3a_n x^{n+r}\right)=0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
4x^{n+r}a_n(n+r)(n+r-1)-4x^{n+r}a_n(n+r)+3a_n x^{n+r}=0
$$

When $n=0$ the above becomes

$$
4x^r a_0 r(-1+r)-4x^r a_0 r+3a_0 x^r=0
$$

Or

$$
\left(4x^r r(-1+r)-4x^r r+3x^r\right)a_0=0
$$

Since $a_0\neq 0$ then the above simplifies to

$$
\left(4r^2-8r+3\right)x^r=0
$$

Since the above is true for all x then the indicial equation becomes

$$
4r^2-8r+3=0
$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \frac{3}{2} \\
r_2 &= \frac{1}{2}
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(4r^2 - 8r + 3)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^{\frac{3}{2}} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \sqrt{x} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$
4a_n(n+r)(n+r-1) - 4a_n(n+r) + 3a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = 0 \tag{4}
$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\rightarrow r_2} 0 &= \lim_{r\rightarrow \frac{1}{2}} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $0 \leq n$ the recursive equation is

$$
4b_n(n+r)(n+r-1) - 4b_n(n+r) + 3b_n = 0 \tag{4}
$$

Which for for the root $r = \frac{1}{2}$ becomes

$$
4b_n\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right) - 4b_n\left(n+\frac{1}{2}\right) + 3b_n = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = 0 \tag{5}
$$

Which for the root $r = \frac{1}{2}$ becomes

$$
b_n = 0 \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{3}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{3}{2}}\left(1 + O(x^6)\right) + c_2\sqrt{x}\left(1 + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{3}{2}}\left(1 + O(x^6)\right) + c_2\sqrt{x}\left(1 + O(x^6)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x^{\frac{3}{2}} \left(1 + O(x^6)\right) + c_2 \sqrt{x}\, \left(1 + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}} \left(1 + O(x^6)\right) + c_2 \sqrt{x}\, \left(1 + O(x^6)\right)$$

Verified OK.

15.18.1 Maple step by step solution

Let's solve

$4y''x^2 - 4y'x + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} - \frac{3y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{3y}{4x^2} = 0$

- Multiply by denominators of the ODE

  $4y''x^2 - 4y'x + 3y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

$y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$4\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right)x^2 - 4\frac{d}{dt}y(t) + 3y(t) = 0$

- Simplify

  $4\frac{d^2}{dt^2}y(t) - 8\frac{d}{dt}y(t) + 3y(t) = 0$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = 2\frac{d}{dt}y(t) - \frac{3y(t)}{4}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + \frac{3y(t)}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - 2r + \frac{3}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)(2r-3)}{4} = 0$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2}, \frac{3}{2}\right)$

- 1st solution of the ODE

  $y_1(t) = e^{\frac{t}{2}}$

- 2nd solution of the ODE

  $y_2(t) = e^{\frac{3t}{2}}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{\frac{t}{2}} + c_2 e^{\frac{3t}{2}}$

- Change variables back using $t = \ln(x)$

  $y = c_1\sqrt{x} + c_2 x^{\frac{3}{2}}$

- Simplify

  $y = \sqrt{x}\,(c_2 x + c_1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 27

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+3/4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\,(c_1 x + c_2) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 20

```
AsymptoticDSolveValue[x^2*y''[x]-x*y'[x]+3/4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 x^{3/2} + c_1 \sqrt{x}$$

15.19 problem 19

Internal problem ID [11920]
Internal file name [`OUTPUT/11929_Saturday_April_13_2024_10_26_28_PM_82600808/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y'x + y(x-1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + y'x + y(x-1) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x-1}{x^2}$$

Table 563: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x-1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + y'x + y(x-1) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x^2 + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) x + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) (x-1) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \\ + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) \\ + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-1} - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{n^2 + 2nr + r^2 - 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{r(r+2)}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(r+2)}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r(r+2)(r+3)(r+1)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(r+2)}$	$-\frac{1}{3}$
a_2	$\frac{1}{r(r+2)(r+3)(r+1)}$	$\frac{1}{24}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{r(r+2)^2(r+3)(r+1)(r+4)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(r+2)}$	$-\frac{1}{3}$
a_2	$\frac{1}{r(r+2)(r+3)(r+1)}$	$\frac{1}{24}$
a_3	$-\frac{1}{r(r+2)^2(r+3)(r+1)(r+4)}$	$-\frac{1}{360}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r(r+2)^2(r+3)^2(r+1)(r+4)(r+5)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{8640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(r+2)}$	$-\frac{1}{3}$
a_2	$\frac{1}{r(r+2)(r+3)(r+1)}$	$\frac{1}{24}$
a_3	$-\frac{1}{r(r+2)^2(r+3)(r+1)(r+4)}$	$-\frac{1}{360}$
a_4	$\frac{1}{r(r+2)^2(r+3)^2(r+1)(r+4)(r+5)}$	$\frac{1}{8640}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{r(r+2)^2(r+3)^2(r+1)(r+4)^2(r+5)(r+6)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{302400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(r+2)}$	$-\frac{1}{3}$
a_2	$\frac{1}{r(r+2)(r+3)(r+1)}$	$\frac{1}{24}$
a_3	$-\frac{1}{r(r+2)^2(r+3)(r+1)(r+4)}$	$-\frac{1}{360}$
a_4	$\frac{1}{r(r+2)^2(r+3)^2(r+1)(r+4)(r+5)}$	$\frac{1}{8640}$
a_5	$-\frac{1}{r(r+2)^2(r+3)^2(r+1)(r+4)^2(r+5)(r+6)}$	$-\frac{1}{302400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 - \frac{x}{3} + \frac{x^2}{24} - \frac{x^3}{360} + \frac{x^4}{8640} - \frac{x^5}{302400} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_2 \\
&= \frac{1}{r(r+2)(r+3)(r+1)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r \to r_2} \frac{1}{r(r+2)(r+3)(r+1)} &= \lim_{r \to -1} \frac{1}{r(r+2)(r+3)(r+1)} \\
&= \textit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2y'' + y'x + y(x-1) = 0$ gives

$$\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x^2 \\
&+ \left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) x \\
&+ \left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right)(x-1) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(\left(y_1''(x)\,x^2+y_1'(x)\,x+y_1(x)\,(x-1)\right)\ln(x)+\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x^2+y_1(x)\right)C \\ +\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x^2 \\ +\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)(x-1)=0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x^2+y_1'(x)\,x+y_1(x)\,(x-1)=0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x^2+y_1(x)\right)C \\ +\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x^2 \\ +\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)(x-1)=0 \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)\,(-1+n+r_2)\right)x^2+2x\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)C \\ +\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)(x-1)=0 \tag{9}$$

Since $r_1=1$ and $r_2=-1$ then the above becomes

$$\left(\sum_{n=0}^{\infty}x^{-3+n}b_n(n-1)\,(-2+n)\right)x^2+2x\left(\sum_{n=0}^{\infty}x^{n}a_n(1+n)\right)C \\ +\left(\sum_{n=0}^{\infty}x^{-2+n}b_n(n-1)\right)x+\left(\sum_{n=0}^{\infty}b_n x^{n-1}\right)(x-1)=0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2-3n+2\right)\right)+\left(\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n (1+n)\right) \\ +\left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right)+\left(\sum_{n=0}^{\infty} b_n x^n\right)+\sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n (1+n)=\sum_{n=2}^{\infty} 2C a_{-2+n} (n-1)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} b_n x^n=\sum_{n=1}^{\infty} b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2-3n+2\right)\right)+\left(\sum_{n=2}^{\infty} 2C a_{-2+n} (n-1)\, x^{n-1}\right) \\ +\left(\sum_{n=0}^{\infty} x^{n-1} b_n (n-1)\right)+\left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right)+\sum_{n=0}^{\infty} \left(-b_n x^{n-1}\right)=0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$b_0-b_1=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1-b_1=0$$

Solving the above for b_1 gives

$$b_1=1$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C+1=0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$4Ca_1 + b_2 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3 + \frac{2}{3} = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{2}{9}$$

For $n = 4$, Eq (2B) gives

$$6Ca_2 + b_3 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 - \frac{25}{72} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{25}{576}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 + b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 + \frac{157}{2880} = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{157}{43200}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2}\left(x\left(1 - \frac{x}{3} + \frac{x^2}{24} - \frac{x^3}{360} + \frac{x^4}{8640} - \frac{x^5}{302400} + O(x^6)\right)\right)\ln(x) + \frac{1 + x - \frac{2x^3}{9} + \frac{25x^4}{576} - \frac{157x^5}{43200} + O(x^6)}{x}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1-\frac{x}{3}+\frac{x^2}{24}-\frac{x^3}{360}+\frac{x^4}{8640}-\frac{x^5}{302400}+O\left(x^6\right)\right) \\
&\quad + c_2 \left(-\frac{1}{2}\left(x\left(1-\frac{x}{3}+\frac{x^2}{24}-\frac{x^3}{360}+\frac{x^4}{8640}-\frac{x^5}{302400}+O\left(x^6\right)\right)\right)\ln\left(x\right) \right. \\
&\quad \left. + \frac{1+x-\frac{2x^3}{9}+\frac{25x^4}{576}-\frac{157x^5}{43200}+O(x^6)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1-\frac{x}{3}+\frac{x^2}{24}-\frac{x^3}{360}+\frac{x^4}{8640}-\frac{x^5}{302400}+O\left(x^6\right)\right) \\
&\quad + c_2 \left(-\frac{x\left(1-\frac{x}{3}+\frac{x^2}{24}-\frac{x^3}{360}+\frac{x^4}{8640}-\frac{x^5}{302400}+O(x^6)\right)\ln\left(x\right)}{2} \right. \\
&\quad \left. + \frac{1+x-\frac{2x^3}{9}+\frac{25x^4}{576}-\frac{157x^5}{43200}+O(x^6)}{x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1-\frac{x}{3}+\frac{x^2}{24}-\frac{x^3}{360}+\frac{x^4}{8640}-\frac{x^5}{302400}+O\left(x^6\right)\right) \\
&\quad + c_2 \left(-\frac{x\left(1-\frac{x}{3}+\frac{x^2}{24}-\frac{x^3}{360}+\frac{x^4}{8640}-\frac{x^5}{302400}+O(x^6)\right)\ln\left(x\right)}{2} \right. \\
&\quad \left. + \frac{1+x-\frac{2x^3}{9}+\frac{25x^4}{576}-\frac{157x^5}{43200}+O(x^6)}{x}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x \left(1 - \frac{x}{3} + \frac{x^2}{24} - \frac{x^3}{360} + \frac{x^4}{8640} - \frac{x^5}{302400} + O(x^6)\right) + c_2 \left(-\frac{x\left(1 - \frac{x}{3} + \frac{x^2}{24} - \frac{x^3}{360} + \frac{x^4}{8640} - \frac{x^5}{302400} + O(x^6)\right) \ln(x)}{2} + \frac{1 + x - \frac{2x^3}{9} + \frac{25x^4}{576} - \frac{157x^5}{43200} + O(x^6)}{x}\right)$$

Verified OK.

15.19.1 Maple step by step solution

Let's solve

$y''x^2 + y'x + y(x-1) = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{(x-1)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{(x-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{x-1}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -1$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x^2 + y'x + y(x-1) = 0$
- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+1)\,(k+r-1) + a_{k-1}\right) x^{k+r}\right) = 0$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)\,(k+r-1) + a_{k-1} = 0$
- Shift index using $k- > k + 1$

  $a_{k+1}(k+2+r)\,(k+r) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+2+r)(k+r)}$
- Recursion relation for $r = -1$

$$a_{k+1} = -\frac{a_k}{(k+1)(k-1)}$$

- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 1$

$$a_{k+1} = -\frac{a_k}{(k+1)(k-1)}$$

- Recursion relation for $r = 1$

$$a_{k+1} = -\frac{a_k}{(k+3)(k+1)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{(k+3)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 63

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(x-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 - \frac{1}{3}x + \frac{1}{24}x^2 - \frac{1}{360}x^3 + \frac{1}{8640}x^4 - \frac{1}{302400}x^5 + O\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(x^2 - \frac{1}{3}x^3 + \frac{1}{24}x^4 - \frac{1}{360}x^5 + O\left(x^6\right.\right.\right.}{x}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 83

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(x-1)*y[x]==0,y[x],{x,0,5}]
```

$$
y(x) \to c_1 \left(\frac{31x^4 - 176x^3 + 144x^2 + 576x + 576}{576x} - \frac{1}{48} x \left(x^2 - 8x + 24\right) \log(x) \right) + c_2 \left(\frac{x^5}{8640} - \frac{x^4}{360} + \frac{x^3}{24} - \frac{x^2}{3} + x \right)
$$

15.20 problem 20

Internal problem ID [11921]
Internal file name [`OUTPUT/11930_Saturday_April_13_2024_10_26_29_PM_87387057/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(x^3 - x\right)y' - 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^3 - x\right)y' - 3y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x^2 - 1}{x}$$
$$q(x) = -\frac{3}{x^2}$$

Table 565: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x^2-1}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = -\frac{3}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + (x^3 - x) y' - 3y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x^2 + (x^3 - x) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)\right) + \sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-3 a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)=\sum_{n=2}^{\infty} a_{n-2}(n+r-2) x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2) x^{n+r}\right) + \sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-3 a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1)-x^{n+r} a_n(n+r)-3 a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)-x^r a_0 r-3 a_0 x^r=0$$

Or

$$\left(x^r r(-1+r)-x^r r-3 x^r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-2 r-3\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2-2 r-3=0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 3 \\
r_2 &= -1
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(r^2 - 2r - 3)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^3 \left(\sum_{n=0}^{\infty} a_n x^n \right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+3} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1} \right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) + a_{n-2}(n+r-2) - a_n(n+r) - 3a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}(n+r-2)}{n^2+2nr+r^2-2n-2r-3} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{a_{n-2}(n+1)}{n\,(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{r}{r^2+2r-3}$$

Which for the root $r = 3$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{r^2+2r-3}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{r^2+2r-3}$	$-\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(2+r)}{r^4 + 8r^3 + 14r^2 - 8r - 15}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{5}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{r^2+2r-3}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{r(2+r)}{r^4+8r^3+14r^2-8r-15}$	$\frac{5}{128}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{r}{r^2+2r-3}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{r(2+r)}{r^4+8r^3+14r^2-8r-15}$	$\frac{5}{128}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^3 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^3 \left(1 - \frac{x^2}{4} + \frac{5x^4}{128} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{r(2+r)}{r^4 + 8r^3 + 14r^2 - 8r - 15} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{r(2+r)}{r^4 + 8r^3 + 14r^2 - 8r - 15} &= \lim_{r\to -1} \frac{r(2+r)}{r^4 + 8r^3 + 14r^2 - 8r - 15} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $x^2y'' + (x^3 - x)\,y' - 3y = 0$ gives

$$\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x^2 + (x^3 - x)\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) - 3Cy_1(x)\ln(x) - 3\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Which can be written as

$$\left(\left(y_1''(x)\,x^2 + (x^3 - x)\,y_1'(x) - 3y_1(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x^2 + \frac{(x^3 - x)\,y_1(x)}{x}\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x^2 + (x^3 - x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) - 3\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x^2 + (x^3 - x)\,y_1'(x) - 3y_1(x) = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x^2 + \frac{(x^3 - x)\,y_1(x)}{x}\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x^2 + (x^3 - x)\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) - 3\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x + (x^2-2)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \\
&+ (x^3-x)\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) - 3\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}
$$

Since $r_1 = 3$ and $r_2 = -1$ then the above becomes

$$
\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{2+n} a_n (n+3)\right) x + (x^2-2)\left(\sum_{n=0}^{\infty} a_n x^{n+3}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-1)(n-2)\right) x^2 \\
&+ (x^3-x)\left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-1)\right) - 3\left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) = 0
\end{aligned} \tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (n+3)\right) + \left(\sum_{n=0}^{\infty} C\, x^{n+5} a_n\right) + \sum_{n=0}^{\infty} \left(-2C\, x^{n+3} a_n\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n^2-3n+2)\right) + \left(\sum_{n=0}^{\infty} x^{n+1} b_n (n-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-3 b_n x^{n-1}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (n+3) = \sum_{n=4}^{\infty} 2C a_{n-4} (n-1)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} C\, x^{n+5} a_n = \sum_{n=6}^{\infty} C a_{n-6} x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-2C\, x^{n+3} a_n\right) = \sum_{n=4}^{\infty} \left(-2C a_{n-4} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} x^{n+1} b_n (n-1) = \sum_{n=2}^{\infty} b_{n-2} (-3+n)\, x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned} \left(\sum_{n=4}^{\infty} 2C a_{n-4} (n-1)\, x^{n-1}\right) + \left(\sum_{n=6}^{\infty} C a_{n-6} x^{n-1}\right) + \sum_{n=4}^{\infty} \left(-2C a_{n-4} x^{n-1}\right) \\ + \left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2 - 3n + 2\right)\right) + \left(\sum_{n=2}^{\infty} b_{n-2} (-3+n)\, x^{n-1}\right) \\ + \sum_{n=0}^{\infty} \left(-x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-3 b_n x^{n-1}\right) = 0 \end{aligned} \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-3b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n = 2$, Eq (2B) gives

$$-b_0 - 4b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-1 - 4b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{1}{4}$$

For $n = 3$, Eq (2B) gives

$$-3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C - \frac{1}{4} = 0$$

Which is solved for C. Solving for C gives

$$C = \frac{1}{16}$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 + 2b_3 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = \frac{1}{16}$ and all b_n, then the second solution becomes

$$y_2(x) = \frac{1}{16}\left(x^3\left(1 - \frac{x^2}{4} + \frac{5x^4}{128} + O(x^6)\right)\right)\ln(x) + \frac{1 - \frac{x^2}{4} + O(x^6)}{x}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^3 \left(1 - \frac{x^2}{4} + \frac{5x^4}{128} + O(x^6)\right) \\
&\quad + c_2 \left(\frac{1}{16} \left(x^3 \left(1 - \frac{x^2}{4} + \frac{5x^4}{128} + O(x^6)\right)\right) \ln(x) + \frac{1 - \frac{x^2}{4} + O(x^6)}{x}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^3 \left(1 - \frac{x^2}{4} + \frac{5x^4}{128} + O(x^6)\right) \\
&\quad + c_2 \left(\frac{x^3 \left(1 - \frac{x^2}{4} + \frac{5x^4}{128} + O(x^6)\right) \ln(x)}{16} + \frac{1 - \frac{x^2}{4} + O(x^6)}{x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^3 \left(1 - \frac{x^2}{4} + \frac{5x^4}{128} + O(x^6)\right) \\
&\quad + c_2 \left(\frac{x^3 \left(1 - \frac{x^2}{4} + \frac{5x^4}{128} + O(x^6)\right) \ln(x)}{16} + \frac{1 - \frac{x^2}{4} + O(x^6)}{x}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^3 \left(1 - \frac{x^2}{4} + \frac{5x^4}{128} + O(x^6)\right) \\
&\quad + c_2 \left(\frac{x^3 \left(1 - \frac{x^2}{4} + \frac{5x^4}{128} + O(x^6)\right) \ln(x)}{16} + \frac{1 - \frac{x^2}{4} + O(x^6)}{x}\right)
\end{aligned}
$$

Verified OK.

15.20.1 Maple step by step solution

Let's solve

$y''x^2 + (x^3 - x)\, y' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y}{x^2} - \frac{(x^2-1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(x^2-1)y'}{x} - \frac{3y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{x^2-1}{x}, P_3(x) = -\frac{3}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -3$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2 + (x^2 - 1)\, xy' - 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y'$ to series expansion for $m = 1..3$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$

- Shift index using $k->k+1-m$

  $x^{m}\cdot y'=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x^{2}\cdot y''$ to series expansion

  $x^{2}\cdot y''=\sum_{k=0}^{\infty} a_{k}(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}(1+r)\,(-3+r)\, x^{r}+a_{1}(2+r)\,(-2+r)\, x^{1+r}+\left(\sum_{k=2}^{\infty} (a_{k}(k+r+1)\,(k+r-3)+a_{k-2}(k-2-$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $(1+r)\,(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{-1,3\}$

- Each term must be 0

  $a_{1}(2+r)\,(-2+r)=0$

- Solve for the dependent coefficient(s)

  $a_{1}=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k}(k+r+1)\,(k+r-3)+a_{k-2}(k-2+r)=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+3+r)\,(k+r-1)+a_{k}(k+r)=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{a_{k}(k+r)}{(k+3+r)(k+r-1)}$

- Recursion relation for $r=-1$

  $a_{k+2}=-\frac{a_{k}(k-1)}{(k+2)(k-2)}$

- Series not valid for $r=-1$, division by 0 in the recursion relation at $k=2$

  $a_{k+2}=-\frac{a_{k}(k-1)}{(k+2)(k-2)}$

- Recursion relation for $r=3$

  $a_{k+2}=-\frac{a_{k}(k+3)}{(k+6)(k+2)}$

- Solution for $r=3$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=-\frac{a_k(k+3)}{(k+6)(k+2)}, a_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+(x^3-x)*diff(y(x),x)-3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1x^4\left(1-\frac{1}{4}x^2+\frac{5}{128}x^4+\mathrm{O}\left(x^6\right)\right)+c_2(\ln\left(x\right)\left(\left(-9\right)x^4+\mathrm{O}\left(x^6\right)\right)+\left(-144+36x^2+\mathrm{O}\left(x^6\right)\right))}{x}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 55

```
AsymptoticDSolveValue[x^2*y''[x]+(x^3-x)*y'[x]-3*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(\frac{5x^7}{128}-\frac{x^5}{4}+x^3\right)+c_1\left(\frac{1}{16}x^3\log(x)-\frac{x^4+16x^2-64}{64x}\right)$$

15.21 problem 21

Internal problem ID [11922]
Internal file name [`OUTPUT/11931_Saturday_April_13_2024_10_26_30_PM_98169401/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - y'x + 8y(x^2 - 1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - y'x + (8x^2 - 8) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{8x^2 - 8}{x^2}$$

Table 567: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{8x^2-8}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - y'x + (8x^2 - 8)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2 - \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x + (8x^2 - 8) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 8x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-8a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 8x^{n+r+2} a_n = \sum_{n=2}^{\infty} 8a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} 8a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-8a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) - 8a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r - 8a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r - 8x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 2r - 8\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 2r - 8 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 4$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 2r - 8\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 6$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^4 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+4}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) - a_n(n + r) + 8a_{n-2} - 8a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{8a_{n-2}}{n^2 + 2nr + r^2 - 2n - 2r - 8} \tag{4}$$

Which for the root $r = 4$ becomes

$$a_n = -\frac{8a_{n-2}}{n(n+6)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{8}{r^2 + 2r - 8}$$

Which for the root $r = 4$ becomes

$$a_2 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{8}{r^2+2r-8}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{8}{r^2+2r-8}$	$-\frac{1}{2}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{64}{(r+4)(r-2)\, r\, (r+6)}$$

Which for the root $r = 4$ becomes

$$a_4 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{8}{r^2+2r-8}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{64}{(r+4)(r-2)r(r+6)}$	$\frac{1}{10}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{8}{r^2+2r-8}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{64}{(r+4)(r-2)r(r+6)}$	$\frac{1}{10}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{512}{(r+4)(r-2)\, r\, (r+6)(r+8)(r+2)}$$

Which for the root $r = 4$ becomes

$$a_6 = -\frac{1}{90}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{8}{r^2+2r-8}$	$-\frac{1}{2}$
a_3	0	0
a_4	$\frac{64}{(r+4)(r-2)r(r+6)}$	$\frac{1}{10}$
a_5	0	0
a_6	$-\frac{512}{(r+4)(r-2)r(r+6)(r+8)(r+2)}$	$-\frac{1}{90}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x^4\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 \dots\right) \\ &= x^4\left(1 - \frac{x^2}{2} + \frac{x^4}{10} - \frac{x^6}{90} + O\left(x^7\right)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 6$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_6(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_6 \\ &= -\frac{512}{(r+4)(r-2)r(r+6)(r+8)(r+2)}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r \to r_2} -\frac{512}{(r+4)(r-2)r(r+6)(r+8)(r+2)} &= \lim_{r \to -2} -\frac{512}{(r+4)(r-2)r(r+6)(r+8)(r+2)} \\ &= \mathit{undefined}\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2y'' - y'x + (8x^2-8)\,y = 0$ gives

$$
\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x^2 \\
&- \left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) x \\
&+ (8x^2-8)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(y_1''(x)\,x^2 - y_1'(x)\,x + (8x^2-8)\,y_1(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x^2\right. \\
&\left.- y_1(x)\right) C + \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x^2 \\
&- \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) x + (8x^2-8)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x^2 - y_1'(x)\,x + \left(8x^2 - 8\right) y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned}
&\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) x^2 - y_1(x)\right) C \\
&+ \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) x^2 \\
&- \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) x + \left(8x^2 - 8\right) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 + 8\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 \\
&- \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x - 8\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}$$

Since $r_1 = 4$ and $r_2 = -2$ then the above becomes

$$\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{3+n} a_n (n+4)\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^{n+4}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} x^{-4+n} b_n (n-2)(-3+n)\right) x^2 + 8\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) x^2 \\
&- \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-2)\right) x - 8\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+4)\right) + \sum_{n=0}^{\infty} \left(-2C a_n x^{n+4}\right) + \left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right) \\
&+ \left(\sum_{n=0}^{\infty} 8 b_n x^n\right) + \sum_{n=0}^{\infty} \left(-x^{n-2} b_n (n-2)\right) + \sum_{n=0}^{\infty} \left(-8 b_n x^{n-2}\right) = 0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+4) &= \sum_{n=6}^{\infty} 2C a_{n-6} (n-2)\, x^{n-2} \\
\sum_{n=0}^{\infty} \left(-2C a_n x^{n+4}\right) &= \sum_{n=6}^{\infty} \left(-2C a_{n-6} x^{n-2}\right) \\
\sum_{n=0}^{\infty} 8 b_n x^n &= \sum_{n=2}^{\infty} 8 b_{n-2} x^{n-2}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$
\begin{aligned}
&\left(\sum_{n=6}^{\infty} 2C a_{n-6} (n-2)\, x^{n-2}\right) + \sum_{n=6}^{\infty} \left(-2C a_{n-6} x^{n-2}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right) + \left(\sum_{n=2}^{\infty} 8 b_{n-2} x^{n-2}\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^{n-2} b_n (n-2)\right) + \sum_{n=0}^{\infty} \left(-8 b_n x^{n-2}\right) = 0
\end{aligned}
\tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$
-5b_1 = 0
$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$
-5b_1 = 0
$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n = 2$, Eq (2B) gives

$$8b_0 - 8b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8 - 8b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = 1$$

For $n = 3$, Eq (2B) gives

$$8b_1 - 9b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-9b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$8b_2 - 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8 - 8b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = 1$$

For $n = 5$, Eq (2B) gives

$$-5b_5 + 8b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

For $n = N$, where $N = 6$ which is the difference between the two roots, we are free to choose $b_6 = 0$. Hence for $n = 6$, Eq (2B) gives

$$6C + 8 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{4}{3}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{4}{3}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{4}{3}\left(x^4\left(1 - \frac{x^2}{2} + \frac{x^4}{10} - \frac{x^6}{90} + O(x^7)\right)\right) \ln(x) + \frac{1 + x^2 + x^4 + O(x^7)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^4\left(1 - \frac{x^2}{2} + \frac{x^4}{10} - \frac{x^6}{90} + O(x^7)\right) \\ &\quad + c_2\left(-\frac{4}{3}\left(x^4\left(1 - \frac{x^2}{2} + \frac{x^4}{10} - \frac{x^6}{90} + O(x^7)\right)\right) \ln(x) + \frac{1 + x^2 + x^4 + O(x^7)}{x^2}\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^4\left(1 - \frac{x^2}{2} + \frac{x^4}{10} - \frac{x^6}{90} + O(x^7)\right) \\ &\quad + c_2\left(-\frac{4x^4\left(1 - \frac{x^2}{2} + \frac{x^4}{10} - \frac{x^6}{90} + O(x^7)\right) \ln(x)}{3} + \frac{1 + x^2 + x^4 + O(x^7)}{x^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x^4\left(1 - \frac{x^2}{2} + \frac{x^4}{10} - \frac{x^6}{90} + O(x^7)\right) \\ &\quad + c_2\left(-\frac{4x^4\left(1 - \frac{x^2}{2} + \frac{x^4}{10} - \frac{x^6}{90} + O(x^7)\right) \ln(x)}{3} + \frac{1 + x^2 + x^4 + O(x^7)}{x^2}\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 x^4 \left(1 - \frac{x^2}{2} + \frac{x^4}{10} - \frac{x^6}{90} + O(x^7)\right) + c_2 \left(-\frac{4x^4 \left(1 - \frac{x^2}{2} + \frac{x^4}{10} - \frac{x^6}{90} + O(x^7)\right) \ln(x)}{3} + \frac{1 + x^2 + x^4 + O(x^7)}{x^2}\right)$$

Verified OK.

15.21.1 Maple step by step solution

Let's solve

$y''x^2 - y'x + (8x^2 - 8) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{8(x^2-1)y}{x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{8(x^2-1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{8(x^2-1)}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = -8$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x^2 - y'x + (8x^2 - 8) y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k- > k - m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)(-4+r)\, x^r + a_1(3+r)(-3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)(k+r-4) + 8a_{k-2})\, x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 4\}$

- Each term must be 0

  $a_1(3+r)(-3+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)(k+r-4) + 8a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $a_{k+2}(k+4+r)(k-2+r) + 8a_k = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{8a_k}{(k+4+r)(k-2+r)}$$

- Recursion relation for $r = -2$

$$a_{k+2} = -\frac{8a_k}{(k+2)(k-4)}$$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 4$

$$a_{k+2} = -\frac{8a_k}{(k+2)(k-4)}$$

- Recursion relation for $r = 4$

$$a_{k+2} = -\frac{8a_k}{(k+8)(k+2)}$$

- Solution for $r = 4$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+2} = -\frac{8a_k}{(k+8)(k+2)}, a_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+8*(x^2-1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^4 \left(1 - \frac{1}{2}x^2 + \frac{1}{10}x^4 + O\left(x^6\right)\right) + \frac{c_2\left(-86400 - 86400x^2 - 86400x^4 + O\left(x^6\right)\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 36

```
AsymptoticDSolveValue[x^2*y''[x]-x*y'[x]+8*(x^2-1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(x^2 + \frac{1}{x^2} + 1\right) + c_2 \left(\frac{x^8}{10} - \frac{x^6}{2} + x^4\right)$$

15.22 problem 22

Internal problem ID [11923]
Internal file name [`OUTPUT/11932_Saturday_April_13_2024_10_26_32_PM_53853751/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + x^2y' - \frac{3y}{4} = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + x^2y' - \frac{3y}{4} = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 1$$
$$q(x) = -\frac{3}{4x^2}$$

Table 569: Table $p(x), q(x)$ singularites.

$p(x) = 1$	
singularity	type

$q(x) = -\frac{3}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + x^2 y' - \frac{3y}{4} = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2 + x^2 \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \frac{3\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)}{4} = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-\frac{3 a_n x^{n+r}}{4}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} a_{n-1} (n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-\frac{3 a_n x^{n+r}}{4}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)\,(n+r-1) - \frac{3 a_n x^{n+r}}{4} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - \frac{3 a_0 x^r}{4} = 0$$

Or

$$\left(x^r r(-1+r) - \frac{3 x^r}{4}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4r^2 - 4r - 3)\, x^r}{4} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - \frac{3}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{3}{2} \\ r_2 &= -\frac{1}{2} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 - 4r - 3)\, x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{3}{2}} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) - \frac{3a_n}{4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-1}(n+r-1)}{4n^2 + 8nr + 4r^2 - 4n - 4r - 3} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = -\frac{a_{n-1}(2n+1)}{2n\,(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{4r}{4r^2 + 4r - 3}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_1 = -\frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4r}{4r^2+4r-3}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16r(1 + r)}{(4r^2 + 4r - 3)(4r^2 + 12r + 5)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = \frac{5}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4r}{4r^2+4r-3}$	$-\frac{1}{2}$
a_2	$\frac{16r(1+r)}{(4r^2+4r-3)(4r^2+12r+5)}$	$\frac{5}{32}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64r(1 + r)(2 + r)}{(4r^2 + 4r - 3)(4r^2 + 12r + 5)(4r^2 + 20r + 21)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = -\frac{7}{192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4r}{4r^2+4r-3}$	$-\frac{1}{2}$
a_2	$\frac{16r(1+r)}{(4r^2+4r-3)(4r^2+12r+5)}$	$\frac{5}{32}$
a_3	$-\frac{64r(1+r)(2+r)}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)}$	$-\frac{7}{192}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256r(1+r)(2+r)(3+r)}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)(4r^2+28r+45)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{7}{1024}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4r}{4r^2+4r-3}$	$-\frac{1}{2}$
a_2	$\frac{16r(1+r)}{(4r^2+4r-3)(4r^2+12r+5)}$	$\frac{5}{32}$
a_3	$-\frac{64r(1+r)(2+r)}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)}$	$-\frac{7}{192}$
a_4	$\frac{256r(1+r)(2+r)(3+r)}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)(4r^2+28r+45)}$	$\frac{7}{1024}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024r(1+r)(2+r)(3+r)(4+r)}{(2r+9)(2r-1)(2r+11)(2r+3)^2(2r+1)(2r+7)^2(2r+5)^2}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = -\frac{11}{10240}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4r}{4r^2+4r-3}$	$-\frac{1}{2}$
a_2	$\frac{16r(1+r)}{(4r^2+4r-3)(4r^2+12r+5)}$	$\frac{5}{32}$
a_3	$-\frac{64r(1+r)(2+r)}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)}$	$-\frac{7}{192}$
a_4	$\frac{256r(1+r)(2+r)(3+r)}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)(4r^2+28r+45)}$	$\frac{7}{1024}$
a_5	$-\frac{1024r(1+r)(2+r)(3+r)(4+r)}{(2r+9)(2r-1)(2r+11)(2r+3)^2(2r+1)(2r+7)^2(2r+5)^2}$	$-\frac{11}{10240}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{3}{2}}\left(1 - \frac{x}{2} + \frac{5x^2}{32} - \frac{7x^3}{192} + \frac{7x^4}{1024} - \frac{11x^5}{10240} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_2 \\
&= \frac{16r(1+r)}{(4r^2+4r-3)(4r^2+12r+5)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} \frac{16r(1+r)}{(4r^2+4r-3)(4r^2+12r+5)} &= \lim_{r\to -\frac{1}{2}} \frac{16r(1+r)}{(4r^2+4r-3)(4r^2+12r+5)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$\frac{d}{dx}y_2(x) = Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$
$$= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)$$
$$\frac{d^2}{dx^2}y_2(x) = Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$
$$= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $x^2y'' + x^2y' - \frac{3y}{4} = 0$ gives

$$\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x^2 + x^2\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) - \frac{3Cy_1(x)\ln(x)}{4} - \frac{3\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)}{4} = 0$$

Which can be written as

$$\left(\left(y_1''(x)x^2 + x^2y_1'(x) - \frac{3y_1(x)}{4}\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x^2 + y_1(x)x\right)C + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x^2 + x^2\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) - \frac{3\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)}{4} = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x^2 + x^2 y_1'(x) - \frac{3y_1(x)}{4} = 0$$

Eq (7) simplifes to

$$\begin{aligned}&\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x^2 + y_1(x)\,x\right)C\\&+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x^2\\&+x^2\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) - \frac{3\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)}{4} = 0\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}&\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right)x + (x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C\\&+\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)x^2\\&+x^2\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) - \frac{3\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)}{4} = 0\end{aligned} \tag{9}$$

Since $r_1 = \frac{3}{2}$ and $r_2 = -\frac{1}{2}$ then the above becomes

$$\begin{aligned}&\left(2\left(\sum_{n=0}^{\infty} x^{n+\frac{1}{2}} a_n \left(n+\frac{3}{2}\right)\right)x + (x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}\right)\right)C\\&+\left(\sum_{n=0}^{\infty} x^{-\frac{5}{2}+n} b_n \left(n-\frac{1}{2}\right)\left(-\frac{3}{2}+n\right)\right)x^2\\&+x^2\left(\sum_{n=0}^{\infty} x^{-\frac{3}{2}+n} b_n \left(n-\frac{1}{2}\right)\right) - \frac{3\left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)}{4} = 0\end{aligned} \tag{10}$$

Expanding $C\,x^{\frac{5}{2}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
C\,x^{\frac{5}{2}} &= C\,x^{\frac{5}{2}} + \dots \\
&= C\,x^{\frac{5}{2}}
\end{aligned}
$$

Expanding $-C\,x^{\frac{3}{2}}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
-C\,x^{\frac{3}{2}} &= -C\,x^{\frac{3}{2}} + \dots \\
&= -C\,x^{\frac{3}{2}}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} C\,x^{n+\frac{3}{2}} a_n (2n+3)\right) + \left(\sum_{n=0}^{\infty} C\,x^{n+\frac{5}{2}} a_n\right) \\
&+ \sum_{n=0}^{\infty} \left(-C\,x^{n+\frac{3}{2}} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n-\frac{1}{2}} b_n (4n^2 - 8n + 3)}{4}\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{x^{n+\frac{1}{2}} b_n (2n-1)}{2}\right) + \sum_{n=0}^{\infty} \left(-\frac{3 b_n x^{n-\frac{1}{2}}}{4}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n - \frac{1}{2}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n-\frac{1}{2}}$ and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} C\,x^{n+\frac{3}{2}} a_n (2n+3) &= \sum_{n=2}^{\infty} C a_{n-2} (2n-1)\, x^{n-\frac{1}{2}} \\
\sum_{n=0}^{\infty} C\,x^{n+\frac{5}{2}} a_n &= \sum_{n=3}^{\infty} C a_{n-3} x^{n-\frac{1}{2}} \\
\sum_{n=0}^{\infty} \left(-C\,x^{n+\frac{3}{2}} a_n\right) &= \sum_{n=2}^{\infty} \left(-C a_{n-2} x^{n-\frac{1}{2}}\right) \\
\sum_{n=0}^{\infty} \frac{x^{n+\frac{1}{2}} b_n (2n-1)}{2} &= \sum_{n=1}^{\infty} \frac{b_{n-1} (-3+2n)\, x^{n-\frac{1}{2}}}{2}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-\frac{1}{2}$.

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} C a_{n-2}(2n-1)\, x^{n-\frac{1}{2}}\right)+\left(\sum_{n=3}^{\infty} C a_{n-3} x^{n-\frac{1}{2}}\right)\\&+\sum_{n=2}^{\infty}\left(-C a_{n-2} x^{n-\frac{1}{2}}\right)+\left(\sum_{n=0}^{\infty} \frac{x^{n-\frac{1}{2}} b_n(4n^2-8n+3)}{4}\right)\\&+\left(\sum_{n=1}^{\infty} \frac{b_{n-1}(-3+2n)\, x^{n-\frac{1}{2}}}{2}\right)+\sum_{n=0}^{\infty}\left(-\frac{3 b_n x^{n-\frac{1}{2}}}{4}\right)=0\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_1-\frac{b_0}{2}=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1-\frac{1}{2}=0$$

Solving the above for b_1 gives

$$b_1=-\frac{1}{2}$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C-\frac{1}{4}=0$$

Which is solved for C. Solving for C gives

$$C=\frac{1}{8}$$

For $n=3$, Eq (2B) gives

$$(a_0+4a_1)\, C+\frac{3b_2}{2}+3b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1}{8}+3b_3=0$$

Solving the above for b_3 gives

$$b_3=\frac{1}{24}$$

For $n = 4$, Eq (2B) gives

$$(a_1 + 6a_2)\, C + \frac{5b_3}{2} + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{61}{384} + 8b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{61}{3072}$$

For $n = 5$, Eq (2B) gives

$$(a_2 + 8a_3)\, C + \frac{7b_4}{2} + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{177}{2048} + 15b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{59}{10240}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = \frac{1}{8}$ and all b_n, then the second solution becomes

$$y_2(x) = \frac{1}{8}\left(x^{\frac{3}{2}}\left(1 - \frac{x}{2} + \frac{5x^2}{32} - \frac{7x^3}{192} + \frac{7x^4}{1024} - \frac{11x^5}{10240} + O\left(x^6\right)\right)\right)\ln(x) + \frac{1 - \frac{x}{2} + \frac{x^3}{24} - \frac{61x^4}{3072} + \frac{59x^5}{10240} + O(x^6)}{\sqrt{x}}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{3}{2}}\left(1 - \frac{x}{2} + \frac{5x^2}{32} - \frac{7x^3}{192} + \frac{7x^4}{1024} - \frac{11x^5}{10240} + O\left(x^6\right)\right) \\ &\quad + c_2\left(\frac{1}{8}\left(x^{\frac{3}{2}}\left(1 - \frac{x}{2} + \frac{5x^2}{32} - \frac{7x^3}{192} + \frac{7x^4}{1024} - \frac{11x^5}{10240} + O\left(x^6\right)\right)\right)\ln(x) \right. \\ &\quad \left. + \frac{1 - \frac{x}{2} + \frac{x^3}{24} - \frac{61x^4}{3072} + \frac{59x^5}{10240} + O(x^6)}{\sqrt{x}}\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{2}} \left(1-\frac{x}{2}+\frac{5 x^2}{32}-\frac{7 x^3}{192}+\frac{7 x^4}{1024}-\frac{11 x^5}{10240}+O\left(x^6\right)\right) \\
&\quad + c_2 \left(\frac{x^{\frac{3}{2}} \left(1-\frac{x}{2}+\frac{5 x^2}{32}-\frac{7 x^3}{192}+\frac{7 x^4}{1024}-\frac{11 x^5}{10240}+O(x^6)\right) \ln \left(x\right)}{8} \right. \\
&\quad \left. + \frac{1-\frac{x}{2}+\frac{x^3}{24}-\frac{61 x^4}{3072}+\frac{59 x^5}{10240}+O(x^6)}{\sqrt{x}}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{3}{2}} \left(1-\frac{x}{2}+\frac{5 x^2}{32}-\frac{7 x^3}{192}+\frac{7 x^4}{1024}-\frac{11 x^5}{10240}+O\left(x^6\right)\right) \\
&\quad + c_2 \left(\frac{x^{\frac{3}{2}} \left(1-\frac{x}{2}+\frac{5 x^2}{32}-\frac{7 x^3}{192}+\frac{7 x^4}{1024}-\frac{11 x^5}{10240}+O(x^6)\right) \ln \left(x\right)}{8} \right. \\
&\quad \left. + \frac{1-\frac{x}{2}+\frac{x^3}{24}-\frac{61 x^4}{3072}+\frac{59 x^5}{10240}+O(x^6)}{\sqrt{x}}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x^{\frac{3}{2}} \left(1-\frac{x}{2}+\frac{5 x^2}{32}-\frac{7 x^3}{192}+\frac{7 x^4}{1024}-\frac{11 x^5}{10240}+O\left(x^6\right)\right) \\
&\quad + c_2 \left(\frac{x^{\frac{3}{2}} \left(1-\frac{x}{2}+\frac{5 x^2}{32}-\frac{7 x^3}{192}+\frac{7 x^4}{1024}-\frac{11 x^5}{10240}+O(x^6)\right) \ln \left(x\right)}{8} \right. \\
&\quad \left. + \frac{1-\frac{x}{2}+\frac{x^3}{24}-\frac{61 x^4}{3072}+\frac{59 x^5}{10240}+O(x^6)}{\sqrt{x}}\right)
\end{aligned}
$$

Verified OK.

15.22.1 Maple step by step solution

Let's solve

$y''x^2 + x^2y' - \frac{3y}{4} = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' + \frac{3y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' - \frac{3y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = 1, P_3(x) = -\frac{3}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\bigg|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\bigg|_{x=0} = -\frac{3}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y' + 4y''x^2 - 3y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^2 \cdot y'$ to series expansion

    $x^2 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+1}$

  ◦ Shift index using $k- >k-1$

$$x^2 \cdot y' = \sum_{k=1}^{\infty} a_{k-1}(k-1+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k-1+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(1+2r)\,(-3+2r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-3)+4a_{k-1}(k-1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$(1+2r)\,(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{-\frac{1}{2}, \frac{3}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$4\left(k+r-\frac{3}{2}\right)\left(k+r+\frac{1}{2}\right) a_k + 4a_{k-1}(k-1+r) = 0$$

- Shift index using $k- >k+1$

  $$4\left(k-\frac{1}{2}+r\right)\left(k+\frac{3}{2}+r\right) a_{k+1} + 4a_k(k+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{4a_k(k+r)}{(2k-1+2r)(2k+3+2r)}$$

- Recursion relation for $r = -\frac{1}{2}$

  $$a_{k+1} = -\frac{4a_k\left(k-\frac{1}{2}\right)}{(2k-2)(2k+2)}$$

- Series not valid for $r = -\frac{1}{2}$, division by 0 in the recursion relation at $k = 1$

  $$a_{k+1} = -\frac{4a_k\left(k-\frac{1}{2}\right)}{(2k-2)(2k+2)}$$

- Recursion relation for $r = \frac{3}{2}$

  $$a_{k+1} = -\frac{4a_k\left(k+\frac{3}{2}\right)}{(2k+2)(2k+6)}$$

- Solution for $r = \frac{3}{2}$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{4a_k\left(k+\frac{3}{2}\right)}{(2k+2)(2k+6)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 65

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x^2*diff(y(x),x)-3/4*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 - \frac{1}{2}x + \frac{5}{32}x^2 - \frac{7}{192}x^3 + \frac{7}{1024}x^4 - \frac{11}{10240}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(-\frac{1}{4}x^2 + \frac{1}{8}x^3 - \frac{5}{128}x^4 + \frac{7}{768}x^5 + \mathrm{O}\right.\right.}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 103

```
AsymptoticDSolveValue[x^2*y''[x]+x^2*y'[x]-3/4*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{7x^{11/2}}{1024} - \frac{7x^{9/2}}{192} + \frac{5x^{7/2}}{32} - \frac{x^{5/2}}{2} + x^{3/2}\right) + c_1\left(\frac{1}{256}x^{3/2}\left(5x^2 - 16x + 32\right)\log(x) - \frac{91x^4 - 224x^3 + 192x^2 + 1536x - 3072}{3072\sqrt{x}}\right)$$

15.23 problem 23

Internal problem ID [11924]
Internal file name [`OUTPUT/11933_Saturday_April_13_2024_10_26_33_PM_41898295/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y' + 2y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{2}{x}$$

Table 571: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2a_n x^{n+r} = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{2a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{2a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{2}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{(r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2
a_2	$\frac{4}{(r+1)^2(r+2)^2}$	1

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{8}{(r+1)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{2}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2
a_2	$\frac{4}{(r+1)^2(r+2)^2}$	1
a_3	$-\frac{8}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{2}{9}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2
a_2	$\frac{4}{(r+1)^2(r+2)^2}$	1
a_3	$-\frac{8}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{2}{9}$
a_4	$\frac{16}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{36}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{32}{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{450}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{2}{(r+1)^2}$	-2
a_2	$\frac{4}{(r+1)^2(r+2)^2}$	1
a_3	$-\frac{8}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{2}{9}$
a_4	$\frac{16}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{36}$
a_5	$-\frac{32}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$-\frac{1}{450}$

Using the above table, then the first solution $y_1(x)$ becomes

$$y_1(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots$$
$$= x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O(x^6)$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$-\frac{2}{(r+1)^2}$	-2	$\frac{4}{(r+1)^3}$	4
b_2	$\frac{4}{(r+1)^2(r+2)^2}$	1	$\frac{-16r-24}{(r+1)^3(r+2)^3}$	-3
b_3	$-\frac{8}{(r+1)^2(r+2)^2(r+3)^2}$	$-\frac{2}{9}$	$\frac{48r^2+192r+176}{(r+1)^3(r+2)^3(r+3)^3}$	$\frac{22}{27}$
b_4	$\frac{16}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{36}$	$-\frac{64(2r^3+15r^2+35r+25)}{(r+1)^3(r+2)^3(r+3)^3(r+4)^3}$	$-\frac{25}{216}$
b_5	$-\frac{32}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$-\frac{1}{450}$	$\frac{320r^4+3840r^3+16320r^2+28800r+17536}{(r+1)^3(r+2)^3(r+3)^3(r+4)^3(r+5)^3}$	$\frac{137}{13500}$

The above table gives all values of b_n needed. Hence the second solution is

$$y_2(x) = y_1(x) \ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots$$
$$= \left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O(x^6)\right) \ln(x) - 3x^2 + 4x + \frac{22x^3}{27} - \frac{25x^4}{216} + \frac{137x^5}{13500}$$
$$+ O(x^6)$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right) \\
&\quad + c_2 \left(\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right) \ln\left(x\right) - 3x^2 + 4x + \frac{22x^3}{27} \right. \\
&\quad \left. - \frac{25x^4}{216} + \frac{137x^5}{13500} + O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right) \\
&\quad + c_2 \left(\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right) \ln\left(x\right) - 3x^2 + 4x + \frac{22x^3}{27} - \frac{25x^4}{216} \right. \\
&\quad \left. + \frac{137x^5}{13500} + O\left(x^6\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right) \\
&\quad + c_2 \left(\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right) \ln\left(x\right) - 3x^2 + 4x + \frac{22x^3}{27} \right. \\
&\quad \left. - \frac{25x^4}{216} + \frac{137x^5}{13500} + O\left(x^6\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 \left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right) \\
&\quad + c_2 \left(\left(x^2 - 2x + 1 - \frac{2x^3}{9} + \frac{x^4}{36} - \frac{x^5}{450} + O\left(x^6\right)\right) \ln\left(x\right) - 3x^2 + 4x + \frac{22x^3}{27} - \frac{25x^4}{216} \right. \\
&\quad \left. + \frac{137x^5}{13500} + O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

15.23.1 Maple step by step solution

Let's solve

$y''x + y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{2y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{2}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x + y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  ◦ Shift index using $k- > k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + 2a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r^2 = 0$$

- Values of r that satisfy the indicial equation

  $$r = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1)^2 + 2a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = -\frac{2a_k}{(k+1)^2}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = -\frac{2a_k}{(k+1)^2}$$

- Solution for $r = 0$

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{2a_k}{(k+1)^2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 59

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_1 + c_2 \ln(x)) \left(1 - 2x + x^2 - \frac{2}{9}x^3 + \frac{1}{36}x^4 - \frac{1}{450}x^5 + \mathrm{O}(x^6)\right) + \left(4x - 3x^2 + \frac{22}{27}x^3 - \frac{25}{216}x^4 + \frac{137}{13500}x^5 + \mathrm{O}(x^6)\right) c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 101

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{x^5}{450} + \frac{x^4}{36} - \frac{2x^3}{9} + x^2 - 2x + 1\right) + c_2 \left(\frac{137x^5}{13500} - \frac{25x^4}{216} + \frac{22x^3}{27} - 3x^2 + \left(-\frac{x^5}{450} + \frac{x^4}{36} - \frac{2x^3}{9} + x^2 - 2x + 1\right) \log(x) + 4x\right)$$

15.24 problem 24

Internal problem ID [11925]
Internal file name [`OUTPUT/11934_Saturday_April_13_2024_10_26_34_PM_44280382/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{2xy'' + 6y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + 6y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = \frac{1}{2x}$$

Table 573: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + 6y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + 6\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6(n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 6(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + 6(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + 6r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + 6r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2r\, x^{-1+r}(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$2r\, x^{-1+r}(2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2a_n(n+r)(n+r-1) + 6a_n(n+r) + a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-1}}{2(n^2 + 2nr + r^2 + 2n + 2r)} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
a_n = -\frac{a_{n-1}}{2n(n+2)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{2r^2 + 8r + 6}$$

Which for the root $r = 0$ becomes

$$a_1 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+8r+6}$	$-\frac{1}{6}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 40r^3 + 140r^2 + 200r + 96}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{96}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+8r+6}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^4+40r^3+140r^2+200r+96}$	$\frac{1}{96}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{8\left(r+3\right)^2 \left(r+1\right) \left(r+4\right) \left(2+r\right) \left(r+5\right)}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{1}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+8r+6}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^4+40r^3+140r^2+200r+96}$	$\frac{1}{96}$
a_3	$-\frac{1}{8(r+3)^2(r+1)(r+4)(2+r)(r+5)}$	$-\frac{1}{2880}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16\left(r+3\right)^2\left(r+1\right)\left(r+4\right)^2\left(2+r\right)\left(r+5\right)\left(r+6\right)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{138240}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+8r+6}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^4+40r^3+140r^2+200r+96}$	$\frac{1}{96}$
a_3	$-\frac{1}{8(r+3)^2(r+1)(r+4)(2+r)(r+5)}$	$-\frac{1}{2880}$
a_4	$\frac{1}{16(r+3)^2(r+1)(r+4)^2(2+r)(r+5)(r+6)}$	$\frac{1}{138240}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{32\left(r+3\right)^2\left(r+1\right)\left(r+4\right)^2\left(2+r\right)\left(r+5\right)^2\left(r+6\right)\left(r+7\right)}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{9676800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+8r+6}$	$-\frac{1}{6}$
a_2	$\frac{1}{4r^4+40r^3+140r^2+200r+96}$	$\frac{1}{96}$
a_3	$-\frac{1}{8(r+3)^2(r+1)(r+4)(2+r)(r+5)}$	$-\frac{1}{2880}$
a_4	$\frac{1}{16(r+3)^2(r+1)(r+4)^2(2+r)(r+5)(r+6)}$	$\frac{1}{138240}$
a_5	$-\frac{1}{32(r+3)^2(r+1)(r+4)^2(2+r)(r+5)^2(r+6)(r+7)}$	$-\frac{1}{9676800}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots \\
&= 1 - \frac{x}{6} + \frac{x^2}{96} - \frac{x^3}{2880} + \frac{x^4}{138240} - \frac{x^5}{9676800} + O(x^6)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_2 \\
&= \frac{1}{4r^4 + 40r^3 + 140r^2 + 200r + 96}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} \frac{1}{4r^4 + 40r^3 + 140r^2 + 200r + 96} &= \lim_{r \to -2} \frac{1}{4r^4 + 40r^3 + 140r^2 + 200r + 96} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right)
$$

Therefore

$$\frac{d}{dx}y_2(x) = Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)$$
$$= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right)$$

$$\frac{d^2}{dx^2}y_2(x) = Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)$$
$$= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)$$

Substituting these back into the given ode $2xy'' + 6y' + y = 0$ gives

$$2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + 6Cy_1'(x)\ln(x) + \frac{6Cy_1(x)}{x} + 6\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Which can be written as

$$\left((2y_1''(x)\,x + y_1(x) + 6y_1'(x))\ln(x) + 2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + \frac{6y_1(x)}{x}\right)C + 2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + 6\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$2y_1''(x)\,x + y_1(x) + 6y_1'(x) = 0$$

Eq (7) simplifes to

$$\left(2\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x+\frac{6y_1(x)}{x}\right)C \\ +2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\ +6\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\frac{\left(4\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x+4\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x} \\ +\frac{2\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x+6\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x}{x} \\ =0 \tag{9}$$

Since $r_1=0$ and $r_2=-2$ then the above becomes

$$\frac{\left(4\left(\sum_{n=0}^{\infty}x^{n-1}a_n n\right)x+4\left(\sum_{n=0}^{\infty}a_n x^{n}\right)\right)C}{x} \\ +\frac{2\left(\sum_{n=0}^{\infty}x^{-4+n}b_n(n-2)(-3+n)\right)x^2+\left(\sum_{n=0}^{\infty}b_n x^{n-2}\right)x+6\left(\sum_{n=0}^{\infty}x^{-3+n}b_n(n-2)\right)x}{x} \\ =0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}4C\,x^{n-1}a_n n\right)+\left(\sum_{n=0}^{\infty}4C\,x^{n-1}a_n\right)+\left(\sum_{n=0}^{\infty}2x^{-3+n}b_n(n-2)(-3+n)\right) \\ +\left(\sum_{n=0}^{\infty}b_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty}6x^{-3+n}b_n(n-2)\right)=0 \tag{2A}$$

The next step is to make all powers of x be $-3+n$ in each summation term. Going over each summation term above with power of x in it which is not already x^{-3+n} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4C\, x^{n-1} a_n n = \sum_{n=2}^{\infty} 4C(n-2)\, a_{n-2} x^{-3+n}$$

$$\sum_{n=0}^{\infty} 4C\, x^{n-1} a_n = \sum_{n=2}^{\infty} 4C a_{n-2} x^{-3+n}$$

$$\sum_{n=0}^{\infty} b_n x^{n-2} = \sum_{n=1}^{\infty} b_{n-1} x^{-3+n}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $-3+n$.

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} 4C(n-2)\, a_{n-2} x^{-3+n}\right) + \left(\sum_{n=2}^{\infty} 4C a_{n-2} x^{-3+n}\right) \\ &+ \left(\sum_{n=0}^{\infty} 2x^{-3+n} b_n (n-2)\,(-3+n)\right) \\ &+ \left(\sum_{n=1}^{\infty} b_{n-1} x^{-3+n}\right) + \left(\sum_{n=0}^{\infty} 6x^{-3+n} b_n (n-2)\right) = 0 \end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-2b_1 + b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_1 + 1 = 0$$

Solving the above for b_1 gives

$$b_1 = \frac{1}{2}$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2 = 0$. Hence for $n=2$, Eq (2B) gives

$$4C + \frac{1}{2} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{8}$$

For $n = 3$, Eq (2B) gives

$$8Ca_1 + b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6b_3 + \frac{1}{6} = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{1}{36}$$

For $n = 4$, Eq (2B) gives

$$12Ca_2 + b_3 + 16b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$16b_4 - \frac{25}{576} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{25}{9216}$$

For $n = 5$, Eq (2B) gives

$$16Ca_3 + b_4 + 30b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$30b_5 + \frac{157}{46080} = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{157}{1382400}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{8}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{8}\left(1 - \frac{x}{6} + \frac{x^2}{96} - \frac{x^3}{2880} + \frac{x^4}{138240} - \frac{x^5}{9676800} + O(x^6)\right)\ln(x) + \frac{1 + \frac{x}{2} - \frac{x^3}{36} + \frac{25x^4}{9216} - \frac{157x^5}{1382400} + O(x^6)}{x^2}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1-\frac{x}{6}+\frac{x^2}{96}-\frac{x^3}{2880}+\frac{x^4}{138240}-\frac{x^5}{9676800}+O\left(x^6\right)\right) \\
&\quad + c_2 \left(-\frac{1}{8}\left(1-\frac{x}{6}+\frac{x^2}{96}-\frac{x^3}{2880}+\frac{x^4}{138240}-\frac{x^5}{9676800}+O\left(x^6\right)\right)\ln\left(x\right) \right. \\
&\quad \left. + \frac{1+\frac{x}{2}-\frac{x^3}{36}+\frac{25x^4}{9216}-\frac{157x^5}{1382400}+O(x^6)}{x^2}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \left(1-\frac{x}{6}+\frac{x^2}{96}-\frac{x^3}{2880}+\frac{x^4}{138240}-\frac{x^5}{9676800}+O\left(x^6\right)\right) \\
&\quad + c_2 \left(\left(-\frac{1}{8}+\frac{x}{48}-\frac{x^2}{768}+\frac{x^3}{23040}-\frac{x^4}{1105920}+\frac{x^5}{77414400}-\frac{O(x^6)}{8}\right)\ln\left(x\right) \right. \\
&\quad \left. + \frac{1+\frac{x}{2}-\frac{x^3}{36}+\frac{25x^4}{9216}-\frac{157x^5}{1382400}+O(x^6)}{x^2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \left(1-\frac{x}{6}+\frac{x^2}{96}-\frac{x^3}{2880}+\frac{x^4}{138240}-\frac{x^5}{9676800}+O\left(x^6\right)\right) \\
&\quad + c_2 \left(\left(-\frac{1}{8}+\frac{x}{48}-\frac{x^2}{768}+\frac{x^3}{23040}-\frac{x^4}{1105920}+\frac{x^5}{77414400}-\frac{O(x^6)}{8}\right)\ln\left(x\right) \right. \\
&\quad \left. + \frac{1+\frac{x}{2}-\frac{x^3}{36}+\frac{25x^4}{9216}-\frac{157x^5}{1382400}+O(x^6)}{x^2}\right) \qquad (1)
\end{aligned}
$$

Verification of solutions

$$y = c_1\left(1-\frac{x}{6}+\frac{x^2}{96}-\frac{x^3}{2880}+\frac{x^4}{138240}-\frac{x^5}{9676800}+O(x^6)\right) + c_2\left(\left(-\frac{1}{8}+\frac{x}{48}-\frac{x^2}{768}+\frac{x^3}{23040}-\frac{x^4}{1105920}+\frac{x^5}{77414400}-\frac{O(x^6)}{8}\right)\ln(x) + \frac{1+\frac{x}{2}-\frac{x^3}{36}+\frac{25x^4}{9216}-\frac{157x^5}{1382400}+O(x^6)}{x^2}\right)$$

Verified OK.

15.24.1 Maple step by step solution

Let's solve

$2y''x+6y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{3y'}{x}-\frac{y}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{3y'}{x}+\frac{y}{2x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3}{x}, P_3(x)=\frac{1}{2x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=3$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

$2y''x+6y'+y=0$

- Assume series solution for y

$y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert y' to series expansion

  $y'=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  - Shift index using $k->k+1$

  $y'=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

  - Convert $x\cdot y''$ to series expansion

  $x\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k->k+1$

  $x\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

$2a_0r(2+r)\, x^{-1+r}+\left(\sum_{k=0}^{\infty} (2a_{k+1}(k+1+r)\,(k+3+r)+a_k)\, x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$2r(2+r)=0$

- Values of r that satisfy the indicial equation

$r\in\{-2,0\}$

- Each term in the series must be 0, giving the recursion relation

$2a_{k+1}(k+1+r)\,(k+3+r)+a_k=0$

- Recursion relation that defines series solution to ODE

$a_{k+1}=-\frac{a_k}{2(k+1+r)(k+3+r)}$

- Recursion relation for $r=-2$

$a_{k+1}=-\frac{a_k}{2(k-1)(k+1)}$

- Series not valid for $r=-2$, division by 0 in the recursion relation at $k=1$

$$a_{k+1} = -\frac{a_k}{2(k-1)(k+1)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k}{2(k+1)(k+3)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{2(k+1)(k+3)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 62

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)+6*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 \left(1 - \frac{1}{6}x + \frac{1}{96}x^2 - \frac{1}{2880}x^3 + \frac{1}{138240}x^4 - \frac{1}{9676800}x^5 + \mathrm{O}\left(x^6\right)\right) x^2 + c_2 \left(\ln\left(x\right)\left(\frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{384}x^4 - \frac{1}{11520}x\right.\right.}{x^2}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 80

```
AsymptoticDSolveValue[2*x*y''[x]+6*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^4}{138240}-\frac{x^3}{2880}+\frac{x^2}{96}-\frac{x}{6}+1\right)+c_1\left(\frac{31x^4-352x^3+576x^2+4608x+9216}{9216x^2}-\frac{1}{768}\left(x^2-16x+96\right)\log(x)\right)$$

15.25 problem 25

Internal problem ID [11926]
Internal file name [`OUTPUT/11935_Saturday_April_13_2024_10_26_35_PM_62176316/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - y'x + (x^2+1)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - y'x + (x^2+1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{x^2+1}{x^2}$$

Table 575: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2+1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' - y' x + (x^2 + 1) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) x^2 - \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) x + (x^2 + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_n(n+r) + a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 2n - 2r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{(1+r)^2}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(1+r)^2 (r+3)^2}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(1+r)^2(r+3)^2}$	$\frac{1}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(1+r)^2(r+3)^2}$	$\frac{1}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$y_1(x) = x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right)$$
$$= x\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$	$\frac{2}{(1+r)^3}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{(1+r)^2(r+3)^2}$	$\frac{1}{64}$	$\frac{-4r-8}{(1+r)^3(r+3)^3}$	$-\frac{3}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= x\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) + x\left(\frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) + x\left(\frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right) \\
&\quad + c_2 \left(x \left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right) \ln(x) + x \left(\frac{x^2}{4}-\frac{3x^4}{128}+O(x^6)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right) \\
&\quad + c_2 \left(x \left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right) \ln(x) + x \left(\frac{x^2}{4}-\frac{3x^4}{128}+O(x^6)\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x \left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right) \\
&\quad + c_2 \left(x \left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right) \ln(x) + x \left(\frac{x^2}{4}-\frac{3x^4}{128}+O(x^6)\right)\right)
\end{aligned}
$$

Verified OK.

15.25.1 Maple step by step solution

Let's solve

$y''x^2 - y'x + (x^2+1) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2+1)y}{x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{(x^2+1)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ○ Define functions

$\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 1$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$y''x^2 - y'x + (x^2+1)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k-m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

  $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

  $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0(-1+r)^2\, x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k+r-1)^2 + a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $a_1 r^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-1)^2 + a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+1+r)^2 + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)^2}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)^2}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+(x^2+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left((c_1 + c_2 \ln(x)) \left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \mathrm{O}\left(x^6\right)\right) c_2\right) x$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 65

```
AsymptoticDSolveValue[x^2*y''[x]-x*y'[x]+(x^2+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(\frac{x^4}{64} - \frac{x^2}{4} + 1\right) + c_2 \left(x \left(\frac{x^2}{4} - \frac{3x^4}{128}\right) + x \left(\frac{x^4}{64} - \frac{x^2}{4} + 1\right) \log(x)\right)$$

15.26 problem 26

Internal problem ID [11927]
Internal file name [`OUTPUT/11936_Saturday_April_13_2024_10_26_36_PM_90976062/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 6, Series solutions of linear differential equations. Section 6.2 (Frobenius). Exercises page 251
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - y'x + (x^2 - 3)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - y'x + (x^2 - 3)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{x^2 - 3}{x^2} \end{aligned}$$

Table 577: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-3}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - y'x + (x^2 - 3)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2 - \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x + (x^2 - 3) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-3 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-3 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) - 3 a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r - 3 a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r - 3x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 2r - 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 2r - 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 2r - 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 4$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^3 \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_n(n+r) + a_{n-2} - 3a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 2n - 2r - 3} \tag{4}$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+4)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 2r - 3}$$

Which for the root $r = 3$ becomes

$$a_2 = -\frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+2r-3}$	$-\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+2r-3}$	$-\frac{1}{12}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 + 8r^3 + 14r^2 - 8r - 15}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{1}{384}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+2r-3}$	$-\frac{1}{12}$
a_3	0	0
a_4	$\frac{1}{r^4+8r^3+14r^2-8r-15}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+2r-3}$	$-\frac{1}{12}$
a_3	0	0
a_4	$\frac{1}{r^4+8r^3+14r^2-8r-15}$	$\frac{1}{384}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^3 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^3 \left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 4$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\rightarrow r_2} a_4(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_4 \\ &= \frac{1}{r^4 + 8r^3 + 14r^2 - 8r - 15} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\rightarrow r_2} \frac{1}{r^4 + 8r^3 + 14r^2 - 8r - 15} &= \lim_{r\rightarrow -1} \frac{1}{r^4 + 8r^3 + 14r^2 - 8r - 15} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= Cy_1'(x) \ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n + r_2)}{x}\right) \\ &= Cy_1'(x) \ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n + r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= Cy_1''(x) \ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n + r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n + r_2)}{x^2}\right) \\ &= Cy_1''(x) \ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n + r_2)(-1 + n + r_2)\right) \end{aligned}$$

Substituting these back into the given ode $x^2 y'' - y' x + (x^2 - 3) y = 0$ gives

$$\left(C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) x^2 - \left(C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right)\right) x + (x^2 - 3) \left(C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0$$

Which can be written as

$$\left(\left(y_1''(x) x^2 - y_1'(x) x + (x^2 - 3) y_1(x)\right) \ln(x) + \left(\frac{2 y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) x^2 - y_1(x)\right) C + \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) x^2 - \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) x + (x^2 - 3) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x) x^2 - y_1'(x) x + (x^2 - 3) y_1(x) = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2 y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) x^2 - y_1(x)\right) C + \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2}\right)\right) x^2 - \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x}\right) x + (x^2 - 3) \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 \\
&- \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x - 3\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}$$

Since $r_1 = 3$ and $r_2 = -1$ then the above becomes

$$\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{2+n} a_n (n+3)\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^{n+3}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) x^2 + \left(\sum_{n=0}^{\infty} x^{-3+n} b_n (n-1)(n-2)\right) x^2 \\
&- \left(\sum_{n=0}^{\infty} x^{n-2} b_n (n-1)\right) x - 3\left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (n+3)\right) + \sum_{n=0}^{\infty} \left(-2C a_n x^{n+3}\right) \\
&+ \left(\sum_{n=0}^{\infty} x^{1+n} b_n\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n (n^2 - 3n + 2)\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-3 b_n x^{n-1}\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (n+3) = \sum_{n=4}^{\infty} 2C a_{n-4} (n-1)\, x^{n-1}$$

$$\sum_{n=0}^{\infty} \left(-2C a_n x^{n+3}\right) = \sum_{n=4}^{\infty} \left(-2C a_{n-4} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n} b_n = \sum_{n=2}^{\infty} b_{n-2} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\left(\sum_{n=4}^{\infty} 2C a_{n-4}(n-1)\, x^{n-1}\right) + \sum_{n=4}^{\infty} \left(-2C a_{n-4} x^{n-1}\right) \\
&+ \left(\sum_{n=2}^{\infty} b_{n-2} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} x^{n-1} b_n \left(n^2 - 3n + 2\right)\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-3 b_n x^{n-1}\right) = 0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-3b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n=2$, Eq (2B) gives

$$b_0 - 4b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 - 4b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{4}$$

For $n = 3$, Eq (2B) gives

$$b_1 - 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-3b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = N$, where $N = 4$ which is the difference between the two roots, we are free to choose $b_4 = 0$. Hence for $n = 4$, Eq (2B) gives

$$4C + \frac{1}{4} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{16}$$

For $n = 5$, Eq (2B) gives

$$6Ca_1 + b_3 + 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{16}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{16}\left(x^3\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right)\right)\ln(x) + \frac{1 + \frac{x^2}{4} + O(x^6)}{x}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^3\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right) \\
&\quad + c_2\left(-\frac{1}{16}\left(x^3\left(1 - \frac{x^2}{12} + \frac{x^4}{384} + O(x^6)\right)\right)\ln(x) + \frac{1 + \frac{x^2}{4} + O(x^6)}{x}\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^3 \left(1-\frac{x^2}{12}+\frac{x^4}{384}+O(x^6)\right) \\
&+ c_2 \left(-\frac{x^3\left(1-\frac{x^2}{12}+\frac{x^4}{384}+O(x^6)\right)\ln(x)}{16}+\frac{1+\frac{x^2}{4}+O(x^6)}{x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 x^3 \left(1-\frac{x^2}{12}+\frac{x^4}{384}+O(x^6)\right) \\
+ c_2 \left(-\frac{x^3\left(1-\frac{x^2}{12}+\frac{x^4}{384}+O(x^6)\right)\ln(x)}{16}+\frac{1+\frac{x^2}{4}+O(x^6)}{x}\right)
\end{aligned}
\tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x^3 \left(1-\frac{x^2}{12}+\frac{x^4}{384}+O(x^6)\right) \\
+ c_2 \left(-\frac{x^3\left(1-\frac{x^2}{12}+\frac{x^4}{384}+O(x^6)\right)\ln(x)}{16}+\frac{1+\frac{x^2}{4}+O(x^6)}{x}\right)
\end{aligned}
$$

Verified OK.

15.26.1 Maple step by step solution

Let's solve

$y''x^2 - y'x + (x^2-3)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x^2-3)y}{x^2} + \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{(x^2-3)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{x^2-3}{x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -1$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = -3$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x^2 - y'x + (x^2 - 3)\, y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\, (k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

$$a_0(1+r)(-3+r)x^r + a_1(2+r)(-2+r)x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+1)(k+r-3) + a_{k-2}\right) x^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+r)(-3+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 3\}$
- Each term must be 0

  $a_1(2+r)(-2+r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+1)(k+r-3) + a_{k-2} = 0$
- Shift index using $k- >k+2$

  $a_{k+2}(k+3+r)(k+r-1) + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+3+r)(k+r-1)}$
- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+2)(k-2)}$
- Series not valid for $r = -1$, division by 0 in the recursion relation at $k = 2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k-2)}$
- Recursion relation for $r = 3$

  $a_{k+2} = -\frac{a_k}{(k+6)(k+2)}$
- Solution for $r = 3$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{a_k}{(k+6)(k+2)}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 47

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+(x^2-3)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^4 \left(1 - \frac{1}{12}x^2 + \frac{1}{384}x^4 + \mathrm{O}\left(x^6\right)\right) + c_2\left(\ln\left(x\right)\left(9x^4 + \mathrm{O}\left(x^6\right)\right) + \left(-144 - 36x^2 + \mathrm{O}\left(x^6\right)\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 52

```
AsymptoticDSolveValue[x^2*y''[x]-x*y'[x]+(x^2-3)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{\left(x^2+8\right)^2}{64x} - \frac{1}{16}x^3 \log(x)\right) + c_2\left(\frac{x^7}{384} - \frac{x^5}{12} + x^3\right)$$

16 Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277

16.1 problem 1

Internal problem ID [11928]
Internal file name [OUTPUT/11937_Saturday_April_13_2024_10_26_37_PM_64304397/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) + y'(t) = 2x(t) + 4y(t) + e^t$$
$$x'(t) + y'(t) = y(t) + e^{4t}$$

The system is

$$x'(t) + y'(t) = 2x(t) + 4y(t) + e^t \tag{1}$$
$$x'(t) + y'(t) = y(t) + e^{4t} \tag{2}$$

Since the left side is the same, this implies

$$2x(t) + 4y(t) + e^t = y(t) + e^{4t}$$
$$y(t) = \frac{e^{4t}}{3} - \frac{2x(t)}{3} - \frac{e^t}{3} \tag{3}$$

Taking derivative of the above w.r.t. t gives

$$y'(t) = \frac{4\,e^{4t}}{3} - \frac{2x'(t)}{3} - \frac{e^t}{3} \tag{4}$$

Substituting (3,4) in (1) to eliminate $y(t)\,, y'(t)$ gives

$$\frac{x'(t)}{3} + \frac{4\,e^{4t}}{3} - \frac{e^t}{3} = -\frac{2x(t)}{3} + \frac{4\,e^{4t}}{3} - \frac{e^t}{3}$$
$$x'(t) = -2x(t) \tag{5}$$

Which is now solved for $x(t)$. Integrating both sides gives

$$\int -\frac{1}{2x}dx = \int dt$$
$$-\frac{\ln(x)}{2} = t + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{x}} = \mathrm{e}^{t+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{x}} = c_2 \mathrm{e}^{t}$$

Given now that we have the solution

$$x(t) = \frac{\mathrm{e}^{-2t}}{c_2^2} \tag{6}$$

Then substituting (6) into (3) gives

$$y(t) = \frac{\left(\mathrm{e}^{6t} c_2^2 - c_2^2 \mathrm{e}^{3t} - 2\right) \mathrm{e}^{-2t}}{3 c_2^2} \tag{7}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
dsolve([diff(x(t),t)+diff(y(t),t)-2*x(t)-4*y(t)=exp(t),diff(x(t),t)+diff(y(t),t)-y(t)=exp(4*
```

$$x(t) = c_1 \mathrm{e}^{-2t}$$
$$y(t) = \frac{\mathrm{e}^{4t}}{3} - \frac{\mathrm{e}^{t}}{3} - \frac{2c_1 \mathrm{e}^{-2t}}{3}$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 52

```
DSolve[{x'[t]+y'[t]-2*x[t]-4*y[t]==Exp[t],x'[t]+y'[t]-y[t]==Exp[4*t]},{x[t],y[t]},t,IncludeS
```

$$x(t) \to \frac{1}{12}(3 + 4c_1)e^{-2t}$$
$$y(t) \to \frac{1}{18}e^{-2t}\left(-6e^{3t} + 6e^{6t} - 3 - 4c_1\right)$$

16.2 problem 2

Internal problem ID [11929]
Internal file name [`OUTPUT/11938_Saturday_April_13_2024_10_26_37_PM_14908028/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) + y'(t) &= x(t) - 2t \\
x'(t) + y'(t) &= t^2 + 3x(t) + y(t)
\end{aligned}
$$

The system is

$$
x'(t) + y'(t) = x(t) - 2t \tag{1}
$$

$$
x'(t) + y'(t) = t^2 + 3x(t) + y(t) \tag{2}
$$

Since the left side is the same, this implies

$$
\begin{aligned}
x(t) - 2t &= t^2 + 3x(t) + y(t) \\
y(t) &= -t^2 - 2x(t) - 2t \qquad (3)
\end{aligned}
$$

Taking derivative of the above w.r.t. t gives

$$
y'(t) = -2t - 2x'(t) - 2 \tag{4}
$$

Substituting (3,4) in (1) to eliminate $y(t), y'(t)$ gives

$$
\begin{aligned}
-x'(t) - 2t - 2 &= x(t) - 2t \\
x'(t) &= -x(t) - 2 \qquad (5)
\end{aligned}
$$

Which is now solved for $x(t)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{-x-2} dx &= \int dt \\
-\ln(-x-2) &= t + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$\frac{1}{-x-2} = e^{t+c_1}$$

Which simplifies to

$$\frac{1}{-x-2} = c_2 e^t$$

Given now that we have the solution

$$x(t) = -\frac{e^{-t}}{c_2} - 2 \tag{6}$$

Then substituting (6) into (3) gives

$$y(t) = -\frac{(t^2 c_2 e^t + 2c_2 t\, e^t - 4c_2 e^t - 2)\, e^{-t}}{c_2} \tag{7}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve([diff(x(t),t)+diff(y(t),t)-x(t)=-2*t,diff(x(t),t)+diff(y(t),t)-3*x(t)-y(t)=t^2],sings
```

$$x(t) = -2 + e^{-t} c_1$$
$$y(t) = -t^2 + 4 - 2\, e^{-t} c_1 - 2t$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 42

```
DSolve[{x'[t]+y'[t]-x[t]==-2*t,x'[t]+y'[t]-3*x[t]-y[t]==t^2},{x[t],y[t]},t,IncludeSingularSo
```

$$x(t) \to -2 - \frac{1}{4} c_1 e^{-t}$$
$$y(t) \to -t^2 - 2t + \frac{c_1 e^{-t}}{2} + 4$$

16.3 problem 3

Internal problem ID [11930]
Internal file name [`OUTPUT/11939_Saturday_April_13_2024_10_26_38_PM_48022610/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x'(t) + y'(t) &= x(t) + 3y(t) + e^t \\ x'(t) + y'(t) &= -x(t) + e^{3t} \end{aligned}$$

The system is

$$x'(t) + y'(t) = x(t) + 3y(t) + e^t \tag{1}$$
$$x'(t) + y'(t) = -x(t) + e^{3t} \tag{2}$$

Since the left side is the same, this implies

$$x(t) + 3y(t) + e^t = -x(t) + e^{3t}$$
$$y(t) = \frac{e^{3t}}{3} - \frac{2x(t)}{3} - \frac{e^t}{3} \tag{3}$$

Taking derivative of the above w.r.t. t gives

$$y'(t) = e^{3t} - \frac{2x'(t)}{3} - \frac{e^t}{3} \tag{4}$$

Substituting (3,4) in (1) to eliminate $y(t), y'(t)$ gives

$$\frac{x'(t)}{3} + e^{3t} - \frac{e^t}{3} = -x(t) + e^{3t}$$
$$x'(t) = -3x(t) + e^t \tag{5}$$

Which is now solved for $x(t)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x'(t) + p(t)x(t) = q(t)$$

Where here

$$
\begin{aligned}
p(t) &= 3 \\
q(t) &= \mathrm{e}^{t}
\end{aligned}
$$

Hence the ode is

$$
x'(t) + 3x(t) = \mathrm{e}^{t}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 3 dt} \\
&= \mathrm{e}^{3t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\left(\mathrm{e}^{t}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{3t} x\right) &= \left(\mathrm{e}^{3t}\right)\left(\mathrm{e}^{t}\right) \\
\mathrm{d}\left(\mathrm{e}^{3t} x\right) &= \mathrm{e}^{4t}\, \mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{3t} x &= \int \mathrm{e}^{4t}\, \mathrm{d}t \\
\mathrm{e}^{3t} x &= \frac{\mathrm{e}^{4t}}{4} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{3t}$ results in

$$
x(t) = \frac{\mathrm{e}^{-3t} \mathrm{e}^{4t}}{4} + \mathrm{e}^{-3t} c_1
$$

which simplifies to

$$
x(t) = \frac{\left(\mathrm{e}^{4t} + 4c_1\right) \mathrm{e}^{-3t}}{4}
$$

Given now that we have the solution

$$
x(t) = \frac{\left(\mathrm{e}^{4t} + 4c_1\right) \mathrm{e}^{-3t}}{4} \tag{6}
$$

Then substituting (6) into (3) gives

$$
y(t) = \frac{\mathrm{e}^{3t}}{3} - \frac{\mathrm{e}^{-3t} \mathrm{e}^{4t}}{6} - \frac{2\, \mathrm{e}^{-3t} c_1}{3} - \frac{\mathrm{e}^{t}}{3} \tag{7}
$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 37

```
dsolve([diff(x(t),t)+diff(y(t),t)-x(t)-3*y(t)=exp(t),diff(x(t),t)+diff(y(t),t)+x(t)=exp(3*t)
```

$$
\begin{aligned}
x(t) &= \frac{e^{t}}{4}+c_1 e^{-3t} \\
y(t) &= \frac{e^{3t}}{3}-\frac{e^{t}}{2}-\frac{2c_1 e^{-3t}}{3}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 55

```
DSolve[{x'[t]+y'[t]-x[t]-3*y[t]==Exp[t],x'[t]+y'[t]+x[t]==Exp[3*t]},{x[t],y[t]},t,IncludeSin
```

$$
\begin{aligned}
x(t) &\to \frac{e^t}{4}+\frac{3}{16}c_1 e^{-3t} \\
y(t) &\to -\frac{e^t}{2}+\frac{e^{3t}}{3}-\frac{1}{8}c_1 e^{-3t}
\end{aligned}
$$

16.4 problem 4

Internal problem ID [11931]
Internal file name [OUTPUT/11940_Saturday_April_13_2024_10_26_38_PM_75029156/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) + y'(t) = x(t) + 2y(t) + 2\,\mathrm{e}^{t}$$
$$x'(t) + y'(t) = 3x(t) + 4y(t) + \mathrm{e}^{2t}$$

The system is

$$x'(t) + y'(t) = x(t) + 2y(t) + 2\,\mathrm{e}^{t} \tag{1}$$
$$x'(t) + y'(t) = 3x(t) + 4y(t) + \mathrm{e}^{2t} \tag{2}$$

Since the left side is the same, this implies

$$x(t) + 2y(t) + 2\,\mathrm{e}^{t} = 3x(t) + 4y(t) + \mathrm{e}^{2t}$$
$$y(t) = -\frac{\mathrm{e}^{2t}}{2} - x(t) + \mathrm{e}^{t} \tag{3}$$

Taking derivative of the above w.r.t. t gives

$$y'(t) = -\mathrm{e}^{2t} - x'(t) + \mathrm{e}^{t} \tag{4}$$

Substituting (3,4) in (1) to eliminate $y(t)\,, y'(t)$ gives

$$-\mathrm{e}^{2t} + \mathrm{e}^{t} = -x(t) - \mathrm{e}^{2t} + 4\,\mathrm{e}^{t}$$
$$x(t) = 3\,\mathrm{e}^{t} \tag{5}$$

Substituting (5) into (3) gives

$$y(t) = -\frac{\mathrm{e}^{2t}}{2} - 2\,\mathrm{e}^{t} \tag{7}$$

Since $x(t) = 3\,e^t$, is missing derivative in x then it is an algebraic equation. Solving for $x(t)$.

$$x(t) = 3\,e^t$$

Given now that we have the solution

$$x(t) = 3\,e^t \tag{6}$$

Then substituting (6) into (3) gives

$$y(t) = -\frac{e^{2t}}{2} - 2\,e^t \tag{7}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([diff(x(t),t)+diff(y(t),t)-x(t)-2*y(t)=2*exp(t),diff(x(t),t)+diff(y(t),t)-3*x(t)-4*y(
```

$$x(t) = 3\,e^t$$
$$y(t) = -\frac{e^{2t}}{2} - 2\,e^t$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 25

```
DSolve[{x'[t]+y'[t]-x[t]-2*y[t]==2*Exp[t],x'[t]+y'[t]-3*x[t]-4*y[t]==Exp[2*t]},{x[t],y[t]},t
```

$$x(t) \to 3e^t$$
$$y(t) \to -\frac{1}{2}e^t(e^t + 4)$$

16.5 problem 5

Internal problem ID [11932]
Internal file name [`OUTPUT/11941_Saturday_April_13_2024_10_26_39_PM_92028239/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 3x(t) + 2y(t) - \mathrm{e}^{t} + \mathrm{e}^{-t} \\
y'(t) &= -5x(t) - 3y(t) + 2\,\mathrm{e}^{t} - \mathrm{e}^{-t}
\end{aligned}
$$

16.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -\mathrm{e}^{t} + \mathrm{e}^{-t} \\ 2\,\mathrm{e}^{t} - \mathrm{e}^{-t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t)+3\sin(t) & 2\sin(t) \\ -5\sin(t) & \cos(t)-3\sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \cos(t)+3\sin(t) & 2\sin(t) \\ -5\sin(t) & \cos(t)-3\sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} (\cos(t)+3\sin(t))\, c_1 + 2\sin(t)\, c_2 \\ -5\sin(t)\, c_1 + (\cos(t)-3\sin(t))\, c_2 \end{bmatrix} \\ &= \begin{bmatrix} (3c_1+2c_2)\sin(t) + c_1\cos(t) \\ (-5c_1-3c_2)\sin(t) + c_2\cos(t) \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \cos(t)-3\sin(t) & -2\sin(t) \\ 5\sin(t) & \cos(t)+3\sin(t) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \cos(t)+3\sin(t) & 2\sin(t) \\ -5\sin(t) & \cos(t)-3\sin(t) \end{bmatrix} \int \begin{bmatrix} \cos(t)-3\sin(t) & -2\sin(t) \\ 5\sin(t) & \cos(t)+3\sin(t) \end{bmatrix} \begin{bmatrix} -e^{t}+e^{-t} \\ 2\,e^{t}-e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} \cos(t)+3\sin(t) & 2\sin(t) \\ -5\sin(t) & \cos(t)-3\sin(t) \end{bmatrix} \begin{bmatrix} -\sin(t)\,(e^{t}-e^{-t}) \\ \frac{(\cos(t)+3\sin(t))\,(e^{t}-e^{-t})}{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \frac{e^{t}}{2} - \frac{e^{-t}}{2} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \left[\begin{array}{c} (3c_1 + 2c_2)\sin(t) + c_1\cos(t) \\ (-5c_1 - 3c_2)\sin(t) + c_2\cos(t) + \frac{e^t}{2} - \frac{e^{-t}}{2} \end{array}\right]
\end{aligned}
$$

16.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\left[\begin{array}{c} x'(t) \\ y'(t) \end{array}\right] = \left[\begin{array}{cc} 3 & 2 \\ -5 & -3 \end{array}\right] \left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] + \left[\begin{array}{c} -e^t + e^{-t} \\ 2\,e^t - e^{-t} \end{array}\right]
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\left[\begin{array}{cc} 3 & 2 \\ -5 & -3 \end{array}\right] - \lambda \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right) = 0
$$

Therefore

$$
\det\left(\left[\begin{array}{cc} 3-\lambda & 2 \\ -5 & -3-\lambda \end{array}\right]\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 + 1 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3+i & 2 \\ -5 & -3+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3+i & 2 & 0 \\ -5 & -3+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{3}{2} - \frac{i}{2}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} 3+i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3+i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{3}{5} + \frac{i}{5}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{3}{5}+\frac{I}{5}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{5}+\frac{i}{5}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{3}{5}+\frac{I}{5}\right)t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{3}{5}+\frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{3}{5}+\frac{I}{5}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{5}+\frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{3}{5}+\frac{I}{5}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -3+i \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} - (i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-i & 2 \\ -5 & -3-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3-i & 2 & 0 \\ -5 & -3-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{3}{2}+\frac{i}{2}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} 3-i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3-i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{3}{5} - \frac{i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{3}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{3}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{3}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{3}{5} - \frac{I}{5}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -3 - i \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\begin{bmatrix} -\frac{3}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} -\frac{3}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} \\ e^{it} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} & \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{5i e^{-it}}{2} & \left(\frac{1}{2} + \frac{3i}{2}\right) e^{-it} \\ -\frac{5i e^{it}}{2} & \left(\frac{1}{2} - \frac{3i}{2}\right) e^{it} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} & \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{5i e^{-it}}{2} & \left(\frac{1}{2} + \frac{3i}{2}\right) e^{-it} \\ -\frac{5i e^{it}}{2} & \left(\frac{1}{2} - \frac{3i}{2}\right) e^{it} \end{bmatrix} \begin{bmatrix} -e^{t} + e^{-t} \\ 2\, e^{t} - e^{-t} \end{bmatrix} dt \\
&= \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} & \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} \left(-\frac{1}{2} + i\right) e^{(-1-i)t} + \left(1 + \frac{i}{2}\right) e^{(1-i)t} \\ \left(-\frac{1}{2} - i\right) e^{(-1+i)t} + \left(1 - \frac{i}{2}\right) e^{(1+i)t} \end{bmatrix} dt \\
&= \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} & \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} \left(\frac{1}{4} + \frac{3i}{4}\right) \left(-e^{(-1-i)t} + e^{(1-i)t}\right) \\ \left(\frac{1}{4} - \frac{3i}{4}\right) \left(-e^{(-1+i)t} + e^{(1+i)t}\right) \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \frac{e^{t}}{2} - \frac{e^{-t}}{2} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{5}-\frac{i}{5}\right) c_1 \mathrm{e}^{it} \\ c_1 \mathrm{e}^{it} \end{bmatrix} + \begin{bmatrix} \left(-\frac{3}{5}+\frac{i}{5}\right) c_2 \mathrm{e}^{-it} \\ c_2 \mathrm{e}^{-it} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\mathrm{e}^{t}}{2} - \frac{\mathrm{e}^{-t}}{2} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{5}-\frac{i}{5}\right) c_1 \mathrm{e}^{it} + \left(-\frac{3}{5}+\frac{i}{5}\right) c_2 \mathrm{e}^{-it} \\ c_1 \mathrm{e}^{it} + c_2 \mathrm{e}^{-it} + \frac{\mathrm{e}^{t}}{2} - \frac{\mathrm{e}^{-t}}{2} \end{bmatrix}$$

16.5.3 Maple step by step solution

Let's solve

$\left[x'(t) = 3x(t) + 2y(t) - \mathrm{e}^{t} + \frac{1}{\mathrm{e}^{t}}, y'(t) = -5x(t) - 3y(t) + 2\,\mathrm{e}^{t} - \frac{1}{\mathrm{e}^{t}}\right]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{3x(t)\mathrm{e}^{t} + 2y(t)\mathrm{e}^{t} - (\mathrm{e}^{t})^2 + 1}{\mathrm{e}^{t}} - 3x(t) - 2y(t) \\ -\frac{5x(t)\mathrm{e}^{t} + 3y(t)\mathrm{e}^{t} - 2(\mathrm{e}^{t})^2 + 1}{\mathrm{e}^{t}} + 5x(t) + 3y(t) \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[-\mathrm{I}, \left[\begin{array}{c} -\frac{3}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} -\frac{3}{5}-\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \left[\begin{array}{c} -\frac{3}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \left[\begin{array}{c} -\frac{3}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \left[\begin{array}{c} -\frac{3}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c} \left(-\frac{3}{5}+\frac{\mathrm{I}}{5}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \left[\begin{array}{c} -\frac{3\cos(t)}{5}+\frac{\sin(t)}{5} \\ \cos(t) \end{array}\right], \vec{x}_2(t) = \left[\begin{array}{c} \frac{3\sin(t)}{5}+\frac{\cos(t)}{5} \\ -\sin(t) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = \left[\begin{array}{c} c_2\left(\frac{3\sin(t)}{5}+\frac{\cos(t)}{5}\right) + c_1\left(-\frac{3\cos(t)}{5}+\frac{\sin(t)}{5}\right) \\ c_1\cos(t) - c_2\sin(t) \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{(-3c_1+c_2)\cos(t)}{5}+\frac{\sin(t)(c_1+3c_2)}{5} \\ c_1\cos(t) - c_2\sin(t) \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{(-3c_1+c_2)\cos(t)}{5} + \frac{\sin(t)(c_1+3c_2)}{5}, y(t) = c_1\cos(t) - c_2\sin(t)\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 48

```
dsolve([2*diff(x(t),t)+diff(y(t),t)-x(t)-y(t)=exp(-t),diff(x(t),t)+diff(y(t),t)+2*x(t)+y(t)=
```

$$x(t) = c_1\sin(t) + c_2\cos(t)$$
$$y(t) = \frac{c_1\cos(t)}{2} - \frac{3c_2\cos(t)}{2} - \frac{3c_1\sin(t)}{2} - \frac{c_2\sin(t)}{2} + \frac{e^t}{2} - \frac{e^{-t}}{2}$$

✓ Solution by Mathematica

Time used: 0.229 (sec). Leaf size: 60

```
DSolve[{2*x'[t]+y'[t]-x[t]-y[t]==Exp[-t],x'[t]+y'[t]+2*x[t]+y[t]==Exp[t]},{x[t],y[t]},t,Incl
```

$$x(t) \to c_1\cos(t) + (3c_1 + 2c_2)\sin(t)$$
$$y(t) \to \frac{1}{2}\left(-e^{-t} + e^t + 2c_2\cos(t) - 2(5c_1 + 3c_2)\sin(t)\right)$$

16.6 problem 6

Internal problem ID [11933]
Internal file name [`OUTPUT/11942_Sunday_April_14_2024_02_31_02_AM_19754612/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -x(t) + t - \mathrm{e}^t \\
y'(t) &= 5x(t) + y(t) - t + 2\,\mathrm{e}^t
\end{aligned}
$$

16.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} t - \mathrm{e}^t \\ -t + 2\,\mathrm{e}^t \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ \frac{5\,e^{t}}{2} - \frac{5\,e^{-t}}{2} & e^{t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^{-t} & 0 \\ \frac{5\,e^{t}}{2} - \frac{5\,e^{-t}}{2} & e^{t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} c_1 \\ \left(\frac{5\,e^{t}}{2} - \frac{5\,e^{-t}}{2}\right) c_1 + e^{t} c_2 \end{bmatrix}\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} e^{t} & 0 \\ -\frac{5\,e^{t}}{2} + \frac{5\,e^{-t}}{2} & e^{-t} \end{bmatrix}\end{aligned}$$

Hence

$$\begin{aligned}\vec{x}_p(t) &= \begin{bmatrix} e^{-t} & 0 \\ \frac{5\,e^{t}}{2} - \frac{5\,e^{-t}}{2} & e^{t} \end{bmatrix} \int \begin{bmatrix} e^{t} & 0 \\ -\frac{5\,e^{t}}{2} + \frac{5\,e^{-t}}{2} & e^{-t} \end{bmatrix} \begin{bmatrix} t - e^{t} \\ -t + 2\,e^{t} \end{bmatrix} dt \\ &= \begin{bmatrix} e^{-t} & 0 \\ \frac{5\,e^{t}}{2} - \frac{5\,e^{-t}}{2} & e^{t} \end{bmatrix} \begin{bmatrix} -\frac{e^{2t}}{2} + e^{t}(t-1) \\ \frac{3(-1-t)e^{-t}}{2} + \frac{5\,e^{2t}}{4} + \frac{5\,e^{t}(1-t)}{2} - \frac{t}{2} \end{bmatrix} \\ &= \begin{bmatrix} t - 1 - \frac{e^{t}}{2} \\ \frac{(-2t+5)e^{t}}{4} - 4t + 1 \end{bmatrix}\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} e^{-t} c_1 + t - 1 - \frac{e^{t}}{2} \\ -\frac{5\,e^{-t} c_1}{2} + \frac{(-2t+10c_1+4c_2+5)e^{t}}{4} - 4t + 1 \end{bmatrix}\end{aligned}$$

16.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} t - e^{t} \\ -t + 2\,e^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 0 \\ 5 & 1-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-1-\lambda)(1-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 5 & 2 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} 5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 5 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 0 & 0 \\ 5 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{5R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{2\,\mathrm{e}^{-t}}{5} \\ \mathrm{e}^{-t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 0 & -\frac{2\,\mathrm{e}^{-t}}{5} \\ \mathrm{e}^{t} & \mathrm{e}^{-t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{5\,\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t} \\ -\frac{5\,\mathrm{e}^{t}}{2} & 0 \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} 0 & -\frac{2\,\mathrm{e}^{-t}}{5} \\ \mathrm{e}^{t} & \mathrm{e}^{-t} \end{bmatrix} \int \begin{bmatrix} \frac{5\,\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t} \\ -\frac{5\,\mathrm{e}^{t}}{2} & 0 \end{bmatrix} \begin{bmatrix} t-\mathrm{e}^{t} \\ -t+2\,\mathrm{e}^{t} \end{bmatrix} dt$$

$$= \begin{bmatrix} 0 & -\frac{2\,\mathrm{e}^{-t}}{5} \\ \mathrm{e}^{t} & \mathrm{e}^{-t} \end{bmatrix} \int \begin{bmatrix} -\frac{1}{2}+\frac{3t\,\mathrm{e}^{-t}}{2} \\ \frac{5\,\mathrm{e}^{t}(-t+\mathrm{e}^{t})}{2} \end{bmatrix} dt$$

$$= \begin{bmatrix} 0 & -\frac{2\,\mathrm{e}^{-t}}{5} \\ \mathrm{e}^{t} & \mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} \frac{(-3t-3)\mathrm{e}^{-t}}{2}-\frac{t}{2} \\ \frac{5\,\mathrm{e}^{2t}}{4}+\frac{5\,\mathrm{e}^{t}(1-t)}{2} \end{bmatrix}$$

$$= \begin{bmatrix} t-1-\frac{\mathrm{e}^{t}}{2} \\ \frac{(-2t+5)\mathrm{e}^{t}}{4}-4t+1 \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t)+\vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 \\ c_1\mathrm{e}^{t} \end{bmatrix} + \begin{bmatrix} -\frac{2c_2\mathrm{e}^{-t}}{5} \\ c_2\mathrm{e}^{-t} \end{bmatrix} + \begin{bmatrix} t-1-\frac{\mathrm{e}^{t}}{2} \\ \frac{(-2t+5)\mathrm{e}^{t}}{4}-4t+1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{2c_2\mathrm{e}^{-t}}{5}+t-1-\frac{\mathrm{e}^{t}}{2} \\ c_2\mathrm{e}^{-t}+\frac{(-2t+4c_1+5)\mathrm{e}^{t}}{4}-4t+1 \end{bmatrix}$$

16.6.3 Maple step by step solution

Let's solve

$[x'(t) = -x(t)+t-\mathrm{e}^{t}, y'(t) = 5x(t)+y(t)-t+2\,\mathrm{e}^{t}]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} t-\mathrm{e}^{t} \\ -t+2\,\mathrm{e}^{t} \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} t - e^{t} \\ -t + 2\,e^{t} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} t - e^{t} \\ -t + 2\,e^{t} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 0 \\ 5 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-t} \cdot \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{t} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $\phi(t) = \begin{bmatrix} -\frac{2\,e^{-t}}{5} & 0 \\ e^{-t} & e^{t} \end{bmatrix}$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

    $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

    $\Phi(t) = \begin{bmatrix} -\frac{2\,e^{-t}}{5} & 0 \\ e^{-t} & e^{t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{2}{5} & 0 \\ 1 & 1 \end{bmatrix}}$

  - Evaluate and simplify to get the fundamental matrix

    $\Phi(t) = \begin{bmatrix} e^{-t} & 0 \\ \frac{5\,e^{t}}{2} - \frac{5\,e^{-t}}{2} & e^{t} \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

    $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

  - Take the derivative of the particular solution

    $\overrightarrow{x}'_p(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

  - Cancel like terms

    $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

  - Multiply by the inverse of the fundamental matrix

$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} -\frac{e^t}{2} + t - 1 + \frac{3e^{-t}}{2} \\ -\frac{15e^{-t}}{4} + \frac{(-2t+11)e^t}{4} - 4t + 1 \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \begin{bmatrix} -\frac{e^t}{2} + t - 1 + \frac{3e^{-t}}{2} \\ -\frac{15e^{-t}}{4} + \frac{(-2t+11)e^t}{4} - 4t + 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{2c_1e^{-t}}{5} - \frac{e^t}{2} + t - 1 + \frac{3e^{-t}}{2} \\ \frac{(4c_1-15)e^{-t}}{4} + \frac{(-2t+4c_2+11)e^t}{4} - 4t + 1 \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{2c_1e^{-t}}{5} - \frac{e^t}{2} + t - 1 + \frac{3e^{-t}}{2}, y(t) = \frac{(4c_1-15)e^{-t}}{4} + \frac{(-2t+4c_2+11)e^t}{4} - 4t + 1\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 42

```
dsolve([2*diff(x(t),t)+diff(y(t),t)-3*x(t)-y(t)=t,diff(x(t),t)+diff(y(t),t)-4*x(t)-y(t)=exp(
```

$$x(t) = t - 1 - \frac{e^t}{2} + c_2e^{-t}$$

$$y(t) = -\frac{5c_2e^{-t}}{2} - 4t + 1 + c_1e^t - \frac{e^tt}{2}$$

✓ Solution by Mathematica

Time used: 0.665 (sec). Leaf size: 72

```
DSolve[{2*x'[t]+y'[t]-3*x[t]-y[t]==t,x'[t]+y'[t]+4*x[t]-y[t]==Exp[t]},{x[t],y[t]},t,IncludeS
```

$$x(t) \to -\frac{t}{7} + \frac{e^t}{6} + c_1 e^{7t} - \frac{1}{49}$$
$$y(t) \to -\frac{4t}{7} - \frac{11}{6} c_1 e^{7t} + \frac{1}{36} e^t (6t - 11 + 66c_1 + 36c_2) - \frac{39}{49}$$

16.7 problem 7

Internal problem ID [11934]
Internal file name [`OUTPUT/11943_Sunday_April_14_2024_02_31_04_AM_22296256/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -t + 2\,\mathrm{e}^{3t} + 6y(t) \\
y'(t) &= x(t) + t - \mathrm{e}^{3t}
\end{aligned}
$$

16.7.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 2\,\mathrm{e}^{3t} - t \\ -\mathrm{e}^{3t} + t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{\sqrt{6}\,t}}{2} + \frac{e^{-\sqrt{6}\,t}}{2} & \frac{\left(-e^{-\sqrt{6}\,t}+e^{\sqrt{6}\,t}\right)\sqrt{6}}{2} \\ \frac{\left(-e^{-\sqrt{6}\,t}+e^{\sqrt{6}\,t}\right)\sqrt{6}}{12} & \frac{e^{\sqrt{6}\,t}}{2} + \frac{e^{-\sqrt{6}\,t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{e^{\sqrt{6}\,t}}{2} + \frac{e^{-\sqrt{6}\,t}}{2} & \frac{\left(-e^{-\sqrt{6}\,t}+e^{\sqrt{6}\,t}\right)\sqrt{6}}{2} \\ \frac{\left(-e^{-\sqrt{6}\,t}+e^{\sqrt{6}\,t}\right)\sqrt{6}}{12} & \frac{e^{\sqrt{6}\,t}}{2} + \frac{e^{-\sqrt{6}\,t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{e^{\sqrt{6}\,t}}{2} + \frac{e^{-\sqrt{6}\,t}}{2}\right) c_1 + \frac{\left(-e^{-\sqrt{6}\,t}+e^{\sqrt{6}\,t}\right)\sqrt{6}\,c_2}{2} \\ \frac{\left(-e^{-\sqrt{6}\,t}+e^{\sqrt{6}\,t}\right)\sqrt{6}\,c_1}{12} + \left(\frac{e^{\sqrt{6}\,t}}{2} + \frac{e^{-\sqrt{6}\,t}}{2}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\left(-c_2\sqrt{6}+c_1\right)e^{-\sqrt{6}\,t}}{2} + \frac{e^{\sqrt{6}\,t}\left(c_2\sqrt{6}+c_1\right)}{2} \\ \frac{\left(-c_1\sqrt{6}+6c_2\right)e^{-\sqrt{6}\,t}}{12} + \frac{e^{\sqrt{6}\,t}\left(c_1\sqrt{6}+6c_2\right)}{12} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \begin{bmatrix} \frac{e^{\sqrt{6}\,t}}{2} + \frac{e^{-\sqrt{6}\,t}}{2} & -\frac{\left(-e^{-\sqrt{6}\,t}+e^{\sqrt{6}\,t}\right)\sqrt{6}}{2} \\ -\frac{\left(-e^{-\sqrt{6}\,t}+e^{\sqrt{6}\,t}\right)\sqrt{6}}{12} & \frac{e^{\sqrt{6}\,t}}{2} + \frac{e^{-\sqrt{6}\,t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$
\vec{x}_p(t) = \left[\begin{array}{cc} \frac{e^{\sqrt{6}\, t}}{2}+\frac{e^{-\sqrt{6}\, t}}{2} & \frac{\left(-e^{-\sqrt{6}\, t}+e^{\sqrt{6}\, t}\right) \sqrt{6}}{2} \\ \frac{\left(-e^{-\sqrt{6}\, t}+e^{\sqrt{6}\, t}\right) \sqrt{6}}{12} & \frac{e^{\sqrt{6}\, t}}{2}+\frac{e^{-\sqrt{6}\, t}}{2} \end{array}\right] \int \left[\begin{array}{cc} \frac{e^{\sqrt{6}\, t}}{2}+\frac{e^{-\sqrt{6}\, t}}{2} & -\frac{\left(-e^{-\sqrt{6}\, t}+e^{\sqrt{6}\, t}\right) \sqrt{6}}{2} \\ -\frac{\left(-e^{-\sqrt{6}\, t}+e^{\sqrt{6}\, t}\right) \sqrt{6}}{12} & \frac{e^{\sqrt{6}\, t}}{2}+\frac{e^{-\sqrt{6}\, t}}{2} \end{array}\right] \left[\begin{array}{c} 2\, e^{3 t}-t \\ -e^{3 t}+t \end{array}\right.
$$

$$
= \left[\begin{array}{cc} \frac{e^{\sqrt{6}\, t}}{2}+\frac{e^{-\sqrt{6}\, t}}{2} & \frac{\left(-e^{-\sqrt{6}\, t}+e^{\sqrt{6}\, t}\right) \sqrt{6}}{2} \\ \frac{\left(-e^{-\sqrt{6}\, t}+e^{\sqrt{6}\, t}\right) \sqrt{6}}{12} & \frac{e^{\sqrt{6}\, t}}{2}+\frac{e^{-\sqrt{6}\, t}}{2} \end{array}\right] \left[\begin{array}{c} -\frac{\sqrt{6}\, e^{-t \left(\sqrt{6}-3\right)}}{6}+\frac{\left(\sqrt{6}\, (t-1)-6 t+1\right) e^{-\sqrt{6}\, t}}{12}+\frac{\sqrt{6}\, e^{t \left(\sqrt{6}+3\right)}}{6}-\frac{e^{\sqrt{6}\, t} \left(\cdots\right.}{\cdots} \\ -\frac{e^{-t \left(\sqrt{6}-3\right)}}{6}+\frac{\left(\left(-6 t+1\right) \sqrt{6}+6 t-6\right) e^{-\sqrt{6}\, t}}{72}-\frac{e^{t \left(\sqrt{6}+3\right)}}{6}+\frac{\left(\left(6 t-1\right)\cdots\right.}{\cdots} \end{array}\right.
$$

$$
= \left[\begin{array}{c} -t+\frac{1}{6} \\ \frac{t}{6}-\frac{e^{3 t}}{3}-\frac{1}{6} \end{array}\right]
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t)+\vec{x}_p(t) \\
&= \left[\begin{array}{c} \frac{\left(-3 c_2 \sqrt{6}+3 c_1\right) e^{-\sqrt{6}\, t}}{6}+\frac{\left(3 c_2 \sqrt{6}+3 c_1\right) e^{\sqrt{6}\, t}}{6}-t+\frac{1}{6} \\ \frac{\left(-c_1 \sqrt{6}+6 c_2\right) e^{-\sqrt{6}\, t}}{12}+\frac{e^{\sqrt{6}\, t} \left(c_1 \sqrt{6}+6 c_2\right)}{12}+\frac{t}{6}-\frac{e^{3 t}}{3}-\frac{1}{6} \end{array}\right]
\end{aligned}
$$

16.7.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\, \vec{x}(t) + \vec{G}(t)
$$

Or

$$
\left[\begin{array}{c} x'(t) \\ y'(t) \end{array}\right] = \left[\begin{array}{cc} 0 & 6 \\ 1 & 0 \end{array}\right] \left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] + \left[\begin{array}{c} 2\, e^{3 t}-t \\ -e^{3 t}+t \end{array}\right]
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\, \vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\, \vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 6 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 6 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \sqrt{6}$$
$$\lambda_2 = -\sqrt{6}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\sqrt{6}$	1	real eigenvalue
$-\sqrt{6}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \sqrt{6}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 6 \\ 1 & 0 \end{bmatrix} - \left(\sqrt{6}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\sqrt{6} & 6 \\ 1 & -\sqrt{6} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\sqrt{6} & 6 & 0 \\ 1 & -\sqrt{6} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{\sqrt{6}\,R_1}{6} \Longrightarrow \left[\begin{array}{cc|c} -\sqrt{6} & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\sqrt{6} & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\sqrt{6}\}$

Hence the solution is

$$\begin{bmatrix} t\sqrt{6} \\ t \end{bmatrix} = \begin{bmatrix} t\sqrt{6} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t\sqrt{6} \\ t \end{bmatrix} = t \begin{bmatrix} \sqrt{6} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t\sqrt{6} \\ t \end{bmatrix} = \begin{bmatrix} \sqrt{6} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\sqrt{6}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 6 \\ 1 & 0 \end{bmatrix} - \left(-\sqrt{6}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{6} & 6 \\ 1 & \sqrt{6} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \sqrt{6} & 6 & 0 \\ 1 & \sqrt{6} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{\sqrt{6}\, R_1}{6} \implies \left[\begin{array}{cc|c} \sqrt{6} & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \sqrt{6} & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\sqrt{6}\}$

Hence the solution is

$$\begin{bmatrix} -t\sqrt{6} \\ t \end{bmatrix} = \begin{bmatrix} -t\sqrt{6} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t\sqrt{6} \\ t \end{bmatrix} = t \begin{bmatrix} -\sqrt{6} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t\sqrt{6} \\ t \end{bmatrix} = \begin{bmatrix} -\sqrt{6} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\sqrt{6}$	1	1	No	$\begin{bmatrix} \sqrt{6} \\ 1 \end{bmatrix}$
$-\sqrt{6}$	1	1	No	$\begin{bmatrix} -\sqrt{6} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\sqrt{6}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\sqrt{6}\, t}\\ &= \left[\begin{array}{c}\sqrt{6}\\ 1\end{array}\right] e^{\sqrt{6}\, t}\end{aligned}$$

Since eigenvalue $-\sqrt{6}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-\sqrt{6}\, t}\\ &= \left[\begin{array}{c}-\sqrt{6}\\ 1\end{array}\right] e^{-\sqrt{6}\, t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c}x(t)\\ y(t)\end{array}\right] = c_1\left[\begin{array}{c}\sqrt{6}\, \mathrm{e}^{\sqrt{6}\, t}\\ \mathrm{e}^{\sqrt{6}\, t}\end{array}\right] + c_2\left[\begin{array}{c}-\sqrt{6}\, \mathrm{e}^{-\sqrt{6}\, t}\\ \mathrm{e}^{-\sqrt{6}\, t}\end{array}\right]$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \left[\begin{array}{cc}\sqrt{6}\, \mathrm{e}^{\sqrt{6}\, t} & -\sqrt{6}\, \mathrm{e}^{-\sqrt{6}\, t}\\ \mathrm{e}^{\sqrt{6}\, t} & \mathrm{e}^{-\sqrt{6}\, t}\end{array}\right]$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{\sqrt{6}\,e^{-\sqrt{6}\,t}}{12} & \frac{e^{-\sqrt{6}\,t}}{2} \\ -\frac{\sqrt{6}\,e^{\sqrt{6}\,t}}{12} & \frac{e^{\sqrt{6}\,t}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \sqrt{6}\,e^{\sqrt{6}\,t} & -\sqrt{6}\,e^{-\sqrt{6}\,t} \\ e^{\sqrt{6}\,t} & e^{-\sqrt{6}\,t} \end{bmatrix} \int \begin{bmatrix} \frac{\sqrt{6}\,e^{-\sqrt{6}\,t}}{12} & \frac{e^{-\sqrt{6}\,t}}{2} \\ -\frac{\sqrt{6}\,e^{\sqrt{6}\,t}}{12} & \frac{e^{\sqrt{6}\,t}}{2} \end{bmatrix} \begin{bmatrix} 2\,e^{3t} - t \\ -e^{3t} + t \end{bmatrix} dt \\
&= \begin{bmatrix} \sqrt{6}\,e^{\sqrt{6}\,t} & -\sqrt{6}\,e^{-\sqrt{6}\,t} \\ e^{\sqrt{6}\,t} & e^{-\sqrt{6}\,t} \end{bmatrix} \int \begin{bmatrix} \frac{\left(\sqrt{6}-3\right)e^{-t\left(\sqrt{6}-3\right)}}{6} - \frac{t\,e^{-\sqrt{6}\,t}\left(\sqrt{6}-6\right)}{12} \\ \frac{\left(-\sqrt{6}-3\right)e^{t\left(\sqrt{6}+3\right)}}{6} + \frac{t\,e^{\sqrt{6}\,t}\left(\sqrt{6}+6\right)}{12} \end{bmatrix} dt \\
&= \begin{bmatrix} \sqrt{6}\,e^{\sqrt{6}\,t} & -\sqrt{6}\,e^{-\sqrt{6}\,t} \\ e^{\sqrt{6}\,t} & e^{-\sqrt{6}\,t} \end{bmatrix} \begin{bmatrix} -\frac{\left(2\,e^{3t}+\left(t-\frac{1}{6}\right)\sqrt{6}-t+1\right)e^{-\sqrt{6}\,t}}{12} \\ -\frac{e^{t\left(\sqrt{6}+3\right)}}{6} + \frac{\left((6t-1)\sqrt{6}+6t-6\right)e^{\sqrt{6}\,t}}{72} \end{bmatrix} \\
&= \begin{bmatrix} -t + \frac{1}{6} \\ \frac{t}{6} - \frac{e^{3t}}{3} - \frac{1}{6} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} c_1\sqrt{6}\,e^{\sqrt{6}\,t} \\ c_1 e^{\sqrt{6}\,t} \end{bmatrix} + \begin{bmatrix} -c_2\sqrt{6}\,e^{-\sqrt{6}\,t} \\ c_2 e^{-\sqrt{6}\,t} \end{bmatrix} + \begin{bmatrix} -t + \frac{1}{6} \\ \frac{t}{6} - \frac{e^{3t}}{3} - \frac{1}{6} \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1\sqrt{6}\,e^{\sqrt{6}\,t} - c_2\sqrt{6}\,e^{-\sqrt{6}\,t} - t + \frac{1}{6} \\ c_1 e^{\sqrt{6}\,t} + c_2 e^{-\sqrt{6}\,t} + \frac{t}{6} - \frac{e^{3t}}{3} - \frac{1}{6} \end{bmatrix}$$

16.7.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -t + 2\left(e^{t}\right)^{3} + 6y(t)\,, y'(t) = x(t) + t - \left(e^{t}\right)^{3}\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 6 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 2(e^t)^3 - t \\ -(e^t)^3 + t \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 6 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 2(e^t)^3 - t \\ -(e^t)^3 + t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 2(e^t)^3 - t \\ -(e^t)^3 + t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 6 \\ 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\sqrt{6}, \begin{bmatrix} \sqrt{6} \\ 1 \end{bmatrix}\right], \left[-\sqrt{6}, \begin{bmatrix} -\sqrt{6} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\sqrt{6}, \begin{bmatrix} \sqrt{6} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{\sqrt{6}\,t} \cdot \begin{bmatrix} \sqrt{6} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\sqrt{6}, \left[\begin{array}{c} -\sqrt{6} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{-\sqrt{6}\,t} \cdot \left[\begin{array}{c} -\sqrt{6} \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{x}_p(t)$$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syste

$$\phi(t) = \left[\begin{array}{cc} \sqrt{6}\,\mathrm{e}^{\sqrt{6}\,t} & -\sqrt{6}\,\mathrm{e}^{-\sqrt{6}\,t} \\ \mathrm{e}^{\sqrt{6}\,t} & \mathrm{e}^{-\sqrt{6}\,t} \end{array}\right]$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \left[\begin{array}{cc} \sqrt{6}\,\mathrm{e}^{\sqrt{6}\,t} & -\sqrt{6}\,\mathrm{e}^{-\sqrt{6}\,t} \\ \mathrm{e}^{\sqrt{6}\,t} & \mathrm{e}^{-\sqrt{6}\,t} \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} \sqrt{6} & -\sqrt{6} \\ 1 & 1 \end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \left[\begin{array}{cc} \frac{\mathrm{e}^{\sqrt{6}\,t}}{2} + \frac{\mathrm{e}^{-\sqrt{6}\,t}}{2} & \frac{\left(-\mathrm{e}^{-\sqrt{6}\,t} + \mathrm{e}^{\sqrt{6}\,t}\right)\sqrt{6}}{2} \\ \frac{\left(-\mathrm{e}^{-\sqrt{6}\,t} + \mathrm{e}^{\sqrt{6}\,t}\right)\sqrt{6}}{12} & \frac{\mathrm{e}^{\sqrt{6}\,t}}{2} + \frac{\mathrm{e}^{-\sqrt{6}\,t}}{2} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$
- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t) = \left[\begin{array}{c} \frac{\left(-3\sqrt{6}-1\right)e^{-\sqrt{6}\,t}}{12} + \frac{\left(3\sqrt{6}-1\right)e^{\sqrt{6}\,t}}{12} - t + \frac{1}{6} \\ \frac{\left(\sqrt{6}+18\right)e^{-\sqrt{6}\,t}}{72} + \frac{\left(-\sqrt{6}+18\right)e^{\sqrt{6}\,t}}{72} + \frac{t}{6} - \frac{e^{3t}}{3} - \frac{1}{6} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \left[\begin{array}{c} \frac{\left(-3\sqrt{6}-1\right)e^{-\sqrt{6}\,t}}{12} + \frac{\left(3\sqrt{6}-1\right)e^{\sqrt{6}\,t}}{12} - t + \frac{1}{6} \\ \frac{\left(\sqrt{6}+18\right)e^{-\sqrt{6}\,t}}{72} + \frac{\left(-\sqrt{6}+18\right)e^{\sqrt{6}\,t}}{72} + \frac{t}{6} - \frac{e^{3t}}{3} - \frac{1}{6} \end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{\left(-1+(-12c_2-3)\sqrt{6}\right)e^{-\sqrt{6}\,t}}{12} + \frac{\left(-1+(12c_1+3)\sqrt{6}\right)e^{\sqrt{6}\,t}}{12} - t + \frac{1}{6} \\ \frac{\left(72c_2+\sqrt{6}+18\right)e^{-\sqrt{6}\,t}}{72} + \frac{\left(72c_1-\sqrt{6}+18\right)e^{\sqrt{6}\,t}}{72} + \frac{t}{6} - \frac{e^{3t}}{3} - \frac{1}{6} \end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x(t) = \frac{\left(-1+(-12c_2-3)\sqrt{6}\right)e^{-\sqrt{6}\,t}}{12} + \frac{\left(-1+(12c_1+3)\sqrt{6}\right)e^{\sqrt{6}\,t}}{12} - t + \frac{1}{6}, y(t) = \frac{\left(72c_2+\sqrt{6}+18\right)e^{-\sqrt{6}\,t}}{72} + \frac{\left(72c_1-\sqrt{6}-\right.}{72}\right.$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 66

```
dsolve([diff(x(t),t)+diff(y(t),t)-x(t)-6*y(t)=exp(3*t),diff(x(t),t)+2*diff(y(t),t)-2*x(t)-6*
```

$$x(t) = e^{\sqrt{6}\,t} c_2 + e^{-\sqrt{6}\,t} c_1 - t + \frac{1}{6}$$
$$y(t) = \frac{\sqrt{6}\,e^{\sqrt{6}\,t} c_2}{6} - \frac{\sqrt{6}\,e^{-\sqrt{6}\,t} c_1}{6} - \frac{1}{6} + \frac{t}{6} - \frac{e^{3t}}{3}$$

✓ Solution by Mathematica

Time used: 8.119 (sec). Leaf size: 142

```
DSolve[{x'[t]+y'[t]-x[t]-6*y[t]==Exp[3*t],x'[t]+2*y'[t]-2*x[t]-6*y[t]==t},{x[t],y[t]},t,Incl
```

$$x(t) \to \frac{1}{6}\left(-6t + 3\left(c_1 - \sqrt{6}c_2\right)e^{-\sqrt{6}t} + 3\left(c_1 + \sqrt{6}c_2\right)e^{\sqrt{6}t} + 1\right)$$
$$y(t) \to \frac{1}{12}e^{-\sqrt{6}t}\left(2e^{\sqrt{6}t}(t-1) - 4e^{\left(3+\sqrt{6}\right)t} + \left(\sqrt{6}c_1 + 6c_2\right)e^{2\sqrt{6}t} - \sqrt{6}c_1 + 6c_2\right)$$

16.8 problem 8

Internal problem ID [11935]
Internal file name [`OUTPUT/11944_Sunday_April_14_2024_02_31_05_AM_57356254/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 6t - 1 + 3y(t) \\
y'(t) &= x(t) - 3t + 1
\end{aligned}
$$

16.8.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 6t - 1 \\ 1 - 3t \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{\sqrt{3}\,t}}{2} + \frac{e^{-\sqrt{3}\,t}}{2} & \frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{2} \\ \frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{6} & \frac{e^{\sqrt{3}\,t}}{2} + \frac{e^{-\sqrt{3}\,t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{e^{\sqrt{3}\,t}}{2} + \frac{e^{-\sqrt{3}\,t}}{2} & \frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{2} \\ \frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{6} & \frac{e^{\sqrt{3}\,t}}{2} + \frac{e^{-\sqrt{3}\,t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{e^{\sqrt{3}\,t}}{2} + \frac{e^{-\sqrt{3}\,t}}{2}\right) c_1 + \frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}\,c_2}{2} \\ \frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}\,c_1}{6} + \left(\frac{e^{\sqrt{3}\,t}}{2} + \frac{e^{-\sqrt{3}\,t}}{2}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\left(-c_2\sqrt{3}+c_1\right)e^{-\sqrt{3}\,t}}{2} + \frac{e^{\sqrt{3}\,t}\left(c_2\sqrt{3}+c_1\right)}{2} \\ \frac{\left(-c_1\sqrt{3}+3c_2\right)e^{-\sqrt{3}\,t}}{6} + \frac{e^{\sqrt{3}\,t}\left(c_1\sqrt{3}+3c_2\right)}{6} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= \left(e^{At}\right)^{-1} \\ &= \begin{bmatrix} \frac{e^{\sqrt{3}\,t}}{2} + \frac{e^{-\sqrt{3}\,t}}{2} & -\frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{2} \\ -\frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{6} & \frac{e^{\sqrt{3}\,t}}{2} + \frac{e^{-\sqrt{3}\,t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \frac{e^{\sqrt{3}\,t}}{2}+\frac{e^{-\sqrt{3}\,t}}{2} & \frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{2} \\ \frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{6} & \frac{e^{\sqrt{3}\,t}}{2}+\frac{e^{-\sqrt{3}\,t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{e^{\sqrt{3}\,t}}{2}+\frac{e^{-\sqrt{3}\,t}}{2} & -\frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{2} \\ -\frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{6} & \frac{e^{\sqrt{3}\,t}}{2}+\frac{e^{-\sqrt{3}\,t}}{2} \end{bmatrix} \begin{bmatrix} 6t-1 \\ 1-3t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^{\sqrt{3}\,t}}{2}+\frac{e^{-\sqrt{3}\,t}}{2} & \frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{2} \\ \frac{\left(-e^{-\sqrt{3}\,t}+e^{\sqrt{3}\,t}\right)\sqrt{3}}{6} & \frac{e^{\sqrt{3}\,t}}{2}+\frac{e^{-\sqrt{3}\,t}}{2} \end{bmatrix} \begin{bmatrix} \frac{\left((4-6t)\sqrt{3}+9t-9\right)e^{-\sqrt{3}\,t}}{6} + e^{\sqrt{3}\,t}\left(\left(t-\frac{2}{3}\right)\sqrt{3}+\frac{3t}{2}-\frac{3}{2}\right) \\ \frac{\left((3t-3)\sqrt{3}-6t+4\right)e^{-\sqrt{3}\,t}}{6} - \frac{e^{\sqrt{3}\,t}\left(\sqrt{3}\,(t-1)+2t-\frac{4}{3}\right)}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3t-3 \\ -2t+\frac{4}{3} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} \frac{\left(-c_2\sqrt{3}+c_1\right)e^{-\sqrt{3}\,t}}{2} + \frac{e^{\sqrt{3}\,t}\left(c_2\sqrt{3}+c_1\right)}{2} + 3t - 3 \\ \frac{\left(-c_1\sqrt{3}+3c_2\right)e^{-\sqrt{3}\,t}}{6} + \frac{e^{\sqrt{3}\,t}\left(c_1\sqrt{3}+3c_2\right)}{6} - 2t + \frac{4}{3} \end{bmatrix} \end{aligned}$$

16.8.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 6t-1 \\ 1-3t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix}0 & 3\\ 1 & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-\lambda & 3\\ 1 & -\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \sqrt{3}$$
$$\lambda_2 = -\sqrt{3}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\sqrt{3}$	1	real eigenvalue
$-\sqrt{3}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}0 & 3\\ 1 & 0\end{bmatrix} - \left(\sqrt{3}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$
$$\begin{bmatrix}-\sqrt{3} & 3\\ 1 & -\sqrt{3}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}-\sqrt{3} & 3 & 0\\ 1 & -\sqrt{3} & 0\end{array}\right]$$

$$R_2 = R_2 + \frac{\sqrt{3}\,R_1}{3} \implies \left[\begin{array}{cc|c} -\sqrt{3} & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\sqrt{3} & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\sqrt{3}\}$

Hence the solution is

$$\begin{bmatrix} t\sqrt{3} \\ t \end{bmatrix} = \begin{bmatrix} t\sqrt{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t\sqrt{3} \\ t \end{bmatrix} = t \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t\sqrt{3} \\ t \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -\sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} - \left(-\sqrt{3}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3} & 3 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \sqrt{3} & 3 & 0 \\ 1 & \sqrt{3} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{\sqrt{3}\,R_1}{3} \implies \left[\begin{array}{cc|c} \sqrt{3} & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \sqrt{3} & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\sqrt{3}\}$

Hence the solution is

$$\begin{bmatrix} -t\sqrt{3} \\ t \end{bmatrix} = \begin{bmatrix} -t\sqrt{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t\sqrt{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t\sqrt{3} \\ t \end{bmatrix} = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\sqrt{3}$	1	1	No	$\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$
$-\sqrt{3}$	1	1	No	$\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\sqrt{3}\,t} \\ &= \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} e^{\sqrt{3}\,t}\end{aligned}$$

Since eigenvalue $-\sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-\sqrt{3}\,t} \\ &= \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix} e^{-\sqrt{3}\,t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \sqrt{3}\,\mathrm{e}^{\sqrt{3}\,t} \\ \mathrm{e}^{\sqrt{3}\,t} \end{bmatrix} + c_2 \begin{bmatrix} -\sqrt{3}\,\mathrm{e}^{-\sqrt{3}\,t} \\ \mathrm{e}^{-\sqrt{3}\,t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \sqrt{3}\,\mathrm{e}^{\sqrt{3}\,t} & -\sqrt{3}\,\mathrm{e}^{-\sqrt{3}\,t} \\ \mathrm{e}^{\sqrt{3}\,t} & \mathrm{e}^{-\sqrt{3}\,t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{\sqrt{3}\,\mathrm{e}^{-\sqrt{3}\,t}}{6} & \frac{\mathrm{e}^{-\sqrt{3}\,t}}{2} \\ -\frac{\sqrt{3}\,\mathrm{e}^{\sqrt{3}\,t}}{6} & \frac{\mathrm{e}^{\sqrt{3}\,t}}{2} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \sqrt{3}\,\mathrm{e}^{\sqrt{3}\,t} & -\sqrt{3}\,\mathrm{e}^{-\sqrt{3}\,t} \\ \mathrm{e}^{\sqrt{3}\,t} & \mathrm{e}^{-\sqrt{3}\,t} \end{bmatrix} \int \begin{bmatrix} \frac{\sqrt{3}\,\mathrm{e}^{-\sqrt{3}\,t}}{6} & \frac{\mathrm{e}^{-\sqrt{3}\,t}}{2} \\ -\frac{\sqrt{3}\,\mathrm{e}^{\sqrt{3}\,t}}{6} & \frac{\mathrm{e}^{\sqrt{3}\,t}}{2} \end{bmatrix} \begin{bmatrix} 6t-1 \\ 1-3t \end{bmatrix} dt \\
&= \begin{bmatrix} \sqrt{3}\,\mathrm{e}^{\sqrt{3}\,t} & -\sqrt{3}\,\mathrm{e}^{-\sqrt{3}\,t} \\ \mathrm{e}^{\sqrt{3}\,t} & \mathrm{e}^{-\sqrt{3}\,t} \end{bmatrix} \int \begin{bmatrix} \frac{\left(\sqrt{3}\,(6t-1)-9t+3\right)\mathrm{e}^{-\sqrt{3}\,t}}{6} \\ \frac{\left((-6t+1)\sqrt{3}-9t+3\right)\mathrm{e}^{\sqrt{3}\,t}}{6} \end{bmatrix} dt \\
&= \begin{bmatrix} \sqrt{3}\,\mathrm{e}^{\sqrt{3}\,t} & -\sqrt{3}\,\mathrm{e}^{-\sqrt{3}\,t} \\ \mathrm{e}^{\sqrt{3}\,t} & \mathrm{e}^{-\sqrt{3}\,t} \end{bmatrix} \begin{bmatrix} -\frac{\left(3\sqrt{3}\,t+\sqrt{3}+6\right)\mathrm{e}^{-\sqrt{3}\,t}\left(6\sqrt{3}\,t-\sqrt{3}-9t+3\right)}{54t+18+18\sqrt{3}} \\ \frac{\left(3\sqrt{3}\,t+\sqrt{3}-6\right)\mathrm{e}^{\sqrt{3}\,t}\left(6\sqrt{3}\,t-\sqrt{3}+9t-3\right)}{-54t-18+18\sqrt{3}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{27t^3-9t^2-24t+6}{9t^2+6t-2} \\ \frac{-54t^3+36t-8}{-9+3(3t+1)^2} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1\sqrt{3}\,\mathrm{e}^{\sqrt{3}\,t} \\ c_1\mathrm{e}^{\sqrt{3}\,t} \end{bmatrix} + \begin{bmatrix} -c_2\sqrt{3}\,\mathrm{e}^{-\sqrt{3}\,t} \\ c_2\mathrm{e}^{-\sqrt{3}\,t} \end{bmatrix} + \begin{bmatrix} \frac{27t^3-9t^2-24t+6}{9t^2+6t-2} \\ \frac{-54t^3+36t-8}{-9+3(3t+1)^2} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1\sqrt{3}\,\mathrm{e}^{\sqrt{3}\,t} - c_2\sqrt{3}\,\mathrm{e}^{-\sqrt{3}\,t} + 3t - 3 \\ c_1\mathrm{e}^{\sqrt{3}\,t} + c_2\mathrm{e}^{-\sqrt{3}\,t} - 2t + \frac{4}{3} \end{bmatrix}$$

16.8.3 Maple step by step solution

Let's solve

$[x'(t) = 6t - 1 + 3y(t)\,, y'(t) = x(t) - 3t + 1]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 6t-1 \\ 1-3t \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 6t-1 \\ 1-3t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 6t-1 \\ 1-3t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\sqrt{3}, \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}\right], \left[-\sqrt{3}, \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[\sqrt{3}, \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{\sqrt{3}\,t} \cdot \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-\sqrt{3}, \left[\begin{array}{c} -\sqrt{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{-\sqrt{3}\,t} \cdot \left[\begin{array}{c} -\sqrt{3} \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \vec{x}_p(t)$$

□ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \left[\begin{array}{cc} \sqrt{3}\, e^{\sqrt{3}\,t} & -\sqrt{3}\, e^{-\sqrt{3}\,t} \\ e^{\sqrt{3}\,t} & e^{-\sqrt{3}\,t} \end{array}\right]$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \left[\begin{array}{cc} \sqrt{3}\, e^{\sqrt{3}\,t} & -\sqrt{3}\, e^{-\sqrt{3}\,t} \\ e^{\sqrt{3}\,t} & e^{-\sqrt{3}\,t} \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} \sqrt{3} & -\sqrt{3} \\ 1 & 1 \end{array}\right]}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \left[\begin{array}{cc} \frac{e^{\sqrt{3}\,t}}{2} + \frac{e^{-\sqrt{3}\,t}}{2} & \frac{\left(-e^{-\sqrt{3}\,t} + e^{\sqrt{3}\,t}\right)\sqrt{3}}{2} \\ \frac{\left(-e^{-\sqrt{3}\,t} + e^{\sqrt{3}\,t}\right)\sqrt{3}}{6} & \frac{e^{\sqrt{3}\,t}}{2} + \frac{e^{-\sqrt{3}\,t}}{2} \end{array}\right]$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t) = \begin{bmatrix} \frac{\left(4\sqrt{3}+9\right)e^{-\sqrt{3}\,t}}{6} + \frac{\left(-4\sqrt{3}+9\right)e^{\sqrt{3}\,t}}{6} + 3t - 3 \\ \frac{\left(-3\sqrt{3}-4\right)e^{-\sqrt{3}\,t}}{6} + \frac{\left(3\sqrt{3}-4\right)e^{\sqrt{3}\,t}}{6} - 2t + \frac{4}{3} \end{bmatrix}$$

- Plug particular solution back into general solution

  $$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \begin{bmatrix} \frac{\left(4\sqrt{3}+9\right)e^{-\sqrt{3}\,t}}{6} + \frac{\left(-4\sqrt{3}+9\right)e^{\sqrt{3}\,t}}{6} + 3t - 3 \\ \frac{\left(-3\sqrt{3}-4\right)e^{-\sqrt{3}\,t}}{6} + \frac{\left(3\sqrt{3}-4\right)e^{\sqrt{3}\,t}}{6} - 2t + \frac{4}{3} \end{bmatrix}$$

- Substitute in vector of dependent variables

  $$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{\left(9+(-6c_2+4)\sqrt{3}\right)e^{-\sqrt{3}\,t}}{6} + \frac{\left(9+(-4+6c_1)\sqrt{3}\right)e^{\sqrt{3}\,t}}{6} + 3t - 3 \\ \frac{\left(6c_2-3\sqrt{3}-4\right)e^{-\sqrt{3}\,t}}{6} + \frac{\left(6c_1+3\sqrt{3}-4\right)e^{\sqrt{3}\,t}}{6} - 2t + \frac{4}{3} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x(t) = \frac{\left(9+(-6c_2+4)\sqrt{3}\right)e^{-\sqrt{3}\,t}}{6} + \frac{\left(9+(-4+6c_1)\sqrt{3}\right)e^{\sqrt{3}\,t}}{6} + 3t - 3, y(t) = \frac{\left(6c_2-3\sqrt{3}-4\right)e^{-\sqrt{3}\,t}}{6} + \frac{\left(6c_1+3\sqrt{3}-4\right)\right.}{6}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 60

```
dsolve([diff(x(t),t)+diff(y(t),t)-x(t)-3*y(t)=3*t,diff(x(t),t)+2*diff(y(t),t)-2*x(t)-3*y(t)=
```

$$x(t) = e^{\sqrt{3}\,t}c_2 + e^{-\sqrt{3}\,t}c_1 + 3t - 3$$
$$y(t) = \frac{\sqrt{3}\,e^{\sqrt{3}\,t}c_2}{3} - \frac{\sqrt{3}\,e^{-\sqrt{3}\,t}c_1}{3} + \frac{4}{3} - 2t$$

✓ Solution by Mathematica

Time used: 6.866 (sec). Leaf size: 137

```
DSolve[{x'[t]+y'[t]-x[t]-3*y[t]==3*t,x'[t]+2*y'[t]-2*x[t]-3*y[t]==1},{x[t],y[t]},t,IncludeSi
```

$$x(t) \to \frac{1}{2}e^{-\sqrt{3}t}\left(6e^{\sqrt{3}t}(t-1) + \left(c_1 + \sqrt{3}c_2\right)e^{2\sqrt{3}t} + c_1 - \sqrt{3}c_2\right)$$
$$y(t) \to \frac{1}{6}e^{-\sqrt{3}t}\left(e^{\sqrt{3}t}(8-12t) + \left(\sqrt{3}c_1 + 3c_2\right)e^{2\sqrt{3}t} - \sqrt{3}c_1 + 3c_2\right)$$

16.9 problem 9

Internal problem ID [11936]
Internal file name [`OUTPUT/11945_Sunday_April_14_2024_02_31_07_AM_5011505/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) + y'(t) = -2y(t) + \sin(t)$$
$$x'(t) + y'(t) = x(t) + y(t)$$

The system is

$$x'(t) + y'(t) = -2y(t) + \sin(t) \tag{1}$$
$$x'(t) + y'(t) = x(t) + y(t) \tag{2}$$

Since the left side is the same, this implies

$$-2y(t) + \sin(t) = x(t) + y(t)$$
$$y(t) = -\frac{x(t)}{3} + \frac{\sin(t)}{3} \tag{3}$$

Taking derivative of the above w.r.t. t gives

$$y'(t) = -\frac{x'(t)}{3} + \frac{\cos(t)}{3} \tag{4}$$

Substituting (3,4) in (1) to eliminate $y(t), y'(t)$ gives

$$\frac{2x'(t)}{3} + \frac{\cos(t)}{3} = \frac{2x(t)}{3} + \frac{\sin(t)}{3}$$
$$x'(t) = x(t) + \frac{\sin(t)}{2} - \frac{\cos(t)}{2} \tag{5}$$

Which is now solved for $x(t)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x'(t) + p(t)x(t) = q(t)$$

Where here

$$\begin{aligned} p(t) &= -1 \\ q(t) &= \frac{\sin(t)}{2} - \frac{\cos(t)}{2} \end{aligned}$$

Hence the ode is

$$-x(t) + x'(t) = \frac{\sin(t)}{2} - \frac{\cos(t)}{2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int(-1)dt} \\ &= e^{-t} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dt}(\mu x) &= (\mu)\left(\frac{\sin(t)}{2} - \frac{\cos(t)}{2}\right) \\ \frac{d}{dt}(e^{-t}x) &= (e^{-t})\left(\frac{\sin(t)}{2} - \frac{\cos(t)}{2}\right) \\ d(e^{-t}x) &= \left(\frac{(\sin(t) - \cos(t))e^{-t}}{2}\right) dt \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-t}x &= \int \frac{(\sin(t) - \cos(t))e^{-t}}{2} dt \\ e^{-t}x &= -\frac{e^{-t}\sin(t)}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-t}$ results in

$$x(t) = -\frac{e^{t}e^{-t}\sin(t)}{2} + c_1e^{t}$$

which simplifies to

$$x(t) = c_1e^{t} - \frac{\sin(t)}{2}$$

Given now that we have the solution

$$x(t) = c_1 \mathrm{e}^t - \frac{\sin(t)}{2} \tag{6}$$

Then substituting (6) into (3) gives

$$y(t) = -\frac{c_1 \mathrm{e}^t}{3} + \frac{\sin(t)}{2} \tag{7}$$

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 27

```
dsolve([diff(x(t),t)+diff(y(t),t)+2*y(t)=sin(t),diff(x(t),t)+diff(y(t),t)-x(t)-y(t)=0],sings
```

$$x(t) = c_1 \mathrm{e}^t - \frac{\sin(t)}{2}$$
$$y(t) = -\frac{c_1 \mathrm{e}^t}{3} + \frac{\sin(t)}{2}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 38

```
DSolve[{x'[t]+y'[t]+2*y[t]==Sin[t],x'[t]+y'[t]-x[t]-y[t]==0},{x[t],y[t]},t,IncludeSingularSo
```

$$x(t) \to \frac{1}{2}\left(-\sin(t) + 3c_1 e^t\right)$$
$$y(t) \to \frac{1}{2}\left(\sin(t) - c_1 e^t\right)$$

16.10 problem 10

Internal problem ID [11937]
Internal file name [`OUTPUT/11946_Sunday_April_14_2024_02_31_07_AM_84888398/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= \frac{3x(t)}{2} - \frac{3y(t)}{2} + \frac{t}{2} + \frac{1}{2} \\
y'(t) &= -\frac{x(t)}{2} + \frac{5y(t)}{2} - \frac{t}{2} + \frac{1}{2}
\end{aligned}
$$

16.10.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \frac{t}{2} + \frac{1}{2} \\ -\frac{t}{2} + \frac{1}{2} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{3\,e^{t}}{4}+\frac{e^{3t}}{4} & -\frac{3\,e^{3t}}{4}+\frac{3\,e^{t}}{4} \\ -\frac{e^{3t}}{4}+\frac{e^{t}}{4} & \frac{e^{t}}{4}+\frac{3\,e^{3t}}{4} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{3\,e^{t}}{4}+\frac{e^{3t}}{4} & -\frac{3\,e^{3t}}{4}+\frac{3\,e^{t}}{4} \\ -\frac{e^{3t}}{4}+\frac{e^{t}}{4} & \frac{e^{t}}{4}+\frac{3\,e^{3t}}{4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{3\,e^{t}}{4}+\frac{e^{3t}}{4}\right) c_1 + \left(-\frac{3\,e^{3t}}{4}+\frac{3\,e^{t}}{4}\right) c_2 \\ \left(-\frac{e^{3t}}{4}+\frac{e^{t}}{4}\right) c_1 + \left(\frac{e^{t}}{4}+\frac{3\,e^{3t}}{4}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-3c_2)e^{3t}}{4}+\frac{3\,e^{t}(c_1+c_2)}{4} \\ \frac{(-c_1+3c_2)e^{3t}}{4}+\frac{e^{t}(c_1+c_2)}{4} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \frac{e^{-3t}(3\,e^{2t}+1)}{4} & \frac{3\,e^{-3t}(e^{2t}-1)}{4} \\ \frac{e^{-3t}(e^{2t}-1)}{4} & \frac{e^{-3t}(e^{2t}+3)}{4} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{3\,e^{t}}{4}+\frac{e^{3t}}{4} & -\frac{3\,e^{3t}}{4}+\frac{3\,e^{t}}{4} \\ -\frac{e^{3t}}{4}+\frac{e^{t}}{4} & \frac{e^{t}}{4}+\frac{3\,e^{3t}}{4} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-3t}(3\,e^{2t}+1)}{4} & \frac{3\,e^{-3t}(e^{2t}-1)}{4} \\ \frac{e^{-3t}(e^{2t}-1)}{4} & \frac{e^{-3t}(e^{2t}+3)}{4} \end{bmatrix} \begin{bmatrix} \frac{t}{2}+\frac{1}{2} \\ -\frac{t}{2}+\frac{1}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{3\,e^{t}}{4}+\frac{e^{3t}}{4} & -\frac{3\,e^{3t}}{4}+\frac{3\,e^{t}}{4} \\ -\frac{e^{3t}}{4}+\frac{e^{t}}{4} & \frac{e^{t}}{4}+\frac{3\,e^{3t}}{4} \end{bmatrix} \begin{bmatrix} \frac{(-6t+1)e^{-3t}}{36}-\frac{3\,e^{-t}}{4} \\ \frac{(6t-1)e^{-3t}}{36}-\frac{e^{-t}}{4} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{13}{18}-\frac{t}{6} \\ -\frac{5}{18}+\frac{t}{6} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{(c_1-3c_2)e^{3t}}{4} + \frac{3\,e^t(c_1+c_2)}{4} - \frac{13}{18} - \frac{t}{6} \\ \frac{(-c_1+3c_2)e^{3t}}{4} + \frac{e^t(c_1+c_2)}{4} - \frac{5}{18} + \frac{t}{6} \end{bmatrix}$$

16.10.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \frac{t}{2} + \frac{1}{2} \\ -\frac{t}{2} + \frac{1}{2} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} \frac{3}{2} - \lambda & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} - \lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{1}{2} & -\frac{3}{2} & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} \frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 3t \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{3}{2} & -\frac{3}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{3} \implies \left[\begin{array}{cc|c} -\frac{3}{2} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^t$$
$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^t$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{3t} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} 3\,\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\mathrm{e}^{3t} & 3\,\mathrm{e}^{t} \\ \mathrm{e}^{3t} & \mathrm{e}^{t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{\mathrm{e}^{-3t}}{4} & \frac{3\,\mathrm{e}^{-3t}}{4} \\ \frac{\mathrm{e}^{-t}}{4} & \frac{\mathrm{e}^{-t}}{4} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -e^{3t} & 3\,e^{t} \\ e^{3t} & e^{t} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{-3t}}{4} & \frac{3\,e^{-3t}}{4} \\ \frac{e^{-t}}{4} & \frac{e^{-t}}{4} \end{bmatrix} \begin{bmatrix} \frac{t}{2}+\frac{1}{2} \\ -\frac{t}{2}+\frac{1}{2} \end{bmatrix} dt$$

$$= \begin{bmatrix} -e^{3t} & 3\,e^{t} \\ e^{3t} & e^{t} \end{bmatrix} \int \begin{bmatrix} -\frac{e^{-3t}(2t-1)}{4} \\ \frac{e^{-t}}{4} \end{bmatrix} dt$$

$$= \begin{bmatrix} -e^{3t} & 3\,e^{t} \\ e^{3t} & e^{t} \end{bmatrix} \begin{bmatrix} \frac{(6t-1)e^{-3t}}{36} \\ -\frac{e^{-t}}{4} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{13}{18}-\frac{t}{6} \\ -\frac{5}{18}+\frac{t}{6} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{3t} \\ c_1 e^{3t} \end{bmatrix} + \begin{bmatrix} 3c_2 e^{t} \\ c_2 e^{t} \end{bmatrix} + \begin{bmatrix} -\frac{13}{18}-\frac{t}{6} \\ -\frac{5}{18}+\frac{t}{6} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{3t} + 3c_2 e^{t} - \frac{13}{18} - \frac{t}{6} \\ c_1 e^{3t} + c_2 e^{t} - \frac{5}{18} + \frac{t}{6} \end{bmatrix}$$

16.10.3 Maple step by step solution

Let's solve

$$\left[x'(t) = \frac{3x(t)}{2} - \frac{3y(t)}{2} + \frac{t}{2} + \frac{1}{2}, y'(t) = -\frac{x(t)}{2} + \frac{5y(t)}{2} - \frac{t}{2} + \frac{1}{2}\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\overrightarrow{x}'(t) = \left[\begin{array}{cc} \frac{3}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{array}\right] \cdot \overrightarrow{x}(t) + \left[\begin{array}{c} \frac{t}{2}+\frac{1}{2} \\ -\frac{t}{2}+\frac{1}{2} \end{array}\right]$$

- System to solve

$$\overrightarrow{x}'(t) = \left[\begin{array}{cc} \frac{3}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{array}\right] \cdot \overrightarrow{x}(t) + \left[\begin{array}{c} \frac{t}{2}+\frac{1}{2} \\ -\frac{t}{2}+\frac{1}{2} \end{array}\right]$$

- Define the forcing function

$$\overrightarrow{f}(t) = \left[\begin{array}{c} \frac{t}{2}+\frac{1}{2} \\ -\frac{t}{2}+\frac{1}{2} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} \frac{3}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c} 3 \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 3 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{t} \cdot \left[\begin{array}{c} 3 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$\vec{x}_2 = e^{3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

  $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $\phi(t) = \begin{bmatrix} 3\,e^{t} & -e^{3t} \\ e^{t} & e^{3t} \end{bmatrix}$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $\Phi(t) = \begin{bmatrix} 3\,e^{t} & -e^{3t} \\ e^{t} & e^{3t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}}$

- Evaluate and simplify to get the fundamental matrix

  $\Phi(t) = \begin{bmatrix} \frac{3\,e^{t}}{4} + \frac{e^{3t}}{4} & -\frac{3\,e^{3t}}{4} + \frac{3\,e^{t}}{4} \\ -\frac{e^{3t}}{4} + \frac{e^{t}}{4} & \frac{e^{t}}{4} + \frac{3\,e^{3t}}{4} \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $\vec{x}_p(t) = \begin{bmatrix} \frac{3\,e^t}{4} - \frac{13}{18} - \frac{e^{3t}}{36} - \frac{t}{6} \\ \frac{e^{3t}}{36} + \frac{t}{6} - \frac{5}{18} + \frac{e^t}{4} \end{bmatrix}$

- Plug particular solution back into general solution

  $\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \begin{bmatrix} \frac{3\,e^t}{4} - \frac{13}{18} - \frac{e^{3t}}{36} - \frac{t}{6} \\ \frac{e^{3t}}{36} + \frac{t}{6} - \frac{5}{18} + \frac{e^t}{4} \end{bmatrix}$

- Substitute in vector of dependent variables

  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3c_1 e^t - c_2 e^{3t} + \frac{3\,e^t}{4} - \frac{13}{18} - \frac{e^{3t}}{36} - \frac{t}{6} \\ c_1 e^t + c_2 e^{3t} + \frac{e^{3t}}{36} + \frac{t}{6} - \frac{5}{18} + \frac{e^t}{4} \end{bmatrix}$

- Solution to the system of ODEs

  $\left\{x(t) = 3c_1 e^t - c_2 e^{3t} + \frac{3\,e^t}{4} - \frac{13}{18} - \frac{e^{3t}}{36} - \frac{t}{6}, y(t) = c_1 e^t + c_2 e^{3t} + \frac{e^{3t}}{36} + \frac{t}{6} - \frac{5}{18} + \frac{e^t}{4}\right\}$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 40

```
dsolve([diff(x(t),t)-diff(y(t),t)-2*x(t)+4*y(t)=t,diff(x(t),t)+diff(y(t),t)-x(t)-y(t)=1],sin
```

$$x(t) = c_2 e^t + c_1 e^{3t} - \frac{t}{6} - \frac{13}{18}$$
$$y(t) = \frac{c_2 e^t}{3} - c_1 e^{3t} - \frac{5}{18} + \frac{t}{6}$$

✓ Solution by Mathematica

Time used: 0.083 (sec). Leaf size: 74

```
DSolve[{x'[t]-y'[t]-2*x[t]+4*y[t]==t,x'[t]+y'[t]-x[t]-y[t]==1},{x[t],y[t]},t,IncludeSingular
```

$$x(t) \to \frac{1}{36}\left(-6t + 9(c_1 - 3c_2)e^{3t} + 27(c_1 + c_2)e^t - 26\right)$$
$$y(t) \to \frac{1}{36}\left(6t - 9(c_1 - 3c_2)e^{3t} + 9(c_1 + c_2)e^t - 10\right)$$

16.11 problem 11

Internal problem ID [11938]
Internal file name [`OUTPUT/11947_Sunday_April_14_2024_02_31_09_AM_68378690/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -2 + x(t) - 3y(t) + 4t \\
y'(t) &= 4 - 3x(t) + y(t) - 4t
\end{aligned}
$$

16.11.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 4t-2 \\ 4-4t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-2t}}{2} + \frac{e^{4t}}{2} & -\frac{e^{4t}}{2} + \frac{e^{-2t}}{2} \\ -\frac{e^{4t}}{2} + \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{e^{4t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-2t}}{2} + \frac{e^{4t}}{2} & -\frac{e^{4t}}{2} + \frac{e^{-2t}}{2} \\ -\frac{e^{4t}}{2} + \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{e^{4t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-2t}}{2} + \frac{e^{4t}}{2}\right) c_1 + \left(-\frac{e^{4t}}{2} + \frac{e^{-2t}}{2}\right) c_2 \\ \left(-\frac{e^{4t}}{2} + \frac{e^{-2t}}{2}\right) c_1 + \left(\frac{e^{-2t}}{2} + \frac{e^{4t}}{2}\right) c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-2t}(c_1+c_2)}{2} + \frac{e^{4t}(c_1-c_2)}{2} \\ \frac{e^{-2t}(c_1+c_2)}{2} - \frac{e^{4t}(c_1-c_2)}{2} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \frac{(e^{6t}+1)e^{-4t}}{2} & \frac{(e^{6t}-1)e^{-4t}}{2} \\ \frac{(e^{6t}-1)e^{-4t}}{2} & \frac{(e^{6t}+1)e^{-4t}}{2} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{e^{-2t}}{2} + \frac{e^{4t}}{2} & -\frac{e^{4t}}{2} + \frac{e^{-2t}}{2} \\ -\frac{e^{4t}}{2} + \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{e^{4t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{(e^{6t}+1)e^{-4t}}{2} & \frac{(e^{6t}-1)e^{-4t}}{2} \\ \frac{(e^{6t}-1)e^{-4t}}{2} & \frac{(e^{6t}+1)e^{-4t}}{2} \end{bmatrix} \begin{bmatrix} 4t-2 \\ 4-4t \end{bmatrix} dt \\
&= \begin{bmatrix} \frac{e^{-2t}}{2} + \frac{e^{4t}}{2} & -\frac{e^{4t}}{2} + \frac{e^{-2t}}{2} \\ -\frac{e^{4t}}{2} + \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{e^{4t}}{2} \end{bmatrix} \begin{bmatrix} \frac{(e^{6t}-2t+1)e^{-4t}}{2} \\ \frac{(e^{6t}+2t-1)e^{-4t}}{2} \end{bmatrix} \\
&= \begin{bmatrix} 1-t \\ t \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \frac{\mathrm{e}^{-2t}(c_1+c_2)}{2} + \frac{\mathrm{e}^{4t}(c_1-c_2)}{2} + 1 - t \\ \frac{\mathrm{e}^{-2t}(c_1+c_2)}{2} + \frac{(-c_1+c_2)\mathrm{e}^{4t}}{2} + t \end{bmatrix}
\end{aligned}
$$

16.11.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 4t-2 \\ 4-4t \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & -3 \\ -3 & 1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 2\lambda - 8 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -3 & 0 \\ -3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ -3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{4t}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{4t} \\ e^{4t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^{-2t} & -e^{4t} \\ e^{-2t} & e^{4t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) \, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{e^{2t}}{2} & \frac{e^{2t}}{2} \\ -\frac{e^{-4t}}{2} & \frac{e^{-4t}}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^{-2t} & -e^{4t} \\ e^{-2t} & e^{4t} \end{bmatrix} \int \begin{bmatrix} \frac{e^{2t}}{2} & \frac{e^{2t}}{2} \\ -\frac{e^{-4t}}{2} & \frac{e^{-4t}}{2} \end{bmatrix} \begin{bmatrix} 4t-2 \\ 4-4t \end{bmatrix} dt$$
$$= \begin{bmatrix} e^{-2t} & -e^{4t} \\ e^{-2t} & e^{4t} \end{bmatrix} \int \begin{bmatrix} e^{2t} \\ e^{-4t}(-4t+3) \end{bmatrix} dt$$
$$= \begin{bmatrix} e^{-2t} & -e^{4t} \\ e^{-2t} & e^{4t} \end{bmatrix} \begin{bmatrix} \frac{e^{2t}}{2} \\ \frac{e^{-4t}(2t-1)}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 1-t \\ t \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} \\ c_1 e^{-2t} \end{bmatrix} + \begin{bmatrix} -c_2 e^{4t} \\ c_2 e^{4t} \end{bmatrix} + \begin{bmatrix} 1-t \\ t \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} - c_2 e^{4t} + 1 - t \\ c_1 e^{-2t} + c_2 e^{4t} + t \end{bmatrix}$$

16.11.3 Maple step by step solution

Let's solve

$[x'(t) = -2 + x(t) - 3y(t) + 4t, y'(t) = 4 - 3x(t) + y(t) - 4t]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 4t-2 \\ 4-4t \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \left[\begin{array}{cc} 1 & -3 \\ -3 & 1 \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} 4t-2 \\ 4-4t \end{array}\right]$$

- Define the forcing function

$$\vec{f}(t) = \left[\begin{array}{c} 4t-2 \\ 4-4t \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} 1 & -3 \\ -3 & 1 \end{array}\right]$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} 1 \\ 1 \end{array}\right]\right], \left[4, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-2t} \cdot \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[4, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{4t} \cdot \left[\begin{array}{c} -1 \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} e^{-2t} & -e^{4t} \\ e^{-2t} & e^{4t} \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} e^{-2t} & -e^{4t} \\ e^{-2t} & e^{4t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \frac{e^{-2t}}{2} + \frac{e^{4t}}{2} & -\frac{e^{4t}}{2} + \frac{e^{-2t}}{2} \\ -\frac{e^{4t}}{2} + \frac{e^{-2t}}{2} & \frac{e^{-2t}}{2} + \frac{e^{4t}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

$\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

- Take the derivative of the particular solution

$\overrightarrow{x}'_p(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- Cancel like terms

$\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t) = \left[\begin{array}{c} 1 - \frac{e^{-2t}}{2} - \frac{e^{4t}}{2} - t \\ \frac{e^{4t}}{2} + t - \frac{e^{-2t}}{2} \end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \left[\begin{array}{c} 1 - \frac{e^{-2t}}{2} - \frac{e^{4t}}{2} - t \\ \frac{e^{4t}}{2} + t - \frac{e^{-2t}}{2} \end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{(2c_1-1)e^{-2t}}{2} + \frac{(-2c_2-1)e^{4t}}{2} - t + 1 \\ \frac{(2c_1-1)e^{-2t}}{2} + \frac{(2c_2+1)e^{4t}}{2} + t \end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x(t) = \frac{(2c_1-1)e^{-2t}}{2} + \frac{(-2c_2-1)e^{4t}}{2} - t + 1, y(t) = \frac{(2c_1-1)e^{-2t}}{2} + \frac{(2c_2+1)e^{4t}}{2} + t\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 40

```
dsolve([2*diff(x(t),t)+diff(y(t),t)+x(t)+5*y(t)=4*t,diff(x(t),t)+diff(y(t),t)+2*x(t)+2*y(t)=
```

$$x(t) = c_2 e^{4t} + c_1 e^{-2t} - t + 1$$
$$y(t) = -c_2 e^{4t} + c_1 e^{-2t} + t$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 80

```
DSolve[{2*x'[t]+y'[t]+x[t]+5*y[t]==4*t,x'[t]+y'[t]+2*x[t]+2*y[t]==2},{x[t],y[t]},t,IncludeSi
```

$$x(t) \to \frac{1}{2}e^{-2t}\left(-2e^{2t}(t-1) + (c_1 - c_2)e^{6t} + c_1 + c_2\right)$$
$$y(t) \to \frac{1}{2}e^{-2t}\left(2e^{2t}t + (c_2 - c_1)e^{6t} + c_1 + c_2\right)$$

16.12 problem 12

Internal problem ID [11939]
Internal file name [`OUTPUT/11948_Sunday_April_14_2024_02_31_09_AM_49739451/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2t^2 - 6y(t) - 2t - 1 \\
y'(t) &= -t^2 + x(t) + y(t) + 2t + 1
\end{aligned}
$$

16.12.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 2t^2 - 2t - 1 \\ -t^2 + 2t + 1 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} e^{\frac{t}{2}} \cos\left(\frac{\sqrt{23}\,t}{2}\right) - \frac{\sqrt{23}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} & -\frac{12\sqrt{23}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} \\ \frac{2\sqrt{23}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} & e^{\frac{t}{2}} \cos\left(\frac{\sqrt{23}\,t}{2}\right) + \frac{\sqrt{23}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} \end{bmatrix}
$$

$$
= \begin{bmatrix} -\frac{e^{\frac{t}{2}}\left(\sqrt{23}\sin\left(\frac{\sqrt{23}\,t}{2}\right) - 23\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right)}{23} & -\frac{12\sqrt{23}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} \\ \frac{2\sqrt{23}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} & \frac{e^{\frac{t}{2}}\left(\sqrt{23}\sin\left(\frac{\sqrt{23}\,t}{2}\right) + 23\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right)}{23} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = e^{At}\vec{c}
$$

$$
= \begin{bmatrix} -\frac{e^{\frac{t}{2}}\left(\sqrt{23}\sin\left(\frac{\sqrt{23}\,t}{2}\right) - 23\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right)}{23} & -\frac{12\sqrt{23}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} \\ \frac{2\sqrt{23}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} & \frac{e^{\frac{t}{2}}\left(\sqrt{23}\sin\left(\frac{\sqrt{23}\,t}{2}\right) + 23\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right)}{23} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
$$

$$
= \begin{bmatrix} -\frac{e^{\frac{t}{2}}\left(\sqrt{23}\sin\left(\frac{\sqrt{23}\,t}{2}\right) - 23\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right)c_1}{23} - \frac{12\sqrt{23}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right)c_2}{23} \\ \frac{2\sqrt{23}\, e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right)c_1}{23} + \frac{e^{\frac{t}{2}}\left(\sqrt{23}\sin\left(\frac{\sqrt{23}\,t}{2}\right) + 23\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right)c_2}{23} \end{bmatrix}
$$

$$
= \begin{bmatrix} -\frac{e^{\frac{t}{2}}\left(\sqrt{23}\,(c_1+12c_2)\sin\left(\frac{\sqrt{23}\,t}{2}\right) - 23\cos\left(\frac{\sqrt{23}\,t}{2}\right)c_1\right)}{23} \\ \frac{2\left(\sqrt{23}\,\left(c_1+\frac{c_2}{2}\right)\sin\left(\frac{\sqrt{23}\,t}{2}\right) + \frac{23\cos\left(\frac{\sqrt{23}\,t}{2}\right)c_2}{2}\right)e^{\frac{t}{2}}}{23} \end{bmatrix}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt
$$

But

$$
e^{-At} = (e^{At})^{-1}
$$

$$
= \begin{bmatrix} \frac{\left(\sqrt{23}\sin\left(\frac{\sqrt{23}\,t}{2}\right) + 23\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right)e^{-\frac{t}{2}}}{23} & \frac{12\sin\left(\frac{\sqrt{23}\,t}{2}\right)\sqrt{23}\, e^{-\frac{t}{2}}}{23} \\ -\frac{2\sin\left(\frac{\sqrt{23}\,t}{2}\right)\sqrt{23}\, e^{-\frac{t}{2}}}{23} & -\frac{\left(\sqrt{23}\sin\left(\frac{\sqrt{23}\,t}{2}\right) - 23\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right)e^{-\frac{t}{2}}}{23} \end{bmatrix}
$$

Hence

$$
\vec{x}_p(t) = \begin{bmatrix} -\frac{e^{\frac{t}{2}}\left(\sqrt{23}\,\sin\left(\frac{\sqrt{23}\,t}{2}\right)-23\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right)}{23} & -\frac{12\sqrt{23}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} \\ \frac{2\sqrt{23}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} & \frac{e^{\frac{t}{2}}\left(\sqrt{23}\,\sin\left(\frac{\sqrt{23}\,t}{2}\right)+23\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right)}{23} \end{bmatrix} \int \begin{bmatrix} \frac{\left(\sqrt{23}\,\sin\left(\frac{\sqrt{23}\,t}{2}\right)+23\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right)e}{23} \\ -\frac{2\sin\left(\frac{\sqrt{23}\,t}{2}\right)\sqrt{23}\,e^{-\frac{t}{2}}}{23} \end{bmatrix}
$$

$$
= \begin{bmatrix} -\frac{e^{\frac{t}{2}}\left(\sqrt{23}\,\sin\left(\frac{\sqrt{23}\,t}{2}\right)-23\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right)}{23} & -\frac{12\sqrt{23}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} \\ \frac{2\sqrt{23}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} & \frac{e^{\frac{t}{2}}\left(\sqrt{23}\,\sin\left(\frac{\sqrt{23}\,t}{2}\right)+23\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right)}{23} \end{bmatrix} \begin{bmatrix} \frac{14\,e^{-\frac{t}{2}}\left(\left(\frac{23}{7}t^2-\frac{23}{6}t-\frac{943}{126}\right)\cos\left(\frac{\sqrt{23}\,t}{2}\right)+\right.}{69} \\ -\frac{5\,e^{-\frac{t}{2}}\left(\left(-\frac{23}{5}t^2+\frac{23}{3}t+\frac{23}{45}\right)\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right.}{69} \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{2}{3}t^2 - \frac{7}{9}t - \frac{41}{27} \\ \frac{1}{3}t^2 - \frac{5}{9}t - \frac{1}{27} \end{bmatrix}
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} -\frac{e^{\frac{t}{2}}\sqrt{23}\,(c_1+12c_2)\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} + e^{\frac{t}{2}}\cos\left(\frac{\sqrt{23}\,t}{2}\right)c_1 + \frac{2t^2}{3} - \frac{7t}{9} - \frac{41}{27} \\ \frac{2\,e^{\frac{t}{2}}\sqrt{23}\,(c_1+\frac{c_2}{2})\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} + e^{\frac{t}{2}}\cos\left(\frac{\sqrt{23}\,t}{2}\right)c_2 + \frac{t^2}{3} - \frac{5t}{9} - \frac{1}{27} \end{bmatrix}
\end{aligned}
$$

16.12.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 2t^2 - 2t - 1 \\ -t^2 + 2t + 1 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$\det\left(\begin{bmatrix}0 & -6\\1 & 1\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix}-\lambda & -6\\1 & 1-\lambda\end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - \lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{1}{2} + \frac{i\sqrt{23}}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{i\sqrt{23}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{1}{2} - \frac{i\sqrt{23}}{2}$	1	complex eigenvalue
$\frac{1}{2} + \frac{i\sqrt{23}}{2}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{1}{2} - \frac{i\sqrt{23}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}0 & -6\\1 & 1\end{bmatrix} - \left(\frac{1}{2} - \frac{i\sqrt{23}}{2}\right)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$
$$\begin{bmatrix}-\frac{1}{2} + \frac{i\sqrt{23}}{2} & -6\\1 & \frac{1}{2} + \frac{i\sqrt{23}}{2}\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c}-\frac{1}{2} + \frac{i\sqrt{23}}{2} & -6 & 0\\1 & \frac{1}{2} + \frac{i\sqrt{23}}{2} & 0\end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{-\frac{1}{2}+\frac{i\sqrt{23}}{2}} \Longrightarrow \left[\begin{array}{cc|c} -\frac{1}{2}+\frac{i\sqrt{23}}{2} & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{2}+\frac{i\sqrt{23}}{2} & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{12t}{i\sqrt{23}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{12t}{\mathrm{I}\sqrt{23}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{12t}{i\sqrt{23}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{12t}{\mathrm{I}\sqrt{23}-1} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{12}{i\sqrt{23}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{12t}{\mathrm{I}\sqrt{23}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{12}{i\sqrt{23}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{12t}{\mathrm{I}\sqrt{23}-1} \\ t \end{bmatrix} = \begin{bmatrix} \frac{12}{i\sqrt{23}-1} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{1}{2} + \frac{i\sqrt{23}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -6 \\ 1 & 1 \end{bmatrix} - \left(\frac{1}{2}+\frac{i\sqrt{23}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2}-\frac{i\sqrt{23}}{2} & -6 \\ 1 & \frac{1}{2}-\frac{i\sqrt{23}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{1}{2}-\frac{i\sqrt{23}}{2} & -6 & 0 \\ 1 & \frac{1}{2}-\frac{i\sqrt{23}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{-\frac{1}{2}-\frac{i\sqrt{23}}{2}} \Longrightarrow \left[\begin{array}{cc|c} -\frac{1}{2}-\frac{i\sqrt{23}}{2} & -6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{1}{2}-\frac{i\sqrt{23}}{2} & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{12t}{1+i\sqrt{23}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{12t}{1+\mathrm{I}\sqrt{23}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{12t}{1+i\sqrt{23}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{12t}{1+\mathrm{I}\sqrt{23}} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{12}{1+i\sqrt{23}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{12t}{1+\mathrm{I}\sqrt{23}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{12}{1+i\sqrt{23}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{12t}{1+\mathrm{I}\sqrt{23}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{12}{1+i\sqrt{23}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$\frac{1}{2}+\frac{i\sqrt{23}}{2}$	1	1	No	$\left[\begin{array}{c} -\frac{6}{\frac{1}{2}+\frac{i\sqrt{23}}{2}} \\ 1 \end{array}\right]$
$\frac{1}{2}-\frac{i\sqrt{23}}{2}$	1	1	No	$\left[\begin{array}{c} -\frac{6}{\frac{1}{2}-\frac{i\sqrt{23}}{2}} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = c_1 \left[\begin{array}{c} -\frac{6\,\mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{23}}{2}\right)t}}{\frac{1}{2}+\frac{i\sqrt{23}}{2}} \\ \mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{23}}{2}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{6\,\mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{23}}{2}\right)t}}{\frac{1}{2}-\frac{i\sqrt{23}}{2}} \\ \mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{23}}{2}\right)t} \end{array}\right]$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \left[\begin{array}{cc} -\frac{6\,\mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{23}}{2}\right)t}}{\frac{1}{2}+\frac{i\sqrt{23}}{2}} & -\frac{6\,\mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{23}}{2}\right)t}}{\frac{1}{2}-\frac{i\sqrt{23}}{2}} \\ \mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{23}}{2}\right)t} & \mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{23}}{2}\right)t} \end{array}\right]$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$
\Phi^{-1} = \left[\begin{array}{cc} -\frac{i\sqrt{23}\,{\mathrm e}^{-\frac{\left(1+i\sqrt{23}\right)t}{2}}}{23} & -\frac{\sqrt{23}\,{\mathrm e}^{-\frac{\left(1+i\sqrt{23}\right)t}{2}}\left(i-\sqrt{23}\right)}{46} \\ \frac{i\sqrt{23}\,{\mathrm e}^{\frac{\left(i\sqrt{23}-1\right)t}{2}}}{23} & \frac{\sqrt{23}\,{\mathrm e}^{\frac{\left(i\sqrt{23}-1\right)t}{2}}\left(\sqrt{23}+i\right)}{46} \end{array}\right]
$$

Hence

$$
\vec{x}_p(t) = \left[\begin{array}{cc} -\frac{6\,{\mathrm e}^{\left(\frac{1}{2}+\frac{i\sqrt{23}}{2}\right)t}}{\frac{1}{2}+\frac{i\sqrt{23}}{2}} & -\frac{6\,{\mathrm e}^{\left(\frac{1}{2}-\frac{i\sqrt{23}}{2}\right)t}}{\frac{1}{2}-\frac{i\sqrt{23}}{2}} \\ {\mathrm e}^{\left(\frac{1}{2}+\frac{i\sqrt{23}}{2}\right)t} & {\mathrm e}^{\left(\frac{1}{2}-\frac{i\sqrt{23}}{2}\right)t} \end{array}\right] \int \left[\begin{array}{cc} -\frac{i\sqrt{23}\,{\mathrm e}^{-\frac{\left(1+i\sqrt{23}\right)t}{2}}}{23} & -\frac{\sqrt{23}\,{\mathrm e}^{-\frac{\left(1+i\sqrt{23}\right)t}{2}}\left(i-\sqrt{23}\right)}{46} \\ \frac{i\sqrt{23}\,{\mathrm e}^{\frac{\left(i\sqrt{23}-1\right)t}{2}}}{23} & \frac{\sqrt{23}\,{\mathrm e}^{\frac{\left(i\sqrt{23}-1\right)t}{2}}\left(\sqrt{23}+i\right)}{46} \end{array}\right] \left[\begin{array}{c} 2t^2-2t-1\cdots \\ -t^2+2t+1\cdots \end{array}\right.
$$

$$
= \left[\begin{array}{cc} -\frac{6\,{\mathrm e}^{\left(\frac{1}{2}+\frac{i\sqrt{23}}{2}\right)t}}{\frac{1}{2}+\frac{i\sqrt{23}}{2}} & -\frac{6\,{\mathrm e}^{\left(\frac{1}{2}-\frac{i\sqrt{23}}{2}\right)t}}{\frac{1}{2}-\frac{i\sqrt{23}}{2}} \\ {\mathrm e}^{\left(\frac{1}{2}+\frac{i\sqrt{23}}{2}\right)t} & {\mathrm e}^{\left(\frac{1}{2}-\frac{i\sqrt{23}}{2}\right)t} \end{array}\right] \int \left[\begin{array}{c} -\frac{3\,{\mathrm e}^{-\frac{\left(1+i\sqrt{23}\right)t}{2}}\left(i\left(t+\frac{1}{3}\right)(t-1)\sqrt{23}+\frac{23t^2}{3}-\frac{46t}{3}-\frac{23}{3}\right)}{46} \\ \frac{3\,{\mathrm e}^{\frac{\left(i\sqrt{23}-1\right)t}{2}}\left(i\left(t+\frac{1}{3}\right)(t-1)\sqrt{23}-\frac{23t^2}{3}+\frac{46t}{3}+\frac{23}{3}\right)}{46} \end{array}\right] dt
$$

$$
= \left[\begin{array}{cc} -\frac{6\,{\mathrm e}^{\left(\frac{1}{2}+\frac{i\sqrt{23}}{2}\right)t}}{\frac{1}{2}+\frac{i\sqrt{23}}{2}} & -\frac{6\,{\mathrm e}^{\left(\frac{1}{2}-\frac{i\sqrt{23}}{2}\right)t}}{\frac{1}{2}-\frac{i\sqrt{23}}{2}} \\ {\mathrm e}^{\left(\frac{1}{2}+\frac{i\sqrt{23}}{2}\right)t} & {\mathrm e}^{\left(\frac{1}{2}-\frac{i\sqrt{23}}{2}\right)t} \end{array}\right] \left[\begin{array}{c} \frac{\sqrt{23}\,{\mathrm e}^{-\frac{\left(1+i\sqrt{23}\right)t}{2}}\left(-36i\sqrt{23}\,t^2+51i\sqrt{23}\,t+43i\sqrt{23}+36t^2-87t+113\right)\left(-3it^2-\sqrt{23}\,t^2\cdots\right.}{16146-1242i(2t+1)\sqrt{23}-19872t^2+32292t} \\ -\frac{\sqrt{23}\,{\mathrm e}^{\frac{\left(i\sqrt{23}-1\right)t}{2}}\left(-36\sqrt{23}\,t^2+51\sqrt{23}\,t+43\sqrt{23}+36it^2-87it+113i\right)\left(3it^2-\sqrt{23}\,t^2\cdots\right.}{(2484t+1242)\sqrt{23}+19872it^2-32292it-16146i} \end{array}\right.
$$

$$
= \left[\begin{array}{c} \frac{-\frac{128}{3}t^6+\frac{1696}{9}t^5-\frac{3328}{27}t^4-\frac{10136}{27}t^3+\frac{6776}{27}t^2+\frac{2960}{9}t+\frac{656}{9}}{\left(-\frac{13}{2}+i\left(t+\frac{1}{2}\right)\sqrt{23}+8t^2-13t\right)\left(\frac{13}{2}+i\left(t+\frac{1}{2}\right)\sqrt{23}-8t^2+13t\right)} \\ \frac{-\frac{64}{3}t^6+\frac{944}{9}t^5-\frac{3848}{27}t^4-\frac{616}{27}t^3+\frac{2536}{27}t^2+\frac{304}{9}t+\frac{16}{9}}{\left(-\frac{13}{2}+i\left(t+\frac{1}{2}\right)\sqrt{23}+8t^2-13t\right)\left(\frac{13}{2}+i\left(t+\frac{1}{2}\right)\sqrt{23}-8t^2+13t\right)} \end{array}\right]
$$

Now that we found particular solution, the final solution is

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

$$
\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} -\frac{6c_1{\mathrm e}^{\left(\frac{1}{2}+\frac{i\sqrt{23}}{2}\right)t}}{\frac{1}{2}+\frac{i\sqrt{23}}{2}} \\ c_1{\mathrm e}^{\left(\frac{1}{2}+\frac{i\sqrt{23}}{2}\right)t} \end{array}\right] + \left[\begin{array}{c} -\frac{6c_2{\mathrm e}^{\left(\frac{1}{2}-\frac{i\sqrt{23}}{2}\right)t}}{\frac{1}{2}-\frac{i\sqrt{23}}{2}} \\ c_2{\mathrm e}^{\left(\frac{1}{2}-\frac{i\sqrt{23}}{2}\right)t} \end{array}\right] + \left[\begin{array}{c} \frac{-\frac{128}{3}t^6+\frac{1696}{9}t^5-\frac{3328}{27}t^4-\frac{10136}{27}t^3+\frac{6776}{27}t^2+\frac{2960}{9}t+\frac{656}{9}}{\left(-\frac{13}{2}+i\left(t+\frac{1}{2}\right)\sqrt{23}+8t^2-13t\right)\left(\frac{13}{2}+i\left(t+\frac{1}{2}\right)\sqrt{23}-8t^2+13t\right)} \\ \frac{-\frac{64}{3}t^6+\frac{944}{9}t^5-\frac{3848}{27}t^4-\frac{616}{27}t^3+\frac{2536}{27}t^2+\frac{304}{9}t+\frac{16}{9}}{\left(-\frac{13}{2}+i\left(t+\frac{1}{2}\right)\sqrt{23}+8t^2-13t\right)\left(\frac{13}{2}+i\left(t+\frac{1}{2}\right)\sqrt{23}-8t^2+13t\right)} \end{array}\right]
$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{32\left(-t^4+\frac{13}{4}t^3-\frac{11}{8}t^2-3t-\frac{3}{4}\right)\left(-\frac{82}{27}+\left(i\sqrt{23}-1\right)c_1 \mathrm{e}^{\frac{\left(1+i\sqrt{23}\right)t}{2}}+c_2\left(-1-i\sqrt{23}\right)\mathrm{e}^{-\frac{\left(i\sqrt{23}-1\right)t}{2}}+\frac{4t^2}{3}-\frac{14t}{9}\right)}{\left(-\frac{13}{2}+i\left(t+\frac{1}{2}\right)\sqrt{23}+8t^2-13t\right)\left(\frac{13}{2}+i\left(t+\frac{1}{2}\right)\sqrt{23}-8t^2+13t\right)} \\ \frac{8\left(-8t^4+26t^3-11t^2-24t-6\right)\left(27c_2 \mathrm{e}^{-\frac{\left(i\sqrt{23}-1\right)t}{2}}+27c_1 \mathrm{e}^{\frac{\left(1+i\sqrt{23}\right)t}{2}}+9t^2-15t-1\right)}{27\left(-\frac{13}{2}+i\left(t+\frac{1}{2}\right)\sqrt{23}+8t^2-13t\right)\left(\frac{13}{2}+i\left(t+\frac{1}{2}\right)\sqrt{23}-8t^2+13t\right)} \end{bmatrix}$$

16.12.3 Maple step by step solution

Let's solve

$[x'(t) = 2t^2 - 6y(t) - 2t - 1, y'(t) = -t^2 + x(t) + y(t) + 2t + 1]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -6 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 2t^2 - 2t - 1 \\ -t^2 + 2t + 1 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -6 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 2t^2 - 2t - 1 \\ -t^2 + 2t + 1 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 2t^2 - 2t - 1 \\ -t^2 + 2t + 1 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -6 \\ 1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\frac{1}{2}-\frac{\mathrm{I}\sqrt{23}}{2}, \left[\begin{array}{c} -\frac{6}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{23}}{2}} \\ 1 \end{array}\right]\right], \left[\frac{1}{2}+\frac{\mathrm{I}\sqrt{23}}{2}, \left[\begin{array}{c} -\frac{6}{\frac{1}{2}+\frac{\mathrm{I}\sqrt{23}}{2}} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{2}-\frac{\mathrm{I}\sqrt{23}}{2}, \left[\begin{array}{c} -\frac{6}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{23}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{23}}{2}\right)t} \cdot \left[\begin{array}{c} -\frac{6}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{23}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{\frac{t}{2}} \cdot \left(\cos\left(\frac{\sqrt{23}\, t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{23}\, t}{2}\right)\right) \cdot \left[\begin{array}{c} -\frac{6}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{23}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{\frac{t}{2}} \cdot \left[\begin{array}{c} -\frac{6\left(\cos\left(\frac{\sqrt{23}\, t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{23}\, t}{2}\right)\right)}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{23}}{2}} \\ \cos\left(\frac{\sqrt{23}\, t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{23}\, t}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t)=\mathrm{e}^{\frac{t}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{\sqrt{23}\, t}{2}\right)}{2}-\frac{\sqrt{23}\,\sin\left(\frac{\sqrt{23}\, t}{2}\right)}{2} \\ \cos\left(\frac{\sqrt{23}\, t}{2}\right) \end{array}\right], \vec{x}_2(t)=\mathrm{e}^{\frac{t}{2}} \cdot \left[\begin{array}{c} -\frac{\sqrt{23}\,\cos\left(\frac{\sqrt{23}\, t}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{23}\, t}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{23}\, t}{2}\right) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$\vec{x}(t)=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)+\vec{x}_p(t)$

- ☐ Fundamental matrix
  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syste

$$\phi(t) = \begin{bmatrix} e^{\frac{t}{2}}\left(-\frac{\cos\left(\frac{\sqrt{23}\,t}{2}\right)}{2} - \frac{\sqrt{23}\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{2}\right) & e^{\frac{t}{2}}\left(-\frac{\sqrt{23}\cos\left(\frac{\sqrt{23}\,t}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{2}\right) \\ e^{\frac{t}{2}}\cos\left(\frac{\sqrt{23}\,t}{2}\right) & -e^{\frac{t}{2}}\sin\left(\frac{\sqrt{23}\,t}{2}\right) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} e^{\frac{t}{2}}\left(-\frac{\cos\left(\frac{\sqrt{23}\,t}{2}\right)}{2} - \frac{\sqrt{23}\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{2}\right) & e^{\frac{t}{2}}\left(-\frac{\sqrt{23}\cos\left(\frac{\sqrt{23}\,t}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{2}\right) \\ e^{\frac{t}{2}}\cos\left(\frac{\sqrt{23}\,t}{2}\right) & -e^{\frac{t}{2}}\sin\left(\frac{\sqrt{23}\,t}{2}\right) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{23}}{2} \\ 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \frac{\left(\sqrt{23}\cos\left(\frac{\sqrt{23}\,t}{2}\right) - \sin\left(\frac{\sqrt{23}\,t}{2}\right)\right)\sqrt{23}\,e^{\frac{t}{2}}}{23} & -\frac{12\sqrt{23}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} \\ \frac{2\sqrt{23}\,e^{\frac{t}{2}}\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{23} & \frac{e^{\frac{t}{2}}\left(\sqrt{23}\sin\left(\frac{\sqrt{23}\,t}{2}\right) + 23\cos\left(\frac{\sqrt{23}\,t}{2}\right)\right)}{23} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} -\frac{53\sqrt{23}\,e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right)}{621} + \frac{41\,e^{\frac{t}{2}} \cos\left(\frac{\sqrt{23}\,t}{2}\right)}{27} + \frac{2t^2}{3} - \frac{7t}{9} - \frac{41}{27} \\ \frac{83\sqrt{23}\,e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right)}{621} + \frac{e^{\frac{t}{2}} \cos\left(\frac{\sqrt{23}\,t}{2}\right)}{27} + \frac{t^2}{3} - \frac{5t}{9} - \frac{1}{27} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \begin{bmatrix} -\frac{53\sqrt{23}\,e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right)}{621} + \frac{41\,e^{\frac{t}{2}} \cos\left(\frac{\sqrt{23}\,t}{2}\right)}{27} + \frac{2t^2}{3} - \frac{7t}{9} - \frac{41}{27} \\ \frac{83\sqrt{23}\,e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right)}{621} + \frac{e^{\frac{t}{2}} \cos\left(\frac{\sqrt{23}\,t}{2}\right)}{27} + \frac{t^2}{3} - \frac{5t}{9} - \frac{1}{27} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{e^{\frac{t}{2}}\left(c_2\sqrt{23}+c_1-\frac{82}{27}\right)\cos\left(\frac{\sqrt{23}\,t}{2}\right)}{2} - \frac{e^{\frac{t}{2}}\left(\left(c_1+\frac{106}{621}\right)\sqrt{23}-c_2\right)\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{2} + \frac{2t^2}{3} - \frac{7t}{9} - \frac{41}{27} \\ -\frac{1}{27} + \frac{(27c_1+1)e^{\frac{t}{2}}\cos\left(\frac{\sqrt{23}\,t}{2}\right)}{27} - e^{\frac{t}{2}}\left(c_2 - \frac{83\sqrt{23}}{621}\right)\sin\left(\frac{\sqrt{23}\,t}{2}\right) + \frac{t^2}{3} - \frac{5t}{9} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{e^{\frac{t}{2}}\left(c_2\sqrt{23}+c_1-\frac{82}{27}\right)\cos\left(\frac{\sqrt{23}\,t}{2}\right)}{2} - \frac{e^{\frac{t}{2}}\left(\left(c_1+\frac{106}{621}\right)\sqrt{23}-c_2\right)\sin\left(\frac{\sqrt{23}\,t}{2}\right)}{2} + \frac{2t^2}{3} - \frac{7t}{9} - \frac{41}{27}, y(t) = -\frac{1}{27} + \frac{(2\right.$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 116

```
dsolve([diff(x(t),t)+diff(y(t),t)-x(t)+5*y(t)=t^2,diff(x(t),t)+2*diff(y(t),t)-2*x(t)+4*y(t)=
```

$$x(t) = e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right) c_2 + e^{\frac{t}{2}} \cos\left(\frac{\sqrt{23}\,t}{2}\right) c_1 + \frac{2t^2}{3} - \frac{7t}{9} - \frac{41}{27}$$

$$y(t) = \frac{t^2}{3} - \frac{e^{\frac{t}{2}} \sin\left(\frac{\sqrt{23}\,t}{2}\right) c_2}{12} - \frac{e^{\frac{t}{2}}\sqrt{23} \cos\left(\frac{\sqrt{23}\,t}{2}\right) c_2}{12} - \frac{e^{\frac{t}{2}} \cos\left(\frac{\sqrt{23}\,t}{2}\right) c_1}{12} + \frac{e^{\frac{t}{2}}\sqrt{23} \sin\left(\frac{\sqrt{23}\,t}{2}\right) c_1}{12} - \frac{5t}{9} - \frac{1}{27}$$

✓ Solution by Mathematica

Time used: 11.178 (sec). Leaf size: 143

```
DSolve[{x'[t]+y'[t]-x[t]+5*y[t]==t^2,x'[t]+2*y'[t]-2*x[t]+4*y[t]==2*t+1},{x[t],y[t]},t,Inclu
```

$$x(t) \to \frac{1}{27}\left(18t^2 - 21t - 41\right) + c_1 e^{t/2}\cos\left(\frac{\sqrt{23}t}{2}\right) - \frac{(c_1 + 12c_2)e^{t/2}\sin\left(\frac{\sqrt{23}t}{2}\right)}{\sqrt{23}}$$

$$y(t) \to \frac{1}{27}\left(9t^2 - 15t - 1\right) + c_2 e^{t/2}\cos\left(\frac{\sqrt{23}t}{2}\right) + \frac{(2c_1 + c_2)e^{t/2}\sin\left(\frac{\sqrt{23}t}{2}\right)}{\sqrt{23}}$$

16.13 problem 13

Internal problem ID [11940]
Internal file name [`OUTPUT/11949_Sunday_April_14_2024_02_31_12_AM_27299389/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -t^2 + x(t) + y(t) + 6t \\
y'(t) &= 3t^2 - 3x(t) - 3y(t) - 8t
\end{aligned}
$$

16.13.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -t^2 + 6t \\ 3t^2 - 8t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{-2t}}{2} + \frac{3}{2} & \frac{1}{2} - \frac{e^{-2t}}{2} \\ -\frac{3}{2} + \frac{3\,e^{-2t}}{2} & \frac{3\,e^{-2t}}{2} - \frac{1}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} -\frac{e^{-2t}}{2} + \frac{3}{2} & \frac{1}{2} - \frac{e^{-2t}}{2} \\ -\frac{3}{2} + \frac{3\,e^{-2t}}{2} & \frac{3\,e^{-2t}}{2} - \frac{1}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(-\frac{e^{-2t}}{2} + \frac{3}{2}\right) c_1 + \left(\frac{1}{2} - \frac{e^{-2t}}{2}\right) c_2 \\ \left(-\frac{3}{2} + \frac{3\,e^{-2t}}{2}\right) c_1 + \left(\frac{3\,e^{-2t}}{2} - \frac{1}{2}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(-c_1-c_2)e^{-2t}}{2} + \frac{3c_1}{2} + \frac{c_2}{2} \\ \frac{(3c_1+3c_2)e^{-2t}}{2} - \frac{3c_1}{2} - \frac{c_2}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{3}{2} - \frac{e^{2t}}{2} & -\frac{e^{2t}}{2} + \frac{1}{2} \\ \frac{3\,e^{2t}}{2} - \frac{3}{2} & -\frac{1}{2} + \frac{3\,e^{2t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} -\frac{e^{-2t}}{2} + \frac{3}{2} & \frac{1}{2} - \frac{e^{-2t}}{2} \\ -\frac{3}{2} + \frac{3\,e^{-2t}}{2} & \frac{3\,e^{-2t}}{2} - \frac{1}{2} \end{bmatrix} \int \begin{bmatrix} \frac{3}{2} - \frac{e^{2t}}{2} & -\frac{e^{2t}}{2} + \frac{1}{2} \\ \frac{3\,e^{2t}}{2} - \frac{3}{2} & -\frac{1}{2} + \frac{3\,e^{2t}}{2} \end{bmatrix} \begin{bmatrix} -t^2 + 6t \\ 3t^2 - 8t \end{bmatrix} dt \\ &= \begin{bmatrix} -\frac{e^{-2t}}{2} + \frac{3}{2} & \frac{1}{2} - \frac{e^{-2t}}{2} \\ -\frac{3}{2} + \frac{3\,e^{-2t}}{2} & \frac{3\,e^{-2t}}{2} - \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{(t-1)^2 e^{2t}}{2} + \frac{5t^2}{2} \\ \frac{3(t-1)^2 e^{2t}}{2} - \frac{5t^2}{2} \end{bmatrix} \\ &= \begin{bmatrix} 2t^2 + t - \frac{1}{2} \\ -t^2 - 3t + \frac{3}{2} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= \begin{bmatrix} \frac{(-c_1-c_2)e^{-2t}}{2} + \frac{3c_1}{2} + \frac{c_2}{2} + 2t^2 + t - \frac{1}{2} \\ \frac{(3c_1+3c_2)e^{-2t}}{2} - \frac{3c_1}{2} - \frac{c_2}{2} - t^2 - 3t + \frac{3}{2} \end{bmatrix}$$

16.13.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -t^2 + 6t \\ 3t^2 - 8t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ -3 & -3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 2\lambda = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 0$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
-2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & 1 & 0 \\ -3 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + 3R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$
0	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{-2t}$$
$$= \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} e^{-2t}$$

Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{0}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{0}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{e^{-2t}}{3} \\ e^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\frac{e^{-2t}}{3} & -1 \\ e^{-2t} & 1 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{3\, e^{2t}}{2} & \frac{3\, e^{2t}}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -\frac{e^{-2t}}{3} & -1 \\ e^{-2t} & 1 \end{bmatrix} \int \begin{bmatrix} \frac{3\,e^{2t}}{2} & \frac{3\,e^{2t}}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -t^2+6t \\ 3t^2-8t \end{bmatrix} dt$$

$$= \begin{bmatrix} -\frac{e^{-2t}}{3} & -1 \\ e^{-2t} & 1 \end{bmatrix} \int \begin{bmatrix} 3\,e^{2t}t(t-1) \\ -5t \end{bmatrix} dt$$

$$= \begin{bmatrix} -\frac{e^{-2t}}{3} & -1 \\ e^{-2t} & 1 \end{bmatrix} \begin{bmatrix} \frac{3(t-1)^2 e^{2t}}{2} \\ -\frac{5t^2}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2t^2+t-\frac{1}{2} \\ -t^2-3t+\frac{3}{2} \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{-2t}}{3} \\ c_1 e^{-2t} \end{bmatrix} + \begin{bmatrix} -c_2 \\ c_2 \end{bmatrix} + \begin{bmatrix} 2t^2+t-\frac{1}{2} \\ -t^2-3t+\frac{3}{2} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{-2t}}{3} - c_2 + 2t^2 + t - \frac{1}{2} \\ c_1 e^{-2t} + c_2 - t^2 - 3t + \frac{3}{2} \end{bmatrix}$$

16.13.3 Maple step by step solution

Let's solve

$[x'(t) = -t^2 + x(t) + y(t) + 6t, y'(t) = 3t^2 - 3x(t) - 3y(t) - 8t]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -t^2+6t \\ 3t^2-8t \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -t^2+6t \\ 3t^2-8t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} -t^2+6t \\ 3t^2-8t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $\phi(t) = \begin{bmatrix} -\frac{e^{-2t}}{3} & -1 \\ e^{-2t} & 1 \end{bmatrix}$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $\Phi(t) = \begin{bmatrix} -\frac{e^{-2t}}{3} & -1 \\ e^{-2t} & 1 \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{3} & -1 \\ 1 & 1 \end{bmatrix}}$

- Evaluate and simplify to get the fundamental matrix

  $\Phi(t) = \begin{bmatrix} -\frac{e^{-2t}}{2} + \frac{3}{2} & \frac{1}{2} - \frac{e^{-2t}}{2} \\ -\frac{3}{2} + \frac{3\,e^{-2t}}{2} & \frac{3\,e^{-2t}}{2} - \frac{1}{2} \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

- Take the derivative of the particular solution

  $\overrightarrow{x}'_p(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t) = \begin{bmatrix} 2t^2 + \frac{e^{-2t}}{2} + t - \frac{1}{2} \\ -t^2 + \frac{3}{2} - \frac{3\,e^{-2t}}{2} - 3t \end{bmatrix}$$

- Plug particular solution back into general solution

  $$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \begin{bmatrix} 2t^2 + \frac{e^{-2t}}{2} + t - \frac{1}{2} \\ -t^2 + \frac{3}{2} - \frac{3\,e^{-2t}}{2} - 3t \end{bmatrix}$$

- Substitute in vector of dependent variables

  $$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{(-2c_1+3)e^{-2t}}{6} + 2t^2 + t - c_2 - \frac{1}{2} \\ c_1 e^{-2t} - t^2 + \frac{3}{2} - \frac{3\,e^{-2t}}{2} - 3t + c_2 \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x(t) = \frac{(-2c_1+3)e^{-2t}}{6} + 2t^2 + t - c_2 - \frac{1}{2}, y(t) = c_1 e^{-2t} - t^2 + \frac{3}{2} - \frac{3\,e^{-2t}}{2} - 3t + c_2\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 43

```
dsolve([2*diff(x(t),t)+diff(y(t),t)+x(t)+y(t)=t^2+4*t,diff(x(t),t)+diff(y(t),t)+2*x(t)+2*y(t
```

$$x(t) = -\frac{c_1 e^{-2t}}{2} + 2t^2 + t + c_2$$
$$y(t) = -t^2 + \frac{3c_1 e^{-2t}}{2} - 3t + 1 - c_2$$

✓ Solution by Mathematica

Time used: 0.101 (sec). Leaf size: 82

```
DSolve[{2*x'[t]+y'[t]+x[t]+y[t]==t^2+4*t,x'[t]+y'[t]+2*x[t]+2*y[t]==2*t^2-2*t},{x[t],y[t]},t
```

$$x(t) \to \frac{1}{2}e^{-2t}\left(e^{2t}\left(4t^2+2t-1+3c_1+c_2\right)-c_1-c_2\right)$$
$$y(t) \to \frac{1}{2}\left(-2t^2-6t+3(c_1+c_2)e^{-2t}+3-3c_1-c_2\right)$$

16.14 problem 14

Internal problem ID [11941]
Internal file name [`OUTPUT/11950_Sunday_April_14_2024_02_31_13_AM_60911303/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -5 - x(t) - t - y(t) \\
y'(t) &= 7 + 2x(t) + 2t + y(t)
\end{aligned}
$$

16.14.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -5-t \\ 2t+7 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) - \sin(t) & -\sin(t) \\ 2\sin(t) & \cos(t) + \sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos(t) - \sin(t) & -\sin(t) \\ 2\sin(t) & \cos(t) + \sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (\cos(t) - \sin(t))\, c_1 - \sin(t)\, c_2 \\ 2\sin(t)\, c_1 + (\cos(t) + \sin(t))\, c_2 \end{bmatrix} \\
&= \begin{bmatrix} (-c_1 - c_2)\sin(t) + c_1 \cos(t) \\ (2c_1 + c_2)\sin(t) + c_2 \cos(t) \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \cos(t) + \sin(t) & \sin(t) \\ -2\sin(t) & \cos(t) - \sin(t) \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \cos(t) - \sin(t) & -\sin(t) \\ 2\sin(t) & \cos(t) + \sin(t) \end{bmatrix} \int \begin{bmatrix} \cos(t) + \sin(t) & \sin(t) \\ -2\sin(t) & \cos(t) - \sin(t) \end{bmatrix} \begin{bmatrix} -5 - t \\ 2t + 7 \end{bmatrix} dt \\
&= \begin{bmatrix} \cos(t) - \sin(t) & -\sin(t) \\ 2\sin(t) & \cos(t) + \sin(t) \end{bmatrix} \begin{bmatrix} (-3 - t)\cos(t) - \sin(t)(t + 4) \\ \sin(t)(2t + 7) - \cos(t) \end{bmatrix} \\
&= \begin{bmatrix} -3 - t \\ -1 \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} (-c_1 - c_2)\sin(t) + c_1\cos(t) - 3 - t \\ (2c_1 + c_2)\sin(t) + c_2\cos(t) - 1 \end{bmatrix}\end{aligned}$$

16.14.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -5 - t \\ 2t + 7 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1 - \lambda & -1 \\ 2 & 1 - \lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
i	1	complex eigenvalue
$-i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1+i & -1 \\ 2 & 1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1+i & -1 & 0 \\ 2 & 1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1+i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} -1+i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+i & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} - \frac{i}{2}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2}-\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2}-\frac{I}{2}\right)t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2}-\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2}-\frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -1-i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1-i & -1 \\ 2 & 1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1-i & -1 & 0 \\ 2 & 1-i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1-i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} -1-i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
i	1	1	No	$\begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} -\frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) e^{it} \\ e^{it} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) e^{-it} \\ e^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) e^{it} & \left(-\frac{1}{2}-\frac{i}{2}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) \, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -i e^{-it} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{-it} \\ i e^{it} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{it} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) e^{it} & \left(-\frac{1}{2}-\frac{i}{2}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} -i e^{-it} & \left(\frac{1}{2}-\frac{i}{2}\right) e^{-it} \\ i e^{it} & \left(\frac{1}{2}+\frac{i}{2}\right) e^{it} \end{bmatrix} \begin{bmatrix} -5-t \\ 2t+7 \end{bmatrix} dt$$

$$= \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) e^{it} & \left(-\frac{1}{2}-\frac{i}{2}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-it}(7+3i+2t)}{2} \\ -\frac{e^{it}(-7+3i-2t)}{2} \end{bmatrix} dt$$

$$= \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) e^{it} & \left(-\frac{1}{2}-\frac{i}{2}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} \frac{e^{-it}(2it+7i-1)}{2} \\ -\frac{e^{it}(2it+7i+1)}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -3-t \\ -1 \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) c_1 e^{it} \\ c_1 e^{it} \end{bmatrix} + \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) c_2 e^{-it} \\ c_2 e^{-it} \end{bmatrix} + \begin{bmatrix} -3-t \\ -1 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) c_1 e^{it} + \left(-\frac{1}{2}-\frac{i}{2}\right) c_2 e^{-it} - 3 - t \\ c_1 e^{it} + c_2 e^{-it} - 1 \end{bmatrix}$$

16.14.3 Maple step by step solution

Let's solve

$[x'(t) = -5 - x(t) - t - y(t), y'(t) = 7 + 2x(t) + 2t + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -5-t \\ 2t+7 \end{bmatrix}$$

- System to solve

$$\overrightarrow{x}'(t) = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} -5-t \\ 2t+7 \end{bmatrix}$$

- Define the forcing function

$$\overrightarrow{f}(t) = \begin{bmatrix} -5-t \\ 2t+7 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\mathrm{I}, \begin{bmatrix} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}t} \cdot \begin{bmatrix} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(t) - \mathrm{I}\sin(t)) \cdot \begin{bmatrix} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} \left(-\frac{1}{2}-\frac{\mathrm{I}}{2}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \begin{bmatrix} -\frac{\cos(t)}{2} - \frac{\sin(t)}{2} \\ \cos(t) \end{bmatrix}, \vec{x}_2(t) = \begin{bmatrix} \frac{\sin(t)}{2} - \frac{\cos(t)}{2} \\ -\sin(t) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_p(t)$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} -\frac{\cos(t)}{2} - \frac{\sin(t)}{2} & \frac{\sin(t)}{2} - \frac{\cos(t)}{2} \\ \cos(t) & -\sin(t) \end{bmatrix}$$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -\frac{\cos(t)}{2} - \frac{\sin(t)}{2} & \frac{\sin(t)}{2} - \frac{\cos(t)}{2} \\ \cos(t) & -\sin(t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \cos(t) - \sin(t) & -\sin(t) \\ 2\sin(t) & \cos(t) + \sin(t) \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

  - Take the derivative of the particular solution

$\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

  - Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

  - Cancel like terms

$\Phi(t)\cdot \vec{v}'(t)=\vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t)=\int_0^t \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t)=\Phi(t)\cdot\left(\int_0^t \frac{1}{\Phi(s)}\cdot \vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t)=\left[\begin{array}{c} -t-3+3\cos(t)-4\sin(t) \\ -1+\cos(t)+7\sin(t) \end{array}\right]$$

- Plug particular solution back into general solution

  $$\vec{x}(t)=c_1\vec{x}_1(t)+c_2\vec{x}_2(t)+\left[\begin{array}{c} -t-3+3\cos(t)-4\sin(t) \\ -1+\cos(t)+7\sin(t) \end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right]=\left[\begin{array}{c} \frac{(6-c_1-c_2)\cos(t)}{2}+\frac{(-c_1+c_2-8)\sin(t)}{2}-t-3 \\ (1+c_1)\cos(t)-1+(-c_2+7)\sin(t) \end{array}\right]$$

- Solution to the system of ODEs

  $$\left\{x(t)=\frac{(6-c_1-c_2)\cos(t)}{2}+\frac{(-c_1+c_2-8)\sin(t)}{2}-t-3, y(t)=(1+c_1)\cos(t)-1+(-c_2+7)\sin(t)\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 42

```
dsolve([3*diff(x(t),t)+2*diff(y(t),t)-x(t)+y(t)=t-1,diff(x(t),t)+diff(y(t),t)-x(t)=t+2],sing
```

$$x(t)=c_2\sin(t)+c_1\cos(t)-3-t$$
$$y(t)=-c_2\cos(t)+c_1\sin(t)-1-c_2\sin(t)-c_1\cos(t)$$

✓ Solution by Mathematica

Time used: 0.106 (sec). Leaf size: 44

```
DSolve[{3*x'[t]+2*y'[t]-x[t]+y[t]==t-1,x'[t]+y'[t]-x[t]==t+2},{x[t],y[t]},t,IncludeSingularS
```

$$x(t) \to -t + c_1 \cos(t) - (c_1 + c_2) \sin(t) - 3$$
$$y(t) \to c_2 \cos(t) + (2c_1 + c_2) \sin(t) - 1$$

16.15 problem 15

Internal problem ID [11942]
Internal file name [`OUTPUT/11951_Sunday_April_14_2024_02_31_14_AM_14031841/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -\frac{7x(t)}{2} - \frac{9y(t)}{2} + \frac{\mathrm{e}^t}{2} \\
y'(t) &= \frac{3x(t)}{2} + \frac{5y(t)}{2} + \frac{\mathrm{e}^t}{2}
\end{aligned}
$$

16.15.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} & -\frac{9}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \frac{\mathrm{e}^t}{2} \\ \frac{\mathrm{e}^t}{2} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{(e^{3t}-3)e^{-2t}}{2} & -\frac{3(e^{3t}-1)e^{-2t}}{2} \\ \frac{(e^{3t}-1)e^{-2t}}{2} & \frac{(3e^{3t}-1)e^{-2t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} -\frac{(e^{3t}-3)e^{-2t}}{2} & -\frac{3(e^{3t}-1)e^{-2t}}{2} \\ \frac{(e^{3t}-1)e^{-2t}}{2} & \frac{(3e^{3t}-1)e^{-2t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{(e^{3t}-3)e^{-2t}c_1}{2} - \frac{3(e^{3t}-1)e^{-2t}c_2}{2} \\ \frac{(e^{3t}-1)e^{-2t}c_1}{2} + \frac{(3e^{3t}-1)e^{-2t}c_2}{2} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{((c_1+3c_2)e^{3t}-3c_1-3c_2)e^{-2t}}{2} \\ \frac{((c_1+3c_2)e^{3t}-c_1-c_2)e^{-2t}}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{3e^{2t}}{2} - \frac{e^{-t}}{2} & \frac{3e^{2t}}{2} - \frac{3e^{-t}}{2} \\ -\frac{e^{2t}}{2} + \frac{e^{-t}}{2} & -\frac{e^{2t}}{2} + \frac{3e^{-t}}{2} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} -\frac{(e^{3t}-3)e^{-2t}}{2} & -\frac{3(e^{3t}-1)e^{-2t}}{2} \\ \frac{(e^{3t}-1)e^{-2t}}{2} & \frac{(3e^{3t}-1)e^{-2t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{3e^{2t}}{2} - \frac{e^{-t}}{2} & \frac{3e^{2t}}{2} - \frac{3e^{-t}}{2} \\ -\frac{e^{2t}}{2} + \frac{e^{-t}}{2} & -\frac{e^{2t}}{2} + \frac{3e^{-t}}{2} \end{bmatrix} \begin{bmatrix} \frac{e^{t}}{2} \\ \frac{e^{t}}{2} \end{bmatrix} dt \\ &= \begin{bmatrix} -\frac{(e^{3t}-3)e^{-2t}}{2} & -\frac{3(e^{3t}-1)e^{-2t}}{2} \\ \frac{(e^{3t}-1)e^{-2t}}{2} & \frac{(3e^{3t}-1)e^{-2t}}{2} \end{bmatrix} \begin{bmatrix} -t + \frac{e^{3t}}{2} \\ t - \frac{e^{3t}}{6} \end{bmatrix} \\ &= \begin{bmatrix} e^{t}\left(\frac{1}{2} - t\right) \\ e^{t}\left(t - \frac{1}{6}\right) \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} -\left(\left(t+\frac{c_1}{2}+\frac{3c_2}{2}-\frac{1}{2}\right)e^{3t}-\frac{3c_1}{2}-\frac{3c_2}{2}\right)e^{-2t} \\ \left(\left(\frac{c_1}{2}+\frac{3c_2}{2}+t-\frac{1}{6}\right)e^{3t}-\frac{c_1}{2}-\frac{c_2}{2}\right)e^{-2t} \end{bmatrix}$$

16.15.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} & -\frac{9}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \frac{e^t}{2} \\ \frac{e^t}{2} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -\frac{7}{2} & -\frac{9}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\frac{7}{2}-\lambda & -\frac{9}{2} \\ \frac{3}{2} & \frac{5}{2}-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + \lambda - 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{7}{2} & -\frac{9}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -\frac{3}{2} & -\frac{9}{2} \\ \frac{3}{2} & \frac{9}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{3}{2} & -\frac{9}{2} & 0 \\ \frac{3}{2} & \frac{9}{2} & 0 \end{array} \right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -\frac{3}{2} & -\frac{9}{2} & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{3}{2} & -\frac{9}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -3t\}$

Hence the solution is

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = \begin{bmatrix} -3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -\frac{7}{2} & -\frac{9}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{9}{2} & -\frac{9}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\frac{9}{2} & -\frac{9}{2} & 0 \\ \frac{3}{2} & \frac{3}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{3} \implies \left[\begin{array}{cc|c} -\frac{9}{2} & -\frac{9}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\frac{9}{2} & -\frac{9}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-2	1	1	No	$\begin{bmatrix} -3 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^t \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -3\,\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -3\,\mathrm{e}^{-2t} & -\mathrm{e}^{t} \\ \mathrm{e}^{-2t} & \mathrm{e}^{t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{\mathrm{e}^{2t}}{2} & -\frac{\mathrm{e}^{2t}}{2} \\ \frac{\mathrm{e}^{-t}}{2} & \frac{3\,\mathrm{e}^{-t}}{2} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -3\,\mathrm{e}^{-2t} & -\mathrm{e}^{t} \\ \mathrm{e}^{-2t} & \mathrm{e}^{t} \end{bmatrix} \int \begin{bmatrix} -\frac{\mathrm{e}^{2t}}{2} & -\frac{\mathrm{e}^{2t}}{2} \\ \frac{\mathrm{e}^{-t}}{2} & \frac{3\,\mathrm{e}^{-t}}{2} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{e}^{t}}{2} \\ \frac{\mathrm{e}^{t}}{2} \end{bmatrix} dt$$

$$= \begin{bmatrix} -3\,\mathrm{e}^{-2t} & -\mathrm{e}^{t} \\ \mathrm{e}^{-2t} & \mathrm{e}^{t} \end{bmatrix} \int \begin{bmatrix} -\frac{\mathrm{e}^{3t}}{2} \\ 1 \end{bmatrix} dt$$

$$= \begin{bmatrix} -3\,\mathrm{e}^{-2t} & -\mathrm{e}^{t} \\ \mathrm{e}^{-2t} & \mathrm{e}^{t} \end{bmatrix} \begin{bmatrix} -\frac{\mathrm{e}^{3t}}{6} \\ t \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{e}^{t}\left(\frac{1}{2} - t\right) \\ \mathrm{e}^{t}\left(t - \frac{1}{6}\right) \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -3c_1\mathrm{e}^{-2t} \\ c_1\mathrm{e}^{-2t} \end{bmatrix} + \begin{bmatrix} -c_2\mathrm{e}^{t} \\ c_2\mathrm{e}^{t} \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{t}\left(\frac{1}{2} - t\right) \\ \mathrm{e}^{t}\left(t - \frac{1}{6}\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\left(\left(t + c_2 - \frac{1}{2}\right)\mathrm{e}^{3t} + 3c_1\right)\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t}\left(\left(c_2 + t - \frac{1}{6}\right)\mathrm{e}^{3t} + c_1\right) \end{bmatrix}$$

16.15.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -\frac{7x(t)}{2} - \frac{9y(t)}{2} + \frac{\mathrm{e}^{t}}{2}, y'(t) = \frac{3x(t)}{2} + \frac{5y(t)}{2} + \frac{\mathrm{e}^{t}}{2}\right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -\frac{7}{2} & -\frac{9}{2} \\ \frac{3}{2} & \frac{5}{2} \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} \frac{\mathrm{e}^{t}}{2} \\ \frac{\mathrm{e}^{t}}{2} \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \left[\begin{array}{cc} -\frac{7}{2} & -\frac{9}{2} \\ \frac{3}{2} & \frac{5}{2} \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} \frac{e^{t}}{2} \\ \frac{e^{t}}{2} \end{array}\right]$$

- Define the forcing function

$$\vec{f}(t) = \left[\begin{array}{c} \frac{e^{t}}{2} \\ \frac{e^{t}}{2} \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} -\frac{7}{2} & -\frac{9}{2} \\ \frac{3}{2} & \frac{5}{2} \end{array}\right]$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \left[\begin{array}{c} -3 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-2, \left[\begin{array}{c} -3 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-2t} \cdot \left[\begin{array}{c} -3 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{t} \cdot \left[\begin{array}{c} -1 \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

  $\vec{x}(t)=c_1\vec{x}_1+c_2\vec{x}_2+\vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $\phi(t)=\begin{bmatrix} -3\,\mathrm{e}^{-2t} & -\mathrm{e}^{t} \\ \mathrm{e}^{-2t} & \mathrm{e}^{t} \end{bmatrix}$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0)=I$ where I is th

  $\Phi(t)=\phi(t)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $\Phi(t)=\begin{bmatrix} -3\,\mathrm{e}^{-2t} & -\mathrm{e}^{t} \\ \mathrm{e}^{-2t} & \mathrm{e}^{t} \end{bmatrix}\cdot\frac{1}{\begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix}}$

- Evaluate and simplify to get the fundamental matrix

  $\Phi(t)=\begin{bmatrix} -\frac{(\mathrm{e}^{3t}-3)\mathrm{e}^{-2t}}{2} & -\frac{3(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{2} \\ \frac{(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{2} & \frac{(3\,\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{2} \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t)=\Phi(t)\cdot\vec{v}(t)$

- Take the derivative of the particular solution

  $\vec{x}_p'(t)=\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A\cdot\Phi(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$

- Cancel like terms

  $\Phi(t)\cdot\vec{v}'(t)=\vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t) = \begin{bmatrix} \frac{(1-2t)e^{-2t}e^{3t}}{2} - \frac{e^{-2t}}{2} \\ \frac{(6t-1)e^{-2t}e^{3t}}{6} + \frac{e^{-2t}}{6} \end{bmatrix}$$

- Plug particular solution back into general solution

  $$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \begin{bmatrix} \frac{(1-2t)e^{-2t}e^{3t}}{2} - \frac{e^{-2t}}{2} \\ \frac{(6t-1)e^{-2t}e^{3t}}{6} + \frac{e^{-2t}}{6} \end{bmatrix}$$

- Substitute in vector of dependent variables

  $$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\left(\left(t + c_2 - \frac{1}{2}\right)e^{3t} + 3c_1 + \frac{1}{2}\right)e^{-2t} \\ \left(\left(c_2 + t - \frac{1}{6}\right)e^{3t} + c_1 + \frac{1}{6}\right)e^{-2t} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x(t) = -\left(\left(t + c_2 - \frac{1}{2}\right)e^{3t} + 3c_1 + \frac{1}{2}\right)e^{-2t}, y(t) = \left(\left(c_2 + t - \frac{1}{6}\right)e^{3t} + c_1 + \frac{1}{6}\right)e^{-2t}\right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 45

```
dsolve([2*diff(x(t),t)+4*diff(y(t),t)+x(t)-y(t)=3*exp(t),diff(x(t),t)+diff(y(t),t)+2*x(t)+2*
```

$$x(t) = c_2 e^t + c_1 e^{-2t} - e^t t$$

$$y(t) = -c_2 e^t - \frac{c_1 e^{-2t}}{3} + e^t t + \frac{e^t}{3}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 76

```
DSolve[{2*x'[t]+4*y'[t]+x[t]-y[t]==3*Exp[t],x'[t]+y'[t]+2*x[t]+2*y[t]==Exp[t]},{x[t],y[t]},t
```

$$x(t) \to \frac{3}{2}(c_1 + c_2)e^{-2t} - \frac{1}{2}e^t(2t - 1 + c_1 + 3c_2)$$
$$y(t) \to \frac{1}{6}e^t(6t - 1 + 3c_1 + 9c_2) - \frac{1}{2}(c_1 + c_2)e^{-2t}$$

16.16 problem 16

Internal problem ID [11943]
Internal file name [`OUTPUT/11952_Sunday_April_14_2024_02_31_15_AM_15795806/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 2x(t) - 2t - t^2 \\
y'(t) &= -3x(t) + y(t) + 2t + 2t^2
\end{aligned}
$$

16.16.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -t^2 - 2t \\ 2t^2 + 2t \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{2t} & 0 \\ -3\,e^{2t}+3\,e^{t} & e^{t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t} & 0 \\ -3\,e^{2t}+3\,e^{t} & e^{t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 \\ (-3\,e^{2t}+3\,e^{t})\,c_1 + e^{t}c_2 \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} e^{-2t} & 0 \\ 3\,e^{-2t}(e^{t}-1) & e^{-t} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{2t} & 0 \\ -3\,e^{2t}+3\,e^{t} & e^{t} \end{bmatrix} \int \begin{bmatrix} e^{-2t} & 0 \\ 3\,e^{-2t}(e^{t}-1) & e^{-t} \end{bmatrix} \begin{bmatrix} -t^2-2t \\ 2t^2+2t \end{bmatrix} dt \\
&= \begin{bmatrix} e^{2t} & 0 \\ -3\,e^{2t}+3\,e^{t} & e^{t} \end{bmatrix} \begin{bmatrix} \frac{e^{-2t}(2t^2+6t+3)}{4} \\ \frac{3(-2t^2-6t-3)e^{-2t}}{4} + e^{-t}(t^2+6t+6) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}t^2+\frac{3}{2}t+\frac{3}{4} \\ -\frac{1}{2}t^2+\frac{3}{2}t+\frac{15}{4} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} e^{2t}c_1 + \frac{t^2}{2} + \frac{3t}{2} + \frac{3}{4} \\ \frac{15}{4} - 3\,e^{2t}c_1 + e^{t}(3c_1+c_2) - \frac{t^2}{2} + \frac{3t}{2} \end{bmatrix}
\end{aligned}$$

16.16.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -t^2 - 2t \\ 2t^2 + 2t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 0 \\ -3 & 1-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(2-\lambda)(1-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ -3 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + 3R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -3 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} -3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} e^{2t} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{t} \end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{e^{2t}}{3} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\frac{e^{2t}}{3} & 0 \\ e^{2t} & e^{t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -3\,e^{-2t} & 0 \\ 3\,e^{-t} & e^{-t} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\frac{e^{2t}}{3} & 0 \\ e^{2t} & e^{t} \end{bmatrix} \int \begin{bmatrix} -3\,e^{-2t} & 0 \\ 3\,e^{-t} & e^{-t} \end{bmatrix} \begin{bmatrix} -t^2 - 2t \\ 2t^2 + 2t \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{e^{2t}}{3} & 0 \\ e^{2t} & e^{t} \end{bmatrix} \int \begin{bmatrix} 3\,e^{-2t}t(t+2) \\ -e^{-t}t(t+4) \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{e^{2t}}{3} & 0 \\ e^{2t} & e^{t} \end{bmatrix} \begin{bmatrix} -\frac{3\,e^{-2t}(2t^2+6t+3)}{4} \\ e^{-t}(t^2+6t+6) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}t^2 + \frac{3}{2}t + \frac{3}{4} \\ -\frac{1}{2}t^2 + \frac{3}{2}t + \frac{15}{4} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{2t}}{3} \\ c_1 e^{2t} \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 e^{t} \end{bmatrix} + \begin{bmatrix} \frac{1}{2}t^2 + \frac{3}{2}t + \frac{3}{4} \\ -\frac{1}{2}t^2 + \frac{3}{2}t + \frac{15}{4} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 e^{2t}}{3} + \frac{t^2}{2} + \frac{3t}{2} + \frac{3}{4} \\ c_1 e^{2t} + c_2 e^{t} - \frac{t^2}{2} + \frac{3t}{2} + \frac{15}{4} \end{bmatrix}$$

16.16.3 Maple step by step solution

Let's solve

$[x'(t) = 2x(t) - 2t - t^2, y'(t) = -3x(t) + y(t) + 2t + 2t^2]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -t^2 - 2t \\ 2t^2 + 2t \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -t^2 - 2t \\ 2t^2 + 2t \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} -t^2 - 2t \\ 2t^2 + 2t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1, \left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right], \left[2, \left[\begin{array}{c} -\frac{1}{3} \\ 1 \end{array}\right]\right]\right]$$
- Consider eigenpair

  $$\left[1, \left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right]$$
- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = e^t \cdot \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$$
- Consider eigenpair

  $$\left[2, \left[\begin{array}{c} -\frac{1}{3} \\ 1 \end{array}\right]\right]$$
- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = e^{2t} \cdot \left[\begin{array}{c} -\frac{1}{3} \\ 1 \end{array}\right]$$
- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

□ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $$\phi(t) = \left[\begin{array}{cc} 0 & -\frac{e^{2t}}{3} \\ e^t & e^{2t} \end{array}\right]$$
  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

    $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$
  - Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} 0 & -\frac{e^{2t}}{3} \\ e^{t} & e^{2t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 0 & -\frac{1}{3} \\ 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^{2t} & 0 \\ -3\,e^{2t} + 3\,e^{t} & e^{t} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- Cancel like terms

$$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} \frac{t^2}{2} - \frac{3\,e^{2t}}{4} + \frac{3t}{2} + \frac{3}{4} \\ -\frac{t^2}{2} + \frac{9\,e^{2t}}{4} - 6\,e^{t} + \frac{3t}{2} + \frac{15}{4} \end{bmatrix}$$

• Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \left[\begin{array}{c} \frac{t^2}{2} - \frac{3\,e^{2t}}{4} + \frac{3t}{2} + \frac{3}{4} \\ -\frac{t^2}{2} + \frac{9\,e^{2t}}{4} - 6\,e^t + \frac{3t}{2} + \frac{15}{4} \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{(-4c_2-9)e^{2t}}{12} + \frac{t^2}{2} + \frac{3t}{2} + \frac{3}{4} \\ \frac{15}{4} + \frac{(4c_2+9)e^{2t}}{4} + (c_1-6)\,e^t - \frac{t^2}{2} + \frac{3t}{2} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{(-4c_2-9)e^{2t}}{12} + \frac{t^2}{2} + \frac{3t}{2} + \frac{3}{4}, y(t) = \frac{15}{4} + \frac{(4c_2+9)e^{2t}}{4} + (c_1-6)\,e^t - \frac{t^2}{2} + \frac{3t}{2}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 45

```
dsolve([2*diff(x(t),t)+diff(y(t),t)-x(t)-y(t)=-2*t,diff(x(t),t)+diff(y(t),t)+x(t)-y(t)=t^2],
```

$$x(t) = \frac{t^2}{2} + \frac{3t}{2} + \frac{3}{4} + c_2e^{2t}$$
$$y(t) = \frac{15}{4} - 3c_2e^{2t} + \frac{3t}{2} - \frac{t^2}{2} + c_1e^t$$

✓ Solution by Mathematica

Time used: 0.215 (sec). Leaf size: 67

```
DSolve[{2*x'[t]+y'[t]-x[t]-y[t]==-2*t,x'[t]+y'[t]+x[t]-y[t]==t^2},{x[t],y[t]},t,IncludeSingu
```

$$x(t) \to \frac{1}{4}\left(2t^2 + 6t + 4c_1e^{2t} + 3\right)$$
$$y(t) \to -\frac{t^2}{2} + \frac{3t}{2} - 3c_1e^{2t} + (3c_1 + c_2)e^t + \frac{15}{4}$$

16.17 problem 17

Internal problem ID [11944]
Internal file name [`OUTPUT/11953_Sunday_April_14_2024_02_31_16_AM_41821047/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.1. Exercises page 277
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 3x(t) - t + 1 \\
y'(t) &= -5x(t) + y(t) + 2t - 1
\end{aligned}
$$

16.17.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 1-t \\ 2t-1 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{3t} & 0 \\ -\frac{5\,e^{3t}}{2} + \frac{5\,e^{t}}{2} & e^{t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^{3t} & 0 \\ -\frac{5\,e^{3t}}{2} + \frac{5\,e^{t}}{2} & e^{t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{3t} c_1 \\ \left(-\frac{5\,e^{3t}}{2} + \frac{5\,e^{t}}{2}\right) c_1 + e^{t} c_2 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} e^{-3t} & 0 \\ \frac{5\,e^{-3t}\left(e^{2t}-1\right)}{2} & e^{-t} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} e^{3t} & 0 \\ -\frac{5\,e^{3t}}{2} + \frac{5\,e^{t}}{2} & e^{t} \end{bmatrix} \int \begin{bmatrix} e^{-3t} & 0 \\ \frac{5\,e^{-3t}\left(e^{2t}-1\right)}{2} & e^{-t} \end{bmatrix} \begin{bmatrix} 1-t \\ 2t-1 \end{bmatrix} dt \\ &= \begin{bmatrix} e^{3t} & 0 \\ -\frac{5\,e^{3t}}{2} + \frac{5\,e^{t}}{2} & e^{t} \end{bmatrix} \begin{bmatrix} \frac{e^{-3t}(-2+3t)}{9} \\ \frac{5(2-3t)e^{-3t}}{18} + \frac{e^{-t}(-2+t)}{2} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{2}{9} + \frac{t}{3} \\ -\frac{4}{9} - \frac{t}{3} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} e^{3t} c_1 - \frac{2}{9} + \frac{t}{3} \\ -\frac{4}{9} - \frac{5\,e^{3t} c_1}{2} + \frac{e^{t}(5c_1+2c_2)}{2} - \frac{t}{3} \end{bmatrix} \end{aligned}$$

16.17.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 1-t \\ 2t-1 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 0 \\ -5 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 0 \\ -5 & 1-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(3-\lambda)(1-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 \\ -5 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 0 \\ -5 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 0 & 0 \\ -5 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{5R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 0 \\ -5 & 1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -5 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(1, 1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} -5 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} -\frac{2}{5} \\ 1 \end{bmatrix} e^{3t} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{t} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{t} \end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{2\,\mathrm{e}^{3t}}{5} \\ \mathrm{e}^{3t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\frac{2\,\mathrm{e}^{3t}}{5} & 0 \\ \mathrm{e}^{3t} & \mathrm{e}^{t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{5\,\mathrm{e}^{-3t}}{2} & 0 \\ \frac{5\,\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\frac{2\,\mathrm{e}^{3t}}{5} & 0 \\ \mathrm{e}^{3t} & \mathrm{e}^{t} \end{bmatrix} \int \begin{bmatrix} -\frac{5\,\mathrm{e}^{-3t}}{2} & 0 \\ \frac{5\,\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} 1-t \\ 2t-1 \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{2\,\mathrm{e}^{3t}}{5} & 0 \\ \mathrm{e}^{3t} & \mathrm{e}^{t} \end{bmatrix} \int \begin{bmatrix} \frac{5\,\mathrm{e}^{-3t}(t-1)}{2} \\ -\frac{\mathrm{e}^{-t}(-3+t)}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{2\,\mathrm{e}^{3t}}{5} & 0 \\ \mathrm{e}^{3t} & \mathrm{e}^{t} \end{bmatrix} \begin{bmatrix} -\frac{5\,\mathrm{e}^{-3t}(-2+3t)}{18} \\ \frac{\mathrm{e}^{-t}(-2+t)}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{2}{9} + \frac{t}{3} \\ -\frac{4}{9} - \frac{t}{3} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{2c_1 e^{3t}}{5} \\ c_1 e^{3t} \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 e^{t} \end{bmatrix} + \begin{bmatrix} -\frac{2}{9} + \frac{t}{3} \\ -\frac{4}{9} - \frac{t}{3} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{2c_1 e^{3t}}{5} - \frac{2}{9} + \frac{t}{3} \\ c_1 e^{3t} + c_2 e^{t} - \frac{4}{9} - \frac{t}{3} \end{bmatrix}$$

16.17.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t) - t + 1, y'(t) = -5x(t) + y(t) + 2t - 1]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & 0 \\ -5 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 1-t \\ 2t-1 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & 0 \\ -5 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 1-t \\ 2t-1 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 1-t \\ 2t-1 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 0 \\ -5 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1, \left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} -\frac{2}{5} \\ 1 \end{array}\right]\right]\right]$$
- Consider eigenpair

  $$\left[1, \left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right]$$
- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_1 = \mathrm{e}^t \cdot \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$$
- Consider eigenpair

  $$\left[3, \left[\begin{array}{c} -\frac{2}{5} \\ 1 \end{array}\right]\right]$$
- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_2 = \mathrm{e}^{3t} \cdot \left[\begin{array}{c} -\frac{2}{5} \\ 1 \end{array}\right]$$
- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(t) = \left[\begin{array}{cc} 0 & -\frac{2\,\mathrm{e}^{3t}}{5} \\ \mathrm{e}^t & \mathrm{e}^{3t} \end{array}\right]$$
- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} 0 & -\frac{2\,e^{3t}}{5} \\ e^t & e^{3t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 0 & -\frac{2}{5} \\ 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^{3t} & 0 \\ -\frac{5\,e^{3t}}{2} + \frac{5\,e^t}{2} & e^t \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- Cancel like terms

$$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\,ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} \frac{2\,e^{3t}}{9} - \frac{2}{9} + \frac{t}{3} \\ -\frac{5\,e^{3t}}{9} - \frac{4}{9} - \frac{t}{3} + e^t \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \begin{bmatrix} \frac{2\,e^{3t}}{9} - \frac{2}{9} + \frac{t}{3} \\ -\frac{5\,e^{3t}}{9} - \frac{4}{9} - \frac{t}{3} + e^t \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} + \frac{2(5-9c_2)e^{3t}}{45} + \frac{t}{3} \\ c_1 e^t + c_2 e^{3t} - \frac{5\,e^{3t}}{9} - \frac{4}{9} - \frac{t}{3} + e^t \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{2}{9} + \frac{2(5-9c_2)e^{3t}}{45} + \frac{t}{3}, y(t) = c_1 e^t + c_2 e^{3t} - \frac{5\,e^{3t}}{9} - \frac{4}{9} - \frac{t}{3} + e^t\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve([2*diff(x(t),t)+diff(y(t),t)-x(t)-y(t)=1,diff(x(t),t)+diff(y(t),t)+2*x(t)-y(t)=t],sin
```

$$x(t) = \frac{t}{3} - \frac{2}{9} + c_2 e^{3t}$$
$$y(t) = -\frac{4}{9} - \frac{5c_2 e^{3t}}{2} - \frac{t}{3} + c_1 e^t$$

✓ Solution by Mathematica

Time used: 0.147 (sec). Leaf size: 58

```
DSolve[{2*x'[t]+y'[t]-x[t]-y[t]==1,x'[t]+y'[t]+2*x[t]-y[t]==t},{x[t],y[t]},t,IncludeSingular
```

$$x(t) \to \frac{t}{3} + c_1 e^{3t} - \frac{2}{9}$$
$$y(t) \to -\frac{t}{3} - \frac{5}{2} c_1 e^{3t} + \left(\frac{5c_1}{2} + c_2\right) e^t - \frac{4}{9}$$

17 Chapter 7, Systems of linear differential equations. Section 7.3. Exercises page 299

17.1 problem 1

Internal problem ID [11945]
Internal file name [`OUTPUT/11954_Sunday_April_14_2024_02_31_17_AM_22819234/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.3. Exercises page 299
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 3x(t) + 4y(t)$$
$$y'(t) = 2x(t) + y(t)$$

With initial conditions

$$[x(0) = 1, y(0) = 2]$$

17.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-t}}{3} + \frac{2\,e^{5t}}{3} & \frac{2\,e^{5t}}{3} - \frac{2\,e^{-t}}{3} \\ \frac{e^{5t}}{3} - \frac{e^{-t}}{3} & \frac{2\,e^{-t}}{3} + \frac{e^{5t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{e^{-t}}{3}+\frac{2\,e^{5t}}{3} & \frac{2\,e^{5t}}{3}-\frac{2\,e^{-t}}{3} \\ \frac{e^{5t}}{3}-\frac{e^{-t}}{3} & \frac{2\,e^{-t}}{3}+\frac{e^{5t}}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} -e^{-t}+2\,e^{5t} \\ e^{5t}+e^{-t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 4 \\ 2 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda - 5 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
5	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{5t}\\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 2\,\mathrm{e}^{5t} \\ \mathrm{e}^{5t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_1\mathrm{e}^{-t} + 2c_2\mathrm{e}^{5t} \\ c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{5t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 1 \\ y(0) = 2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -c_1 + 2c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 1 \\ c_2 = 1 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\mathrm{e}^{-t} + 2\,\mathrm{e}^{5t} \\ \mathrm{e}^{5t} + \mathrm{e}^{-t} \end{bmatrix}$$

The following is the phase plot of the system.

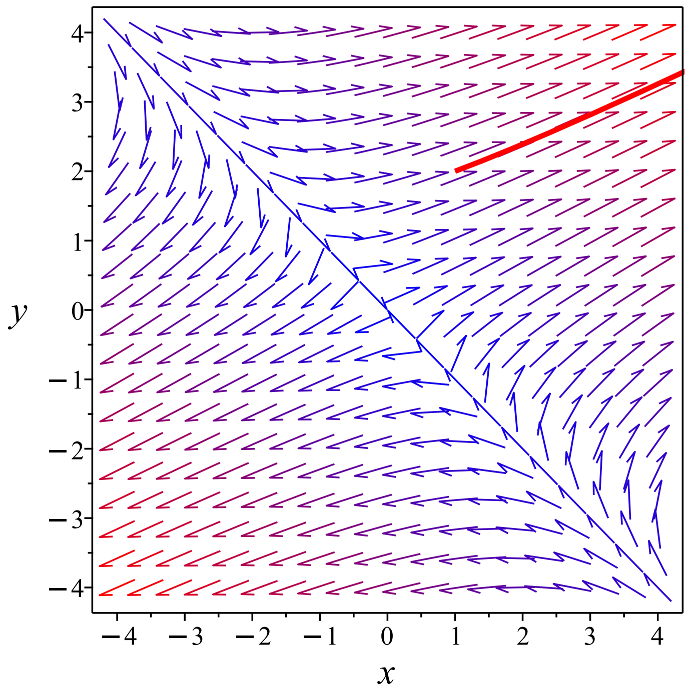
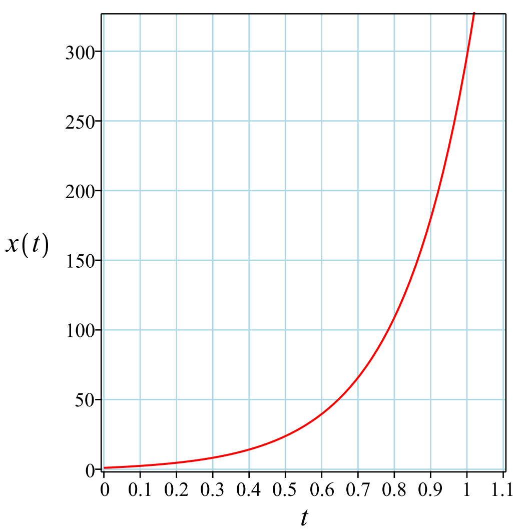
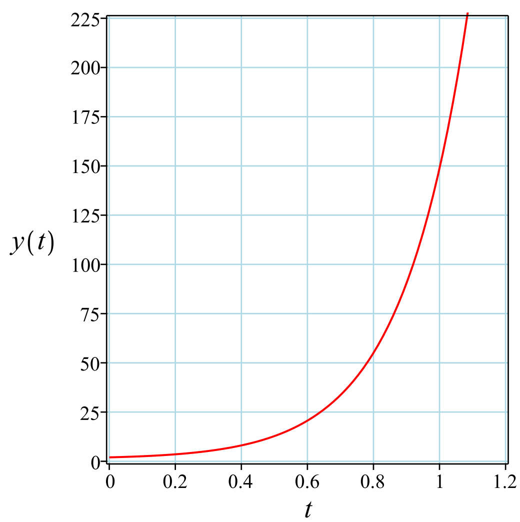


Figure 560: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 30

```
dsolve([diff(x(t),t) = 3*x(t)+4*y(t), diff(y(t),t) = 2*x(t)+y(t), x(0) = 1, y(0) = 2], sings
```

$$x(t) = 2\,\mathrm{e}^{5t} - \mathrm{e}^{-t}$$
$$y(t) = \mathrm{e}^{5t} + \mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 34

```
DSolve[{x'[t]==3*x[t]+4*y[t],y'[t]==2*x[t]+y[t]},{x[0]==1,y[0]==2},{x[t],y[t]},t,IncludeSing
```

$$x(t) \to e^{-t}\left(2e^{6t} - 1\right)$$
$$y(t) \to e^{-t} + e^{5t}$$

17.2 problem 2

Internal problem ID [11946]
Internal file name [OUTPUT/11955_Sunday_April_14_2024_02_31_18_AM_92513105/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.3. Exercises page 299
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 5x(t) + 3y(t) \\
y'(t) &= 4x(t) + y(t)
\end{aligned}
$$

With initial conditions

$$[x(0) = 0, y(0) = 8]$$

17.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-t}}{4} + \frac{3\,e^{7t}}{4} & \frac{3\,e^{7t}}{8} - \frac{3\,e^{-t}}{8} \\ \frac{e^{7t}}{2} - \frac{e^{-t}}{2} & \frac{3\,e^{-t}}{4} + \frac{e^{7t}}{4} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{e^{-t}}{4} + \frac{3\,e^{7t}}{4} & \frac{3\,e^{7t}}{8} - \frac{3\,e^{-t}}{8} \\ \frac{e^{7t}}{2} - \frac{e^{-t}}{2} & \frac{3\,e^{-t}}{4} + \frac{e^{7t}}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} 3\,e^{7t} - 3\,e^{-t} \\ 6\,e^{-t} + 2\,e^{7t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

17.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & 3 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda - 7 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 7$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
7	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 3 \\ 4 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & 3 & 0 \\ 4 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{3} \Longrightarrow \left[\begin{array}{cc|c} 6 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 7$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 3 \\ 4 & 1 \end{bmatrix} - (7) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 3 & 0 \\ 4 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
7	1	1	No	$\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\
&= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

Since eigenvalue 7 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{7t} \\
&= \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} e^{7t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{\mathrm{e}^{-t}}{2} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{3\,\mathrm{e}^{7t}}{2} \\ \mathrm{e}^{7t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_1 \mathrm{e}^{-t}}{2} + \frac{3 c_2 \mathrm{e}^{7t}}{2} \\ c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{7t} \end{bmatrix}
$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$
\begin{bmatrix} x(0) = 0 \\ y(0) = 8 \end{bmatrix} \tag{1}
$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$
\begin{bmatrix} 0 \\ 8 \end{bmatrix} = \begin{bmatrix} -\frac{c_1}{2} + \frac{3c_2}{2} \\ c_1 + c_2 \end{bmatrix}
$$

Solving for the constants of integrations gives

$$\left[\begin{array}{c} c_1 = 6 \\ c_2 = 2 \end{array}\right]$$

Substituting these constants back in original solution in Eq. (1) gives

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} 3\,e^{7t} - 3\,e^{-t} \\ 6\,e^{-t} + 2\,e^{7t} \end{array}\right]$$

The following is the phase plot of the system.

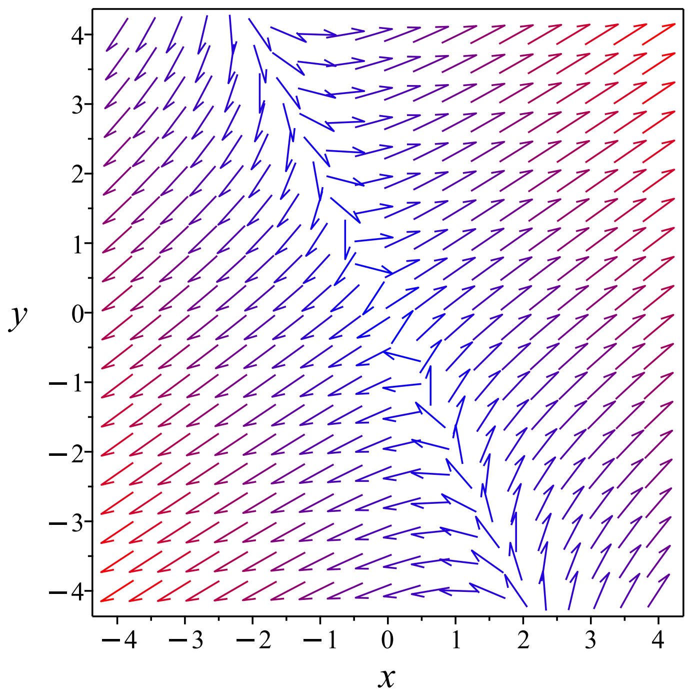
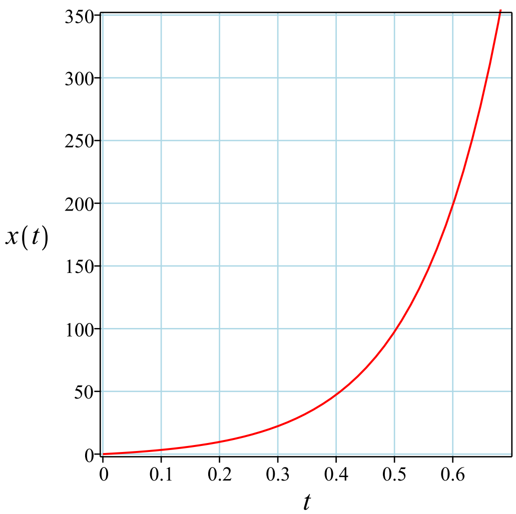
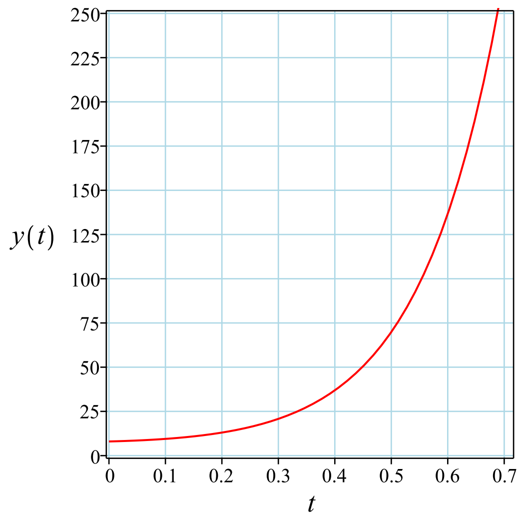


Figure 561: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 34

```
dsolve([diff(x(t),t) = 5*x(t)+3*y(t), diff(y(t),t) = 4*x(t)+y(t), x(0) = 0, y(0) = 8], sings
```

$$x(t) = 3\,\mathrm{e}^{7t} - 3\,\mathrm{e}^{-t}$$
$$y(t) = 2\,\mathrm{e}^{7t} + 6\,\mathrm{e}^{-t}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 36

```
DSolve[{x'[t]==5*x[t]+3*y[t],y'[t]==4*x[t]+y[t]},{x[0]==0,y[0]==8},{x[t],y[t]},t,IncludeSing
```

$$x(t) \to 3e^{-t}\left(e^{8t} - 1\right)$$
$$y(t) \to 2e^{-t}\left(e^{8t} + 3\right)$$

17.3 problem 3

Internal problem ID [11947]
Internal file name [`OUTPUT/11956_Sunday_April_14_2024_02_31_19_AM_54968245/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.3. Exercises page 299
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 5x(t) + 2y(t) + 5t \\
y'(t) &= 3x(t) + 4y(t) + 17t
\end{aligned}
$$

17.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 5t \\ 17t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,e^{2t}}{5} + \frac{3\,e^{7t}}{5} & \frac{2\,e^{7t}}{5} - \frac{2\,e^{2t}}{5} \\ \frac{3\,e^{7t}}{5} - \frac{3\,e^{2t}}{5} & \frac{3\,e^{2t}}{5} + \frac{2\,e^{7t}}{5} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \frac{2\,e^{2t}}{5} + \frac{3\,e^{7t}}{5} & \frac{2\,e^{7t}}{5} - \frac{2\,e^{2t}}{5} \\ \frac{3\,e^{7t}}{5} - \frac{3\,e^{2t}}{5} & \frac{3\,e^{2t}}{5} + \frac{2\,e^{7t}}{5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{2\,e^{2t}}{5} + \frac{3\,e^{7t}}{5}\right) c_1 + \left(\frac{2\,e^{7t}}{5} - \frac{2\,e^{2t}}{5}\right) c_2 \\ \left(\frac{3\,e^{7t}}{5} - \frac{3\,e^{2t}}{5}\right) c_1 + \left(\frac{3\,e^{2t}}{5} + \frac{2\,e^{7t}}{5}\right) c_2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(2c_1-2c_2)e^{2t}}{5} + \frac{3\,e^{7t}\left(c_1+\frac{2c_2}{3}\right)}{5} \\ \frac{(-3c_1+3c_2)e^{2t}}{5} + \frac{3\,e^{7t}\left(c_1+\frac{2c_2}{3}\right)}{5} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \frac{e^{-7t}(2\,e^{5t}+3)}{5} & -\frac{2\,e^{-7t}(e^{5t}-1)}{5} \\ -\frac{3\,e^{-7t}(e^{5t}-1)}{5} & \frac{e^{-7t}(3\,e^{5t}+2)}{5} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \frac{2\,e^{2t}}{5} + \frac{3\,e^{7t}}{5} & \frac{2\,e^{7t}}{5} - \frac{2\,e^{2t}}{5} \\ \frac{3\,e^{7t}}{5} - \frac{3\,e^{2t}}{5} & \frac{3\,e^{2t}}{5} + \frac{2\,e^{7t}}{5} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-7t}(2\,e^{5t}+3)}{5} & -\frac{2\,e^{-7t}(e^{5t}-1)}{5} \\ -\frac{3\,e^{-7t}(e^{5t}-1)}{5} & \frac{e^{-7t}(3\,e^{5t}+2)}{5} \end{bmatrix} \begin{bmatrix} 5t \\ 17t \end{bmatrix} dt \\ &= \begin{bmatrix} \frac{2\,e^{2t}}{5} + \frac{3\,e^{7t}}{5} & \frac{2\,e^{7t}}{5} - \frac{2\,e^{2t}}{5} \\ \frac{3\,e^{7t}}{5} - \frac{3\,e^{2t}}{5} & \frac{3\,e^{2t}}{5} + \frac{2\,e^{7t}}{5} \end{bmatrix} \begin{bmatrix} \frac{(-7t-1)e^{-7t}}{5} + \frac{6(2t+1)e^{-2t}}{5} \\ \frac{(-7t-1)e^{-7t}}{5} + \frac{9(-2t-1)e^{-2t}}{5} \end{bmatrix} \\ &= \begin{bmatrix} 1+t \\ -5t-2 \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \frac{(2c_1-2c_2)e^{2t}}{5} + \frac{e^{7t}(3c_1+2c_2)}{5} + t + 1 \\ \frac{(-3c_1+3c_2)e^{2t}}{5} + \frac{e^{7t}(3c_1+2c_2)}{5} - 5t - 2 \end{bmatrix}$$

17.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 5t \\ 17t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 9\lambda + 14 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 7$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
7	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & 2 & 0 \\ 3 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 7$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} - (7) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$
7	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the

corresponding eigenvector solution is

$$\vec{x}_1(t) = \vec{v}_1 e^{2t}$$
$$= \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} e^{2t}$$

Since eigenvalue 7 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{7t}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -\frac{2\,e^{2t}}{3} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{7t} \\ e^{7t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\frac{2\,e^{2t}}{3} & e^{7t} \\ e^{2t} & e^{7t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{3\,e^{-2t}}{5} & \frac{3\,e^{-2t}}{5} \\ \frac{3\,e^{-7t}}{5} & \frac{2\,e^{-7t}}{5} \end{bmatrix}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} -\frac{2\,e^{2t}}{3} & e^{7t} \\ e^{2t} & e^{7t} \end{bmatrix} \int \begin{bmatrix} -\frac{3\,e^{-2t}}{5} & \frac{3\,e^{-2t}}{5} \\ \frac{3\,e^{-7t}}{5} & \frac{2\,e^{-7t}}{5} \end{bmatrix} \begin{bmatrix} 5t \\ 17t \end{bmatrix} dt$$
$$= \begin{bmatrix} -\frac{2\,e^{2t}}{3} & e^{7t} \\ e^{2t} & e^{7t} \end{bmatrix} \int \begin{bmatrix} \frac{36t\,e^{-2t}}{5} \\ \frac{49\,e^{-7t}t}{5} \end{bmatrix} dt$$
$$= \begin{bmatrix} -\frac{2\,e^{2t}}{3} & e^{7t} \\ e^{2t} & e^{7t} \end{bmatrix} \begin{bmatrix} -\frac{9(2t+1)e^{-2t}}{5} \\ -\frac{(7t+1)e^{-7t}}{5} \end{bmatrix}$$
$$= \begin{bmatrix} 1+t \\ -5t-2 \end{bmatrix}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{2c_1e^{2t}}{3} \\ c_1e^{2t} \end{bmatrix} + \begin{bmatrix} c_2e^{7t} \\ c_2e^{7t} \end{bmatrix} + \begin{bmatrix} 1+t \\ -5t-2 \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{2c_1e^{2t}}{3} + c_2e^{7t} + 1 + t \\ c_1e^{2t} + c_2e^{7t} - 5t - 2 \end{bmatrix}$$

17.3.3 Maple step by step solution

Let's solve

$[x'(t) = 5x(t) + 2y(t) + 5t, y'(t) = 3x(t) + 4y(t) + 17t]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 5t \\ 17t \end{bmatrix}$$

- System to solve

$$\overrightarrow{x}'(t) = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 5t \\ 17t \end{bmatrix}$$

- Define the forcing function

$$\overrightarrow{f}(t) = \begin{bmatrix} 5t \\ 17t \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}\right], \left[7, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{2t} \cdot \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[7, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{7t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $\phi(t) = \begin{bmatrix} -\frac{2\,e^{2t}}{3} & e^{7t} \\ e^{2t} & e^{7t} \end{bmatrix}$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

  $\Phi(t) = \begin{bmatrix} -\frac{2\,e^{2t}}{3} & e^{7t} \\ e^{2t} & e^{7t} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{2}{3} & 1 \\ 1 & 1 \end{bmatrix}}$

- Evaluate and simplify to get the fundamental matrix

  $\Phi(t) = \begin{bmatrix} \frac{2\,e^{2t}}{5} + \frac{3\,e^{7t}}{5} & \frac{2\,e^{7t}}{5} - \frac{2\,e^{2t}}{5} \\ \frac{3\,e^{7t}}{5} - \frac{3\,e^{2t}}{5} & \frac{3\,e^{2t}}{5} + \frac{2\,e^{7t}}{5} \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_p(t) = \Phi(t) \cdot \overrightarrow{v}(t)$

- Take the derivative of the particular solution

  $\overrightarrow{x}_p'(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}'(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}'(t) = \overrightarrow{f}(t)$

- Multiply by the inverse of the fundamental matrix

$$\overrightarrow{v}'(t) = \frac{1}{\Phi(t)} \cdot \overrightarrow{f}(t)$$

- Integrate to solve for $\overrightarrow{v}(t)$

$$\overrightarrow{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$$

- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

$$\overrightarrow{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{x}_p(t) = \left[\begin{array}{c} t+1-\frac{6\,\mathrm{e}^{2t}}{5}+\frac{\mathrm{e}^{7t}}{5} \\ \frac{\mathrm{e}^{7t}}{5}-5t-2+\frac{9\,\mathrm{e}^{2t}}{5} \end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \left[\begin{array}{c} t+1-\frac{6\,\mathrm{e}^{2t}}{5}+\frac{\mathrm{e}^{7t}}{5} \\ \frac{\mathrm{e}^{7t}}{5}-5t-2+\frac{9\,\mathrm{e}^{2t}}{5} \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{(-10c_1-18)\mathrm{e}^{2t}}{15}+\frac{(15c_2+3)\mathrm{e}^{7t}}{15}+t+1 \\ \frac{(9+5c_1)\mathrm{e}^{2t}}{5}+\frac{(1+5c_2)\mathrm{e}^{7t}}{5}-5t-2 \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{(-10c_1-18)\mathrm{e}^{2t}}{15}+\frac{(15c_2+3)\mathrm{e}^{7t}}{15}+t+1, y(t) = \frac{(9+5c_1)\mathrm{e}^{2t}}{5}+\frac{(1+5c_2)\mathrm{e}^{7t}}{5}-5t-2\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 41

```
dsolve([diff(x(t),t)=5*x(t)+2*y(t)+5*t,diff(y(t),t)=3*x(t)+4*y(t)+17*t],singsol=all)
```

$$x(t) = c_2\mathrm{e}^{7t} + c_1\mathrm{e}^{2t} + t + 1$$
$$y(t) = c_2\mathrm{e}^{7t} - \frac{3c_1\mathrm{e}^{2t}}{2} - 2 - 5t$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 84

```
DSolve[{x'[t]==5*x[t]+2*y[t]+5*t,y'[t]==3*x[t]+4*y[t]+17*t},{x[t],y[t]},t,IncludeSingularSol
```

$$x(t) \to t + \frac{1}{5}\left(2(c_1 - c_2)e^{2t} + (3c_1 + 2c_2)e^{7t} + 5\right)$$
$$y(t) \to -5t - \frac{3}{5}(c_1 - c_2)e^{2t} + \frac{1}{5}(3c_1 + 2c_2)e^{7t} - 2$$

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18.1 problem 1

Internal problem ID [11948]
Internal file name [`OUTPUT/11957_Sunday_April_14_2024_02_31_20_AM_21165008/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.4. Exercises page 309
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 5x(t) - 2y(t) \\
y'(t) &= 4x(t) - y(t)
\end{aligned}
$$

18.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -e^{t} + 2\,e^{3t} & -e^{3t} + e^{t} \\ 2\,e^{3t} - 2\,e^{t} & 2\,e^{t} - e^{3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\mathrm{e}^{t}+2\,\mathrm{e}^{3t} & -\mathrm{e}^{3t}+\mathrm{e}^{t} \\ 2\,\mathrm{e}^{3t}-2\,\mathrm{e}^{t} & 2\,\mathrm{e}^{t}-\mathrm{e}^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} (-\mathrm{e}^{t}+2\,\mathrm{e}^{3t})\,c_1+(-\mathrm{e}^{3t}+\mathrm{e}^{t})\,c_2 \\ (2\,\mathrm{e}^{3t}-2\,\mathrm{e}^{t})\,c_1+(2\,\mathrm{e}^{t}-\mathrm{e}^{3t})\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (2c_1-c_2)\,\mathrm{e}^{3t}-(c_1-c_2)\,\mathrm{e}^{t} \\ (2c_1-c_2)\,\mathrm{e}^{3t}-2(c_1-c_2)\,\mathrm{e}^{t} \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & -2 \\ 4 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 4 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 - 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	1	1	No	$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$
3	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{t} \\ &= \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{t}\end{aligned}$$

Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{3t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{t}}{2} \\ e^{t} \end{bmatrix} + c_2 \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{t}}{2} + c_2 e^{3t} \\ c_1 e^{t} + c_2 e^{3t} \end{bmatrix}$$

The following is the phase plot of the system.

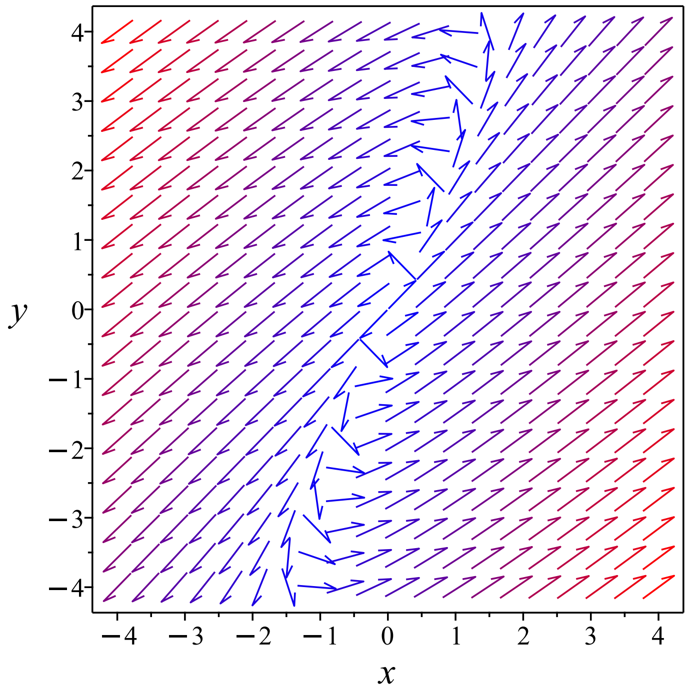


Figure 562: Phase plot

18.1.3 Maple step by step solution

Let's solve

$[x'(t) = 5x(t) - 2y(t), y'(t) = 4x(t) - y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[1, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

  $$\left[1, \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_1 = e^t \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

  $$\left[3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

  $$\vec{x}_2 = e^{3t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

  $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

- Substitute solutions into the general solution

  $$\vec{x} = c_1 e^t \cdot \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} + c_2 e^{3t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^t}{2} + c_2 e^{3t} \\ c_1 e^t + c_2 e^{3t} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x(t) = \frac{c_1 e^t}{2} + c_2 e^{3t}, y(t) = c_1 e^t + c_2 e^{3t}\right\}$$

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
dsolve([diff(x(t),t)=5*x(t)-2*y(t),diff(y(t),t)=4*x(t)-y(t)],singsol=all)
```

$$x(t) = c_1 e^t + c_2 e^{3t}$$
$$y(t) = 2c_1 e^t + c_2 e^{3t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 63

```
DSolve[{x'[t]==5*x[t]-2*y[t],y'[t]==4*x[t]-y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to e^t \left(c_1 \left(2e^{2t} - 1\right) - c_2 \left(e^{2t} - 1\right)\right)$$
$$y(t) \to e^t \left(2c_1 \left(e^{2t} - 1\right) - c_2 \left(e^{2t} - 2\right)\right)$$

18.2 problem 2

Internal problem ID [11949]
Internal file name [`OUTPUT/11958_Sunday_April_14_2024_02_31_20_AM_33601750/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.4. Exercises page 309
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 5x(t) - y(t)$$
$$y'(t) = 3x(t) + y(t)$$

18.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -\frac{e^{2t}}{2} + \frac{3\,e^{4t}}{2} & -\frac{e^{4t}}{2} + \frac{e^{2t}}{2} \\ \frac{3\,e^{4t}}{2} - \frac{3\,e^{2t}}{2} & \frac{3\,e^{2t}}{2} - \frac{e^{4t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -\frac{e^{2t}}{2}+\frac{3\,e^{4t}}{2} & -\frac{e^{4t}}{2}+\frac{e^{2t}}{2} \\ \frac{3\,e^{4t}}{2}-\frac{3\,e^{2t}}{2} & \frac{3\,e^{2t}}{2}-\frac{e^{4t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(-\frac{e^{2t}}{2}+\frac{3\,e^{4t}}{2}\right)c_1+\left(-\frac{e^{4t}}{2}+\frac{e^{2t}}{2}\right)c_2 \\ \left(\frac{3\,e^{4t}}{2}-\frac{3\,e^{2t}}{2}\right)c_1+\left(\frac{3\,e^{2t}}{2}-\frac{e^{4t}}{2}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_2-c_1)e^{2t}}{2}+\frac{3\,e^{4t}\left(-\frac{c_2}{3}+c_1\right)}{2} \\ \frac{(-3c_1+3c_2)e^{2t}}{2}+\frac{3\,e^{4t}\left(-\frac{c_2}{3}+c_1\right)}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$
$$\lambda_2 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	1	1	No	$\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{2t} \\ &= \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} e^{2t}\end{aligned}$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{4t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{2t}}{3} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{2t}}{3} + c_2 e^{4t} \\ c_1 e^{2t} + c_2 e^{4t} \end{bmatrix}$$

The following is the phase plot of the system.

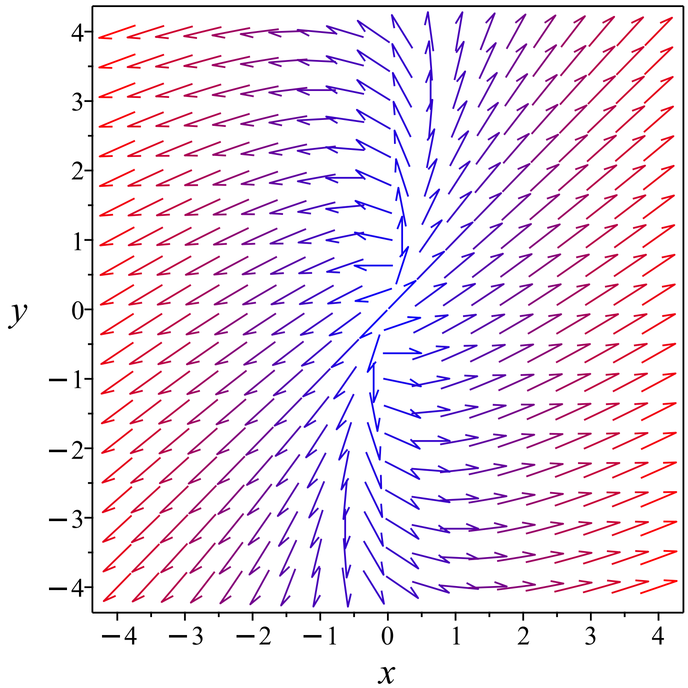


Figure 563: Phase plot

18.2.3 Maple step by step solution

Let's solve

$[x'(t) = 5x(t) - y(t), y'(t) = 3x(t) + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{4t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = c_1 \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} + c_2 \mathrm{e}^{4t} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{c_1 e^{2t}}{3} + c_2 e^{4t} \\ c_1 e^{2t} + c_2 e^{4t} \end{bmatrix}$$

- Solution to the system of ODEs

  $$\left\{x(t) = \frac{c_1 e^{2t}}{3} + c_2 e^{4t}, y(t) = c_1 e^{2t} + c_2 e^{4t}\right\}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve([diff(x(t),t)=5*x(t)-y(t),diff(y(t),t)=3*x(t)+y(t)],singsol=all)
```

$$x(t) = c_1 e^{4t} + c_2 e^{2t}$$
$$y(t) = c_1 e^{4t} + 3c_2 e^{2t}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 73

```
DSolve[{x'[t]==5*x[t]-y[t],y'[t]==3*x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> Tru
```

$$x(t) \to \frac{1}{2} e^{2t} \left(c_1 \left(3e^{2t} - 1\right) - c_2 \left(e^{2t} - 1\right)\right)$$
$$y(t) \to \frac{1}{2} e^{2t} \left(3c_1 \left(e^{2t} - 1\right) - c_2 \left(e^{2t} - 3\right)\right)$$

18.3 problem 23

Internal problem ID [11950]
Internal file name [OUTPUT/11959_Sunday_April_14_2024_02_31_21_AM_78993495/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.4. Exercises page 309
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= -2x(t) + 7y(t) \\
y'(t) &= 3x(t) + 2y(t)
\end{aligned}
$$

With initial conditions

$$
[x(0) = 9, y(0) = -1]
$$

18.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \frac{7\,\mathrm{e}^{-5t}}{10} + \frac{3\,\mathrm{e}^{5t}}{10} & \frac{7\,\mathrm{e}^{5t}}{10} - \frac{7\,\mathrm{e}^{-5t}}{10} \\ \frac{3\,\mathrm{e}^{5t}}{10} - \frac{3\,\mathrm{e}^{-5t}}{10} & \frac{3\,\mathrm{e}^{-5t}}{10} + \frac{7\,\mathrm{e}^{5t}}{10} \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{7\,e^{-5t}}{10} + \frac{3\,e^{5t}}{10} & \frac{7\,e^{5t}}{10} - \frac{7\,e^{-5t}}{10} \\ \frac{3\,e^{5t}}{10} - \frac{3\,e^{-5t}}{10} & \frac{3\,e^{-5t}}{10} + \frac{7\,e^{5t}}{10} \end{bmatrix} \begin{bmatrix} 9 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 7\,e^{-5t} + 2\,e^{5t} \\ 2\,e^{5t} - 3\,e^{-5t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -2 & 7 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -2-\lambda & 7 \\ 3 & 2-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 25 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$
$$\lambda_2 = -5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-5	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 7 \\ 3 & 2 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & 7 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & 7 & 0 \\ 3 & 7 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & 7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{7t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 7 \\ 3 & 2 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 7 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -7 & 7 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{7} \Longrightarrow \left[\begin{array}{cc|c} -7 & 7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -7 & 7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	1	1	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
-5	1	1	No	$\begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 5 is real and distinct then the

corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{5t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}\end{aligned}$$

Since eigenvalue -5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-5t} \\ &= \begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix} e^{-5t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{7\,e^{-5t}}{3} \\ e^{-5t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{5t} - \frac{7 c_2 e^{-5t}}{3} \\ c_1 e^{5t} + c_2 e^{-5t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 9 \\ y(0) = -1 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 9 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 - \frac{7c_2}{3} \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 2 \\ c_2 = -3 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 7\,e^{-5t} + 2\,e^{5t} \\ 2\,e^{5t} - 3\,e^{-5t} \end{bmatrix}$$

The following is the phase plot of the system.

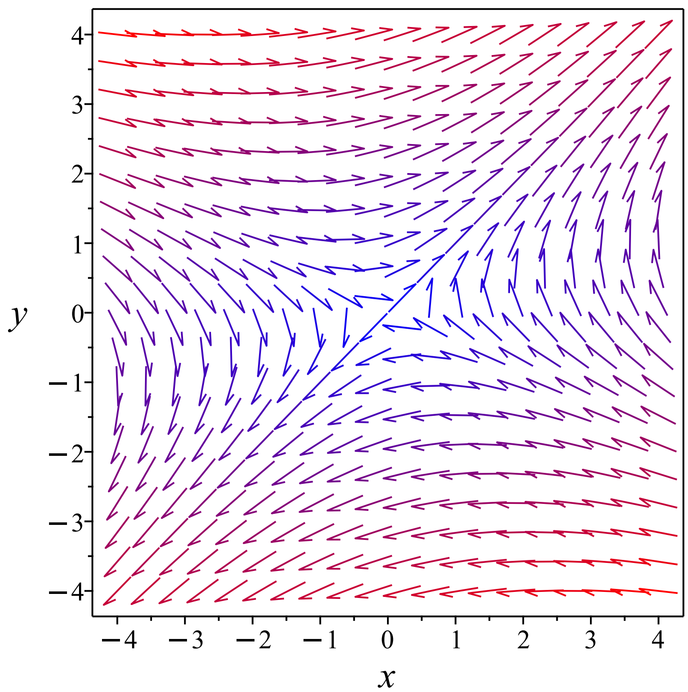
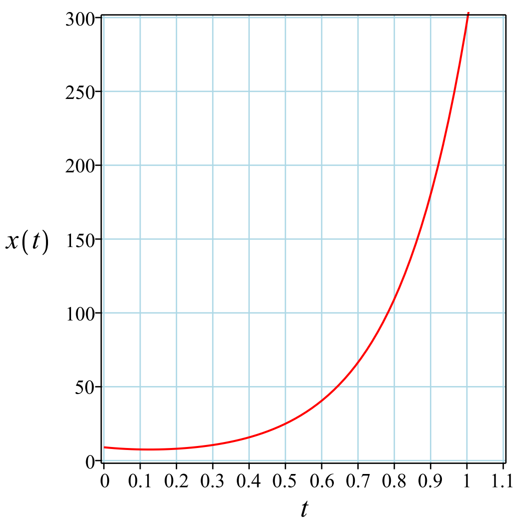
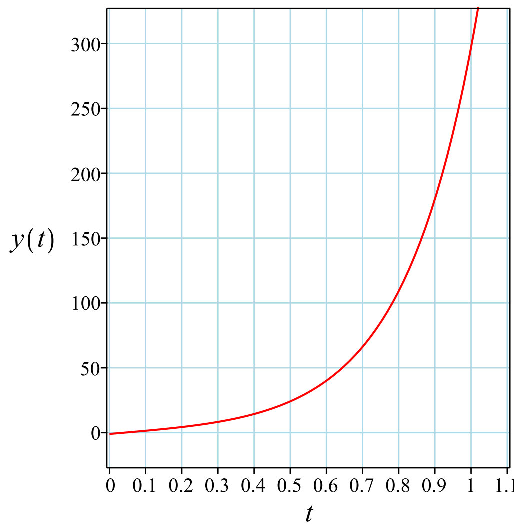


Figure 564: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
dsolve([diff(x(t),t) = -2*x(t)+7*y(t), diff(y(t),t) = 3*x(t)+2*y(t), x(0) = 9, y(0) = -1], s
```

$$x(t) = 2\,\mathrm{e}^{5t} + 7\,\mathrm{e}^{-5t}$$
$$y(t) = 2\,\mathrm{e}^{5t} - 3\,\mathrm{e}^{-5t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 38

```
DSolve[{x'[t]==-2*x[t]+7*y[t],y'[t]==3*x[t]+2*y[t]},{x[0]==9,y[0]==-1},{x[t],y[t]},t,Include
```

$$x(t) \to 7e^{-5t} + 2e^{5t}$$
$$y(t) \to 2e^{5t} - 3e^{-5t}$$

18.4 problem 24

Internal problem ID [11951]
Internal file name [`OUTPUT/11960_Sunday_April_14_2024_02_31_22_AM_81973480/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.4. Exercises page 309
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -2x(t) + y(t)$$
$$y'(t) = 7x(t) + 4y(t)$$

With initial conditions

$$[x(0) = 6, y(0) = 2]$$

18.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(e^{8t}+7)e^{-3t}}{8} & \frac{(e^{8t}-1)e^{-3t}}{8} \\ \frac{7(e^{8t}-1)e^{-3t}}{8} & \frac{(7e^{8t}+1)e^{-3t}}{8} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{x}_0 \\
&= \begin{bmatrix} \frac{(e^{8t}+7)e^{-3t}}{8} & \frac{(e^{8t}-1)e^{-3t}}{8} \\ \frac{7(e^{8t}-1)e^{-3t}}{8} & \frac{(7e^{8t}+1)e^{-3t}}{8} \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{3(e^{8t}+7)e^{-3t}}{4} + \frac{(e^{8t}-1)e^{-3t}}{4} \\ \frac{21(e^{8t}-1)e^{-3t}}{4} + \frac{(7e^{8t}+1)e^{-3t}}{4} \end{bmatrix} \\
&= \begin{bmatrix} (e^{8t}+5)\,e^{-3t} \\ (7\,e^{8t}-5)\,e^{-3t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

18.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -2 & 1 \\ 7 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -2-\lambda & 1 \\ 7 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda - 15 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ 7 & 4 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 7 & 7 & 0 \end{array}\right]$$

$$R_2 = R_2 - 7R_1 \Longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -2 & 1 \\ 7 & 4 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 1 \\ 7 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -7 & 1 & 0 \\ 7 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -7 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -7 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{7}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{7} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{7} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{7} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{7} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{7} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{7} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{7} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	1	1	No	$\begin{bmatrix} \frac{1}{7} \\ 1 \end{bmatrix}$
-3	1	1	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{5t}\\ &= \begin{bmatrix} \frac{1}{7} \\ 1 \end{bmatrix} e^{5t}\end{aligned}$$

Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-3t}\\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{5t}}{7} \\ e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-3t} \\ e^{-3t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{(c_1 e^{8t} - 7c_2) e^{-3t}}{7} \\ (c_1 e^{8t} + c_2) e^{-3t} \end{bmatrix}$$

Since initial conditions are given, the solution above needs to be updated by solving for the constants of integrations using the given initial conditions

$$\begin{bmatrix} x(0) = 6 \\ y(0) = 2 \end{bmatrix} \tag{1}$$

Substituting initial conditions into the above solution at $t = 0$ gives

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{c_1}{7} - c_2 \\ c_1 + c_2 \end{bmatrix}$$

Solving for the constants of integrations gives

$$\begin{bmatrix} c_1 = 7 \\ c_2 = -5 \end{bmatrix}$$

Substituting these constants back in original solution in Eq. (1) gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{(7\,\mathrm{e}^{8t}+35)\,\mathrm{e}^{-3t}}{7} \\ (7\,\mathrm{e}^{8t}-5)\,\mathrm{e}^{-3t} \end{bmatrix}$$

The following is the phase plot of the system.

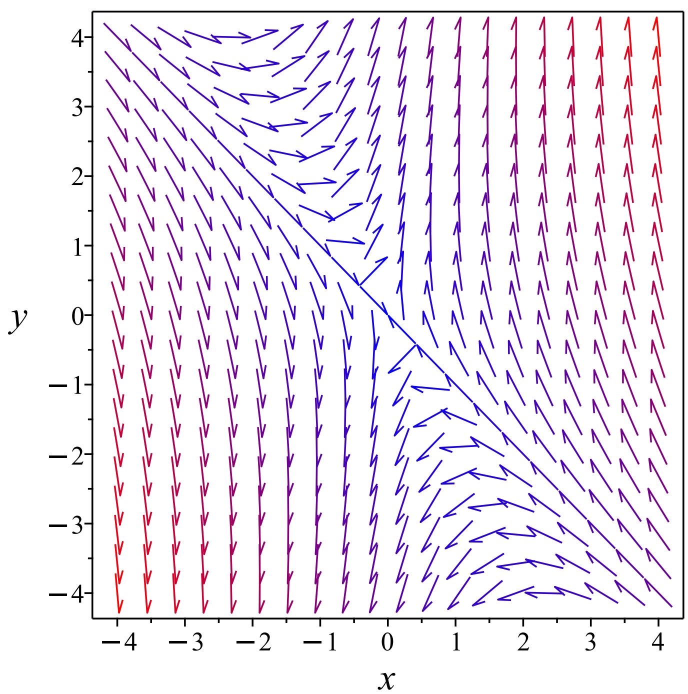
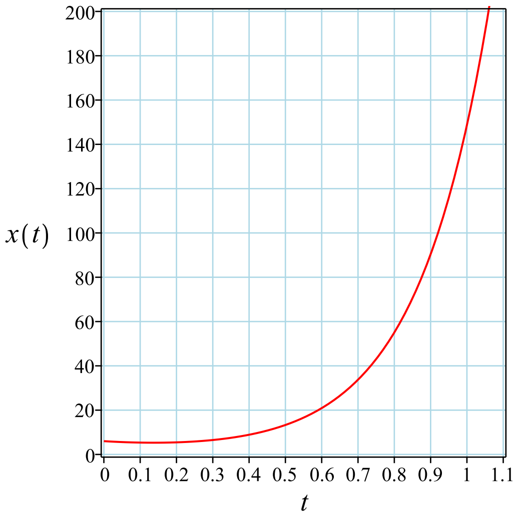
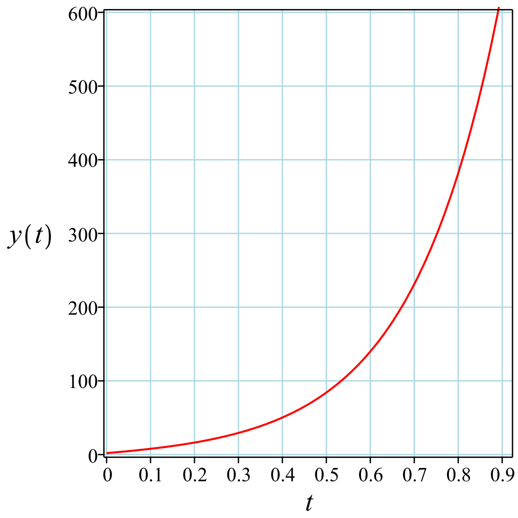


Figure 565: Phase plot

The following are plots of each solution.





✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 32

```
dsolve([diff(x(t),t) = -2*x(t)+y(t), diff(y(t),t) = 7*x(t)+4*y(t), x(0) = 6, y(0) = 2], sing
```

$$x(t) = e^{5t} + 5\,e^{-3t}$$
$$y(t) = 7\,e^{5t} - 5\,e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 36

```
DSolve[{x'[t]==-2*x[t]+y[t],y'[t]==7*x[t]+4*y[t]},{x[0]==6,y[0]==2},{x[t],y[t]},t,IncludeSin
```

$$x(t) \to e^{-3t}\left(e^{8t} + 5\right)$$
$$y(t) \to e^{-3t}\left(7e^{8t} - 5\right)$$

19 Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345

19.1 problem 1

Internal problem ID [11952]
Internal file name [`OUTPUT/11961_Sunday_April_14_2024_02_31_23_AM_64925071/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 3\lambda - 4 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 4$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
4	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 4$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{2t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-1	1	2	No	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
4	1	2	No	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}^{-1}$$

19.2 problem 2

Internal problem ID [11953]
Internal file name [`OUTPUT/11962_Sunday_April_14_2024_02_31_23_AM_658154/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det \begin{bmatrix} 3-\lambda & 2 \\ 6 & -1-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 2\lambda - 15 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 5$$
$$\lambda_2 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-3	1	real eigenvalue
5	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 6 & 2 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 6 & 2 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 6 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 6 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{3}\right\}$

Hence the solution is

$$
\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Considering $\lambda = 5$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -2 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 6 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 + 3R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-3	1	2	No	$\begin{bmatrix} -1 \\ 3 \end{bmatrix}$
5	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}^{-1}$$

19.3 problem 3

Internal problem ID [11954]
Internal file name [`OUTPUT/11963_Sunday_April_14_2024_02_31_24_AM_18998694/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 3 & 1 \\ 12 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 3 & 1 \\ 12 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix} 3-\lambda & 1 \\ 12 & 2-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 5\lambda - 6 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 6$$
$$\lambda_2 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
6	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 3 & 1 \\ 12 & 2 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 3 & 1 \\ 12 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 4 & 1 \\ 12 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & 1 & 0 \\ 12 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 - 3R_1 \Longrightarrow \left[\begin{array}{cc|c} 4 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -\frac{t}{4}\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Considering $\lambda = 6$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 3 & 1 \\ 12 & 2 \end{bmatrix} - (6) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 3 & 1 \\ 12 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -3 & 1 \\ 12 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & 1 & 0 \\ 12 & -4 & 0 \end{array}\right]$$

$$R_2 = R_2 + 4R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-1	1	2	No	$\begin{bmatrix} -1 \\ 4 \end{bmatrix}$
6	1	2	No	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 1 \\ 4 & 3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 3 & 1 \\ 12 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 4 & 3 \end{bmatrix}^{-1}$$

19.4 problem 4

Internal problem ID [11955]
Internal file name [`OUTPUT/11964_Sunday_April_14_2024_02_31_24_AM_23223417/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} -2 & 7 \\ 3 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -2 & 7 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} -2-\lambda & 7 \\ 3 & 2-\lambda \end{bmatrix} = 0$$

$$\lambda^2 - 25 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 5$$
$$\lambda_2 = -5$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-5	1	real eigenvalue
5	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -5$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} -2 & 7 \\ 3 & 2 \end{bmatrix} - (-5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} -2 & 7 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 3 & 7 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 3 & 7 & 0 \\ 3 & 7 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 3 & 7 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{7t}{3}\right\}$

Hence the solution is

$$
\begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{7}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}$$

Considering $\lambda = 5$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} -2 & 7 \\ 3 & 2 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} -2 & 7 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -7 & 7 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -7 & 7 & 0 \\ 3 & -3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{7} \Longrightarrow \left[\begin{array}{cc|c} -7 & 7 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -7 & 7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-5	1	2	No	$\begin{bmatrix} -7 \\ 3 \end{bmatrix}$
5	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$P = \begin{bmatrix} -7 & 1 \\ 3 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} -2 & 7 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -7 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -7 & 1 \\ 3 & 1 \end{bmatrix}^{-1}$$

19.5 problem 5

Internal problem ID [11956]
Internal file name [OUTPUT/11965_Sunday_April_14_2024_02_31_25_AM_38090367/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$
$$\det\left(\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det \begin{bmatrix} 3-\lambda & 4 \\ 5 & 2-\lambda \end{bmatrix} = 0$$
$$\lambda^2 - 5\lambda - 14 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 7$$
$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
7	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 5 & 4 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 5 & 4 & 0 \\ 5 & 4 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} 5 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 5 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -\frac{4t}{5}\}$

Hence the solution is

$$
\begin{bmatrix} -\frac{4t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4t}{5} \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{4t}{5} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{4}{5} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{4t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{4t}{5} \\ t \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

Considering $\lambda = 7$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} - (7) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -4 & 4 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 5 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{5R_1}{4} \Longrightarrow \left[\begin{array}{cc|c} -4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-2	1	2	No	$\begin{bmatrix} -4 \\ 5 \end{bmatrix}$
7	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 7 \end{bmatrix}$$

$$P = \begin{bmatrix} -4 & 1 \\ 5 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 5 & 1 \end{bmatrix}^{-1}$$

19.6 problem 6

Internal problem ID [11957]
Internal file name [`OUTPUT/11966_Sunday_April_14_2024_02_31_25_AM_34181949/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 3-\lambda & -5 \\ -4 & 2-\lambda \end{bmatrix} = 0$$

$$\lambda^2 - 5\lambda - 14 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 7$$

$$\lambda_2 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
7	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 5 & -5 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{cc|c} 5 & -5 & 0 \\ -4 & 4 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + \frac{4R_1}{5} \Longrightarrow \left[\begin{array}{cc|c} 5 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 5 & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t\}$

Hence the solution is

$$
\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 7$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\ \left(\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} - (7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -4 & -5 \\ -4 & -5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{cc|c} -4 & -5 & 0 \\ -4 & -5 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{cc|c} -4 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{5t}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{5t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{5t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{5}{4} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{5t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5}{4} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{5t}{4} \\ t \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-2	1	2	No	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
7	1	2	No	$\begin{bmatrix} -5 \\ 4 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 7 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -5 \\ 1 & 4 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 1 & 4 \end{bmatrix}^{-1}$$

19.7 problem 7

Internal problem ID [11958]
Internal file name [`OUTPUT/11967_Sunday_April_14_2024_02_31_26_AM_18554227/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det \begin{bmatrix} 1-\lambda & 1 & -1 \\ 2 & 3-\lambda & -4 \\ 4 & 1 & -4-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 7\lambda - 6 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
-3	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$A\boldsymbol{v} = \lambda \boldsymbol{v}$$
$$A\boldsymbol{v} - \lambda \boldsymbol{v} = \boldsymbol{0}$$
$$(A - \lambda I)\boldsymbol{v} = \boldsymbol{0}$$
$$\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & -4 \\ 4 & 1 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 2 & 2 & -4 & 0 \\ 4 & 1 & -5 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 2 & 2 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 4 & 1 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 2 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 2 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = -3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 4 & 1 & -1 \\ 2 & 6 & -4 \\ 4 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 4 & 1 & -1 & 0 \\ 2 & 6 & -4 & 0 \\ 4 & 1 & -1 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & 1 & -1 & 0 \\ 0 & \frac{11}{2} & -\frac{7}{2} & 0 \\ 4 & 1 & -1 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4 & 1 & -1 & 0 \\ 0 & \frac{11}{2} & -\frac{7}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 4 & 1 & -1 \\ 0 & \frac{11}{2} & -\frac{7}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{11}, v_2 = \frac{7t}{11}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -4 \\ 4 & 1 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 2 & 1 & -4 & 0 \\ 4 & 1 & -6 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 3 & -6 & 0 \\ 4 & 1 & -6 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 5 & -10 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - \frac{5R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
-3	1	3	No	$\begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 7 & 2 \\ 1 & 11 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 7 & 2 \\ 1 & 11 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 7 & 2 \\ 1 & 11 & 1 \end{bmatrix}^{-1}$$

19.8 problem 8

Internal problem ID [11959]
Internal file name [OUTPUT/11968_Sunday_April_14_2024_02_31_26_AM_33178477/index.tex]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 1 \\ -3 & -6 & 6-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 10\lambda^2 - 31\lambda + 30 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 5$$
$$\lambda_2 = 2$$
$$\lambda_3 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue
5	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & -6 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ -3 & -6 & 4 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -6 & 4 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - 3R_1 \implies \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 7 & 0 \end{array}\right]
$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$
\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & -3 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -1 & -1 & -1 \\ 0 & -3 & 7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{10t}{3}, v_2 = \frac{7t}{3}\right\}$

Hence the solution is

$$
\begin{bmatrix} -\frac{10t}{3} \\ \frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{10t}{3} \\ \frac{7t}{3} \\ t \end{bmatrix}
$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} -\frac{10t}{3} \\ \frac{7t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{10}{3} \\ \frac{7}{3} \\ 1 \end{bmatrix}
$$

Or, by letting $t = 1$ then the eigenvector is

$$
\begin{bmatrix} -\frac{10t}{3} \\ \frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{10}{3} \\ \frac{7}{3} \\ 1 \end{bmatrix}
$$

Which can be normalized to

$$\begin{bmatrix} -\frac{10t}{3} \\ \frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \\ 3 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -3 & -6 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & -6 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -3 & -6 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{9}{2} & \frac{9}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 - 9R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -1 & -1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Considering $\lambda = 5$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -4 & -1 & -1 \\ 1 & -2 & 1 \\ -3 & -6 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ -3 & -6 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & -1 & -1 & 0 \\ 0 & -\frac{9}{4} & \frac{3}{4} & 0 \\ -3 & -6 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & -1 & -1 & 0 \\ 0 & -\frac{9}{4} & \frac{3}{4} & 0 \\ 0 & -\frac{21}{4} & \frac{7}{4} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{7R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} -4 & -1 & -1 & 0 \\ 0 & -\frac{9}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -1 & -1 \\ 0 & -\frac{9}{4} & \frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{3}, v_2 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
2	1	3	No	$\begin{bmatrix} -10 \\ 7 \\ 3 \end{bmatrix}$
3	1	3	No	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$
5	1	3	No	$\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$P = \begin{bmatrix} -10 & -1 & -1 \\ 7 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} = \begin{bmatrix} -10 & -1 & -1 \\ 7 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -10 & -1 & -1 \\ 7 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}^{-1}$$

19.9 problem 9

Internal problem ID [11960]
Internal file name [`OUTPUT/11969_Sunday_April_14_2024_02_31_27_AM_2632022/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 1 \\ -3 & 1 & -1-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 3\lambda^2 + 4\lambda - 12 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 3 & -1 & -1 \\ 1 & 5 & 1 \\ -3 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 1 & 5 & 1 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 0 & \frac{16}{3} & \frac{4}{3} & 0 \\ -3 & 1 & 1 & 0 \end{array}\right]
$$

$$R_3 = R_3 + R_1 \implies \left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 0 & \frac{16}{3} & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -1 & -1 \\ 0 & \frac{16}{3} & \frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{4}, v_2 = -\frac{t}{4}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{4} \\ -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{4} \\ -\frac{t}{4} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{4} \\ -\frac{t}{4} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} \frac{t}{4} \\ -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} \frac{t}{4} \\ -\frac{t}{4} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ -3 & 1 & -3 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 1 & -3 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]
$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$
\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -3 & 1 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 1 & -4 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + \frac{R_1}{2} \implies \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -3 & 1 & -4 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - \frac{3R_1}{2} \implies \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{5}{2} & -\frac{5}{2} & 0 \end{array}\right]
$$

$$
R_3 = R_3 + 5R_2 \implies \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -1 & -1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-2	1	3	No	$\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
3	1	3	No	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 4 & 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 4 & 1 & 1 \end{bmatrix}^{-1}$$

19.10 problem 10

Internal problem ID [11961]
Internal file name [`OUTPUT/11970_Sunday_April_14_2024_02_31_28_AM_40149345/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue
2	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]
$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = -2t\}$

Hence the solution is

$$\begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]
$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$
\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-1	1	3	No	$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$
1	1	3	No	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}$$

19.11 problem 11

Internal problem ID [11962]
Internal file name [`OUTPUT/11971_Sunday_April_14_2024_02_31_28_AM_20775540/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} 1 & 3 & -6 \\ 0 & 2 & 2 \\ 0 & -1 & 5 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & 3 & -6 \\ 0 & 2 & 2 \\ 0 & -1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1-\lambda & 3 & -6 \\ 0 & 2-\lambda & 2 \\ 0 & -1 & 5-\lambda \end{bmatrix} = 0$$

$$-(-1+\lambda)\left(\lambda^2 - 7\lambda + 12\right) = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 4$$
$$\lambda_3 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
3	1	real eigenvalue
4	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 3 & -6 \\ 0 & 2 & 2 \\ 0 & -1 & 5 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 3 & -6 \\ 0 & 2 & 2 \\ 0 & -1 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 3 & -6 \\ 0 & 1 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 0 & 3 & -6 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 0 & 3 & -6 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right]
$$

$$R_3 = R_3 + \frac{R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} 0 & 3 & -6 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} 0 & 3 & -6 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 3 & -6 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} 1 & 3 & -6 \\ 0 & 2 & 2 \\ 0 & -1 & 5 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} 1 & 3 & -6 \\ 0 & 2 & 2 \\ 0 & -1 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -2 & 3 & -6 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} -2 & 3 & -6 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 3 & -6 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} -2 & 3 & -6 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Considering $\lambda = 4$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\ \left(\begin{bmatrix} 1 & 3 & -6 \\ 0 & 2 & 2 \\ 0 & -1 & 5 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left(\begin{bmatrix} 1 & 3 & -6 \\ 0 & 2 & 2 \\ 0 & -1 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -3 & 3 & -6 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -3 & 3 & -6 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} -3 & 3 & -6 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 3 & -6 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	3	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
3	1	3	No	$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$
4	1	3	No	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} 1 & 3 & -6 \\ 0 & 2 & 2 \\ 0 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

19.12 problem 12

Internal problem ID [11963]
Internal file name [`OUTPUT/11972_Sunday_April_14_2024_02_31_29_AM_78741034/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} -5-\lambda & -12 & 6 \\ 1 & 5-\lambda & -1 \\ -7 & -10 & 8-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 8\lambda^2 - 19\lambda + 12 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$
$$\lambda_3 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
3	1	real eigenvalue
4	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -6 & -12 & 6 \\ 1 & 4 & -1 \\ -7 & -10 & 7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} -6 & -12 & 6 & 0 \\ 1 & 4 & -1 & 0 \\ -7 & -10 & 7 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + \frac{R_1}{6} \Longrightarrow \left[\begin{array}{ccc|c} -6 & -12 & 6 & 0 \\ 0 & 2 & 0 & 0 \\ -7 & -10 & 7 & 0 \end{array}\right]
$$

$$R_3 = R_3 - \frac{7R_1}{6} \Longrightarrow \left[\begin{array}{ccc|c} -6 & -12 & 6 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} -6 & -12 & 6 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6 & -12 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -8 & -12 & 6 \\ 1 & 2 & -1 \\ -7 & -10 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -8 & -12 & 6 & 0 \\ 1 & 2 & -1 & 0 \\ -7 & -10 & 5 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{8} \Longrightarrow \left[\begin{array}{ccc|c} -8 & -12 & 6 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ -7 & -10 & 5 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{7R_1}{8} \Longrightarrow \left[\begin{array}{ccc|c} -8 & -12 & 6 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -8 & -12 & 6 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -8 & -12 & 6 \\ 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = 0, v_2 = \frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} 0 \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Considering $\lambda = 4$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -9 & -12 & 6 \\ 1 & 1 & -1 \\ -7 & -10 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -9 & -12 & 6 & 0 \\ 1 & 1 & -1 & 0 \\ -7 & -10 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{9} \Longrightarrow \left[\begin{array}{ccc|c} -9 & -12 & 6 & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -7 & -10 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{7R_1}{9} \Longrightarrow \left[\begin{array}{ccc|c} -9 & -12 & 6 & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{2}{3} & -\frac{2}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} -9 & -12 & 6 & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -9 & -12 & 6 \\ 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
1	1	3	No	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
3	1	3	No	$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$
4	1	3	No	$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} -5 & -12 & 6 \\ 1 & 5 & -1 \\ -7 & -10 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}^{-1}$$

19.13 problem 13

Internal problem ID [11964]
Internal file name [`OUTPUT/11973_Sunday_April_14_2024_02_31_30_AM_38495058/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} -2 & 5 & 5 \\ -1 & 4 & 5 \\ 3 & -3 & 2 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -2 & 5 & 5 \\ -1 & 4 & 5 \\ 3 & -3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det \begin{bmatrix} -2-\lambda & 5 & 5 \\ -1 & 4-\lambda & 5 \\ 3 & -3 & 2-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 4\lambda^2 - \lambda - 6 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 2$$
$$\lambda_2 = 3$$
$$\lambda_3 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} -2 & 5 & 5 \\ -1 & 4 & 5 \\ 3 & -3 & 2 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} -2 & 5 & 5 \\ -1 & 4 & 5 \\ 3 & -3 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -1 & 5 & 5 \\ -1 & 5 & 5 \\ 3 & -3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} -1 & 5 & 5 & 0 \\ -1 & 5 & 5 & 0 \\ 3 & -3 & 3 & 0 \end{array}\right]
$$

$$
R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & -3 & 3 & 0 \end{array}\right]
$$

$$R_3 = R_3 + 3R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 12 & 18 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -1 & 5 & 5 & 0 \\ 0 & 12 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 5 & 5 \\ 0 & 12 & 18 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{5t}{2}, v_2 = -\frac{3t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{5t}{2} \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5t}{2} \\ -\frac{3t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{5t}{2} \\ -\frac{3t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{5}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -\frac{5t}{2} \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

Which can be normalized to

$$\begin{bmatrix} -\frac{5t}{2} \\ -\frac{3t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix}$$

Considering $\lambda = 2$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ A\boldsymbol{v} - \lambda\boldsymbol{v} &= \boldsymbol{0} \\ (A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \end{aligned}$$

$$\left(\begin{bmatrix} -2 & 5 & 5 \\ -1 & 4 & 5 \\ 3 & -3 & 2 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} -2 & 5 & 5 \\ -1 & 4 & 5 \\ 3 & -3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 5 & 5 \\ -1 & 2 & 5 \\ 3 & -3 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & 5 & 5 & 0 \\ -1 & 2 & 5 & 0 \\ 3 & -3 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & 5 & 5 & 0 \\ 0 & \frac{3}{4} & \frac{15}{4} & 0 \\ 3 & -3 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & 5 & 5 & 0 \\ 0 & \frac{3}{4} & \frac{15}{4} & 0 \\ 0 & \frac{3}{4} & \frac{15}{4} & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} -4 & 5 & 5 & 0 \\ 0 & \frac{3}{4} & \frac{15}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & 5 & 5 \\ 0 & \frac{3}{4} & \frac{15}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -5t, v_2 = -5t\}$

Hence the solution is

$$\begin{bmatrix} -5t \\ -5t \\ t \end{bmatrix} = \begin{bmatrix} -5t \\ -5t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -5t \\ -5t \\ t \end{bmatrix} = t \begin{bmatrix} -5 \\ -5 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -5t \\ -5t \\ t \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ 1 \end{bmatrix}$$

Considering $\lambda = 3$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda\boldsymbol{v} \\
A\boldsymbol{v} - \lambda\boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} -2 & 5 & 5 \\ -1 & 4 & 5 \\ 3 & -3 & 2 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} -2 & 5 & 5 \\ -1 & 4 & 5 \\ 3 & -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -5 & 5 & 5 \\ -1 & 1 & 5 \\ 3 & -3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & 5 & 5 & 0 \\ -1 & 1 & 5 & 0 \\ 3 & -3 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 5 & 5 & 0 \\ 0 & 0 & 4 & 0 \\ 3 & -3 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{3R_1}{5} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 5 & 5 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{2} \Longrightarrow \left[\begin{array}{ccc|c} -5 & 5 & 5 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -5 & 5 & 5 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-1	1	3	No	$\begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix}$
2	1	3	No	$\begin{bmatrix} -5 \\ -5 \\ 1 \end{bmatrix}$
3	1	3	No	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} -5 & -5 & 1 \\ -3 & -5 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} -2 & 5 & 5 \\ -1 & 4 & 5 \\ 3 & -3 & 2 \end{bmatrix} = \begin{bmatrix} -5 & -5 & 1 \\ -3 & -5 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -5 & -5 & 1 \\ -3 & -5 & 1 \\ 2 & 1 & 0 \end{bmatrix}^{-1}$$

19.14 problem 14

Internal problem ID [11965]
Internal file name [`OUTPUT/11974_Sunday_April_14_2024_02_31_31_AM_74535894/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.5. Matrices and vectors. Exercises page 345
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"find eigenvalues and eigenvectors"**

Find the eigenvalues and associated eigenvectors of the matrix

$$\begin{bmatrix} -2 & 6 & -18 \\ 12 & -23 & 66 \\ 5 & -10 & 29 \end{bmatrix}$$

The first step is to determine the characteristic polynomial of the matrix in order to find the eigenvalues of the matrix A. This is given by

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -2 & 6 & -18 \\ 12 & -23 & 66 \\ 5 & -10 & 29 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} -2-\lambda & 6 & -18 \\ 12 & -23-\lambda & 66 \\ 5 & -10 & 29-\lambda \end{bmatrix} = 0$$

$$-\lambda^3 + 4\lambda^2 + \lambda - 4 = 0$$

The eigenvalues are the roots of the above characteristic polynomial. Solving for the roots gives

$$\lambda_1 = 1$$
$$\lambda_2 = 4$$
$$\lambda_3 = -1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue
4	1	real eigenvalue

For each eigenvalue λ found above, we now find the corresponding eigenvector. Considering $\lambda = -1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} -2 & 6 & -18 \\ 12 & -23 & 66 \\ 5 & -10 & 29 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} -2 & 6 & -18 \\ 12 & -23 & 66 \\ 5 & -10 & 29 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -1 & 6 & -18 \\ 12 & -22 & 66 \\ 5 & -10 & 30 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 6 & -18 & 0 \\ 12 & -22 & 66 & 0 \\ 5 & -10 & 30 & 0 \end{array}\right]$$

$$R_2 = R_2 + 12R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 6 & -18 & 0 \\ 0 & 50 & -150 & 0 \\ 5 & -10 & 30 & 0 \end{array}\right]$$

$$R_3 = R_3 + 5R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 6 & -18 & 0 \\ 0 & 50 & -150 & 0 \\ 0 & 20 & -60 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{2R_2}{5} \Longrightarrow \left[\begin{array}{ccc|c} -1 & 6 & -18 & 0 \\ 0 & 50 & -150 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 6 & -18 \\ 0 & 50 & -150 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 3t\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 3t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 3t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 0 \\ 3t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

Considering $\lambda = 1$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \mathbf{0} \\
(A - \lambda I)\boldsymbol{v} &= \mathbf{0} \\
\left(\begin{bmatrix} -2 & 6 & -18 \\ 12 & -23 & 66 \\ 5 & -10 & 29 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} -2 & 6 & -18 \\ 12 & -23 & 66 \\ 5 & -10 & 29 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -3 & 6 & -18 \\ 12 & -24 & 66 \\ 5 & -10 & 28 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} -3 & 6 & -18 & 0 \\ 12 & -24 & 66 & 0 \\ 5 & -10 & 28 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} -3 & 6 & -18 & 0 \\ 0 & 0 & -6 & 0 \\ 5 & -10 & 28 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + \frac{5R_1}{3} \Longrightarrow \left[\begin{array}{ccc|c} -3 & 6 & -18 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} -3 & 6 & -18 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & 6 & -18 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Considering $\lambda = 4$

We need now to determine the eigenvector $\boldsymbol{v}$ where

$$
\begin{aligned}
A\boldsymbol{v} &= \lambda \boldsymbol{v} \\
A\boldsymbol{v} - \lambda \boldsymbol{v} &= \boldsymbol{0} \\
(A - \lambda I)\boldsymbol{v} &= \boldsymbol{0} \\
\left(\begin{bmatrix} -2 & 6 & -18 \\ 12 & -23 & 66 \\ 5 & -10 & 29 \end{bmatrix} - (4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\left(\begin{bmatrix} -2 & 6 & -18 \\ 12 & -23 & 66 \\ 5 & -10 & 29 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -6 & 6 & -18 \\ 12 & -27 & 66 \\ 5 & -10 & 25 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will now do Gaussian elimination in order to solve for the eigenvector. The augmented matrix is

$$
\left[\begin{array}{ccc|c} -6 & 6 & -18 & 0 \\ 12 & -27 & 66 & 0 \\ 5 & -10 & 25 & 0 \end{array}\right]
$$

$$
R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -6 & 6 & -18 & 0 \\ 0 & -15 & 30 & 0 \\ 5 & -10 & 25 & 0 \end{array}\right]
$$

$$
R_3 = R_3 + \frac{5R_1}{6} \Longrightarrow \left[\begin{array}{ccc|c} -6 & 6 & -18 & 0 \\ 0 & -15 & 30 & 0 \\ 0 & -5 & 10 & 0 \end{array}\right]
$$

$$
R_3 = R_3 - \frac{R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} -6 & 6 & -18 & 0 \\ 0 & -15 & 30 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -6 & 6 & -18 \\ 0 & -15 & 30 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Or, by letting $t = 1$ then the eigenvector is

$$\begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

The following table summarises the result found above.

λ	algebraic multiplicity	geometric multiplicity	defective eigenvalue?	associated eigenvectors
-1	1	3	No	$\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$
1	1	3	No	$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$
4	1	3	No	$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Since the matrix is not defective, then it is diagonalizable. Let P the matrix whose columns are the eigenvectors found, and let D be diagonal matrix with the eigenvalues at its diagonal. Then we can write

$$A = PDP^{-1}$$

Where

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 2 & -1 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} -2 & 6 & -18 \\ 12 & -23 & 66 \\ 5 & -10 & 29 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$

20 Chapter 7, Systems of linear differential equations. Section 7.7. Exercises page 375

20.1 problem 1

Internal problem ID [11966]
Internal file name [`OUTPUT/11975_Sunday_April_14_2024_02_31_32_AM_73858456/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.7. Exercises page 375
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) + y(t) - z(t) \\
y'(t) &= 2x(t) + 3y(t) - 4z(t) \\
z'(t) &= 4x(t) + y(t) - 4z(t)
\end{aligned}
$$

20.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{(-4\,\mathrm{e}^{5t}+15\,\mathrm{e}^{4t}-1)\mathrm{e}^{-3t}}{10} & \mathrm{e}^{2t}-\mathrm{e}^{t} & \frac{(-6\,\mathrm{e}^{5t}+5\,\mathrm{e}^{4t}+1)\mathrm{e}^{-3t}}{10} \\ \frac{(-8\,\mathrm{e}^{5t}+15\,\mathrm{e}^{4t}-7)\mathrm{e}^{-3t}}{10} & -\mathrm{e}^{t}+2\,\mathrm{e}^{2t} & \frac{(-12\,\mathrm{e}^{5t}+5\,\mathrm{e}^{4t}+7)\mathrm{e}^{-3t}}{10} \\ \frac{(-4\,\mathrm{e}^{5t}+15\,\mathrm{e}^{4t}-11)\mathrm{e}^{-3t}}{10} & \mathrm{e}^{2t}-\mathrm{e}^{t} & \frac{(-6\,\mathrm{e}^{5t}+5\,\mathrm{e}^{4t}+11)\mathrm{e}^{-3t}}{10} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \begin{bmatrix} \frac{(-4\,\mathrm{e}^{5t}+15\,\mathrm{e}^{4t}-1)\mathrm{e}^{-3t}}{10} & \mathrm{e}^{2t}-\mathrm{e}^{t} & \frac{(-6\,\mathrm{e}^{5t}+5\,\mathrm{e}^{4t}+1)\mathrm{e}^{-3t}}{10} \\ \frac{(-8\,\mathrm{e}^{5t}+15\,\mathrm{e}^{4t}-7)\mathrm{e}^{-3t}}{10} & -\mathrm{e}^{t}+2\,\mathrm{e}^{2t} & \frac{(-12\,\mathrm{e}^{5t}+5\,\mathrm{e}^{4t}+7)\mathrm{e}^{-3t}}{10} \\ \frac{(-4\,\mathrm{e}^{5t}+15\,\mathrm{e}^{4t}-11)\mathrm{e}^{-3t}}{10} & \mathrm{e}^{2t}-\mathrm{e}^{t} & \frac{(-6\,\mathrm{e}^{5t}+5\,\mathrm{e}^{4t}+11)\mathrm{e}^{-3t}}{10} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(-4\,\mathrm{e}^{5t}+15\,\mathrm{e}^{4t}-1)\mathrm{e}^{-3t}c_1}{10} + (\mathrm{e}^{2t}-\mathrm{e}^{t})\,c_2 + \frac{(-6\,\mathrm{e}^{5t}+5\,\mathrm{e}^{4t}+1)\mathrm{e}^{-3t}c_3}{10} \\ \frac{(-8\,\mathrm{e}^{5t}+15\,\mathrm{e}^{4t}-7)\mathrm{e}^{-3t}c_1}{10} + (-\mathrm{e}^{t}+2\,\mathrm{e}^{2t})\,c_2 + \frac{(-12\,\mathrm{e}^{5t}+5\,\mathrm{e}^{4t}+7)\mathrm{e}^{-3t}c_3}{10} \\ \frac{(-4\,\mathrm{e}^{5t}+15\,\mathrm{e}^{4t}-11)\mathrm{e}^{-3t}c_1}{10} + (\mathrm{e}^{2t}-\mathrm{e}^{t})\,c_2 + \frac{(-6\,\mathrm{e}^{5t}+5\,\mathrm{e}^{4t}+11)\mathrm{e}^{-3t}c_3}{10} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3\left(\left(c_1-\frac{2c_2}{3}+\frac{c_3}{3}\right)\mathrm{e}^{4t}+\left(-\frac{4c_1}{15}+\frac{2c_2}{3}-\frac{2c_3}{5}\right)\mathrm{e}^{5t}-\frac{c_1}{15}+\frac{c_3}{15}\right)\mathrm{e}^{-3t}}{2} \\ \frac{3\left(\left(c_1-\frac{2c_2}{3}+\frac{c_3}{3}\right)\mathrm{e}^{4t}+\left(-\frac{8c_1}{15}+\frac{4c_2}{3}-\frac{4c_3}{5}\right)\mathrm{e}^{5t}-\frac{7c_1}{15}+\frac{7c_3}{15}\right)\mathrm{e}^{-3t}}{2} \\ \frac{3\,\mathrm{e}^{-3t}\left(\left(c_1-\frac{2c_2}{3}+\frac{c_3}{3}\right)\mathrm{e}^{4t}+\left(-\frac{4c_1}{15}+\frac{2c_2}{3}-\frac{2c_3}{5}\right)\mathrm{e}^{5t}-\frac{11c_1}{15}+\frac{11c_3}{15}\right)}{2} \end{bmatrix}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 1 & -1 \\ 2 & 3-\lambda & -4 \\ 4 & 1 & -4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 7\lambda + 6 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$
$$\lambda_2 = 1$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue
-3	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 6 & -4 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 1 & -1 & 0 \\ 2 & 6 & -4 & 0 \\ 4 & 1 & -1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & 1 & -1 & 0 \\ 0 & \frac{11}{2} & -\frac{7}{2} & 0 \\ 4 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 4 & 1 & -1 & 0 \\ 0 & \frac{11}{2} & -\frac{7}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 1 & -1 \\ 0 & \frac{11}{2} & -\frac{7}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t}{11}, v_2 = \frac{7t}{11}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix} = \begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{t}{11} \\ \frac{7t}{11} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -1 \\ 2 & 2 & -4 \\ 4 & 1 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 2 & 2 & -4 & 0 \\ 4 & 1 & -5 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 2 & 2 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 4 & 1 & -5 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 2 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 2 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 2 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 \\ 2 & 1 & -4 \\ 4 & 1 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 2 & 1 & -4 & 0 \\ 4 & 1 & -6 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 3 & -6 & 0 \\ 4 & 1 & -6 & 0 \end{array}\right]$$

$$R_3 = R_3 + 4R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 5 & -10 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{5R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = 2t\}$

Hence the solution is

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	1	1	No	$\begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-3t} \\ &= \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix} e^{-3t} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{t} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t}
\end{aligned}
$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{2t} \\
&= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{-3t}}{11} \\ \frac{7\,e^{-3t}}{11} \\ e^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{t} \\ e^{t} \\ e^{t} \end{bmatrix} + c_3 \begin{bmatrix} e^{2t} \\ 2\,e^{2t} \\ e^{2t} \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{(11c_3 e^{5t} + 11c_2 e^{4t} + c_1)e^{-3t}}{11} \\ \frac{(22c_3 e^{5t} + 11c_2 e^{4t} + 7c_1)e^{-3t}}{11} \\ (c_3 e^{5t} + c_2 e^{4t} + c_1)\,e^{-3t} \end{bmatrix}
$$

20.1.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) + y(t) - z(t)\,, y'(t) = 2x(t) + 3y(t) - 4z(t)\,, z'(t) = 4x(t) + y(t) - 4z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = e^{-3t} \cdot \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = e^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\begin{bmatrix} 2, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \end{bmatrix}$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_3 = e^{2t} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\overrightarrow{x} = c_1 \overrightarrow{x}_1 + c_2 \overrightarrow{x}_2 + c_3 \overrightarrow{x}_3$$

- Substitute solutions into the general solution

$$\overrightarrow{x} = e^{-3t} c_1 \cdot \begin{bmatrix} \frac{1}{11} \\ \frac{7}{11} \\ 1 \end{bmatrix} + c_2 e^{t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{2t} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \frac{(11c_3 e^{5t}+11c_2 e^{4t}+c_1) e^{-3t}}{11} \\ \frac{(22c_3 e^{5t}+11c_2 e^{4t}+7c_1) e^{-3t}}{11} \\ (c_3 e^{5t} + c_2 e^{4t} + c_1) e^{-3t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{(11c_3 e^{5t}+11c_2 e^{4t}+c_1) e^{-3t}}{11}, y(t) = \frac{(22c_3 e^{5t}+11c_2 e^{4t}+7c_1) e^{-3t}}{11}, z(t) = (c_3 e^{5t} + c_2 e^{4t} + c_1) e^{-3t}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 65

```
dsolve([diff(x(t),t)=x(t)+y(t)-z(t),diff(y(t),t)=2*x(t)+3*y(t)-4*z(t),diff(z(t),t)=4*x(t)+y(
```

$$\begin{aligned} x(t) &= c_1 e^{t} + c_2 e^{2t} + c_3 e^{-3t} \\ y(t) &= c_1 e^{t} + 2c_2 e^{2t} + 7c_3 e^{-3t} \\ z(t) &= c_1 e^{t} + c_2 e^{2t} + 11c_3 e^{-3t} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 198

```
DSolve[{x'[t]==x[t]+y[t]-z[t],y'[t]==2*x[t]+3*y[t]-4*z[t],z'[t]==4*x[t]+y[t]-4*z[t]},{x[t],y
```

$$\begin{aligned} x(t) &\to \frac{1}{10} e^{-3t} \left(c_1 \left(15e^{4t} - 4e^{5t} - 1\right) + 2(5c_2 - 3c_3)e^{5t} + 5(c_3 - 2c_2)e^{4t} + c_3\right) \\ y(t) &\to \frac{1}{10} e^{-3t} \left(c_1 \left(15e^{4t} - 8e^{5t} - 7\right) + 4(5c_2 - 3c_3)e^{5t} + 5(c_3 - 2c_2)e^{4t} + 7c_3\right) \\ z(t) &\to \frac{1}{10} e^{-3t} \left(c_1 \left(15e^{4t} - 4e^{5t} - 11\right) + 2(5c_2 - 3c_3)e^{5t} + 5(c_3 - 2c_2)e^{4t} + 11c_3\right) \end{aligned}$$

20.2 problem 2

Internal problem ID [11967]
Internal file name [`OUTPUT/11976_Sunday_April_14_2024_02_31_33_AM_94077692/index.tex`]

Book: Differential Equations by Shepley L. Ross. Third edition. John Willey. New Delhi. 2004.
Section: Chapter 7, Systems of linear differential equations. Section 7.7. Exercises page 375
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= x(t) - y(t) - z(t)\\
y'(t) &= x(t) + 3y(t) + z(t)\\
z'(t) &= -3x(t) - 6y(t) + 6z(t)
\end{aligned}
$$

20.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{10\,e^{2t}}{3} - 3\,e^{3t} + \frac{2\,e^{5t}}{3} & \frac{7\,e^{5t}}{6} - \frac{9\,e^{3t}}{2} + \frac{10\,e^{2t}}{3} & -\frac{e^{5t}}{2} + \frac{e^{3t}}{2} \\ 3\,e^{3t} - \frac{7\,e^{2t}}{3} - \frac{2\,e^{5t}}{3} & -\frac{7\,e^{2t}}{3} - \frac{7\,e^{5t}}{6} + \frac{9\,e^{3t}}{2} & \frac{e^{5t}}{2} - \frac{e^{3t}}{2} \\ 3\,e^{3t} - e^{2t} - 2\,e^{5t} & -\frac{7\,e^{5t}}{2} + \frac{9\,e^{3t}}{2} - e^{2t} & \frac{3\,e^{5t}}{2} - \frac{e^{3t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{10\,e^{2t}}{3} - 3\,e^{3t} + \frac{2\,e^{5t}}{3} & \frac{7\,e^{5t}}{6} - \frac{9\,e^{3t}}{2} + \frac{10\,e^{2t}}{3} & -\frac{e^{5t}}{2} + \frac{e^{3t}}{2} \\ 3\,e^{3t} - \frac{7\,e^{2t}}{3} - \frac{2\,e^{5t}}{3} & -\frac{7\,e^{2t}}{3} - \frac{7\,e^{5t}}{6} + \frac{9\,e^{3t}}{2} & \frac{e^{5t}}{2} - \frac{e^{3t}}{2} \\ 3\,e^{3t} - e^{2t} - 2\,e^{5t} & -\frac{7\,e^{5t}}{2} + \frac{9\,e^{3t}}{2} - e^{2t} & \frac{3\,e^{5t}}{2} - \frac{e^{3t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{10\,e^{2t}}{3} - 3\,e^{3t} + \frac{2\,e^{5t}}{3}\right) c_1 + \left(\frac{7\,e^{5t}}{6} - \frac{9\,e^{3t}}{2} + \frac{10\,e^{2t}}{3}\right) c_2 + \left(-\frac{e^{5t}}{2} + \frac{e^{3t}}{2}\right) c_3 \\ \left(3\,e^{3t} - \frac{7\,e^{2t}}{3} - \frac{2\,e^{5t}}{3}\right) c_1 + \left(-\frac{7\,e^{2t}}{3} - \frac{7\,e^{5t}}{6} + \frac{9\,e^{3t}}{2}\right) c_2 + \left(\frac{e^{5t}}{2} - \frac{e^{3t}}{2}\right) c_3 \\ \left(3\,e^{3t} - e^{2t} - 2\,e^{5t}\right) c_1 + \left(-\frac{7\,e^{5t}}{2} + \frac{9\,e^{3t}}{2} - e^{2t}\right) c_2 + \left(\frac{3\,e^{5t}}{2} - \frac{e^{3t}}{2}\right) c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(-6c_1 - 9c_2 + c_3)e^{3t}}{2} + \frac{(4c_1 + 7c_2 - 3c_3)e^{5t}}{6} + \frac{10\,e^{2t}(c_1 + c_2)}{3} \\ \frac{(6c_1 + 9c_2 - c_3)e^{3t}}{2} + \frac{(-4c_1 - 7c_2 + 3c_3)e^{5t}}{6} - \frac{7\,e^{2t}(c_1 + c_2)}{3} \\ \frac{(6c_1 + 9c_2 - c_3)e^{3t}}{2} + \frac{(-4c_1 - 7c_2 + 3c_3)e^{5t}}{2} - e^{2t}(c_1 + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

20.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 1 \\ -3 & -6 & 6-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$
$$\lambda_2 = 5$$
$$\lambda_3 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue
3	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & -6 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ -3 & -6 & 4 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \implies \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -6 & 4 & 0 \end{array}\right]$$

$$R_3 = R_3 - 3R_1 \implies \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 7 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & -3 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & -3 & 7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{10t}{3}, v_2 = \frac{7t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{10t}{3} \\ \frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{10t}{3} \\ \frac{7t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{10t}{3} \\ \frac{7t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{10}{3} \\ \frac{7}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{10t}{3} \\ \frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{10}{3} \\ \frac{7}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{10t}{3} \\ \frac{7t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \\ 3 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} - (3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -3 & -6 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & -6 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -3 & -6 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{9}{2} & \frac{9}{2} & 0 \end{array}\right]$$

$$R_3 = R_3 - 9R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -1 & -1 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -1 & -1 \\ 1 & -2 & 1 \\ -3 & -6 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & -1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ -3 & -6 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & -1 & -1 & 0 \\ 0 & -\frac{9}{4} & \frac{3}{4} & 0 \\ -3 & -6 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{4} \Longrightarrow \left[\begin{array}{ccc|c} -4 & -1 & -1 & 0 \\ 0 & -\frac{9}{4} & \frac{3}{4} & 0 \\ 0 & -\frac{21}{4} & \frac{7}{4} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{7R_2}{3} \Longrightarrow \left[\begin{array}{ccc|c} -4 & -1 & -1 & 0 \\ 0 & -\frac{9}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -4 & -1 & -1 \\ 0 & -\frac{9}{4} & \frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{3}, v_2 = \frac{t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{t}{3} \\ \frac{t}{3} \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
3	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$
5	1	1	No	$\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} -\frac{10}{3} \\ \frac{7}{3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{3t} \\ &= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{3t} \end{aligned}$$

Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{5t} \\ &= \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} e^{5t} \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{2t}$$
$$= \begin{bmatrix} -\frac{10}{3} \\ \frac{7}{3} \\ 1 \end{bmatrix} e^{2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -e^{3t} \\ e^{3t} \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{e^{5t}}{3} \\ \frac{e^{5t}}{3} \\ e^{5t} \end{bmatrix} + c_3 \begin{bmatrix} -\frac{10\,e^{2t}}{3} \\ \frac{7\,e^{2t}}{3} \\ e^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -c_1 e^{3t} - \frac{c_2 e^{5t}}{3} - \frac{10 c_3 e^{2t}}{3} \\ c_1 e^{3t} + \frac{c_2 e^{5t}}{3} + \frac{7 c_3 e^{2t}}{3} \\ c_1 e^{3t} + c_2 e^{5t} + c_3 e^{2t} \end{bmatrix}$$

20.2.3 Maple step by step solution

Let's solve

$[x'(t) = x(t) - y(t) - z(t), y'(t) = x(t) + 3y(t) + z(t), z'(t) = -3x(t) - 6y(t) + 6z(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & -6 & 6 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2, \begin{bmatrix} -\frac{10}{3} \\ \frac{7}{3} \\ 1 \end{bmatrix}\right], \left[3, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right], \left[5, \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} -\frac{10}{3} \\ \frac{7}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{2t} \cdot \begin{bmatrix} -\frac{10}{3} \\ \frac{7}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[3, \left[\begin{array}{c} -1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{3t} \cdot \left[\begin{array}{c} -1 \\ 1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[5, \left[\begin{array}{c} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = \mathrm{e}^{5t} \cdot \left[\begin{array}{c} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 \mathrm{e}^{2t} \cdot \left[\begin{array}{c} -\frac{10}{3} \\ \frac{7}{3} \\ 1 \end{array}\right] + c_2 \mathrm{e}^{3t} \cdot \left[\begin{array}{c} -1 \\ 1 \\ 1 \end{array}\right] + c_3 \mathrm{e}^{5t} \cdot \left[\begin{array}{c} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y(t) \\ z(t) \end{array}\right] = \left[\begin{array}{c} -\frac{10 c_1 \mathrm{e}^{2t}}{3} - c_2 \mathrm{e}^{3t} - \frac{c_3 \mathrm{e}^{5t}}{3} \\ \frac{7 c_1 \mathrm{e}^{2t}}{3} + c_2 \mathrm{e}^{3t} + \frac{c_3 \mathrm{e}^{5t}}{3} \\ c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{3t} + c_3 \mathrm{e}^{5t} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{10 c_1 \mathrm{e}^{2t}}{3} - c_2 \mathrm{e}^{3t} - \frac{c_3 \mathrm{e}^{5t}}{3}, y(t) = \frac{7 c_1 \mathrm{e}^{2t}}{3} + c_2 \mathrm{e}^{3t} + \frac{c_3 \mathrm{e}^{5t}}{3}, z(t) = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{3t} + c_3 \mathrm{e}^{5t}\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 74

```
dsolve([diff(x(t),t)=x(t)-y(t)-z(t),diff(y(t),t)=x(t)+3*y(t)+z(t),diff(z(t),t)=-3*x(t)-6*y(t
```

$$x(t) = c_1 e^{3t} + c_2 e^{2t} + c_3 e^{5t}$$
$$y(t) = -c_1 e^{3t} - \frac{7c_2 e^{2t}}{10} - c_3 e^{5t}$$
$$z(t) = -c_1 e^{3t} - \frac{3c_2 e^{2t}}{10} - 3c_3 e^{5t}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 217

```
DSolve[{x'[t]==x[t]-y[t]-z[t],y'[t]==x[t]+3*y[t]+z[t],z'[t]==3*x[t]-6*y[t]+6*z[t]},{x[t],y[t
```

$$x(t) \to -\frac{1}{45} e^{2t} \left(5(c_1 + 10c_2)e^{2t} \cos\left(\sqrt{5}t\right) + \sqrt{5}(7c_1 - 11c_2 + 9c_3)e^{2t} \sin\left(\sqrt{5}t\right) - 50(c_1 + c_2)\right)$$
$$y(t) \to \frac{1}{45} e^{2t} \left(5(c_1 + 10c_2)e^{2t} \cos\left(\sqrt{5}t\right) + \sqrt{5}(7c_1 - 11c_2 + 9c_3)e^{2t} \sin\left(\sqrt{5}t\right) - 5(c_1 + c_2)\right)$$
$$z(t) \to (c_1 + c_2)\left(-e^{2t}\right) + (c_1 + c_2 + c_3)e^{4t} \cos\left(\sqrt{5}t\right) + \frac{(c_1 - 8c_2 + 2c_3)e^{4t} \sin\left(\sqrt{5}t\right)}{\sqrt{5}}$$